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Treatment of singularities using the boundary element method

Junhua Steven He
Lehigh University

TREATMENT OF SINGULARITIES USING THE BOUNDARY ELEMENT METHOD

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JUN STEVEN HE

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Date

J DA Walker

J. D. A. Walker
Thesis Advisor

F. Erdogan

F. Erdogan
Chairman, Department of Mechanical Engineering and Mechanics

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ABSTRACT

A modification of the classical boundary element method is presented for the numerical solution of the Laplace equation when singularities occur on the boundary. The approach provides a local treatment of boundary singularities by incorporating the analytical nature of the solution near the singularity directly in the numerical algorithm. The modified method is applied to some typical examples and numerical results are given.

1. Introduction

Problems involving Laplace or Poisson equations arise in a wide variety of engineering applications, and in recent years the boundary element method has become a popular solution technique. In this method, the governing field equations are recast into a system of coupled integral equations which apply only on the boundary of the solution domain. The method uses the known boundary data to compute the unknown boundary data and then the solution at interior points, if required. This means that the system of algebraic equations generated by the boundary integral equation method is considerably smaller than that generated by an equivalent finite difference method or finite element method. It follows that the boundary integral equation method is an effective tool for the numerical solution of the Laplace equation.

It is also well known that the presence of boundary singularities tends to degrade the accuracy of the numerical solution. Consequently, considerable attention has been given in recent times to seeking modifications of the classical method in which special treatment is afforded to singular points. In particular, Symm (1977) showed how the classical boundary integral equation method could be globally modified to incorporate the analytical nature of a singularity whenever it occured on the boundary of the solution domain. Later on, Xanthis et al. (1980) suggested a method in which the analytical nature of the singularity is incorporated into the boundary integral method by introduction of special functional behavior over those segments of the boundary nearest singular point on the boundary. More recently, Ingham and Kelmanson (1984) developed an alternative method of local treatment of the boundary singularity.

In this thesis, another method of local treatment of boundary singularities is presented. It is illustrated by an application to some typical examples of two dimensional steady state heat transfer. Analytical results for all the integrations associated with the boundary integral method that are generated by incorporating the analytical nature of the singularity have been obtained. Numerical results obtained with the new procedure are found to compare with those obtained from Symm's (1977) method.

The basic ideas behind the boundary integral method are described in Chapter 2. In Chapter 3, the singularities due to a sudden change of boundary condition are discussed and some current methods used to treat the singularities are described. In Chapter 4, the ~~analytic~~ results needed for local treatment of singularities are presented.

In Chapter 5, the method is applied to two relatively difficult problems and the results are shown to compare quite favorably with previous numerical solutions of the problem.

2. The Boundary Integral Method

2.1 The Laplace Equation

The Laplace equation is usually the mathematical statement of some conservation principle. For example, in steady state heat conduction problems, the material temperature T(x,y) satisfies:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \tag{2.1}$$

which is obtained by applying the principle of conservation of energy to a differential element of the material. A typical application is illustrated in figure 2.1(a) where two sides of a rectangular region are at a given temperature, one side is insulated and the bottom face is exposed to a free convective flow. The objective in this problem is to determine the temperature distribution within the solid by solving the Laplace equation (2.1) subject to the given thermal boundary conditions. Another classical example is to evaluate the deflection of a membrane which is stretched over some region D of the x-y plane bounded by a curve C (figure 2.1(b)). The displacement $\omega(x,y)$ is governed by

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = 0, \tag{2.2}$$

with given ω on the boundary curve C.

In general, the Laplace equation is

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{2.3}$$

for two dimensional problems, or

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \tag{2.4}$$

for three dimensional problems. Boundary conditions for a well-posed problem must be specified on a closed curve in two dimensions and on a closed surface in three dimensions. Generally, such boundary conditions involve either known values of ϕ or its normal derivative $\frac{\partial \phi}{\partial n}$. The first type of problem is known as the "Dirichlet Problem" for a domain D wherein the value of ϕ is specified everywhere on the boundary C (figure 2.2). One physical interpretation of the solution $\phi(x,y)$ is that it is the temperature

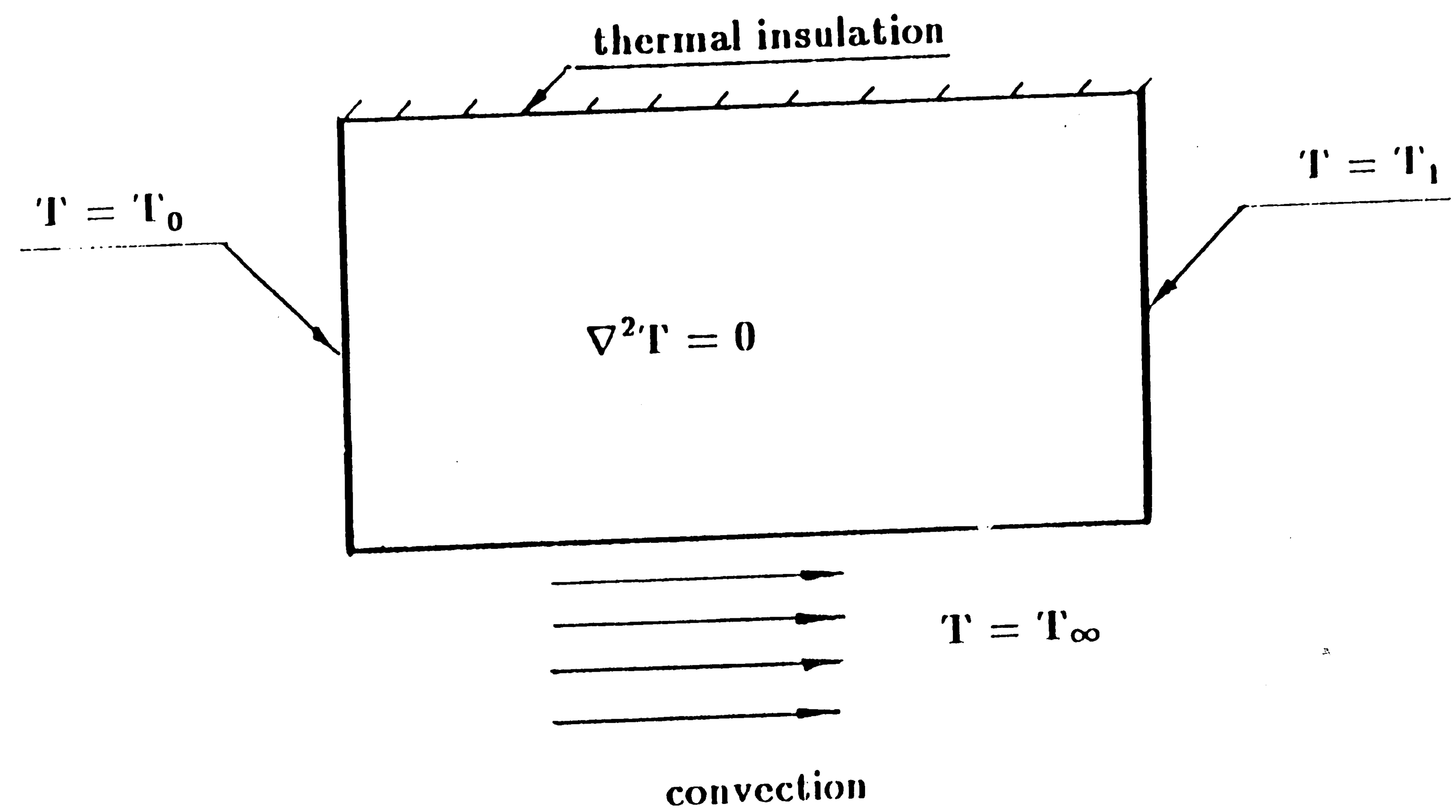


Figure 2. 1 (a) A typical thermal problem

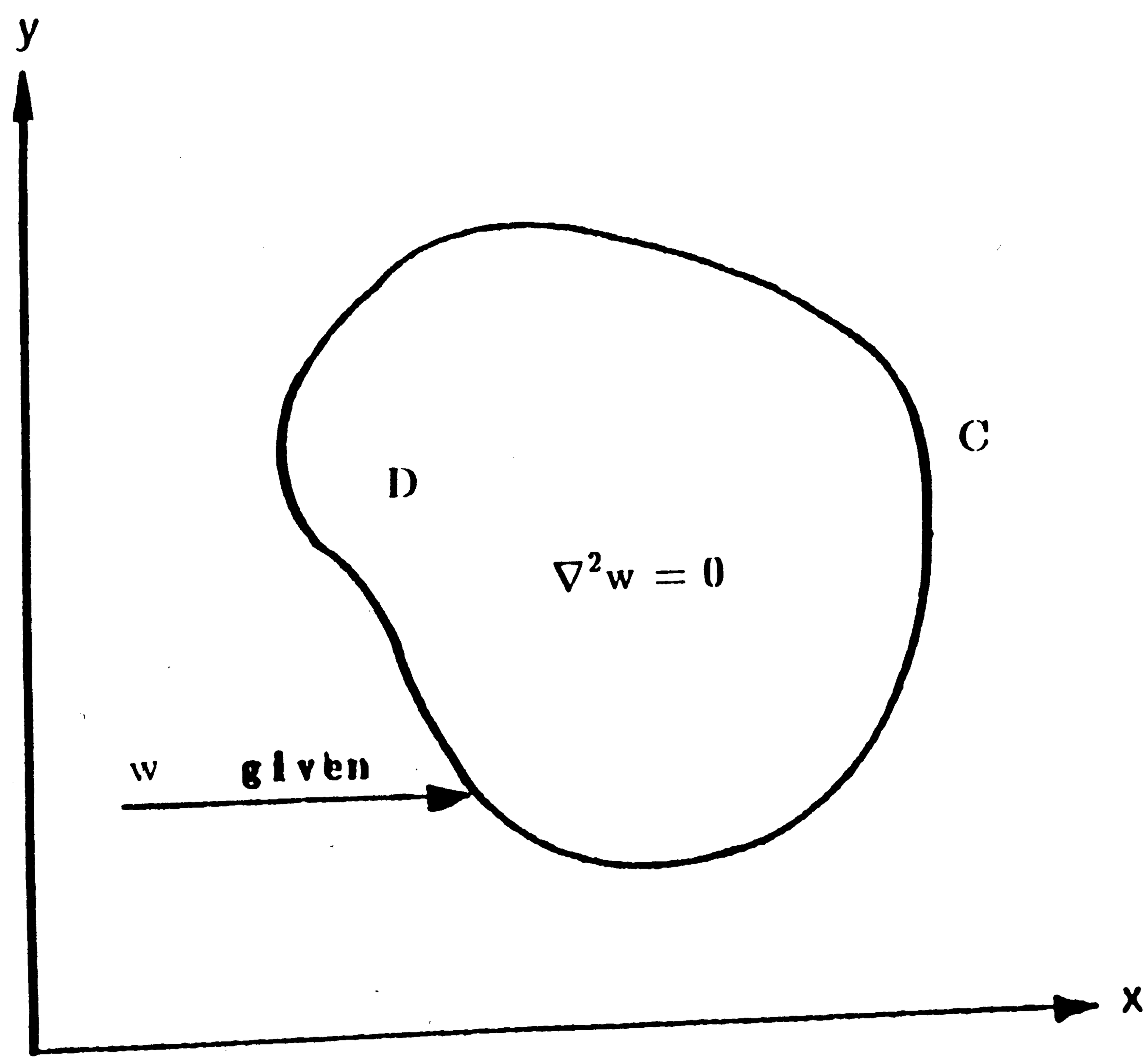


Figure 2. 1 (b) Stretched membrane.

distribution in a uniform heat-conducting body occupying the domain D, when the temperature distribution on the boundary C is fixed.

The second type of problem is known as a "Neumann Problem" for the domain D when the value of the normal derivative of ϕ is specified at all locations on the boundary C. (figure 2.3). It is common to denote the outward normal derivative of ϕ on the boundary as $\phi_n = \partial\phi/\partial n$ and call it the flux since physically it represents a flow of some quantity across the boundary. In general, it is also necessary to specify the value of ϕ for at least one point on the boundary to render the solution unique.

The third type of problem is known as "Robin's problem" when a linear combination of ϕ and ϕ_n is specified on the boundary. A typical example of this boundary condition occurs in convection heat transfer where a surface is cooled or heated by a moving fluid stream (figure 2.1(a)). Nonlinear boundary conditions are also possible but these are outside of the scope of this thesis.

In engineering practice, it is often necessary to deal with "mixed" boundary conditions where ϕ is specified along a portion of the boundary and ϕ_n is given along the other parts. In this thesis, such "mixed" boundary conditions are of interest since a singularity often occurs at the location where the boundary condition changes from one form to another.

2.2 Green's Indentity and Green's Functions

Consider now any two scalar functions $\phi(x,y,z)$ and $\psi(x,y,z)$ having continuous first derivatives and defined in a domain D. For any vector $\vec{Q}$ the Gauss divergence theorem applied for a volume V yields

$$\int_V \nabla \cdot \vec{Q}\, dV = \int_s \vec{n} \cdot \vec{Q}\, ds, \tag{2.5}$$

where the first integral is carried out over the volume V and the second is over the bounding surface s(c.f. figure 2.4(a)); in addition, $\vec{n}$ is the outward normal vector to s. Substituting $\vec{Q} = \phi\nabla\psi$ into equation (2.5) gives

$$\int_V \nabla\cdot(\phi\nabla\psi)dV = \int_s \vec{n}\cdot(\phi\nabla\psi)\, ds. \tag{2.6}$$

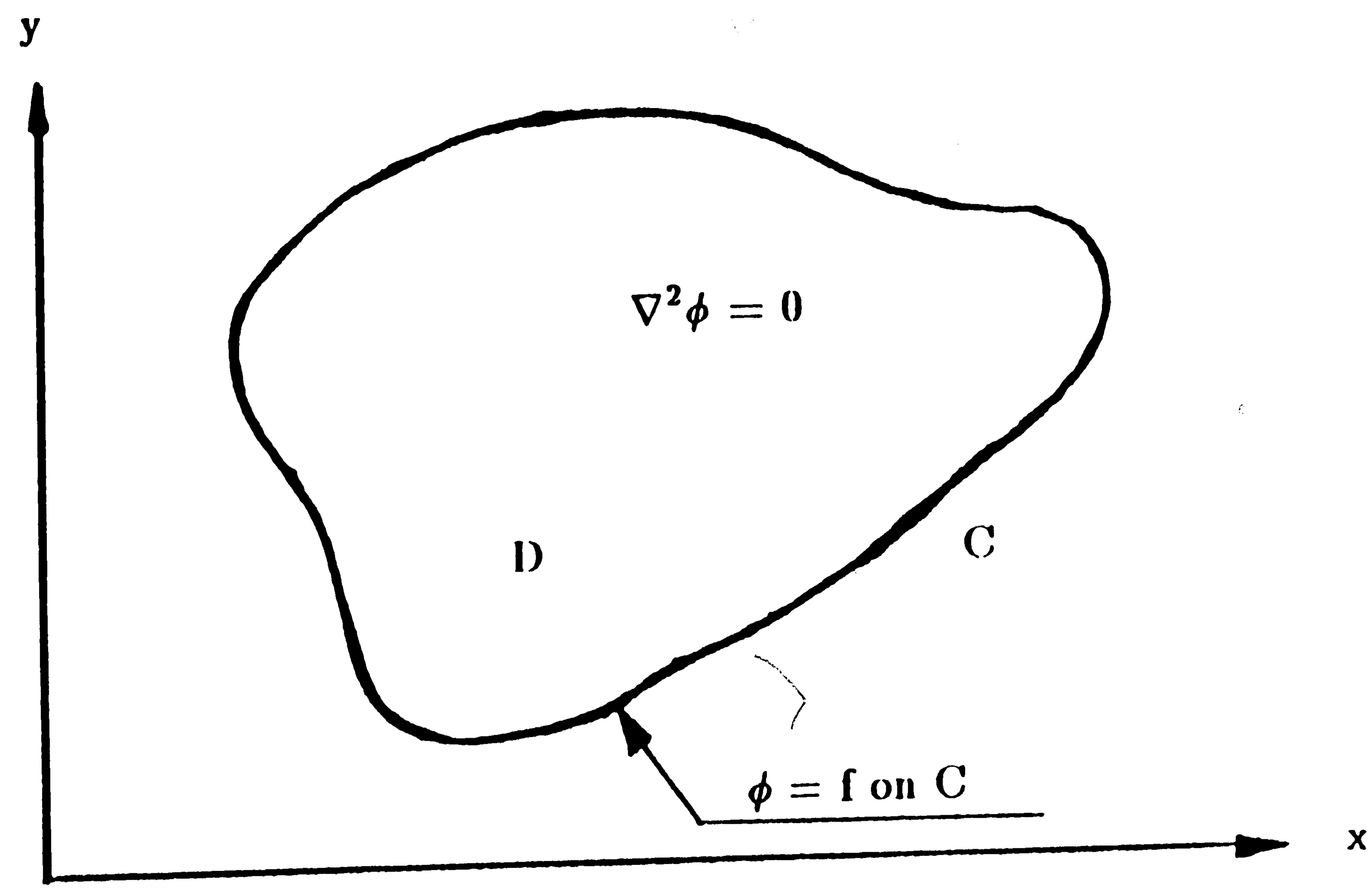


Figure 2. 2. Dirichlet problem.

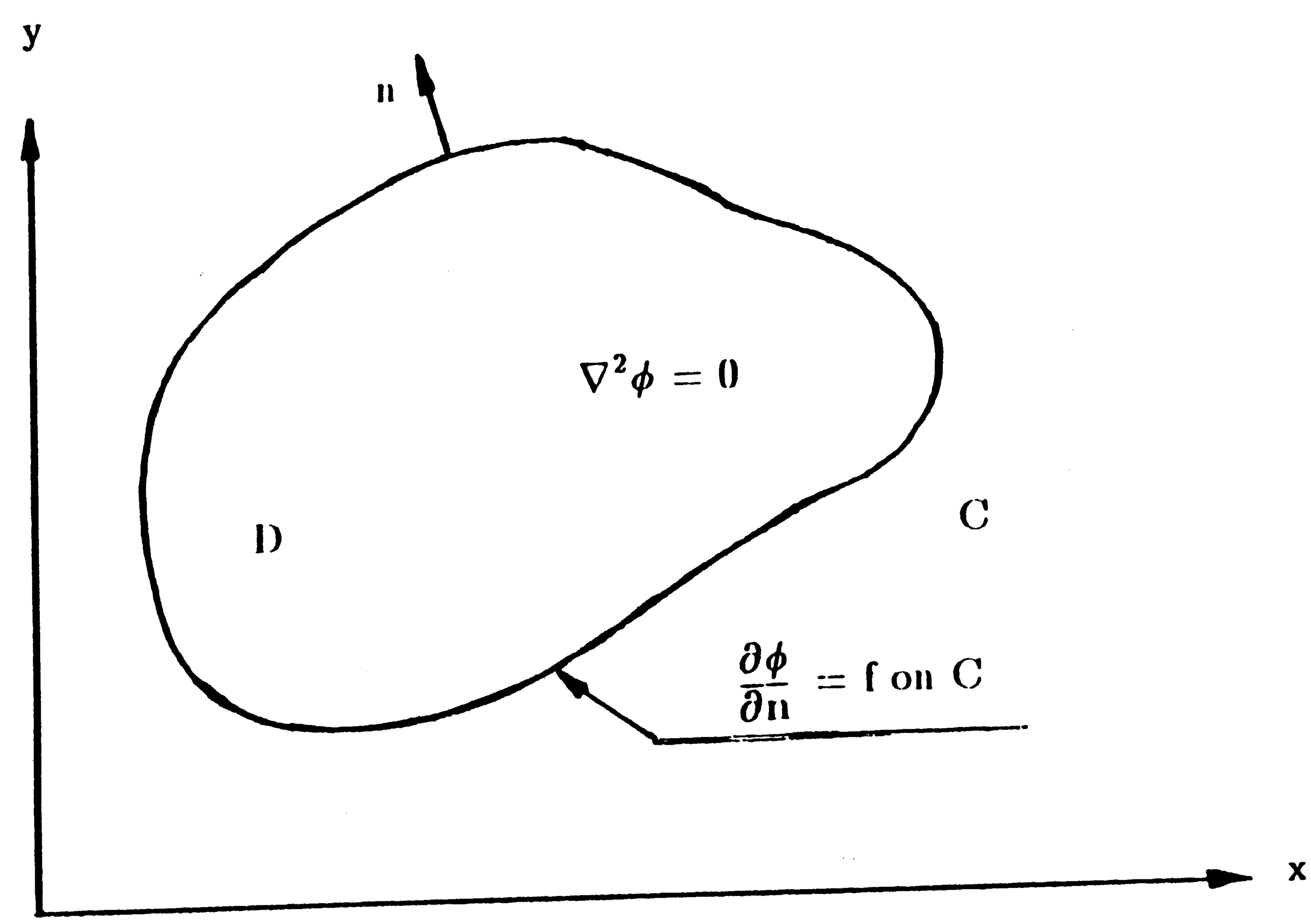


Figure 2. 3. Neumann problem.

Since

$$\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla\cdot\psi, \tag{2.7}$$

and

$$\vec{n}\cdot(\phi\nabla\psi) = \phi\,\vec{n}\cdot\nabla\psi = \phi\,\frac{\partial\psi}{\partial n}, \tag{2.8}$$

it follows that equation (2.6) can be written as

$$\int_V [\nabla\phi\cdot\nabla\psi + \phi\nabla^2\cdot\psi]\,dV = \int_s \phi\,\frac{\partial\psi}{\partial n}\,ds. \tag{2.9}$$

This equation is known as "Green's First Identity". But ϕ and ψ can be interchanged to give

$$\int_V [\nabla\psi\cdot\nabla\phi + \psi\nabla^2\cdot\phi]\,dV = \int_s \psi\,\frac{\partial\psi}{\partial n}\,ds. \tag{2.10}$$

Subtracting this equation from (2.9), yields "Green's Second Identity", viz.

$$\int_V [\phi\nabla^2\psi - \psi\nabla^2\,\phi]\,dV = \int_s (\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\psi}{\partial n})ds. \tag{2.11}$$

For two dimensional problems, the volume integral may be written as an area (x−y plane) integral and the surface integral may be converted into a contour integral, according to

$$\int_A [\phi\nabla^2\psi - \psi\nabla^2\,\phi]\,dA = \int_C (\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\psi}{\partial n})ds, \tag{2.12}$$

where the area A is bounded by the closed curve C in the x−y plane (see figure (2.4.b)).

The two dimensional Green's function is defined as a solution of the differential equation

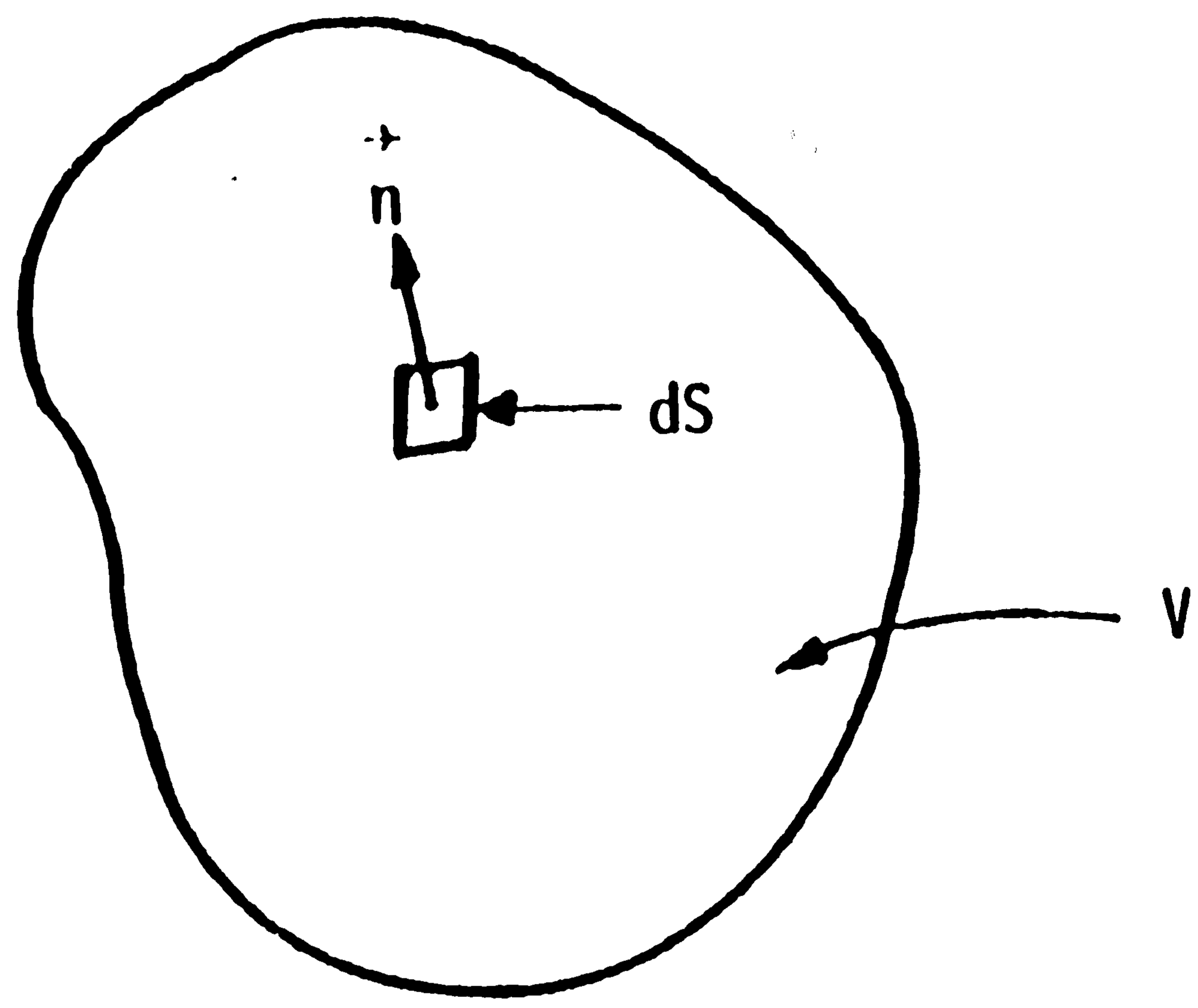


Figure 2. 4 (a) Gauss divergence theorem

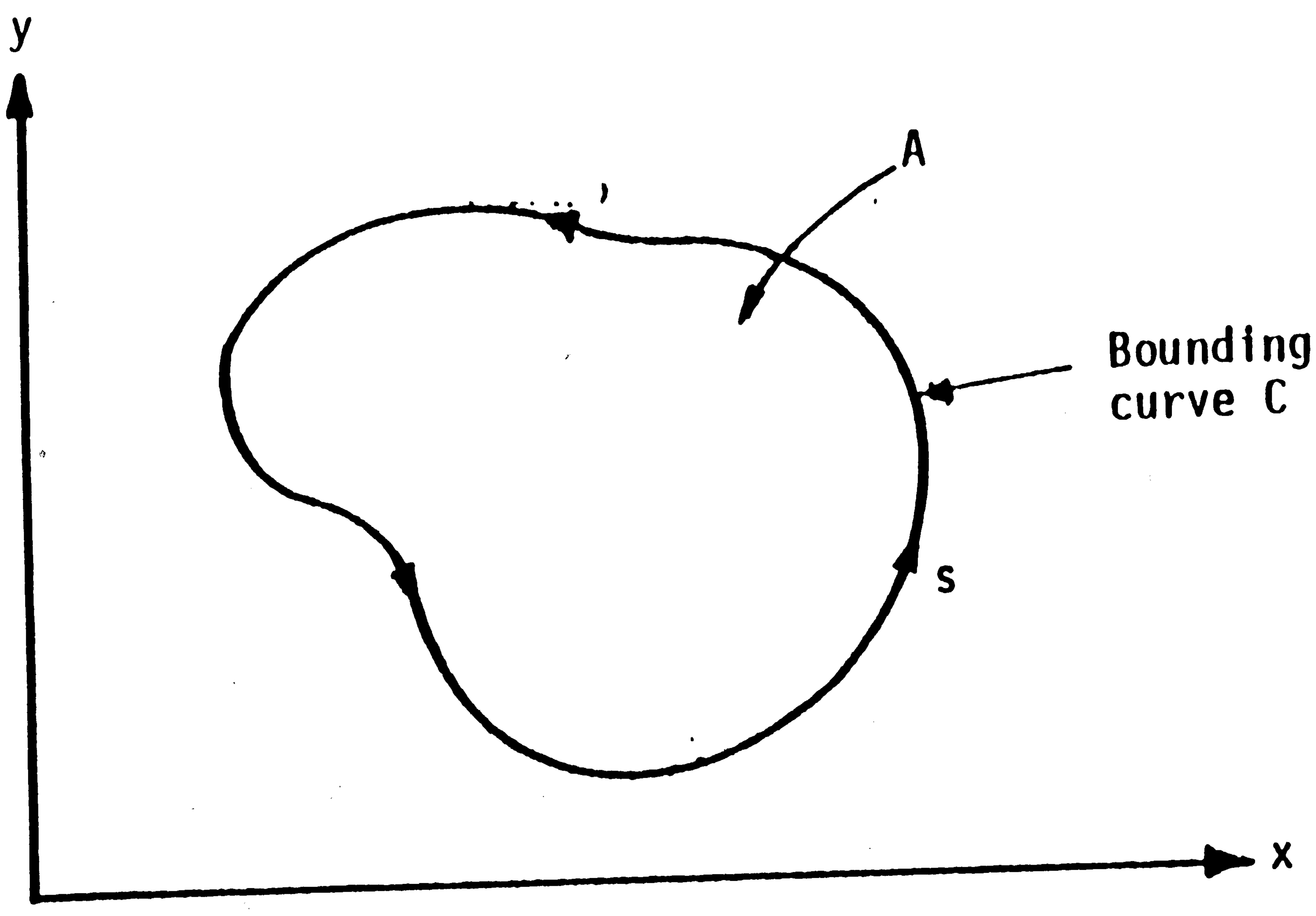


Figure 2. 4 (b) Area enclosed by contour C.

$$\nabla^2 G(x,y) = \delta(x - x_0, y - y_0), \tag{2.13}$$

where $\delta(\xi,\eta)$ is the two-dimensional delta function defined by

$$\delta(x - x_0, y - y_0) = 0, \quad \text{for} \quad (x,y) \neq (x_0,y_0), \tag{2.14}$$

and

$$\iint_A \delta\,(x-x_0, y-y_0)\,ds = \begin{cases} 0, & (x_0, y_0) \text{ not in } A \\ 1, & (x_0,y_0) \epsilon A \end{cases} . \tag{2.15}$$

The delta function has the so-called "sifting property",

$$\iint_A \delta\,(x-x_0, y-y_0)\,f(x,y)ds = f(x_0,y_0), \tag{2.16}$$

and may be interpreted as a point source of unit strength. Consider the radially symmetric solution of the differential equation (2.13), which in polar coordinates (R,θ) is,

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial G}{\partial R}\right) + \frac{1}{R^2}\left(\frac{\partial^2 G}{\partial \theta^2}\right) = \delta\,(R) \tag{2.17}$$

where $R = \sqrt{(x-x_0)^2 + (y-y_0)^2}$ measures radial distance from the point (x_0,y_0) as indicated in figure 2.5.

For a symmetric solution,

$$\frac{\partial G}{\partial \theta} = 0,$$

and equation becomes

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial G}{\partial R}\right) = \delta\,(R), \tag{2.18}$$

Unless $R = 0$, (i.e. $x=x_0$, $y=y_0$),

$$\delta(R) = 0 \quad \text{for} \quad R>0.$$

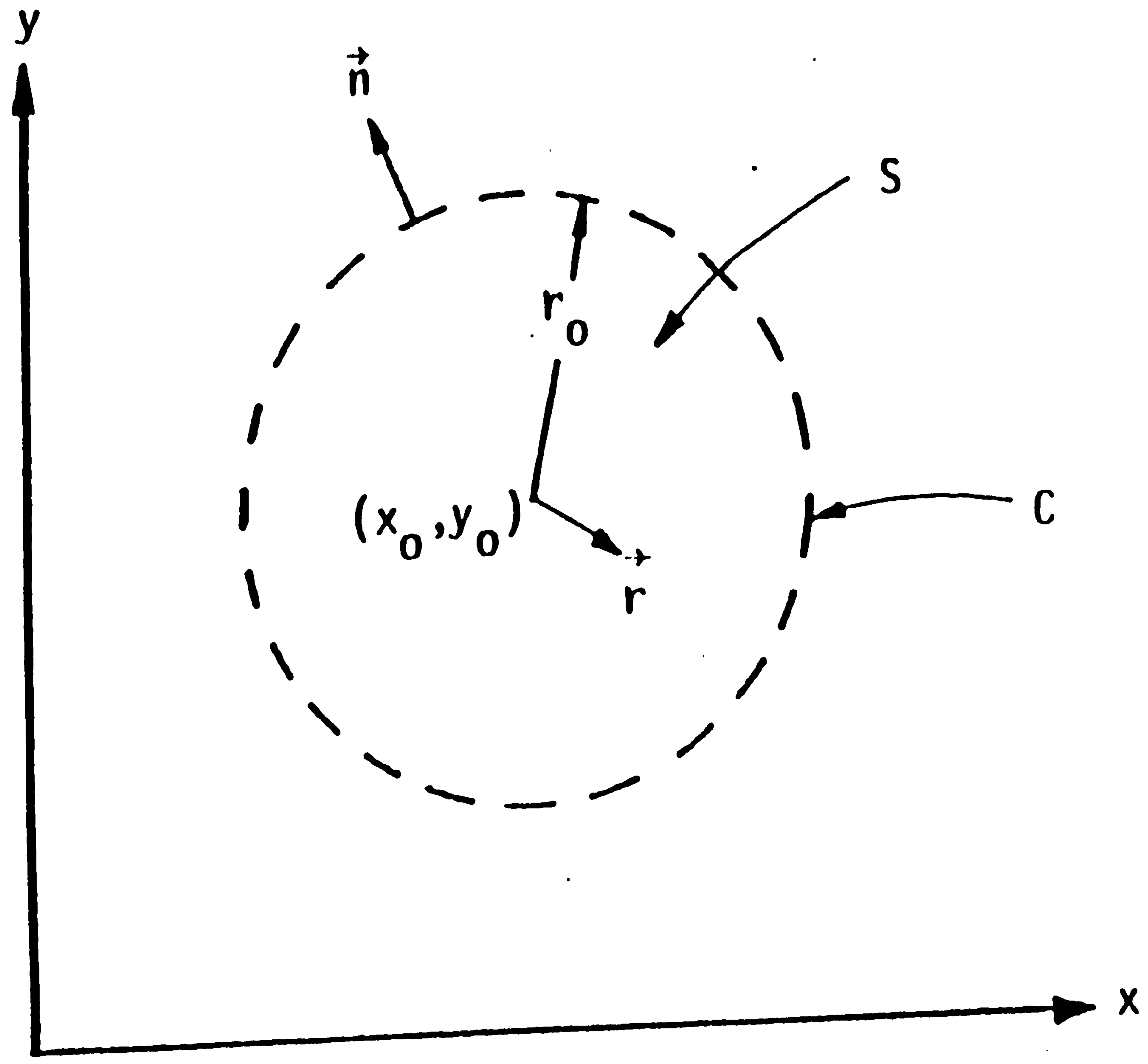


Figure 2. 5. Coordinate system near the sources point.

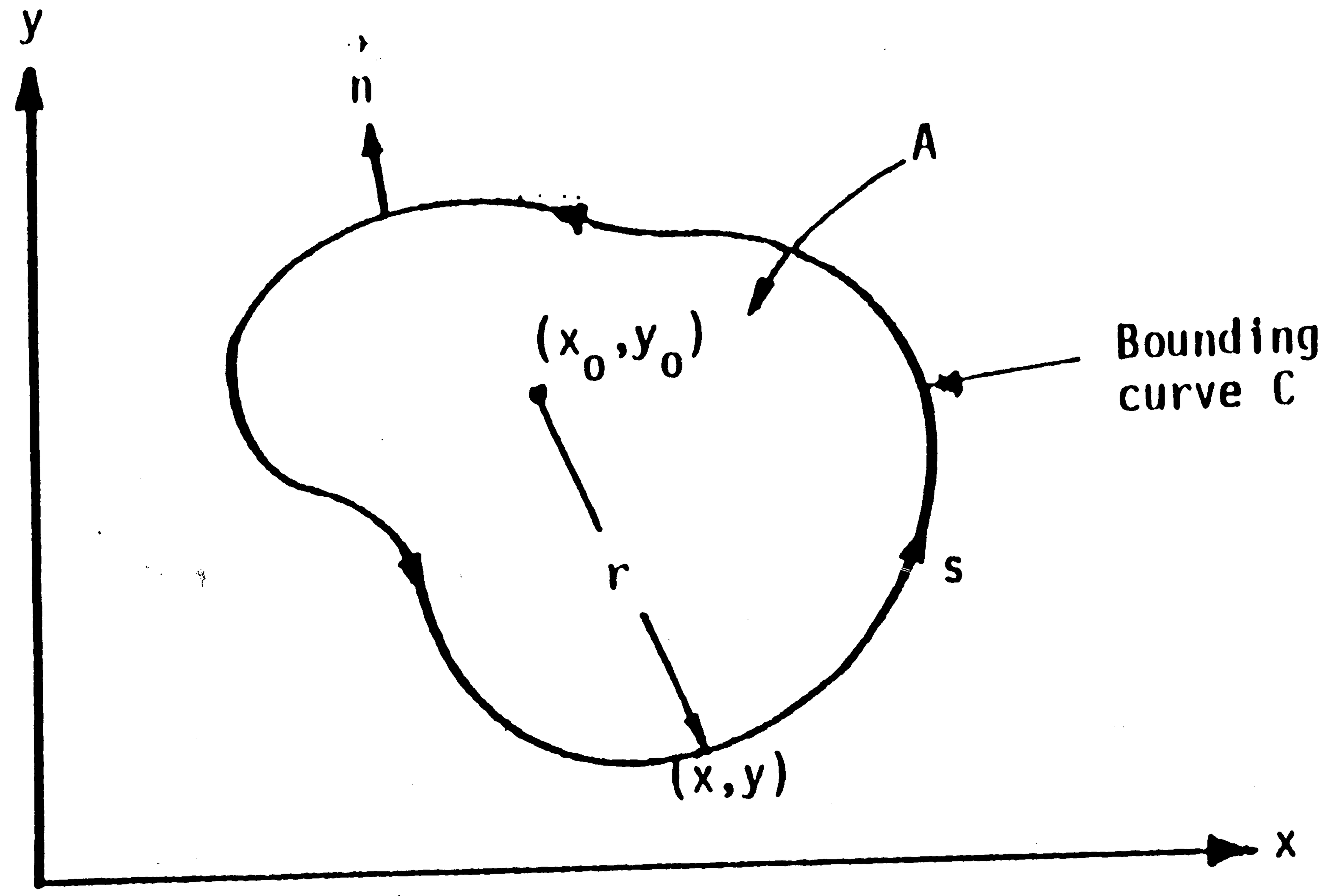


Figure 2. 6. Geometry associated with the boundary integral formula.

Thus, the general solution is given by

$$G(R) = A \log R + B, \tag{2.19}$$

where A and B are constants to be determined. To this end, equation (2.18) can be integrated over a circle C centered at (x_0, y_0) with arbitrary radius $R_0 \neq 0$ to obtain

$$\iint_A \nabla^2 G \, dx \, dy = \iint_A \delta(x-x_0, y-y_0) dx \, dy = 1. \tag{2.20}$$

The left hand can be simplified by using the two dimensional version of the divergence theorem, viz.

$$\iint_A \nabla \cdot \nabla G \, dx \, dy = \int_C \vec{n} \cdot \nabla G \, ds = \int_C \frac{\partial G}{\partial n} ds, \tag{2.21}$$

where $\vec{n}$ is the unit normal pointing outward from the to curve C, $\partial G/\partial n$ is the normal derivative in the direction $\vec{n}$ and ds is a differential element along C. In the present case, we have $ds = R_0 d\theta$ and

$$\int_C \frac{\partial G}{\partial n} ds = \int_0^{2\pi} \frac{\partial G}{\partial R} \Big|_{R=R_0} R_0 d\theta$$

$$= \int_0^{2\pi} \frac{\partial}{\partial R} (A \log R + B) \Big|_{R=R_0} R_0 d\theta,$$

$$= \int_0^{2\pi} \frac{A}{R_0} R_0 d\theta = 2\pi A.$$

Thus

$$A = \frac{1}{2\pi}. \tag{2.22}$$

On the other hand, B remains arbitrary. Since we are mainly interested in the singular part of Green's function G, it is natural to choose $B = 0$. Therefore,

$$G(R) = \frac{1}{2\pi} \log R \tag{2.23}$$

which is called the principal Green's function or the fundamental solution. Very often, Green's functions are constructed to satisfy specific boundary conditions for certain geometric situations. However, all Green's functions in two-dimensions contain a singular part and a regular part. The singular part is the fundamental solution while the regular part is constructed to satisfy certain boundary conditions for specific geometrical situations. In most applications of the boundary integral method, however, only the principal Green's function is needed.

2.3 The Boundary Integral Formulation in Two Dimensions

At this stage, Green's identities and the principal Green's function can be employed to obtain the boundary integral formulation in two dimensions. In equation(2.13), we take $\psi(x,y)$ to be the principal Green's function G such that

$$\nabla^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta\,(x-x_0,\, y-y_0),$$

and

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Then it follows that

$$\int_A [\phi \nabla^2 G - G \nabla^2 \phi]\, dA = \int_C (\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}) ds. \tag{2.24}$$

However, the left side of equation (2.24) becomes

$$\int_A \phi\, \delta\,(x-x_0,\, y-y_0)\, dA = \phi(x_0,y_0),$$

and consequently

$$\phi(x_0,y_0) = \int_C [\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}]\, ds, \tag{2.25}$$

where $(x_0,\, y_0)$ is an arbitrary point within the area A. C is the boundary curve of A and $\frac{\partial}{\partial n}$ denotes the derivative in the outward normal direction to curve C (see figure 2.6).

Equation (2.25) yields a formula to compute ϕ at any point $(x_0,\, y_0)$ in area A in terms of a line integral around the boundary curve C, provided that both ϕ or $\phi_n = \frac{\partial \phi}{\partial n}$ are known on C. In general, however, either ϕ or ϕ_n is given on each segment of the boundary curve C, but not both. An algorithm must be found to determine the unknown boundary data from the given boundary data. To this end, the principal Green's function in equation (2.23) is substituted into equation (2.25) which becomes

$$\phi(x_0,y_0) = \frac{1}{2\pi} \int_C [\phi \frac{\partial}{\partial n}\log R - \log R \frac{\partial \phi}{\partial n}]\, ds \tag{2.26}$$

for (x_0,y_0) within A. Now, let the point (x_0,y_0) approach the contour C from the interior and suppose for the moment that the limiting location of (x_0,y_0) is at a "corner" on C, with θ_c measuring the interior angle of the contour as indicated in figure 2.7 . For a smooth contour, it is easy to see that $\theta_c = \pi$. Then let C_ϵ be the arc of the small circle with radius ϵ and C' be the remaining portion of C. The integration along C can be divided into two parts, viz.

$$\int_C = \lim_{\epsilon \to 0} [\int_{C'} + \int_{C_\epsilon}] . \tag{2.27}$$

First, along C_ϵ, we have

$$\frac{\partial}{\partial n}\log R \,|_{R=\epsilon} = \frac{\partial}{\partial R} \log R \,|_{R=\epsilon} = \frac{1}{\epsilon} , \tag{2.28}$$

and $ds = \epsilon \, d\theta'$, where θ' is the polar angle ranging from 0 to $2\pi - \theta_c$. Thus,

$$\lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{C_\epsilon} [\, \phi \frac{\partial}{\partial n}\log R - \log R \frac{\partial \phi}{\partial R} \,]\, ds$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi-\theta_c} [\, \frac{\phi}{\epsilon} - \log \epsilon \frac{\partial \phi}{\partial n} \,]\, \epsilon \, d\theta',$$

$$= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi-\theta_c} \phi\, (x_0,y_0)\, d\theta',$$

$$= \frac{2\pi - \theta_c}{2\pi} \phi\, (x_0,y_0) . \tag{2.29}$$

Substituting in equation(2.26) and letting $\epsilon \to 0$ to yields

$$\theta_c \, \phi\, (x_0,y_0) = \int_C [\, \phi \frac{\partial}{\partial n}\log R - \log R \frac{\partial \phi}{\partial R} \,]\, ds \tag{2.30}$$

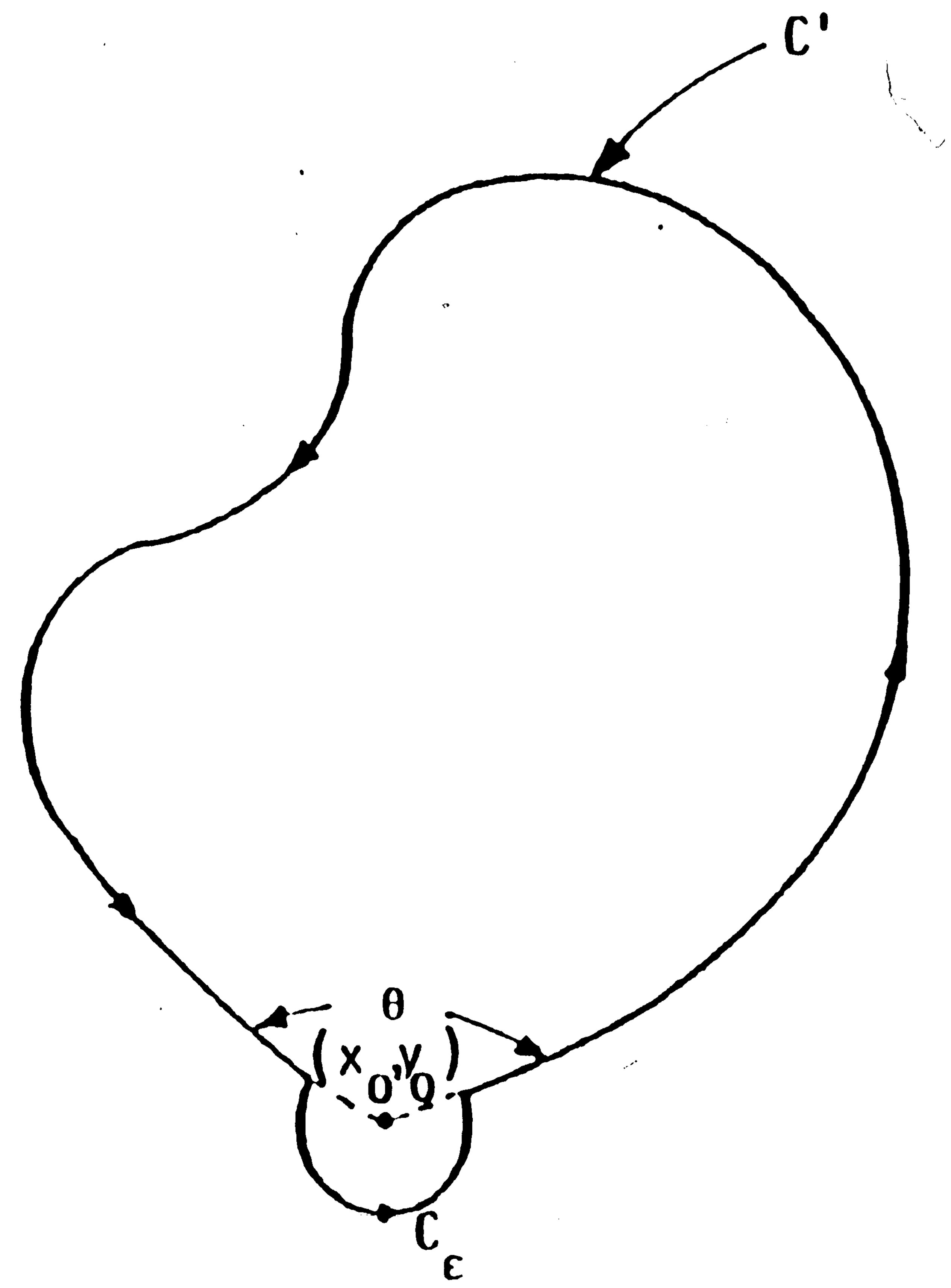


Figure 2. 7. Indenting the contour near the field point when (x_0, y_0) is at a corner of the contour.

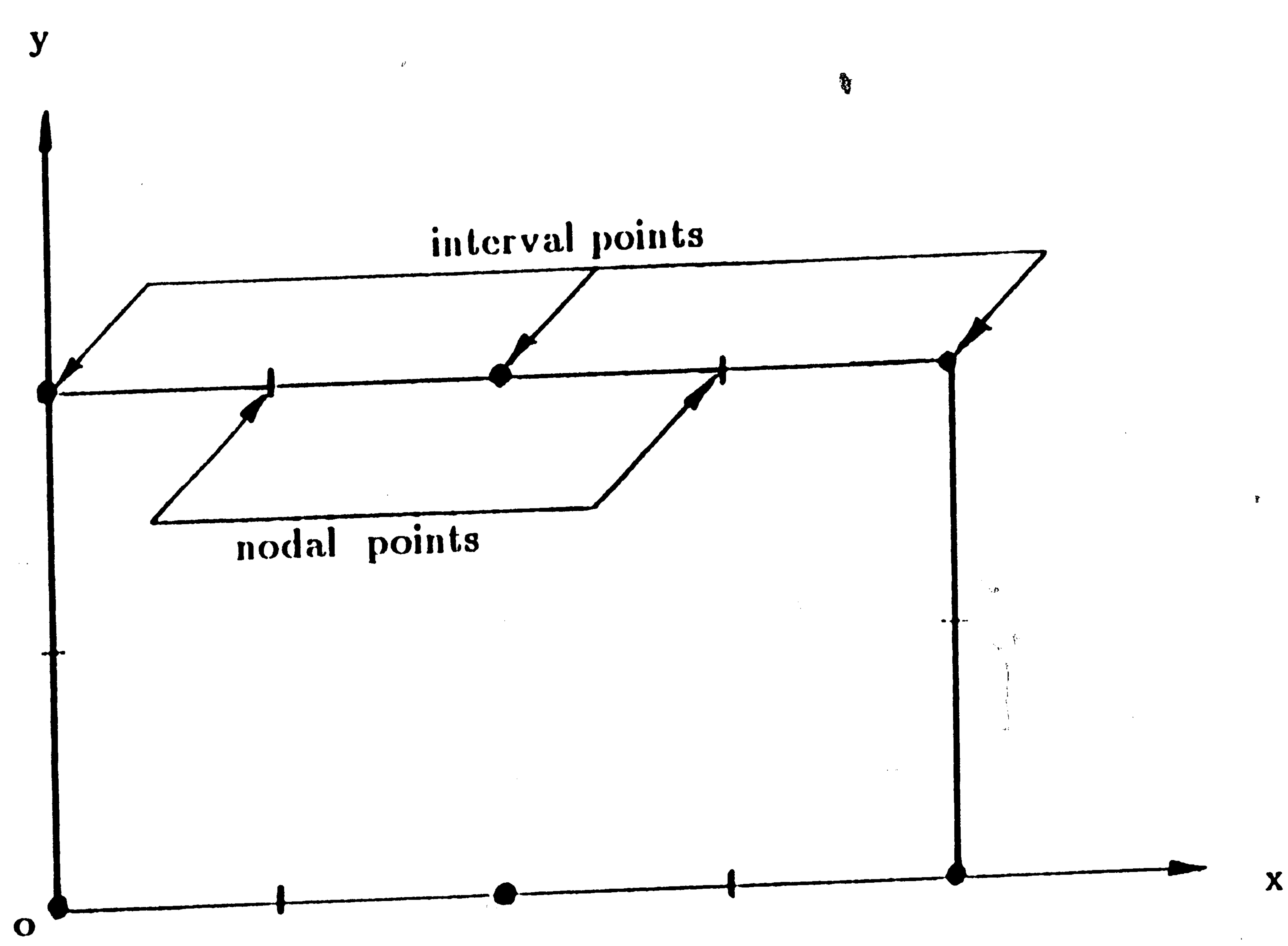


Figure 2. 8. Nodal points and interval points associated with CBEM.

for (x_0,y_0) on C at a corner. In the case of smooth contour C at (x_0,y_0), we have

$$\pi\phi\,(x_0,y_0) = \int_C [\,\phi\,\frac{\partial}{\partial n}\log R - \log R\,\frac{\partial\phi}{\partial R}\,]\,ds, \tag{2.31}$$

for (x_0,y_0) on C only. It is worthwhile to note that in equation (2.30) and (2.31), C is actually the contour C′ in the limit $\epsilon \to 0$ and, consequently, the integrals must be interpreted as Cauchy principal value integrals.

The main task of the boundary integral method is then to discretize the integral equation (2.30) to determine the unknown boundary data. Once the boundary data are fully determined (ie, all values of $\phi(x,y)$ and ϕ_n are known along C), the values of $\phi(x,y)$ in the interior may be evaluated by equation(2.26), viz.

$$\phi\,(x_0,y_0) = \frac{1}{2\pi}\int_C [\,\phi\,\frac{\partial}{\partial n}\log R - \log R\,\frac{\partial\phi}{\partial R}\,]\,ds \tag{2.32}$$

for (x_0,y_0) within A.

2.4 The Classical Boundary Element Method

There is a variety of ways to discretize the boundary contour C. In the classical boundary element method (CBEM), the contour C is split into a number of equal segments, the ends of which will be referred as interval points as indicated in figure 2.8; all corners or ends of a segment of a specific boundary condition are taken at interval points. In this study, the geometries of main interest are such that C consists of straight lines which meet at right angled corners. Then, all the corner points coincide with interval points. On the other hand, the nodal points are defined as the midpoints of each interval as indicated schematically in figure 2.8.

On each segment of the contour C, it is assumed that either ϕ or ϕ_n is known. Let s_j denote the jth interval point with coordinates (x_j,y_j); the next interval point is s_{j+1}, with coordinate $(x_{j+1},\, y_{j+1})$. The values of ϕ and ϕ_n at the nodal point of this segment are denoted by ϕ_j and $\phi_j{}'$ as shown in figure 2.9. The main idea in the CBEM is to assume ϕ and ϕ' are approximately constant over each interval and equal to their values at nodal point. Thus the values of ϕ_j and ϕ'_j may be removed from the integral over each piece of the boundary, viz.

$$\int_{s_j}^{s_{j+1}} \phi\,\frac{\partial}{\partial n}\,(\log R)\,ds \simeq \phi_j \int_{s_j}^{s_{j+1}} \frac{\partial\phi}{\partial R}\log R\,ds, \tag{2.33}$$

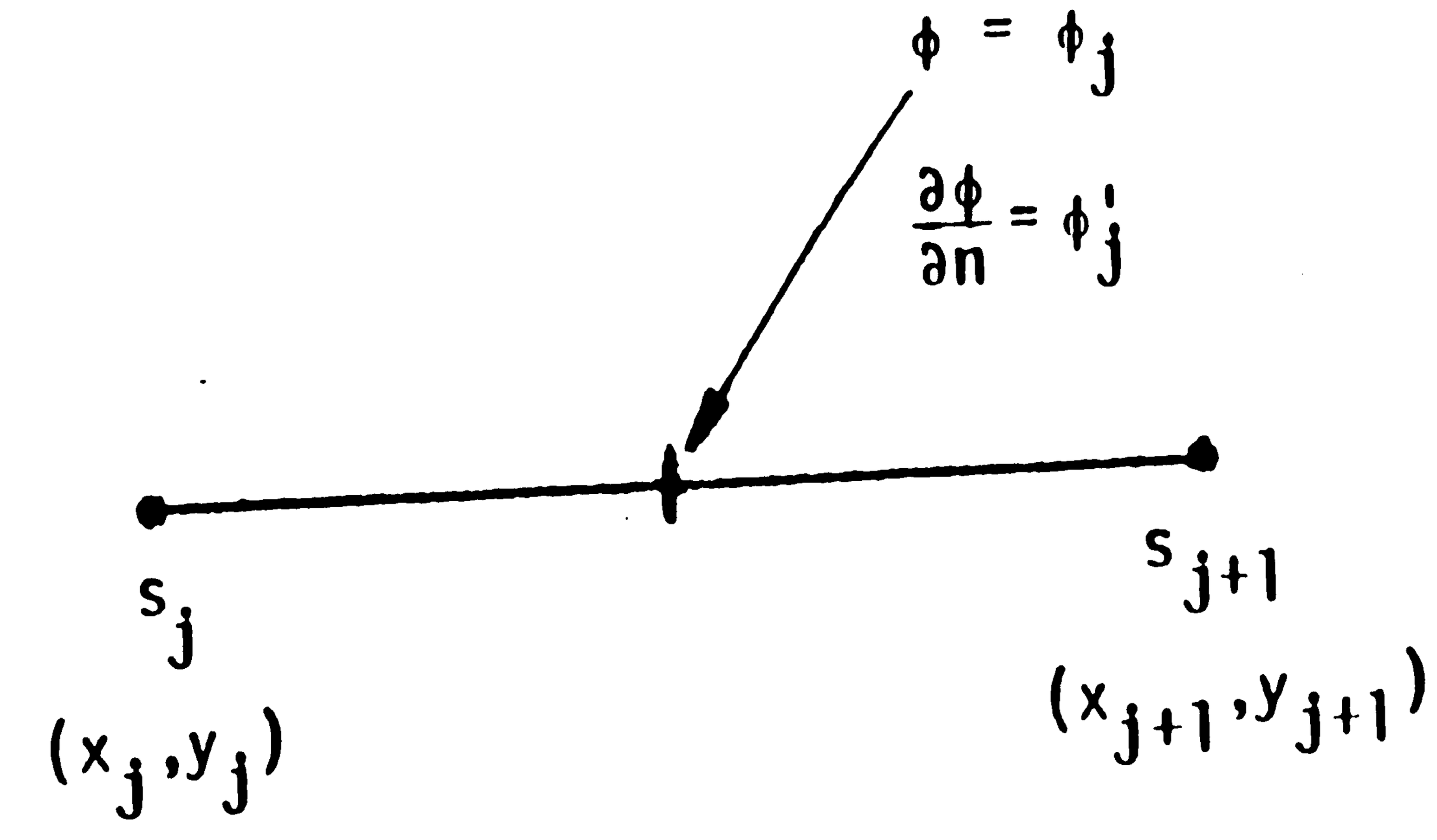


Figure 2. 9. The j th segment on the boundary.

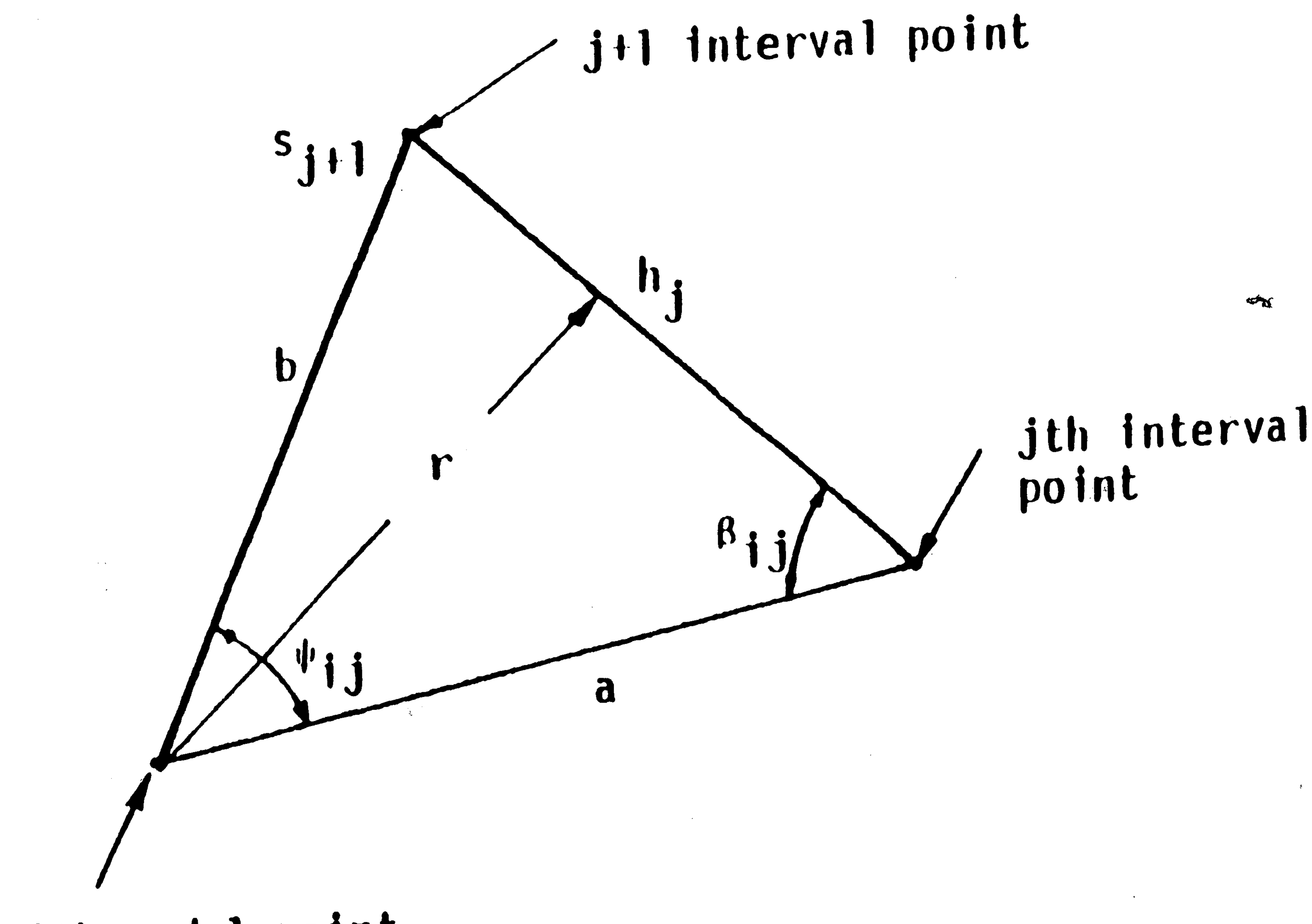


Figure 2. 10. Geometry and notation for analytical evaluation of coefficients in CBEM.

$$\int_{s_j}^{s_{j+1}} (\log R) \frac{\partial \phi}{\partial n} \, ds \simeq \phi_j' \int_{s_j}^{s_{j+1}} (\log R) \, ds. \tag{2.34}$$

It is worthwhile to recall that ϕ_j' is just a shorthand notation to denote the outward normal gradient of ϕ at nodal point of the jth segment; R is the radial distance from some other boundary point to the jth segment on the boundary.

Recall that equation (2.30) is valid for any boundary point (x_0,y_0) and, in the classical boundary element method, it is chosen to be the ith nodal point. The discretized form of the equation (2.30) is

$$\pi\phi_i = \sum_{j=1}^{n} \left(\phi_j \int_{s_j}^{s_{j+1}} \frac{\partial}{\partial n}(\log R_i) ds - \phi_j' \int_{s_j}^{s_{j+1}} (\log R_i) ds \right). \tag{2.35}$$

$$(i= 1,2,...,n)$$

Now define constants α_{ij} and β_{ij} by

$$\beta_{ij} = \int_{s_j}^{s_{j+1}} \frac{\partial}{\partial n}(\log R_i) ds, \tag{2.36}$$

$$\alpha_{ij} = \int_{s_j}^{s_{j+1}} (\log R_i) ds. \tag{2.37}$$

Then, equation (2.35) becomes

$$\pi\phi_i = \sum_{j=1}^{n} \beta_{ij} \, \phi_j - \sum_{j=1}^{n} \alpha_{ij} \, \phi_j'. \tag{2.38}$$

$$(i=1,2,...n)$$

The coefficients α_{ij} and β_{ij} may be obtained analytically for the contours which are composed of straight lines and, in this case, it can be shown that (Jaswon and Symm, 1977)

$$\alpha_{ij} = a_i \cos \beta_j \log(\frac{a_i}{b_i}) + h_j(\log b_i - 1) + a_i \psi_i \sin(\beta_j), \tag{2.39}$$

and

$$\beta_{ij} = \psi, \tag{2.40}$$

where a and b are distances from the ith nodal point to the ends of the the interval as indicated in Figure 2.10; ψ and β are respectively interval angles at nodal point P_i and interval point s_j; h_j is the mesh length of the jth interval.

For a more concise form of equation (2.38), we set

$$B_{ij} = \beta_{ij} - \pi\,\delta_{ij} \tag{2.41}$$

and

$$A_{ij} = \alpha_{ij} \tag{2.42}$$

where δ_{ij} is Kronecker delta. Then,

$$\sum_{j=1}^{N} B_{ij}\,\phi_j - \sum_{j=1}^{N} A_{ij}\,\phi_j{}' = 0 \tag{2.43}$$

$$(i=1,2,\ldots N)$$

or in matrix form

$$[B]\,\vec{\phi} = [A]\,\vec{\phi}' \tag{2.44}$$

When the element coefficients A_{ij} and B_{ij} have been evaluated, a rearrangement of equation (2.44) can be made in such a way so that all of the unknowns (either the ϕ_i or the ϕ_i') are put on the left side and with the known boundary data on the righthand side. The resulting form is

$$C\,\vec{X} = \vec{d}, \tag{2.45}$$

which represents N linear equations with N unknowns. The solution of the equation (2.45) is easily obtained using standard methods.

Note that as the mesh is decreased by a factor of 2, on each successive mesh the nodal points in the previous mesh become the interval points in the following mesh. To compare the accuracy of each calculation, we can compute values of ϕ and ϕ' on the contour at specific locations via the formula.

$$\pi\phi(s) = \sum_{j=1}^{n} \beta_{ij}\,\phi_j - \sum_{j=1}^{n} \alpha_{ij}\,\phi_j{}', \tag{2.46}$$

where s is any point on the boundary except for a corner point; ϕ and ϕ' are assumed to have been obtained from a solution of equation (2.45). The coefficients α_{sj} and β_{sj} can be evaluated for each point s on the boundary from equations(2.39) and (2.40). In a similar manner, we can compute the ϕ values in the interior points in terms of boundary values of ϕ and ϕ', according to

$$2\pi\phi(p) = \sum_{j=1}^{n} \beta_{pj}\,\phi_j - \sum_{j=1}^{n} \alpha_{pj}\,\phi_j{}' \tag{2.47}$$

where P is any interior point.

3. Singularities

3.1 Basic problems and mathematical model

In the solution of the Laplace equation, the occurence of singular fluxes or irregular behavior in ϕ near the boundary is quite common. In such regions, the variation in ϕ may be quite intense and the fluxes may be quite large and vary sharply. The term irregularity is used to imply that there is a neighborhood near a particular point where the solution is varying very rapidly. As a consequence, gradients of the solution will be large near this point and the normal derivative may be very large or even undefined. Generally speaking, singularities are expected at any point where there is a corner in the boundary contour or at any point where there is an abrupt change in boundary conditions as shown in figure 3.1.

Since most numerical methods are based on the assumption that the solution may be represented locally by a Taylor series expansion, relatively large numerical errors may be incurred in the vicinity of singularities. In problems with corners, the boundary condition is also often different on each side of the corner. In addition, even in cases where the boundary conditions are continuous at the corner from one side to another, the geometry itself can cause trouble in the numerical scheme.

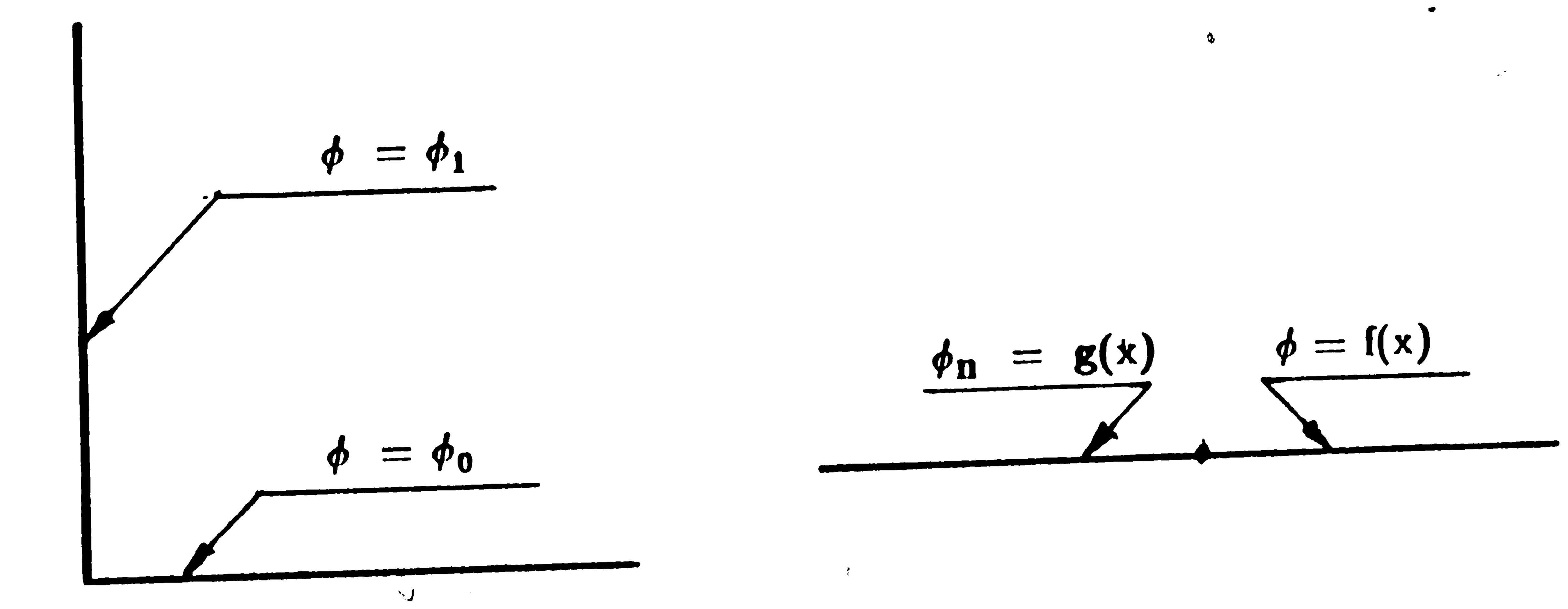
There is a wide variety of possible types of changes in boundary conditions along the boundary that may be encountered when solving Laplace equations. In this paper, however, our attention will be focussed on the situation where the boundary conditions at $x=x_0$ change from a given value $\phi=\phi_0$ for $x<x_0$ to an insulated condition for $x > x_0$ on a straight line as indicated in figure 3.2. It is expected that there must be a region near x_0 with intense variation in the solution for ϕ. To determine the nature of this variation, a polar coordinate system is chosen such that x_0 is the center of the coordinate system as indicated in figure 3.2. Then, the Laplace equation in polar form is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0. \tag{3.1}$$

Applying the method of separation of variables, viz.

$$\phi(r, \theta) = P(\theta)\, R(r), \tag{3.2}$$

and substituting into equation (3.1) yields



(a) Corner singularity

(b) Abrupt change in boundary conditions.

Figuure 3. 1. Expected singular points

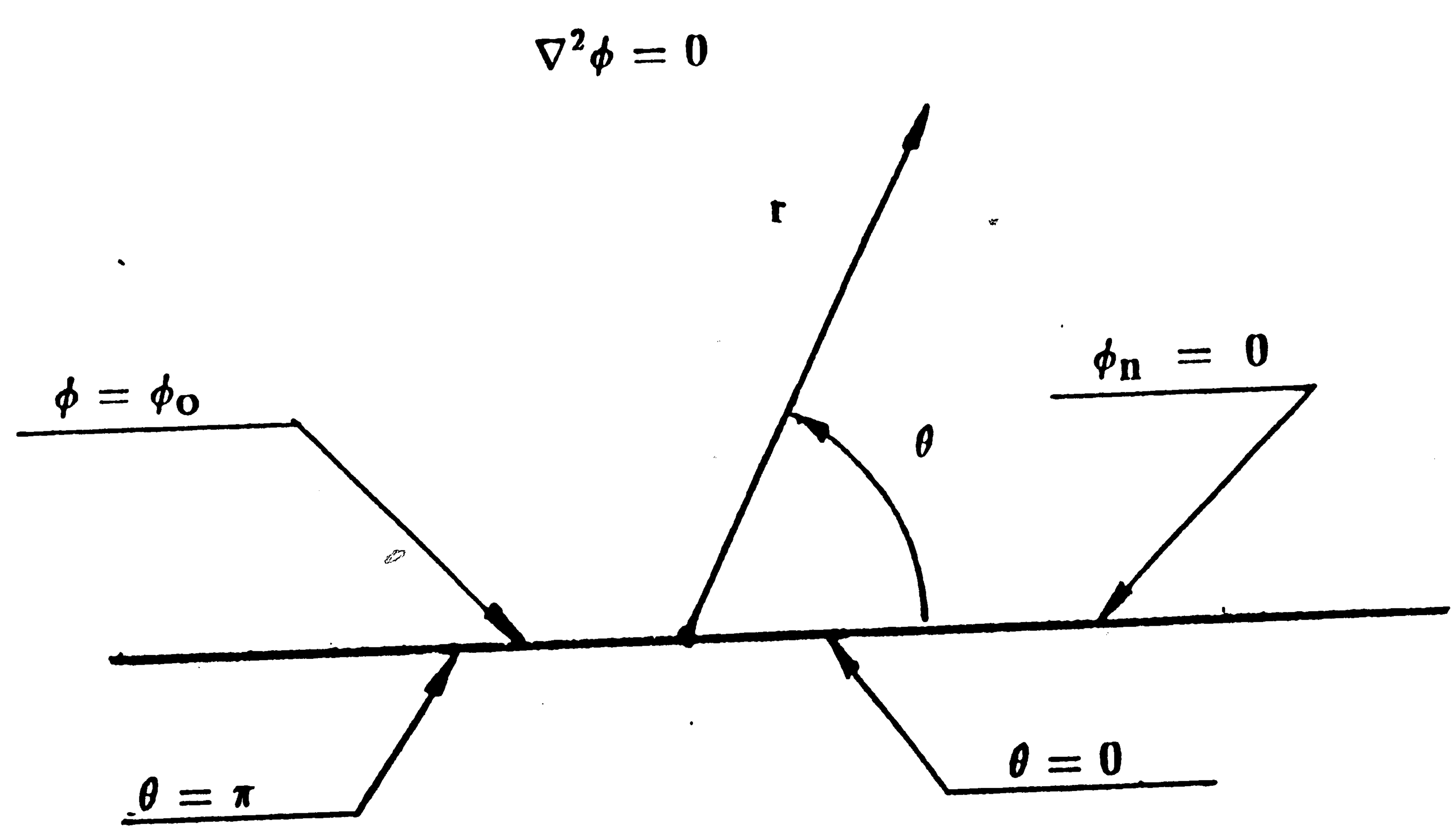


Figure 3. 2. Local coordinate system near a point where the boundary conditions change abruptly on a straight line.

$$P(\theta)\, R''(r) + \frac{P(\theta)}{r} R'(r) + \frac{1}{r^2} R\, P''(\theta) = 0, \tag{3.3}$$

and accordingly

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = - \frac{P''}{P} = \lambda^2, \tag{3.4}$$

where λ is the real separation constant. The solutions for R(r) equation are of the form,

$$R(r) = C\, r^{\lambda} + D\, r^{-\lambda}, \tag{3.5}$$

and it is necessary to take D = 0 to get bounded solution as $r \to 0$. Thus,

$$R(r) = C\, r^{\lambda}. \tag{3.6}$$

The solutions for the θ equation are of the form,

$$P(\theta) = A \cos \lambda\theta + B \sin \lambda\theta, \tag{3.7}$$

where A and B are arbitrary constants. The objective is now to find out a functional form near the singularity (r=0) which satisfies not only the Laplace equation, but also the boundary conditions. For simplicity, make the transformation

$$\hat{\phi} = \phi - \phi_0. \tag{3.8}$$

Then, $\hat{\phi}$ satisfies the Laplace equation and has boundary conditions

$$\hat{\phi} = 0 \qquad \text{for } \theta = \pi,\ r > 0, \tag{3.9}$$

$$\frac{\partial \hat{\phi}}{\partial \theta} = 0 \qquad \text{for } \theta = 0,\ r > 0. \tag{3.10}$$

equation (3.9) is obvious from the condition $\phi = \phi_0$ at $\theta = \pi$ while equation (3.10) follows from the fact $\frac{\partial \phi}{\partial y} = 0$ at $\theta = 0$. Since

$$\frac{\partial \phi}{\partial x} = \cos\theta \frac{\partial \phi}{\partial r} - \frac{\sin\theta}{r} \frac{\partial \phi}{\partial \theta}, \tag{3.11}$$

$$P(\theta)\, R''(r) + \frac{P(\theta)}{r} R'(r) + \frac{1}{r^2} R\, P''(\theta) = 0, \tag{3.3}$$

and accordingly

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = - \frac{P''}{P} = \lambda^2, \tag{3.4}$$

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$$\frac{\partial \phi}{\partial x} = \cos\theta \frac{\partial \phi}{\partial r} - \frac{\sin\theta}{r} \frac{\partial \phi}{\partial \theta}, \tag{3.11}$$

$$\frac{\partial \phi}{\partial y} = \sin\theta \frac{\partial \phi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \phi}{\partial \theta}, \tag{3.12}$$

on $\theta = 0$ and $r > 0$, it follows that

$$\frac{\partial \phi}{\partial y} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0.$$

Thus,

$$\frac{\partial \phi}{\partial \theta} = 0 \text{ for } \theta = 0, \; r > 0, \tag{3.13}$$

and therefore

$$\frac{\partial \hat{\phi}}{\partial \theta} = 0 .$$

Now from equation (3.7), we have

$$\frac{\partial P}{\partial \theta} = -\lambda \text{ A} \sin\theta + \lambda \text{ B} \cos\lambda\theta, \tag{3.14}$$

and to satisfy (3.10), B = 0. Therefore

$$P(\theta) = A\cos\lambda\theta, \tag{3.15}$$

and to satisfy equation (3.9), $P(\pi) = 0$ which results in

$$A\cos\lambda\pi = 0. \tag{3.16}$$

It follows that in order to have nontrival solution, we must have

$$\lambda\pi = (n + \tfrac{1}{2})\,\pi, \qquad n = 0,1,2,... \tag{3.17}$$

and consequently

$$\hat{\phi}(r,\theta) = \sum_{n=0}^{\infty} C_n r^{(n+\frac{1}{2})} \cdot \cos(n+\tfrac{1}{2})\,\theta, \tag{3.18}$$

or, in terms of ϕ

$$\phi(r,\theta) = \phi_o + \sum_{n=0}^{\infty} C_n r^{(n+\frac{1}{2})} \cdot \cos\,(n+\tfrac{1}{2})\,\theta. \tag{3.19}$$

The gradient is given by

$$\frac{\partial\phi}{\partial\theta} = -\sum_{n=0}^{\infty} C_n r^{(n+\frac{1}{2})} \cdot (n+\tfrac{1}{2})\,\sin\,(n+\tfrac{1}{2})\,\theta, \tag{3.20}$$

and this (along with equation (3.19)) gives a local solution in the vicinity of the singular point x_0.

Since our particular interest is in how ϕ and ϕ' behave along the boundary, it is of interest to examine the local solution on each side of $x{=}x_o$. On the right side of x_0, $(\theta = 0)$, we have from (3.19) and (3.20):

$$\phi\,(r,0) = \phi_0 + C_0\, r^{\frac{1}{2}} + C_1\, r^{\frac{3}{2}} + \cdots, \tag{3.21}$$

$$\phi_\theta \equiv 0, \tag{3.22}$$

where $r = x-x_0$; $(x > x_0)$. On the left side of x_0, $(\theta{=}\pi)$, we have

$$\phi = \phi_0, \tag{3.23}$$

$$\phi_\theta = -\tfrac{1}{2}\, Co\, \frac{1}{r^{1/2}} + \tfrac{3}{2}\, C_1\, r^{1/2} - \tfrac{5}{2}\, C_2\, r^{3/2} + \cdots, \tag{3.24}$$

where

$$r = (x_0-x); \qquad (x_0>x).$$

It is worthwhile to note that equations (3.21) and (3.24) show the functional form of the solution near the singularity. However, the coefficients C_n are unknown constants which are not easily evaluated and depend on the rest of the boundary conditions in the problem. To develop an accurate numerical algorithm, it is necessary to find a way to calculate these constants in the course of the solution of the boundary integral equations.

3.2 Previous Studies

Many efforts have been made to deal with singularities on the boundary. One of the first and most effective methods is due to Symm (1973). The basis of Symm's method is to deal with a global pertubation function consisting of the actual function minus the singular parts. This global transformation will produce a function which is "almost" regular near the singularity. In this manner, good accuracy can be achieved near the irregularity.

To illustrate Symm's method, consider the model problem example shown in figure 3.3. We wish to solve the Laplace equation with given boundary conditions. It is obvious that there is an abrupt change in boundary conditions at point 0, and a singularity is anticipated there. We first examine the numerical results of the boundary integral method obtained by ignoring the presence of the singularity. Applying the classical boundary integral method, we start with the very coarse mesh consisting of $n = 6$ intervals and all the corners being interval points as shown in figure 3.4. We may then successively refine the mesh with $n=12$, 24, 48 and 96. Some results obtained by Symm (1973) are shown in figure 3.5. Note that there is a very slow convergence on successive meshes, particularly near the singularity at point 0. The values near 0 have changed almost 3% from $n=48$ to $n=96$.

Symm's treatment of the singularity will now be considered. Beginning with isolating the algebraic form of the singularity given in equations (3.19), we have the following expansions near O:

$$\phi(r,\theta) = \phi_0 + \sum_{n=0}^{\infty} C_n r^{(n+\frac{1}{2})} \cdot \cos\,(n+\tfrac{1}{2})\,\theta$$

$$\simeq 500 + C_0\, r^{1/2}\cos(\tfrac{\theta}{2}) + C_1\, r^{3/2}\cos\tfrac{3\theta}{2} + \ldots, \tag{3.27}$$

where the constants C_0, C_1, ... are unknowns. Next, introduce a global perturbation function which is the actual function ϕ minus first terms of the local irregularity, according to,

$$\Phi \simeq \phi - C_0\, r^{1/2}\cos(\tfrac{\theta}{2}) - C_1\, r^{3/2}\sin(\tfrac{\theta}{2}), \tag{3.28}$$

It is worthwhile to note that the dominant irregularities in the solution near O have subtracted out while the other high order terms have been omitted. Thus, Φ is actually

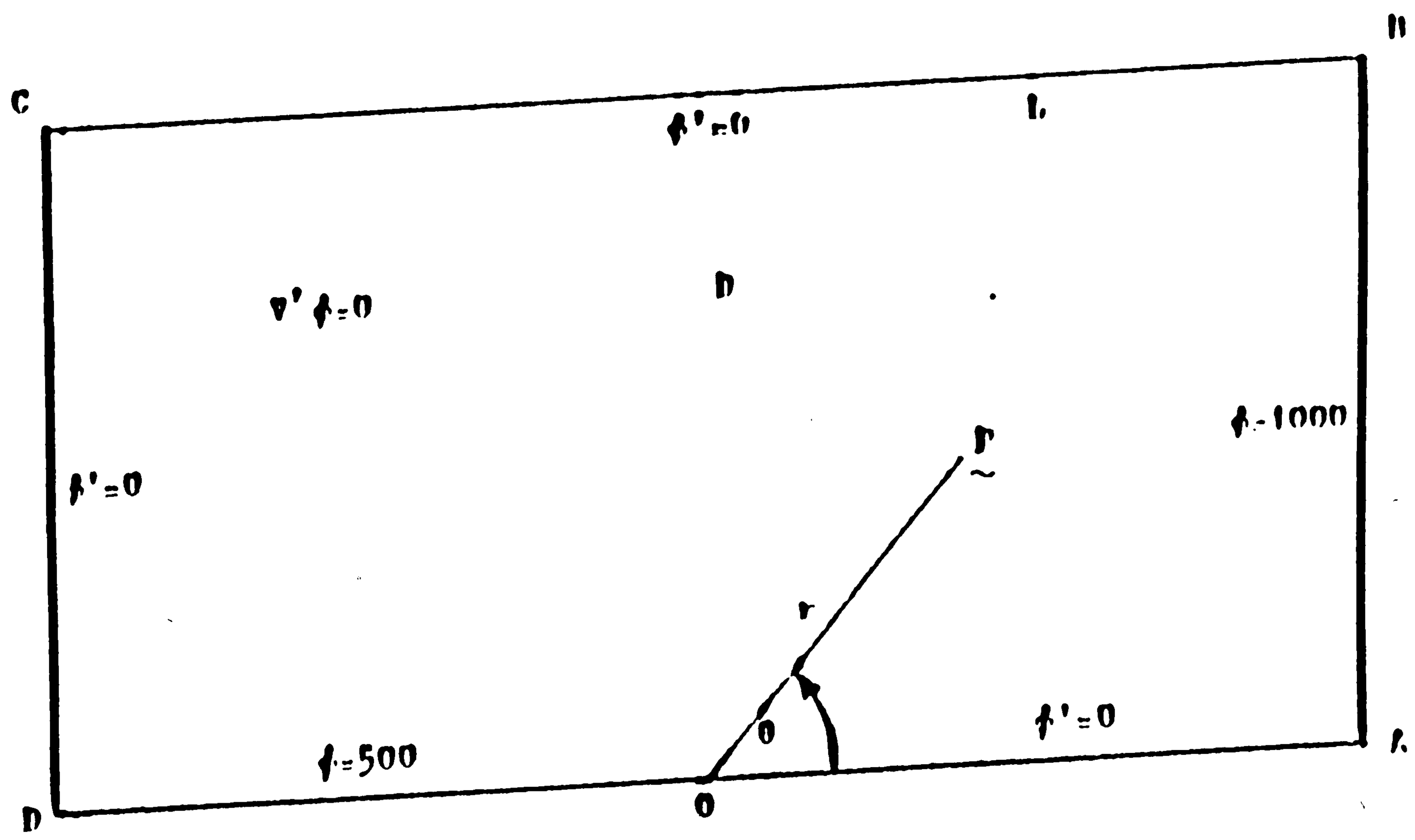

Figure 3. 3. Model problem to illustrate Symm's method.

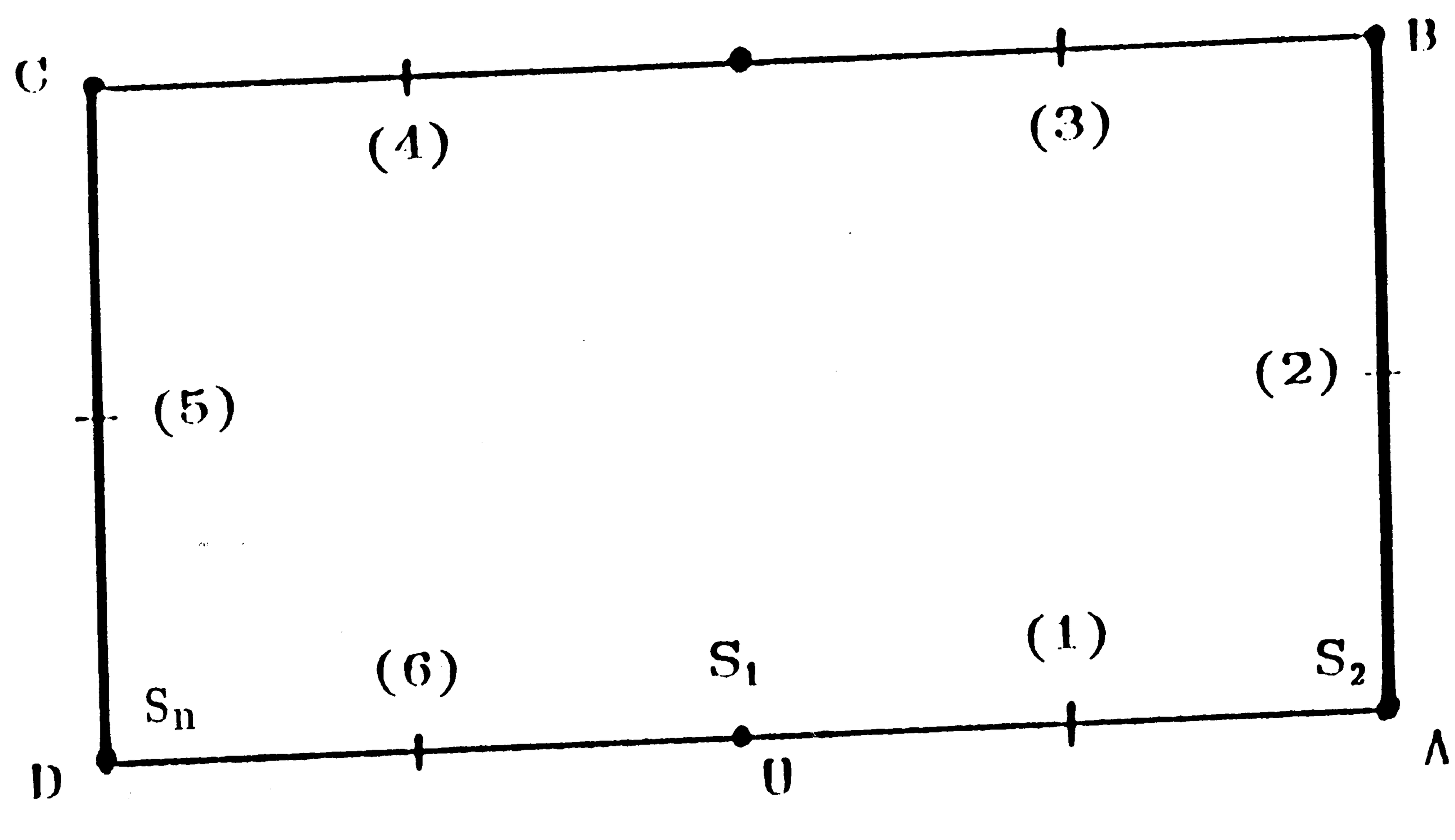

Figure 3. 4. Coarse mesh for the model problem (n = 6).

C B

	595 593 592	602 600 599	614 611 610	630 627 626	651 648 647	678 674 673	709 705 704	744 741 739	783 779 778	824 821 819	867 864 863	911 909 908	956 955 954
	588 587 586	595 593 592	606 604 603	623 620 618	644 641 639	671 667 665	703 699 697	739 735 733	779 775 774	821 818 816	865 862 861	910 908 907	955 954 953
	577 576 575	583 582 581	594 592 591	610 607 606	631 628 626	658 654 653	692 688 686	730 726 724	773 768 767	817 813 812	862 859 858	908 906 905	954 953 952
	562 561 560	567 566 565	577 575 574	591 589 587	611 608 607	639 635 634	676 671 669	718 713 711	764 759 757	811 807 805	859 856 854	906 904 903	953 952 951
	543 543 542	547 546 546	555 553 553	566 564 563	584 581 580	612 608 606	653 647 644	703 697 693	755 749 746	806 801 799	855 852 850	904 902 901	952 951 950
	522 522 522	524 524 524	528 528 527	535 534 534	546 545 544	570 566 564	623 612 608	686 678 674	746 740 736	801 796 793	853 849 847	903 900 899	952 951 950
	500 500 500	499 500 500	501 500 500	499 500 500	498 500 500	495 499 500	578 555 539	677 667 662	743 736 732	800 794 791	853 848 846	903 900 899	953 951 950

D O A

n=24
n=48
n=96

Figure 3. 5. Results obtained by Symm (1973) using the classical method for the model problem of figure 3. 3.

"almost" regular near O.

The next aspect of Symm's method is that it is necessary to construct the boundary conditions for Φ around the entire contour; while this task is easily accomplished for one singularity, it can be very tedious for two or more singularities. In the present problem, we have

$$(1) \text{ on OA: } \Phi' = 0, \tag{3.29}$$

$$(2) \text{ on AB: } \Phi = 1000 - C_0\, r^{1/2}\cos(\tfrac{\theta}{2}) - C_1\, r^{3/2}\cos\tfrac{3\theta}{2}, \tag{3.30}$$

$$(3) \text{ on BC: } \Phi' = -\tfrac{1}{2}\, C_0\, r^{-1/2}\sin(\tfrac{\theta}{2}) + \tfrac{3}{2}\, C_1\, r^{1/2}\sin(\tfrac{\theta}{2}), \tag{3.31}$$

$$(4) \text{ on CD: } \Phi = \tfrac{1}{2}\, C_0\, r^{-1/2}\cos(\tfrac{\theta}{2}) - \tfrac{3}{2}\, C_1\, r^{1/2}\cos(\tfrac{\theta}{2}), \tag{3.32}$$

$$(5) \text{ on DO: } \Phi = 500. \tag{3.33}$$

Since Φ satisfies the Laplace equation, it also satisfies the discretized boundary integral formula (2.38). By substituting the known boundary conditions for Φ, we readily obtain (c.f. figure 3.4)

$$\begin{aligned}\pi\Phi_i = &\sum_{\mathrm{OA,BC,CD}} \beta_{ij}\, \Phi_j + 500 \sum_{\mathrm{DO}} \beta_{ij} \\ &+ 1000 \sum_{\mathrm{AB}} \beta_{ij} - \sum_{\mathrm{AB,DO}} \alpha_{ij}\Phi_j' \\ &+ C_0\, E_{oi} + C_1\, E_{oi},\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}E_{1i} = &- \sum_{\mathrm{AB}} \beta_{ij}\, r^{1/2}\cos(\tfrac{\theta}{2}) \\ &+ \tfrac{1}{2} \sum_{\mathrm{BC}} \alpha_{ij}\, r^{-1/2}\sin(\tfrac{\theta}{2}) \\ &- \tfrac{1}{2} \sum_{\mathrm{CD}} \alpha_{ij}\, r^{-1/2}\cos(\tfrac{\theta}{2}),\end{aligned} \tag{3.35}$$

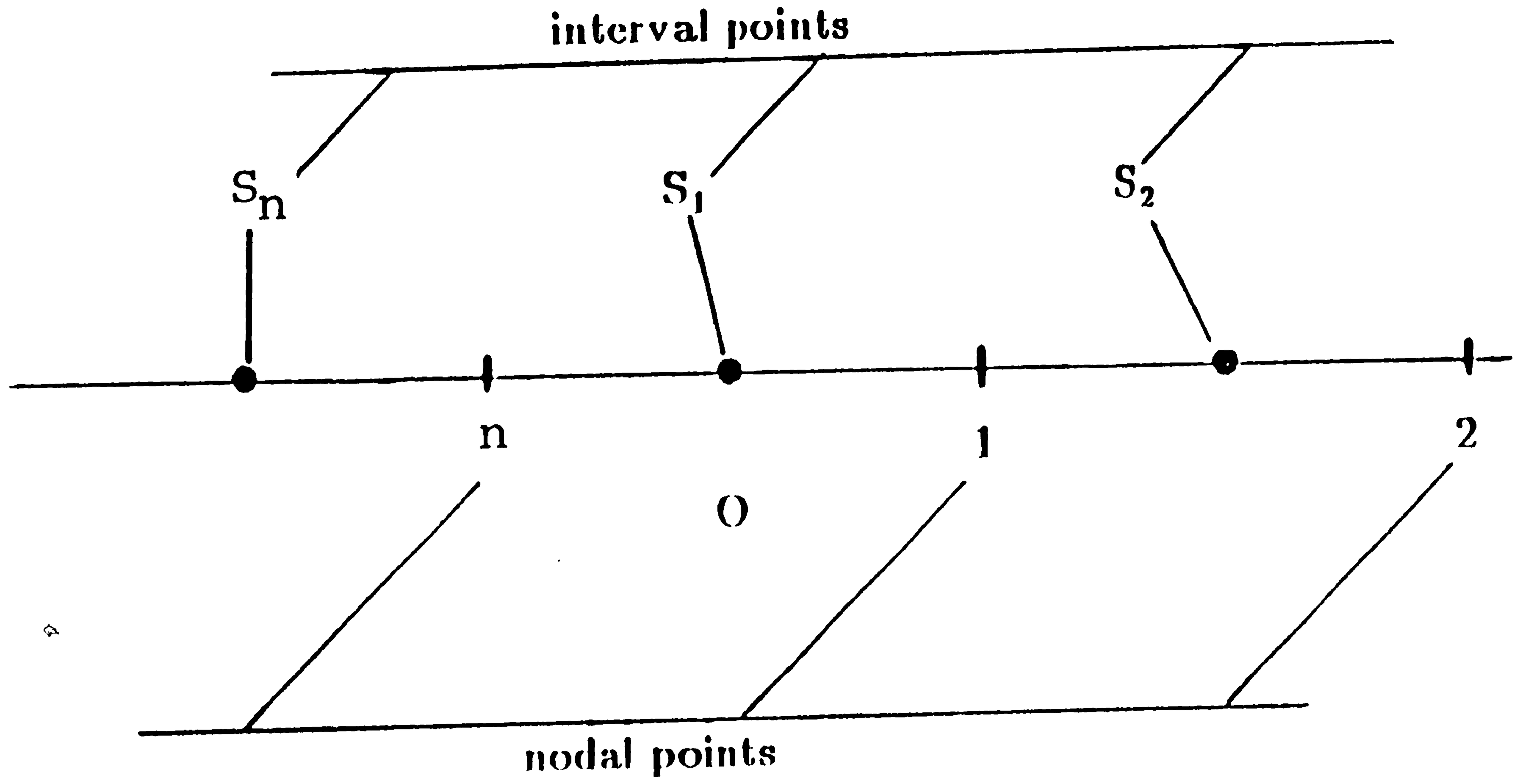

Figure 3. 6. Numbering scheme near singularity.

$$E_{oi} = -\sum_{AB} \beta_{ij}\, r^{3/2} \cos(\tfrac{3\theta}{2})$$

$$-\tfrac{3}{2} \sum_{BC} \alpha_{ij}\, r^{1/2} \sin(\tfrac{\theta}{2})$$

$$-\tfrac{3}{2} \sum_{CD} \alpha_{ij}\, r^{1/2} \cos(\tfrac{\theta}{2}), \tag{3.36}$$

for i = 1,2,..., n. Since C_0 and C_1 are also unknowns, equation (3.34) now consists of N equations for n + 2 unknowns. Two additional equations are necessary to solve the system. In Symm's method, those two additional equations are given by:

$$\Phi_1 = 500 \tag{3.37}$$

$$\Phi_n{}' = 0 \tag{3.38}$$

where the subscripts are numbered in a counter-clock wise direction as shown in figure 3.6.

At this stage it is possible to carry out the numerical calculation to solve for the global perturbation function Φ, and then the actual function ϕ. Some results obtained by Symm (1973) are shown in figure 3.7, which are believed to be accurate to at least five significant figures. The results are compared in figure 3.7 with an independent and highly accurate computation based on a conformal mapping method due to Whiteman and Papamichael (1971). It is worthwhile to note that now there is a dramatic increase of accuracy near the singularity. Also, the constants C_o and C_i are given in figure 3.8 where it may be noted that rapid converge takes place with increasing n.

Symm (1973) also solved another example problem using the same technique which has a weaker singularity near O and is shown in figure 3.9. The constants associated with the singularity are presented in figure 3. 10(a).

Symm's method is very effective in producing accurate results for problems with strong singularities. On the other hand, however, this approach can become tedious if more than one singular point occurs. The approach is tedious to program for the problems with multiple singularities because of the global transformations required for each singular point.

Another approach has been presented by Xanthis, Bernal and Atkinson (1981) where the singularity is locally treated near the point in question without introducing a

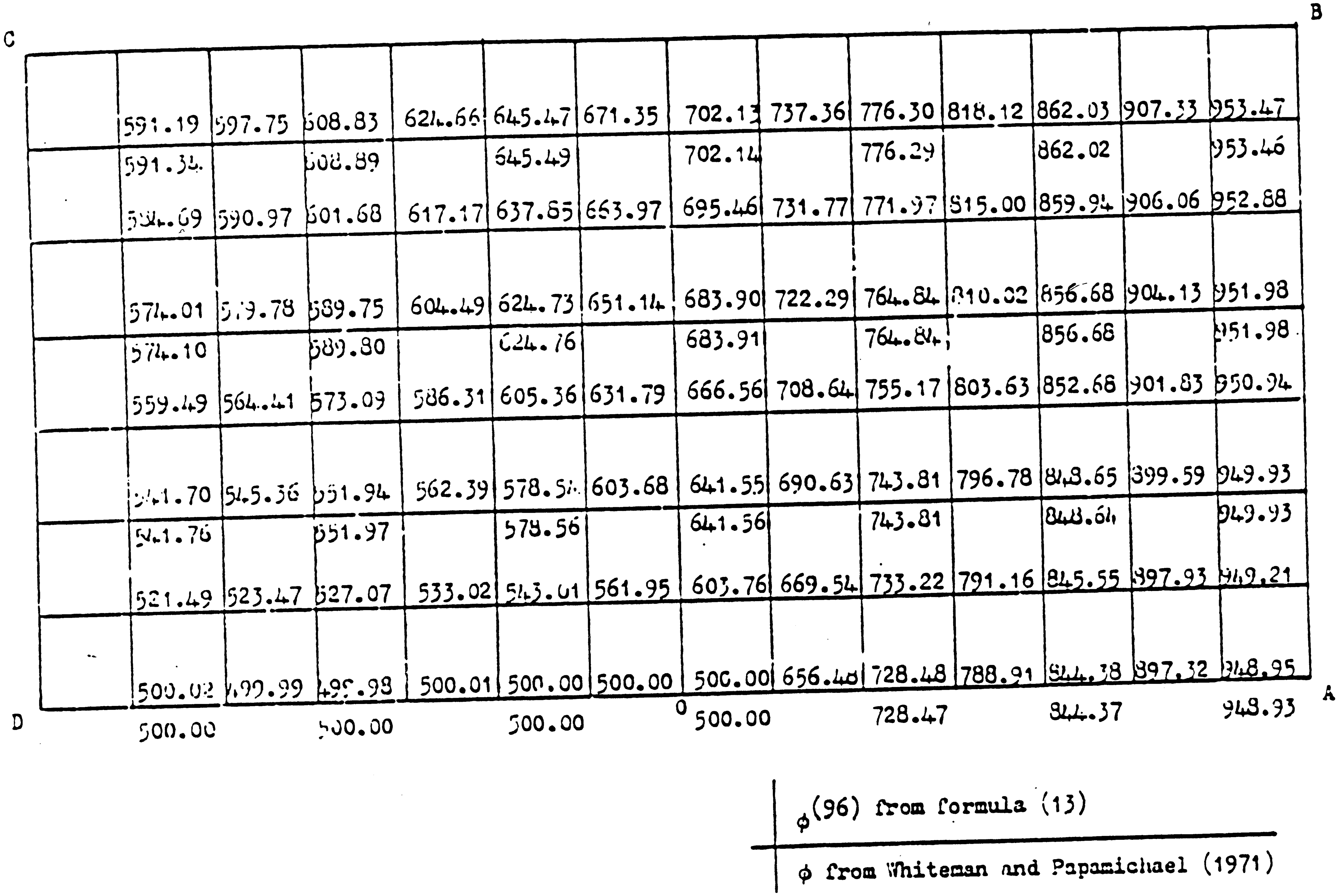

Figure 3.7. Solution obtained by Symm (1973) which accounts for the singularity at O.

n	C_o	C_1
6	157.05	3.59
12	152.13	4.61
24	151.75	4.68
48	151.65	4.69
96	151.63	4.71

Figure 3.8. The constants associated with the singularity problem of figure 3.3.

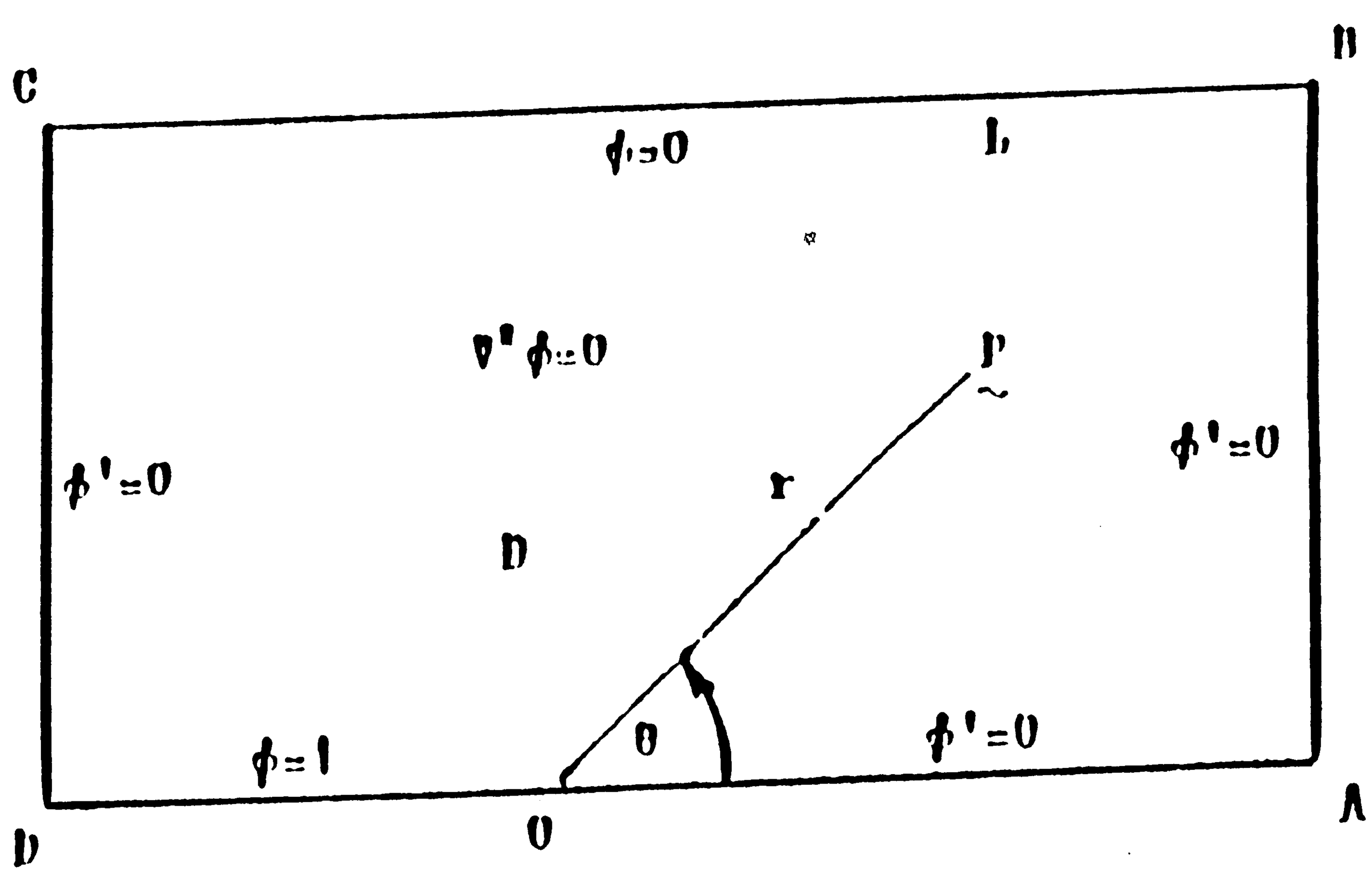

Figure 3. 9 . Another example problem having a boundary singularity.

global transformation. The local approach is a more convenient method when there are more than one singularity.

To illustrate this local approach, we again consider the example shown in figure 3.3. The solution near the singularity 0 is of the form:

$$\phi(r, \theta) \sim 500 + C_0\, r^{1/2} \cos(\tfrac{\theta}{2}) - C_1\, r^{3/2} \cos\tfrac{3\theta}{2} + \cdots, \tag{3.39}$$

$$\phi'(r,\theta) \sim -\tfrac{1}{2}\, C_0\, r^{-1/2} \sin(\tfrac{\theta}{2}) + \tfrac{3}{2}\, C_1\, r^{1/2} \sin(\tfrac{3\theta}{2}) + \cdots, \tag{3.40}$$

where C_0 and C_1 are unknowns. For $\theta = 0$, we have

$$\phi_1 = 500 + C_0\, r^{1/2} + C_1\, r^{3/2} + \cdots \tag{3.41}$$

$$\phi_1' = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \Big|_{\theta=0} = 0 \tag{3.42}$$

For $\theta = \pi$, we have

$$\phi_n = 500, \tag{3.43}$$

$$\phi_n' = -\tfrac{1}{2}\, C_0\, r^{-1/2} + \tfrac{3}{2}\, C_1\, r^{1/2} + \cdots. \tag{3.44}$$

Here, only two terms expansion are used and higher order terms which are small as $r \to 0$ are omitted. The main idea of the local treatment is to use the above expansion about the singularity to represent ϕ_1 and ϕ_n' over the intervals on either side of the singular point O as shown in figure 3.6. The unknowns on the two intervals on either side of O now are C_0 and C_1 instead of ϕ_1 and ϕ_n'.

Next, substituting them into the boundary integral equation formula (2.31) yields

$$\pi \phi_i = \int_{s_1}^{s_2} [\, 500 + C_0\, r^{1/2} + C_1\, r^{3/2}] \left(\frac{\partial}{\partial n} \log R \right) ds$$

$$+ \sum_{j=2}^{n} \phi_j \int_{s_j}^{s_{j+1}} \frac{\partial}{\partial n} \log R \, ds - \sum_{j=1}^{n-1} \phi_j' \int_{s_j}^{s_{j+1}} \log R \, ds$$

$$- \int_{s_n}^{s_1} [-\frac{1}{2} C_0 r^{-1/2} + \frac{3}{2} C_1 r^{1/2}] \log R \, ds \tag{3.45}$$

(i = 1,2,...,n)

The integers over the intervals surrounding the singular point O can be written in the form

$$\int_{s_1}^{s_2} [\, 500 + C_0 r^{1/2} + C_1 r^{3/2}] \frac{\partial}{\partial n} \log R \, ds = 500\, \beta_{i1} + C_0 \epsilon_{i0} + C_1 \epsilon_{i1}, \tag{3.46}$$

where

$$\epsilon_{i0} = \int_{s_1}^{s_2} r^{1/2} \frac{\partial}{\partial n} \log R \, ds \,, \tag{3.47}$$

$$\epsilon_{i1} = \int_{s_1}^{s_2} r^{3/2} \frac{\partial}{\partial n} \log R \, ds, \tag{3.48}$$

and

$$\int_{s_N}^{s_1} [-\frac{1}{2} C_0 r^{-1/2} + \frac{3}{2} C_1 r^{1/2}] \log R \, ds = -\frac{1}{2} C_0 \eta_{i0} + \frac{3}{2} C_1 \eta_{i1} \tag{3.49}$$

where,

$$\eta_{i0} = \int_{s_N}^{s_1} r^{-1/2} \log R \, ds, \tag{3.50}$$

$$\eta_{i1} = \int_{s_n}^{s_1} r^{1/2} \log R \, ds. \tag{3.51}$$

Now, equation (3.45) may be rewritten as

$$\begin{aligned} \pi \, \phi_i = & \sum_{j=2}^{n} \beta_{ij} \phi_j + 500\, \beta_{i1} + C_0 \epsilon_{i0} + C_1 \epsilon_{i1} \\ & - \sum_{j=1}^{n-1} \phi_j' \, \alpha_{ij} + \frac{1}{2} C_0 \eta_{i0} - \frac{3}{2} C_1 \eta_{i1}, \end{aligned} \tag{3.52}$$

for (i = 1,2,...,n).

$$- \int_{s_n}^{s_1} [-\tfrac{1}{2} C_0 r^{-1/2} + \tfrac{3}{2} C_1 r^{1/2}] \log R \, ds \tag{3.45}$$

(i = 1,2,...,n)

The integers over the intervals surrounding the singular point O can be written in the form

$$\int_{s_1}^{s_2} [\, 500 + C_0 r^{1/2} + C_1 r^{3/2}] \frac{\partial}{\partial n} \log R \, ds = 500\, \beta_{i1} + C_0 \epsilon_{i0} + C_1 \epsilon_{i1}, \tag{3.46}$$

where

$$\epsilon_{i0} = \int_{s_1}^{s_2} r^{1/2} \frac{\partial}{\partial n} \log R \, ds \,, \tag{3.47}$$

$$\epsilon_{i1} = \int_{s_1}^{s_2} r^{3/2} \frac{\partial}{\partial n} \log R \, ds, \tag{3.48}$$

and

$$\int_{s_N}^{s_1} [-\tfrac{1}{2} C_0 r^{-1/2} + \tfrac{3}{2} C_1 r^{1/2}] \log R \, ds = -\tfrac{1}{2} C_0 \eta_{i0} + \tfrac{3}{2} C_1 \eta_{i1} \tag{3.49}$$

where,

$$\eta_{i0} = \int_{s_N}^{s_1} r^{-1/2} \log R \, ds, \tag{3.50}$$

$$\eta_{i1} = \int_{s_n}^{s_1} r^{1/2} \log R \, ds. \tag{3.51}$$

Now, equation (3.45) may be rewritten as

$$\begin{aligned} \pi \, \phi_i = & \sum_{j=2}^{n} \beta_{ij} \phi_j + 500\, \beta_{i1} + C_0 \epsilon_{i0} + C_1 \epsilon_{i1} \\ & - \sum_{j=1}^{n-1} \phi_j{}' \, \alpha_{ij} + \tfrac{1}{2} C_0 \eta_{i0} - \tfrac{3}{2} C_1 \eta_{i1}, \end{aligned} \tag{3.52}$$

for (i = 1,2,...,n).

This gives n linear equations for n unknowns including the constants C_0 and C_1 associated with the singularity. The solution of equation (3.52) would be straightforward provided that the integrals in equations (3.47) to (3.51) can be evaluated. At present, analytical results have not been obtained and the integrals were evaluated numerically using Gaussian quadrature. Ingham and Kelmanson (1984) solved the same example as that shown in figure 3.9 using this type of procedure. His results of constants associated with the singularity are presented in figure 3. 10(b).

3. 3 Another Approach of Local Treatment

The method of Ingham and Kelmanson (1982) uses two terms in an expansion of analytical representation of the boundary singularity. When the procedure is incorporated into the classical boundary integral equation method, both the potential term on the righthand side of O and the flux term on the lefthand side of O are treated. In this study, another arrangement was tried such that only the flux terms are incorporated into the classical method; this is because the major trouble associated with slow convergence around the singular point is caused by the large flux magnitudes near the singularity. In this procedure, two terms in the expansion for the flux are still used but only in the two intervals to the left of the point 0 (where $\phi = 500$ is given). Thus in these intervals

$$\phi' = -\frac{1}{2} C_0 r^{-1/2} + \frac{3}{2} C_1 r^{1/2} + \cdots, \tag{3.53}$$

where r varies from 0 to h for the interval n (closest to 0) and from h to 2h over interval (n - 1) (one interval to the right of 0).

Substitution of this expression into the classical boundary integral formula (2.31) yields

$$\pi \phi_i = \sum_{j=1}^{n} \beta_{ij}\phi_j - \sum_{j=1}^{n-2} \phi_j' \alpha_{ij}$$
$$+ \frac{1}{2} C_0(\eta_{i0} + \eta_{i2}) - \frac{3}{2} C_1(\eta_{i1} + \eta_{i3}), \tag{3.54}$$

where

$$\eta_{i0} = \int_{s_n}^{s_1} r^{-1/2} \log R \, ds, \tag{3.55}$$

(a) Symm's method

n	C_o	C_1
30	−0.48358	0.02987
60	−0.48353	0.02988

(b) Ingham's method

n	C_o	C_1
30	−0.4843	0.0313
60	−0.4844	0.0314

Figure 3. 10. Computed results for constants associated with the singularity for the example of figure 3. 9.

$$\eta_{i1} = \int_{s_n}^{s_1} r^{1/2} \log R \, ds, \tag{3.56}$$

$$\eta_{i2} = \int_{s_{n-1}}^{s_n} r^{-1/2} \log R \, ds, \tag{3.57}$$

$$\eta_{i3} = \int_{s_{n-1}}^{s_n} r^{1/2} \log R \, ds. \tag{3.58}$$

Equation (3.54) can be readily solved for the model problems provided the integrals (3.55) - (3.58) can be evaluated. Analytical solutions for these integrals will be obtained.

4. Analytical Results of Local Treatment of Singularties

4.1 Introduction

As we discussed in Chapter 3, the local approach of treatment of singularities has definite advantages over Symm's (1973) method since we do not need to modify the boundary integral method globally. This process is probably the easist way to deal with singularities from the point of view of programming. In this chapter, the objective is to complete analytic solutions of the integrals defined from (3.55) to (3.58).

First, we rewrite the integrals in question again as:

$$\eta_{i0} = \int_{s_n}^{s_1} r^{-1/2} \log R_i \, ds, \tag{4.1}$$

$$\eta_{i1} = \int_{s_n}^{s_1} r^{1/2} \log R_i \, ds, \tag{4.2}$$

$$\eta_{i2} = \int_{s_{n-1}}^{s_n} r^{-1/2} \log R_i \, ds, \tag{4.3}$$

$$\eta_{i3} = \int_{s_{n-1}}^{s_n} r^{1/2} \log R_i \, ds, \tag{4.4}$$

for (i= 1,2,..., n). Here R_i is the distance from the ith nodal point to the interval in question; ds is the differential element of the contour and r is the distance from the singular point as shown in figure 4.1. The nodal points are numbered in a similar way to interval points, as indicated in figure 4.2.

The major difficulty associated with the integrations in equations (4.1) - (4.4) is that it is not possible to give simple analytic results for all nodal positions. Instead, care must be taken for each part of the contour. As far as the given example problem is concerned, the whole contour C was split into 3 parts as shown in figure 4.2. These are as follows:

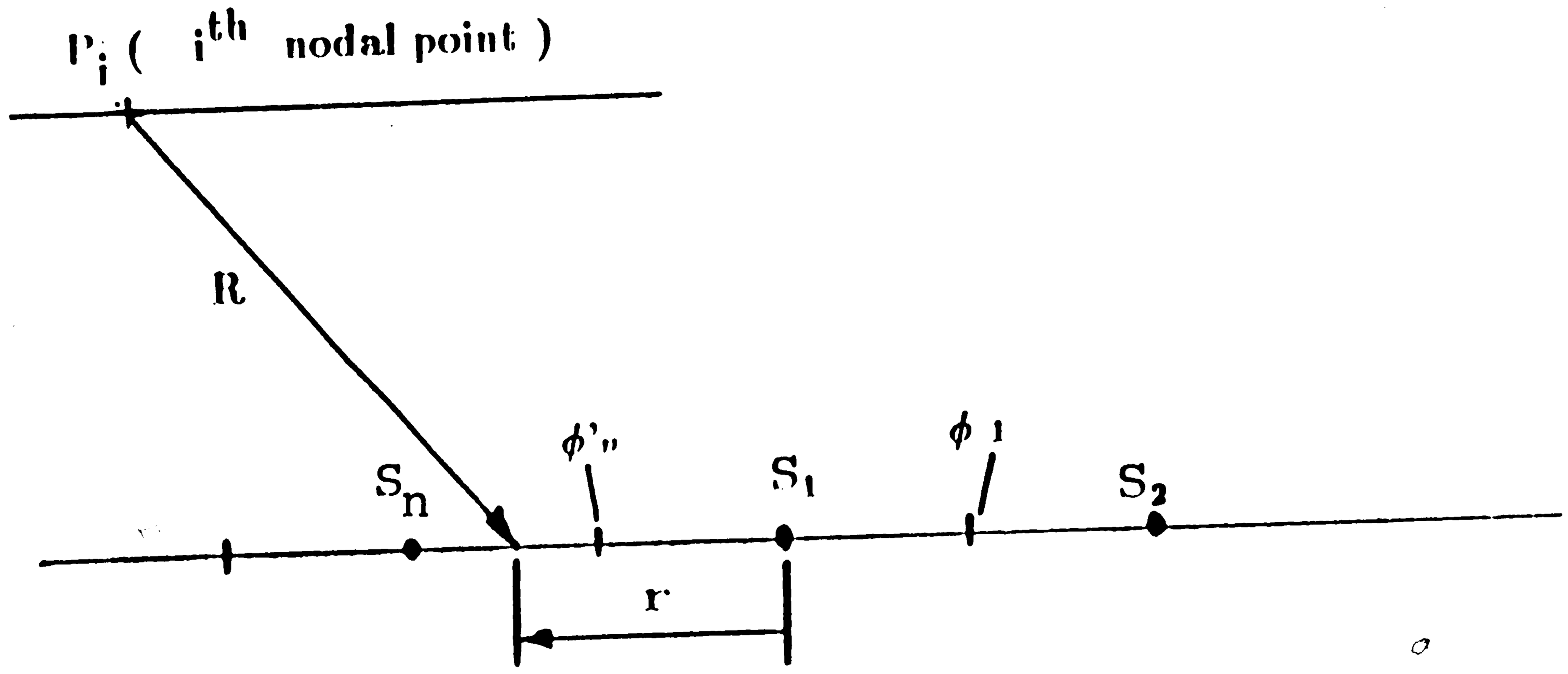

Figure 4. 1. Geometry and notation for line integral.

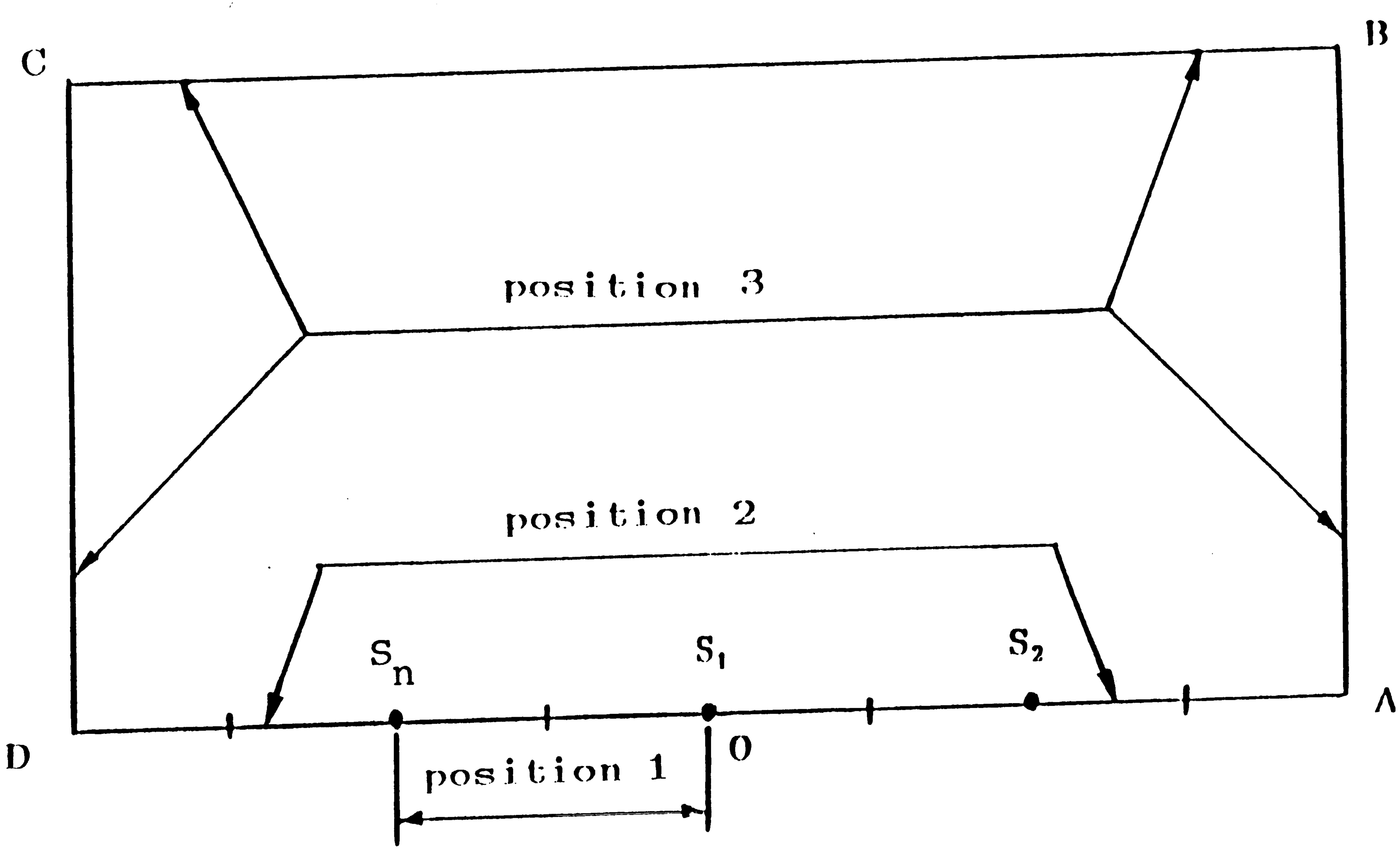

Figure 4.2. Calculation scheme for line integral and three types of positions for the i^{th} nodal point.

(1) When the ith nodal point is on the interval closest to singularity (i.e., at position 1).

(2) When the ith nodal point is at any other position on line DA, ($\alpha = 0$ or $\alpha = \pi$).

(3) The rest of the contour (i.e., when the ith nodal point is not collinear with 0).

It is also very important to note that we have three different variables, i.e., R, r and s. To carry the calculation out, we need to make suitable transformations with care. In the following two sections, each type of integration will be treated separately. In section 4.2, the integrations for positions of the type (1) and (2) above will be carried out while the third type will be treated in section 4.3.

4.2 Integrals Along Intervals Collinear With the Singular Point

First, we can have a general formula for integral (4.1) and (4.2), such as

$$\eta_i = \int_{s_n}^{s_1} r^{\gamma} \log R_i \, ds, \quad i = 1,2,\ldots, n, \tag{4.5}$$

where $\gamma = -1/2$ and $1/2$ respectively. s_n denotes the interval point just before the singular point. s_1 is the singular point (as shown in figure 4.3), r is the distance from the singular point and R_i is the distance from the ith nodal point to the interval $[s_n, s_1]$. The line integral is along the contour in the counter-clockwise direction. From the cosine law, we have

$$R_i = (b^2 + r^2 - 2br \cos\alpha)^{1/2}, \tag{4.6}$$

and with s the integration variable define

$$\xi = | s - s_n | , \tag{4.7}$$

then,

$$\xi = h - r, \tag{4.8}$$

and

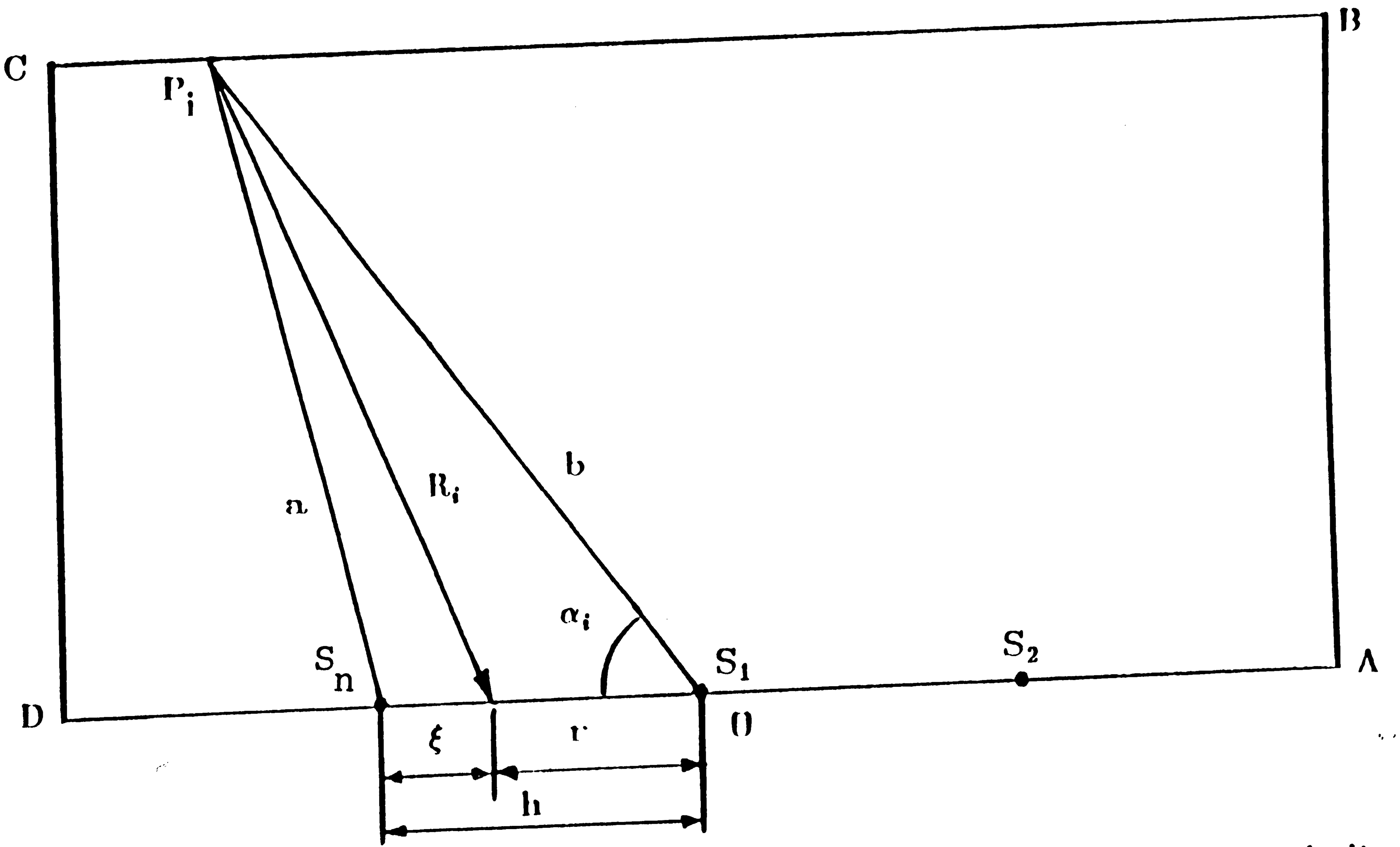

Figure 4. 3. Geometry for the integration over the interval adjacent to the singularity.

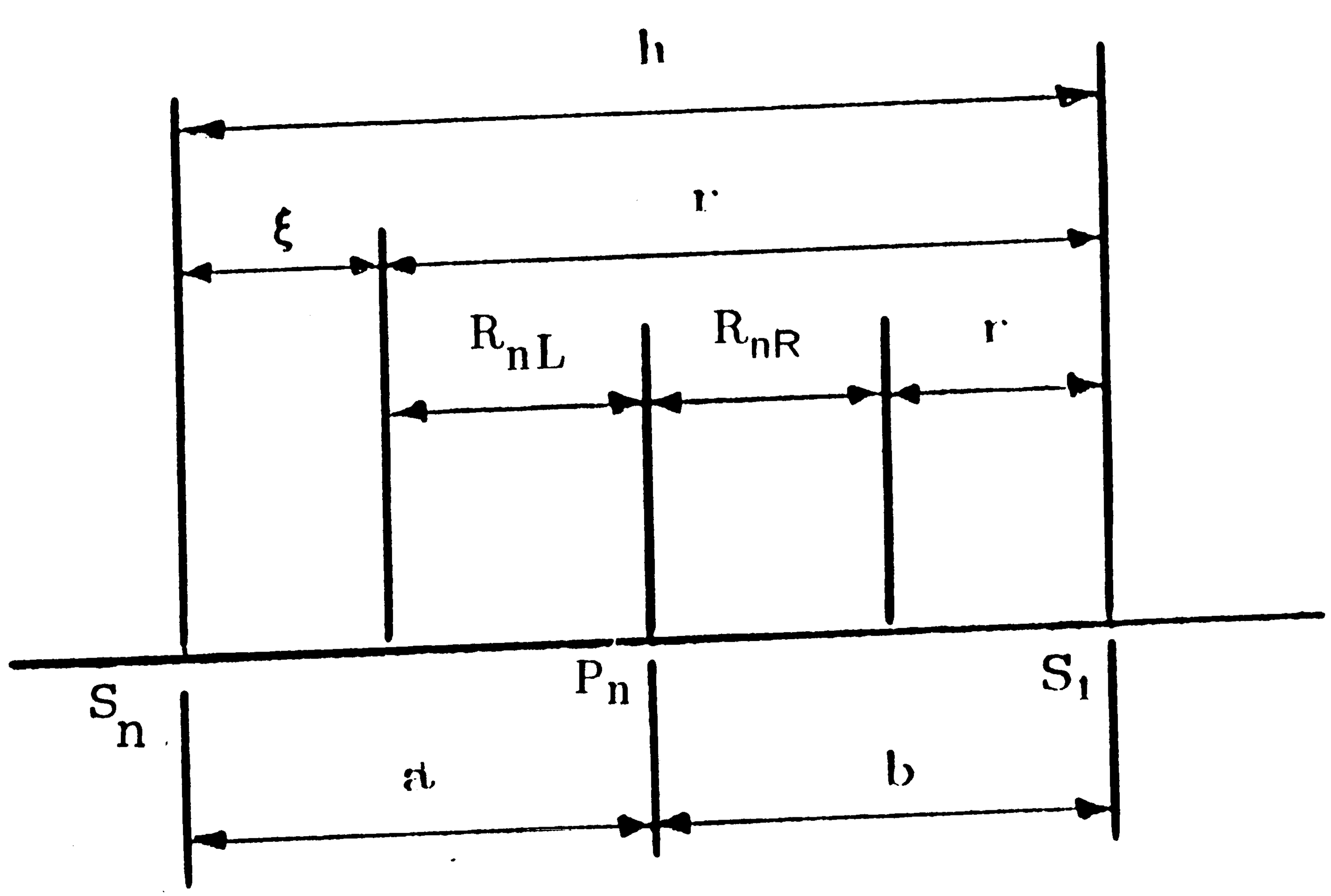

Figure 4.4. Geometry for the calculation when the nodal point lies in the interval adjacent to the singularity.

$$ds = d\xi = -\, dr, \tag{4.9}$$

As s moves from S_n to S_1, r changes from h to 0, so that

$$\eta_i = \int_0^h r^\gamma \log (b^2 + r^2 - 2br\cos\alpha)^{1/2} dr, \tag{4.10}$$

for i= 1,2,..., n which is valid for all the nodal points except for i = n when the nodal point lies on the interval of integration; this special case will now be treated separately.

When the nth nodal point is inside the interval from S_n to S_1 as shown in figure 4.4, the line integral is split into two parts, according to

$$\eta_n = \int_{S_n}^{P_n} r^\gamma \log R_{nL}\, ds_L + \int_{P_n}^{S_1} r^\gamma \log R_{nR}\, ds_R \tag{4.11}$$

where R_{nL} and R_{nR} represent R_n on the left side and right side respectively, as shown in figure 4.4. We have

$$R_{nL} = (r - b) = r - \frac{h}{2}, \tag{4.12}$$

$$R_{nR} = (b - r) = \frac{h}{2} - r, \tag{4.13}$$

and ds = $-$dr; thus

$$\int_{S_n}^{P_n} r^\gamma \log R_{nL}\, ds = \int_{h/2}^{h} r^\gamma \log (r - \tfrac{h}{2})dr, \tag{4.14}$$

and

$$\int_{P_n}^{S_1} r^\gamma \log R_{nR}\, ds = \int_0^{h/2} r^\gamma \log (\tfrac{h}{2} - r)\, dr \tag{4.15}$$

For $\gamma = -1/2$, we have

$$\eta_{nO} = \int_0^h r^{-1/2} \log r \, dr + \int_0^{h/2} r^{-1/2} \log\left(\frac{h}{2r}-1\right) dr$$

$$+ \int_{h/2}^h r^{-1/2} \log\left(1 - \frac{h}{2r}\right) dr. \tag{4.16}$$

It can be shown (see Appendix A)

$$\int_0^h r^{-1/2} \log r \, dr = 2h^{1/2} (\log h - 2), \tag{4.17}$$

$$\int_0^{h/2} r^{-1/2} \log\left(\frac{h}{2r} - 1\right) dr = 2^{3/2}(\log 2)\, h^{1/2}, \tag{4.18}$$

$$\int_{h/2}^h r^{-1/2} \log\left(1- \frac{h}{2r}\right) dr$$

$$= -h^{1/2}\left\{ (1+2^{1/2})\, 2 \log 2 + 2^{1/2}\log\left(\frac{2^{1/2}-1}{2^{1/2}+1}\right) \right\}. \tag{4.19}$$

and therefore

$$\eta_{nO} = h^{1/2}\left\{ 2 \log h - 4 - 2\log 2 - 2^{1/2} \log\left(\frac{2^{1/2}-1}{2^{1/2}+1}\right) \right\} \tag{4.20}$$

Let

$$K_0 = 4 + 2 \log 2 + 2^{1/2} \log\left(\frac{2^{1/2}-1}{2^{1/2}+1}\right) \}$$

$$= 2.89339340055896856 52, \tag{4.21}$$

and finally we have,

$$\eta_{nO} = h^{1/2} \{ 2\log h - K_0\} \tag{4.22}$$

Consider next the integral for $\gamma = 1/2$ for which

$$\eta_{n1} = \int_0^h r^{1/2} \log r \, dr + \int_0^{h/2} r^{1/2} \log\left(\frac{h}{2r} - 1\right) dr$$

$$+ \int_{h/2}^{h} r^{1/2} \log\left(1 - \frac{h}{2r}\right) dr. \tag{4.23}$$

The necessary integrals are evaluated in Appendix B and it follows that

$$\eta_{n1} = \frac{1}{3} h^{3/2} \{ 2 \log h - K_1 \} \tag{4.24}$$

where K_1 is a constant defined by

$$K_1 = \frac{4}{3} + 2 + 2^{1/2} \log \tan\left(\frac{\pi}{8}\right) + 2 \log 2$$

$$= 3.47317721417276 29253 \tag{4.25}$$

Thus far, the two line integrals have been derived for the interval adjacent to the singularity when the point P lies along a line collinear with the segment but P lies inside the segment. Now consider those line integrals with $\alpha = 0$ or π; that is, when P lies outside the segment adjacent to O. When the nodal point is on the right hand side of the singular point, $\alpha = \pi$. When it is on the left hand side, α=0. (c.f. figure 4.5).

From equation (4.10), if $\alpha = 0$, we have then,

$$\eta_i = \int_0^h r^\gamma \log (b^2 + r^2 - 2br)^{1/2} dr$$

$$= \int_0^h r^\gamma \log (b - r) dr \tag{4.26}$$

For $\gamma = -1/2$ and for α=0, we have

$$\eta_i = \int_0^h r^{-1/2} \log (b - r) dr \tag{4.27}$$

and it is shown in Appendix C that

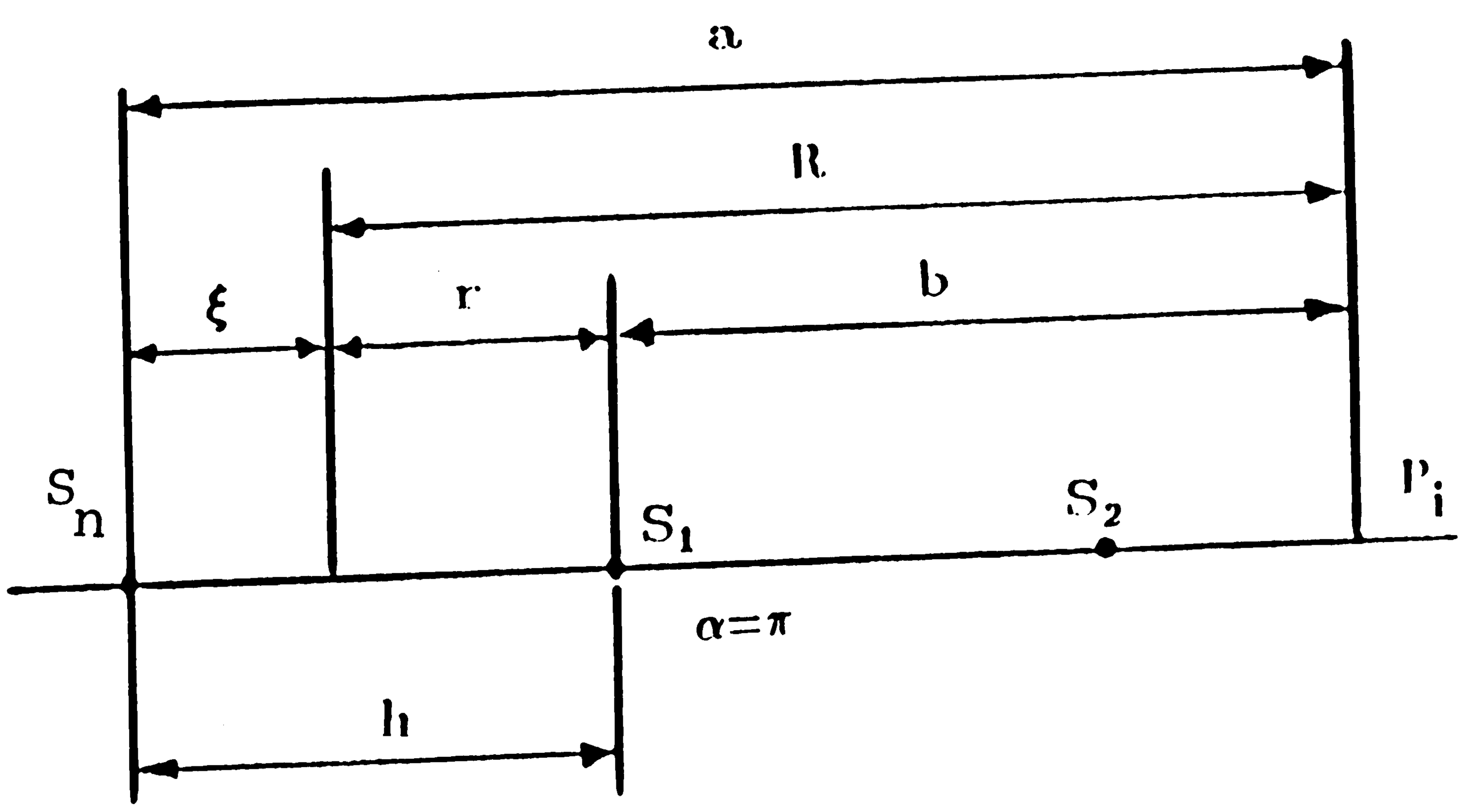

Figure 4. 5 (a) P_i is to the right of the singularity.

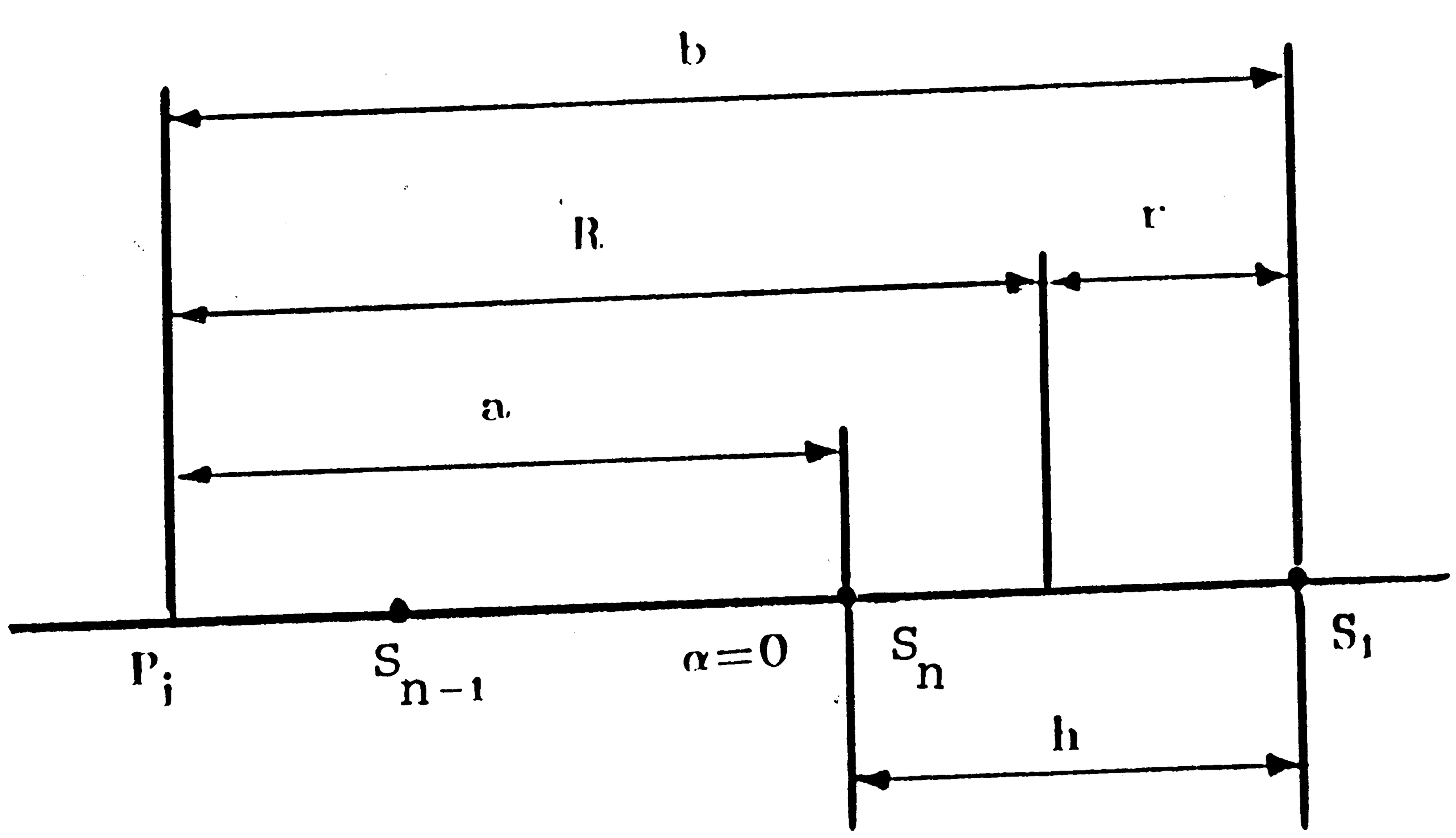

Figure 4. 5 (b) P_i is to the left of the singularity.

$$\eta_{io} = 2\, h^{1/2} \log (b-h) - 4\, h^{1/2}$$

$$- 2\, b^{1/2} \log \left(\frac{b^{1/2} - h^{1/2}}{b^{1/2} + h^{1/2}} \right), \text{ for } \alpha=0 \qquad (4.28)$$

For $\gamma = 1/2$, we have

$$\eta_{io} = \int_0^h r^{1/2} \log (b - r)\, dr \qquad (4.29)$$

and it is shown in Appendix C that for $\alpha=0$

$$\eta_{i1} = \tfrac{2}{3}[h^{3/2} \log (b-h) - b^{3/2} \log \left(\frac{b^{1/2} - h^{1/2}}{b^{1/2} + h^{1/2}}\right) - 2bh^{1/2} - \tfrac{2}{3}\, h^{3/2}], \text{ for } \alpha=0. \qquad (4.30)$$

Now consider nodal points on the right side of the singular point, so that $\alpha=\pi$. We have

$$\eta_i = \int_0^h r^{\gamma} \log (b^2 + r^2 + 2br)^{1/2} dr$$

$$= \int_0^h r^{\gamma} \log (b+r)\, dr, \qquad (4.31)$$

and for $\gamma=-1/2$,

$$\eta_{io} = \int_0^h r^{-1/2} \log (b+r) dr \qquad (4.32)$$

It is shown in Appendix B that

$$\eta_{io} = 2h^{1/2} \log (b+h) - 4h^{1/2} + 4b^{1/2} \tan^{-1} \frac{h^{1/2}}{b^{1/2}}, \text{ for } \alpha=\pi \qquad (4.33)$$

For $\gamma=1/2$,

$$\eta_{i1} = \int_0^h r^{1/2} \log(b+r) dr \qquad (4.34)$$

and it is shown in Appendix B that

$$\eta_{i1} = \frac{2}{3}\{ h^{3/2}\log(b+h)+2bh^{1/2}- 2b^{3/2}\tan^{-1}(\frac{h^{1/2}}{b^{1/2}})-\frac{2}{3}h^{3/2}\}. \tag{4.35}$$

So far, line integrals along the straight line containg the field point have been evaluated. Now situations will be considered where the field point in not collinear with the interval under consideration. We have now (for $\alpha\neq 0$ or π)

$$\begin{aligned}\eta_i &= \int_0^h r^\gamma \log(b^2 + r^2 - 2br\cos\alpha)^{1/2} dr \\ &= \frac{1}{2}\int_0^h r^\gamma \log(b^2 + r^2 - 2br\cos\alpha)\, dr .\end{aligned} \tag{4.36}$$

Integrating by parts yields

$$\eta_i = \frac{1}{2(\gamma+1)} r^{\gamma+1}\log(b^2 + r^2 - 2br\cos\alpha)\Big|_0^h$$

$$- \frac{1}{(\gamma+1)}\int_0^h \frac{r^{\gamma+1}(r-b\cos\alpha)}{b^2 + r^2 - 2br\cos\alpha} dr ,$$

and thus

$$\eta_i = \frac{1}{2(\gamma+1)} h^{\gamma+1}\log(b^2 + h^2 - 2bh\cos\alpha)$$

$$- \frac{1}{(\gamma+1)}\int_0^h \frac{r^{\gamma+2} - b\cos\alpha\cdot r^{\gamma+1}}{b^2 + r^2 - 2br\cos\alpha} dr$$

$$= I_1 - \frac{1}{(\gamma+1)}(I_2 - b\cos\alpha\ I_3) \tag{4.37}$$

where

$$I_1 = \frac{1}{2(\gamma+1)} h^{\gamma+1}\log(b^2 + h^2 - 2bh\cos\alpha), \tag{4.38}$$

$$I_2 = \int_0^h \frac{r^{\gamma+2}}{b^2 + r^2 - 2br\cos\alpha} dr , \tag{4.39}$$

$$I_3 = \int_0^h \frac{r^{\gamma+1}}{b^2 + r^2 - 2br\cos\alpha}\, dr \quad , \tag{4.40}$$

and $\gamma = -1/2$ or $1/2$.

The integrals in equations (4.39) and (4.40) can be expressed in terms of and integral $J_{1/2}(\alpha)$ defined by

$$J_{1/2}(\alpha) = \int_0^h \frac{x^{1/2}}{x^2 - 2b\cos\alpha\, x + b^2}\, dx \tag{4.41}$$

To evaluate this integral define

$$\xi = b\cos\alpha,\ \eta = b\sin\alpha \tag{4.42}$$

$$\tau = \xi + i\eta = be^{i\alpha} \tag{4.43}$$

and then it can be shown (see appendix E) that

$$J_{1/2}(\alpha) = \frac{1}{b^{1/2}\sin\alpha}\left\{ \sin\tfrac{\alpha}{2}\, J_L(\alpha) + \cos\tfrac{\alpha}{2}\, J_T(\alpha)\right\} \tag{4.44}$$

where $J_L(\alpha)$ and $J_T(\alpha)$ are defined by

$$J_L(\alpha) = \tfrac{1}{2}\log\left(\frac{h - 2\cos\frac{\alpha}{2}(bh)^{1/2} + b}{h + 2\cos\frac{\alpha}{2}(bh)^{1/2} + b}\right) \tag{4.45}$$

$$J_T(\alpha) = \tan^{-1}\left(\frac{h^{1/2} + b^{1/2}\cos\frac{\alpha}{2}}{b^{1/2}\sin\frac{\alpha}{2}}\right) - \tan^{-1}\left(\frac{b^{1/2}\cos\frac{\alpha}{2} - h^{1/2}}{b^{1/2}\sin\frac{\alpha}{2}}\right) \tag{4.46}$$

Two other integrals occur in the evaluation of equation (4.39) and (4.40). The first of these

$$J_{3/2}(\alpha) = \int_0^h \frac{x^{3/2}}{x^2 - 2b\cos\alpha\, x + b^2}\, dx \tag{4.47}$$

is evaluated in Appendix F and is given by

$$J_{3/2}(\alpha) = \frac{b^{1/2}}{\sin\alpha}\{\sin(\tfrac{3\alpha}{2})\, J_L(\alpha) + \cos(\tfrac{3\alpha}{2})\, J_T(\alpha)\} + 2h^{1/2} \tag{4.48}$$

The second of these integrals is also evaluated in Appendix F and is gaven by

$$J_{5/2}(\alpha) = \int_0^h \frac{x^{5/2}}{x^2 - 2b\cos\alpha\, x + b^2}\, dx$$

$$= \frac{b^{3/2}}{\sin\alpha}\{\sin(\tfrac{5\alpha}{2})\, J_L(\alpha) + \cos(\tfrac{5\alpha}{2})\, J_T(\alpha)\}$$

$$+ 4b\cos\alpha\, h^{1/2} + \tfrac{2}{3} h^{3/2} \tag{4.49}$$

We may write the results for equation (4.37) in a concise form by also defining

$$J_0(\alpha) = \log(b^2 + h^2 - 2bh\cos\alpha) \tag{4.50}$$

Thus, for $\gamma = -1/2$, we have from equation (4.37)

$$\eta_{i0} = h^{1/2} J_0(\alpha) - 2(J_{3/2}(\alpha) \text{- } b\cos\alpha\, J_{1/2}(\alpha)) \tag{4.51}$$

Now for $\gamma = 1/2$, we have from equation (4.37)

$$\eta_{i1} = \tfrac{1}{3} h^{3/2} J_0(\alpha) - \tfrac{2}{3}(J_{5/2}(\alpha) \text{- } b\cos\alpha\, J_{3/2}(\alpha)) \tag{4.52}$$

Note that the $J(\alpha)$ functions depend upon the geometry of the boundary contour only.

4.3 Line Integrals Over Intevals Remote From The Singular Point

Based upon the analytical results for line integrals over the interval nearest the singular point, we can easily carry out those integrals over the intervals remote from the singular point . We can simply use the idea of superposition of line integrals. Begining with

$$\eta_{i2} = \int_{s_{n-1}}^{s_n} r^{-1/2} \log R \, ds \tag{4.53}$$

$$= \int_{s_{n-1}}^{s_1} r^{-1/2} \log R \, ds - \int_{s_n}^{s_1} r^{-1/2} \log R \, ds$$

$$= \int_{s_{n-1}}^{s_1} r^{-1/2} \log R \, ds - \eta_{i0}$$

Thus,

$$\eta_{i0} + \eta_{i2} = \int_{s_{n-1}}^{s_1} r^{-1/2} \log R \, ds \tag{4.54}$$

Similarly, we have

$$\eta_{i1} + \eta_{i3} = \int_{s_{n-1}}^{s_1} r^{1/2} \log R \, ds \tag{4.55}$$

These two integrals are of the same form as those we treated in the last section . The only difference is that the integral interval is now from 0 to 2h instead of from 0 to h.

5. Numerical Results and Conclusions

5.1 Numerical Results

In this section, we apply the modified boundary integral equation method with the present local treatment of singularities to the earlier examples shown in figure 3.3 and figure 3.9. We will use both two terms and three terms in the expansion near the singular point.

In figure 5.1 , the constants associated with the singualrity for the example in figure 3.9 are presented for the mesh size n = 30 and n = 60. It is easy to see that the rate of convergence is rapid enough to be comparable with that of Symm's (1973) method as well as that of Ingham and Kelmanson (1984) . As a matter of fact, the first constant C_0 obtained with the present method using a two term expansion is only 0.14 % in error from that of Symm's method. However, the second singularity constant C_1 of the present method differs by 3 % from Symm's result, but is almost the same as that of Ingham and kelmarson . On the other hand, better accuracy is achieved for both Co and C_1 using the present method with a three term expansion. The maximum error is only 0.03 % as compared to Symm's results.

In figure 5.2, the singularity constants for the example shown in figure 3.3 are presented. Even though the singularity is now much stronger at the origin O , the maximum error for both constants obtained by the present method with a three term expansion is only 0.3 %.

5. 2 Conclusions

A modification of the classical boundary integral equation method has been presented which enables accurate treatment of Laplace equations containing boundary singularities. This method requires a slight modification of the classical method with the reward of a dramatic inprovement in the rate of convergence of results throughout the entire solution. Since the analytical results of those integrals for the analytical nature of the singularity have been obtained, the present method provides a very simple and efficient approch to treat boundary singularities locally. In particular, whenever there are more than one boundary singularity, this method can be simply applied at each singular point with very little extra effort. From the point of view of programming, the present method is relatively easy to impliment.

(a) Two term expansion

n	C_o	C_1
30	-0.48265	0.03015
60	-0.48267	0.03118

(b) Three term expansion

n	C_o	C_1	C_2
30	-0.4836	0.02988	-0.00007
60	-0.4834	0.02987	-0.00003

Figure 5. 1. Computed results for constants associated with the singularity for the example of figure 3. 9.

(a) Two term expansion

n	C_o	C_1
48	151.31	3.99
96	151.36	3.88

(b) Three term expansion

n	C_o	C_1	C_2
48	150.97	4.754	0.157
96	151.58	4.713	0.134

Figure 5. 2. Computed results for constants associated with the singularity for the example of figure 3. 3.

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Appendix A

In this appendix, the analytical formulae for the integral in equation (4.16) are derived. We have

$$\int_{s_n}^{s_1} r^{-1/2} \log R_i \, ds$$

$$= \int_{s_n}^{P_n} r^{-1/2} \log R_i \, ds \; + \int_{P_n}^{s_1} r^{-1/2} \log R_i \, ds$$

$$= \int_0^h r^{-1/2} \log r \, dr + \int_0^b r^{-1/2} \log \left(\frac{b}{r} - 1 \right) dr$$

$$+ \int_b^h r^{-1/2} \log \left(1 - \frac{b}{r} \right) dr \, , \tag{A.1}$$

where $b = h/2$. Each of these integrals will be denoted by I_1, I_2 and I_3 respectively. The evaluation of I_1 is immediate and gives

$$I_1 = \int_0^h r^{-1/2} \log r \, dr = 2h^{1/2} \log h - 4h^{1/2} \tag{A.2}$$

The second integral is

$$I_2 = \int_0^b r^{-1/2} \log \left(\frac{b}{r} - 1 \right) dr \tag{A.3}$$

and to evaluate, let $r = b \cos^2 x$ for which

$$dr = -2b \cos x \cdot \sin x \, dx. \tag{A.4}$$

Therefore,

$$I_2 = \int_0^{\pi/2} 4b^{1/2} \cdot \sin x \, \log (\tan x) dx$$

$$= -4b^{1/2}\{ \lim_{\epsilon\to 0} [- \cos\epsilon \cdot \log (\tan\epsilon) + \log\tan (\tfrac{\epsilon}{2})]$$

$$+ \lim_{\eta\to\pi/2} [\cos\eta \cdot \log (\tan\eta) - \log\tan (\tfrac{\eta}{2})]$$

$$= -4b^{1/2}\{ \lim_{\epsilon\to 0} [- \log (\tan\epsilon) + \log\tan (\tfrac{\epsilon}{2})]$$

$$= -4b^{1/2}\log 2. \tag{A.5}$$

Lastly, consider the integral

$$I_3 = \int_b^h r^{-1/2} \log (1 - \tfrac{b}{r})dr = \int_b^{2b} \frac{1}{r^{1/2}} \log (1 - \tfrac{b}{r}) \, dr \, . \tag{A.6}$$

To solve, let

$$r = b \sec^2 x, \qquad (b \leq x \leq 2b) \tag{A.7}$$

so that

$$dr = 2b \sec^2 x \tan x \cdot dx, \tag{A.8}$$

and consequently

$$I_3 = \int_0^{\pi/4} 4b^{1/2} \cdot \log (\sin x) \cdot d (\sec x)$$

$$= 4b^{1/2} \{ \sec x \cdot \log \text{din } x - \int_0^{\pi/4} \csc x \, dx \}$$

$$= 4b^{1/2} \lim_{\epsilon\to 0} \{ \sec x \log \sin x - \log\tan (x/2)\}$$

$$= 4b^{1/2}\{ [2^{1/2}\log\sin (\pi/4) - \log [\tan (\pi/8)]\}$$

$$- \lim_{\epsilon \to 0} \{ \sec\epsilon \log (\sin\epsilon) \ - \log \tan (\epsilon/2) \} \tag{A.9}$$

Note that

$$\lim_{\epsilon \to 0} \{ \sec\epsilon \log (\sin \epsilon) \ - \log \tan (\epsilon/2) \} = 0 \tag{A.10}$$

$$\lim_{\epsilon \to 0} \{ \log \epsilon - \log (\epsilon/2) \} = + \log 2 \tag{A.11}$$

and it follows that

$$I_3 \ = 4b^{1/2}\{ \ 2^{1/2}\log \frac{2^{1/2}}{2} - \log [\tan (\pi/8) - \log 2 \}. \tag{A.12}$$

Consequently, we have (summing all three contributions)

$$\int_{s_N}^{s_1} r^{-1/2} \log R_i \, ds = 2h^{1/2}(\log h - 2)$$

$$+ 4b^{1/2}\{ - \frac{2^{1/2}}{2} \log 2 - \log [\tan (\pi/8)\} \ . \tag{A.13}$$

This expression may be simplified somewhat by noting that

$$\log \tan(\tfrac{\pi}{8}) = \log \tan [(\tfrac{\pi}{4})/2] = \tfrac{1}{2} \log (\frac{2^{1/2} - 1}{2^{1/2} + 1}) \ , \tag{A.14}$$

and thus

$$\int_{s_N}^{s_1} r^{-1/2} \log R_i \, ds = h^{1/2}\{ 2 \log h - K_0\} \ , \tag{A.15}$$

where

$$K_0 = 4 - 2 \log 2 - 2^{1/2}\log(\frac{2^{1/2} - 1}{2^{1/2} + 1})\}. \tag{A.16}$$

Appendix B

In this appendix, the analytical formulae for the integral in equation (4.23) are derived. We have

$$\int_{s_n}^{s_1} r^{-1/2} \log R_i \, ds$$

$$= \int_0^h r^{-1/2} \log r \, dr + \int_0^b r^{-1/2} \log \left(\tfrac{b}{r} - 1 \right) dr + \int_b^h r^{-1/2} \log \left(1 - \tfrac{b}{r} \right) dr, \qquad \text{(B.1)}$$

where $b = h/2$. Each integral in equation (B.1) will be denoted by I_1, I_2 and I_3 respectively. By an integration by parts, it follows that

$$I_1 = \int_0^h r^{-1/2} \log r \, dr = \tfrac{2}{3} h^{3/2} \{ \log h - \tfrac{2}{3} \}. \qquad \text{(B.2)}$$

The second integral is

$$I_2 = \int_0^b r^{1/2} \log \left(\tfrac{b}{r} - 1 \right) dr ,$$

and setting $r = b \cos^2 x$, it follows that

$$I_2 = \int_0^{\pi/2} b^{1/2} \cdot \cos x \cdot \log (\tan^2 x) \cdot b \cdot 2 \sin x \cos x \, dx$$

$$= -4 b^{1/2} \cdot b \int_0^{\pi/2} - \log(\tan x) \cdot d\left(\frac{\cos^3 x}{3}\right). \qquad \text{(B.3)}$$

Integrating by parts gives

$$I_2 = -\tfrac{4}{3} b^{3/2} [\cos^3 x \cdot \log(\tan x)]_0^{\pi/2} - \int_0^{\pi/2} \frac{\cos x}{\tan x} dx \qquad \text{(B.4)}$$

and since

$$\int \frac{\cos x \, dx}{\tan x} = \log |\tan \tfrac{x}{2}| + \cos x , \qquad \text{(B.5)}$$

$$I_2 = \frac{2^{1/2}}{3} h^{3/2} (\log 2 - 1). \qquad \text{(B.6)}$$

Lastly consider

$$I_3 = \int_b^h r^{1/2} \log \left(1 - \frac{b}{r} \right) dr \, , \tag{B.7}$$

and since b = h/2,

$$I_3 = \int_b^{2b} r^{1/2} \log \left(1 - \frac{b}{r} \right) dr. \tag{B.8}$$

Introduce the variable x defined by

$$r = b \sec^2 x; \qquad (b \leq r \leq 2b). \tag{B.9}$$

and equation (B.8) becomes

$$I_3 = \int_0^{\pi/4} b^{1/2} \sec x \cdot \log (1 - \cos^2 x) \, 2b \sec^2 x \cdot \tan x \cdot dx$$

$$= 4b^{1/2} \cdot b \int_0^{\pi/4} \log(\sin x) \cdot \sec^2 x \cdot d \cdot \sec x$$

$$= \frac{4}{3} b^{3/2} \{ \sec^3 x \cdot \log(\sin x) - \log(\tan(\tfrac{x}{2})) - \sec x \}_0^{\pi/2}. \tag{B.10}$$

Taking the limits indicated, it can be shown that

$$I_3 = - \frac{1}{3} h^{3/2} \{ 2\log 2 + 2^{1/2} \log 2 + 2^{1/2} \log \tan\frac{\pi}{8} + 2 - 2^{1/2} \} \tag{B.11}$$

Consequently, it follows that

$$\int_{s_n}^{s_1} r^{-1/2} \log R_i \, ds = I_1 + I_2 + I_3 = \frac{1}{3} h^{3/2} \{ 2\log h - K_1 \} \tag{B.12}$$

where the constant K_1 is given by

$$K_1 = \frac{4}{3} + 2 + 2^{1/2} \log \tan\frac{\pi}{8} + 2 \log 2$$

$$= 3.4731772141727629253. \tag{B.13}$$

Appendix C

In this appendix, the evaluation of equation (4.27) is considered. For $\alpha = 0$ and $\gamma = -1/2$, we have

$$\eta_{io} = \int_0^h r^{-1/2} \log (b^2 - 2br + r^2)^{1/2} dr$$

$$= 2 \int_b^h \log (b - r) dr^{1/2}$$

$$= 2h^{1/2} \log (b - h) + 2I_0, \tag{C.1}$$

where

$$I_0 = \int_b^h \frac{r^{1/2}}{b - r} dr \,. \tag{C.2}$$

Let $r = b \sin^2 x$; then, $\sin x = (\frac{r}{b})^{1/2}$ and

$r = 0$ at $x = 0$ and at $r = h$, $x = \hat{x} = \arcsin (\frac{h}{b})^{1/2}$. Therefore,

$$I_0 = 2b^{1/2} \int_0^{\hat{x}} \frac{\sin^2 x}{\cos x} dx$$

$$= 2b^{1/2} \int_0^{\hat{x}} (\frac{1 - \cos^2 x}{\cos x}) dx$$

$$= 2b^{1/2} \int_0^{\hat{x}} (\sec x - \cos x) \, dx$$

$$= 2b^{1/2} [\log(\sec\hat{x} + \tan\hat{x}) \Big|_0^{\hat{x}} - \sin x \Big|_0^{\hat{x}}]$$

$$= 2b^{1/2} [\log(\sec\hat{x} + \tan\hat{x}) - \sin x] \,. \tag{C.3}$$

Since $\sec\hat{x} = (\frac{b}{b-h})^{1/2}$, $\tan\hat{x} = (\frac{h}{b-h})^{1/2}$,

we have

$$(\sec\hat{x} + \tan\hat{x}) = \frac{b^{1/2} + h^{1/2}}{\sqrt{b - h}} = \left(\frac{b^{1/2} + h^{1/2}}{b^{1/2} - h^{1/2}} \right)^{1/2}.$$

Then,

$$\log(\sec\hat{x} + \tan\hat{x}) = \frac{1}{2}\log \frac{b^{1/2} + h^{1/2}}{b^{1/2} - h^{1/2}}$$

and I_0 can be written

$$I_0 = 2b^{1/2}\left\{ \frac{1}{2}\log \frac{b^{1/2} + h^{1/2}}{b^{1/2} - h^{1/2}} - \frac{h^{1/2}}{b^{1/2}} \right\}$$

$$= b^{1/2}\left\{ \frac{1}{2}\log \frac{b^{1/2} + h^{1/2}}{b^{1/2} - h^{1/2}} - 2h^{1/2} \right\}. \qquad \text{(C.4)}$$

Thus, it follows that

$$\eta_{io} = 2h^{1/2}\log(b - h) + 2I_0$$

$$= 2h^{1/2}\log(b - h) + 4h^{1/2} - 2b^{1/2}\log\left(\frac{b^{1/2} + h^{1/2}}{b^{1/2} - h^{1/2}} \right). \qquad \text{(C.5)}$$

Next, consider the situation $\alpha = 0$, $\gamma = +\frac{1}{2}$, for which

$$\eta_{i1} = \int_0^h r^{1/2} \log(b - r)dr$$

$$= \frac{2}{3}\int_0^h \log(b - r)dr^{3/2} \qquad \text{(C.6)}$$

By an integration by parts, it follows

$$\eta_{i1} = \frac{2}{3}[r^{3/2}\log(b - r)]\Big|_0^h + \frac{2}{3}\int_0^h \frac{r^{3/2}}{b - r}dr$$

$$= \frac{2}{3}h^{3/2}\log(b - h) + \frac{2}{3}I_1, \qquad \text{(C.7)}$$

where

$$I_1 = \int_0^h \frac{r^{3/2}}{b - r}\, dr = \int_0^h r^{1/2} \left(\frac{r}{b - r}\right) dr$$

$$= b \int_0^h \left(\frac{r^{1/2}}{b - r}\right) dr - \int_0^h r^{1/2}\, dr$$

$$= b\, I_0 - \frac{2}{3}\, h^{3/2} \tag{C.8}$$

Therefore, we have

$$I_1 = bI_0 - \frac{2}{3}h^{3/2}$$

$$= b^{3/2}\left\{ \log \frac{b^{1/2} + h^{1/2}}{b^{1/2} - h^{1/2}} - 2bh^{1/2} - \frac{2}{3}h^{3/2} \right\} \tag{C.9}$$

and consequently

$$\eta_{i1} = \int_0^h r^{1/2} \log (b - r)dr$$

$$= \frac{2}{3}h^{3/2}\log(b - h) + \frac{2}{3}\, I_1 \tag{C.10}$$

Appendix D

In this appendix, the evaluation of equation (4.32) is considered. For $\alpha = \pi$, and $\gamma = -1/2$, we have

$$\eta_{i0} = \int_0^h r^{-1/2} \log (b + r)dr \quad .$$

$$= 2 \int_0^h \log (b + r)dr^{1/2} \tag{D.1}$$

By an integration by parts, we have

$$\eta_{i0} = 2r^{1/2}\log(b + r)] \Big|_0^h + 2 \int_0^h \frac{r^{1/2}}{b + r} dr$$

$$= 2h^{1/2}\log(b + h) + 2\, I_2 \tag{D.2}$$

where,

$$I_2 = \int_0^h \frac{r^{1/2}}{b + r} dr \tag{D.3}$$

To evaluate the integral (D.3), let $r = b \tan^2 x$ and thus $x = \tan^{-1}(\frac{r}{b})^{1/2}$. Letteing $\hat{x} = \tan^{-1}(\frac{h}{b})^{1/2}$, we have

$$I_2 = \int_0^{\hat{x}} 2b^{1/2}\tan^2 x \cdot dx$$

$$= 2b^{1/2} \int_0^{\hat{x}} (\sec^2 x - 1)dx \tag{D.4}$$

By integration,

$$I_2 = 2b^{1/2} \{ \tan x - x \} \Big|_0^{\hat{x}=\tan^{-1}(\frac{h}{b})^{1/2}}$$

$$= 2b^{1/2}\left\{\frac{h^{1/2}}{b^{1/2}} - \arctan\left(\frac{h^{1/2}}{b^{1/2}}\right)\right\}$$

$$= 2h^{1/2} - 2b^{1/2}\tan^{-1}\left(\frac{h^{1/2}}{b^{1/2}}\right) \qquad (D.5)$$

By substitution of I_2 into equation (D.2), we have

$$\eta_{io} = 2h^{1/2}\log(b + h) + 2I_2$$

$$= 2h^{1/2}\log(b + h) - 4h^{1/2} + 4b^{1/2}\tan^{-1}\left(\frac{h^{1/2}}{b^{1/2}}\right) . \qquad (D.6)$$

Lastly for the case of $\alpha = \pi$ and $\gamma = \frac{1}{2}$, we have

$$\eta_{i1} = \int_0^h r^{1/2} \log (b + r)dr$$

$$= \frac{2}{3}\int_0^h \log (b + r)dr^{3/2} \qquad (D.7)$$

By an integration by parts, it follows that

$$\eta_{i1} = \frac{2}{3}\left[r^{3/2}\log(b + r)\Big|_0^h + \frac{2}{3}\int_0^h \frac{r^{3/2}}{b + r}\, dr \right]$$

$$= \frac{2}{3}h^{3/2}\log(b + h) + \frac{2}{3}\, I_3 , \qquad (D.8)$$

where

$$I_3 = \int_0^h \frac{r^{3/2}}{b + r}\, dr \ . \qquad (D.9)$$

This integral can be evaluated as follows :

$$I_3 = \int_0^h r^{1/2} \left(\frac{r}{b + r}\right) dr ,$$

$$= \int_0^h r^{1/2}\left(1 - \frac{b}{b+r}\right) dr ,$$

$$= \frac{2}{3}h^{3/2}\Big|_0^h - b\ I_2 \ , \qquad \text{(D.10)}$$

where $I_2 = \int_0^h \frac{r^{1/2}}{b+r} dr$ is known from equation (D.5). Thus, we have

$$I_3 = \frac{2}{3}h^{3/2} - b[2h^{1/2} - 2b^{1/2}\tan^{-1}(\frac{h}{b})^{1/2}]$$

$$= \frac{2}{3}h^{3/2} - 2bh^{1/2} + 2b^{3/2}\tan^{-1}(\frac{h}{b})^{1/2}. \qquad \text{(D.11)}$$

Substituting this result into the equation (D.8), we have

$$\eta_{i1} = \int_0^h r^{1/2} \log(b + r) dr$$

$$= \frac{2}{3}h^{3/2}\log(b + h) - \frac{2}{3}\ I_3$$

$$= \frac{2}{3}\ h^{3/2}\log(b + h) - \frac{2}{3}h^{3/2} + 2h^{1/2}b - 2b^{3/2}\tan^{-1}(\frac{h}{b})^{1/2}\ . \qquad \text{(D.12)}$$

Appendix E

In this appendix, the analytical formula for the integral in equation (4.41) is derived. We have for $\xi = b \cos \alpha$, $\eta = b \sin \alpha$

$$J_{1/2}(\alpha) = \int_0^x \frac{t^{1/2}}{(\xi - t)^2 + \eta^2} dt \tag{E.1}$$

where the subscript 1/2 is a convenient notation associated with the power of t in the numerator. By multiplying a common factor $(\eta t^{1/2})$ through both numerator and denominator, we have

$$J_{1/2}(\alpha) = \int_0^x \frac{\eta t \cdot dt}{(\eta t^{1/2}) [(\xi - t)^2 + \eta^2]}$$

$$= I_m \int_0^x \left(\frac{-1}{\eta t^{1/2}} \right) \frac{(-i\eta t)}{[(\xi - t) + i\eta][(\xi - t) - i\eta]} dt \ . \tag{E.2}$$

where I_m denotes the imaginary part. By adding a real term and then making simplifications it can be shown that

$$J_{1/2}(\alpha) = -I_m \int_0^x \left(\frac{1}{\eta t^{1/2}} \right) \frac{\left(\xi(\xi - t) + \eta^2\right) - i\eta t)}{[(\xi - t) + i\eta][(\xi - t) - i\eta]} dt$$

$$= -I_m \int_0^x \left(\frac{1}{\eta t^{1/2}} \right) \frac{(\xi + i\eta)[(\xi - t) - i\eta]}{[(\xi - t) + i\eta][(\xi - t) - i\eta]} dt$$

$$= -I_m \int_0^x \left(\frac{1}{\eta t^{1/2}} \right) \frac{\xi + i\eta}{\xi + i\eta - t} dt \ . \tag{E. 3}$$

To evaluate the integral first define $\tau = \xi + i\eta$ and equation (3) becomes

$$J_{1/2}(\alpha) = I_m \left\{ \int_0^x \left(-\frac{1}{\eta t^{1/2}} \right) \left(\frac{\tau}{\tau - t} \right) dt \right\}$$

$$= I_m \left\{ \int_0^x -\frac{1}{2t^{1/2}} \frac{\tau^{1/2}}{\eta} \left[\frac{1}{\tau^{1/2} - t^{1/2}} + \frac{1}{\tau^{1/2} + t^{1/2}} dt \right\}$$

$$= I_m \left\{ \frac{\tau^{1/2}}{\eta} \int_0^x \left[\frac{1}{\tau^{1/2} - t^{1/2}} \left(- \frac{1}{2t^{1/2}} \right) - \frac{1}{\tau^{1/2} + t^{1/2}} \frac{1}{2t^{1/2}} \right] dt \right\}$$

$$= I_m \left\{ \frac{\tau^{1/2}}{\eta} \left(\ln (\tau^{1/2} - t^{1/2}) - \ln (\tau^{1/2} + t^{1/2}) \right) \Big|_0^x \right\}$$

$$= I_m \left\{ \frac{\tau^{1/2}}{\eta} \left[\ln \frac{\tau^{1/2} - x^{1/2}}{\tau^{1/2} + x^{1/2}} \right] \right\} \tag{E.4}$$

The result may be writen in real form by first recalling that

$$\tau = \xi + i\eta = b \cos \alpha + ib \sin \alpha = b\, e^{i\alpha}$$

Therefore

$$J_{1/2}(\alpha) = I_m \left\{ \frac{\tau^{1/2}}{\eta} \ln \left(\frac{\tau^{1/2} - x^{1/2}}{\tau^{1/2} + x^{1/2}} \right) \right\}$$

$$= I_m \left\{ \frac{\tau^{1/2}}{\eta} \ln \left(\frac{i\,(b^{1/2}\cos(\alpha/2) + ib^{1/2}\sin(\alpha/2) - x^{1/2})}{i\,(b^{1/2}\cos(\alpha/2) + ib^{1/2}\sin(\alpha/2) + x^{1/2})} \right) \right\}$$

$$= I_m \left\{ \frac{\tau^{1/2}}{\eta} \left[\ln (b^{1/2}\sin(\alpha/2) + i(b^{1/2}\cos(\alpha/2) - x^{1/2}) \right. \right.$$

$$\left. - \ln (-b^{1/2}\sin(\alpha/2) + i(b^{1/2}\cos(\alpha/2) + x^{1/2}) \right\}$$

$$= I_m \left\{ \frac{\tau^{1/2}}{\eta} \left[\log \left(\frac{b - 2\cos(\alpha/2) b^{1/2} x^{1/2} + x}{b + 2\cos(\alpha/2) b^{1/2} x^{1/2} + x} \right)^{1/2} \right. \right.$$

$$\left. \left. + i \left[\tan^{-1} \left(\frac{x^{1/2} + b^{1/2}\cos(\alpha/2)}{b^{1/2}\sin(\alpha/2)} \right) - \tan \left(\frac{b^{1/2}\cos(\alpha/2) - x^{1/2}}{b^{1/2}\sin(\alpha/2)} \right) \right] \right\}. \tag{E.5}$$

Now, we define the following real functions:

$J_L(\alpha) + i\, J_T(\alpha)$

$$J_L(\alpha) = \frac{1}{2} \log \left(\frac{b - 2\cos(\alpha/2) b^{1/2} x^{1/2} + x}{b + 2\cos(\alpha/2) b^{1/2} x^{1/2} + x} \right), \tag{E.6}$$

$$J_T(\alpha) = \tan^{-1}\left(\frac{x^{1/2} + b^{1/2}\cos(\alpha/2)}{b^{1/2}\sin(\alpha/2)} \right) - \tan\left(\frac{b^{1/2}\cos(\alpha/2) - x^{1/2}}{b^{1/2}\sin(\alpha/2)} \right). \tag{E.7}$$

equation (E.5) can be writen according to

$$J_{1/2}(\alpha) = \mathrm{Im} \left\{ \frac{\tau^{1/2}}{\eta} \left(J_L(\alpha) + i\, J_T(\alpha) \right) \right\}$$

$$= \mathrm{Im} \left\{ \frac{b^{1/2} e^{i\alpha/2}}{b \sin\alpha} \left(J_L(\alpha) + i\, J_T(\alpha) \right) \right\}$$

$$= \frac{1}{b^{1/2} \sin\alpha} \left(\cos\tfrac{\alpha}{2}\, J_L(\alpha) + \sin\tfrac{\alpha}{2}\, J_T(\alpha) \right). \tag{E.8}$$

Appendix F

In this appendix, the analytical formulae for the integrals (4.47) and (4.49) are derived based on the results obtained in Appendix E. We define integrals $J_{3/2}(\alpha)$ and $J_{5/2}(\alpha)$ as follows :

$$J_{3/2}(\alpha) = \int_0^x \frac{t^{3/2}}{(\xi - t)^2 + \eta^2} dt \ , \tag{F.1}$$

$$J_{5/2}(\alpha) = \int_0^x \frac{t^{5/2}}{(\xi - t)^2 + \eta^2} dt \ . \tag{F.2}$$

Again, the subscript 3/2 or 5/2 is just a notation corresponding to the power of t in the numerator. Beginning with the equation (F.1) and using the same method as in Appendix E,

$$J_{3/2}(\alpha) = \int_0^x \frac{t^{3/2}}{(t - \xi)^2 + \eta^2} dt = \int_0^x t^{1/2} \frac{t}{(t - \xi)^2 + \eta^2} dt$$

$$= \int_0^x \frac{t^{1/2}}{\eta} \cdot \frac{\eta t}{[(\xi - t) + i\eta][(\xi - t) - i\eta]} dt$$

$$= -\mathrm{Im} \int_0^x \left(\frac{t^{1/2}}{\eta} \right) \frac{-i\eta t}{[(\xi - t) + i\eta][(\xi - t) - i\eta]} dt$$

$$= \mathrm{Im} \int_0^x \left(-\frac{t^{1/2}}{\eta} \right) \frac{\xi + i\eta}{\xi + i\eta - t} dt \tag{F.3}$$

To evaluate this integral, let $\tau = \xi + i\eta$ and then

$$J_{3/2}(\alpha) = \mathrm{Im} \int_0^x \left(-\frac{t^{1/2}}{\eta} \right) \frac{\tau}{\tau - t} dt$$

$$= \mathrm{Im} \int_0^x \left(-\frac{\tau}{\eta t^{1/2}} \right) \frac{t}{\tau - t} dt$$

$$= \mathrm{Im} \int_0^x \left(- \frac{\tau}{\eta t^{1/2}} \right) \frac{\tau - (\tau - t)}{\tau - t} \, dt$$

$$= \mathrm{Im} \left\{ \int_0^x \left(- \frac{\tau}{\eta t^{1/2}} \right) \frac{\tau}{\tau - t} \, dt + \int_0^x \left(\frac{\tau}{\eta t^{1/2}} \right) dt \right\} \tag{F.4}$$

Denoting the two integrals as J_1 and J_2 according to

$$J_1 = \tau \int_0^x \left(- \frac{1}{\eta t^{1/2}} \right) \frac{\tau}{\tau - t} \, dt = \tau \; J_{1/2}(\alpha)$$

$$= \frac{\tau^{3/2}}{\eta} \ln \left(\frac{\tau^{1/2} - x^{1/2}}{\tau^{1/2} + x^{1/2}} \right), \tag{F.5}$$

$$J_2 = \int_0^x \frac{\tau}{\eta t^{1/2}} \, dt = 2x^{1/2} \frac{\tau}{\eta} . \tag{F.6}$$

It follows that

$$J_{3/2}(\alpha) = \int_0^x \frac{t^{3/2}}{(t - \xi)^2 + \eta^2} dt$$

$$= \mathrm{Im} \left\{ \frac{\tau^{3/2}}{\eta} \ln \left(\frac{\tau^{1/2} - x^{1/2}}{\tau^{1/2} + x^{1/2}} \right) + 2x^{1/2} \frac{\tau}{\eta} \right\} \tag{F.7}$$

In a similar manner,

$$J_{5/2}(\alpha) = \int_0^x \frac{t^{5/2}}{(\xi - t)^2 + \eta^2} dt = \int_0^x t^{3/2} \frac{t}{(t - \xi)^2 + \eta^2} dt$$

$$= \mathrm{Im} \int_0^x \left(- \frac{t^{3/2}}{\eta} \right) \cdot \left(\frac{\tau}{\tau - t} \right) dt$$

$$= \mathrm{Im} \int_0^x \left(- \frac{t\tau}{\eta t^{1/2}} \right) \left(\frac{t}{\tau - t} \right) dt$$

$$= \mathrm{Im} \left\{ \int_0^x \left(- \frac{t\tau}{\eta t^{1/2}} \right) \left(\frac{\tau - (\tau - t)}{\tau - t} \right) dt \right.$$

$$= \mathrm{Im}\left\{ \int_0^x \left(- \frac{t\tau}{\eta t^{1/2}} \right)\left(\frac{\tau}{\tau - t}\right) dt + \int_0^x \left(\frac{t\tau}{\eta t^{1/2}} \right) dt \right\}$$

$$= \mathrm{Im}\left\{ \int_0^x \left(- \frac{\tau^2}{\eta t^{1/2}} \right)\left(\frac{t}{\tau - t}\right) dt + \int_0^x \left(\frac{t^2\tau}{\eta} \right) dt \right\}$$

$$= \mathrm{Im}\left\{ \int_0^x \left(- \frac{\tau^2}{\eta t^{1/2}} \right)\left(\frac{\tau - (\tau - t)}{\tau - t}\right) dt + \frac{2}{3}\frac{\tau}{\eta} x^{3/2} dt \right\}$$

$$= \mathrm{Im}\left\{ \int_0^x \left(- \frac{\tau^2}{\eta t^{1/2}} \right)\left(\frac{\tau}{\tau - t} - 1\right) dt + \frac{2}{3}\frac{\tau}{\eta} x^{3/2} dt \right\}$$

$$= \mathrm{Im}\,(J_3 + J_4 + J_5) \tag{F.8}$$

where J_3 , J_4 and J_5 are defined as follows :

$$J_3 = \int_0^x \left(- \frac{\tau^3}{\eta t^{1/2}} \right) \frac{1}{\tau - t}\, dt$$

$$= \frac{\tau^{5/2}}{\eta} \ln \left(\frac{\tau^{1/2} - x^{1/2}}{\tau^{1/2} + x^{1/2}}\right) , \tag{F.9}$$

$$J_4 = \int_0^x \frac{\tau^2}{\eta t^{1/2}}\, dt = 2x^{1/2}\frac{\tau^2}{\eta} , \tag{F.10}$$

$$J_5 = \frac{2}{3}x^{3/2}\frac{\tau}{\eta} . \tag{F.11}$$

Finally, we have the analytic form for the integral is

$$J_{5/2}(\alpha) = \int_0^x \frac{t^{5/2}}{(t - \xi)^2 + \eta^2} dt = \mathrm{Im}\,(J_3 + J_4 + J_5)$$

$$= \frac{\tau^{5/2}}{\eta} \ln \left(\frac{\tau^{1/2} - x^{1/2}}{\tau^{1/2} + x^{1/2}}\right) + 2x^{1/2}\frac{\tau^2}{\eta} + \frac{2}{3}x^{3/2}\frac{\tau}{\eta} \tag{F.12}$$

Vita

BIOGRAPHY OF THE CANDIDATE

NAME : Jun Steven He

DATE OF BIRTH : September 15, 1958

PLACE OF BIRTH: Yangchow, Jiangsu province, P. R. China

PARANTS NAME : Zhongzheng He and Wenling Guo

EDUCATION :

7 / 85	M.S. in Civil Engineering, East China Technical University of Water Resources.
8 / 82	B.S. in Civil Engineering, East China Technical University of Water Resources.