1988

A statistical model of induction hardening /

Jack M. Kloeber Jr.
Lehigh University

A STATISTICAL MODEL OF INDUCTION HARDENING

by

Jack M. Kloeber Jr.

A Thesis

Presented to the Graduate Committee

of Lehigh University

in candidacy for the degree of

Master of Science in Industrial Engineering.

Lehigh University

1988

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science in Industrial Engineering.

date: May 20, 1988

John W. Adams
Advisor in Charge

N. E. Kerr
IE Department Chairperson

Acknowledgments

I would like to give my special thanks to Professor John Adams who, as my advisor, always supported and directed me during my research. It has been a great pleasure to work with him as he was always available for advice, discussion and direction. Without his statistical expertise, this thesis would never have been possible.

I would also like to express my thanks to Dr. Gary Miller, Assistant Director of the Materials Research Center, for his insight into the materials science aspect of my research, his help in obtaining the samples used in the experiments, and for coordinating many aspects of the actual running of the experiment.

I would like to thank Mr. Steve Stinson, and all the people from the Materials Science Department of Caterpillar Tractor plant in York, Pennsylvania, for taking the time to produce the required samples used in the research, and for being available for discussions and problem solving sessions.

Lastly, I would like to thank my wife, Elena, and my children Mary, Jack, and Sophia for their patience, understanding and support during my many long hours in front of the computer.

Table of Contents

Abstract

This thesis determines the mathematical relationships involved in the magnetic induction hardening process of steel bars used in various industrial applications. The factors identified as the most significant variables are the bar diameter, coil diameter, composition of the steel, travel speed of the induction coil, and power (voltage) used during the induction. To improve the quality and reliability of the hardening process, both the level of the main effects and the level and nature of the interaction effects on the hardness throughout the steel bar need to be known. Knowledge of the depth at which Rockwell C 50 and Rockwell C 30 are achieved would be sufficient to control the quality and reliability. A model denoting the relationships stated above was found with 95% confidence intervals of approximately 1.1 mm and 1.4 mm (respectively). However, hardness was found to depend not only on the five factors stated above and on the depth from the surface of the bar, but, also on the angular position around the interior of the bar. In Chapter Two, it is shown that a dependence of the hardness on Θ exists but the nature of that dependence is not determined. In Chapter Three, the model that best describes the effect of Θ on the hardness values is shown to be an exponential model that includes the cosine(Θ) within the framework of a 4th order polynomial as a function of depth. This model gives estimates of hardness that are extremely close to the measured values and also yields residual plots that support the assumptions of independent, normally distributed error, with μ=0.0, and constant σ^2. Areas defining research topics extending from this work are discussed.

Chapter 1
AN INITIAL MODEL

1.1 Description of the Problem

1.1.1 Origin of the Problem

Induction hardening is a popular method of improving the mechanical properties of round steel bars used in various high-wear and high stress applications. Induction hardening occurs when an electric coil is placed around a steel rod. The application of power to to the electric coil will induce an electric current in the steel rod which, as a conductor, will heat up as the steel resists the electric current. The bar is subsequently water quenched to produce martensite-a hard microconstituant-at the surface. As the power is increased, more induced current is converted to heat, and there is a greater depth of hardening after quenching. This heat can be used either to harden the rod at the surface(case hardening) or to completely harden the rod(through hardening).

Because the steel bars are used in a variety of applications, they require a variety of mechanical properties and microstructures. In general, the mechanical properties and microstructures are controlled by composition and heat treatment. Induction hardening can play an important role in economically achieving a desired set of properties. Since it can be costly to to produce products that do not meet specifications, it behooves a parts manufacturer to understand the quantitative relationships between desired properties and the process control variables.

One way to measure the effect of the induction hardening is to know the depth beneath the surface at which a certain hardness is obtained after induction hardening. Typically, Rockwell "C"(RC) hardness values of 50 and 30 are used in specifications to insure that an adequate depth of hardneing has been achieved.

Considering quality assurance, a problem that arises is that there is no non-destructive method of explicitly determining depth for a given hardness which is sufficiently accurate. Presently, to be sure that a given depth of hardness is achieved, one must cut a section of the bar, polish it, and make several hardness measurements-a very laborious process. Therefore, a method is needed for accurately predicting the depth to a given hardness as a function of the variables in the induction process.

It is known that steel composition greatly affects hardenability for through hardening and would be expected to have a prominent influence in induction hardening as well. Carbon is especially important reguarding hardenbility. Also important are the percentages of the following elements: titanium, a chromium, molybdenum, vanadium, tungsten, niobium, copper, managanese, and phosphorus. Rather than deal with the effects of each element individually, their influence is lumped together in a parameter called DI. DI is based on the Grosman hardenability factors[1] and is the product of composition and element factor for each element.

$$DI=(\div C)\times f_C\times(\div Mn)\times f_{Mn}\times....$$

A steel with a high value for DI will harden to a deeper depth than will a steel described by a low DI.

The applied power(voltage) is known to be a significant factor in the induction process. Its direct relationship with the amount of hardening that occurs is well documented [2]. Time of exposure to the induction coil is another obvious factor that must be taken into account. Since the process under consideration involves a coil that travels the length of the rod, the speed of travel of the coil is the factor that controls the time of exposure to the coil at any given point on the bar.

Other factors that may affect the hardening process are the diameter of the steel bar to be hardened and the distance of the coil from the steel bar(that is, the inner diameter of the induction coil).

The problem now becomes a more straight-forward one of finding a mathematical model of the induction process which enables one to predict the depth to hardness values of RC 50 and RC 30 as a function of the control variables, DI, power, speed of the coil, and coil and bar diameter.

1.2 Past Work

Some guidance for this thesis comes from an experiment reported in February 1987 entitled,"Modelling the Induction Hardening Process", by Drs. G. A. Miller and J. W. Adams. Their final model defined a relationship between the three independent variables- input power, coil travel speed, and DI (composition)-- and the two dependent variables-- depth of hardness for RC 50 and RC 30. The quenching process, bar diameter, and coil diameter were all held constant for the first half of their experiment. In the second half of the experiment, the bar diameter was shifted. A summary of their results follows:

$$y(11) = 5.00 + 0.88*X1 + 0.35*X2 - 0.31*X3 + 0.26*X1*X3 - 0.27*X2*X3 - 0.57*X1**2 - 0.11*X3^2$$

$$y(20) = 7.72 + 1.42*X1 + 0.56*X2 - 0.24*X3 + 0.03*X1*X2 + 0.28*X1*X3 - 0.51*X2*X3$$

y(11) = The depth to achieve hardness of Rockwell C 50
y(20) = The depth to achieve hardness of Rockwell C 30
X1 = DI (as discussed above)
X2 = input power, % of 240kw
X3 = coil travel speed, inch/second

Variables X1, X2, and X3 were transformed to values between -1 and +1, to facilitate calculations of the statistical analysis.

The effect of time was not found to be significant at the 5% level for α(Type one error). In other words, due to the results obtained in the statistical analysis, one could not be 95% certain that time was a significant factor in the experiment. This was an important finding since measurements taken over several months could now be included in the same analysis and model. The following rcommendations were made:

1. The operating domain for input power, coil travel speed, and bar diameter must be specified.

2. Data should be generated for 2 1/2-inch and possibly 2 1/4-inch diameter bars. The inclusion of a third bar size will allow us to include the effect of bar diameter in the model in a general way that will facilitate checking for curvature.

3. To date, we have relied upon single estimates of depth to a given hardness. We need to make additional measurements on samples already tested to expand the database and allow for measurement error in the model. In essence, results obtained thus far assume that measurements of the depth of hardening do not vary. This is not true and needs to be accounted for in a general model.

1.3 Scope of the Experiment

Even with the previous work done in this field, not enough is known about the exact numerical relationships between the several variables discussed above and the actual effect on the induction process. Especially unclear are the interaction effects, if any, and the curvature effects, if any, of the various 'independent' variables. Here the word 'independent' refers only to the fact that these variables are chosen or assigned to conduct the experiment rather than measured after the fact like the 'dependent' variable, depth of hardness. Independence is not intended to imply that there is no correlation between the variables.

This thesis will focus on the following unanswered questions.

1. It is difficult to find the exact depth for a given hardness using direct measurement because measurements should not be taken closer together than one-sixteenth of an inch. This is due to the fact that at closer distances one measurement may affect its neighbor. Therefore, a technique must be devised to closely estimate the actual average depth to RC 50 and RC 30 hardness. This will be done by approximating the functional relation between hardness and depth.

2. What is the mathematical model that will predict the depth of hardness with a minimum (and acceptable level) of error. This model should include any or all of the following independent variables that are thought to affect the induction hardening process:

    a. Steel Composition (DI)

    b. Diameter of the Steel Bar

    c. Diameter of the Induction Coil

    d. Power Applied to the Induction Coil

    e. Induction Coil Travel Speed

f. Quenching Process (This is normally held constant at 30-35 psi and 80-85 degrees and will not be listed as a factor for this experiment.)

3. In the problem considered above, it is assumed that the actual depth for a given hardness, is independent of location around the bar circumference. Is this a valid assumption? If it is not valid then how exactly does the actual depth of a given hardness vary with the angular position in the bar? Can a single, closed-form function be found that explains the distribution of hardness with respect to both depth and angular distance from a known index point?

4. Assuming all of the above goals are met, can valid confidence intervals be specified for the estimates of depth for a given hardness?

1.4 Experimental Design

Ideally, such an experiment would be designed to be completely orthogonal in all five independent variables. The ranges would have to be found for the five variables and then appropriate replications would be taken for each five factor 'cell'.

One design for this type of experiment is a complete factorial design. Since a factorial design is normally meant for discrete levels of each independent variable, an adequate number of levels for each variable should be chosen and the same number of repetitions produced at all levels of each variable. Then, classical techniques of ANOVA would be appropriate in exploring the interactions and effects of the five identified factors.

As is frequently the case in industrial experiments, the ranges of certain independent variables are subject to practical restrictions. The DI and bar diameter were restricted by the bar stock that was already part of the inventory. The levels of power and speed could be varied easily within a range limited only by the practicality of the values needed to obtain usable steel bars. The coil diameter was limited to the sizes that were on hand at the time of testing, and also limited by the size of the steel bar to be hardened. A table of values that were tested is below.

FACTOR \ LEVELS	1	2	3	4	5	6
Compostion of Steel(DI)	.86				1.34	
Induction Power(% of 240KW)	.75		.85	.90	.95	
Induction Speed(inch/second)	.05		.10	.14	.15	
Diameter of the Bar(inch)	2.25		2.50		2.75	
Coil Diameter(inch)	2.50	2.80	2.90	3.00	3.21	3.40

Table 1.1

Since a full factorial was not possible, a regression model was viewed as the best alternative. Because of the restriction of the number of levels available in each variable, a regression model that included linear terms, linear interaction terms, and quadratic terms would have to suffice in explaining the relationships.

1.5 Procedure

Bar stock was chosen that had the appropriate DI value listed on its heat card. This bar stock was turned to the desired diameter. Then, the steel bar was cut into lengths of 10 to 12 inches long. These lengths were randomly assigned to different levels of coil travel speed, power, and different coil diameter according to the design. The samples were induction hardened and quenched. Each bar was a separate replicate. These disks, about 1 inch in thickness, were marked with the sample number. The "fishtail" location was also indicated with a white line on the outer surface of the disk. This "fishtail", hereafter termed the index point for the rest of this thesis, indicates the location of the power in and power out cords on the induction coil. It was suspected that the magnetic field was different at this point. A change in the induction process could translate into a difference in the depth of a given hardness. Induction hardening was performed at the Caterpillar Tractor plant in York, Pennsylvania. The samples, now 1 inch slices of steel bars, were then shipped to Lehigh University for extensive radial hardness testing.

Since, in the first report by Dr. Miller and Dr. Adams, it was suggested that there may be a difference in the depth of hardness for a given circumferential position around the disk, the hardness measurements were designed to detect this variation, if it existed. The machine to be used was a RC hardness machine which, when correctly calibrated, was accurate to + 1 point of RC hardness. Then, since the points of special interest were hardness values RC 50 and RC 30, measurements were taken and recorded starting from 1/16 inch depth from the surface to approximately a depth at which Rockwell C 20 was reached. The measurements were made at 1/16 inch intervals on a radial line from the surface towards the center of the disk. RC 20 was set as the lower limit because the machine starts to lose its accuracy at readings below RC 20. The first line of measurements was made at the marked index point and was labelled as 0 degrees. The disk was then rotated thirty degrees and another line of measurements was made. The testing continued in this manner until there were twelve lines of tests on the disk.

To complete measurements, each disk took approximately 45 minutes. Therefore, about 6 to 10 disks were measured each day. To maintain independence of the variation of the measurements over time, the machine's calibration was checked each day, using test blocks, before the measurements began.

1.6 Raw Data and Initial Results

Since the first goal of the experiment was to determine a method of closely estimating the average depth of RC 50 and RC 30, I decided to combine all the values for each disk and run a polynomial regression to determine the curve which best describes the relationship between depth and hardness.

$$y = c0 + c1*X + c2*X^2 + c3*X^3 + c4*X^4 \ldots.$$

y = the hardness in terms of Rockwell C
X = the depth in 1/16 inch from the surface of the disk

The following pages contain the measurements for the 62 samples. These data were measured by two different individuals from late June 1987 to early August 1987. Note that some samples were tested to 11/16 inch while others could only be tested to 6/16 inch because of lower hardenability.

SAMPLE-- H16

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16	7/16	8/16	9/16
0	57.7	57.3	55.2	45.1	40.9	35.9	30.0	28.2	27.7
30	58.3	57.8	52.6	47.0	40.8	34.3	30.2	27.3	24.7
60	58.8	55.9	55.3	50.0	36.5	34.0	32.0	30.7	26.3
90	58.6	58.0	57.2	52.1	40.8	33.7	32.4	31.0	27.3
120	58.7	57.8	56.2	49.8	39.3	34.3	32.1	29.3	27.3
150	57.2	57.7	54.8	51.3	39.9	35.0	30.8	30.3	27.9
180	58.3	58.1	56.0	47.0	39.5	36.0	32.0	30.3	28.9
210	58.9	58.3	56.2	47.7	39.7	34.0	32.0	29.1	28.0
240	58.2	58.6	56.3	46.8	40.2	33.3	31.5	29.7	28.1
270	57.5	58.1	56.0	48.2	40.1	35.1	33.6	30.0	28.1
300	58.0	58.0	57.1	50.7	41.2	36.3	32.0	29.9	27.1
330	56.8	57.0	55.1	45.0	35.7	33.3	30.9	28.2	24.6

SAMPLE-- H17

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16	7/16	8/16	9/16
0	58.3	58.1	57.3	52.8	40.7	32.2	29.8	29.3	26.8
30	58.0	57.6	56.0	48.8	39.4	32.0	30.3	28.9	27.2
60	57.8	57.7	56.6	52.3	35.2	33.5	31.3	28.7	26.8
90	58.0	57.5	54.3	47.3	39.6	31.3	31.0	29.0	28.0
120	58.7	58.4	55.2	49.6	39.9	34.0	30.1	27.6	25.3
150	58.3	56.7	55.3	49.7	38.2	34.8	30.0	28.3	26.5
180	58.3	57.1	55.7	47.7	38.9	33.8	30.8	28.5	26.9
210	58.3	57.0	56.0	47.3	37.3	33.8	31.8	28.7	27.9
240	58.4	57.3	55.7	47.8	41.0	33.3	31.5	27.7	28.9
270	58.9	58.0	55.7	47.3	39.5	31.3	30.0	29.6	26.0
300	58.1	58.0	54.8	49.7	39.7	31.5	31.0	27.3	27.3
330	58.8	57.4	56.3	51.7	40.7	33.0	29.3	28.9	27.9

SAMPLE-- H18

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16	7/16	8/16
0	58.3	57.7	55.1	47.2	36.5	32.3	28.0	27.5
30	57.9	58.0	56.2	51.3	39.2	32.8	30.3	29.3
60	58.9	58.0	54.8	49.3	38.5	34.2	29.8	28.7
90	58.1	57.3	55.4	49.5	38.5	33.5	29.9	28.9
120	58.7	57.0	55.8	47.6	38.3	32.7	30.7	27.8
150	57.5	57.0	55.3	47.3	40.8	34.2	29.5	29.7
180	58.8	58.3	56.1	51.9	37.3	32.3	30.3	26.3
210	58.1	57.2	53.9	49.1	41.4	32.1	30.8	28.8
240	57.9	57.1	55.0	50.2	41.8	32.2	29.1	27.2
270	60.0	58.0	56.4	48.3	37.0	33.2	29.2	27.8
300	58.2	59.4	56.2	48.9	38.0	31.0	28.8	28.3
330	58.3	58.0	56.2	46.2	37.0	33.8	28.2	28.3

SAMPLE-- H19

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16	7/16	8/16
0	57.9	57.8	52.8	47.0	38.0	32.0	30.0	26.0
30	58.4	57.3	55.5	52.2	43.2	32.2	29.8	27.9
60	58.0	57.0	54.5	49.0	36.5	32.2	31.0	27.5
90	58.2	57.2	55.0	50.1	41.2	33.6	29.0	27.1
120	58.9	57.3	55.0	47.5	39.6	33.9	30.0	28.8
150	58.7	57.5	55.8	51.0	41.2	33.3	30.2	29.0
180	58.1	58.2	55.4	46.5	41.8	34.0	31.7	28.7
210	58.7	57.6	55.0	49.0	38.9	33.2	31.9	29.1
240	58.0	56.8	54.9	50.0	38.8	34.5	31.2	28.1
270	57.8	57.3	55.8	48.8	40.1	34.5	31.2	28.9
300	57.0	57.5	54.8	47.8	39.7	32.9	29.3	27.8
330	58.4	58.2	56.2	48.0	38.2	34.2	30.9	28.0

SAMPLE-- H20

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16	7/16	8/16
0	58.8	58.6	56.9	50.3	40.8	33.0	30.0	28.1
30	58.2	57.2	56.0	49.2	38.2	32.7	29.6	27.2
60	58.2	57.0	53.8	50.8	40.7	33.3	29.3	26.9
90	58.0	57.8	55.7	46.1	39.0	34.1	29.3	28.9
120	58.3	57.1	54.8	49.8	41.0	33.2	30.4	29.8
150	58.2	57.8	53.6	47.7	39.9	34.3	30.1	29.4
180	58.0	57.2	53.8	48.9	40.0	35.1	32.1	28.3
210	58.1	57.5	55.0	50.3	41.0	35.2	29.0	26.3
240	55.4	57.7	55.8	46.1	41.3	35.2	29.3	26.9
270	58.7	57.6	53.1	48.7	38.6	34.9	29.1	27.9
300	58.5	57.6	53.9	50.2	40.3	31.8	29.2	28.7
330	58.3	57.4	55.7	46.1	42.1	33.0	29.7	27.7

SAMPLE-- H21

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	58.0	57.8	55.5	50.4	38.1	31.0
30	58.2	57.3	54.1	46.0	35.1	30.9
60	58.1	57.8	55.1	49.8	37.4	29.6
90	58.2	57.0	53.7	46.5	34.2	28.8
120	58.0	57.5	55.9	48.6	37.1	31.6
150	58.3	57.5	56.1	50.7	39.0	31.8
180	58.1	58.0	56.3	50.1	37.6	29.7
210	58.6	57.5	56.6	50.1	37.8	30.2
240	58.1	57.6	56.5	50.3	38.2	30.9
270	57.9	57.1	55.2	47.1	37.1	31.7
300	58.1	56.8	55.1	49.3	38.7	31.5
330	58.1	57.3	55.1	49.1	37.9	31.6

SAMPLE-- H22

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	57.2	56.4	52.8	42.5	33.9	29.8
30	57.1	57.1	55.0	47.3	34.8	29.8
60	57.3	57.1	55.5	49.6	36.6	31.0
90	57.6	57.0	55.7	48.9	37.0	30.5
120	57.9	57.2	55.4	50.7	38.0	30.4
150	57.5	57.4	56.1	53.1	38.8	31.0
180	57.6	57.1	56.1	52.0	38.9	30.9
210	57.7	57.1	56.8	51.6	38.5	31.1
240	57.1	56.9	55.9	52.7	40.1	30.9
270	57.0	57.0	55.9	50.0	38.2	31.0
300	57.3	57.0	55.2	48.0	37.1	30.5
330	57.5	57.3	54.8	46.5	35.5	30.0

SAMPLE-- H23

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	57.8	57.5	56.1	49.3	38.1	31.7
30	58.0	57.3	54.6	46.4	36.2	31.5
60	58.6	57.3	54.6	46.8	36.1	30.7
90	58.3	57.3	54.6	47.2	36.1	31.0
120	58.4	57.6	54.7	47.2	37.0	30.7
150	58.3	57.9	55.5	49.3	38.3	31.1
180	58.0	57.7	55.9	50.0	37.6	31.4
210	58.1	57.5	56.0	50.0	37.0	30.7
240	57.9	57.5	55.9	50.4	36.9	30.5
270	57.6	57.5	56.3	50.9	38.1	31.5
300	58.0	57.0	55.0	47.3	36.4	30.8
330	57.9	57.0	55.1	47.5	35.4	31.0

SAMPLE-- H24

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	58.0	57.0	53.8	43.7	35.6	29.6
30	57.9	57.1	52.5	41.2	33.6	30.5
60	57.0	56.2	52.7	45.7	36.0	28.4
90	57.4	56.6	54.0	44.9	35.7	29.3
120	58.0	57.3	54.7	46.9	36.0	31.0
150	57.7	57.1	53.9	48.3	37.0	32.8
180	58.2	57.8	55.1	46.9	36.2	32.7
210	58.1	57.1	54.6	45.1	34.2	30.5
240	58.1	57.4	55.6	46.4	33.9	30.9
270	57.9	57.3	54.2	45.6	36.6	31.7
300	58.3	57.0	54.1	44.9	34.8	29.3
330	58.0	57.3	54.2	44.4	34.1	29.9

SAMPLE-- H25

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	58.1	57.0	54.7	45.1	34.6	29.7
30	58.1	57.3	53.6	45.2	34.6	30.5
60	58.1	57.4	54.4	46.0	36.2	30.2
90	58.1	57.3	55.7	47.6	36.4	29.9
120	58.3	57.8	55.5	49.4	36.9	30.8
150	58.5	57.9	56.3	51.0	38.5	31.1
180	58.1	57.9	56.0	49.5	36.4	30.8
210	57.4	56.5	54.9	47.4	35.5	30.0
240	57.9	57.1	55.0	46.3	34.1	29.1
270	58.2	57.3	54.3	43.8	33.6	28.4
300	57.7	56.5	52.6	44.3	32.9	29.1
330	58.3	57.1	53.9	44.9	35.0	30.0

SAMPLE-- H26

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	58.5	56.3	49.7	36.3	28.9	25.2
30	58.3	57.4	53.3	38.9	30.4	26.1
60	57.9	57.2	52.7	37.7	31.2	26.8
90	57.8	56.8	52.3	38.0	30.5	27.1
120	58.8	56.8	51.4	37.9	30.7	26.9
150	57.5	56.3	52.1	38.1	31.0	26.6
180	58.4	56.9	52.8	39.6	31.5	27.0
210	58.1	57.0	54.3	42.1	32.4	27.1
240	58.1	56.9	53.3	41.2	33.2	28.3
270	58.8	57.3	54.6	42.2	32.1	28.2
300	58.0	57.2	54.4	39.6	31.2	26.3
330	58.2	57.5	54.2	39.6	30.5	26.6

SAMPLE-- H30

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	58.0	56.5	51.4	36.7	29.9	25.4
30	58.5	57.1	50.7	33.5	28.3	24.6
60	58.5	57.2	52.1	37.5	30.2	25.8
90	58.1	56.5	50.2	37.1	30.8	26.4
120	58.6	57.3	52.2	39.1	31.2	26.4
150	58.2	56.4	52.0	38.5	30.1	25.5
180	58.2	56.9	52.6	38.6	31.4	26.6
210	57.9	56.8	52.3	38.1	31.6	26.6
240	57.5	57.3	54.0	39.0	30.7	26.6
270	57.5	57.1	52.9	37.6	29.3	25.8
300	57.7	56.8	52.7	36.5	29.4	24.8
330	57.4	56.5	52.9	36.2	29.5	25.0

SAMPLE-- H31

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	58.0	56.7	49.1	37.6	31.4	27.1
30	57.9	55.9	49.1	37.1	30.4	26.1
60	57.6	56.4	50.0	36.8	30.3	26.3
90	58.3	56.5	48.6	36.1	29.8	25.7
120	58.4	56.7	50.8	37.4	30.5	26.1
150	58.3	55.9	48.5	36.8	29.5	25.5
180	57.9	56.8	51.6	39.1	31.3	26.6
210	58.3	56.1	50.2	38.4	32.5	26.6
240	57.8	56.3	50.9	38.8	31.5	26.9
270	58.2	56.7	50.4	39.1	31.2	26.7
300	58.0	56.1	48.9	37.2	31.0	26.3
330	57.9	56.2	51.0	38.7	30.5	26.5

SAMPLE-- H32

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16	7/16
0	55.3	54.2	45.6	33.6	27.8	23.0	18.4
30	56.5	54.9	45.4	34.4	28.3	23.7	19.4
60	56.5	55.3	47.5	34.0	29.7	24.2	19.5
90	57.2	56.7	52.1	37.3	29.5	25.0	20.1
120	57.5	57.5	53.3	38.2	30.2	25.4	20.8
150	58.2	57.3	53.9	39.2	29.7	25.5	20.9
180	57.6	57.0	53.7	39.0	29.6	25.2	20.5
210	56.7	56.0	51.5	37.2	29.4	25.8	20.0
240	57.2	55.9	50.4	36.9	29.3	25.3	20.6
270	57.8	56.6	49.8	35.5	29.7	25.3	20.5
300	56.6	55.5	46.8	34.4	29.0	24.5	19.7
330	55.6	53.6	43.8	32.6	27.7	22.9	18.9

SAMPLE-- H33

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	53.7	49.6	39.8	31.0	22.5	13.1
30	53.7	46.4	35.1	27.0	20.0	10.7
60	56.2	45.0	33.7	26.4	19.8	10.4
90	54.5	48.0	37.2	28.9	20.6	11.8
120	55.8	47.9	36.9	27.3	20.4	11.2
150	53.4	49.5	38.4	28.7	21.8	12.6
180	55.3	46.1	36.4	28.7	22.3	12.0
210	56.3	49.6	39.2	30.5	23.2	13.5
240	55.8	49.1	38.2	29.4	23.1	13.7
270	55.4	48.4	37.5	29.5	22.0	12.8
300	54.9	47.5	37.5	28.9	21.8	12.7
330	56.3	48.3	35.9	27.4	20.5	10.5

SAMPLE-- H34

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	52.4	44.9	35.6	27.9	20.7	11.2
30	56.2	46.6	35.7	28.2	20.7	11.3
60	56.5	49.0	37.2	27.2	21.4	11.6
90	55.9	49.8	38.4	29.8	21.6	10.0
120	55.6	50.0	40.3	31.2	23.0	13.1
150	57.2	50.4	39.3	30.2	23.4	14.0
180	56.5	48.0	37.8	29.9	22.3	12.7
210	55.6	46.0	35.8	27.5	21.1	11.2
240	54.9	45.5	35.4	27.0	20.6	11.3
270	55.7	44.2	35.2	26.7	20.4	10.3
300	57.0	47.4	37.5	29.3	21.9	11.1
330	55.6	49.1	37.6	27.9	20.7	10.4

SAMPLE-- H51

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16	7/16
0	57.6	57.2	54.8	46.5	38.3	33.3	31.9
30	58.0	57.2	54.3	46.4	38.6	34.1	32.0
60	57.9	57.0	54.5	46.2	37.2	33.5	32.2
90	58.0	57.0	55.7	48.3	38.7	34.0	31.9
120	57.8	57.5	55.8	47.2	38.8	33.9	31.1
150	58.7	57.9	55.3	48.0	38.7	34.1	31.8
180	58.1	57.1	55.1	46.9	38.1	34.7	31.5
210	57.6	56.9	55.3	47.5	37.9	33.8	31.3
240	58.0	57.2	55.8	47.9	37.6	34.1	31.2
270	58.0	57.3	55.6	46.2	38.2	33.0	32.3
300	58.0	57.0	54.0	45.0	36.2	32.8	31.5
330	57.0	57.1	54.2	46.1	37.8	33.3	31.3

SAMPLE-- H52

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16	7/16	8/16
0	57.7	57.2	54.9	47.7	38.1	32.9	30.7	30.0
30	57.2	56.3	52.3	45.0	37.2	32.7	29.1	28.3
60	57.7	56.5	54.0	47.2	37.9	32.8	30.0	28.1
90	57.8	57.5	55.8	46.7	37.7	33.9	31.2	29.5
120	57.7	57.2	55.8	50.8	40.9	33.0	29.8	28.9
150	58.0	57.4	55.2	49.2	40.6	33.3	30.3	29.4
180	58.6	57.1	55.0	49.1	40.2	34.1	30.9	30.0
210	58.0	57.5	55.8	49.1	40.2	34.1	30.9	30.0
240	57.7	57.2	54.9	48.5	39.7	34.0	31.2	29.9
270	57.3	57.2	55.0	48.8	39.0	33.0	31.2	30.0
300	57.9	56.9	54.0	46.9	39.8	34.0	31.9	28.8
330	58.0	56.7	54.0	46.2	39.2	32.8	30.3	29.7

SAMPLE-- H53

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16	7/16
0	58.5	57.7	55.3	46.2	38.0	34.8	31.1
30	58.4	57.7	55.9	48.2	40.8	35.9	32.2
60	58.7	57.7	55.0	47.2	39.3	33.0	32.7
90	58.2	57.3	56.2	48.2	38.1	33.6	31.7
120	58.0	57.6	55.9	50.3	39.8	34.6	33.7
150	58.7	57.3	54.7	49.1	43.2	36.0	32.3
180	58.8	57.7	55.3	48.9	38.4	32.5	30.8
210	58.1	57.9	56.9	52.9	42.4	36.2	32.1
240	57.9	57.1	56.9	47.3	39.4	34.1	31.6
270	57.7	58.0	56.2	48.1	38.7	35.5	32.5
300	57.8	57.1	55.2	45.3	36.9	33.4	31.2
330	58.6	57.4	54.7	49.0	40.5	35.1	32.0

DEPTH-- DEGREES	8/16	9/16	10/16	11/16
0	32.0	29.4	29.5	29.4
30	51.1	29.7	29.2	27.1
60	50.4	31.9	30.8	27.7
90	31.0	31.0	30.5	29.5
120	31.5	30.1	29.7	27.4
150	31.5	31.8	32.8	28.0
180	31.0	30.8	29.5	28.1
210	32.0	28.5	29.2	28.8
240	30.8	28.8	29.8	27.3
270	30.7	30.0	28.8	27.4
300	30.7	30.3	28.0	27.9
330	29.4	29.7	30.1	27.1

SAMPLE-- H54

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	57.6	56.6	52.5	44.4	35.0	30.7
30	57.5	56.7	52.8	42.7	35.0	30.4
60	57.9	57.3	52.6	44.9	34.9	30.2
90	57.9	56.8	54.3	46.0	35.5	30.3
120	57.6	57.0	54.6	46.4	37.2	31.9
150	58.1	56.8	54.5	48.3	38.0	31.1
180	57.6	56.5	54.0	46.7	36.6	32.1
210	58.0	57.2	55.0	44.8	36.8	30.7
240	58.3	57.2	54.0	46.3	36.6	30.8
270	57.7	57.2	54.2	44.7	34.4	30.1
300	57.6	57.4	54.0	46.3	35.1	30.7
330	58.3	57.0	54.0	45.2	35.1	30.6

SAMPLE-- H55

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	55.1	48.8	39.4	30.0	25.1	19.0
30	57.1	46.8	35.8	29.1	24.0	18.0
60	57.0	48.0	35.4	28.0	22.6	14.9
90	57.0	48.5	36.4	28.8	22.7	16.9
120	57.0	49.5	37.8	29.6	24.2	18.1
150	56.5	48.9	36.6	29.2	24.1	17.8
180	56.7	48.1	36.5	29.7	23.7	16.9
210	56.2	47.0	35.3	28.0	23.0	15.6
240	56.9	49.1	36.6	29.3	23.1	16.2
270	56.7	49.2	48.0	30.1	24.2	17.5
300	56.1	49.0	38.1	30.3	24.2	17.3
330	56.3	47.8	36.3	29.1	23.1	16.0

SAMPLE-- H56

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	52.3	42.9	30.2	14.9	9.8	8.5
30	54.0	41.3	28.2	13.9	8.9	8.1
60	54.9	38.4	26.8	14.6	9.8	8.8
90	54.3	41.9	30.0	18.5	11.2	9.0
120	54.4	41.5	29.4	19.0	9.2	9.8
150	54.6	42.2	29.6	19.5	10.8	9.6
180	51.0	37.6	27.4	16.1	10.4	10.1
210	50.0	35.8	27.4	16.3	10.1	9.2
240	53.1	37.0	26.9	12.6	9.3	9.1
270	53.6	40.1	27.6	13.3	9.0	8.1
300	52.9	36.7	25.9	12.5	10.2	9.2
330	55.0	40.1	28.0	14.3	10.6	9.1

SAMPLE-- H57

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	51.5	34.7	24.8	11.1	7.4	6.6
30	53.0	36.5	25.4	11.1	8.0	7.2
60	54.4	36.0	25.6	11.1	8.6	7.9
90	53.5	36.0	26.1	12.2	8.7	7.8
120	54.1	38.1	26.9	13.1	8.3	7.9
150	53.4	36.7	27.3	13.6	8.2	8.2
180	52.9	39.0	28.6	15.1	9.4	9.0
210	51.8	37.4	28.2	14.6	9.2	8.4
240	51.6	34.7	26.9	12.7	9.0	8.5
270	51.5	36.4	27.2	12.4	8.9	8.7
300	51.0	36.0	27.0	12.5	8.5	8.4
330	48.5	34.5	24.9	10.1	8.6	7.9

SAMPLE-- H58

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	47.2	38.1	28.4	12.3	7.7	6.6
30	54.1	37.1	25.4	10.9	8.5	6.9
60	53.4	33.7	23.5	9.7	8.5	6.9
90	54.9	35.9	25.0	11.0	8.3	7.1
120	52.4	33.7	25.1	11.0	8.5	7.6
150	55.3	38.8	27.2	13.9	7.8	7.1
180	54.6	38.2	27.3	13.5	8.3	7.4
210	54.5	36.9	26.3	13.0	8.4	7.8
240	53.9	36.9	27.5	13.1	7.9	7.1
270	55.0	36.8	25.9	13.1	7.9	8.0
300	53.9	34.0	24.8	10.8	8.1	7.5
330	49.8	34.1	23.9	9.4	7.5	7.5

SAMPLE-- H59

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16	7/16
0	57.5	57.6	57.2	56.0	53.9	50.3	41.4
30	57.7	58.0	56.0	55.0	52.0	46.1	40.0
60	57.4	58.7	56.0	54.7	53.3	45.7	39.5
90	58.1	57.1	56.0	55.1	52.9	47.1	42.3
120	57.8	57.5	56.1	55.1	54.9	48.8	44.5
150	57.5	57.3	56.6	55.6	54.5	50.2	42.3
180	57.2	57.0	56.9	56.1	54.4	51.0	43.2
210	58.2	57.1	57.0	56.0	55.0	49.9	42.6
240	57.1	58.7	57.1	54.1	53.3	50.1	42.8
270	58.0	57.8	57.0	54.7	55.1	51.3	42.7
300	58.0	57.5	57.2	55.0	53.6	50.7	42.3
330	57.2	57.2	56.0	55.1	52.1	48.6	41.2

DEPTH-- DEGREES :	8/16	9/16	10/16	11/16
0	35.0	33.3	31.0	30.9
30	35.2	34.7	30.1	28.6
60	33.4	33.1	33.6	28.8
90	36.0	33.1	30.8	28.3
120	36.0	32.8	31.0	28.9
150	36.2	34.6	30.8	28.5
180	37.5	33.9	30.7	30.1
210	37.2	34.2	30.8	30.0
240	35.7	32.8	30.9	29.1
270	34.7	32.2	30.1	28.9
300	35.4	32.7	29.9	29.0
330	35.6	32.3	31.0	29.0

SAMPLE-- H60

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	57.6	57.4	56.7	55.8	49.8	34.4
30	57.1	57.1	56.6	54.0	44.9	32.8
60	57.6	56.6	55.9	53.8	44.4	32.1
90	57.3	56.9	55.6	54.8	47.4	33.7
120	57.7	56.8	55.9	55.0	48.4	33.8
150	57.0	56.6	56.4	55.1	50.0	35.5
180	57.4	57.1	55.9	55.3	50.4	36.2
210	57.5	57.4	56.5	54.9	50.4	37.5
240	57.0	56.9	56.6	55.5	50.9	38.8
270	57.5	56.9	56.8	55.4	51.2	38.3
300	57.4	57.2	55.9	54.7	48.9	34.0
330	57.5	57.0	56.4	55.1	47.2	33.1

SAMPLE-- H61

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	52.5	48.9	39.3	28.7	24.2	22.2
30	51.8	50.2	39.4	28.3	24.6	22.6
60	53.0	50.4	38.2	27.8	24.2	22.3
90	52.8	50.6	40.4	28.5	24.5	22.3
120	51.3	50.4	38.8	28.3	24.4	22.4
150	53.8	51.9	41.2	29.9	24.8	22.9
180	53.7	51.7	40.5	29.0	24.5	22.6
210	53.2	50.7	42.3	30.0	25.4	23.3
240	53.1	52.0	42.7	29.6	24.9	22.5
270	53.0	51.3	39.5	28.3	24.2	22.4
300	52.8	50.6	39.2	27.5	24.5	22.0
330	52.4	49.1	37.0	27.0	23.5	21.7

SAMPLE-- H62

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	56.8	56.2	55.5	54.9	47.5	33.8
30	56.6	56.6	55.5	54.5	47.3	33.9
60	57.1	57.1	55.9	54.7	49.0	35.1
90	57.1	56.7	56.1	54.9	48.6	35.2
120	56.9	56.6	56.0	55.0	51.3	37.5
150	57.1	56.7	56.6	55.2	51.4	38.2
180	57.2	56.6	56.4	55.6	52.0	40.0
210	56.9	56.8	55.9	55.5	52.0	41.2
240	57.0	56.5	56.2	54.9	52.0	40.3
270	57.0	57.0	56.0	55.0	50.6	37.9
300	56.9	56.9	56.3	54.6	49.8	35.0
330	57.1	56.2	55.1	54.9	49.1	33.9

SAMPLE-- H63

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	58.0	54.9	46.7	36.3	30.1	25.9
30	58.5	56.9	47.4	36.7	30.0	25.5
60	58.0	55.3	46.2	34.7	29.5	25.0
90	57.9	56.3	47.9	36.3	30.2	25.1
120	58.6	55.9	47.9	37.3	30.7	25.0
150	57.7	55.7	46.9	35.6	29.4	25.0
180	58.2	55.6	46.3	35.7	29.3	25.2
210	58.7	56.3	48.3	36.7	29.9	25.5
240	58.0	56.1	49.0	37.1	30.1	25.8
270	58.0	55.2	45.3	35.8	30.3	24.9
300	58.0	56.1	47.4	36.8	30.0	25.3
330	57.9	54.7	46.2	36.8	30.0	26.0

SAMPLE-- H64

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	58.7	56.9	48.1	39.0	31.7	26.9
30	58.3	56.0	48.8	33.7	28.5	25.8
60	58.7	56.6	50.9	40.7	31.0	26.2
90	58.3	57.1	51.9	39.4	30.5	24.9
120	56.9	57.1	51.5	39.9	31.0	27.7
150	58.0	55.1	49.1	41.8	31.7	26.9
180	57.7	56.3	52.0	36.2	31.3	27.7
210	58.8	56.1	50.2	41.9	32.3	27.5
240	57.9	56.3	52.5	40.7	32.1	26.0
270	58.9	56.9	51.3	40.2	34.0	27.9
300	58.7	55.8	49.5	36.7	29.9	26.7
330	58.8	56.6	50.9	39.6	31.3	28.0

SAMPLE-- H65

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	58.6	55.9	48.3	34.2	28.8	26.3
30	58.5	56.7	51.0	39.3	31.9	27.0
60	58.7	56.7	50.1	36.5	30.0	26.1
90	59.2	55.8	48.6	39.4	30.5	27.3
120	59.3	56.1	50.0	38.3	30.2	25.0
150	59.3	55.8	49.3	38.9	32.8	28.3
180	57.4	54.9	47.9	36.7	31.0	25.4
210	58.3	56.9	51.8	40.1	32.9	28.0
240	58.7	56.9	50.8	42.3	31.2	28.7
270	58.7	56.1	52.9	42.2	32.7	28.2
300	58.5	57.0	50.9	38.4	31.3	26.2
330	58.1	57.6	52.6	38.1	31.6	28.2

SAMPLE-- H66

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	58.0	53.6	44.5	34.7	27.2	20.5
30	57.8	53.3	42.8	34.2	26.1	18.6
60	57.7	51.6	42.4	35.5	27.5	18.8
90	57.8	52.0	41.1	32.2	24.9	17.9
120	58.2	54.6	42.6	33.4	26.0	18.5
150	58.4	54.9	44.1	35.5	25.9	20.0
180	58.0	54.6	43.2	33.8	25.3	20.0
210	58.6	54.0	41.9	34.3	27.3	20.7
240	58.0	53.8	42.9	33.8	27.0	19.9
270	58.3	54.1	42.5	33.6	25.3	19.0
300	58.3	54.6	43.1	32.8	26.5	19.4
330	58.1	52.8	43.4	34.1	26.1	17.7

SAMPLE-- L1

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	53.0	51.4	40.8	29.7	25.3	23.2
30	53.0	51.5	42.0	29.8	25.9	23.4
60	53.0	52.2	44.0	30.9	25.7	23.7
90	53.0	51.9	43.9	30.5	26.0	23.8
120	53.0	52.0	45.5	30.7	26.1	24.1
150	53.3	52.4	44.3	30.6	27.8	24.7
180	53.3	53.0	48.4	33.8	26.9	24.9
210	53.5	52.5	45.3	31.4	26.3	24.7
240	53.0	52.0	43.9	30.6	25.5	24.0
270	52.8	52.0	43.5	29.7	25.8	24.2
300	53.7	52.0	43.5	31.0	25.9	24.6
330	53.0	51.8	43.9	30.7	26.0	23.9

SAMPLE-- L2

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	53.3	52.0	43.0	30.3	25.7	23.0
30	53.2	52.0	42.5	30.7	25.9	22.8
60	53.1	52.3	46.3	32.0	26.6	23.7
90	53.6	52.9	46.1	31.0	25.7	23.0
120	53.3	52.3	45.5	31.3	26.3	24.4
150	52.5	52.6	45.9	31.7	26.2	24.2
180	53.2	52.4	45.0	31.0	25.5	23.7
210	53.3	52.8	45.8	31.2	26.3	23.7
240	53.0	52.6	45.0	31.2	26.0	23.6
270	53.1	52.3	45.2	31.5	26.2	23.6
300	53.4	52.6	44.6	30.4	26.0	24.3
330	53.3	52.1	45.6	31.5	26.0	24.2

SAMPLE-- L3

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	53.8	52.3	43.0	30.7	24.9	23.0
30	53.8	51.3	40.0	30.3	24.3	22.6
60	53.9	52.8	44.0	30.4	24.5	23.0
90	53.3	52.9	46.2	30.9	25.8	23.9
120	53.1	52.6	45.7	30.6	25.0	24.0
150	52.9	52.8	48.2	32.0	26.1	23.2
180	53.4	52.9	49.9	33.3	25.9	23.2
210	53.8	53.1	48.6	32.0	25.7	23.2
240	53.6	52.9	48.1	32.0	25.7	23.3
270	53.2	52.6	46.3	31.0	25.8	24.0
300	53.3	52.4	46.3	30.9	25.8	24.3
330	53.2	52.9	45.4	30.4	25.0	23.1

SAMPLE-- L4

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	53.8	52.0	40.5	29.3	24.4	21.6
30	53.9	50.1	38.6	28.0	23.5	20.8
60	53.7	51.0	41.3	30.2	25.2	22.0
90	53.5	52.4	44.1	32.3	26.3	22.8
120	54.0	52.2	46.0	32.8	26.1	22.7
150	54.2	52.0	42.2	30.6	25.1	22.1
180	53.7	52.3	45.4	33.8	26.8	22.7
210	53.0	51.8	43.0	30.9	25.8	21.2
240	53.9	52.0	45.0	33.5	26.8	23.7
270	53.7	51.2	42.0	30.9	26.0	22.4
300	53.8	51.2	43.0	31.2	26.0	22.5
330	53.0	51.2	41.3	30.7	25.4	20.3

SAMPLE-- L5

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	53.1	50.0	38.6	30.2	25.3	20.5
30	54.3	51.2	42.0	29.8	26.3	21.7
60	53.9	51.0	40.4	29.8	25.8	21.9
90	53.8	50.9	40.9	29.8	25.7	23.0
120	54.0	50.9	40.9	31.3	26.7	22.8
150	53.8	51.8	42.6	31.2	26.1	22.8
180	53.8	52.1	44.1	32.0	26.9	23.0
210	54.1	52.0	44.2	30.6	25.6	22.3
240	53.8	52.2	44.1	31.8	26.2	23.5
270	53.7	52.1	43.0	31.0	25.9	23.0
300	53.9	51.9	42.6	30.1	25.2	22.1
330	53.2	50.3	39.6	30.1	25.0	20.5

SAMPLE-- L6

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	53.0	52.0	39.1	27.5	23.5	20.0
30	53.0	52.1	42.1	29.0	25.1	22.6
60	52.8	52.3	41.4	27.9	24.1	22.1
90	53.1	52.0	40.7	28.1	24.7	22.9
120	52.9	52.6	40.7	28.5	23.9	21.1
150	52.6	52.6	43.6	29.5	25.4	22.4
180	53.2	52.5	43.4	28.6	24.0	21.0
210	53.1	53.1	44.6	29.7	24.5	21.5
240	53.1	52.9	44.3	30.4	25.0	21.9
270	53.2	53.0	42.7	28.7	23.9	21.2
300	53.3	52.5	41.2	28.1	23.5	20.8
330	53.1	50.7	38.5	28.1	24.0	20.2

SAMPLE-- L7

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	52.1	49.9	38.0	28.0	24.2	22.0
30	52.8	49.6	37.4	27.5	24.5	22.0
60	52.9	51.0	39.3	28.8	24.4	22.0
90	53.6	51.2	39.1	28.1	25.0	22.0
120	53.1	51.0	39.6	28.5	25.1	22.0
150	53.7	51.0	40.1	28.6	25.0	22.3
180	53.7	51.4	41.0	29.0	24.6	22.6
210	53.0	50.9	39.1	28.7	24.4	21.0
240	53.3	50.7	39.0	28.7	25.1	22.4
270	54.0	50.5	38.4	28.2	24.6	22.0
300	53.4	50.0	37.0	27.7	24.1	21.4
330	53.2	50.4	38.1	27.6	23.9	21.5

SAMPLE-- L8

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	52.7	49.3	35.1	26.9	22.5	19.3
30	52.9	50.6	38.4	28.4	25.1	22.0
60	53.0	49.5	37.4	27.9	24.8	22.0
90	52.7	50.3	37.2	27.0	23.5	20.2
120	52.5	51.0	40.0	29.4	25.0	22.5
150	53.0	51.1	40.5	29.7	25.3	21.3
180	53.4	50.9	40.5	29.3	25.1	22.6
210	53.1	50.8	39.7	28.8	24.4	21.9
240	52.8	50.5	40.4	28.8	25.0	22.9
270	52.9	50.6	39.8	28.2	24.5	20.5
300	53.8	50.7	39.7	29.5	24.5	20.5
330	52.9	50.4	38.4	27.0	22.8	19.6

SAMPLE-- L9

DEPTH--DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	53.1	50.3	41.3	28.8	24.1	20.4
30	53.0	50.0	37.8	27.1	22.4	19.2
60	52.7	50.4	40.0	28.9	24.4	20.6
90	52.6	50.8	41.5	30.4	24.2	19.9
120	52.6	50.3	40.0	28.6	24.9	20.8
150	52.4	50.0	38.6	28.1	23.7	21.1
180	52.7	50.5	41.8	29.6	23.9	20.4
210	53.0	51.3	40.5	28.5	22.9	20.0
240	53.3	50.5	40.9	30.7	23.6	19.9
270	52.9	50.6	42.3	30.3	23.1	20.0
300	53.0	50.0	38.7	27.9	22.8	19.8
330	52.5	50.5	41.0	28.6	24.2	20.3

SAMPLE-- L10

DEPTH--DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	52.2	48.6	36.5	27.3	24.1	20.9
30	52.8	49.7	37.1	27.6	23.7	21.5
60	53.2	51.0	40.7	29.0	24.9	21.1
90	52.6	51.0	40.4	28.9	24.5	21.9
120	53.4	51.3	40.9	28.1	24.0	21.4
150	52.7	50.4	38.5	27.7	24.0	21.8
180	52.8	50.4	39.5	28.9	24.5	20.7
210	52.9	50.4	38.2	27.4	24.1	21.9
240	52.8	50.5	37.8	27.5	23.2	20.3
270	53.0	49.2	37.6	27.2	23.6	21.6
300	52.9	49.1	37.3	26.4	23.2	20.8
330	52.8	49.0	36.1	27.0	23.7	20.9

SAMPLE-- L11

DEPTH--DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	52.6	50.4	39.9	28.5	23.9	21.5
30	53.3	51.4	40.0	27.9	23.6	20.6
60	53.6	50.9	38.2	27.2	23.0	20.0
90	53.7	50.6	38.6	28.1	23.3	20.1
120	53.1	51.1	38.5	27.3	23.5	20.4
150	53.4	50.7	37.9	27.0	23.2	20.5
180	52.7	51.2	38.1	28.0	23.9	20.7
210	53.1	52.4	41.4	29.0	24.2	21.3
240	53.4	52.2	42.9	28.8	24.1	20.6
270	53.1	51.6	41.0	28.2	23.8	21.1
300	52.6	52.3	42.0	28.5	24.1	21.2
330	52.3	51.2	40.5	28.1	23.6	21.3

SAMPLE-- L12

DEPTH--1/16 DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	52.9	49.9	36.5	26.3	21.9	18.6
30	53.0	49.6	36.5	26.6	22.2	19.1
60	53.2	50.4	37.7	26.9	23.2	20.0
90	53.4	50.4	38.6	27.5	23.4	20.3
120	53.4	50.6	37.5	27.3	23.0	19.5
150	53.2	50.6	38.3	27.5	23.5	19.8
180	53.3	50.2	37.6	27.3	23.3	20.0
210	53.7	50.4	37.4	26.9	22.6	20.2
240	52.8	50.9	39.4	28.0	23.9	20.9
270	53.1	50.4	39.2	27.9	23.8	21.1
300	53.6	51.6	40.4	28.4	24.2	21.6
330	53.0	50.5	39.3	27.5	23.1	19.5

SAMPLE-- L13

DEPTH--1/16 DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	52.6	50.5	38.3	27.5	23.4	20.4
30	53.2	49.3	36.2	26.3	22.5	19.6
60	53.1	49.5	36.3	26.6	22.9	19.5
90	54.0	51.2	38.1	28.0	23.8	20.6
120	53.9	51.3	39.5	28.4	23.5	20.7
150	53.2	51.4	37.9	26.6	23.3	21.0
180	53.5	51.2	39.4	27.5	23.0	20.6
210	54.0	52.2	42.7	28.4	24.1	20.9
240	53.8	51.8	41.7	27.9	23.8	20.9
270	53.8	51.6	40.1	28.4	24.2	20.7
300	53.4	50.4	37.3	27.0	23.2	20.5
330	53.5	50.1	34.6	26.2	23.2	20.3

SAMPLE-- L14

DEPTH--1/16 DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	52.4	47.5	32.6	25.3	21.6	18.5
30	53.0	48.2	32.9	24.8	22.0	18.9
60	53.7	49.9	35.8	26.3	22.5	19.3
90	53.4	50.5	37.1	27.3	23.5	20.0
120	53.3	51.6	40.4	27.5	23.7	20.9
150	53.4	51.2	38.5	27.3	23.5	20.6
180	53.4	51.4	39.8	27.5	23.4	20.5
210	53.6	50.7	38.0	27.8	23.4	20.3
240	52.9	51.4	39.5	28.3	23.4	20.2
270	53.2	49.8	35.4	26.2	22.2	19.6
300	52.1	49.3	33.6	25.9	21.6	18.9
330	51.6	48.3	33.8	25.5	22.1	19.2

SAMPLE-- L15

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	52.5	48.7	34.4	25.1	22.1	19.0
30	53.1	50.1	36.7	26.5	22.7	19.3
60	52.8	50.5	36.7	26.1	22.3	19.6
90	52.4	50.8	38.5	26.3	22.1	19.4
120	52.7	50.8	37.9	26.2	22.4	18.7
150	53.0	50.0	35.7	24.9	21.3	18.3
180	52.6	50.4	37.4	26.1	22.2	19.1
210	53.7	51.0	37.9	26.7	22.1	19.5
240	53.8	50.3	37.2	25.7	22.1	18.6
270	53.1	50.1	36.9	25.5	22.7	20.1
300	53.7	51.2	38.4	26.0	22.6	19.6
330	53.3	49.4	35.2	26.0	22.2	19.2

SAMPLE-- L35

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	53.2	52.1	41.0	31.2	26.0	24.9
30	52.9	52.2	46.0	33.2	27.0	24.6
60	53.3	53.0	46.4	32.0	26.3	23.8
90	52.9	52.9	46.9	33.7	26.3	23.8
120	53.2	53.0	49.7	35.4	26.5	23.8
150	53.8	53.0	47.4	34.8	26.4	23.2
180	54.0	52.9	42.9	34.8	26.7	24.3
210	53.9	50.9	45.9	34.0	27.0	25.0
240	53.2	52.3	45.5	32.8	27.1	25.3
270	53.8	52.7	44.2	31.0	25.9	23.6
300	54.0	52.9	47.0	35.5	27.1	24.3
330	52.9	50.8	40.1	29.0	24.5	23.0

SAMPLE-- L37

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	48.6	44.9	36.4	25.8	22.2	19.5
30	48.4	46.5	37.0	25.5	21.0	18.0
60	48.1	47.4	38.3	26.5	21.0	18.8
90	48.2	46.0	36.0	25.8	21.1	18.3
120	45.5	45.0	36.8	24.9	20.0	17.8
150	48.0	46.7	37.1	25.5	21.2	18.1
180	47.4	45.6	36.1	24.8	21.6	18.5
210	49.2	47.3	38.0	25.9	21.9	19.0
240	48.1	47.2	38.2	26.1	21.8	19.9
270	48.1	46.6	38.2	26.1	21.1	18.6
300	48.0	47.5	36.5	25.0	20.7	18.7
330	49.0	46.1	35.7	25.4	21.3	18.7

SAMPLE-- L38

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	50.3	47.2	36.3	28.8	24.6	21.9
30	52.9	47.8	36.8	26.9	23.8	22.0
60	52.7	50.1	39.6	28.2	24.3	21.9
90	53.7	50.8	41.0	29.1	24.8	22.3
120	54.9	51.5	41.0	29.8	25.0	22.7
150	53.3	51.1	41.0	29.9	25.4	23.1
180	53.0	51.1	42.0	30.1	25.6	23.6
210	52.7	50.5	40.9	29.8	25.0	22.7
240	53.5	51.5	41.8	30.0	25.5	22.8
270	52.1	49.0	41.7	30.1	25.4	23.2
300	52.9	51.7	43.3	31.6	25.8	22.8
330	52.9	50.1	40.0	31.1	25.5	23.0

SAMPLE-- L39

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	42.5	34.8	27.8	23.0	19.8	16.8
30	50.0	34.4	24.9	21.6	18.8	15.2
60	48.2	32.8	24.9	21.5	18.8	15.0
90	48.1	34.0	25.8	21.7	19.4	15.1
120	47.4	32.9	25.0	21.4	19.4	15.1
150	46.8	33.3	25.0	21.7	19.0	16.0
180	46.2	31.0	24.8	21.3	18.8	15.2
210	48.1	31.8	24.2	21.0	17.9	14.0
240	46.7	30.4	24.0	21.2	18.0	14.0
270	48.3	34.8	25.5	22.4	19.8	15.8
300	47.8	32.9	25.0	21.2	18.3	14.8
330	51.0	36.8	26.0	21.9	19.0	15.6

SAMPLE-- L40

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	49.6	35.7	26.1	21.2	18.7	15.8
30	44.5	33.2	24.0	20.7	18.2	15.0
60	49.7	34.0	24.3	20.1	17.9	13.9
90	45.8	31.3	24.4	21.0	18.0	13.5
120	49.1	32.0	24.3	21.1	17.4	13.9
150	47.4	31.2	24.4	20.1	17.6	13.3
180	49.2	33.6	26.0	23.0	19.1	15.4
210	46.7	31.1	23.3	19.8	17.8	14.2
240	48.1	31.4	24.4	22.0	19.1	16.0
270	47.3	30.6	23.8	20.9	18.8	15.3
300	46.0	30.2	24.0	21.4	19.0	14.9
330	46.2	28.7	22.3	19.2	17.0	13.0

SAMPLE-- L41

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	48.7	33.9	25.7	21.3	18.5	15.2
30	46.2	34.1	26.2	21.2	19.0	15.2
60	48.2	32.8	24.1	21.8	18.7	15.0
90	47.2	32.8	24.3	21.6	18.6	14.9
120	49.8	34.4	25.8	22.0	19.1	15.2
150	48.1	32.1	24.5	21.9	18.2	15.1
180	48.9	33.6	25.5	21.4	18.2	16.0
210	48.2	33.8	25.6	21.8	18.6	15.8
240	50.3	37.4	26.7	22.2	19.8	16.2
270	47.6	32.6	24.6	21.5	18.5	14.8
300	48.3	32.2	25.0	21.5	18.7	15.0
330	44.3	29.4	23.7	20.8	17.3	15.1

SAMPLE-- L42

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	37.6	24.3	19.6	14.0	9.5	8.8
30	38.7	25.0	20.1	14.7	9.7	9.2
60	43.4	27.4	21.3	15.0	9.7	9.3
90	42.8	28.5	22.6	16.6	9.4	9.4
120	44.8	27.9	22.4	17.2	10.0	9.7
150	42.4	28.8	22.1	16.8	10.1	9.9
180	44.5	28.7	22.8	18.9	10.9	9.9
210	45.3	29.1	22.4	18.3	10.5	9.3
240	42.2	26.9	21.0	16.8	9.9	9.5
270	42.7	26.0	20.9	17.2	9.5	8.8
300	42.6	28.3	22.3	18.2	10.4	9.6
330	41.5	26.8	21.3	15.0	10.0	9.8

SAMPLE-- L43

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16	
0	53.7	53.0	51.0	47.3	41.0	33.7	28.4
30	53.0	52.6	51.7	49.2	44.9	35.8	28.4
60	53.0	52.2	51.1	47.8	41.1	33.0	29.0
90	52.3	52.8	50.9	48.0	41.9	32.9	29.3
120	53.3	52.6	50.8	47.0	40.4	33.1	27.9
150	53.5	52.9	51.7	47.0	40.0	33.1	27.9
180	53.1	52.0	51.1	48.1	42.2	34.9	33.7
210	53.2	52.0	51.1	45.9	40.8	33.0	27.4
240	52.8	53.0	51.9	50.7	42.4	34.7	28.2
270	53.0	52.8	52.0	47.5	40.1	31.7	27.4
300	54.0	52.3	50.1	46.7	39.0	31.5	27.0
330	53.0	52.2	51.3	47.3	39.2	31.3	27.7

SAMPLE-- L44

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16	7/16
0	53.3	53.0	52.4	50.0	44.9	34.3	28.0
30	53.8	52.0	52.5	49.0	43,0	34.3	29.2
60	52.4	52.3	50.8	47.8	41.2	33.2	28.6
90	53.0	52.9	52.3	50.3	43.3	33.5	28.5
120	53.4	53.3	52.8	49.7	43.6	34.8	29.3
150	51.8	53.6	52.2	49.3	44.0	35.3	29.2
180	52.9	53.3	57.7	49.8	43.2	34.8	29.7
210	53.6	53.6	52.2	50.9	46.2	35.7	30.2
240	53.5	52.3	57.8	50.3	44.4	34.8	30.0
270	52.7	52.8	52.3	49.4	43.5	33.0	30.4
300	52.9	51.8	51.7	48.4	41.1	33.0	28.4
330	52.8	53.0	52.3	48.2	40.2	30.7	28.3

SAMPLE-- L45

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16	7/16
0	52.7	52.0	52.2	50.3	42.2	32.7	28.2
30	53.3	52.3	51.9	48.8	41.4	32.3	27.6
60	51.8	52.4	51.7	48.7	40.7	31.4	26.9
90	52.9	52.9	50.9	47.9	41.1	31.7	27.5
120	52.8	52.5	51.1	48.7	42.3	34.3	28.1
150	52.6	52.7	50.7	48.3	43.0	32.8	27.1
180	52.6	52.6	51.7	49.1	44.2	35.1	28.8
210	48.2	52.0	51.3	49.6	43.3	36.0	29.2
240	52.0	52.3	51.2	48.7	42.7	33.7	27.7
270	52.8	52.6	51.2	48.0	41.3	33.4	28.3
300	53.0	52.6	51.7	50.0	43.2	33.5	28.2
330	53.2	52.7	51.9	49.1	40.9	32.2	29.4

SAMPLE-- L46

DEPTH--1/16 DEGREES :		2/16	3/16	4/16	5/16	6/16
0	52.5	52.5	50.5	44.9	32.1	24.4
30	53.0	52.4	51.0	46.3	35.0	25.6
60	53.3	52.5	50.1	45.2	34.1	25.7
90	52.0	52.0	51.0	46.1	35.6	26.7
120	52.5	52.5	51.0	46.3	35.4	27.0
150	52.4	53.0	51.1	46.9	36.1	27.6
180	53.1	53.0	51.4	47.6	36.4	27.5
210	53.2	52.2	50.9	47.2	36.0	27.0
240	53.1	52.0	49.3	45.3	33.5	25.7
270	53.3	52.1	51.3	47.6	36.4	26.7
300	52.8	52.5	51.7	47.4	35.9	26.6
330	52.9	52.9	50.9	46.0	33.8	25.5

SAMPLE-- L47

DEPTH-- DEGREES:	1/16	2/16	3/16	4/16	5/16	6/16
0	52.0	45.3	33.4	24.2	19.6	17.7
30	52.9	45.9	32.7	26.0	21.6	18.9
60	53.3	47.2	33.0	23.9	19.3	17.9
90	52.7	45.9	32.7	23.6	19.0	16.8
120	52.6	47.9	33.9	24.0	19.5	16.9
150	53.8	46.9	30.9	23.1	19.9	17.5
180	52.7	48.4	34.7	25.1	20.7	18.6
210	52.4	46.2	32.1	24.0	21.0	18.9
240	51.9	44.7	30.9	23.5	21.8	19.3
270	52.9	44.2	28.9	21.8	19.2	16.8
300	52.3	43.7	31.9	24.7	20.1	17.3
330	52.0	44.7	31.2	22.3	19.1	16.3

SAMPLE-- L48

DEPTH-- DEGREES:	1/16	2/16	3/16	4/16	5/16	6/16
0	44.2	34.5	25.1	22.8	18.6	14.9
30	50.3	37.4	25.7	21.2	18.3	15.9
60	50.3	41.4	28.3	23.5	19.4	15.6
90	50.0	42.8	28.7	22.6	19.1	15.7
120	51.9	46.0	30.3	23.0	20.0	16.2
150	53.7	46.0	30.2	23.5	20.0	16.4
180	51.5	46.4	31.9	23.6	20.5	16.4
210	51.5	47.0	32.3	23.7	20.0	16.4
240	52.0	45.8	30.6	24.0	20.2	16.9
270	51.5	45.0	30.3	23.5	20.0	16.4
300	51.3	42.7	27.5	23.0	19.7	16.1
330	51.2	40.1	25.9	22.2	18.9	14.1

SAMPLE-- L49

DEPTH-- DEGREES:	1/16	2/16	3/16	4/16	5/16	6/16
0	44.2	28.9	21.0	18.3	14.7	9.0
30	44.0	31.2	21.3	17.2	13.4	7.8
60	46.9	32.3	21.5	17.7	13.9	7.7
90	47.2	33.5	23.4	19.5	15.6	9.8
120	47.8	35.0	23.4	19.3	16.1	10.2
150	47.6	35.4	23.9	19.0	15.3	10.4
180	48.1	38.0	25.3	20.1	13.4	10.1
210	46.5	36.4	22.4	17.6	14.8	11.0
240	45.6	34.2	22.5	15.0	7.0	1.7
270	45.3	34.8	23.1	17.8	15.1	11.0
300	48.5	37.2	23.9	18.7	15.5	10.9
330	46.1	31.7	21.8	17.5	14.6	10.5

SAMPLE-- L50

DEPTH-- DEGREES	1/16	2/16	3/16	4/16	5/16	6/16
0	50.2	34.5	24.0	20.2	17.5	12.9
30	51.3	38.1	24.6	20.0	16.4	11.3
60	51.0	37.3	25.6	20.5	17.4	12.5
90	51.9	39.4	26.3	21.1	17.9	14.0
120	51.7	39.0	25.9	20.7	17.6	13.3
150	50.9	39.6	26.4	20.8	17.5	13.9
180	52.2	43.1	29.0	22.5	18.5	15.0
210	52.6	42.6	28.3	22.1	18.3	14.2
240	52.2	42.9	28.3	22.1	18.0	14.7
270	48.1	41.3	26.9	20.8	19.5	14.4
300	51.3	41.0	27.1	21.6	17.7	13.6
330	52.5	39.3	25.5	21.0	17.1	12.7

Table 1.2

Before a polynomial regression model could be run, an order for the polynomial had to be decided upon. The depth vs. hardness plots for each disk should contain a clue as to what order would be appropriate. Two typical plots of depth vs. hardness are found in Appendix A.

The plots seemed to indicate that a third order polynomial could adequately explain the data. Multiple linear regression using IMSL[2] was used to identify the four parameters needed for this model. Below are listed the results of these regressions(one per sample).

COEFFICIENTS

CONSTANT	DEPTH	$DEPTH^2$	$DEPTH^3$	SAMPLE
49.5083	12.5438	-4.6714	.3597	H16
51.2167	10.1967	-3.7272	.2512	H17
50.6500	11.2432	-4.1722	.2988	H18
52.1639	8.8917	-3.3762	.2274	H19
52.3167	8.6402	-3.3220	.2271	H20
50.1722	11.3308	-4.0795	.2730	H21
48.8861	11.8484	-4.0737	.2624	H22
49.2083	12.6849	-4.6136	.3301	H23
47.8889	14.9535	-5.8170	.4723	H24

48.2917	14.2876	-5.3866	.4139	H25
45.3778	19.9738	-8.2176	.7289	H26
44.8389	20.8495	-8.7107	.7850	H30
49.1972	15.1647	-6.9895	.6387	H31
45.6861	18.5156	-8.0886	.7374	H32
63.6472	-7.8832	-.3349	.0376	H33
66.2111	-10.6007	.4910	-.0392	H34
48.5500	13.8157	-5.2899	.4278	H51
51.0889	9.9761	-3.8303	.2781	H52
50.3806	11.5784	-4.3756	.3329	H53
48.8972	13.4896	-5.3113	.4253	H54
67.1111	-10.2118	.0009	.0532	H55
64.2972	-9.3968	-2.0201	.3422	H56
65.7167	-12.9502	-1.0909	.2748	H57
69.1750	-16.1261	-.3000	.2118	H58
58.7250	-1.3842	.5132	-.0907	H59
61.8611	-7.0399	3.1087	-.4466	H60
62.2806	-8.1245	3.4578	-.4677	H62
52.9833	10.6426	-5.8611	.5546	H63
49.7278	14.4648	-6.6132	.5946	H64
50.5167	13.9388	-6.5690	.5993	H65
59.7306	1.6374	-3.2277	.3063	H66
42.4944	18.5205	-8.5724	.8309	L1
41.2167	20.1756	-8.9868	.8579	L2
39.6806	22.5032	-9.7292	.9223	L3
47.2389	12.7192	-6.6971	.6480	L4
48.9417	10.7146	-6.1701	.6093	L5
43.4417	17.8042	-8.6357	.8458	L6
50.2278	8.9785	-6.0512	.6309	L7
49.8528	9.0010	-5.9593	.6136	L8
47.5417	11.5009	-6.5421	.6463	L9
50.0389	8.6700	-5.9584	.6216	L10
46.9444	13.2204	-7.3314	.7363	L11
50.3472	8.9269	-6.1226	.6350	L12
50.1833	9.6243	-6.3880	.6627	L13
52.1306	6.4840	-5.4970	.5892	L14
50.7306	8.5847	-6.2275	.6568	L15
42.4278	18.4597	-8.2480	.7779	L35
47.2583	12.2086	-6.9695	.7100	L36
41.3194	13.3578	-6.9994	.6927	L37
48.3944	10.4150	-6.2233	.6307	L38
71.1194	-28.9398	5.8637	-.4323	L39
73.2250	-32.0934	6.7855	-.5116	L40
71.8889	-29.4133	5.9182	-.4316	L41
60.3528	-21.8279	3.5006	-.2156	L42
51.9417	1.6500	-.4957	-.0505	L43
51.6083	1.3266	.0299	-.1241	L44
51.3611	.9556	.0741	-.1232	L45
47.7667	6.8254	-2.1542	.0681	L46
59.0667	-3.4364	-2.7777	.3694	L47
62.8056	-11.3449	-.0994	.1170	L48
66.9194	-23.8085	3.9475	-.2630	L49
70.1444	-20.7945	2.6268	-.1204	L50

Unfortunately, these sum of squares appeared to be much too large for the kind of accuracy that was desired so the order of the polynomial was increased by one and the 5 coefficients for each sample were found. The IMSL subroutine RPOLY was again used and much lower sum of squares were found. The fourth order polynomial was the highest order that could be tried without overfitting the data because the smallest(and most common) number of depths for a given disk was six. A fourth order polynomial model would leave 1 degree of freedom for estimating the Mean Squared Error(variance), but a fifth order polynomial would leave no degrees of freedom for checking goodness of fit. The average R^2 for the fourth order model was approximately and the average sum of squares was ---quite an improvement.

COEFFICIENTS

CONSTANT	DEPTH	$DEPTH^2$	$DEPTH^3$	$DEPTH^4$	SAMPLE
63.0083	-10.7062	7.8375	-2.2653	.1875	H16
67.9042	-18.5429	11.7352	-2.9936	.2318	H17
66.8375	-16.6353	10.8269	-2.8487	.2248	H18
63.8014	-11.1507	7.4070	-2.0355	.1616	H19
60.6792	-5.7619	4.4266	-1.3990	.1161	H20
70.3222	-23.3720	14.5912	-3.6451	.2799	H21
70.8736	-26.0190	16.2996	-4.0129	.3054	H22
69.9333	-23.0082	14.5899	-3.6998	.2878	H23
63.0514	-11.1597	8.2323	-2.4760	.2106	H24
67.7667	-19.2527	12.6587	-3.3729	.2705	H25
53.0778	6.7127	-1.0829	-.7684	.1069	H26
47.3389	16.5440	-6.3942	.2989	.0347	H30
46.5472	19.7286	-9.4449	1.1540	-.0368	H31
41.8736	25.0816	-11.6212	1.4787	-.0530	H32
49.2847	16.8522	-13.6430	2.8303	-.1995	H33
53.5111	11.2715	-11.2766	2.4302	-.1764	H34
63.1750	-11.3718	8.2615	-2.4160	.2031	H51
62.6389	-9.9156	6.8718	-1.9677	.1604	H52
64.6181	-12.9417	8.8167	-2.4355	.1977	H53
62.4472	-9.8465	7.2440	-2.2094	.1882	H54
51.5486	16.5903	-14.4192	3.0792	-.2161	H55
65.5597	-11.5711	-.8503	.0967	.0175	H56
71.2542	-22.4870	4.0401	-.8020	.0769	H57
73.6375	-23.8115	3.8349	-.6559	.0620	H58
53.1250	8.2603	-4.6757	.9981	-.0778	H59
58.6861	-1.5719	.1668	.1708	-.0441	H60
55.6056	3.3714	-2.7272	.8302	-.0927	H62

42.8333	28.1231	-15.2660	2.5282	-.1410	H63
52.0528	10.4606	-4.4589	.1425	.0323	H64
51.9292	11.5062	-5.2602	.3247	.0196	H65
46.7306	24.0262	-15.2734	2.8341	-.1806	H66
31.2319	37.9170	-19.0080	3.0209	-.1564	L1
32.1167	35.8478	-17.4188	2.6273	-.1264	L2
35.4931	29.7150	-13.6093	1.7365	-.0582	L3
34.9639	33.8595	-18.0709	3.0348	-.1705	L4
33.0167	38.1410	-20.9260	3.7058	-.2212	L5
21.8417	55.0042	-28.6500	5.0458	-.3000	L6
25.9153	50.8501	-28.5789	5.3584	-.3377	L7
26.4028	49.3871	-27.6877	5.1733	-.3257	L8
31.5167	39.0995	-21.3906	3.7623	-.2226	L9
26.5139	49.1853	-27.7564	5.1959	-.3267	L10
24.0444	52.6593	-28.5502	5.1890	-.3181	L11
25.5222	51.6811	-29.1251	5.4621	-.3448	L12
24.5958	53.6916	-30.0970	5.6381	-.3554	L13
23.2556	56.2132	-32.2522	6.2038	-.4010	L14
21.3306	59.2180	-33.4692	6.3735	-.4083	L15
39.3028	23.8417	-11.1436	1.3856	-.0434	L35
27.3708	46.4593	-25.3970	4.5770	-.2762	L36
24.8944	41.6453	-22.2186	3.8865	-.2281	L37
33.2944	36.4205	-20.2148	3.5668	-.2097	L38
70.9444	-28.6384	5.7015	-.3982	-.0024	L39
76.0500	-36.9587	9.4031	-1.0609	.0392	L40
73.6139	-32.3841	7.5166	-.7671	.0240	L41
84.7028	-63.7640	26.0630	-4.9503	.3382	L42
57.0417	-7.1334	4.2299	-1.0421	.0708	L43
57.3833	-8.6193	5.3809	-1.2470	.0802	L44
54.9486	-5.2228	3.3982	-.8208	.0498	L45
64.5667	-22.1079	13.4125	-3.1986	.2333	L46
35.8167	36.6053	-24.3208	4.8903	-.3229	L47
37.5306	32.1842	-23.5189	5.0316	-.3510	L48
56.4569	-5.7898	-5.7469	1.7713	-.1453	L49
50.9569	12.2507	-15.1521	3.6105	-.2665	L50

Once the coefficients were known for a given regression model, a reverse prediction had to be made to find the estimated depth to achieve a given hardness. The IMSL subroutine ZRPOLY was used to find the four roots of the fourth order polynomial. For example, the following equation had to be solved by ZRPOLY to find the root which represented the estimated depth at which hardness RC 50 could be achieved, for sample H16.

$$50.0 = 63.0083 - 10.7062(X) + 7.8375(X^2) - 2.2653(X^3) + .1875(X^4)$$

OR

$$0 = 13.0083 - 10.7062(X) + 7.8375(X^2) - 2.2653(X^3) + .1875(X^4)$$

Four roots were calculated and the one root that was in the area of interest was chosen as the estimate of the depth of RC 50 for this disk. In the same manner, all of the estimates for depth of RC 50 and Rockwell C 30 were chosen. The following is a list of these calculated estimates.

SAMPLE	RC 30 ESTIMATE	RC 50 ESTIMATE
H16	.0000	6.0894
H17	10.0794	6.2147
H18	10.1722	6.1298
H19	10.2588	6.1133
H20	10.4563	6.0645
H21	9.5765	6.1291
H22	9.5253	6.1792
H23	9.6464	6.0624
H24	11.3258	5.6092
H25	9.5358	5.8182
H26	10.2181	4.9983
H30	7.7882	4.8105
H31	8.0156	4.5822
H32	7.6017	4.4762
H33	6.0372	2.8282
H34	6.0480	2.7899
H51	.0000	5.8519
H52	10.4910	5.9624
H53	.0000	6.0980
H54	11.6725	5.6026
H55	6.0965	2.9153
H56	4.4061	1.9842
H57	4.0656	1.7910
H58	3.9992	1.8660
H59	11.7036	9.3212
H60	9.9067	7.7133
H62	10.0116	7.9612
H63	7.8506	4.2463
H64	8.1390	4.7213
H65	8.1274	4.6683
H66	7.0908	3.7489
L1	6.7315	3.6970
L2	6.8007	3.8007
L3	6.7927	3.8972
L4	6.7063	3.5479
L5	6.6228	3.4796
L6	6.2771	3.6216
L7	6.0473	3.2928
L8	6.0744	3.2708
L9	6.2512	3.3001
L10	5.9587	3.2065
L11	6.0818	3.4222
L12	5.8857	3.2599
L13	5.9085	3.3229
L14	5.6704	3.1590
L15	5.6439	3.2231

L35	7.0281	3.8198
L36	6.1684	3.3258
L37	5.6957	.0000
L38	6.3770	3.2596
L39	3.7103	1.3877
L40	3.4711	1.4046
L41	3.6933	1.4248
L42	2.7169	1.1692
L43	10.2182	5.4839
L44	10.2105	6.2584
L45	10.0213	5.8316
L46	8.7343	5.3686
L47	5.1224	2.5468
L48	4.7388	1.9797
L49	3.6832	1.1335
L50	4.3522	1.8307

The zeroes indicate values that were outside of the desired region such as sample L37 which does not contain any values equal to or greater than RC 50. The samples which have a zero as one of the estimates were not used in determining the final regression model because the estimates indicate that the specific five-variable combinations were not favorable ones for producing the desired properties in the steel bars.

Other samples, in addition to the ones that contained '0000.0' estimates in the above table, were discarded because of various unwanted chemical or microstructural properties such as excessive grain size which weakens the bar, through hardening which was not desired(only case hardening), or incomplete hardening. The samples that remained were valid either for determining a RC 30 model or a RC 50 model or valid for both models.

Samples for RC 30		Samples for RC 50	
H16	L1	H17	L1
H17	L2	H18	L2
H18	L3	H19	L3
H19	L4	H20	L4
H20	L5	H21	L5
H21	L6	H22	L6
H22	L7	H23	L7
H23	L8	H24	L8
H24	L9	H25	L9
H25	L10	H26	L10
H26	L11	H29	L11
H29	L12	H30	L12
H30	L13	H31	L13
H31	L14	H32	L14
H32	L15	H33	L15
H33	L35	H34	L35
H34	L36	H51	L36
H51	L37	H52	L38
H52	L38	H53	L47
H53	L39	H54	L47
H54	L40	H55	
H55	L41	H56	
H56	L42	H57	
H57	L43	H58	
H58	L44	H62	
H59	L45	H63	
H60	L46	H64	
H62	L47	H65	
H63	L48	H66	
H64	L49		
H65	L50		
H66			

1.7 The Regression

A regression model(initial) was decided upon which took into account all five linear effects of the five significant factors, all five quadratic effects of the five significant factors, and all the possible interaction effects between the five linear factors. The original model, containing all of these effects, is shown below.

X1 -- DI(STEEL COMPOSITION)
X2 -- BAR DIAMETER
X3 -- COIL DIAMETER
X4 -- INDUCTION POWER
X5 -- SPEED OF INDUCTION COIL
X6 -- $X1^2$
X7 -- $X2^2$

X8 -- $X3^2$
X9 -- $X4^2$
X10-- $X5^2$
X11-- X1*X2
X12-- X1*X3
X13-- X1*X4
X14-- X1*X5
X15-- X2*X3
X16-- X2*X4
X17-- X2*X5
X18-- X3*X4
X19-- X3*X5
X20-- X4*X5

Y50 -- DEPTH OF HARDNESS RC 50
Y30 -- DEPTH OF HARDNESS RC 30

1.8 Model

$$
\begin{aligned}
Y50_i = C0 &+ C1*X1_i + C2*X2_i + C3*X3_i + C4*X4_i + C5*X5_i + C6*X6_i \\
&+ C7*X7_i + C8*X8_i + C9*X9_i + C10*X10_i + C11*X11_i \\
&+ C12*X12_i + C13*X13_i + C14*X14_i + C15*X15_i + C16*X16_i \\
&+ C17*X17_i + C18*X18_i + C19*X19_i + C20*X20_i \\
&+ RES50_i
\end{aligned}
$$

$$
\begin{aligned}
Y30_i = C0 &+ C1*X1_i + C2*X2_i + C3*X3_i + C4*X4_i + C5*X5_i + C6*X6_i \\
&+ C7*X7_i + C8*X8_i + C9*X9_i + C10*X10_i + C11*X11_i \\
&+ C12*X12_i + C13*X13_i + C14*X14_i + C15*X15_i + C16*X16_i \\
&+ C17*X17_i + C18*X18_i + C19*X19_i + C20*X20_i \\
&+ RES30_i
\end{aligned}
$$

The program used to execute the multiple linear regression uses the least squares algorithm, where the sum of the squared errors is minimized. The IMSL subroutine CORVC first calculates the means for the twenty variables, the covariance of the twenty variables, and subtracts the appropriate mean from each of the twenty variables for each observation (i). Then, the IMSL subroutine RSTEP is used to perform a stepwise regression. The actual algorithm is a forward stepwise regression. The significance level for accepting a new variable into the least squares solution was set at .05. In other words, variables that were found to be significant in minimizing the sum of squares of the residuals were included in the model if the level of significance was .05 or less. However, these same variables would not be ejected from the model simply because another variable or two were added that caused a slight increase in the first variable's level of significance. After all of the twenty variables had been analyzed in this manner, the following model was arrived at.

Model for Depth of Hardness for RC 50

FORWARD SELECTION

DEPENDENT VARIABLE	R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR
1	96.638	96.332	.2709

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	4	92.78	23.20	316.169	.0000
ERROR	44	3.23	.07		
TOTAL	48	96.01			

* * * INFERENCE ON COEFFICIENTS * * *
(CONDITIONAL ON THE SELECTED MODEL)

VARIABLE	COEF. ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER T
6	-35.34	1.3	-26.931	.0000
9	-.12	.0	-2.727	.0091
12	-1.15	.1	-10.064	.0000
14	8.63	.3	25.026	.0000

* * * STATISTICS FOR VARIABLES NOT IN THE MODEL * * *

VARIABLE	COEF. ESTIMATE	STANDARD ERROR	T-STATISTIC TO ENTER	PROB. OF LARGER T
2	.13	.6	.207	.8370
3	.01	.3	.040	.9685
4	.44	10.5	.042	.9671
5	.54	.8	.647	.5211
7	.06	.3	.207	.8370
8	.00	.1	-.003	.9979
10	.34	.5	.678	.5013
11	-25.15	38.5	-.653	.5170
13	.02	.2	.101	.9204
15	-8.51	4.4	-1.942	.0587

16	.00	.1	.018	.9859
17	.06	.2	.379	.7065
18	-1.53	2.9	-.531	.5980
19	.16	.2	.646	.5220
20	4.36	105.0	.042	.9671
21	-9.07	8.3	-1.098	.2782

* * * FORWARD SELECTION SUMMARY * * *

VARIABLE	STEP ENTERED
6	2
9	4
12	3
14	1

THIS IS THE ESTIMATE OF THE REGRESSION CURVE FOR RC50

$$Y = 4.3106 - 35.3367(X5) - .1156(X3)^2 - 1.1473(X1)(X2) + 8.6291(X1)(X4)$$

SAMPLE	RESIDUAL
H34	-.4344E+00
H51	-.1651E+00
H52	-.5460E-01
H53	.8100E-01
H54	.3543E+00
H55	.4320E+00
H56	.2695E+00
H57	.7635E-01
H58	.1513E+00
H62	.4003E+00
H63	-.5496E+00
H64	-.7462E-01
H65	-.1276E+00
H66	-.2783E+00
L1	-.1631E+00
L2	-.5942E-01
L3	.3708E-01
L4	-.9322E-01
L5	-.1632E-01
L6	.8149E-02
L7	-.1017E+00
L8	-.1237E+00
L9	-.9436E-01
L10	-.4275E-01
L11	.2744E+00
L12	.1121E+00
L13	.3203E+00
L14	.1564E+00
L15	.2205E+00
L35	-.3298E+00
L36	-.4363E+00
L38	-.3967E+00
L46	.2281E+00
L47	.4467E+00

THE VARIANCE OF THE RESIDUAL IS .733648E-01

The results for determining the significant effects and their appropriate coefficients for RC 50 were obtained by using the same regression package as described above for the RC 50 model. However, in the stepwise regression portion of the program, Backward Elimination was used instead of Forward Regression. The difference in the two techniques is that, in Backward Elimination, all of the effects are included in the initial model. One by one, the effects are dropped from the model if their significance is found to be greater than .05. This significance level is arbitrary and was chosen at the author's discretion. The Backward Elimination method sometimes yields a greater adjusted R^2 than the Forward Regression method, as in this case.

Model for Depth of Hardness for RC 30

BACKWARD ELIMINATION

DEPENDENT VARIABLE	R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR
1	93.046	92.566	.7747

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	4	465.8	116.5	194.015	.0000
ERROR	58	34.8	.6		
TOTAL	62	500.6			

* * * INFERENCE ON COEFFICIENTS * * *
(CONDITIONAL ON THE SELECTED MODEL)

VARIABLE	COEF. ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER T
2	19.49	1.05	18.537	.0000
5	10.03	1.31	7.668	.0000
12	-3.04	.38	-8.036	.0000
15	-46.87	2.44	-19.217	.0000

* * * STATISTICS FOR VARIABLES NOT IN THE MODEL * * *

VARIABLE	COEF. ESTIMATE	STANDARD ERROR	T-STATISTIC TO ENTER	PROB. OF LARGER T
3	1.08	2.02	.536	.5942

4	.08	.67	.125	.9008
6	-3.07	12.97	-.237	.8137
7	.00	.00	....	....
8	.23	.40	.566	.5733
9	.01	.11	.139	.8897
10	25.24	26.18	.964	.3389
11	24.87	52.26	.476	.6359
13	-.07	.63	-.110	.9128
14	-2.47	5.49	-.450	.6542
16	.07	.24	.286	.7759
17	.16	2.11	.075	.9403
18	2.80	4.55	.616	.5401
19	.00	.00	....	....
20	.00	.00	....	....
21	-6.52	13.26	-.491	.6250

* * * BACKWARD ELIMINATION SUMMARY * * *

VARIABLE	STEP REMOVED
3	10
4	4
6	12
8	2
9	9
10	3
11	7
13	6
14	1
16	8
17	11
18	13
21	5

RESIDUALS PRODUCED BY THE ABOVE REGRESSION CURVE.

DISK	RESIDUAL
H16	.1454E+01
H17	-.7671E+00
H18	-.6743E+00
H19	-.5877E+00
H20	-.3902E+00
H21	-.2500E+00
H22	-.3012E+00
H23	-.1801E+00
H24	.1499E+01
H25	-.2907E+00
H26	.1412E+01
H29	-.6544E-02
H30	-.1018E+01
H31	-.2896E+00

H32	-.7035E+00
H33	.2444E+00
H34	.2552E+00
H51	-.4827E+00
H52	-.1992E+01
H53	.2157E+01
H54	.1230E+01
H55	-.1054E+00
H56	.2441E+00
H57	-.9638E-01
H58	-.1628E+00
H59	.2232E+01
H60	-.1468E+01
H62	-.1378E+01
H63	-.3569E+00
H64	-.6848E-01
H65	-.8008E-01
H66	.9232E+00
L1	-.2995E+00
L2	-.2303E+00
L3	-.2383E+00
L4	-.3247E+00
L5	-.4082E+00
L6	-.9927E-01
L7	-.3291E+00
L8	-.3020E+00
L9	-.1252E+00
L10	-.4177E+00
L11	.3600E+00
L12	.1639E+00
L13	.1867E+00
L14	-.5136E-01
L15	-.7786E-01
L35	-.5142E+00
L36	-.9305E-01
L37	-.5374E+00
L38	.1439E+00
L39	.1990E+00
L40	-.4022E-01
L41	.1820E+00
L42	.6879E+00
L43	.6704E+00
L44	.6627E+00
L45	.4735E+00
L46	.6014E+00
L47	-.3945E+00
L48	.5311E+00
L49	-.5245E+00
L50	.1445E+00

THE VARIANCE OF THE RESIDUAL IS .600231E+00

THIS IS THE FINAL ESTIMATE OF THE TRUE REGRESSION CURVE FOR RC30

$$Y=-8.8302+19.4866*(X1)+10.0279*(X4)-3.0447*(X1)*(X2)+ -46.87143*(X1)*(X5)$$

Given the estimates of the depths at which the hardness of RC 50 or RC 30 were achieved, the above regression equations are the best combinations of the variables allowed to enter the least squares solution.

1.8.1 The Confidence Interval

At each point in the 21 factor space defined by the variables' values, the confidence interval changes as the distance of each variable increases or decreases from the location of its mean value. The confidence interval depends completely upon the variance of the estimated depth. This variance is defined mathematically with the following expression.

$$VAR\{a_1X_1+a_2X_2...a_nX_n\} = a^2{}_1VAR\{X_1\}+a^2{}_2VAR\{X_2\}+...+a^2{}_nVAR\{X_n\} + 2\sum_i\sum_j a_ia_jCOV\{X_iX_j\}$$

Even with only four or five variables in the regression solution, the above expression can be very complicated. Fortunately, the IMSL subroutine RSTEP and RCASE provide the user with the complete variance-covariance matrix for the 21 variables. The appropriate values can be picked out of this matrix to make a new variance-covariance matrix which includes only those values that are needed to compute the variance of the depth estimate. The IMSL subroutine BLINF takes the distance of each variable from its mean value and nthe variance-covariance matrix desribed above and computes the variance of the estimated depth. With an assumption that the estimate follows a Student's T distribution, the confidence interval can now be easily calculated. The results are shown below.

CI Results for RC 50

THIS IS THE VARIANCE/COVARIANCE MATRIX.

.234669E+02	.639153E-02	.145512E+00	-.152605E+01
.639153E-02	.244955E-01	-.239533E-01	.372465E-01
.145512E+00	-.239533E-01	.177145E+00	-.437574E+00
-.152605E+01	.372465E-01	-.437574E+00	.162055E+01

VARIABLE	MEAN
X5	.101633E+00
X8	.109310E+02
X11	.287827E+01
X13	.992082E+00

CI Results for RC 30

THIS IS THE VARIANCE/COVARIANCE MATRIX.

.184113E+01	-.718064E-01	-.587396E+00	-.765534E+00
-.718064E-01	.284955E+01	.507973E-01	-.132053E+00
-.587396E+00	.507973E-01	.239171E+00	-.134780E+00
-.765534E+00	-.132053E+00	-.134780E+00	.991096E+01
-.718064E-01	-.587396E+00	-.765534E+00	-I

VARIABLE		MEAN VALUE
5	=	.870635E+00
12	=	.276484E+01
15	=	.113146E+00

THIS THE TABLE OF VALUES OF INDEP AND DEP VARIABLES

DI	BDIA	CDIA	POWER	SPEED	MEAN	CI	95%PRED
1.34	2.25	2.90	.90	.10	10.846470	1.352010	9.494461
1.34	2.25	3.21	.90	.10	10.846470	1.352010	9.494461
1.34	2.25	3.21	.90	.10	10.846470	1.352010	9.494461
1.34	2.25	3.21	.90	.10	10.846470	1.352010	9.494461
1.34	2.25	3.40	.90	.10	10.846470	1.352010	9.494461
1.34	2.50	2.90	.90	.10	9.826507	1.323964	8.502544
1.34	2.50	3.21	.90	.10	9.826507	1.323964	8.502544
1.34	2.50	3.40	.90	.10	9.826507	1.323964	8.502544
1.34	2.50	3.40	.90	.10	9.826507	1.323964	8.502544
1.34	2.50	3.40	.90	.10	9.826507	1.323964	8.502544
1.34	2.75	3.21	.90	.10	8.806544	1.351867	7.454678
1.34	2.75	3.21	.90	.10	8.806544	1.351867	7.454678
1.34	2.75	3.40	.90	.10	8.806544	1.351867	7.454678

1.34	2.75	3.40	.85	.10	8.305151	1.347790	6.957361
1.34	2.75	3.40	.85	.10	8.305151	1.347790	6.957361
1.34	2.75	3.40	.85	.14	5.792842	1.366535	4.426307
1.34	2.75	3.40	.85	.14	5.792842	1.366535	4.426307
1.34	2.25	3.40	.75	.05	12.482675	1.436541	11.046134
1.34	2.25	3.40	.75	.05	12.482675	1.436541	11.046134
1.34	2.25	3.40	.75	.05	12.482675	1.436541	11.046134
1.34	2.75	3.40	.75	.05	10.442749	1.438240	9.004509
1.34	2.25	3.40	.75	.15	6.201903	1.438642	4.763261
1.34	2.75	3.40	.75	.15	4.161977	1.416715	2.745261
1.34	2.75	3.40	.75	.15	4.161977	1.416715	2.745261
1.34	2.75	3.40	.75	.15	4.161977	1.416715	2.745261
1.34	2.25	3.40	.95	.05	14.488250	1.414252	13.073999
1.34	2.75	3.40	.95	.05	12.448324	1.429322	11.019003
1.34	2.75	3.40	.95	.05	12.448324	1.429322	11.019003
1.34	2.25	3.40	.95	.15	8.207478	1.409401	6.798077
1.34	2.25	3.40	.95	.15	8.207478	1.409401	6.798077
1.34	2.25	3.40	.95	.15	8.207478	1.409401	6.798077
1.34	2.75	3.40	.95	.15	6.167552	1.400632	4.766921
.86	2.25	2.90	.90	.10	7.030971	1.334468	5.696503
.86	2.25	2.90	.90	.10	7.030971	1.334468	5.696503
.86	2.25	2.90	.90	.10	7.030971	1.334468	5.696503
.86	2.25	3.21	.90	.10	7.030971	1.334468	5.696503
.86	2.25	3.40	.90	.10	7.030971	1.334468	5.696503
.86	2.50	2.90	.90	.10	6.376368	1.323459	5.052909
.86	2.50	3.21	.90	.10	6.376368	1.323459	5.052909
.86	2.50	3.21	.90	.10	6.376368	1.323459	5.052909
.86	2.50	3.21	.90	.10	6.376368	1.323459	5.052909
.86	2.50	3.40	.90	.10	6.376368	1.323459	5.052909
.86	2.75	3.21	.90	.10	5.721765	1.335646	4.386119
.86	2.75	3.21	.90	.10	5.721765	1.335646	4.386119
.86	2.75	3.40	.90	.10	5.721765	1.335646	4.386119
.86	2.75	3.40	.90	.10	5.721765	1.335646	4.386119
.86	2.75	3.40	.90	.10	5.721765	1.335646	4.386119
.86	2.25	3.40	.75	.05	7.542261	1.395511	6.146750
.86	2.75	3.40	.75	.05	6.233055	1.395070	4.837985
.86	2.75	3.40	.75	.05	6.233055	1.395070	4.837985
.86	2.75	3.40	.75	.05	6.233055	1.395070	4.837985
.86	2.25	3.40	.75	.15	3.511318	1.403288	2.108030
.86	2.25	3.40	.75	.15	3.511318	1.403288	2.108030
.86	2.25	3.40	.75	.15	3.511318	1.403288	2.108030
.86	2.75	3.40	.75	.15	2.202112	1.392906	.809206
.86	2.25	3.40	.95	.05	9.547836	1.365538	8.182298
.86	2.25	3.40	.95	.05	9.547836	1.365538	8.182298
.86	2.25	3.40	.95	.05	9.547836	1.365538	8.182298
.86	2.75	3.40	.95	.05	8.238630	1.373983	6.864647
.86	2.25	3.40	.95	.15	5.516893	1.368865	4.148028
.86	2.75	3.40	.95	.15	4.207687	1.367160	2.840526
.86	2.75	3.40	.95	.15	4.207687	1.367160	2.840526
.86	2.75	3.40	.95	.15	4.207687	1.367160	2.840526

The question remains, "Are these regression equations adequate estimates of the true relationships between the independent variables?" The reliability of the model and its underlying assumptions are questioned in Chapter Two.

Chapter 2
A QUESTION OF UNIFORMITY

2.1 The Hypothesis Test

All of the work done up to now has assumed that the hardening process performs uniformly throughout the steel rod. In the case of a disk cut from a hardened steel rod, the assumption is that the effect of the hardening process depends only on depth from the surface and not on the angular location of the point of interest. However, this assumption has not been shown to be true. The purpose of this chapter is to analyze this hypothesis and determine whether it is supported by the data or not.

NULL HYPOTHESIS: The hardness of a given disk of induction hardened steel does not depend upon its angular location from a set index point, but only upon its distance from the surface of the disk.

ALTERNATIVE HYPOTHESIS: The hardness of a given disk of induction hardened steel does vary significantly, depending upon its angular location from a set index point.

Data is available to test this hypothesis since the measurements of depth and hardness were taken at 30 degree intervals around each sample disk. The index point is the "fishtail" point that was marked on each disk. The "fishtail" point is the location of the power-in and power-out cords that extend from the induction coil. These cords could affect the magnetic field in and around the disk which, in turn, could affect the hardening of the disk. Below is an illustration of the measurements taken in relation to distance from the surface of the disk and angular distance from a radial line at the index point.

Figure 2.1

2.2 Experimental Design

A design had to be chosen that would test the effect of T on the hardness but would block out the effect of the six other known factors that effect the hardness of these sample disks. These factors are :

DEPTH

DI

BAR DIAMETER

COIL DIAMETER

INDUCTION POWER

TRAVEL SPEED OF THE COIL

Several problems were immediately recognized. First, no more samples could be taken than had already been produced due to cost, time, and production restrictions. Also, the last five factors listed above, were of various levels and these levels were not uniformly spread throughout the five dimensional space so as to allow a full factorial analysis of the data. Fortunately, a complete factorial analysis could be done with respect to depth and Θ. A way had to be found to either discount the effects of the aforementioned five factors or to fully take account of them during the analysis of the data.

Reviewing the levels of each factor, DI was observed only at two levels, Bar Diameter was observed at three levels, Coil Diameter was observed at three levels, Induction Power was observed at four levels, and Induction Speed was observed at four levels. A full factorial design of this experiment would involve 2 * 3 * 3 * 4 * 4 or 288 samples with only one repetition at each level. Only 66 samples were taken with four being discarded immediately as inappropriate samples. Many others were repetitions of the same levels of each of the five factors. A greatly reduced design had to be constructed.

I immediately started looking for a reduced factorial designed experiment but none was found. If each of the factors was assumed to be linear in its effect on hardness, which is not at all a valid assumption as seen in the model produced in Chapter 1, a 2^5 design might yield useable results. However, even samples at these 2^5 levels were not available due to the restricted method of setting up the original experiment. A fractional factorial design, 2^{5-3} seemed to be the most reasonable in terms of ability to attain the desired levels for the five factors and to maintain some level of orthogonality. Using Montgomery's[3] method of designing fractional factorial experiments, the table below was developed.

FACTOR--	A	B	C	D=AB	E=AC	RUN
	−	−	−	+	+	de
	+	−	−	−	−	a
	−	+	−	−	+	be
	+	+	−	+	−	abd
	−	−	+	+	−	cd
	+	−	+	−	+	ace
	−	+	+	−	−	bc
	+	+	+	+	+	abcde

The five factors were assigned to the above factor names A,B,C,D, and E. In terms of actual factor levels, the runs column can be interpreted as the following table. The (-) sign indicates a factor's lowest level and a (+) sign indicates a factor's highest level.

RUN	FACTORS--DI	POWER	SPEED	BAR DIA	COIL DIA	AVAILABLE?
	A	B	C	D	E	
de	.86	.75	.05	2.75	3.90	no
a	1.34	.75	.05	2.25	2.90	no
be	.86	.95	.05	2.25	3.40	yes
abd	1.34	.95	.05	2.75	3.40	yes
cd	.86	.75	.15	2.75	3.40	yes
ace	1.34	.75	.15	2.25	3.40	yes
bc	.86	.95	.15	2.25	2.90	no
abcde	1.34	.95	.15	2.75	3.90	no

It is obvious that with this proposed design, not enough samples were available to reasonably conduct the experiment. The above results are typical of the results found when assigning the five factors in different ways to the five columns defined in Montgomery's fractional factorial design. In each case, there were samples missing from the required samples for that particular design and mix of factors. It appeared that another method of blocking out the other significant factors had to be found.

In order to minimize the effect of the five factors, it seemed reasonable to choose samples that were most representative of the conditions under which the majority of steel rods would be made during actual production. In other words, an attempt was made to minimize the distance between the factors' values for the range of values that would be used during normal production and the values for the samples used for this experiment. Another important aspect of choosing the five factor location of the samples was a location which had more than one representative in the pool of available samples. This was needed so that more credibility could be given to the results and so that an applicable test might be run at a given significance level to indicate acceptance or rejection of the null hypothesis.

For four of the five factors, samples were available for more than two levels of each factor. A sample with values in the middle of these four ranges would satisfy the above mentioned minimization. For the factor of Steel Composition, or DI, this could not be done. So, a sample location was chosen that contained centrally located values for the four other significant factors and the 1.34 DI level. Another sample was chosen that again contained centrally located values for the four multilevel factors, and the .86 DI level. Both of these points in the five factor space had 3 representatives in the sample pool. Below are listed the levels for each of the five factors to be blocked out.

SAMPLE	DI	BAR DIA	COIL DIA	POWER	SPEED	REPETITIONS
'LOWCOMP'	.86	2.50	3.21	.90	.10	3
'HICOMP'	1.34	2.50	3.40	.90	.10	3

2.3 The Experiment

The samples used for this experiment were H16, H17, and H18. Also used in the low DI range were L1, L2, and L3. The data for these samples are listed below.

SAMPLE-- H23

DEPTH--1/16	2/16	3/16	4/16	5/16	6/16
DEGREES					
:					
0 57.8	57.5	56.1	49.3	38.1	31.7

30	58.0	57.3	54.6	46.4	36.2	31.5
60	58.6	57.3	54.6	46.8	36.1	30.7
90	58.3	57.3	54.6	47.2	36.1	31.0
120	58.4	57.6	54.7	47.2	37.0	30.7
150	58.3	57.9	55.5	49.3	38.3	31.1
180	58.0	57.7	55.9	50.0	37.6	31.4
210	58.1	57.5	56.0	50.0	37.0	30.7
240	57.9	57.5	55.9	50.4	36.9	30.5
270	57.6	57.5	56.3	50.9	38.1	31.5
300	58.0	57.0	55.0	47.3	36.4	30.8
330	57.9	57.0	55.1	47.5	35.4	31.0

SAMPLE-- H24

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	58.0	57.0	53.8	43.7	35.6	29.6
30	57.9	57.1	52.5	41.2	33.6	30.5
60	57.0	56.2	52.7	45.7	36.0	28.4
90	57.4	56.6	54.0	44.9	35.7	29.3
120	58.0	57.3	54.7	46.9	36.0	31.0
150	57.7	57.1	53.9	48.3	37.0	32.8
180	58.2	57.8	55.1	46.9	36.2	32.7
210	58.1	57.1	54.6	45.1	34.2	30.5
240	58.1	57.4	55.6	46.4	33.9	30.9
270	57.9	57.3	54.2	45.6	36.6	31.7
300	58.3	57.0	54.1	44.9	34.8	29.3
330	58.0	57.3	54.2	44.4	34.1	29.9

SAMPLE-- H25

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	58.1	57.0	54.7	45.1	34.6	29.7
30	58.1	57.3	53.6	45.2	34.6	30.5
60	58.1	57.4	54.4	46.0	36.2	30.2
90	58.1	57.3	55.7	47.6	36.4	29.9
120	58.3	57.8	55.5	49.4	36.9	30.8
150	58.5	57.9	56.3	51.0	38.5	31.1
180	58.1	57.9	56.0	49.5	36.4	30.8
210	57.4	56.5	54.9	47.4	35.5	30.0
240	57.9	57.1	55.0	46.3	34.1	29.1
270	58.2	57.3	54.3	43.8	33.6	28.4
300	57.7	56.5	52.6	44.3	32.9	29.1
330	58.3	57.1	53.9	44.9	35.0	30.0

SAMPLE-- L7

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	52.1	49.9	38.0	28.0	24.2	22.0

30	52.8	49.6	37.4	27.5	24.5	22.0
60	52.9	51.0	39.3	28.8	24.4	22.0
90	53.6	51.2	39.1	28.1	25.0	22.0
120	53.1	51.0	39.6	28.5	25.1	22.0
150	53.7	51.0	40.1	28.6	25.0	22.3
180	53.7	51.4	41.0	29.0	24.6	22.6
210	53.0	50.9	39.1	28.7	24.4	21.0
240	53.3	50.7	39.0	28.7	25.1	22.4
270	54.0	50.5	38.4	28.2	24.6	22.0
300	53.4	50.0	37.0	27.7	24.1	21.4
330	53.2	50.4	38.1	27.6	23.9	21.5

SAMPLE-- L8

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	52.7	49.3	35.1	26.9	22.5	19.3
30	52.9	50.6	38.4	28.4	25.1	22.0
60	53.0	49.5	37.4	27.9	24.8	22.0
90	52.7	50.3	37.2	27.0	23.5	20.2
120	52.5	51.0	40.0	29.4	25.0	22.5
150	53.0	51.1	40.5	29.7	25.3	21.3
180	53.4	50.9	40.5	29.3	25.1	22.6
210	53.1	50.8	39.7	28.8	24.4	21.9
240	52.8	50.5	40.4	28.8	25.0	22.9
270	52.9	50.6	39.8	28.2	24.5	20.5
300	53.8	50.7	39.7	29.5	24.5	20.5
330	52.9	50.4	38.4	27.0	22.8	19.6

SAMPLE-- L9

DEPTH-- DEGREES :	1/16	2/16	3/16	4/16	5/16	6/16
0	53.1	50.3	41.3	28.8	24.1	20.4
30	53.0	50.0	37.8	27.1	22.4	19.2
60	52.7	50.4	40.0	28.9	24.4	20.6
90	52.6	50.8	41.5	30.4	24.2	19.9
120	52.6	50.3	40.0	28.6	24.9	20.8
150	52.4	50.0	38.6	28.1	23.7	21.1
180	52.7	50.5	41.8	29.6	23.9	20.4
210	53.0	51.3	40.5	28.5	22.9	20.0
240	53.3	50.5	40.9	30.7	23.6	19.9
270	52.9	50.6	42.3	30.3	23.1	20.0
300	53.0	50.0	38.7	27.9	22.8	19.8
330	52.5	50.5	41.0	28.6	24.2	20.3

2.4 Analysis

ANOVA was used to analyze the above data. The choice of this type of analysis was based on the form of the data, the simple hypothesis that was to be tested, and upon the applicability of the assumptions that must be made whenever using Analysis of Variance techniques. Once the samples were chosen, the data were in a complete factorial design in depth and Θ. The depth could esaily be blocked out. Since only a test of a simple hypothsis was required, and no regression coeffcients were necessary or wanted, ANOVA was appropriate. Lastly, the assumptions were accepted that the error or variation of the observations within each cell are random variations and distributed as independent, NORMAL random variables $(0.0,\sigma^2)$. If these assumptions were correct, then the results from an ANOVA could be used in rejecting or accepting the null hypothesis.

In this analysis, the relationship of depth to hardness was already known to be an extremely dependent one so that the actual significance of this effect was not of particular interest. However, the interaction effect between the depth and Θ was of interest so depth was not just a blocking factor but, in fact, a factor of interest.

A two-way ANOVA program was used to analyze the two sets of data. These two sets were entered and analyzed seperately. The analysis included only four depths for every value of Θ for the low compositions samples because the area of interest includes readings down to RC 30. Readings from greater depths would have been less reliable because the machine loses some of its accuracy when the hàrdness is under RC 20. In the high composition samples, however, all six depths were used because the hardness readings never approached RC 20. There were 48 useable readings from each low compositon sample and 72 useable readings from the high composition samples. The ANOVA results for the low composition data are listed below.

TWO-WAY FIXED EFFECTS ANALYSIS OF VARIANCE

SOURCE	SUM OF SQUARES	DF	MEAN SQUARE	F	SIGNIFICANCE
ANGLE	28.46875	11	2.588068	2.953	0.0021
DEPTH	13625.03	3	4541.677	5182.776	.0000
INTERACTION	16.4375	33	.4981061	.568	.9663
ERROR	84.125	96	.8763021		
TOTAL	13754.06	143			

This analysis shows that with a significance of .0021, the Θ is extremely important in explaining the variation in hardness. The interaction, on the other hand, seems to have almost no effect on the prediction of the hardness(INTERACTION significance=.9663). The ANOVA results for the high compostion data are shown below and compare favorably with the low composition results.

TWO-WAY FIXED EFFECTS ANALYSIS OF VARIANCE

SOURCE	SUM OF SQUARES	DF	MEAN SQUARE	F	SIGNIFICANCE
ANGLE	72.15625	11	6.559659	4.718	.00000
DEPTH	24528.53	5	4905.706	3528.249	.00000
INTERACTION	62.8125	55	1.142046	.821	.7964
ERROR	200.2188	144	1.390408		

TOTAL	24863.72	215

Again, the importance of the variable Θ was revealed by the significance of factor A =.00000. Also, as with the results from the low composition data, the interaction effect is nowhere near the level of significance that would indicate a relationship between depth and Θ.

2.5 Assumptions

One must ask the question,"Are these results reliable?" In the analysis, the MSE which was used to determine the significance of the factors in question had 96 and 144 degrees of freedom, respectively, for the low and high composition data. The relatively large number of degrees of freedom lends credibility to the F statistic which was used to determine the levels of significance and therefore lends credibility to the conclusions made from this statistic. There were three assumptions made about the error that should now be plotted and tested for validity.

2.5.1 Normality of Error

Normal probability plots made on the errors, or residuals are shown below. The low composition residuals have a plot which veers off the straight line near the tails. This type of plot would indicate a slight deviation from the standard normal distribution. The high composition plot is more randomly scattered around a straight line and indicates a better fit of the normal distribution. When the tails veer off from the striaght line, this indicates heavy tails is the problem and certainly is an indication of non-normality.

2.5.2 Independence

In Appendix B, plots of residuals vs. Depth, residuals vs. Θ, and plots of residuals vs. Predicted Hardness are attached. The plots for residuals vs. Depth for both high composition and low compostion steel samples were difficult to interpret. The four or six depths that were plotted, seemed to indicate that the variance of the residuals was relatively small near the surface of the steel rod and closer to the center of the steel rod (greater than 3/16 inch in depth), but the variance was greater near depths of 2/16 inch and 3/16 inch. Since there were so few depths plotted, it is hard to say that this possible trend was, in actuality, a dependency.

Since the predicted hardness is tied so closely with the value of depth, it was no surprise that the plots of residuals vs. Predicted Hardness for both high and low composition steel samples looked very much like the plots for residuals vs. Depth. The variance seemed to be smaller when the predicted hardness was at values of RC 50 and RC 20 but greater for predicted hardness values of RC 30 and RC 40.

The plots of residual vs. Θ did have enough points of Θ to detect a trend if one did, in fact, exist. The trend that was evident was one of relatively stable and constant mean but varying variance. The variance seemed to decrease and increase cyclically as Θ increased from 0 degrees to 330 degrees. Most of the classical transformations are not applicable to a trend that is cyclical. Just by judging from the plots, the frequency of the trend appeared to be about 90 degrees so that the variance was at a minimum when Θ had values of 0 degrees and 180 degrees.

2.5.3 Constant Variance

The same plots that were used to check for independence were used to check for constant variance. The same comments about the various plots of the residuals apply to check this assumption except that the mean of the residuals is no longer an important issue. Since the mean stayed fairly constant and centered around zero for all of the plots, the question of independence became a question of constant variance. This question must be answered that the variance is definitely not constant.

2.5.4 Conclusions

The assumption of normality of the residuals is questionable in the case of low compostion steel samples but relatively valid for the high composition steel samples. The assumption of independence as far as the mean remaining constant and close to zero is valid in all cases. However, the assumption of independence in relation to the variance and, of course, the assumption of constant variance must be seriously questioned. All of the plots of the residuals show some kind of dependence of the size of the variance on the various variables with which independence is most important. Predicted hardness, depth, and Θ all affect the size of the variance even though the exact relationship is not clear in any of the cases. With these results, it is very dubious as to whether the ANOVA results can be applied and, consequently, whether the conclusions of the F test can still be made with any degree of certainty.

It is apparent that some other test that does not depend on the assumptions listed above for the ANOVA test must be used in order to arrive at a conclusion that can be trusted. A non-parametric test is a natural option when the normal assumptions of independence and constant variance can not be made.

2.6 Non-Parametric Test

The non-parametric test called the Friedman's test [4] was chosen as the correct test for this particular case. It was chosen for the following reasons.

1. It does not need the assumption of an underlying normal distribution which was a questionable assumption in the low composition steel samples.

2. The Friedman test checks two samples for independence. The two variables that we need to check the independence of are Θ and Hardness.

3. This test is easily adaptable to blocking out one factor, namely (Depth), since the interaction effect is negligible.

4. The Friedman statistic is easy to calculate and the table is readily available as opposed to other non-parametric tests that would have to have tables developed specifically for this set of data.

5. The effect, if there exists an effect, need not be linear to be detected by the Friedman test. We have already seen that there are some definite non-linear factors at work in this data set and a test that would check for only linear effects could well yield an erroneous conclusion.

The Friedman test consists of ranking the observations within each block of data and then performing what is essentially an ANOVA on the ranks rather than on the actual values(hardness readings). The blocks would be the four or six depths used in the low composition and high composition samples, respectively. Of course, the F statistic is not used to test the significance of the results, the Chi Square statistic is used. It has been shown that the Chi Square distribution very closely approximates the distribution of the Friedman statistic under the null hypothesis when the number of samples is relatively large.

Low Composition Samples

Θ\DEPTH	1	2	3	4	TOT	1	2	3	4	TOT	1	2	3	4	TOT
0	1	2	3	4	10	3	1	1	1	6	11	4	9	7	31
30	2	1	2	1	6	5	7	4	5	21	10	3	1	1	15
60	3	8	9	11	31	9	2	3	6	20	6	6	4	8	24
90	9	11	8	5	33	2	3	2	3	10	3	11	10	11	35
120	5	10	10	7	32	1	11	9	10	31	4	5	5	6	20
150	10	9	11	8	38	8	12	11	12	43	1	2	2	3	8
180	11	12	12	12	47	11	10	10	9	40	5	7	11	9	32
210	4	7	7	10	28	10	9	7	7	33	8	12	6	4	30
240	7	6	6	9	28	4	5	12	8	29	12	9	7	12	40
270	12	5	5	6	28	7	6	8	4	25	7	10	12	10	39
300	8	3	1	3	15	12	8	6	11	37	9	1	3	2	15
330	6	4	4	2	16	6	4	5	2	17	2	8	8	5	23

2.6.1 Calculations for the Low DI Samples

K = Number of treatments(angles) = 12

N = Number of Blocks(Depths) = 4

$$R_{..} = (K+1)/2 = (12+1)/2 = 6.5$$

$$S_{L7} = 12N/(K(K+1)) \times \sum_{j=1}^{4} (R_{.j} - R_{..})^2$$

$$S_{L7} = (12/(NK(K+1))) \times \sum_{j} R_{.j}^{\,2} \quad -3N(K+1)$$

$$S_{L7} = (12/(4x12x13)) \times (9696) \quad -3 \times 4 \times 13 = 30.46$$

$$S_{L8} = (12/(4x12x13)) \times (9600) \quad -3 \times 4 \times 13 = 28.62$$

$$S_{L9} = (12/(4x12x13)) \times (9250) \quad -3 \times 4 \times 13 = 21.885$$

Use $X^2_{(12-1)}$ for evaluating the Friedman statistic since (NK) is relatively large.

The P-Value for L7 = $P\{X^2_{(11)} > 30.46\} < .005$

The P-Value for L8 = $P\{X^2_{(11)} > 28.62\} < .005$

The P-Value for L9 = $P\{X^2_{(11)} > 21.885\} = .025$

High Composition Samples

Θ\DEPTH	1	2	3	4	5	6	TOT	1	2	3	4	5	6	TOT
0	2	9	11	8	10	12	52	8	4	3	2	6	4	27
30	6	4	1	1	4	11	27	4	5	1	1	1	7	19
60	12	3	2	2	3	3	25	1	1	2	8	9	1	22
90	9	5	3	3	2	7	29	2	2	5	4	7	2	22
120	11	10	4	4	7	2	38	6	8	10	10	8	9	51
150	10	12	7	7	12	8	56	3	6	4	12	12	12	49
180	7	11	9	9	9	9	54	11	12	11	11	10	11	67
210	8	7	10	10	8	4	47	9	7	9	6	4	6	41
240	3	8	8	11	6	1	37	10	11	12	9	2	8	52
270	1	6	12	12	11	10	52	5	10	7	7	11	10	50
300	5	2	5	5	5	5	27	12	3	6	5	5	3	34
330	4	1	6	6	1	6	23	7	9	8	3	3	5	35

Θ\DEPTH	1	2	3	4	5	6	TOT
0	6	3	6	4	5	4	28
30	4	7	2	5	4	9	31
60	5	9	5	6	8	8	41
90	8	6	10	9	9	5	47
120	11	10	9	10	11	11	62
150	12	11	12	12	12	12	71
180	7	12	11	11	10	10	61
210	1	2	7	8	7	7	32
240	3	5	8	7	3	2	28
270	9	8	4	1	2	1	25
300	2	1	1	2	1	3	10
330	10	4	3	3	6	6	32

2.6.2 Calculations for High DI Samples

K = Number of treatments(angles) = 12

N = Number of Blocks(Depths) = 6

$R_{..} = (K+1)/2 = (12+1)/2 = 6.5$

$$S_{H23}=12N/(K(K+1))\times\sum_j (R_{.j}-R_{..})^2$$

$$S_{H23}=(12/(NK(K+1)))\times\sum_j R_{.j}^{\,2} \;-3N(K+1)$$

$$S_{H23}=(12/(6\times 12\times 13))\times(19935) \;-3\times 6\times 13=21.577$$

$$S_{H24}=(12/(6\times 12\times 13))\times(20815) \;-3\times 4\times 13=32.86$$

$$S_{H25}=(12/(6\times 12\times 13))\times(21798) \;-3\times 6\times 13=45.46$$

Use $X^2_{(12-1)}$ for evaluating the Friedman statistic since (NK) is relatively large.

The P-Value for H23 = $P\{X^2_{(11)} > 21.577\} = .034$

The P-Value for H24 = $P\{X^2_{(11)} > 32.86\} < .005$

The P-Value for H25 = $P\{X^2_{(11)} > 45.46\} < .005$

2.7 Conclusions About the Importance of THETA

The Friedman Test arrived at the same conclusions as the parametric test except that now, the nagging assumptions of independence and constant variance can be ignored. With a great deal of confidence, it can be asserted that the hardness depends upon the value of Θ for both the low composition samples (p-value less than or equal to .025) and for the high composition samples (p-value less than or equal to .034). This conclusion, however, is strictly valid for only the values of the five significant factors discussed in the beginning of this chapter and not for all values or all ranges of values of these factors. The values of the five factors were chosen, however, to be representative of the operating conditions normally used in this particular magnetic induction hardening process. The effect of Θ was so significant, it would not be expected to significantly decrease in the neighborhood of the tested values of the factors. Most important is the fact that the effect of Θ was significant at one point in the five dimensional space. If an accurate model is desired for explaining the depth of a particular hardness, Θ should most definitely be included in this model. In other words, the assumption that Θ has no effect on our ability to predict the depth of hardness for a given set of operating conditions is incorrect.

It should also be noted that the interaction effect between Θ and Depth is almost non-existent. This is a surprise. It was thought that the most logical reason for hardness to depend on Θ was the nonuniformity of the induction field due to the "fishtail" effect of the induction coil. However, if this was the main cause, one would expect to find a decreasing effect of Θ on the hardness readings as the depth increased because the induction hardness process itself has a decreasing effect on the hardness of the steel rod as the depth increases. As stated above, this is not the case. Perhaps another cause of the non-uniformity of the hardness should be explored. The composition of the steel may be a major cause of the Θ effect, or perhaps the physical set-up of the induction hardening operation, to include the quenching, is responsible.

The nature of the effect is a question that has not been answered at all in this chapter. An additional set of plots was produced to perhaps lend some insight into the actual relationship of hardness vs. Θ. Appendix B contains two plots of the average hardness readings vs. Θ. These averages completely ignore the changes due to depth and any interaction between Θ and Depth. From the plots it is even more apparent that the effect of Θ is cyclical. By the very nature of the variable Θ, a cyclical effect of length 360 degrees was expected. However, the cycle evident in the plots is approximately 180 degrees. A 180 degree cycle would point towards a functional relationship involving the transcendental functions Sine, Cosine, or a combination of both. This relationship is the starting point for the next area of study-developing a regression model that includes a Sine(Θ), or Cosine(Θ) term in the model.

Chapter 3
INCLUDING ANGLE IN THE MODEL

3.1 Testing for Linearity and Symmetry

Because of the results found in the last chapter, it was concluded that any model used to predict the depth of some given hardness on a steel disk must include an angular position variable(Θ). There was no indication, though, as to how to include this new variable. Since, the last plots made in chapter two showed a pattern of something close to symmetry around the 0-180 degree line, it was determined that the first test should be to check symmetry for all the disks.

Dr. Robert Storer assisted me in developing a test that would test both the symmetry of the hardness readings on the disk and the linearity of the dependency of hardness on Θ. This test was based on the following model:

HARDNESS(i,j)= B0 + B1*$X1_j$ + B2*$X2_j$ + B3*$X3_j$

HARDNESS(i,j)=Hardness reading at depth i, and Θ j.

$X1_j$= The shortest angular distance from Θ_j and the index point. $0<X1_j<180$

$X2_j$= 1 if $0<\Theta_j<180$
0 if $180<\Theta_j<360$

$X3_j$= 1 if $180<\Theta_j<360$
0 if $0<\Theta_j<180$

This particular design allows the simultaneous testing of two hpotheses.

HO_1: There is no difference between the hardness of the side of the disk labelled 0-180 degrees and the hardness of the disk labelled 180-360 degrees.

Ha_1: There is a significant difference between the hardness on the two sides of the disk.

HO_2: A linear model is not an adequate model to explain the dependence of Hardness on Θ.

Ha_2: A linear model is adequate in explaining the dependence of hardness on Θ.

The test was set up as follows:

1. Identify the hardness readings for a given depth (say 1/16 inch) for the values of Θ from 0 degrees to 180 degrees. Assign the Θ_j values to $X1_j$.
2. For each j in (1) above, assign $X2_j=1$, and, assign $X3_j=0$.
3. Identify the hardness readings for the same depth for the values of Θ from 180 to 360 degrees. Assign $(360-\Theta_j)$ to $X1_j$.
4. For each j in (3) above, assign $X3_j=1$, and, assign $X2_j=0$.
5. Use multiple linear regression to find the best coefficients for the three variables and the constant.
6. Use the F ratio test generated by the regression package to determine acceptance or rejection of the Null Hypothesis_2.
7. Use the p-values of the Student's t test done on the coefficients of the variables X2 and X3 to determine acceptance or rejection of the Null Hypothesis_1.
8. Repeat steps 1 through 7 for all six depths and for all sample disks.

This method of testing the symmetry of the hardness has several advantages. First, not only is the symmetry being tested, but, also the linearity of the relationship between Θ and hardness can be checked through the goodness of fit ratio inherent in any linear regression, the F test. The second advantage is that the depth effect can be totally ignored since all of the tests are being conducted at the same depth. The final advantage is that a natural output statistic for a regression is the MSE or the estimate of the variance of the error for that regression. This estimate can then be used to adjust the difference in the coefficients in the model so that the results for one test can be compared quite readily to the results of a test done at a different depth.

The results for all of the 62 sample disks are listed below. The most important numbers in the results are the adjusted R^2 values for each linear regression which indicate the goodness of fit of the linear model along with the F statistic. Also important, of course, are the p-values of the T-test results for the coefficients of X2 and X3, which are used to determine the acceptance or rejection of the Null Hypothesis_1. A p-value associated with an F-ratio that was less than .10, was considered cause for rejecting the Null Hypothesis_2, and accepting the Alternate Hypothesis_2, that the model was linear. If both p-values for the T statistics for the coefficients of X2 and X3 were below .10, then that was considered cause for rejecting the Null Hypothesis_1, that the hardness was symmetric around the 0-180 degree line.The results below are only for the sample disks H17 and L7, but are indicative of the results for all 62 samples disks.

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
45.367	18.050	.3362	58.33	.5763

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	.563	.1877	1.661	.2730
RESIDUAL	6	.678	.1130		
CORRECTED TOTAL	9	1.241			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	58.71	.3526	166.5	.0000	11.0
2	.00	.0035	-.7	.5347	2.0
3	-1.00	.4986	-2.0	.0917	5.5
4	.01	.0050	1.5	.1937	6.5

THE RESULTS FOR SAMPLE H17 AT 1 SIXTEENTHS OF AN INCH DEPTH.

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
15.459	.000	.5648	57.56	.9812

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	.350	.1167	.366	.7807
RESIDUAL	6	1.914	.3190		
CORRECTED TOTAL	9	2.264			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	57.99	.5924	97.90	.0000	11.0
2	-.01	.0060	-.84	.4332	2.0
3	-.08	.8377	-.10	.9270	5.5
4	.00	.0084	.16	.8794	6.5

THE RESULTS FOR SAMPLE H17 AT 2 SIXTEENTHS OF AN INCH DEPTH.

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
20.730	.000	.7632	55.59	1.373

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	.914	.3047	.523	.6822
RESIDUAL	6	3.495	.5825		
CORRECTED TOTAL	9	4.409			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	55.61	.800	69.47	.0000	11.0
2	.00	.008	.12	.9051	2.0
3	.71	1.132	.63	.5536	5.5

4	-.01	.011	-.91	.3988	6.5

THE RESULTS FOR SAMPLE H17 AT 3 SIXTEENTHS OF AN INCH DEPTH.

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
44.323	16.485	1.653	49.15	3.363

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	13.05	4.350	1.592	.2870
RESIDUAL	6	16.39	2.732		
CORRECTED TOTAL	9	29.44			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	51.97	1.734	29.98	.0000	11.0
2	-.04	.017	-2.05	.0866	2.0
3	-2.16	2.452	-.88	.4122	5.5
4	.03	.025	1.33	.2332	6.5

THE RESULTS FOR SAMPLE H17 AT 4 SIXTEENTHS OF AN INCH DEPTH.

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
26.148	.000	1.82	39.05	4.66

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	7.04	2.345	.708	.5814
RESIDUAL	6	19.87	3.312		
CORRECTED TOTAL	9	26.91			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	41.29	1.909	21.63	.0000	11.0
2	-.02	.019	-.96	.3761	2.0
3	-3.52	2.699	-1.30	.2400	5.5
4	.03	.027	.96	.3749	6.5

THE RESULTS FOR SAMPLE H17 AT 5 SIXTEENTHS OF AN INCH DEPTH.

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
39.971	9.957	1.185	32.85	3.606

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	5.61	1.869	1.332	.3490
RESIDUAL	6	8.42	1.403		

CORRECTED TOTAL	9	14.03

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	31.56	1.242	25.40	.0000	11.0
2	.01	.012	.91	.3990	2.0
3	-.27	1.757	-.15	.8829	5.5
4	.01	.018	.51	.6285	6.5

THE RESULTS FOR SAMPLE H17 AT 6 SIXTEENTHS OF AN INCH DEPTH.

RESULTS FOR L7

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
53.054	29.581	.428	50.63	.8453

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	1.242	.4140	2.260	.1817
RESIDUAL	6	1.099	.1832		
CORRECTED TOTAL	9	2.341			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	49.99	.4489	111.4	.0000	11.0
2	.01	.0045	1.3	.2558	2.0
3	-.07	.6348	-.1	.9158	5.5
4	.00	.0064	.6	.5864	6.5

THE RESULTS FOR SAMPLE L7 AT 2 SIXTEENTHS OF AN INCH DEPTH.

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
74.338	61.507	.6054	38.71	1.564

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	6.370	2.123	5.794	.0332
RESIDUAL	6	2.199	.366		
CORRECTED TOTAL	9	8.569			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	37.12	.6349	58.46	.0000	11.0

2	.01	.0064	2.09	.0817	2.0
3	.27	.8979	.30	.7738	5.5
4	.01	.0090	.63	.5532	6.5

THE RESULTS FOR SAMPLE L7 AT 3 SIXTEENTHS OF AN INCH DEPTH.

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
64.474	46.711	.3612	28.24	1.279

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	1.421	.4737	3.630	.0839
RESIDUAL	6	.783	.1305		
CORRECTED TOTAL	9	2.204			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	27.22	.3789	71.84	.0000	11.0
2	.01	.0038	2.80	.0311	2.0
3	.51	.5358	.95	.3779	5.5
4	.00	.0054	-.80	.4517	6.5

THE RESULTS FOR SAMPLE L7 AT 4 SIXTEENTHS OF AN INCH DEPTH.

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
63.675	45.512	.316	24.61	1.284

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	1.050	.3500	3.506	.0893
RESIDUAL	6	.599	.0998		
CORRECTED TOTAL	9	1.649			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	23.82	.3314	71.88	.0000	11.0
2	.01	.0033	2.00	.0922	2.0
3	.47	.4687	1.00	.3546	5.5
4	.00	.0047	-.21	.8389	6.5

THE RESULTS FOR SAMPLE L7 AT 5 SIXTEENTHS OF AN INCH DEPTH.

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
26.202	.000	.4524	21.86	2.07

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F

REGRESSION	3	.436	.1453	.710	.5804
RESIDUAL	6	1.228	.2047		
CORRECTED TOTAL	9	1.664			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	21.66	.4745	45.65	.0000	11.0
2	.00	.0048	.00	1.0000	2.0
3	.22	.6710	.33	.7542	5.5
4	.00	.0067	.30	.7768	6.5

THE RESULTS FOR SAMPLE L7 AT 6 SIXTEENTHS OF AN INCH DEPTH.

It is apparent from the above results, that very few of the tests of Null $Hypothesis_1$ resulted in rejection. In fact, 37 of 372 test resulted in a p-value large enough to reject the null hypothesis. This percentage(10%), would have been quite smaller had the cutoff limit for accepting or rejecting the Null $Hypothesis_1$ had been set at .05 instead of .10. Therefore, the overall Null $Hypothesis_1$ can be accepted with a great deal of confidence and we can now assume that, generally, the hardness is symmetric about the 0-180 degree axis.

The second conclusion from the testing of the data in this way, can be drawn by looking at the p-values of the F statistics. Out of 372 tests, 275 tests resulted in p-values greater than .10. A large p-value supports the Null $Hypothesis_2$ and suggests that a linear model is not an adequate model for explaining the relationship between $\Theta$and the hardness. In 275 out of 372 tests, or 74% of the time, the results did not allow rejection of the Null $Hypothesis_2$. The overall conclusion must be that, in general, the linear model is not an adequate model in explaining the desired dependence. The effect of Θ is something other than linear.

The majority of plots from the previous chapter indicated a possible cyclic relationship between Θ and hardness. A cyclic relationship can very easily be thought to imply a relationship involving cosine or sine or both. Such a relationlship also is attractive simply because of the physical layout of the problem. We have a round disk in which a property(hardness) varies according to some relationship with both depth and Θ. The natural step is to include the transcendental functions of sine and cosine in the relationship with Θ. But, what form should be tried?

3.2 Introducing the Fourier Series

If the results from the previous test of hypothesis are used, a direction of investigation can be determined. Since we know that the relationship is not linear but is cyclical and is probably symmetric around the 0-180 degree diameter line, the simple cosine function appears to be the logical place to start the search. The cosine function is symmetric around the 0-180 degree line whereas the sine function is symmetric only around the 90-270 degree line.

It is known that any cyclical function can be approximated as closely as is desired with a Fourier[5] series. Also, it was estimated that a fourth order polynomial was the best curve for describing the relationship between depth and hardness. Possibly there is a way to combine these two concepts and come up with a general model that will explain both the depth and Θ relationships with hardness in a single equation. This was the original model that was attempted. The above intuitive approach is represented by the following regression model.

```
y =c0 + c1*radius          +c2*radius*cos(Θ)     +c3*radius*cos(2Θ)+
   c4*radius*cos(3Θ)       +c5*radius*cos(4Θ)    +c6*radius*cos(5Θ)+
   c7*radius*sin(Θ)        +c8*radius*sin(2Θ)    +c9*radius*sin(3Θ)+
   c10*radius*sin(4Θ)      +c11*radius*sin(5Θ)   +
```

$$\begin{aligned}&c12*radius^2 \quad +c13*radius^2*cos(\Theta) \; +c14*radius^2*cos(2\Theta)+\\&c15*radius^2*cos(3\Theta)+c16*radius^2*cos(4\Theta)+c17*radius^2*cos(5\Theta)+\\&c18*radius^2*sin(\Theta) \; +c19*radius^2*sin(2\Theta)+c20*radius^2*sin(3\Theta)+\\&c21*radius^2*sin(4\Theta)+c22*radius^2*sin(5\Theta)+\end{aligned}$$

$$\begin{aligned}&c23*radius^3 \quad +c24*radius^3*cos(\Theta) \; +c25*radius^3*cos(2\Theta)+\\&c26*radius^3*cos(3\Theta)+c27*radius^3*cos(4\Theta)+c28*radius^3*cos(5\Theta)+\\&c29*radius^3*sin(\Theta) \; +c30*radius^3*sin(2\Theta)+c31*radius^3*sin(3\Theta)+\\&c32*radius^3*sin(4\Theta)+c33*radius^3*sin(5\Theta)+\end{aligned}$$

$$\begin{aligned}&c34*radius^4 \quad +c35*radius^4*cos(\Theta) \; +c36*radius^4*cos(2\Theta)+\\&c37*radius^4*cos(3\Theta)+c38*radius^4*cos(4\Theta)+c39*radius^4*cos(5\Theta)+\\&c40*radius^4*sin(\Theta) \; +c41*radius^4*sin(2\Theta)+c42*radius^4*sin(3\Theta)+\\&c43*radius^4*sin(4\Theta)+c44*radius^4*sin(5\Theta)+e(i)\end{aligned}$$

where y = Hardness at distance from the center of the disk(radius) and Θ is the angular distance from the index point

and e = Random error distributed as Normal$(0,\sigma^2)$

Note that the Fourier series is applied to each term of depth(or radius), specifically, to radius, $radius^2$, $radius^3$ and, $radius^4$. This is because we do not know which, if any, of the terms of depth is the dominant term. Also note that the Fourier series is taken only to sin and cos of $5*\Theta$. This was done for two reasons. First, the model presented above already has 45 terms. As the length of the Fourier series increases by one(ie. $5*\Theta$ to $6*\Theta$), the number of degrees of freedom available to determine the Mean Square Error decreases by eight. Therefore, the level of the Fourier series should be held to a minimum. The second reason for limiting the series to $5*\Theta$, is that the readings of hardness were taken at thirty degree intervals around the sample disks. Since the effect of Θ seems to be symmetric around the 0-180 degree diameter, any term containing a cosine of $6*\Theta$ or greater would be merely redundant.

This model was run in Fortran using the IMSL subroutine RGIVN to compute the least squares estimates of all 45 coefficients. All terms were left in the model, initially, in order to observe the entire model and to check the overall goodness of fit using all of the terms. Clearly, if this model using all forty-five terms could not adequately explain the variation in the data, then it would not be worth trying to pick out the most significant of the coefficients to retain in the final model.

THESE ARE THE RESULTS FOR SAMPLE H17

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
99.284	98.185	1.323	48.84	2.709

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	43	6799.	158.1	90.331	.0000
RESIDUAL	28	49.	1.8		
CORRECTED TOTAL	71	6848.			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	264224.	46699.	5.658	.0000	8.971E+10
2	0.	0.	....	....	1.265+322
3	-519058.	92234.	-5.628	.0000	3.987E+09
4	-1921.	1135.	-1.692	.1017	1.097E+08
5	-60.	1135.	-.053	.9579	1.097E+08
6	-1205.	1135.	-1.061	.2977	1.097E+08
7	-667.	1135.	-.588	.5615	1.097E+08
8	599.	1135.	.528	.6018	1.097E+08
9	899.	1135.	.792	.4351	1.097E+08
10	1446.	1135.	1.273	.2133	1.097E+08
11	246.	1135.	.217	.8299	1.097E+08
12	223.	1135.	.196	.8458	1.097E+08
13	14.	46.	.295	.7699	1.814E+05
14	381612.	68245.	5.592	.0000	3.604E+10
15	2775.	1679.	1.652	.1097	1.004E+09
16	83.	1679.	.049	.9610	1.004E+09
17	1782.	1679.	1.061	.2977	1.004E+09
18	977.	1679.	.582	.5653	1.004E+09
19	-902.	1679.	-.537	.5954	1.004E+09
20	-1332.	1679.	-.793	.4345	1.004E+09
21	-2141.	1679.	-1.275	.2129	1.004E+09
22	-364.	1679.	-.217	.8299	1.004E+09
23	-325.	1679.	-.193	.8481	1.004E+09
24	-13.	45.	-.289	.7745	7.357E+05

25	-124442.	22420.	-5.551	.0000	3.628E+10
26	-1333.	827.	-1.612	.1181	1.029E+09
27	-37.	827.	-.045	.9648	1.029E+09
28	-877.	827.	-1.060	.2981	1.029E+09
29	-477.	827.	-.577	.5688	1.029E+09
30	452.	827.	.546	.5893	1.029E+09
31	656.	827.	.794	.4340	1.029E+09
32	1055.	827.	1.275	.2127	1.029E+09
33	179.	827.	.217	.8299	1.029E+09
34	157.	827.	.190	.8508	1.029E+09
35	3.	11.	.281	.7808	1.876E+05
36	15189.	2759.	5.505	.0000	4.067E+09
37	213.	136.	1.573	.1271	1.181E+08
38	5.	136.	.039	.9691	1.181E+08
39	144.	136.	1.059	.2987	1.181E+08
40	77.	136.	.572	.5721	1.181E+08
41	-75.	136.	-.555	.5834	1.181E+08
42	-108.	136.	-.795	.4335	1.181E+08
43	-173.	136.	-1.275	.2127	1.181E+08
44	-29.	136.	-.217	.8301	1.181E+08
45	-25.	136.	-.186	.8538	1.181E+08

* * * CASE ANALYSIS * * *

OBS.	OBSERVED	RESIDUAL
1	58.3000	-.0313
2	58.0000	.0971
3	57.8000	.2561
4	58.0000	-.0628
5	58.7000	-.3896
6	58.3000	.3490
7	58.3000	-.0144
8	58.3000	.1949
9	58.4000	-.0690
10	58.9000	-.2853
11	58.1000	-.1596
12	58.8000	-.0006
13	58.1000	.1902
14	57.6000	-.1849
15	57.7000	-.7298
16	57.5000	.4532
17	58.4000	1.0246
18	56.7000	-.8167
19	57.1000	-.0832
20	57.0000	-.8194
21	57.3000	.2655
22	58.0000	.4832
23	58.0000	.8214
24	57.4000	-.0267
25	57.3000	-.1049
26	56.0000	-.2876
27	56.6000	.0703
28	54.3000	-.8934
29	55.2000	-.3112
30	55.3000	-.3549
31	55.7000	.8429
32	56.0000	.6351
33	55.7000	.5526
34	55.7000	.0321

	35	54.8000	-.7770
	36	56.3000	-.5588
	37	52.8000	1.1794
	38	48.8000	-.7191
Y	39	52.3000	3.3585
	40	47.3000	-1.1637
	41	49.6000	.4174
	42	49.7000	.7622
	43	47.7000	-.1945
	44	47.3000	-.5978
	45	47.8000	-.9222
	46	47.3000	-1.7842
	47	49.7000	.6387
	48	51.7000	.1800
	49	40.7000	-.3569
	50	39.4000	.2652
Y	51	35.2000	-3.0018
	52	39.6000	1.2210
	53	39.9000	.2373
	54	38.2000	-1.2281
	55	38.9000	.5568
	56	37.3000	-.7002
Y	57	41.0000	1.7647
	58	39.5000	.6710
	59	39.7000	.8612
	60	40.7000	-.8676
	61	32.2000	.2902
	62	32.0000	-.3374
Y	63	33.5000	1.2134
	64	31.3000	-.7211
	65	34.0000	.1883
	66	34.8000	.1218
	67	33.8000	.0590
	68	33.8000	.1208
	69	33.3000	-.4251
	70	31.3000	-.2833
	71	31.5000	-.2181
	72	33.0000	.1070

THE SUM OF SQUARES ERROR = .4901E+02

THESE ARE THE RESULTS FOR SAMPLE L7

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
99.943	99.854	.4731	36.25	1.305

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	43	10902.	253.5	1132.878	.0000
RESIDUAL	28	6.	.2		
CORRECTED TOTAL	71	10908.			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	-565175.	26598.	-21.25	.0000	2.276E+11
2	0.	0.	....	....	1.265+322
3	1007356.	46746.	21.55	.0000	8.010E+09
4	-503.	512.	-.98	.3340	2.198E+08
5	532.	512.	1.04	.3072	2.198E+08
6	-215.	512.	-.42	.6782	2.198E+08
7	411.	512.	.80	.4283	2.198E+08
8	-223.	512.	-.44	.6663	2.198E+08
9	673.	512.	1.31	.1992	2.198E+08
10	-54.	512.	-.11	.9164	2.198E+08
11	-317.	512.	-.62	.5402	2.198E+08
12	-603.	512.	-1.18	.2491	2.198E+08
13	29.	19.	1.58	.1244	2.879E+05
14	-672574.	30785.	-21.85	.0000	7.234E+10
15	689.	674.	1.02	.3157	2.005E+09
16	-710.	674.	-1.05	.3010	2.005E+09
17	283.	674.	.42	.6779	2.005E+09
18	-541.	674.	-.80	.4289	2.005E+09
19	287.	674.	.43	.6739	2.005E+09
20	-899.	674.	-1.33	.1928	2.005E+09
21	65.	674.	.10	.9240	2.005E+09
22	431.	674.	.64	.5276	2.005E+09
23	799.	674.	1.19	.2459	2.005E+09
24	-26.	16.	-1.57	.1269	1.164E+06
25	199350.	9003.	22.14	.0000	7.272E+10
26	-313.	296.	-1.06	.2982	2.045E+09
27	316.	296.	1.07	.2947	2.045E+09
28	-124.	296.	-.42	.6782	2.045E+09
29	237.	296.	.80	.4295	2.045E+09
30	-123.	296.	-.41	.6815	2.045E+09
31	400.	296.	1.35	.1865	2.045E+09
32	-25.	296.	-.09	.9319	2.045E+09
33	-195.	296.	-.66	.5156	2.045E+09
34	-353.	296.	-1.19	.2426	2.045E+09
35	6.	4.	1.56	.1299	2.957E+05
36	-22130.	987.	-22.43	.0000	8.136E+09
37	47.	43.	1.10	.2817	2.332E+08
38	-47.	43.	-1.08	.2885	2.332E+08
39	18.	43.	.42	.6791	2.332E+08
40	-35.	43.	-.80	.4301	2.332E+08
41	17.	43.	.40	.6891	2.332E+08
42	-59.	43.	-1.37	.1802	2.332E+08
43	3.	43.	.08	.9403	2.332E+08
44	29.	43.	.68	.5042	2.332E+08
45	52.	43.	1.20	.2395	2.332E+08

* * * CASE ANALYSIS * * *

OBS.	OBSERVED	RESIDUAL
1	52.1000	-.1568
2	52.8000	.1512
3	52.9000	-.1528
4	53.6000	.1721
5	53.1000	-.1796
6	53.7000	.3072
7	53.7000	.0602

	8	53.0000	.1512
	9	53.3000	-.1048
	10	54.0000	.1372
	11	53.4000	-.2094
	12	53.2000	.1382
	13	49.9000	.0372
	14	49.6000	-.0277
	15	51.0000	-.0489
	16	51.2000	-.1030
	17	51.0000	-.0393
Y	18	51.0000	-.6018
Y	19	51.4000	-.7515
	20	50.9000	-.0953
	21	50.7000	-.0923
	22	50.5000	-.0965
	23	50.0000	.3100
	24	50.4000	-.0605
	25	38.0000	.1971
	26	37.4000	-.0931
	27	39.3000	.3625
	28	39.1000	.0808
	29	39.6000	.6056
	30	40.1000	.6215
Y	31	41.0000	1.1499
	32	39.1000	.0163
	33	39.0000	.3274
	34	38.4000	.0841
	35	37.0000	-.2229
	36	38.1000	.0099
	37	28.0000	.0445
	38	27.5000	-.3653
	39	28.8000	.0368
	40	28.1000	-.6506
	41	28.5000	-.4832
	42	28.6000	-.6222
	43	29.0000	-.2490
	44	28.7000	-.2736
	45	28.7000	-.1001
	46	28.2000	-.3237
	47	27.7000	.1816
	48	27.6000	-.3340
	49	24.2000	.1047
	50	24.5000	.2149
	51	24.4000	-.0521
	52	25.0000	.4611
	53	25.1000	.3542
	54	25.0000	.1015
	55	24.6000	-.0971
	56	24.4000	-.0265
	57	25.1000	.2619
	58	24.6000	-.0554
	59	24.1000	.2631
	60	23.9000	.0390
	61	22.0000	.1066
	62	22.0000	-.2134
	63	22.0000	.1879
	64	22.0000	-.2938
	65	22.0000	.0756

66	22.3000	-.1395
67	22.6000	.2207
68	21.0000	-.1054
69	22.4000	.0412
70	22.0000	-.0789
71	21.4000	.0109
72	21.5000	-.1259

THE SUM OF SQUARES ERROR = .6266E+01

This model ended up explaining 98.185 percent of the variation in the data for sample H17 and 99.854 percent of the variation of the data for sample L7. In itself, this adjusted R^2 does not tell the whole story. The majority of the estimates of the coefficients had a confidence interval that included zero. This is the same as saying that the coefficient's p-value was greater than .05. In fact, the majority of coefficient estimates had very large p-values, greater than .10. If a model uses many terms which have coefficients that may actually be zero, then the model is a misleading one. The reduction in variation of the data is also misleading. The sum of squares may be somewhat smaller but the ability of the model to explain or predict the hardness using calculated relationships between the variables is seriously in question.

The residual plots are usually an excellent way to check the model for adequacy and to check the assumptions implied in using least squares estimates. The plots of the residuals vs. depth for the above model and for for several samples are found in Appendix C. It appears that, like the ANOVA plots in Chapter 2, the variance of the residuals, for the high DI steels samples, varies with depth. For the low DI steel samples, both the mean and variance of the residuals vary quite noticeably. The normality plot also is badly skewed, indicating some other forces at work than those already in the model. The residual vs. Θ plot, however, shows very little variation in mean or variance, indicating that the model adequately explains the relationship between hardness and Θ.

3.3 The Exponential Relationship

Since the relationship between the hardness and depth is now in question, perhaps another look at the plots of hardness vs. depth is warranted. Also, since a regression equation is desired to explain the hardness function over the entire surface of the disk, distance from the center of the disk will be used instead of depth for any further regressions. This change will force the effect of Θ to go to zero as the distance from the center of the disk goes to zero. This distance will be referred to as the radius for the rest of the thesis.

Several typical plots of hardness vs. radius are shown in Appendix A. One possible relationship that appears likely is an exponential one. A model such as Hardness $= e^{C1*radius}$ would have both advantages and disadvantages. The shape of such a dependence would naturally assume a shape close to what is found in the plots. However, with the above exponential model, there is only one coefficient to estimate to improve the fit of the model, whereas there are five coefficients to be estimated in the fourth order polynomial model. The exponential model still needs terms that include Θ, so the first try at this exponential model is listed below.

```
y = exp(c0 +    c1*radius   + c2*radius*cos(Θ)  + c3*radius*cos(2Θ)+
        c4*radius*cos(3Θ)  + c5*radius*cos(4Θ)+ c6*radius*cos(5Θ)+
        c7*radius*sin(Θ)   + c8*radius*sin(2Θ)+ c9*radius*sin(3Θ)+
        c10*radius*sin(4Θ)+ c11*radius*sin(5Θ))
```

where y = hardness, Θ=angle, radius = distance from the center of the disk, and c1-c11 are the coefficients that must be estimated.

This model was estimated using the IMSL subroutine RGIVN and the results from this regression are listed below. Note that even though the number of estimated parameters was reduced considerably from the 45 parameters used in the straight Fourier series model, the fit(adjusted R^2) is almost as good, at least for the low DI steel samples. The F statistic is significantly higher for the low DI steel samples also. Note that the high DI steel samples, H17 is typical, do not fit this model as well as with the non-exponential, Fourier model.

THESE ARE THE RESULTS FOR SAMPLE H17

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
86.872	84.720	.08561	3.866	2.214

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	10	2.959	.2959	40.367	.0000
RESIDUAL	61	.447	.0073		
CORRECTED TOTAL	71	3.406			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	.013	.1923	.07	.9457	3.631E+02
2	.000	.0000	....	....	1.265+322
3	1.897	.0945	20.07	.0000	1.000E+00
4	.003	.0070	.36	.7167	1.000E+00
5	.004	.0070	.50	.6167	1.000E+00
6	.004	.0070	.55	.5839	1.000E+00
7	-.001	.0070	-.19	.8496	1.000E+00
8	.000	.0070	-.07	.9463	1.000E+00
9	.000	.0070	-.05	.9638	1.000E+00
10	-.003	.0070	-.41	.6818	1.000E+00
11	.000	.0070	-.01	.9945	1.000E+00
12	-.002	.0070	-.30	.7616	1.000E+00

THESE ARE THE RESULTS FOR SAMPLE L7

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
96.412	95.823	.07082	3.532	2.005

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	10	8.221	.8221	163.898	.0000
RESIDUAL	61	.306	.0050		
CORRECTED TOTAL	71	8.527			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	-3.685	.1786	-20.63	.0000	4.578E+02
2	.000	.0000	....	....	1.265+322
3	3.163	.0782	40.45	.0000	1.000E+00
4	-.007	.0052	-1.28	.2059	1.000E+00
5	-.001	.0052	-.15	.8845	1.000E+00
6	.000	.0052	-.01	.9912	1.000E+00
7	.002	.0052	.34	.7383	1.000E+00
8	-.001	.0052	-.23	.8183	1.000E+00
9	.003	.0052	.51	.6116	1.000E+00
10	.000	.0052	.07	.9443	1.000E+00
11	.001	.0052	.19	.8535	1.000E+00
12	-.003	.0052	-.56	.5802	1.000E+00

With another look at the plots of hardness vs. radius, there is sometimes a portion of the curve, near the surface of the disk(near the maximum value for radius) where the slope actually becomes negative. Unfortunately, a function of the form $y = e^x$ will never be able to approximate a negative slope function. If, however, a quadratic term of x is included in the exponential function, then a curve that looks something like the normal probability function would result. If the domain is limited properly, then the resulting function should approximate the actual relationship that is evident in the hardness vs. radius plots. Keeping the Fourier series concept, the resulting proposed model is as listed below.

```
y =exp(c0 + c1*radius         +c2*radius*cos(Θ)       +c3*radius*cos(2Θ)+
    c4*radius*cos(3Θ)      +c5*radius*cos(4Θ)      +c6*radius*cos(5Θ)+
    c7*radius*sin(Θ)       +c8*radius*sin(2Θ)      +c9*radius*sin(3Θ)+
    c10*radius*sin(4Θ)     +c11*radius*sin(5Θ)     +

    c12*radius^2          +c13*radius^2*cos(Θ)  +c14*radius^2*cos(2Θ)+
    c15*radius^2*cos(3Θ)+c16*radius^2*cos(4Θ)+c17*radius^2*cos(5Θ)+
    c18*radius^2*sin(Θ)  +c19*radius^2*sin(2Θ)+c20*radius^2*sin(3Θ)+
    c21*radius^2*sin(4Θ)+c22*radius^2*sin(5Θ))
```

Below, are the regression results of this quadratic model for the samples H17 and L7.

THESE ARE THE RESULTS FOR SAMPLE H17

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
97.821	96.906	.03853	3.866	.9965

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	21	3.332	.1587	106.885	.0000
RESIDUAL	50	.074	.0015		
CORRECTED TOTAL	71	3.406			

THESE ARE THE RESULTS FOR SAMPLE L7

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
96.490	95.016	.07737	3.532	2.191

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	21	8.228	.3918	65.456	.0000
RESIDUAL	50	.299	.0060		
CORRECTED TOTAL	71	8.527			

The fit for both of these samples seems to be very good with an adjusted R^2 of 95.016 and 96.906. However, when the size of the sum of squared residuals is evaluated, it is apparent that a better model is needed.

THE SUM OF SQUARES ERROR FOR L7 = .5323E+03
THE SUM OF SQUARES ERROR FOR H17 = .1370E+03

This type of value for the SSE means that the average residual could be off from the estimate by between 1 and 3 Rockwell C hardness points. Also, it would not be uncommon to find estimates of hardness that would be off by 3 or more. This size of an error would not lend itself to being called a 'good model'.

3.4 Simplifying the Model

In examining the list of 23 estimated coefficients, it appeared that there was a pattern in the t statistics and p-values as shown below for sample H16 and H17. This pattern was exhibited in most of the samples analyzed with this model. Note that only a few coefficients did not include zero in their confidence intervals. If one looks at which terms in the regression model these coefficients amplified, a pattern is evident.

COEFFICIENTS FOR SAMPLE H16

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	-24.09	2.026	-11.89	.0000	1.787E+05
2	.00	.000	....	....	1.265+322
3	25.83	1.999	12.92	.0000	1.981E+03
4	-.01	.064	-.12	.9033	3.705E+02
5	.01	.064	.10	.9178	3.705E+02
6	-.03	.064	-.42	.6798	3.705E+02
7	.06	.064	.99	.3283	3.705E+02
8	.07	.064	1.02	.3138	3.705E+02
9	-.01	.064	-.14	.8870	3.705E+02
10	-.05	.064	-.84	.4031	3.705E+02
11	.09	.064	1.34	.1878	3.705E+02
12	.06	.064	1.00	.3241	3.705E+02
13	.01	.003	1.52	.1353	1.000E+00
14	-5.92	.492	-12.04	.0000	1.981E+03
15	.00	.031	.05	.9598	3.705E+02
16	-.01	.031	-.17	.8642	3.705E+02
17	.01	.031	.39	.6962	3.705E+02
18	-.03	.031	-.94	.3519	3.705E+02
19	-.03	.031	-.96	.3435	3.705E+02
20	.00	.031	.15	.8788	3.705E+02
21	.03	.031	.82	.4178	3.705E+02
22	-.04	.031	-1.31	.1966	3.705E+02
23	-.03	.031	-.91	.3663	3.705E+02

COEFFICIENTS FOR SAMPLE L7

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	-30.15	1.919	-15.71	.0000	1.787E+05
2	.00	.000	....	....	1.265+322
3	31.67	1.894	16.73	.0000	1.981E+03
4	-.03	.061	-.56	.5790	3.705E+02
5	.07	.061	1.11	.2708	3.705E+02

6	.05	.061	.83	.4083	3.705E+02
7	-.04	.061	-.72	.4725	3.705E+02
8	-.02	.061	-.27	.7845	3.705E+02
9	.01	.061	.18	.8596	3.705E+02
10	-.04	.061	-.67	.5056	3.705E+02
11	-.01	.061	-.11	.9118	3.705E+02
12	-.04	.061	-.69	.4947	3.705E+02
13	.00	.003	-.55	.5846	1.000E+00
14	-7.33	.466	-15.73	.0000	1.981E+03
15	.02	.030	.60	.5503	3.705E+02
16	-.03	.030	-1.06	.2957	3.705E+02
17	-.02	.030	-.77	.4441	3.705E+02
18	.02	.030	.70	.4854	3.705E+02
19	.01	.030	.27	.7902	3.705E+02
20	-.01	.030	-.18	.8553	3.705E+02
21	.02	.030	.62	.5356	3.705E+02
22	.00	.030	.11	.9124	3.705E+02
23	.02	.030	.65	.5163	3.705E+02

The terms whose coefficients are significantly different from zero are: constant,radius*cos(Θ), radius^2*cos(Θ). It is obvious that these particular terms are much more significant than the other terms in the model. This pattern of significant terms held true for every single high DI steel sample that was analyzed, but only for a few of the low DI steel samples that were analyzed. If these terms were significant, were there other terms that would significantly improve the model that also fit into the pattern that seemed to be developing? To test these suspicions, a regression was run using the same model except the radius went up to fourth order. If there really was a simple, even though as yet unexplained, pattern, this model should give us the following significant terms:

constant
radius*cos(Θ)
radius^2*cos(Θ)
radius^3*cos(Θ)
radius^4*cos(Θ)

The results for this fourth order-exponential-Fourier series model are below.

THESE ARE THE RESULTS FOR SAMPLE H17

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
99.218	98.018	.03084	3.866	.7976

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	43	3.379	.07859	82.641	.0000
RESIDUAL	28	.027	.00095		
CORRECTED TOTAL	71	3.406			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	5493.	1088.	5.046	.0000	8.971E+10
2	0.	0.	....	....	1.265+322
3	-10829.	2150.	-5.037	.0000	3.987E+09
4	-51.	26.	-1.930	.0638	1.097E+08
5	4.	26.	.169	.8667	1.097E+08
6	-32.	26.	-1.191	.2435	1.097E+08
7	-23.	26.	-.887	.3828	1.097E+08
8	18.	26.	.670	.5085	1.097E+08
9	26.	26.	.995	.3284	1.097E+08
10	34.	26.	1.283	.2101	1.097E+08
11	4.	26.	.169	.8672	1.097E+08
12	-3.	26.	-.095	.9252	1.097E+08
13	0.	1.	.014	.9892	1.814E+05
14	7993.	1591.	5.025	.0000	3.604E+10
15	74.	39.	1.890	.0692	1.004E+09
16	-7.	39.	-.168	.8681	1.004E+09
17	47.	39.	1.192	.2432	1.004E+09
18	34.	39.	.877	.3879	1.004E+09
19	-26.	39.	-.677	.5040	1.004E+09
20	-39.	39.	-.993	.3294	1.004E+09
21	-50.	39.	-1.284	.2098	1.004E+09
22	-7.	39.	-.169	.8670	1.004E+09
23	4.	39.	.094	.9259	1.004E+09
24	0.	1.	-.010	.9921	7.357E+05
25	-2616.	523.	-5.006	.0000	3.628E+10
26	-36.	19.	-1.850	.0749	1.029E+09
27	3.	19.	.167	.8688	1.029E+09
28	-23.	19.	-1.192	.2433	1.029E+09
29	-17.	19.	-.868	.3929	1.029E+09
30	13.	19.	.684	.4998	1.029E+09
31	19.	19.	.990	.3304	1.029E+09
32	25.	19.	1.283	.2099	1.029E+09
33	3.	19.	.169	.8669	1.029E+09
34	-2.	19.	-.093	.9262	1.029E+09
35	0.	0.	.005	.9962	1.876E+05
36	321.	64.	4.983	.0000	4.067E+09
37	6.	3.	1.811	.0809	1.181E+08
38	-1.	3.	-.167	.8689	1.181E+08

39	4.	3.	1.191	.2437	1.181E+08
40	3.	3.	.859	.3976	1.181E+08
41	-2.	3.	-.690	.4958	1.181E+08
42	-3.	3.	-.988	.3315	1.181E+08
43	-4.	3.	-1.283	.2102	1.181E+08
44	-1.	3.	-.169	.8669	1.181E+08
45	0.	3.	.094	.9261	1.181E+08

THE SUM OF SQUARES ERROR = .5563E+02

THESE ARE THE RESULTS FOR SAMPLE L7

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
99.848	99.614	.02154	3.532	.6098

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	43	8.514	.1980	426.946	.0000
RESIDUAL	28	.013	.0005		
CORRECTED TOTAL	71	8.527			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	-10685.	1211.	-8.825	.0000	2.276E+11
2	0.	0.	....	....	1.265+322
3	19157.	2128.	9.002	.0000	8.010E+09
4	-2.	23.	-.069	.9455	2.198E+08
5	17.	23.	.735	.4684	2.198E+08
6	-8.	23.	-.347	.7313	2.198E+08
7	14.	23.	.604	.5507	2.198E+08
8	-11.	23.	-.456	.6519	2.198E+08
9	13.	23.	.569	.5737	2.198E+08
10	-4.	23.	-.188	.8519	2.198E+08
11	-2.	23.	-.070	.9447	2.198E+08
12	-18.	23.	-.760	.4535	2.198E+08
13	1.	1.	1.092	.2842	2.879E+05
14	-12864.	1401.	-9.179	.0000	7.234E+10
15	3.	31.	.092	.9271	2.005E+09
16	-23.	31.	-.741	.4648	2.005E+09
17	11.	31.	.349	.7296	2.005E+09
18	-18.	31.	-.600	.5531	2.005E+09
19	14.	31.	.447	.6585	2.005E+09
20	-18.	31.	-.577	.5686	2.005E+09
21	6.	31.	.183	.8560	2.005E+09
22	3.	31.	.085	.9327	2.005E+09
23	23.	31.	.762	.4525	2.005E+09
24	-1.	1.	-1.081	.2888	1.164E+06
25	3835.	410.	9.356	.0000	7.272E+10
26	-2.	13.	-.116	.9084	2.045E+09

27	10.	13.	.747	.4611	2.045E+09
28	-5.	13.	-.351	.7284	2.045E+09
29	8.	13.	.597	.5553	2.045E+09
30	-6.	13.	-.438	.6650	2.045E+09
31	8.	13.	.585	.5633	2.045E+09
32	-2.	13.	-.178	.8603	2.045E+09
33	-1.	13.	-.100	.9212	2.045E+09
34	-10.	13.	-.764	.4513	2.045E+09
35	0.	0.	1.070	.2938	2.957E+05
36	-428.	45.	-9.532	.0000	8.136E+09
37	0.	2.	.140	.8898	2.332E+08
38	-1.	2.	-.753	.4576	2.332E+08
39	1.	2.	.352	.7275	2.332E+08
40	-1.	2.	-.594	.5574	2.332E+08
41	1.	2.	.429	.6715	2.332E+08
42	-1.	2.	-.593	.5577	2.332E+08
43	0.	2.	.172	.8648	2.332E+08
44	0.	2.	.114	.9102	2.332E+08
45	2.	2.	.766	.4501	2.332E+08

THE SUM OF SQUARES ERROR = .1694E+02

As predicted, the terms that included COS(Θ). proved to be significantly more important than the other terms. Now, when one looks at the t-statistic or p-value terms, the pattern is obvious. In every single regression, the most significant terms were the same. Only one other pattern emerged from this investigation. All of the terms that involved cos(2Θ) also had markedly higher t statistics for the majority of the samples analyzed. One would suppose that the correct step would be to use the recognition of the pattern to develop an adequate model for the regression. Unfortunately, the terms whose coefficients are significant in a very large model, are not always that important when the "insignificant terms" are removed. This is due to the fact that many of the terms in a large regression can be correlated to many of the other terms in the model. The result can be that several very important terms in the 'true' model of the relationship can have very low t statistics in the large model regression. If the many correlated terms were removed, the few really important terms would emerge as very significant.

The next step in finding the 'true' model to describe the dependency was to run another regression. This time, the entire exponential-fourth order Fourier series model was trimmed down to include only the terms listed above. The results for several of the samples are shown below. What was described in the previous paragraph is exactly what occurred. The adjusted R^2 for all of the regressions below are very low. The level of success of this model was not even close to that expected by the results leading up to this last trial. Apparently, some term or terms that had not shown up in the larger regression model was much more important than indicated by the p-values.

THESE ARE THE RESULTS FOR SAMPLE H17

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
.283	.000	.2006	3.968	5.055

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	3	.008	.00259	.064	.9785
RESIDUAL	68	2.736	.04023		
CORRECTED TOTAL	71	2.743			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	3.968	.024	167.9	.0000	1.000E+00
2	.000	.000	....	....	1.265+322
3	-2.844	7.000	-.4	.6858	1.814E+05
4	2.790	6.892	.4	.6869	7.357E+05
5	-.682	1.693	-.4	.6885	1.876E+05

THE SUM OF SQUARES ERROR = .6896E+04

THESE ARE THE RESULTS FOR SAMPLE L7

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
.117	.000	.3085	3.675	8.396

* * * ANALYSIS OF VARIANCE * * *

SUM OF MEAN PROB. OF

SOURCE	DF	SQUARES	SQUARE	OVERALL F	LARGER F
REGRESSION	3	.008	.00254	.027	.9941
RESIDUAL	68	6.472	.09518		
CORRECTED TOTAL	71	6.480			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	3.675	.04	101.1	.0000	1.000E+00
2	.000	.00	....	....	1.265+322
3	1.450	12.08	.1	.9048	2.879E+05
4	-1.278	10.59	-.1	.9043	1.164E+06
5	.280	2.32	.1	.9042	2.957E+05

THE SUM OF SQUARES ERROR = .1113E+05

Because it was known that there was an extremely strong relationship between depth and hardness, it was unexpected to find no significant terms of the radius or radius ^{2}, etc. Guessing that some combination of sine and cosine had rendered the linear and quadratic and cubic terms of radius insignificant, I added these terms into the reduced model, in hopes of improving the performance of the model and its intuitive physical justification. The following model was introduced and tested.

$$
\begin{aligned}
y = \exp(c0 + {} & c1*radius && + c2*radius*\cos(\Theta) && + \\
& c3*radius^2 && + c4*radius^2*\cos(\Theta) && + \\
& c5*radius^3 && + c6*radius^3*\cos(\Theta) && + \\
& c7*radius^4 && + c8*radius^4*\cos(\Theta))
\end{aligned}
$$

This model turned out to be a much better predicter of hardness values than the same model without the linear, quadratic, cubic, and quartic terms. The adjusted R^2 for the samples averaged about 98% and compared very favorably with the full fourth order Fourier model that had 45 parameters. With only eight parameters, the standard error of the regression had fallen to very acceptable levels. The sample H17 had a sum of squares error for all 72 measurements of 86.51. The full model had an SSE for sample H17 of 55.63. The MSE(mean square error) which is the estimate of $sigma^2$ for the reduced model for H17 was .0007. For the full model, the MSE was .0095, which is over 35% higher. The F statistic for the full model of 45 parameters was 82.64. However, because of the dramatic drop in terms, the F statistic for the reduced model was 654.28. The actual estimates of the 9 coefficients are listed below.

THESE ARE THE RESULTS FOR SAMPLE H17

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
98.649	98.501	.02406	3.968	.6065

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	7	2.706	.3866	667.608	.0000
RESIDUAL	64	.037	.0006		
CORRECTED TOTAL	71	2.743			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	4996.	849.	5.881	.0000	8.971E+10
2	0.	0.	....	....	1.265+322
3	-9843.	1678.	-5.867	.0000	3.987E+09
4	-3.	1.	-3.387	.0012	1.814E+05
5	7262.	1241.	5.850	.0000	3.604E+10
6	3.	1.	3.374	.0013	7.357E+05
7	-2376.	408.	-5.827	.0000	3.628E+10
8	-1.	0.	-3.356	.0013	1.876E+05
9	291.	50.	5.797	.0000	4.067E+09

THE SUM OF SQUARES ERROR = .8578E+02

THESE ARE THE RESULTS FOR SAMPLE L7

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
99.758	99.732	.01564	3.675	.4257

* * * ANALYSIS OF VARIANCE * * *

SUM OF	MEAN	PROB. OF

SOURCE	DF	SQUARES	SQUARE	OVERALL F	LARGER F
REGRESSION	7	6.464	.9235	3773.593	.0000
RESIDUAL	64	.016	.0002		
CORRECTED TOTAL	71	6.480			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	-9866.	880.	-11.22	.0000	2.276E+11
2	0.	0.	....	....	1.265+322
3	17676.	1546.	11.44	.0000	8.010E+09
4	1.	1.	2.37	.0210	2.879E+05
5	-11860.	1018.	-11.65	.0000	7.234E+10
6	-1.	1.	-2.38	.0203	1.164E+06
7	3532.	298.	11.87	.0000	7.272E+10
8	0.	0.	2.38	.0202	2.957E+05
9	-394.	33.	-12.08	.0000	8.136E+09

THE SUM OF SQUARES ERROR = .2441E+02

Similar results were achieved throughout the sample set. In many cases, even the sum of squared error approached the SSE of the full model. One unexpected outcome noted in the above results is that the linear term 'radius' had a coefficient estimate of zero. This was true in every regression that was run but in this case, with only eight other terms in the model, it was even more surprising. This means that the linear value of the radius has absolutely no importance in predicting the hardness. In fact, if the coefficient of this is anything other than zero, the model will be less successful in explaining the data. In every sample, the terms that contain $\cos(\Theta)$ were very significant, ie. had a p-value less than .05. The other terms containing just the radius to certain powers were, in some cases, quite significant and in other cases not as significant. The p-values for these power terms of radius ranged from .30 down to .0001.

The following question must be investigated. If results that are this successful can be obtained from a fourth order model, what kind of results can be expected from a fifth order, sixth order, or even an eighth order analogous model of the reduced one? To test this, regressions were run, again using the IMSL subroutine RGIVN, for the fifth order, sixth order, seventh order, and eighth order models. The results for the fifth and eighth order models are listed below, but they are completely indicative of the results obtained from the sixth, and seventh order models.

THE RESULTS FOR SAMPLE H17 USING THE FIFTH ORDER MODEL

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
98.758	98.577	.02344	3.968	.5909

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	9	2.709	.3010	547.681	.0000
RESIDUAL	62	.034	.0005		
CORRECTED TOTAL	71	2.743			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	37596.	32226.	1.167	.2478	1.360E+14
2	0.	0.	....	....	1.265+322
3	-90337.	79560.	-1.135	.2606	9.447E+12
4	-45.	20.	-2.239	.0288	1.097E+08
5	86700.	78509.	1.104	.2737	1.519E+14
6	65.	30.	2.192	.0321	1.004E+09
7	-41544.	38707.	-1.073	.2873	3.444E+14
8	-31.	15.	-2.145	.0359	1.029E+09
9	9940.	9535.	1.042	.3012	1.546E+14
10	5.	2.	2.099	.0399	1.181E+08
11	-950.	939.	-1.012	.3155	9.791E+12

THE SUM OF SQUARES ERROR = .7603E+02

THE RESULTS FOR SAMPLE L7 USING THE FIFTH ORDER MODEL

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
99.856	99.836	.01225	3.675	.3333

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F

REGRESSION	9	6.471	.7190	4792.154	.0000
RESIDUAL	62	.009	.0002		
CORRECTED TOTAL	71	6.480			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	-205731.	30127.	-6.829	.0000	4.356E+14
2	0.	0.	....	....	1.265+322
3	448011.	66187.	6.769	.0000	2.395E+13
4	-3.	13.	-.215	.8308	2.198E+08
5	-389826.	58128.	-6.706	.0000	3.847E+14
6	4.	17.	.251	.8028	2.005E+09
7	169417.	25510.	6.641	.0000	8.709E+14
8	-2.	8.	-.288	.7746	2.045E+09
9	-36774.	5594.	-6.573	.0000	3.902E+14
10	0.	1.	.324	.7469	2.332E+08
11	3190.	490.	6.503	.0000	2.464E+13

THE SUM OF SQUARES ERROR = .1406E+02

THE RESULTS FOR SAMPLE H17 USING THE EIGHTH ORDER MODEL

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
98.943	98.683	.02265	3.98	.5691

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	14	2.737	.1955	381.101	.0000
RESIDUAL	57	.029	.0005		
CORRECTED TOTAL	71	2.766			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	-3.448E+11	1.288E+11	-2.678	.0097	2.328E+27
2	0.000E+00	0.000E+00	....	....	1.265+322
3	8.644E+11	3.158E+11	2.737	.0082	1.595E+26
4	-2.788E+09	1.618E+10	-.172	.8638	7.599E+25
5	-8.079E+11	2.927E+11	-2.761	.0077	2.262E+27
6	8.259E+09	4.792E+10	.172	.8638	2.789E+27
7	2.931E+11	1.390E+11	2.109	.0393	4.757E+27
8	-1.019E+10	5.910E+10	-.172	.8638	1.794E+28
9	2.374E+10	7.544E+10	.315	.7541	1.037E+28
10	6.698E+09	3.886E+10	.172	.8638	3.312E+28
11	-4.808E+10	3.334E+10	-1.442	.1547	1.323E+28
12	-2.476E+09	1.436E+10	-.172	.8638	1.952E+28
13	1.107E+10	6.082E+09	1.820	.0739	2.661E+27
14	4.878E+08	2.830E+09	.172	.8638	3.298E+27
15	0.000E+00	0.000E+00	....	....	1.265+322
16	-4.003E+07	2.322E+08	-.172	.8638	9.745E+25
17	-1.848E+08	8.863E+07	-2.085	.0415	1.791E+25

THE SUM OF SQUARES ERROR = .9925E+02

THE RESULTS FOR SAMPLE L7 FOR THE EIGHTH ORDER MODEL

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
99.870	99.838	.01213	3.676	.33

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	14	6.456	.4611	3133.732	.0000
RESIDUAL	57	.008	.0001		
CORRECTED TOTAL	71	6.464			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	2.938E+10	6.575E+10	.447	.6567	2.115E+27
2	0.000E+00	0.000E+00	....	....	1.265+322
3	-8.045E+10	1.518E+11	-.530	.5982	1.285E+26
4	-7.506E+09	7.787E+09	-.964	.3391	7.736E+25
5	9.297E+10	1.363E+11	.682	.4979	2.156E+27
6	1.979E+10	2.052E+10	.964	.3391	2.828E+27
7	-5.850E+10	5.992E+10	-.976	.3331	4.899E+27
8	-2.172E+10	2.253E+10	-.964	.3391	1.808E+28
9	2.149E+10	2.131E+10	1.009	.3175	5.772E+27
10	1.271E+10	1.319E+10	.964	.3391	3.314E+28
11	-4.556E+09	9.128E+09	-.499	.6196	8.701E+27
12	-4.183E+09	4.339E+09	-.964	.3391	1.935E+28
13	5.054E+08	2.085E+09	.242	.8093	3.448E+27
14	7.338E+08	7.612E+08	.964	.3391	3.234E+27
15	-2.159E+07	1.812E+08	-.119	.9055	1.875E+26
16	-5.361E+07	5.561E+07	-.964	.3391	9.446E+25
17	0.000E+00	0.000E+00	....	....	1.265+322

THE SUM OF SQUARES ERROR = .1566E+02

It seems that once an order above the fourth order is introduced, the model falls apart. In other words, as seen in the results above, all of the coefficients' confidence intervals contain zero. The p-value for all of the coefficients sky-rocketed to .30 and higher. This did not happen gradually as the fifth order was added, then the sixth, seventh and finally the eighth. The values rose to .30 and more with the first step to fifth order. Even the constant lost its significance with a p-value of .4051 in the eighth order model. Obviously, anything over fourth order is not warranted.

Before looking at the plots of residuals, two questions should be asked. The first question must be, are there any other terms that should be included in model? The second question is, once a final model is decided upon, does it make sense? Do the terms that seem to explain the data the best, also show a relationship between the variables concerned that are possible, physically?

The only other terms that consistently had small p-values throughout the various regressions, were the terms that contained cos(2Θ). Even though the p-values were not as small as those associated with the radius and radius*cos(Θ), the radius*cos(2Θ) terms were consistently some of the lowest p-values. A regression was run on a model that incorporated all the radius, radius*cos(Θ), and radius*cos(2Θ) terms. The results for H17 and L7 are listed below.

THESE ARE THE RESULTS FOR SAMPLE H17

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
98.837	98.624	.02306	3.968	.5811

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	11	2.711	.2465	463.646	.0000
RESIDUAL	60	.032	.0005		
CORRECTED TOTAL	71	2.743			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	4996.	814.	6.138	.0000	8.971E+10

2	0.	0.	....	....	1.265+322
3	-9843.	1607.	-6.124	.0000	3.987E+09
4	-45.	20.	-2.276	.0264	1.097E+08
5	0.	1.	.032	.9744	1.814E+05
6	7262.	1189.	6.106	.0000	3.604E+10
7	65.	29.	2.229	.0296	1.004E+09
8	0.	1.	.007	.9946	7.357E+05
9	-2376.	391.	-6.081	.0000	3.628E+10
10	-31.	14.	-2.181	.0331	1.029E+09
11	0.	0.	-.042	.9670	1.876E+05
12	291.	48.	6.051	.0000	4.067E+09
13	5.	2.	2.134	.0369	1.181E+08

THE SUM OF SQUARES ERROR = .7205E+02

THESE ARE THE RESULTS FOR SAMPLE L7

R-SQUARED (PERCENT)	ADJUSTED R-SQUARED	EST. STD. DEV. OF MODEL ERROR	MEAN	COEFFICIENT OF VAR. (PERCENT)
99.761	99.718	.01605	3.675	.4369

* * * ANALYSIS OF VARIANCE * * *

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	OVERALL F	PROB. OF LARGER F
REGRESSION	11	6.464	.5877	2280.314	.0000
RESIDUAL	60	.015	.0003		
CORRECTED TOTAL	71	6.480			

* * * INFERENCE ON COEFFICIENTS * * *

COEF.	ESTIMATE	STANDARD ERROR	T-STATISTIC	PROB. OF LARGER \T\	VARIANCE INFLATION
1	-9866.	903.	-10.93	.0000	2.276E+11
2	0.	0.	....	....	1.265+322
3	17676.	1586.	11.14	.0000	8.010E+09
4	-3.	17.	-.16	.8705	2.198E+08
5	0.	1.	-.60	.5498	2.879E+05
6	-11860.	1045.	-11.35	.0000	7.234E+10
7	4.	23.	.19	.8489	2.005E+09
8	0.	1.	.60	.5479	1.164E+06
9	3532.	306.	11.56	.0000	7.272E+10
10	-2.	10.	-.22	.8271	2.045E+09
11	0.	0.	-.61	.5455	2.957E+05
12	-394.	33.	-11.77	.0000	8.136E+09
13	0.	1.	.25	.8055	2.332E+08

THE SUM OF SQUARES ERROR = .2386E+02

As seen in the above figures, with an increase of 4 more terms, the H17 sample had a sum of squared error of 72.05 vs a sum of squared error of 85.78 with the 10 term regression that does not include the cos(2Θ) term. The adjusted R^2 for H17 increased only slightly from 98.501 to 98.624. The results for the low DI steel sample, L7, show a very slight decrease of R^2 from 99.732 to 99.718. The sum of squared error decreased also very slightly from 24.41 to 23.86. The slight contradiction in the results and the overall negligible decreases in the sum of squared error leads me to believe that the addition of the cos(2Θ) terms was not warranted.

Therefore, it seems that the best model available to explain the variation in hardness in all samples taken is the following.

$$
\begin{aligned}
y = \exp(c0 + c1*radius \quad & + c2*radius*\cos(\Theta) \quad + \\
c3*radius^2 \quad & + c4*radius^2*\cos(\Theta) \quad + \\
c5*radius^3 \quad & + c6*radius^3*\cos(\Theta) \quad + \\
c7*radius^4 \quad & + c8*radius^4*\cos(\Theta))
\end{aligned}
$$

3.5 Conclusions

For each sample the coefficients c0 to c8 must be estimated using least squares estimation. Before the Fourier series form of the model was even attempted, it was known that the majority of the samples had a Θ effect that was fairly symmetric around the 0-180 degree line. It is perfectly logical then, that in the process of determining the best model, all of the sine terms were discarded. That the cos(Θ) terms were the only ones kept from the Fourier series terms, makes the model much more elegant. With only a cos(Θ) term in the model, it has to be symmetric around the 0-180 degree line. The exponential relationship between depth and hardness is again a logical element in the model that is supported by the statistics. A log transformation is one of the classical ways of lowering dependency of error on the levels of the independent variables. Finally, the fact that a fourth order model was undoubtedly the best simply supports the reasoning from the beginning of the analysis that the curvature of the hardness vs. depth curve should be able to be explained by at most a fourth order polynomial. Therefore, the model as presented above is logical conclusion both statistically and physcially. The size of the sum of squares also makes sense because, with an average of 80 for the SSE, that averages out to about 1.1 RC points. The machine that did the testing for Rockwell C hardness, is only accurate to +or - 1 RC point. Therefore, any attempt to explain the variation closer than 1 RC point would be trying to overfit the model. Overfitting usually ends with poorer predictions of the dependent variable in between and beyond the points that were actually used to form the model's coefficients.

The final step in any regression where the model is the desired result is analyzing the plots of residuals. First, the normality plot of the residuals will be analyzed. The plot can be found on the first page in Appendix C. This shows that the model yields residuals that are very close to normal. For the H17 sample, the plot also clearly points to an outlier. The normality plots for various other samples show similar results. For the low DI samples, the results are the same. In analyzing the plots of radius vs. residuals, there still seems to be a slight increase in variance as the value of radius nears the center of its range. The plots of Θ vs. residuals show an expected mean of zero and a variance that stays relatively constant. Generally, the model produces residuals that closely adhere to the assumptions used in least squares regeression. These assumptions are:

1. Error is a random variable distributed as Normal$(0,\sigma^2)$
2. Independence of error with respect to the variables in the regression.
3. Constant variance$(\sigma^2)$of error with respect to the variables in the regression.

Chapter 4
CONCLUSIONS

4.1 Initial Model

4.1.1 Range of Validity

Based upon the assumption of uniformity of hardness around the steel bar, a model that will yield a good estimate of the depths at which RC 50 and RC 30 can be achieved is shown at the end of Chater One. Using the confidence intervals listed on page 44, one can be 95% confident that the depth at which a hardness will be achieved will meet or exceed the listed depth. These confidence intervals increase as any of the values of the control variables strays from the mean value found in the experiment. Also, the two models are valid only within the limits that were defined in the original design of the experiment found in Table 1.1. Extrapolation outside of these limits becomes extremely risky and the level of confidence in the calculated estimate drops significantly.

One serious problem with the two models is that the variable DI was evaluated at only two levels. This number of levels would be adequate if the relationship between DI and depth of a given hardness were linear. However, if DI has a true quadratic effect, then the model will predict erroneous depths at which RC 50 or RC 30 are achieved if the sample is made from a median DI(for example 1.00) steel.

4.1.2 Sensitivity

The factors in the final RC 50 model in Chapter One have coefficients that support the known or suspected relationships between the five factors.

ROCKWELL C 50

VARIABLE	10% INCREASE	CHANGE IN PRED DEPTH
X1	.115	+ .74818
X2	.250	- .11473
X3	.331	- 2.29550
X4	.086	+ .86291
X5	.010	- 3.53370

Through this type of analysis is possible to see which variables have the most dramatic effect on the predicted depth at which RC 50 may be achieved. Certainly X3 and X5 (coil diameter and speed) seem to have the biggest impact for a 10% increase in their values. The variables with negative coefficients, X2, X3, and X5, simply confirm the relationships that were either suspectd or known as discussed in the first pages of Chapter One. It lends credibilty to the model that the sign of the coefficients support the known relationships.

The factors in the final RC 30 model also have coefficients that support the suspected relationships between the five factors.

ROCKWELL C 30

VARIABLE	10% INCREASE	CHANGE IN PRED DEPTH
X1	.115	- 3.042
X2	.250	- .30447
X4	.086	+ 1.00279
X5	.010	- 4.687

This analysis is slightly more difficult to interpret. Again, speed has the most dramatic effect upon the predicted depth. Strangely, it seems that with a 10% increase in X1(DI), a drop of over 3 mm will occur for the predicted depth of hardness of RC 30. This is due, however, to the large negative coefficient of the interactive effect X1*X5. The linear term that shows the main effect of X1 would cause a 1.949 mm increase in the predicted depth of hardness for RC 30. It is interesting that there is no representation of the effect of coil size in this model. It seems that as the depth is increased, the coil size has less impact on the depth at which a given hardness can be achieved. Again, the signs of the changes to predicted depth of hardness seem to logically support the known or suspected relationships and lends credibility to the model.

4.1.3 Summary

In summary, the models developed in Chapter One are limited in range, especially in the range of bar diameter, DI, and coil diameter. The testing of DI at only two levels restricts the model to a linear one(in DI), which is a very large, and probably incorrect, assumption. The last problem with the models in Chapter One is that the models were based upon the assumption of independent and normally distributed residuals. The plots and analysis in Chapter Two indicate that this is an invalid assumption.

4.2 Dependence Upon Theta

In Chapter Two, two different points in the five-dimensional space were chosen to be the indicators of whether or not there was some sort of dependence of hardness on Θ, the angular distance of a tested point from the index point. These two points were chosen do to the non-orthogonality of the sample space, and to the relative closeness of these two points to the 'center', or median point of the five dimensional space.

Both a classic parametric(ANOVA) and non-parametric(Friedman) test for significance were conducted and the results were basically the same. Both in the high DI and low DI samples, the data showed that there was a very significant($\alpha <$ .05) dependence of hardness on Θ. The exact nature of that dependence was not at all obvious and required further investigation.

4.3 Including Theta

In a attempt to find a model that would accurately predict the hardness at any depth, attention was focused not on the five-dimensional space and all 62 samples, but on finding a model that included Θ and depth for a single disk. It was hoped that a general model could be found, to be applied to any sample, that would yield coefficients that would be an accurate representation of the physical dependence of hardness on both depth and Θ.

The guage that used to determine whether any given model was an accurate estimate of the actual relationships between the factors involved was three- fold. Adjusted R^2 was a very important factor in objectively comparing various models since this value accounts the possibility of overfitting. The absolute size of the sum of squares was also looked at as an important, but not all encompassing, guage of goodness of fit. The ability of the plots of the residuals against the independent variables, against predicted hardness, and as a normal probability plot to support the assumptions of independent, normal residuals with constant variance was the final test of the many models that were tried.

Though a 45 parameter model that used the Fourier series in Θ was originally attempted, a log transform model with simplified Fourier series terms became the model that best fit all of the samples' data. The final model yielded much improved plots against radius and Θ, and the normal probability plots also improved(Appendix C).

4.4 Recommendations

The first recommendation is that if the models in Chapter One must be used, they should be used only within the limits described in the first paragraph in this Chapter. Any attempt to verify the accuracy of the model should be done by measuring the average depth at which RC 50 or RC 30 is achieved, and not by measuring this depth at some randomly selected Θ. Since Θ is significant, it makes sense that quite different results will occur when different values for Θ are chosen.

It is also recommended that more samples be taken, specifically, more samples that have DI values near the center of the range(around a DI value of 1.00). This would allow for the possibility of a quadratic effect from DI to enter the model. Without this additional sampling, I would not recommend using the RC 50 and RC 30 models to predict depth of hardness for steel bars that are composed of median DI steel.

With more values of DI available, another direction for continued research is indicated. Perhaps a single general model to fit all samples' data is not what should be attempted. Perhaps the correct or 'true' general model should change as the level of DI changes. This could be accomplished either by including the value of DI in the model or, more likely, by developing a general model for each discrete level of DI. Of course, these levels are initially undefined and would have to be chosen very discriminately. Presence of a difference in appropriate models due to the difference in the DI values was slightly evident in the plots found in Chapter Three and could become more evident as various levels of DI are introduced.

It is strongly recommend using the model developed in Chapter Three to predict hardness at various depths and Θ's rather than using the bracketing method at a randomly selected Θto estimate the hardness. The developed model is also preferrable over using the fourth degree polynomial in predicting the hardness at a given depth, which was used in Chapter One, because this polynomial model incorrectly assumes uniformity around the disk.

Lastly, the most logical direction for continuing analysis would be to develop a model that would combine the results of Chapter One and Chapter Two. What is ultimately needed is a regression equation that would take the five independent factors of DI, bar diameter, coil diameter, input power level, and coil travel speed and predict the hardness values throughout the steel bar. One of the difficulties in doing this would be that even though a general model has been found that performs quite well in predicting the hardness values throughout a sample disk, a new specific model would have to be calculated for each different combination of the five variables before a prediction could made of the hardness values. This would entail doing a regression within a regression. A possible alternative to performing the regression described above would be to analyze the entire induction process to try to identify and eliminate all major sources of variation that tie hardness to Θ. Subsequently, the method used in Chapter One could be used, validly now, to calculate an adequate regression model without consideration of Θ.

References

[1] Semiatin, S.L., and D.E. Stutz, Induction Heat Treatment of Steel, American Society for Metals, Metals Park, Ohio, 1986.

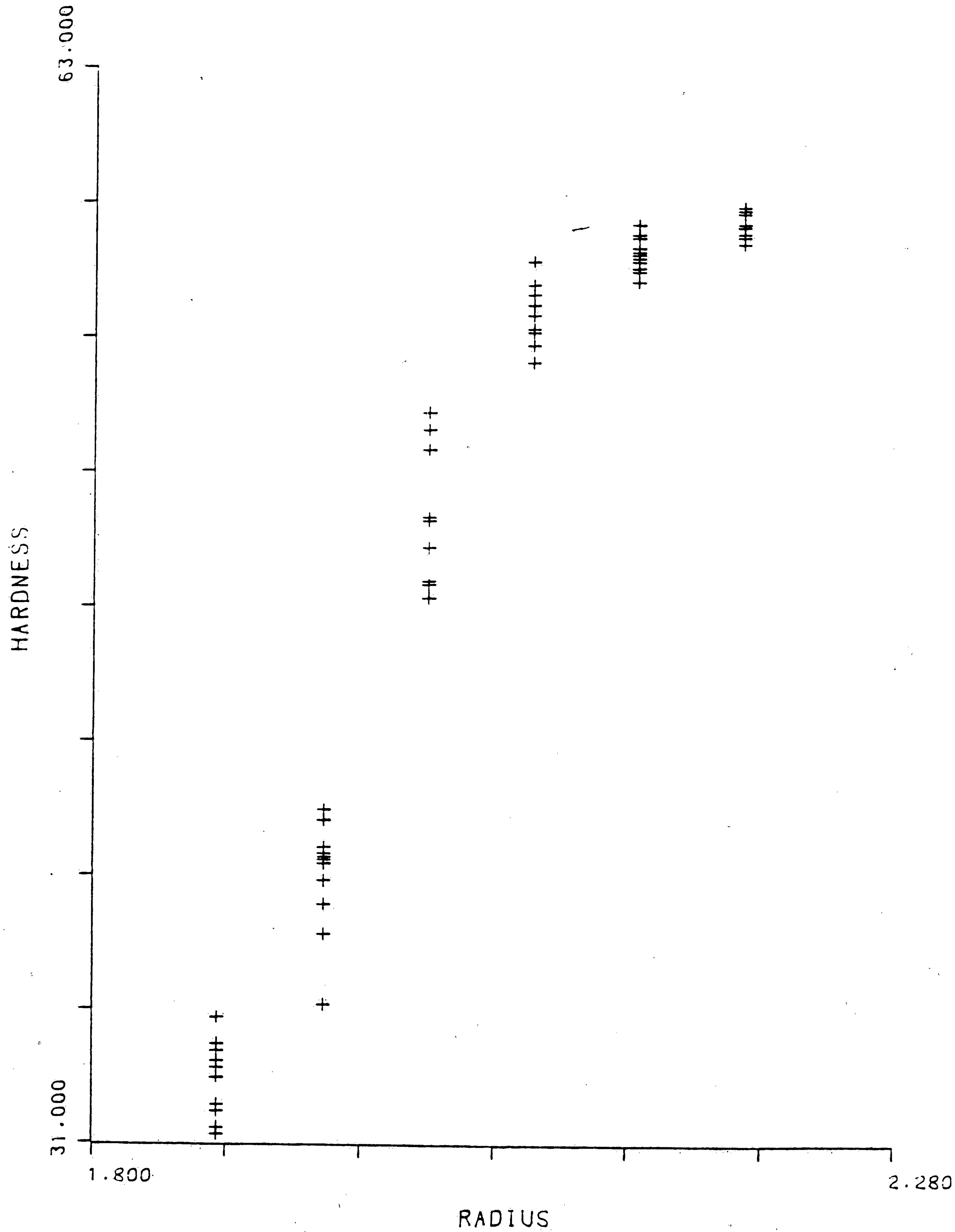
[2] Problem Solving Software Systems, IMSL User's Manual, IMSL Inc., New York, 1987.

[3] Montgomery, D.C., Design and Analysis of Experiments, John Wiley and Sons, New York, 1984.

[4] Neter, J., Wasserman, W., and Kutner, M.H., Applied Linear Statistical Methods, Irwin, Homewood, Illinois, 1985.

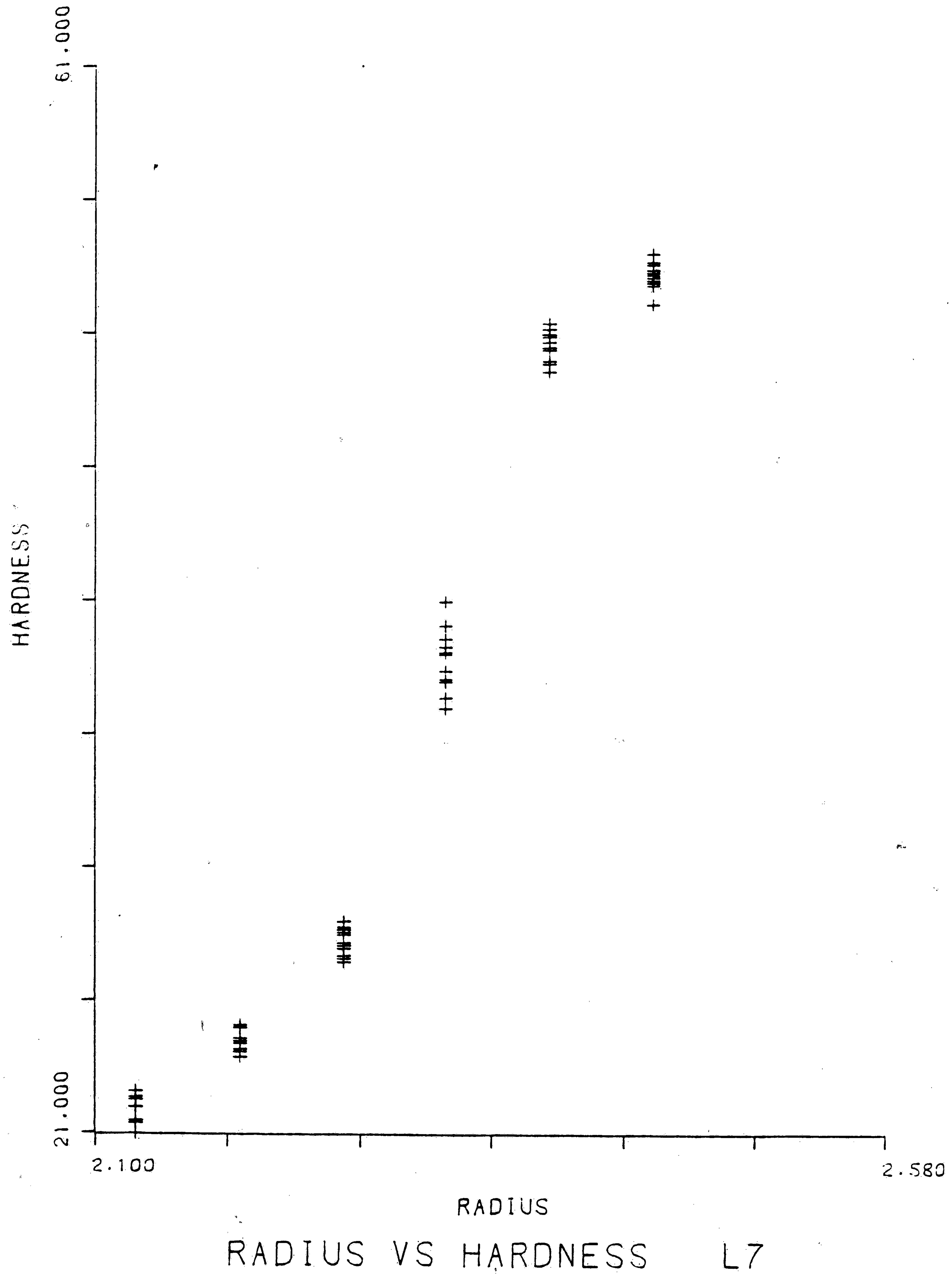
[5] Bloomfield, P., Fourier Analysis of Time Series: An Introduction, John Wiley and Sons, New York, 1976.

Appendix A



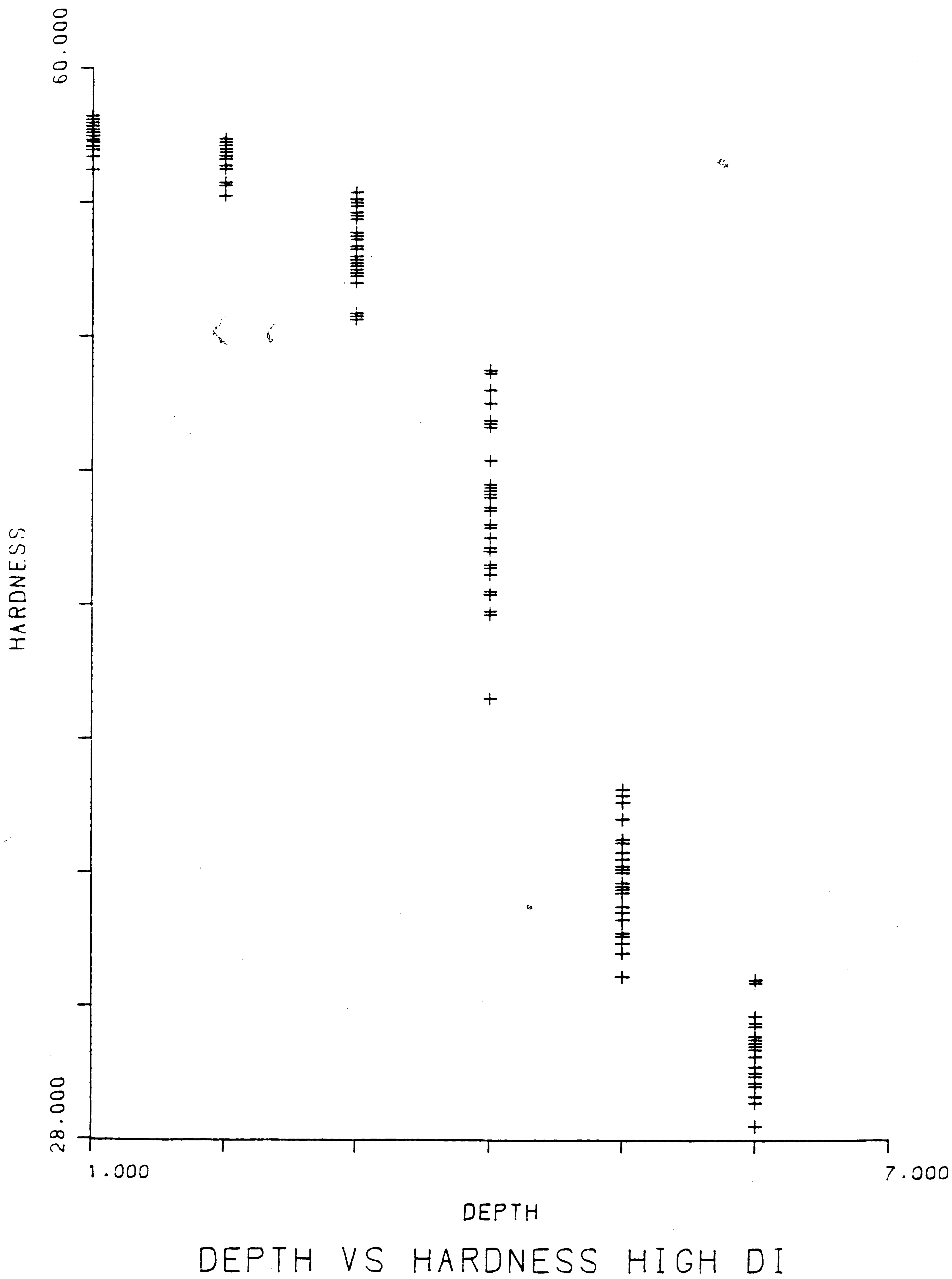
RADIUS VS HARDNESS H17

HARDNESS PLOTS

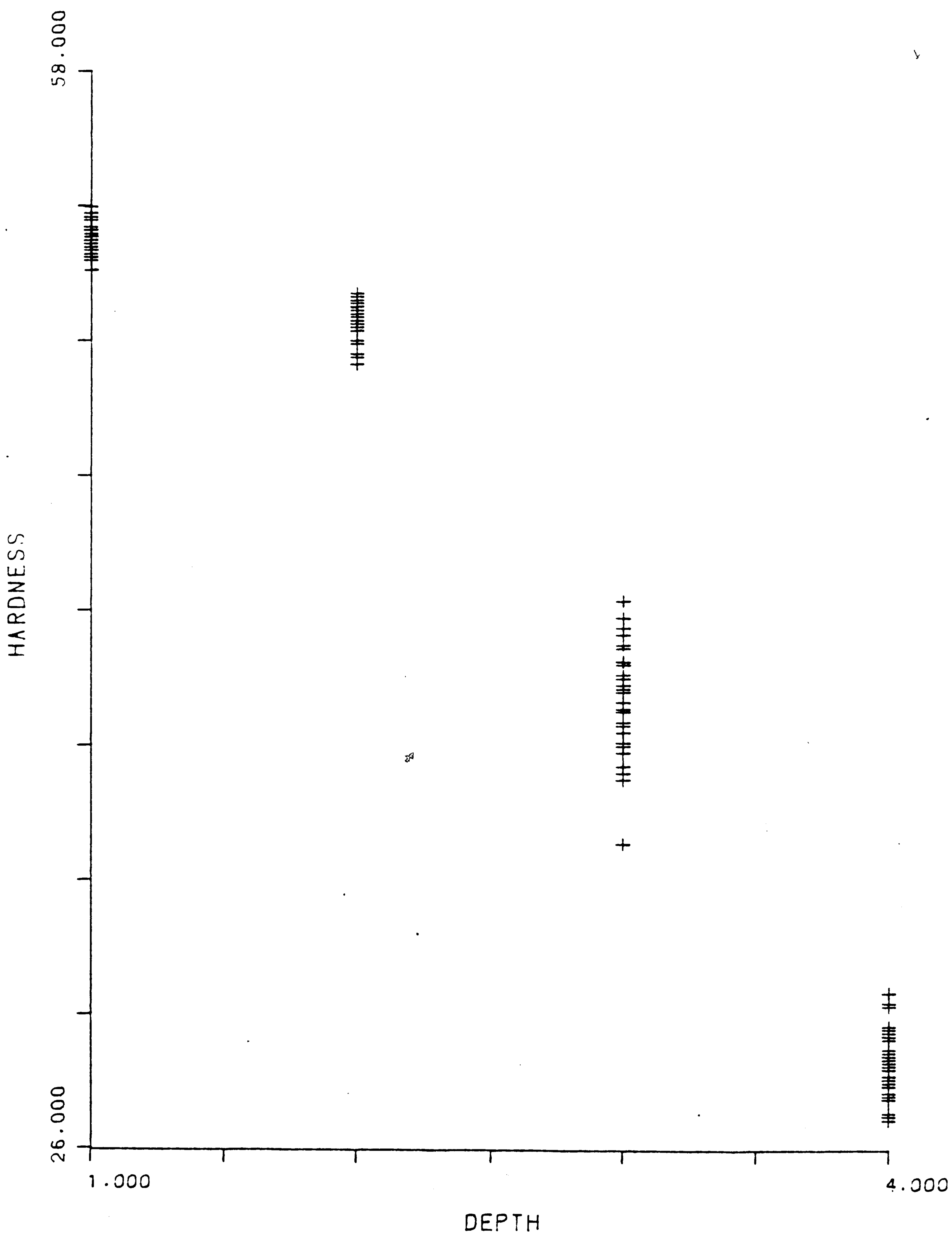


HARDNESS PLOTS

Appendix B

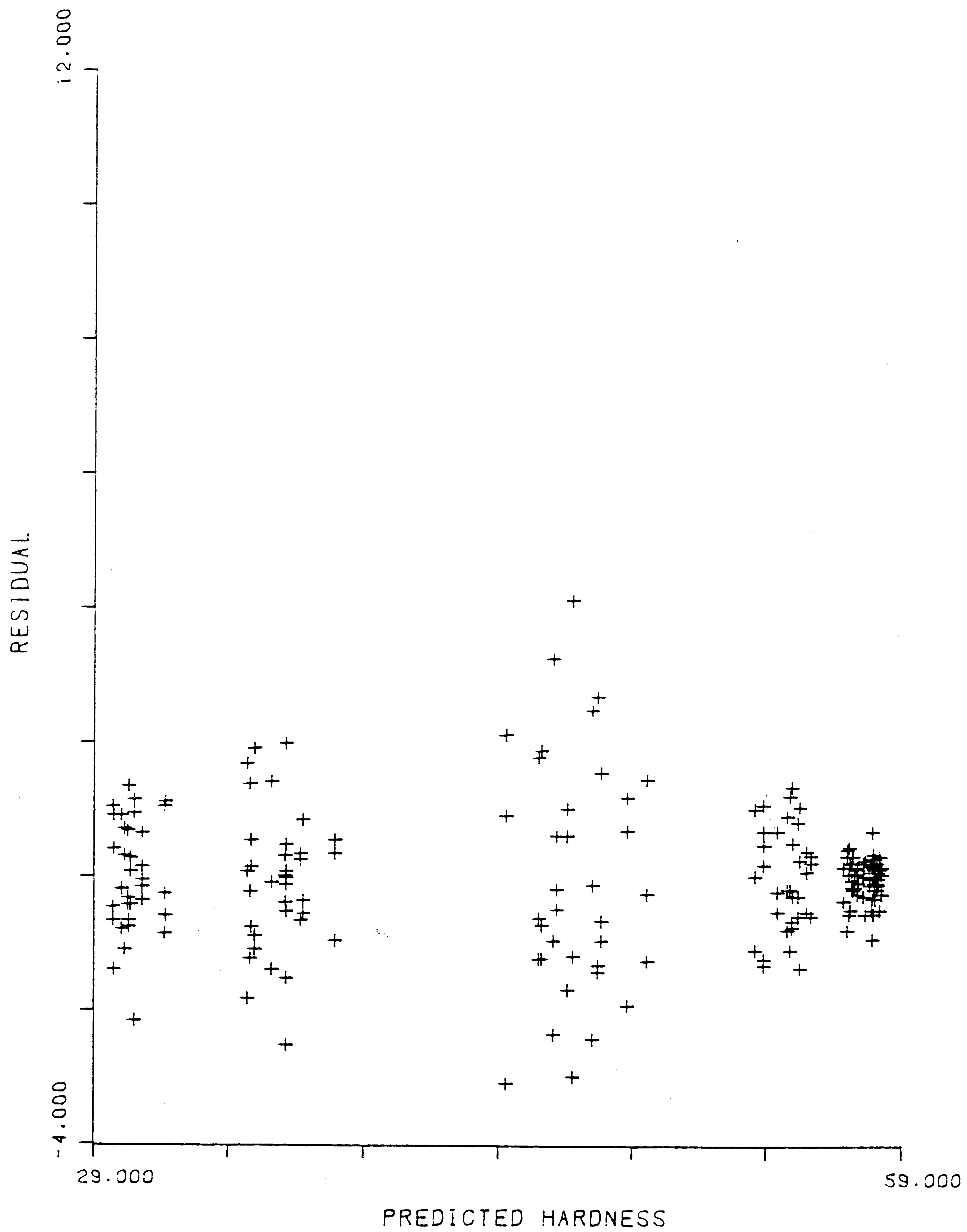


HIGH/LOW HARDNESS PLOTS



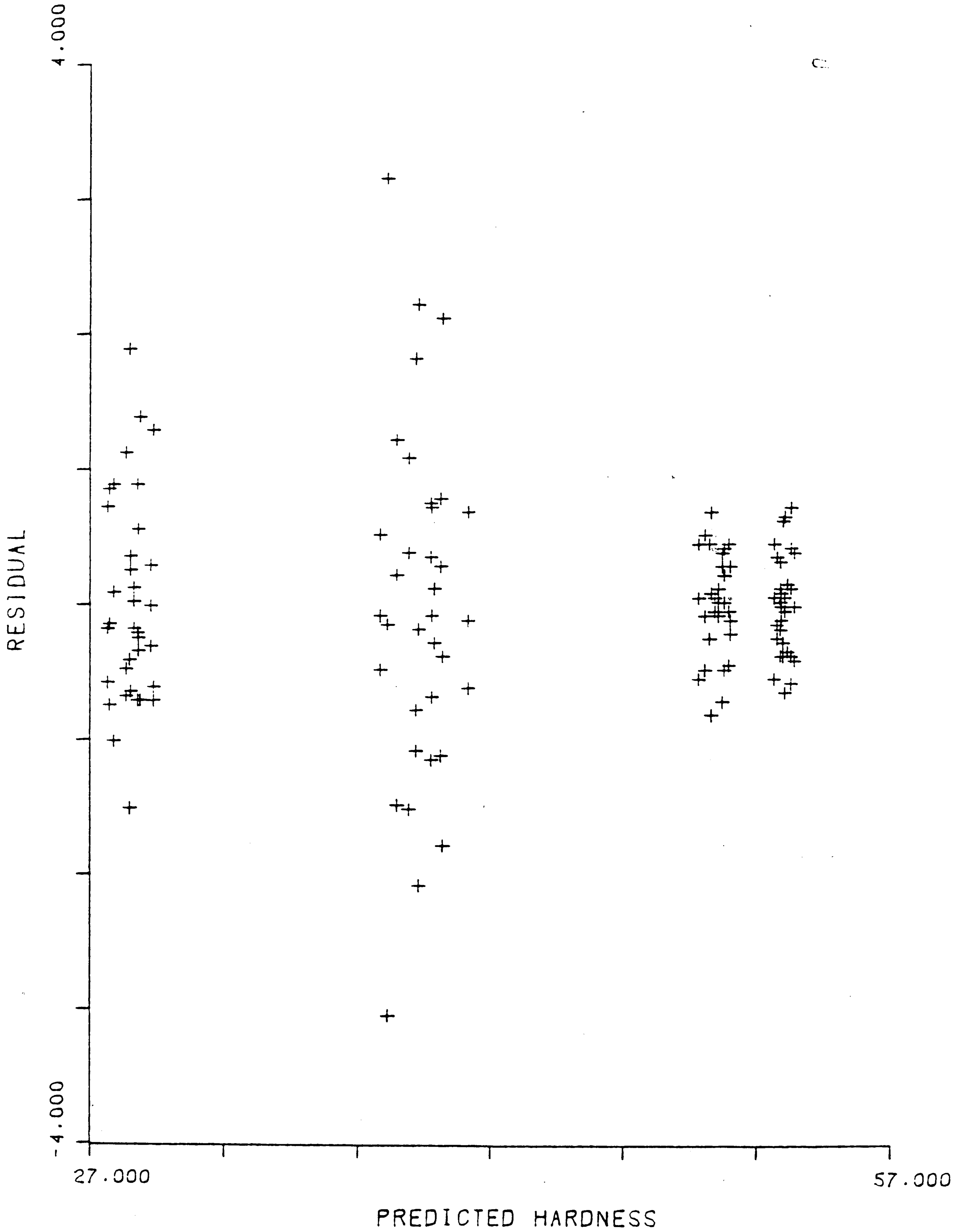
DEPTH VS HARDNESS LOW DI

HIGH/LOW HARDNESS PLOTS



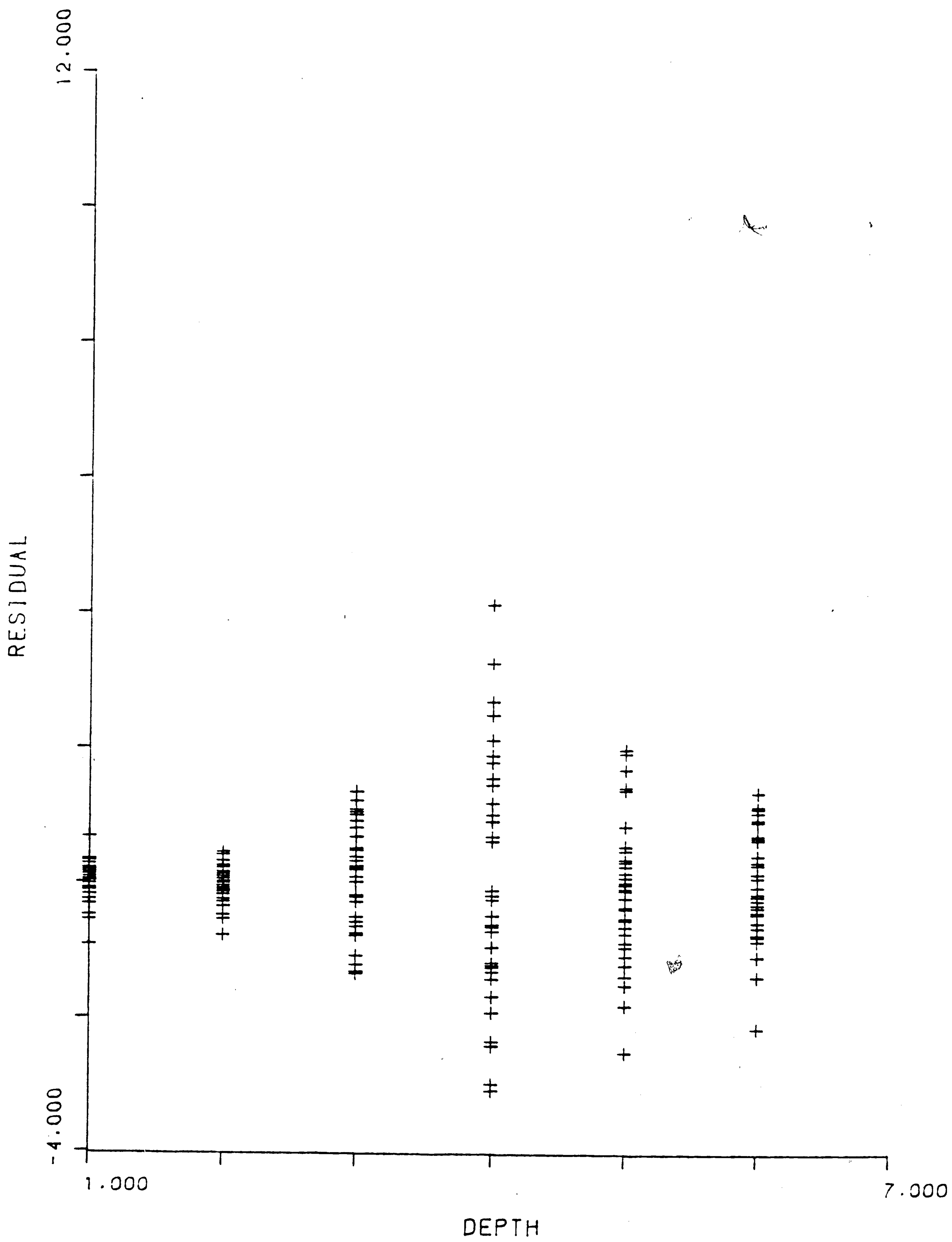
PREDICTED HARDNESS VS RESIDUAL HIGH DI

PREDICTED HARDNESS PLOTS



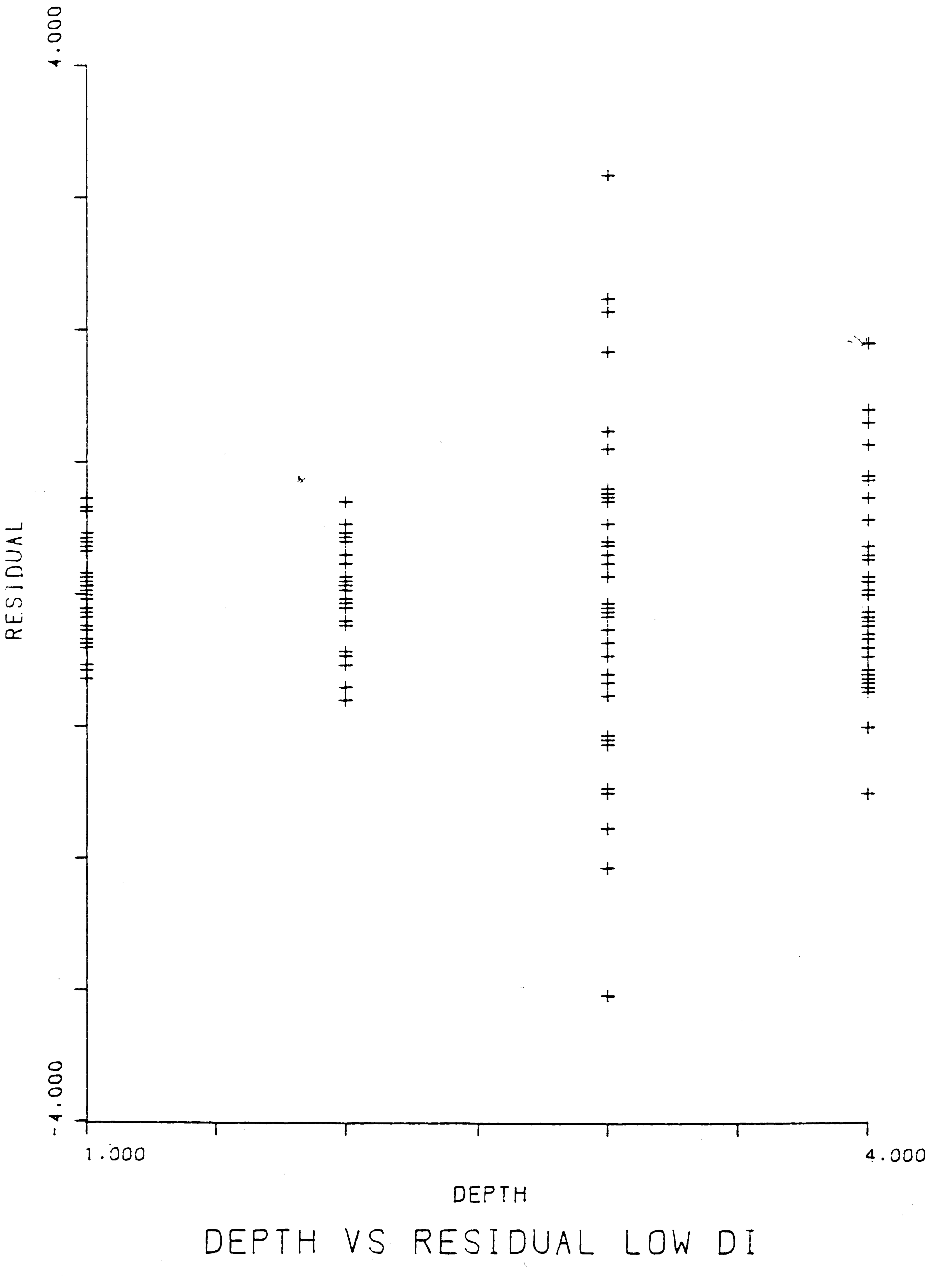
PREDICTED HARDNESS VS RESIDUAL LOW DI

PREDICTED HARDNESS PLOTS

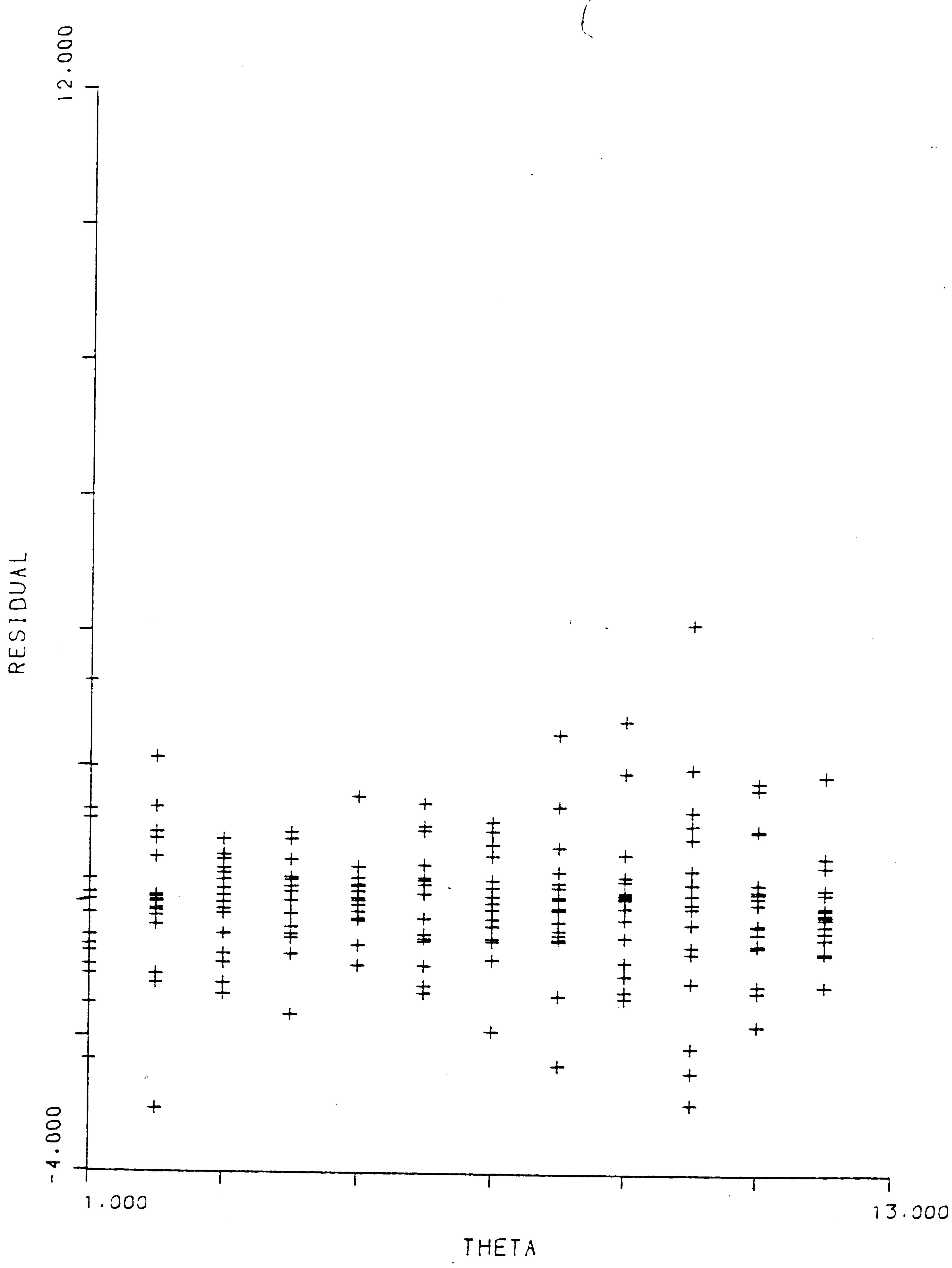


DEPTH VS RESIDUAL HIGH DI

DEPTH PLOTS

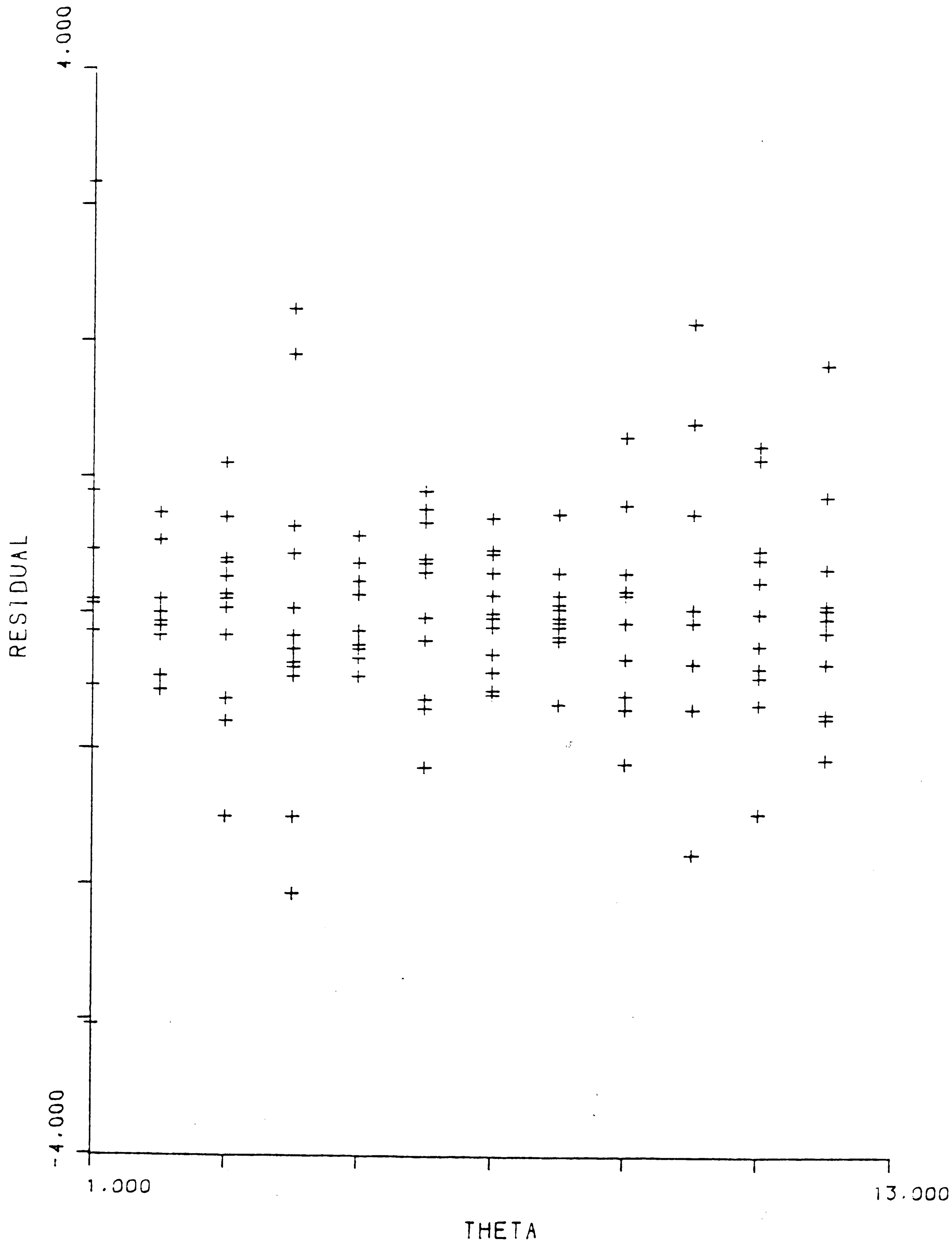


DEPTH PLOTS



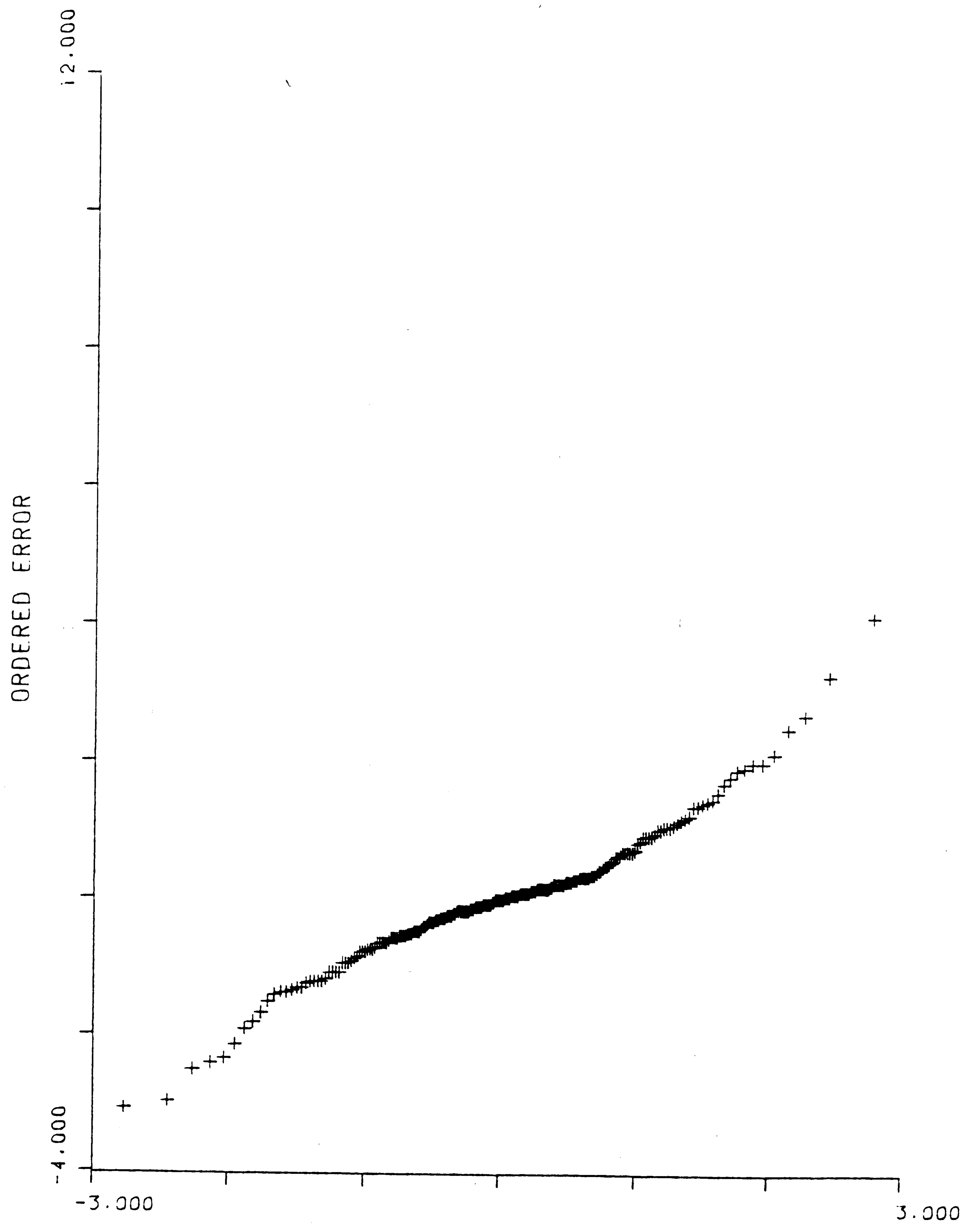
THETA VS RESIDUAL HIGH DI

THETA PLOTS



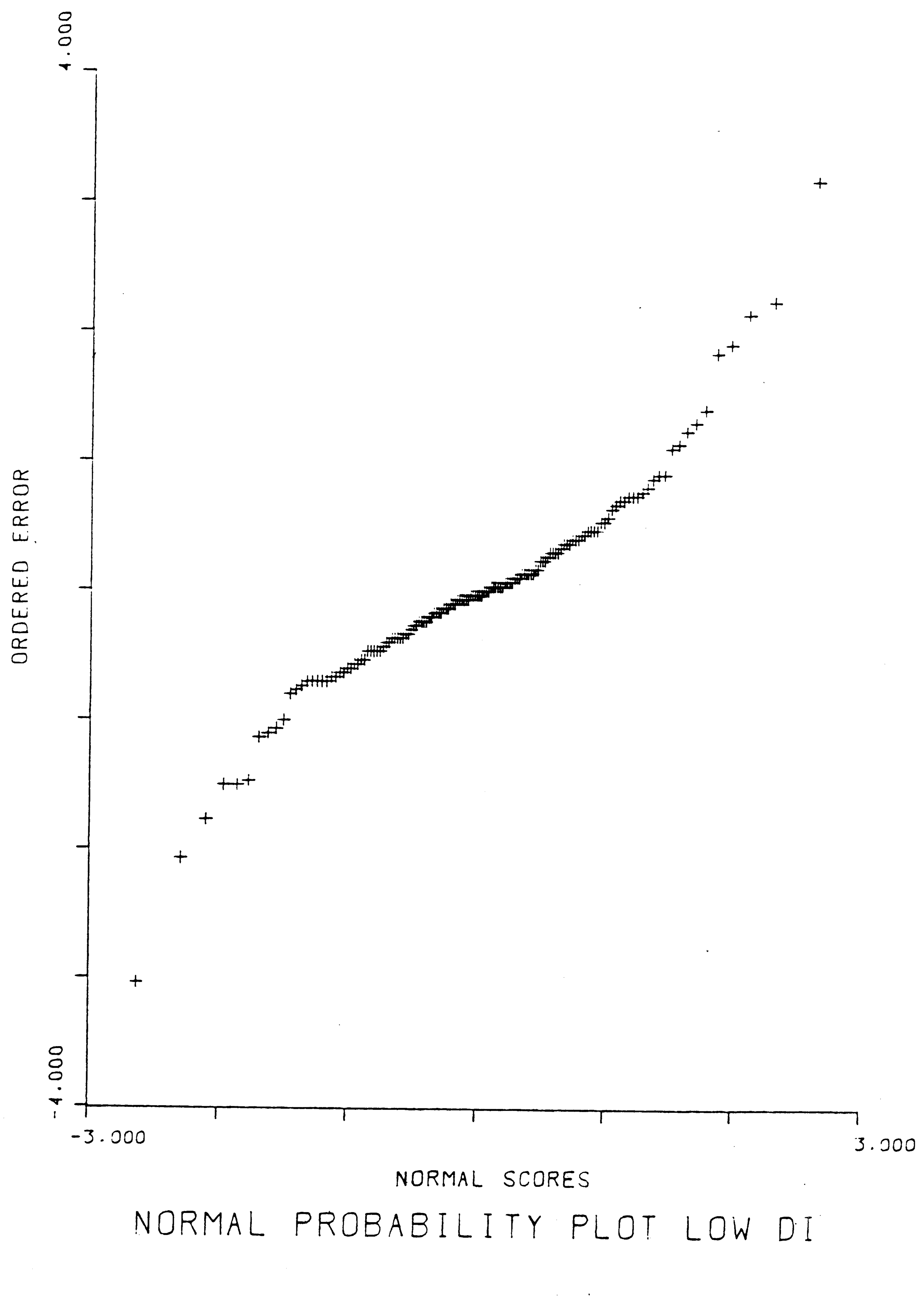
THETA VS RESIDUAL LOW DI

THETA PLOTS



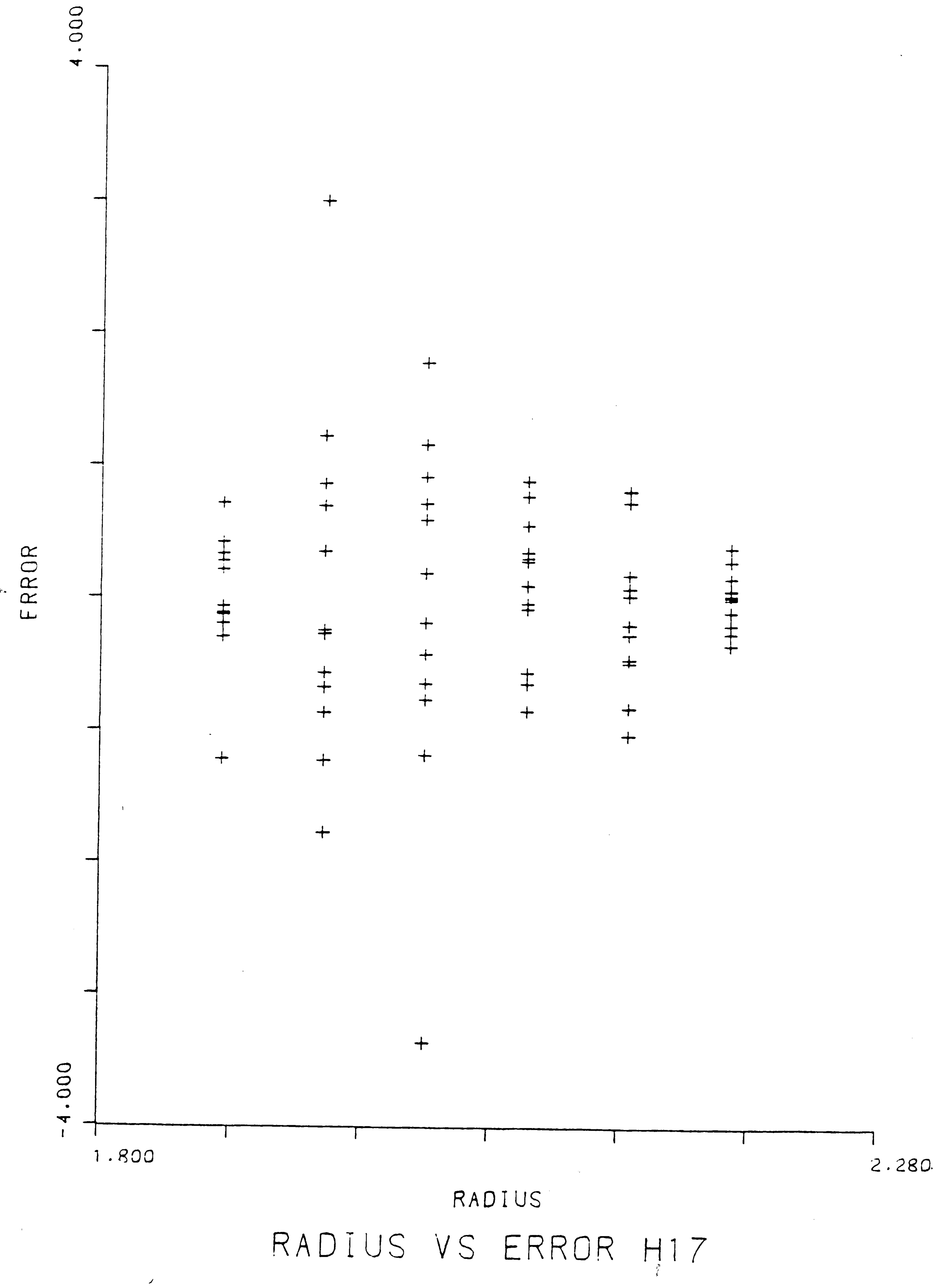
NORMAL PROBABILITY PLOT HIGH DI

NORMAL PROBABILITY PLOTS

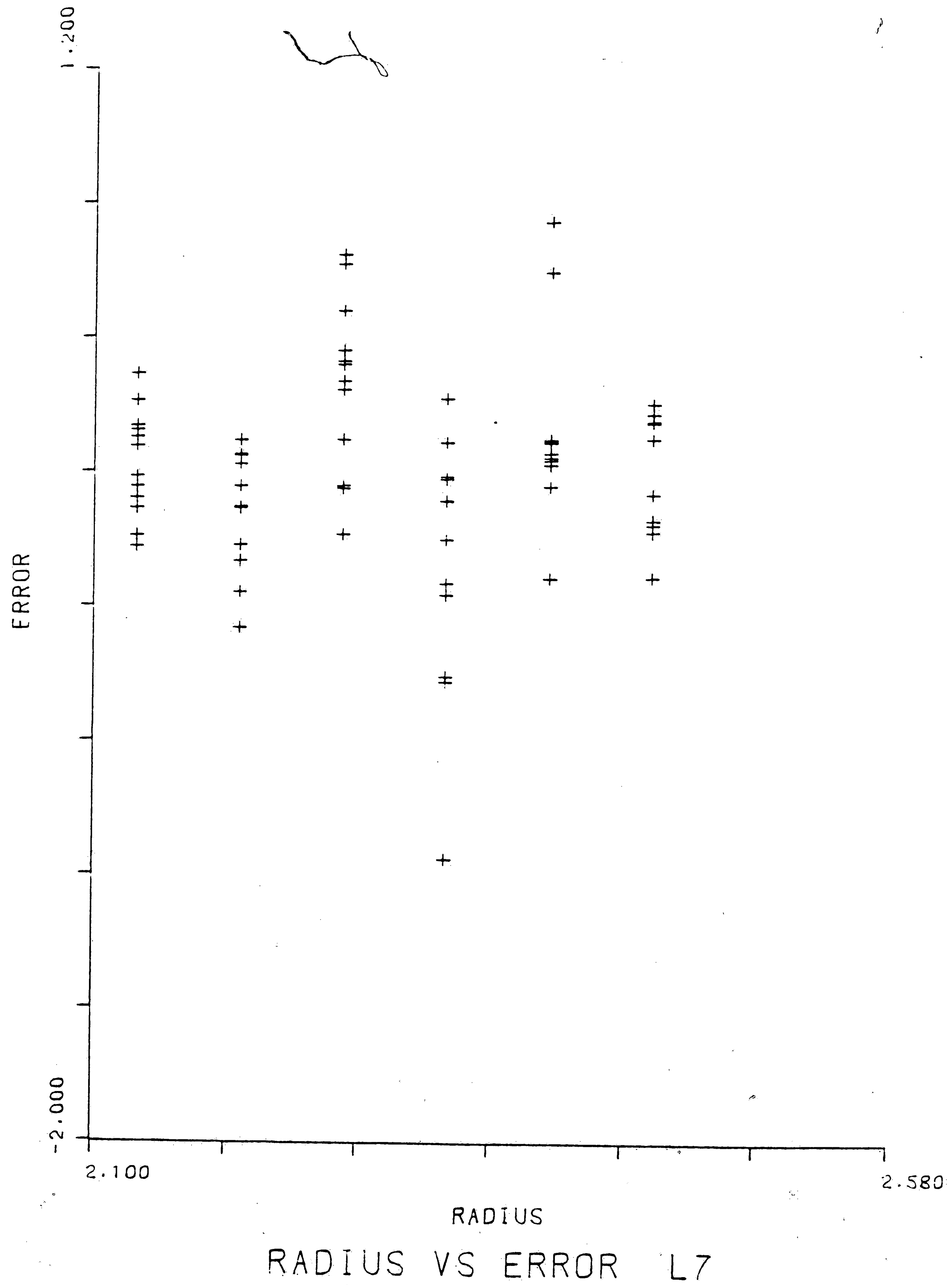


NORMAL PROBABILITY PLOTS

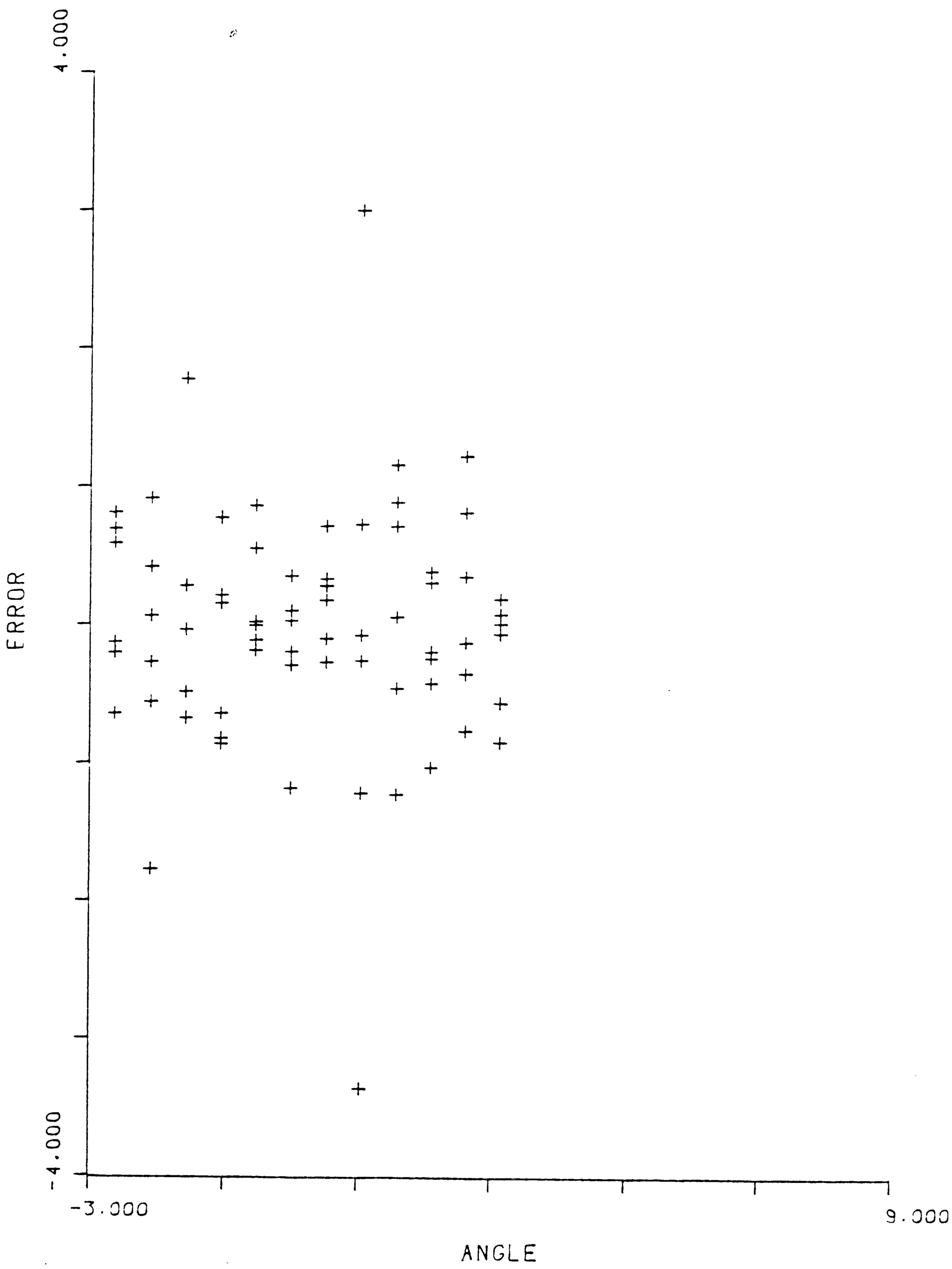
Appendix C



ORIGINAL FULL FOURIER MODEL

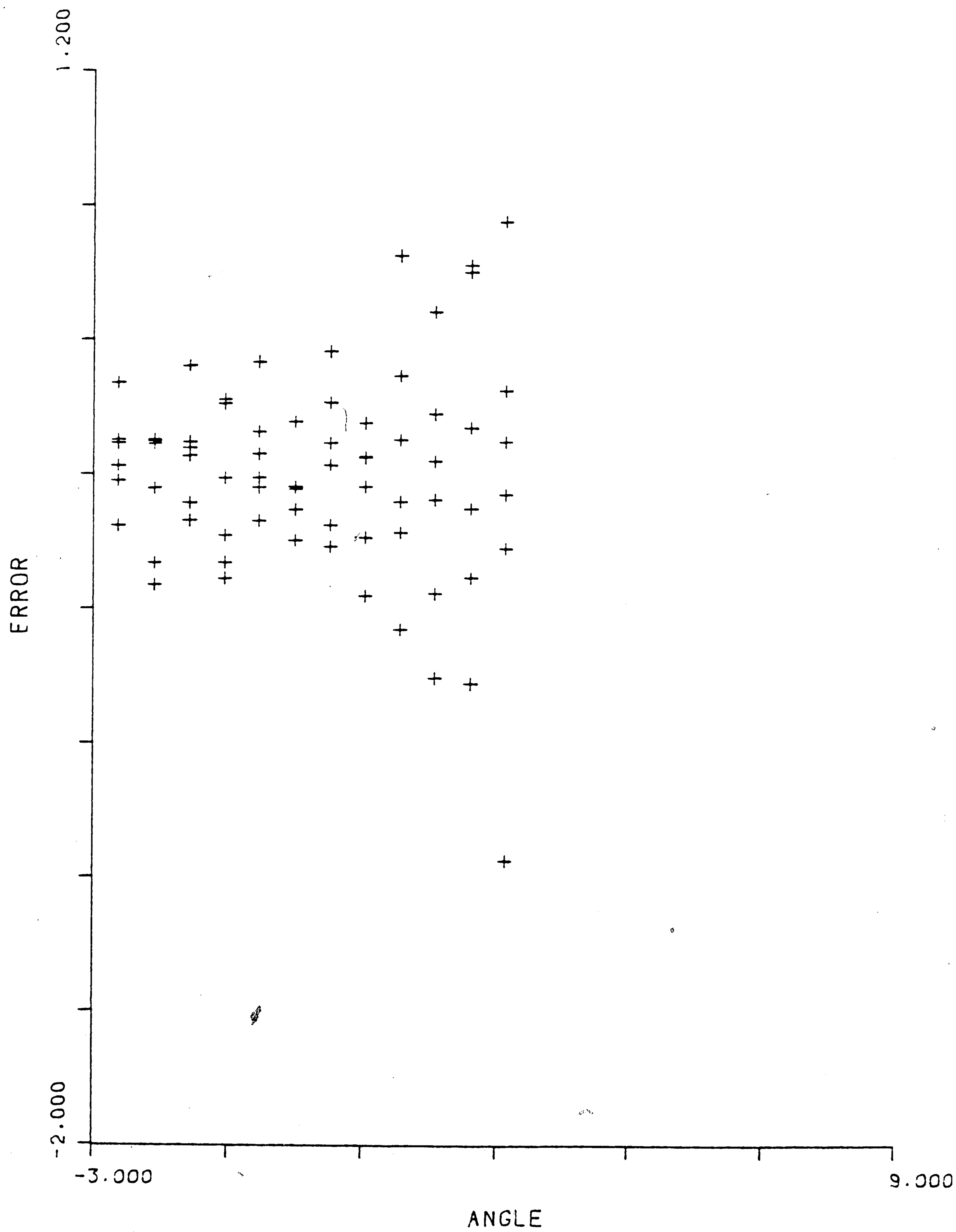


ORIGINAL FULL FOURIER MODEL



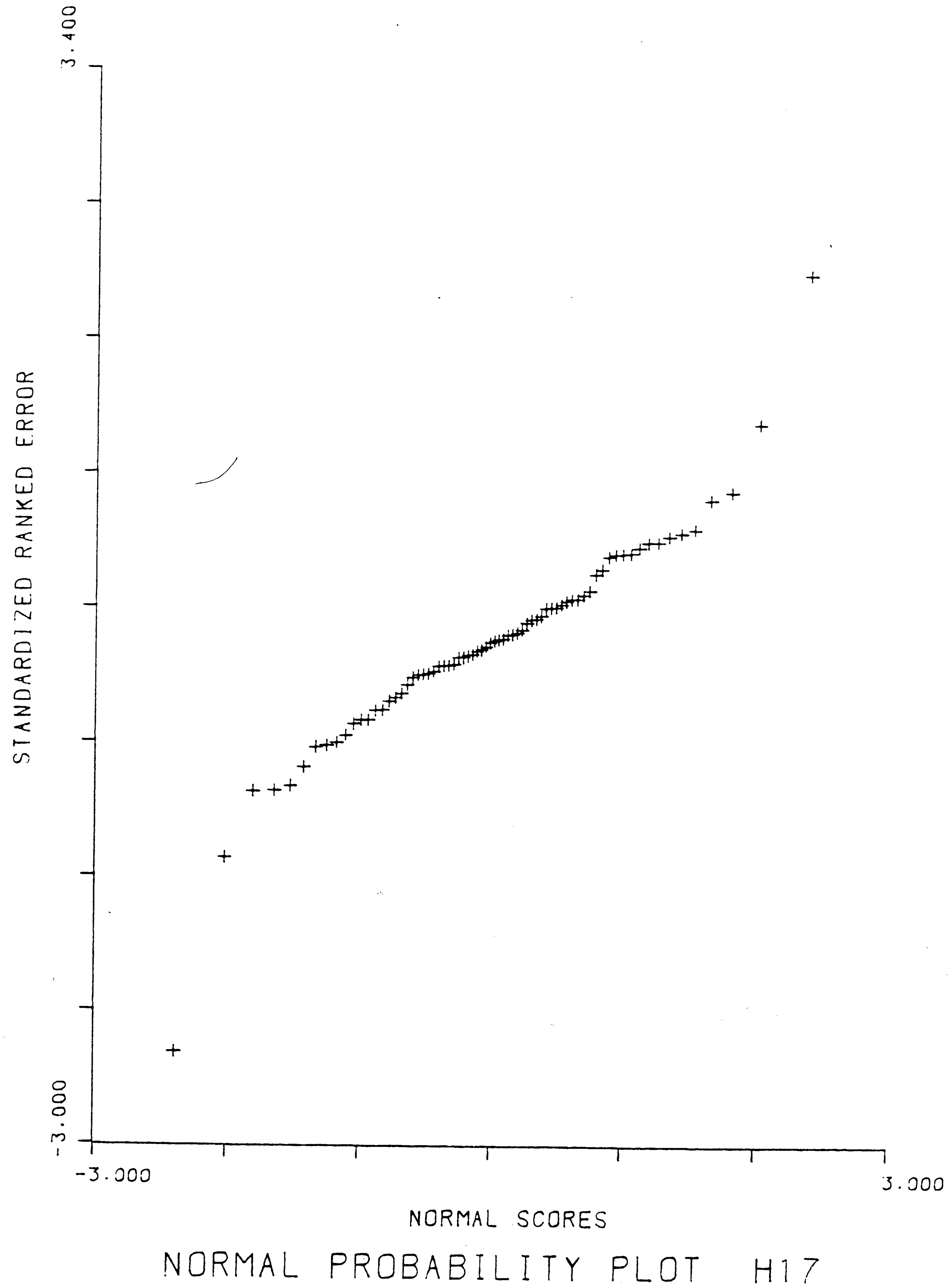
ANGLE VS ERROR H17

ORIGINAL FULL FOURIER MODEL

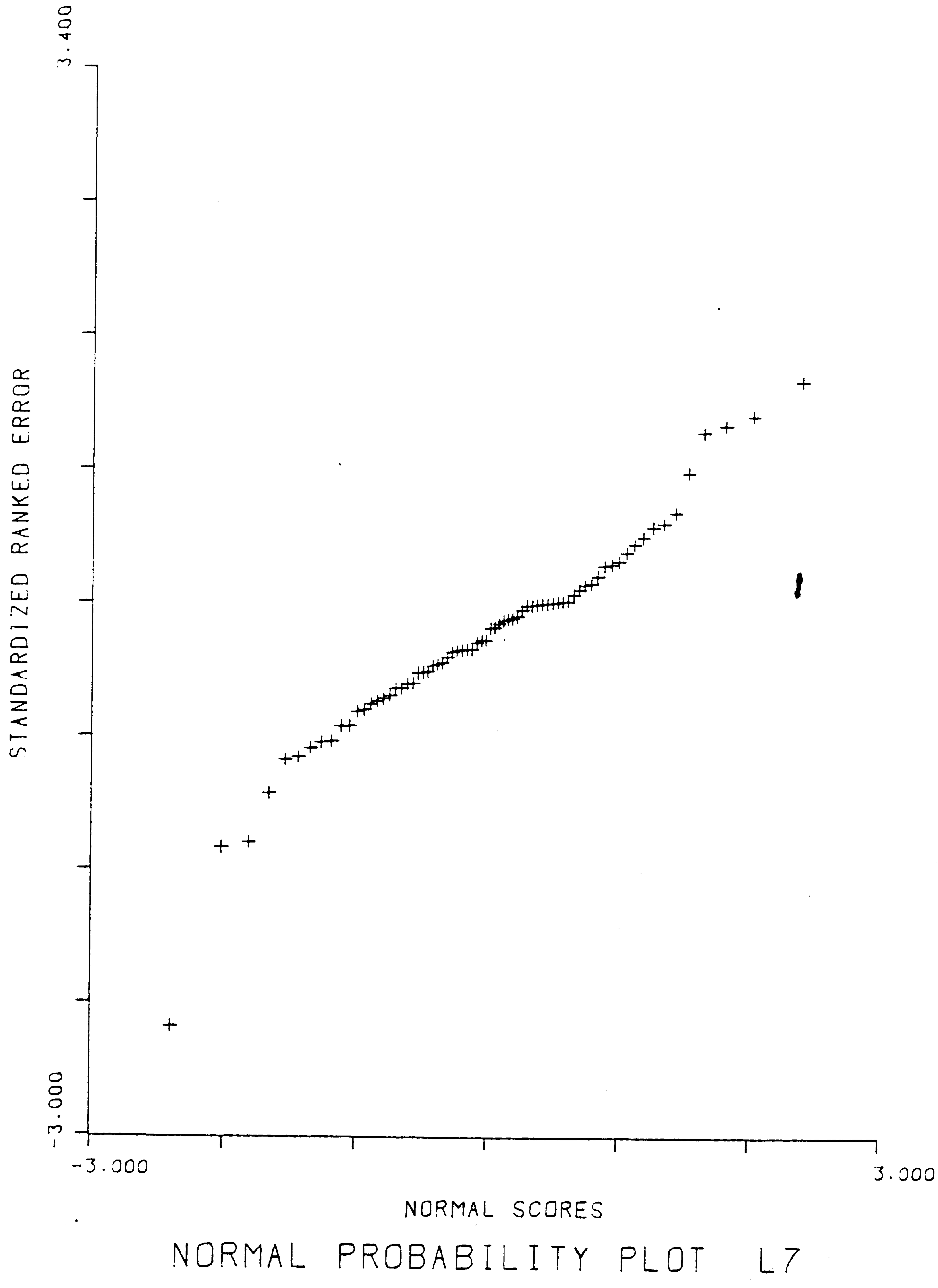


ANGLE VS ERROR L7

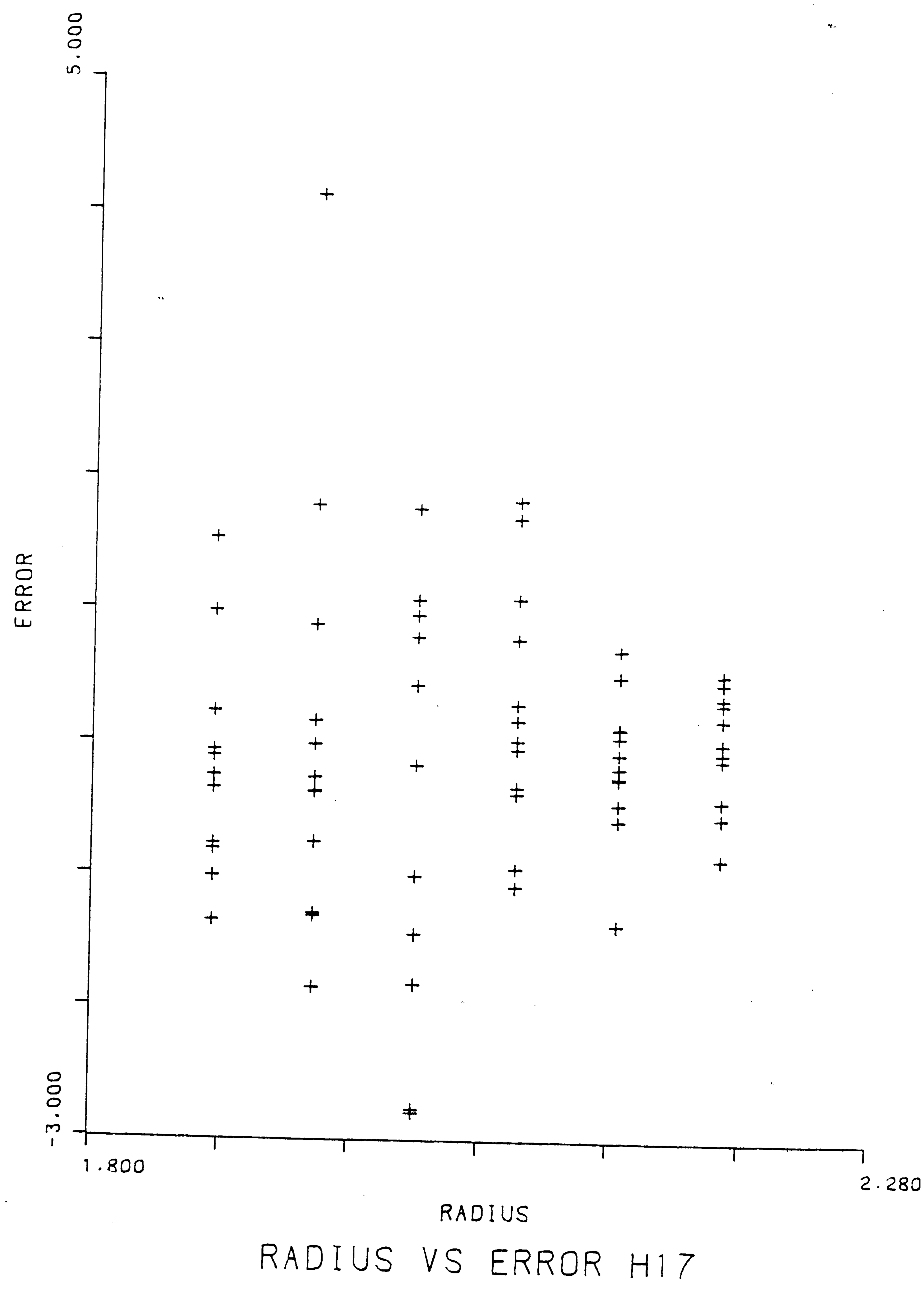
ORIGINAL FULL FOURIER MODEL



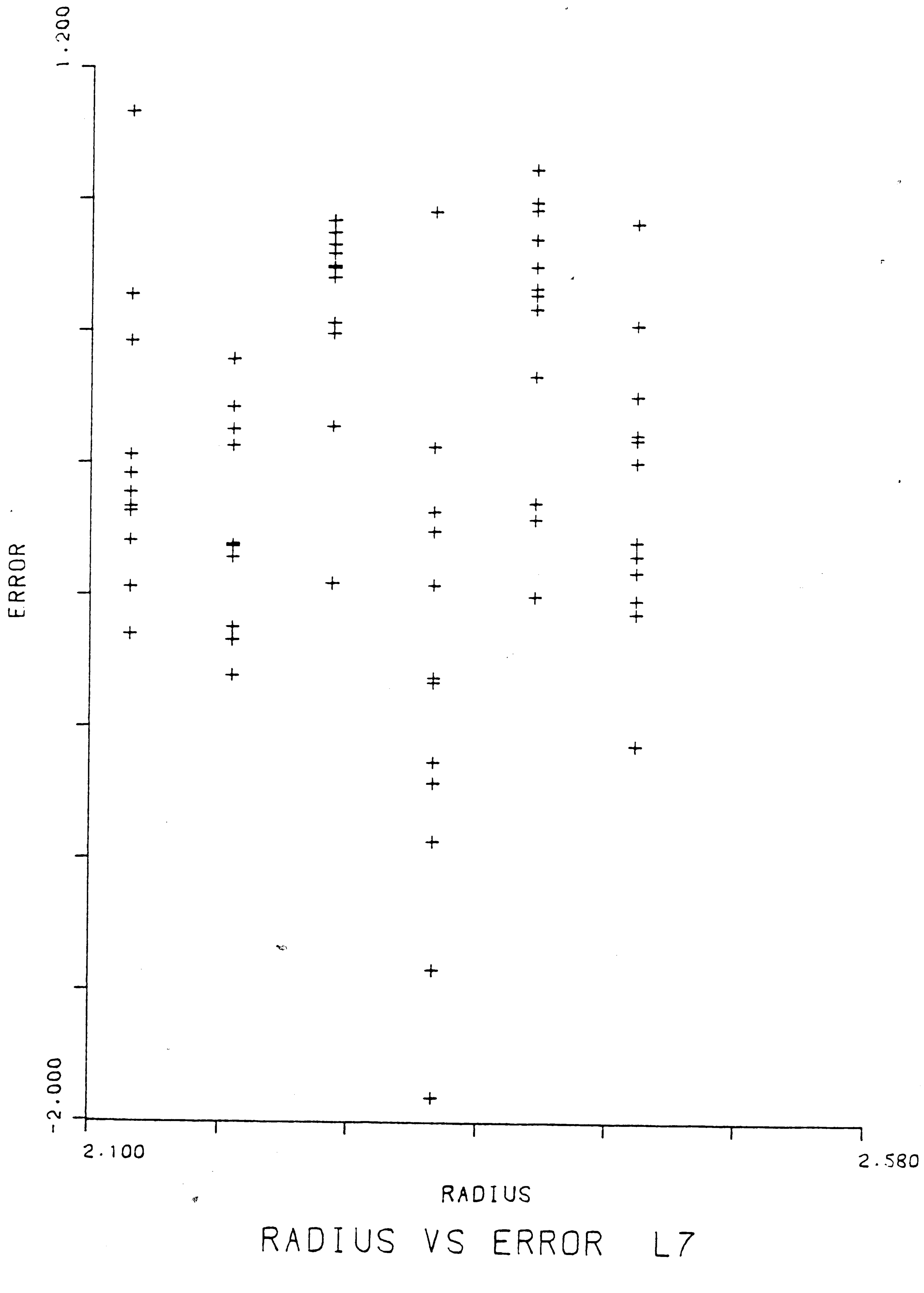
ORIGINAL FULL FOURIER MODEL



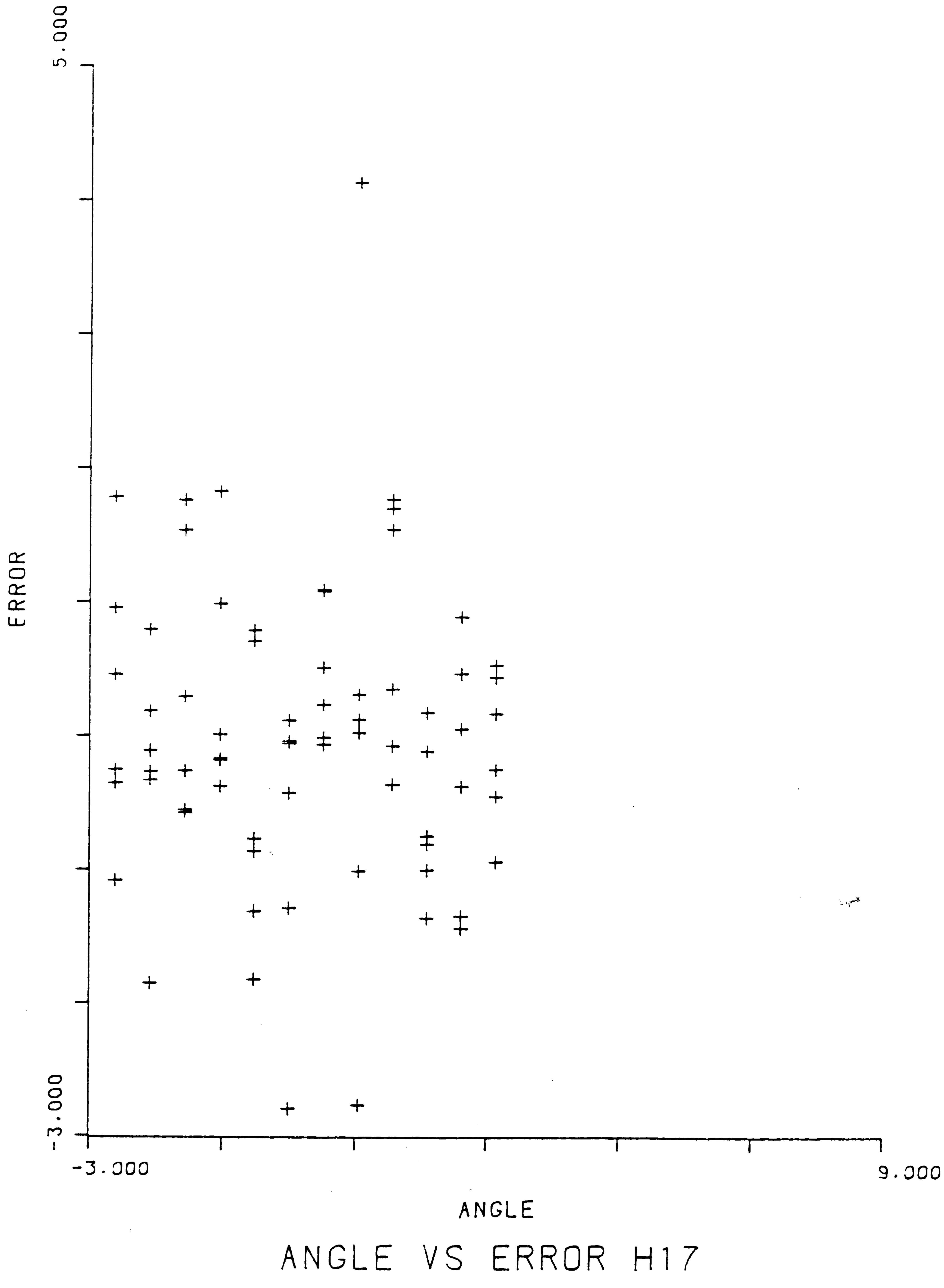
ORIGINAL FULL FOURIER MODEL



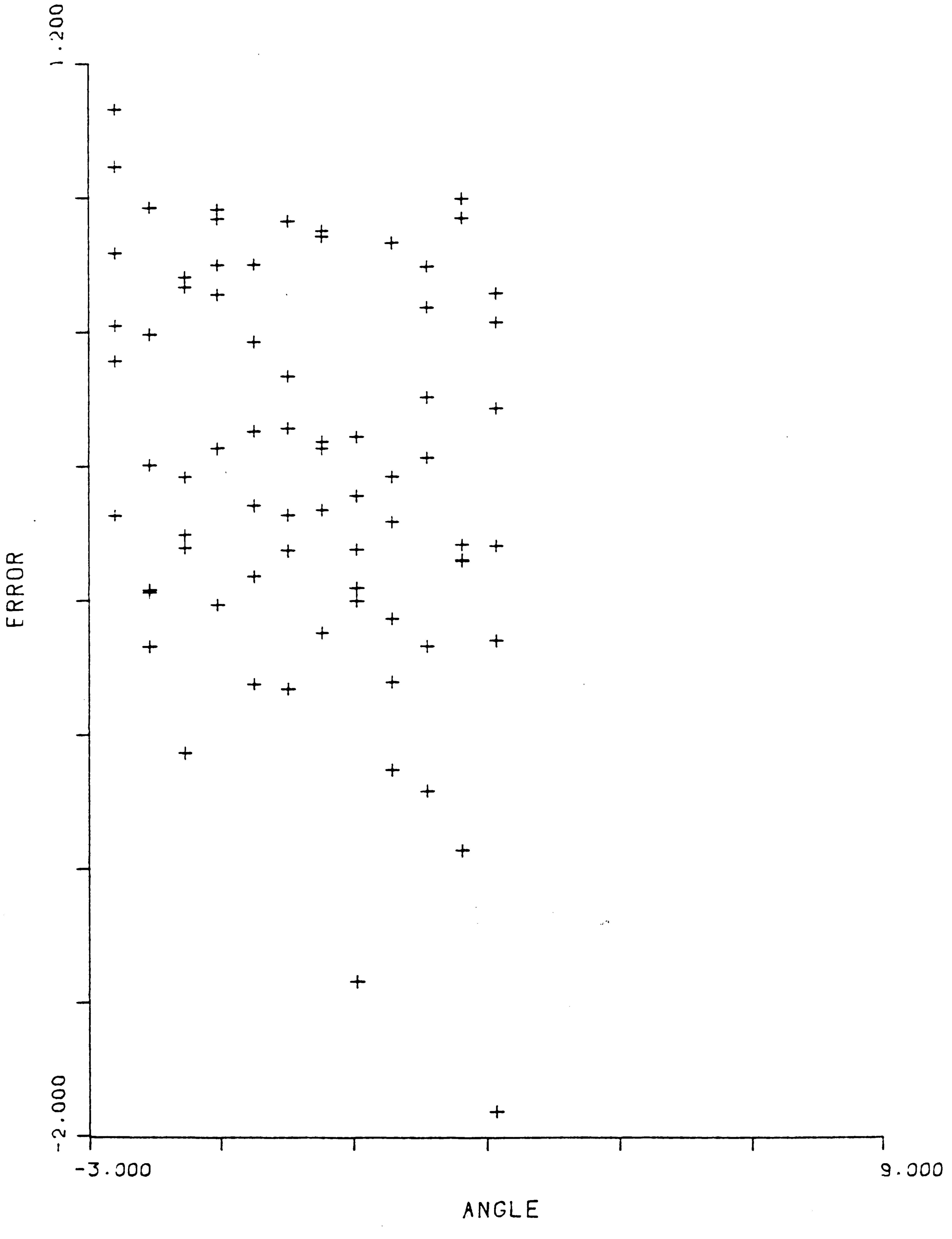
4TH ORDER EXP MODEL



4TH ORDER EXP MODEL

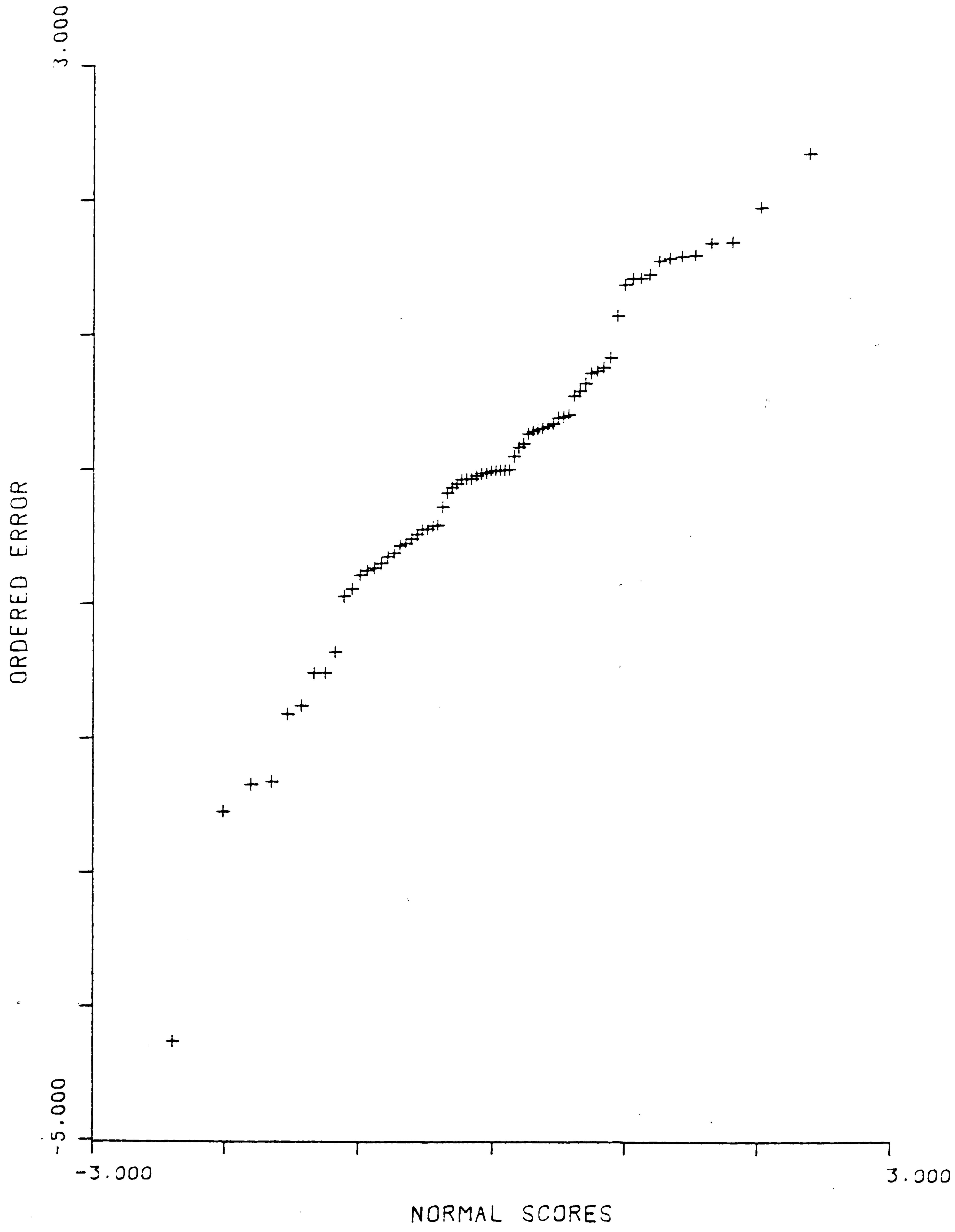


4TH ORDER EXP MODEL



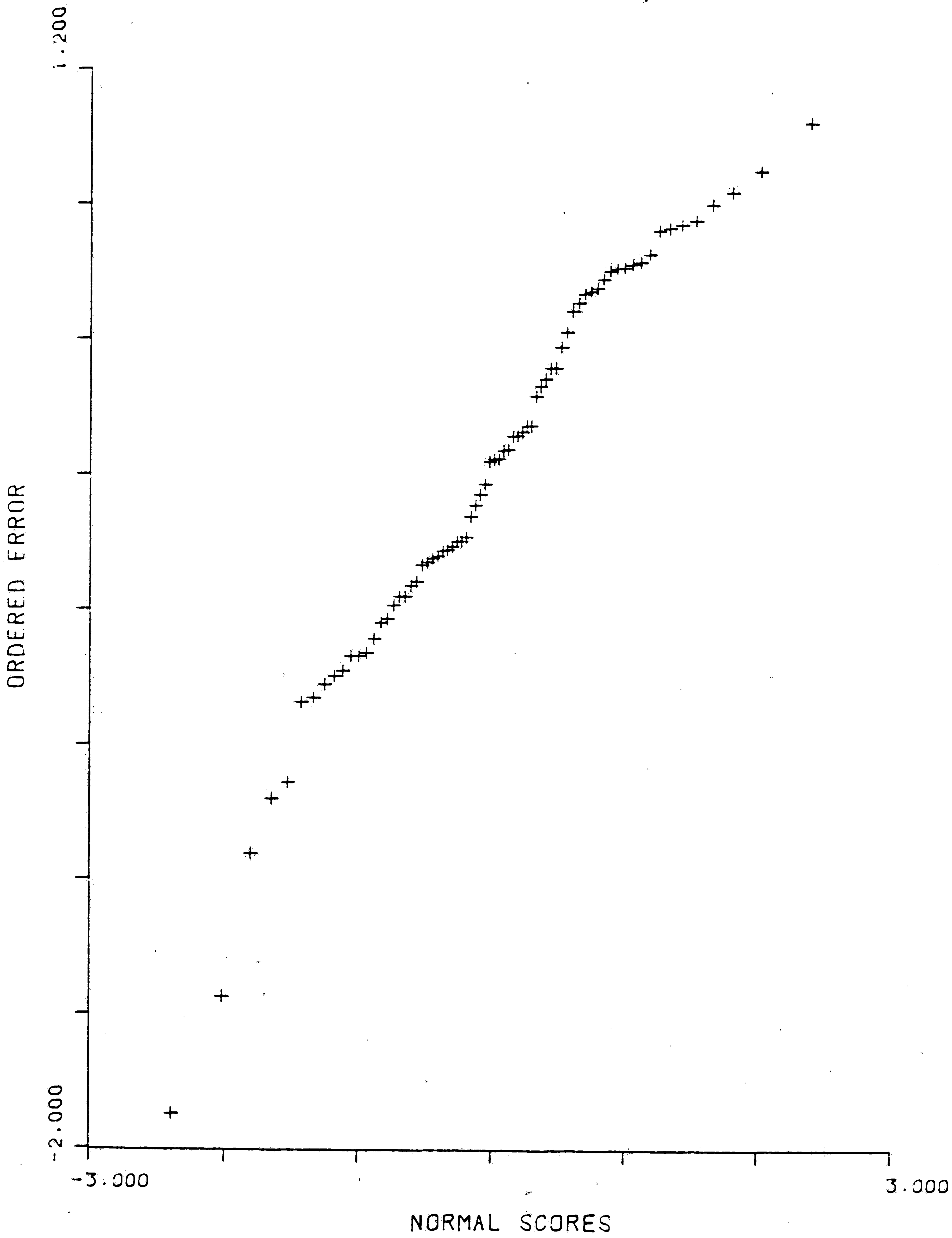
ANGLE VS ERROR L7

4TH ORDER EXP MODEL



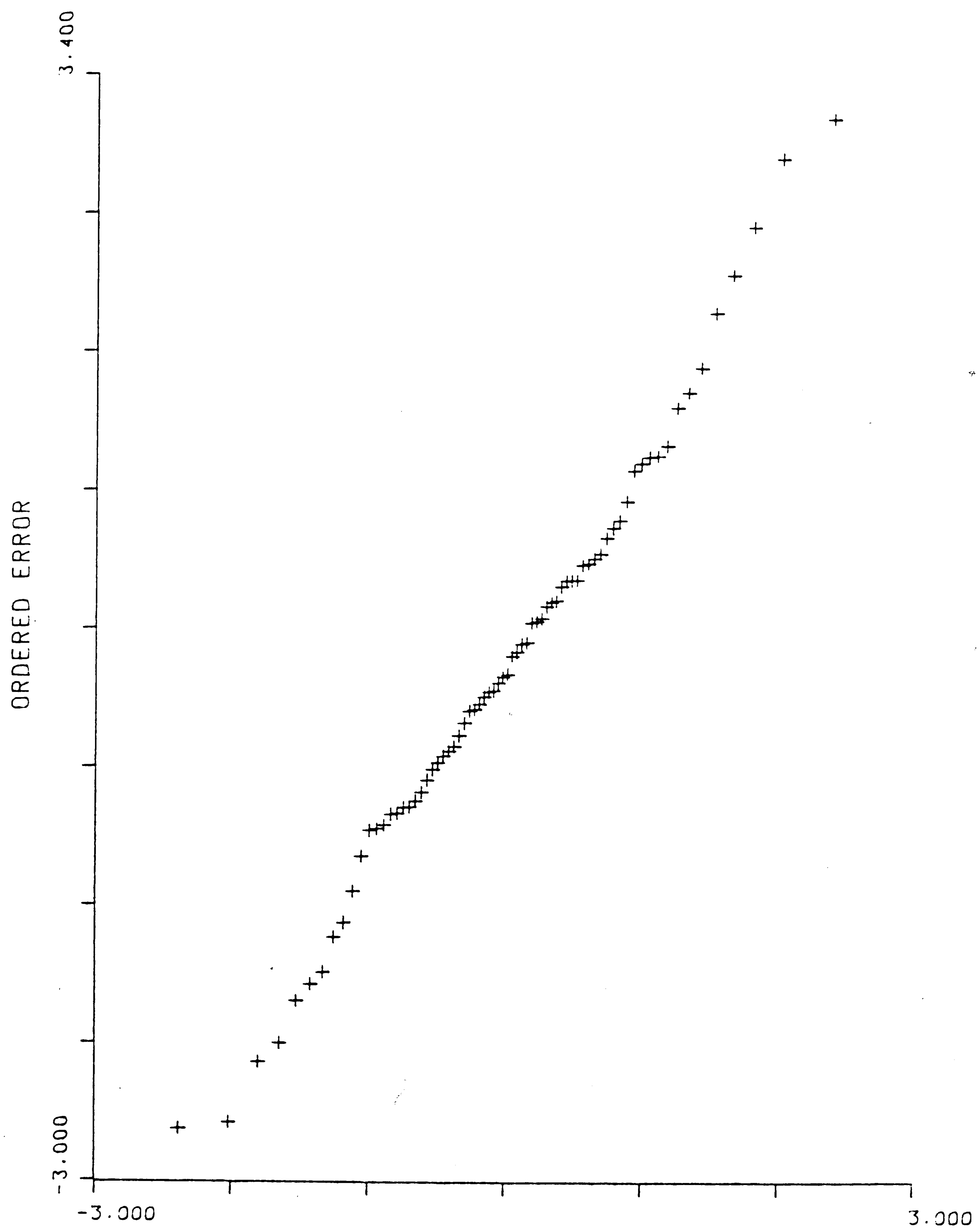
NORMAL PROBABILITY PLOT L1

4TH ORDER EXP MODEL



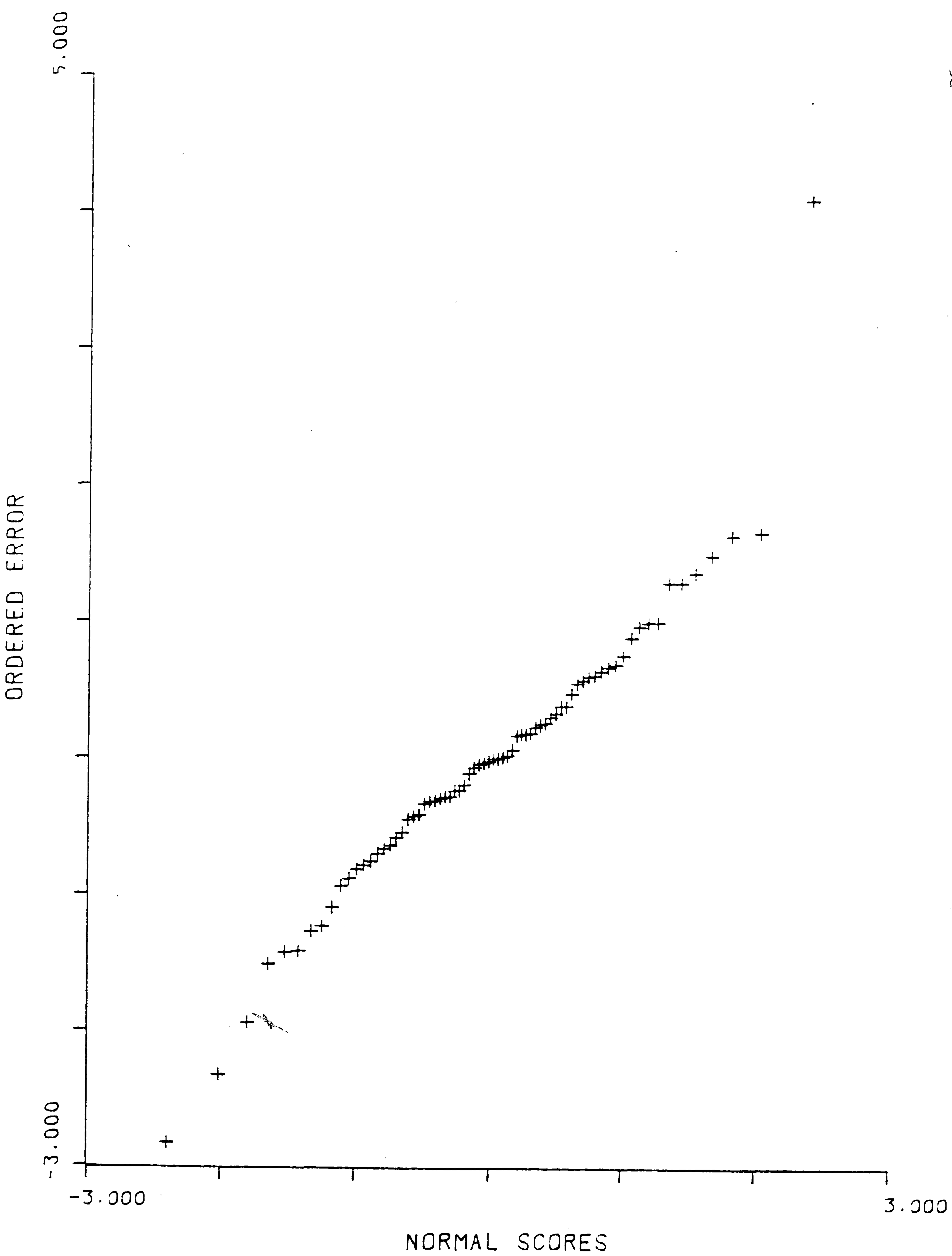
NORMAL PROBABILITY PLOT L7

4TH ORDER EXP MODEL



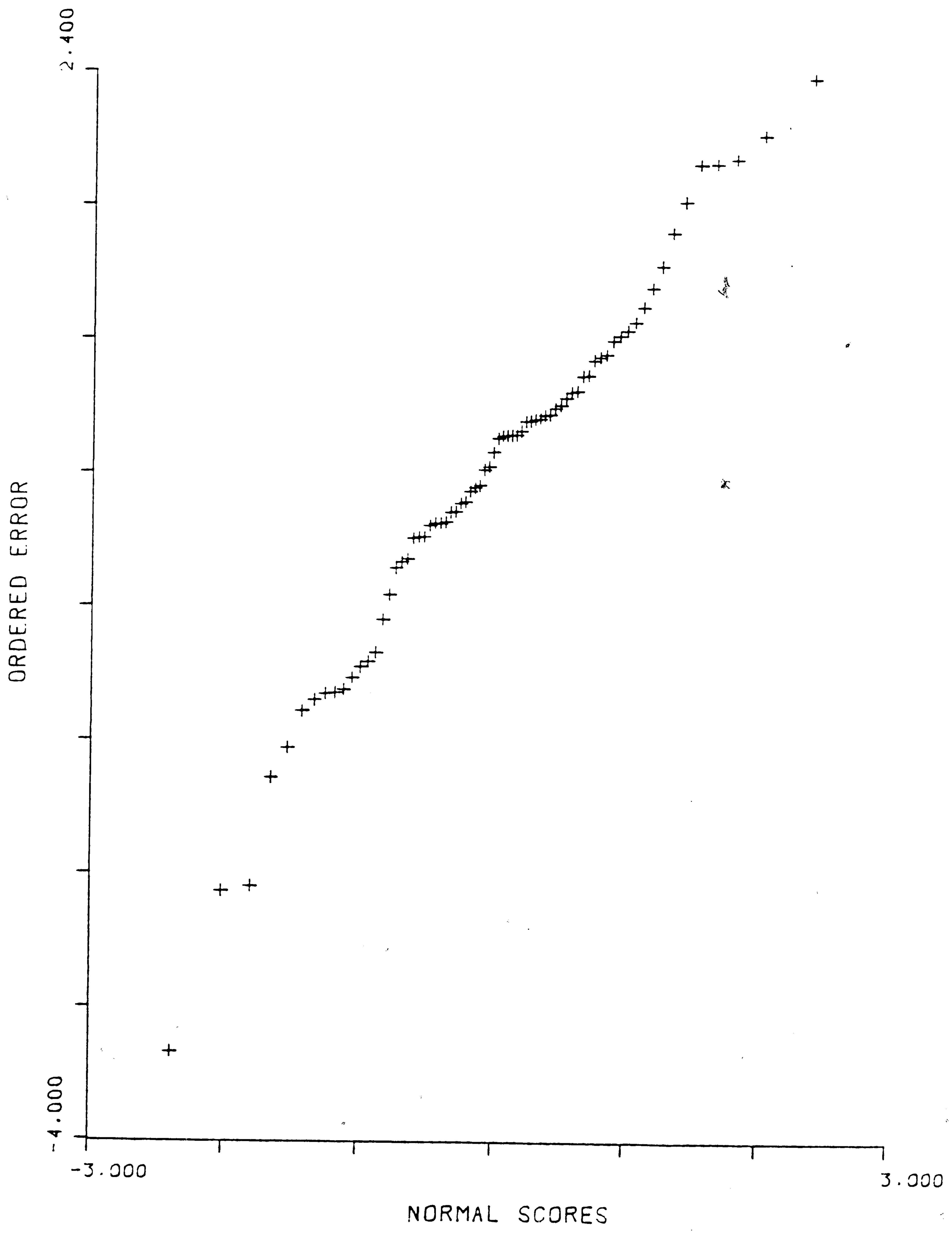
NORMAL PROBABILITY PLOT H16

4TH ORDER EXP MODEL



NORMAL PROBABILITY PLOT H17

4TH ORDER EXP MODEL



NORMAL PROBABILITY PLOT H18

4TH ORDER EXP MODEL

Vita

Jack M. Kloeber Jr., the son of Jack M. and Mary D. Kloeber, was born in Kingston, Pennsylvania in 1955. In 1977 he received a Bachelor of Science degree in Industrial Engineering from Lehigh University. Since 1977 he has served as an active duty officer in the Field Artillery Branch of the United States Army. He has held the rank of Captain since 1981. He will receive the degree of Master of Science in Industrial Engineering from Lehigh University in June 1988. His follow-on assignment in the United States Army will be as a Mathematics and Operations Research Instructor at the United States Military Academy at West Point, New York.