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A brief introduction to some aspects of petri net theory /

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A Brief Introduction to Some Aspects of Petri Net Theory

by

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Table of Contents:

Abstract

The following paper is intended to be an intoduction to petri net theory. The definitions of petri nets, coverability trees and coverability graphs are covered, along with the basic properties of same. It is shown that petri nets with regular firing languages exist and that it is decidable whether or not the firing language of a given petri net is regular.

Section 1 - Preliminary Results:

Segment 1.0 - Introduction:

The first section of this paper contains the definitions and theorems upon which our introduction to petri net theory is based. Its contents should be familiar to most readers, however the reader should be conversant with the specific statements of the definitions and theorems within before proceeding to the second section. For those who are not familiar with the theorems which follow, explicit proofs have been provided. Since this material is preliminary to the main thrust of this paper, the following definitions and results are listed with little or no comment.

Segment 1.1 - Graph Theory and Trees:

This segment contains the basic definitions from graph theory which we will require in section two. It also contains a definition of trees as a subclass of directed graphs. We prove that the standard properties of trees hold for our definition.

Def 1.1.1 General Graph (gg):

A general graph G is a system consisting of:

1) a non-empty set V of objects called vertices,
2) a set E of objects called edges,
3) a function μ defined on E with values consisting of

subsets of V having one or two elements.

We write $G = (V,E,\mu)$ to represent a gg.

We say that a gg is finite iff V and E are finite.

If $e \in E$, $v,v' \in V$ and $\mu(e) = \{v,v'\}$, we call v and v' the *end points* of e.

Note: We avoid the usual notation and do not insist that the end points determine the edges.

Fig 1-1 Some General Graphs:

Def 1.1.2 Connects:

Let $G = (V,E,\mu)$ be a gg,

$v,v' \in V$,

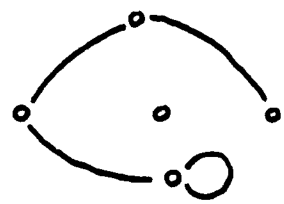
$e \in E$.

If $\mu(e) = \{v,v'\}$, then we say that e *connects* v and v'.

Def 1.1.3 Graph:

Let $G = (V,E,\mu)$ be a gg.

Then G is said to be a *graph* iff $\forall\, e,e' \in E,\ \mu(e) = \mu(e') \Longrightarrow e = e'$.



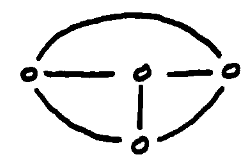


Fig 1-2 Some Graphs:

<u>Def 1.1.4 Path</u>:

A <u>path</u> in a gg $G = (V,E,\mu)$ is a sequence

$$\pi = v_0 e_1 v_1 \ldots v_{k-1} e_k v_k, \quad k \in N,$$

where $v_i \in V \; \forall i \in \{0,\ldots,k\}$,

$e_i \in E \; \forall i \in \{1,\ldots,k\}$,

e_i connects v_{i-1} and $v_i \; \forall i \in \{1,\ldots,k\}$.

v_0 is said to be the <u>initial vertex</u> or <u>initial point</u> of π.

v_k is said to be the <u>final vertex</u> or <u>final point</u> of π.

Given π, we define

$$\pi^R = v_k e_k v_{k-1} \ldots v_1 e_1 v_0.$$

Thus the initial vertex of π is the final vertex of π^R and vice versa.

If π and π' are paths in some gg such that

$$\pi = v_0 e_1 v_1 \ldots v_{k-1} e_k v_k, \quad k \in N,$$

$$\pi' = v'_0 e'_1 v'_1 \ldots v'_{j-1} e'_j v'_j, \quad j \in N,$$

and $v_k = v'_0$, (i.e. The final vertex of π equals the initial vertex of π'.)

Then we define $\pi\pi'$ as follows:

$$\pi\pi' = v_0 e_1 v_1 \ldots v_{k-1} e_k v_k e'_1 v'_1 \ldots v'_{j-1} e'_j v'_j.$$

Note that the initial vertex of π is the initial vertex of $\pi\pi'$ and that the final vertex of π' is the final vertex of $\pi\pi'$.

Observe that:

$$(\pi\pi')^R = \pi'^R\pi^R.$$

Finally, if $\pi\pi'$ is defined and $\pi'\pi''$ is defined, then so are $(\pi\pi')\pi''$ and $\pi(\pi'\pi'')$, and further,

$$(\pi\pi')\pi'' = \pi(\pi'\pi'').$$

Def 1.1.5 Length of a Path:

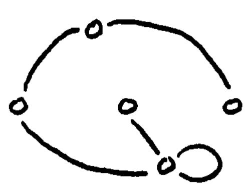
Let $G = (V,E,\mu)$ be a gg,

$\pi = v_0e_1v_1\ldots v_{k-1}e_kv_k$, $k \in N$, be a path in G.

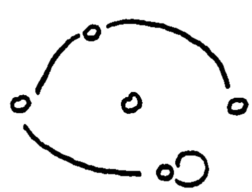
Then π is said to have length k. In other words, the length of π is equal to the number of edges in π.

Def 1.1.6 Connected:

A graph, general or otherwise, is said to be connected iff $\forall\, v,v' \in V$, $v \neq v'$, $\exists$ a path π such that v is the initial vertex of π and v' is the final vertex.



A Connected Graph:



An Un-connected Graph:

Fig 1-3:

Def 1.1.7 Bipartite General Graph:

A gg $G = (V,E,\mu)$ is said to be bipartite iff $\exists\, V',V'' \subset V$ such that:

1) $V = V' \cup V''$,

2) $V' \cap V'' = \emptyset$,

3) $V' \neq \emptyset$; $V'' \neq \emptyset$,

4) $\forall e \in E$, $\mu(e) \cap V' \neq \emptyset$,

$\mu(e) \cap V'' \neq \emptyset$.

<u>Def 1.1.8 General Directed Graph (gdg)</u>:

A <u>general directed graph</u> is a system consisting of:

1) a non-empty set V of objects called vertices,

2) a set E of objects called edges,

3) two functions $\iota, \varphi: E \dashrightarrow V$. Given $e \in E$, $\iota(e)$ is said to be the <u>initial vertex</u> of e and $\varphi(e)$ is said to be the <u>final vertex</u> of e.

We write $D = (V, E, \iota, \varphi)$ to represent a gdg.

We say that a gdg is finite iff V and E are finite.

Note that every gdg $D = (V, E, \iota, \varphi)$ defines a gg $G(D) = (V, E, \mu)$ where:

$$\mu(e) = \{\iota(e), \varphi(e)\} \quad \forall e \in E.$$

We call G(D) the gg associated with D.

Fig 1-4 Some General Directed Graphs:

Def 1.1.8 Directed Path in a General Directed Graph:

A directed path in a gdg $D = (V,E,\tau,\varphi)$ is a sequence

$$\Delta = v_0 e_1 v_1 \ldots v_{k-1} e_k v_k, \quad k \in N,$$

where $v_i \in V \quad \forall i \in \{0,\ldots,k\}$,

$e_i \in V \quad \forall i \in \{1,\ldots,k\}$ and

$\tau(e_i) = v_{i-1};\ \varphi(e_i) = v_i \quad \forall i \in \{1,\ldots,k\}$.

Note that Δ defines a path in G(D) and that a path in G(D) need not define a directed path in D.

We define the length of Δ to equal the length of the path in G(D) defined by Δ.

Def 1.1.10 Loop:

A path, directed or otherwise, is said to be a loop iff the initial vertex is also the final vertex.

A loop is said to be simple iff no vertex other that the initial & final vertex occurs more than once in the loop.

A path, directed or otherwise, is said to contain a loop iff one vertex occurs more than once.

Def 1.1.11 Directed Graph (dg):

Let $D = (V,E,\tau,\varphi)$ be a gdg.

If $\forall e,e' \in E$, $((\tau(e) = \tau(e')$ and $\varphi(e) = \varphi(e')) \implies e = e')$

Then D is said to be a directed graph.

Note that if we redefine E to be the set of ordered pairs

$$E = \{(\tau(e),\varphi(e)) \mid e \in E\},$$

we can write:

$D = (V,E)$

to fully describe D.

Fig 1-5 Some Directed Graphs:

Def 1.1.12 Bipartite General Directed Graph:

A gdg D is bipartite iff G(D) is a bipartite gg.

Fig 1-6 Some Bipartite General Directed Graphs:

Def 1.1.13 $^{\bullet}v$ and $v^{\bullet}$:

Let $D = (V,E)$ be a dg,

$v \in V$.

Then $^{\bullet}v = \{v' \mid v' \in V,\ (v',v) \in E\}$ and

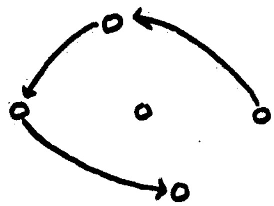
$v^{\bullet} = \{v' \mid v' \in V,\ (v,v') \in E\}$.

Def 1.1.14 Pure:

Let $D = (V,E)$ be a dg.

Then D is said to be pure iff for all $v, v' \in V$,

$(v,v') \in V \Longrightarrow (v',v) \notin V.$



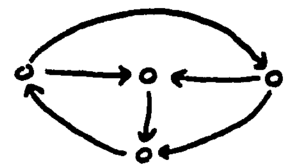


Fig 1-7 Some Pure Directed Graphs:

Def 1.1.15 Tree:

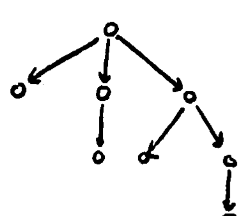
Let $T = (V,E,\dot{\tau},\varphi)$ be a directed graph with the following properties:

1) a unique vertex $r \in V$, called the root vertex, with the following properties:

   a) $\nexists e \in E \ni \varphi(e) = r$. i.e. no edge enters r.

   b) $v \in V \Longrightarrow \exists \Delta = v_0 e_1 v_1 \ldots v_{k-1} e_k v_k$, $k \in \mathbb{N}$, in T such that $v_0 = r$ and $v_k = v$.

2) $v \in V \setminus \{r\} \Longrightarrow |{}^{\cdot}v| = 1$. i.e. $\forall v \in V \setminus \{r\}$ $\exists$ one and only one $e \in E$ such that $\varphi(e) = v$.



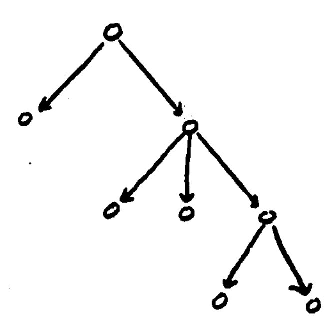


Fig 1-8 Some Trees:

Thm 1.1.16:

Let $T = (V,E,\tau,\varphi)$ be a tree.

Then there is no directed path Δ in T such that Δ is a loop.

Pf: by contradiction

Suppose Δ is a loop in T.

Let $v \in V$ be a vertex in Δ.

Since T is a tree, there exists a root vertex $r \in V$ and a path Δ' such that:

$$\Delta' = v_0 e_1 v_1 \ldots v_{k-1}, e_k v_k, \quad k \in N,$$

$$v_0 = r \quad \text{and}$$

$$v_k = v.$$

Since each vertex in $V \setminus \{r\}$ has one and only one edge entering it, Δ must contain Δ'.

But r has no edges entering it.

Thus, while r must be the initial vertex of Δ, it cannot be the final vertex of Δ.

Hence Δ is not a loop.

Thm 1.1.17:

Let $T = (V,E,\tau,\varphi)$ be a tree,

$v, v' \in V$ and

$\Delta = v_0 e_1 v_1 \ldots v_{k-1} e_k v_k$, $k \in N$, $v_0 = v$, $v_k = v'$ be a directed path in T.

Then Δ is unique.

Pf: by contradiction

Suppose Δ is not unique.

Then there exists $\Delta' = v'_0 e'_1 v'_1 \ldots v'_{j-1} e'_j v'_j$, $j \in N$, such that:

$v'_0 = v_0 = v$,

$v'_j = v_k = v'$ and

either $(j = k)$ or $(\exists i \in \{1,\ldots,k\} \ni v_i \neq v'_i)$.

But this implies that there exists some vertex $v'' \in V$ which has two edges entering it.

But, by def of tree, this cannot occur.

Thus Δ is unique.

<u>Def 1.1.18 Parent, Child, Sibling and Leaf</u>:

Let $T = (V,E,\tau,\varphi)$ be a tree,

$v,v',v'' \in V$; $e',e'' \in E$ such that

$\tau(e') = \tau(e'') = v$,

$\varphi(e') = v'$ and

$\varphi(e'') = v''$.

Then v is said to be the <u>parent</u> of both v' and v". Likewise, both v' and v" are said to be <u>children</u> of v. v' and v" are said to be <u>siblings</u>. Further, if $\nexists e \in E \ni \tau(e) = v'$, then v' is said to be a <u>leaf</u>.

<u>Def 1.1.19 Depth(v)</u>:

Let $T = (V,E,\tau,\varphi)$ be a tree,

$v \in V$.

Then the <u>depth</u> of v, written Depth(v), is defined to be the length of the directed path Δ such that r is the initial vertex of Δ and v is the final vertex. Since Δ is unique, so is Depth(v).

Def 1.1.20 Finitely Branching:

Let $T = (V,E,\dot{\tau},\varphi)$ be a tree,

$E_v = \{e \mid e \in E, \dot{\tau}(e) = v\} \; \forall v \in V.$

Then $|E_v| < \infty \; \forall v \in V$ <==> T is finitely branching.

Def 1.1.21 Infinite:

Let $T = (V,E,\dot{\tau},\varphi)$ be a tree.

Then $|V| = \infty$ <==> T is infinite.

Def 1.1.22 Subtree:

Let $T = (V,E,\dot{\tau},\varphi)$ be a tree,

$x \in V.$

Define the subtree $T_x = (V_x, E_x, \dot{\tau}_x, \varphi_x)$ as follows:

$V_x = \{v \mid v \in V, \exists$ a directed path Δ in T $\ni$ x is the initial vertex of Δ and v is the final vertex$\}$,

$E_x = \{e \mid e \in E, \exists v_x, v'_x \in V_x \ni \dot{\tau}(e) = v_x$ and $\varphi(e) = v'_x\}$,

$\dot{\tau}_x = \dot{\tau}$ restricted to E_x,

$\varphi_x = \varphi$ restricted to E_x and

x is defined to be the root vertex of T_x.

Note that by virtue of its definition, T_x fulfills the definition of a tree. Specifically:

1) no edge enters x,

2) by def of V_x, $\forall v_x \in V_x$, $\exists$ a directed path in T_x with initial vertex x and final vertex v_x,

3) $E_x \subseteq E$ and the definition of E_x above together imply that for all $v_x \in V_x \setminus \{x\}$, there exists a unique $e_x \in E_x$ such that $\varphi(e_x) = v_x$.

<u>Thm 1.1.23 Konig's Lemma</u>:

If $T = (V,E,\dot{\tau},\varphi)$ is a finitely branching, infinite tree, Then T contains an infinite path.

Pf: We construct such a path via the following induction.

Base step:

Let $v_0 = r$.

Then the subtree $T_v = T$, and hence is both finitely branching and infinite.

Let $\Delta_0 = v_0$ be a directed path in T of length zero.

Note that Δ_0 has initial vertex r and final vertex v_0.

Induction step:

Suppose that for $i \in N$, $i \geq 0$ we have found a finitely branching, infinite subtree $T_{v_i} = (V_{v_i}, E_{v_i}, \dot{\tau}_{v_i}, \varphi_{v_i})$ in T and a directed path

$$\Delta_i = v_0 e_1 v_1 \ldots v_{i-1} e_i v_i, \quad v_0 = r,$$

also in T.

Since T_{v_i} is finitely branching, v_i must have a finite number of children.

Define $C_{v_i} \subseteq V_{v_i}$ to be the set of children of v_i:

$$C_{v_i} = \{c_1, c_2, \ldots, c_j\}, \quad j \in N.$$

Since T_{v_i} is infinite, $\exists\, k \in N$, $1 \leq k \leq j \ni$ the subtree T_{c_k} is a finitely branching, infinite tree.

Define $v_{i+1} = c_k$.

Thus $T_{v_{i+1}} = T_{c_k}$.

By def of tree, $\exists\, e_{i+1} \in E_{v_i} \ni \tau(e_{i+1}) = v_i$ and $\varphi(e_{i+1}) = v_{i+1}$.

Thus $\Delta_{i+1} = \Delta_i e_{i+1} v_{i+1} = v_0 e_1 v_1 \ldots v_i e_{i+1} v_{i+1}$ is a directed path with initial vertex $v_0 = r$ and final vertex v_{i+1}.

By the above induction, Δ_i can be defined for arbitrarily large i. Hence T contains an infinite path.

Segment 1.2 - Language Theory:

This segment contains the basic definitions and theorems from language theory which we will require in the second segment of section 2. The reader should pay particular attention to the definition of the finite recognition automaton and its relation to regular languages and right linear grammars.

<u>Def 2.1.1 String</u>:

Let $A \neq \emptyset$ be a set,

$\alpha = a_1 a_2 \ldots$ be a sequence, finite or infinite, of elements of A.

Then α is said to be a <u>string</u> of elements of A.

Note that if $\alpha = a_1 a_2 \ldots a_n$, $n \in \mathbf{N}$, is a finite string, then α is said to have <u>length</u> n.

<u>Def 1.2.2 Nul String, Positive Closure and Closure:</u>

Let $A \neq \emptyset$ be a set,

$n \in \mathbf{N}$.

For $n > 0$, define A^n to be the set of strings of elements of A of length n.

Define:

1) Λ to be the string of zero length and call it the <u>nul string</u> or the <u>empty string</u>.
2) $A^0 = \{\Lambda\}$.
3) $A^+ = \bigcup_{n=1}^{\infty} A^n$, the set of all non-empty strings of elements of A, to be the <u>positive closure</u> of A.
4) $A^* = \bigcup_{n=0}^{\infty} A^n = A^+ \cup \{\Lambda\}$, the set of all strings of elements of A, to be the <u>closure</u> of A.

Note that by definition, $\emptyset^0 = \{\Lambda\}$ and $\emptyset^n = \emptyset$ for $n > 0$.

Thus $\emptyset^* = \{\Lambda\}$ and $\emptyset^+ = \emptyset$.

<u>Def 1.2.3 Concatination:</u>

Let $A \neq \emptyset$ be a set.

$\alpha, \beta \in A^*$,

$\alpha = a_1 \ldots a_m$, $m \in \mathbf{N}$,

$\beta = b_1 \ldots b_n, \quad n \in N.$

Then $\alpha\beta = a_1 \ldots a_m b_1 \ldots b_n$ is said to be the concatination of α and β.

Note that $\Lambda\alpha = \alpha = \alpha\Lambda$.

Further, if $\gamma \in A^*$,

$(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

Concatination is also defined for sets of strings:

Let $B, C \subseteq A^*$.

Then $BC = \{\beta\gamma | \beta \in B, \gamma \in C\}$.

<u>Def 1.2.4 Regular Expression</u>:

Let $A \neq \emptyset$ be a set.

Define the set of <u>regular expressions</u> on A as follows:

1) $\emptyset$ is a regular expression on A.
2) Λ is a regular expression on A.
3) If $a \in A$, then a is a regular expression on A.
4) If r and r' are regular expressions on A, then so are (rr') and $(r \cup r')$. Note that $(r \cup r')$ is frequently written r|r' or r+r'.
5) If r is a regular expression on A, then so is r*.

<u>Def 1.1.5 Regular Language</u>:

Let $A \neq \emptyset$ be a set,

r,r' be regular expressions on A.

Then r defines a <u>regular language</u> $L(r) \subseteq A^*$ as follows:

1) $L(\emptyset) = \emptyset$.

2) $L(\Lambda) = \{\Lambda\}$.

3) If $a \in A$, then $L(a) = \{a\}$.

4) $L((rr')) = L(r)L(r')$.

5) $L((r \vee r')) = L(r) \cup L(r')$.

6) $L(r^*) = L(r)^*$.

<u>Def 1.2.6 Length of a Regular Expression</u>:

Let $A \neq \emptyset$ be a set,

r, r' be regular expressions on A.

Then the <u>length</u> of the regular expression r on A, written $\bar{l}(r)$, is defined as follows:

$\bar{l}(\emptyset) = 1$,

$\bar{l}(\Lambda) = 1$,

$\bar{l}(a) = 1 \quad \forall\, a \in A$,

$\bar{l}((rr')) = \bar{l}(r) + \bar{l}(r') + 2$,

$\bar{l}((r \vee r') = \bar{l}(r) + \bar{l}(r') + 3$,

$\bar{l}(r^*) = \bar{l}(r) + 1$.

D<u>Def 1.2.7 Finite Recognition Automaton</u>:

A <u>finite recognition automaton</u> is a system consisting of:

1) a gdg $D = (V, E, \tau, \varphi)$, where both V and E are finite,

2) a set A,

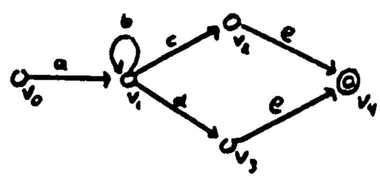
3) an A-labeling $l: E \dashrightarrow A^*$,

4) two subsets $S, F \subseteq V \ni S = \{v_0\}$, $v_0 \in V$; $F \neq \emptyset$.

We write

$\mathcal{a} = (D,A,l.S,F)$

to denote a finite recognition automaton. S and F are called the start and finish sets respectively



$A = \{a,b,c,d,e\}$

$V = \{v_0,\ldots,v_4\}$

$S = \{v_0\}$

$F = \{v_4\}$

Fig 1-9 A Finite Recognition Automaton Recognizing

$(a(b^*((c \cup d)e)))$

Def 1.2.8 Admissable Path:

Let $\mathcal{a} = (D.A.l.S,F)$ be a finite recognition automaton.

An admissable path in $\mathcal{a}$ is a directed path in D with initial vertex in S and final vertex in F.

Def 1.2.9 Language Recognized by a Finite Recognition Automaton:

Let $\mathcal{a} = (D,A,l,S,F)$, $D = (V,E,\iota,\phi)$, be a finite recognition automaton,

$\Delta = v_0 e_1 v_1 \ldots v_{k-1} e_k v_k$, $k \in \mathbb{N}$, be a directed path in D.

Define $l(\Delta) = l(e_1)l(e_2)\ldots l(e_k)$. Note that $l(\Delta) \in A^*$.

The language recognized by $\mathcal{a}$, written $L(\mathcal{a})$, is defined as follows:

$L(\mathcal{a}) = \{l(\Delta) \mid \Delta \text{ is an admissable path in } \mathcal{a}\}$.

Thm 1.2.10:

Let A be a finite set,

$L \subseteq A^*$.

Then L is a regular language iff there exists a finite recognition automaton $\mathcal{A}$ with an A-labeling such that $L(\mathcal{A}) = L$.

Pf: (==>) by construction

Suppose L is a regular language.

Then there exists a regular expression r on A such that

$L(r) = L$.

We now proceed by induction on the length of r to construct a finite recognition automaton $\mathcal{A}_r$ to recognize L.

Base step:

Suppose $\bar{l}(r) = 1$.

Then by definition of $\bar{l}(r)$, r must be equal to either $\emptyset$, or a, where $a \in A$. For each case we construct $\mathcal{A}_r$ as follows:

$r = \emptyset$: $\mathcal{A}_r$: 

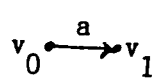
$S = \{v_0\}$ $F = \{v_1\}$

$L(\mathcal{A}_r) = \emptyset = L(r)$.

r = : $\mathcal{A}_r$: 

$S = \{v_0\}$ $F = \{v_1\}$

$L(\mathcal{A}_r) = \{ \} = L(r)$.

r = a: $\mathcal{A}_r$: 

$S = \{v_0\}$ $F = \{v_1\}$

$L(\mathcal{A}_r) = \{a\} = L(r)$.

Thus for all regular expressions r on A of length 1, we can construct a finite recognition automaton recognizing L(r).

Induction Step:

Suppose that for any regular expression on A of length less than k, $k \in \mathbb{N}$, $k > 1$, we can construct a finite recognition automaton recognizing it. Further suppose that $\bar{l}(r) = k$.

Then r must be of one of the forms (pq), $(p \cup q)$ or p* where p and q are regular expressions on A.

By definition of length of a regular expression,

$$\bar{l}(p) < k.$$

Likewise, if $r \neq p^*$,

$$\bar{l}(q) < k$$

as well.

Thus, by the induction hypothesis, we can construct a finite recognition automaton

$$\mathcal{A}_p = (D_p, A, 1_p, S_p, F_p), \quad D_p = (V_p, E_p, \tau_p, \varphi_p),$$

such that $L(\mathcal{A}_p) = L(p)$. As above if $r \neq p^*$, we can also construct a finite recognition automaton

$$\mathcal{A}_q = (D_q, A, 1_q, S_q, F_q), \quad D_q = (V_q, E_q, \tau_q, \varphi_q),$$

such that $L(\mathcal{A}_q) = L(q)$. Further, we can choose $\mathcal{A}_p$ and $\mathcal{A}_q$ such that they have no vertices in common and no edges connecting them.

We now consider the above three cases individually:

1) r = (pq):

We form the finite recognition automaton

$$\mathcal{A}_r = (D_r, A, 1_r, S_r, F_r), \quad D_r = (V_r, E_r, \tau_r, \varphi_r),$$

as follows:

Let $V_r = V_p \cup V_q$.

We form E_r, τ_r and φ_r as follows:

Initially, let $E_r = E_p \cup E_q$,

$$\tau_r(e) = \begin{cases} \tau_p(e) & \forall e \in E_p \\ \tau_q(e) & \forall e \in E_q, \end{cases}$$

$$\varphi_r(e) = \begin{cases} \varphi_p(e) & \forall e \in E_p \\ \varphi_q(e) & \forall e \in E_q. \end{cases}$$

We then expand E_r, τ_r and φ_r as follows:

For each $v_f \in F_p$, we introduce a new edge

$e_{v_f} \in E_r$ such that

$\tau_r(e_{v_f}) = v_f$ and

$\varphi_r(e_{v_f}) = v_s$, $v_s \in S_q$.

We complete our definition of α_r with the following:

Let

$$l_r(e) = \begin{cases} l_p(e) & \text{if } e \in E_p \\ l_q(e) & \text{if } e \in E_q \\ \Lambda & \text{otherwise,} \end{cases}$$

$S_r = S_p$ and

$F_r = F_q$.

Having defined α_r, we must now show that

$L(\alpha_r) = L(\alpha_p)L(\alpha_q) = L(p)L(q) = L(r)$.

Suppose $\omega \in L(\alpha_r)$.

Then we can find an admissable path Δ in α_r such that

$\Delta = v_0 e_1 v_1 \ldots v_{i-1} e_i v_i$, $i \in N$,

where $\{v_0\} = S_r$,

$v_i \in F_r$ and

$l_r(\Delta) = l_r(e_1) l_r(e_2) \ldots l_r(e_i) = \omega.$

Let j be the least integer such that $1 \leq j \leq i$ and

$v_j \in V_q$.

Since $v_0 \in S_p$ and $v_i \in F_q$, j must exist.

Consider the edge e_j:

$\tau_r(e_j) = v_{j-1} \in V_p.$

By construction of $\mathcal{a}_r$, the only edges which can have both vertices not in the same set V_p or V_q are the edges e_{v_f}, where $v_f \in F_p$.

Thus $e_j = e_{v_f}$ where $v_f \in F_p$.

Hence $v_{j-1} = v_f \in F_p$,

$\{v_j\} = S_q$ and

$l_r(e_j) = \Lambda.$

Since the only edges connecting a vertex in V_p with a vertex in V_q are the edges $e_{v_f} \in E_r$, $v_f \in F_p$, and no edge in E_r connects a vertex in V_q to a vertex in V_p, it follows that:

$v_0, v_1, \ldots, v_{j-1} \in V_p$ and

$v_j, v_{j+1}, \ldots, v_i \in V_q$.

Further, since the $e_{v_f} \in E_r$, $v_f \in F_p$, are the only edges not in $E_p \cup E_q$, we have that

$\Delta_1 = v_0 e_1 v_1 \ldots v_{j-2} e_{j-1} v_{j-1}$ and

$\Delta_2 = v_j e_{j+1} v_{j+1} \ldots v_{i-1} e_i v_i$

are admissable paths in $\mathcal{a}_p$ and $\mathcal{a}_q$ respectively.

Thus $\Delta = \Delta_1 e_j \Delta_2$ and

$$l_r(\Delta) = l_p(\Delta_1) \Lambda l_q(\Delta_2) = l_p(\Delta_1) l_q(\Delta_2) = \omega.$$

Hence $\omega \in L(\mathcal{A}_p)L(\mathcal{A}_q)$.

Therefore $L(\mathcal{A}_r) \subseteq L(\mathcal{A}_p)L(\mathcal{A}_q)$.

Now suppose $\xi \in L(\mathcal{A}_p)L(\mathcal{A}_q)$,

$$\xi = \Theta_1 \Theta_2$$

where $\Theta_1 \in L(\mathcal{A}_p)$ and

$\Theta_2 \in L(\mathcal{A}_q)$.

Thus there exist admissable paths Δ_1 and Δ_2 in $\mathcal{A}_p$ and $\mathcal{A}_q$ respectively, such that

$l_p(\Delta_1) = \Theta_1$ and

$l_q(\Delta_2) = \Theta_2$.

By definition of an admissable path, Δ_1 must have its final vertex v_f in F_p. Further, the initial vertex of Δ_2 must be an element of S_q.

By construction of $\mathcal{A}_r$, there exists an edge $e_{v_f} \in E_r$ connecting the final vertex of Δ_1 to the initial vertex of Δ_2 such that $l_r(e_{v_f}) = \Lambda$.

Since $S_r = S_p$ and $F_r = F_q$, the directed path

$$\Delta = \Delta_1 e_{v_f} \Delta_2$$

is an admissable path in $\mathcal{A}_r$ such that

$$l_r(\Delta) = l_p(\Delta_1)\ l_q(\Delta_2) = \Theta_1 \Theta_2 = \xi.$$

Hence $l_r(\Delta) = \xi \in L(\mathcal{A}_r)$.

Therefore $L(\mathcal{A}_p)L(\mathcal{A}_q) \subseteq L(\mathcal{A}_r)$.

Combining the above with the previous result, we obtain:

$$L(\mathcal{A}_r) = L(\mathcal{A}_p)L(\mathcal{A}_q) = L(p)L(q) = L(r).$$

2) $r = (p \cup q)$:

We form the finite recognition automaton

$$\mathcal{A}_r = (D_r, A, I_r, S_r, F_r), \quad D_r = (V_r, E_r, \dot{\tau}_r, \varphi_r),$$

as follows:

Let $V_r = V_p \cup V_q \cup \{v_i, v_f\}$,

where $(V_p \cup V_q) \cap \{v_i, v_f\} = \emptyset$

and $v_i \neq v_f$.

We form E_r, $\dot{\tau}_r$ and φ_r as follows:

Initially let $E_r = E_p \cup E_q$,

$$\dot{\tau}_r(e) = \begin{cases} \dot{\tau}_p(e) & \text{if } e \in E_p \\ \dot{\tau}_q(e) & \text{if } e \in E_q, \end{cases}$$

$$\varphi_r(e) = \begin{cases} \varphi_p(e) & \text{if } e \in E_p \\ \varphi_q(e) & \text{if } e \in E_q. \end{cases}$$

We then expand E_r, $\dot{\tau}_r$ and φ_r as follows:

For each $v \in S_p \cup S_q$ we introduce a new edge e_v to E_r such that:

$\dot{\tau}_r(e_v) = v_i$ and

$\varphi_r(e_v) = v$.

For each $v \in F_p \cup F_q$ we introduce a new edge e_v to E_r such that:

$\dot{\tau}_r(e_v) = v$ and

$\varphi_r(e_v) = v_f$.

We complete our definition of $\mathcal{A}_r$ with the following:

Let

$$l_r(e) = \begin{cases} l_p(e) & \text{if } e \in E_p \\ l_q(e) & \text{if } e \in E_q \\ \Lambda & \text{otherwise,} \end{cases}$$

$$S_r = \{v_i\} \quad \text{and}$$

$$F_r = \{v_f\}.$$

Having constructed $\mathcal{A}_r$, we must now show that

$$L(\mathcal{A}_r) = L(\mathcal{A}_p) \cup L(\mathcal{A}_q) = L(p) \cup L(q) = L(r).$$

Suppose $\omega \in L(\mathcal{A}_r)$.

Then we can find an admissable path Δ in $\mathcal{A}_r$ such that

$$\Delta = v_0 e_1 v_1 \ldots v_{j-1} e_j v_j, \quad j \in \mathbb{N},$$

where $l_r(\Delta) = \omega$,

$$v_0 = v_i \quad \text{and}$$

$$v_j = v_f.$$

By definition of $\mathcal{A}_r$, $v_1 \in S_p \cup S_q$ and $v_{j-1} \in F_p \cup F_q$.

Since $S_p \cap S_q = \emptyset$ and $F_p \cap F_q = \emptyset$, v_1 and v_{j-1} must be in either V_p or V_q, not both.

Since there are no paths in $\mathcal{A}_r$ connecting a vertex of V_p with one of V_q, or vice versa, $v_1, v_2, \ldots v_{j-1}$ must all be in either V_p or V_q, not both. Further, $e_2, e_3, \ldots, e_{j-1}$ must all be in either E_p or E_q, not both, since the only edges in E_r which connect two edges of V_p or V_q are in E_p and E_q respectively.

Thus $\Delta' = v_1 e_2 v_2 \ldots v_{j-2} e_{j-1} v_{j-1}$ is an admissable path in either $\mathcal{A}_p$ or $\mathcal{A}_q$.

Since $\Delta = v_0 e_1 \Delta' e_j v_j$ and

$$l_r(\Delta) = l_r(e_l)l_r(\Delta')l_r(e_j) = \Lambda l_r(\ ')\Delta = \omega,$$

We have that either

$$\omega \in L(\alpha_p) \text{ or } \omega \in L(\alpha_q).$$

Thus $\omega \in L(\alpha_p) \cup L(\alpha_q)$.

Therefore $L(\alpha_r) \subseteq L(\alpha_p) \cup L(\alpha_q)$.

Now suppose $\xi \in L(\alpha_p) \cup L(\alpha_q)$.

Thus we can find an addmissable path Δ' in either α_p or α_q such that:

$$l(\Delta') = \xi.$$

Without loss of generality, assume that Δ' is an admissable path in α_p.

Then Δ' has initial vertex $v_{i_p} \in S_p$ and final vertex $v_{f_p} \in F_p$.

By definition of α_r, there exists an edge $e_i \in E_r$ such that:

$$\tau_r(e_i) = v_i,$$
$$\varphi_r(e_i) = v_{i_p} \quad \text{and}$$
$$l_r(e_i) = \Lambda.$$

Likewise, there exists an edge $e_f \in E_r$ such that:

$$\tau_r(e_f) = v_{f_p},$$
$$\varphi_r(e_f) = v_f \quad \text{and}$$
$$l_r(e_f) = \Lambda.$$

Thus we can define the admissable path

$$\Delta = v_i e_i \Delta' e_f v_f$$

in α_r where $l_r(\Delta) = l(e_i)l_p(\Delta')l_r(e_f)$

$$= \Lambda 1_p(\Delta')\Lambda = 1_p(\Delta') = \mathfrak{Z}.$$

Since Δ is an admissable path in $\mathcal{A}_r$, it follows that

$$\mathfrak{Z} = 1_p(\Delta') = 1_r(\Delta) \in L(\mathcal{A}_r).$$

Thus $L(\mathcal{A}_p) \cup L(\mathcal{A}_q) \subseteq L(\ _r)$.

Combining the above with the previous result, we obtain:

$$L(\mathcal{A}_r) = L(\mathcal{A}_p) \cup L(\mathcal{A}_q) = L(p) \cup L(q) = L(r).$$

3) $r = p^*$:

We form the finite recognition automaton

$$\mathcal{A}_r = (D_r, A, 1_r, S_r, F_r), \quad D_r = (V_r, E_r, \tau_r, \varphi_r),$$

as follows:

Let $V_r = V_p \cup \{v_i, v_f\}$ where $V_p \cap \{v_i, v_f\} = \emptyset$ and $v_i \neq v_f$.

We form E_r, τ_r and φ_r as follows:

Initially let $E_r = E_p$,

$\tau_r = \tau_p$ and

$\varphi_r = \varphi_p$.

We then expand E_r, τ_r and φ_r as follows:

For each $v \in F_p$ we introduce a new edge $e_f \in E_r$ such that:

$\tau_r(e_f) = v$ and

$\varphi_r(e_f) = v_f$.

For each $v \in S_p$ we introduce a new edge $e_s \in E_r$ such that:

$\tau_r(e_s) = v_i$ and

$\varphi_r(e_s) = v$.

We introduce two new edges e_n and e_r to E_r such that:

$$\tau_r(e_n) = v_i \qquad \varphi_r(e_n) = v_f \quad \text{and}$$
$$\tau_r(e_r) = v_f \qquad \varphi_r(e_r) = v_i.$$

We complete our definition of $\mathcal{A}_r$ as follows:

Let
$$l_r(e) = \begin{cases} l_p(e) & \text{if } e \in E_p \\ \Lambda & \text{otherwise,} \end{cases}$$

$S_r = \{v_i\}$ and

$F_r = \{v_r\}$.

Having constructed $\mathcal{A}_r$, we must show that

$$L(\mathcal{A}_r) = L(\mathcal{A}_p)^* = L(p)^* = L(r).$$

Suppose $\omega \in L(\mathcal{A}_r)$.

If $\omega = \Lambda$,

Then, since Λ is an element of the closure of any set,

$$\omega = \Lambda \in L(\mathcal{A}_p)^*.$$

Suppose $\omega \neq \Lambda$.

Then we can find an admissable path Δ in $\mathcal{A}_r$ with no occurences of $e_n \in E_r$ such that:

$$\Delta = v_0 e_1 v_1 \ldots v_{k-1} e_k v_k, \quad k \in N, \quad \text{and}$$
$$l_r(\Delta) = \omega \in L(\mathcal{A}_r).$$

By definition of $\mathcal{A}_r$ and Δ,

$v_0 = v_i$,

$v_1 \in S_p$,

$v_{k-1} \in F_p$ and

$v_k = v_f$.

Since $\mathcal{A}_r$ was constructed around $\mathcal{A}_p$, and since v_i and v_f are connected to $\mathcal{A}_p$ only via S_p and F_p respectively, Δ must contain one or more subpaths $\Delta_1, \Delta_2, \ldots, \Delta_n$, $n \in \mathbb{N}$, which are admissable in $\mathcal{A}_p$.

Thus we can rewrite Δ as follows:

$$\Delta = v_i e_s \Delta_1 e_f v_f e_r v_i e_s \Delta_2 \ldots \Delta_n e_s v_f$$

where

$$\begin{aligned} l_r(\Delta) &= \Lambda l_p(\Delta_1)\Lambda\Lambda\Lambda l_p(\Delta_2)\ldots l_p(\Delta_n)\Lambda \\ &= l(\Delta_1)l(\Delta_2)\ldots l(\Delta_n) = \omega. \end{aligned}$$

Since Δ_j is an admissable path in $\mathcal{A}_p$ for all $j \in \{1,\ldots,n\}$,

$$l_p(\Delta_j) \in L(\ _p) \quad \forall\ j \in \{1,\ldots,n\}.$$

Thus $\omega \in L(\mathcal{A}_p)^*$.

Hence $L(\mathcal{A}_r) \subseteq L(\mathcal{A}_p)^*$.

Now suppose $\xi \in L(\mathcal{A}_p)^*$.

If $\xi = \Lambda$,

Then $\Delta = v_i e_n v_f$ is an admissable path in $\mathcal{A}_r$ such that

$$l_r(\Delta) = \Lambda = \xi,$$

and thus $\xi = \Lambda \in L(\mathcal{A}_r)$.

Suppose $\xi \neq \Lambda$.

Then ξ is of the form

$$\xi = \xi_1 \xi_2 \ldots \xi_n, \quad n \in \mathbb{N},$$

where $\xi_j \in L(\mathcal{A}_p)\ \forall\, j \in \{1,\ldots,n\}$

Thus for each ξ_j, $j \in \{1,\ldots,n\}$, we can find an admissable path Δ_j in $\mathcal{A}_p$ such that:

$l_p(\Delta_j) = \mathcal{S}_j$.

Define $\Delta = v_i e_s \Delta_1 e_f v_f e_r v_i e_s \Delta_2 \ldots \Delta_n e_f v_f$.

Since $\mathcal{A}_r$ is an expansion of $\mathcal{A}_p$, for each $j \in \{1,\ldots,n\}$,

Δ_j is a path in D_r with initial vertex $v' \in S_p$ and final vertex $v \in F_p$.

Since $\varphi_r(e_s) = v'$ and

$\tau_r(e_f) = v$,

Δ is an admissable path in $\mathcal{A}_r$.

Thus $l_r(\Delta)$ is defined, and may be written as follows:

$$\begin{aligned} l_r(\Delta) &= l_r(e_s) l_r(\Delta_1) l_r(e_f) l_r(e_r) l_r(e_s) l_r(\Delta_1) \ldots \\ &\quad \ldots l_r(\Delta_n) l_r(e_s) \\ &= \Lambda\, l_r(\Delta_1) \Lambda \Lambda \Lambda\, l_r(\Delta_2) \ldots l_r(\Delta_n) \Lambda \\ &= l_r(\Delta_1) l_r(\Delta_2) \ldots l_r(\Delta_n) \\ &= \mathcal{S}_1 \mathcal{S}_2 \ldots \mathcal{S}_n = \mathcal{S}. \end{aligned}$$

Therefore $\mathcal{S} \in L(\mathcal{A}_r)$ and

$L(\mathcal{A}_p)^* \subseteq L(\mathcal{A}_r)$.

Combining the above with the previous result, we obtain

$$L(\mathcal{A}_r) = L(\mathcal{A}_p)^* = L(p)^* = L(r).$$

This completes both the induction and the first half of the proof.

(<==)

Let $\mathcal{A} = (D, A, l, S, F)$, $D = (V, E, \tau, \varphi)$ be a finite recognition automaton where:

$V = \{v_1, v_2, \ldots, v_r\}$, $r \in N$ and

$S = \{v_1\}$.

We must show that $L(\alpha)$ is a regular language.

Let i,j and k be integers such that $1 \leq i,j \leq r$ and $1 \leq k \leq r+1$.

Define an (i,j,k) path to be a path

$$\Delta = w_0 e_1 w_1 \ldots w_{s-1} e_s v_s, \quad s \in N,$$

in D such that $w_0 = v_i$,

$$w_s = v_j,$$

and for all $p \in \{1,\ldots,s-1\}$,

$$(w_p = v_t) \Longrightarrow (t < k), \quad t \in N.$$

Let $\mathcal{E}^k_{i,j} = \{l(\Delta) \mid \Delta \text{ is an } (i,j,k) \text{ path}\}$.

We now show inductively that $\mathcal{E}^k_{i,j}$ is a regular language.

Base step:

Consider $\mathcal{E}^0_{i,j}$, $1 \leq i,j \leq r$.

Then the associated (i,j,k) paths must be of the form:

$$\Delta = v_i e\, v_j, \quad e \in E.$$

Since E is finite, so is

$$\mathcal{E}^0_{i,j} = \{l(\Delta) \mid \Delta \text{ is an } (i,j,0) \text{ path}\}$$
$$= \{l(e_1), l(e_2), \ldots, l(e_n)\}, \quad n \in N.$$

Thus $\mathcal{E}^0_{i,j}$ is a regular language for all $i,j \in N$, $1 \leq i,j \leq r$.

Induction step:

Suppose that for $k \in N$, $k \geq 0$, and for all $i,j \in N$, $1 \leq i,j \leq r$ we have shown that $\mathcal{E}^k_{i,j}$ is a regular language.

Then $\mathcal{E}^{k+1}_{i,j} = (\mathcal{E}^k_{i,j} \cup (\mathcal{E}^k_{i,k+1}(\mathcal{E}^k_{k+1,k+1}{}^* \, \mathcal{E}^k_{k+1,j})))$

and hence is a regular language.

Let $j_1, j_2, \ldots, j_b \in N$, $b \in N$, be the indices of the vertices in F. i.e. $v_{j_c} \in F \ \forall c \in \{1,\ldots,b\}$.

Then $L(\alpha) = \bigcup_{c=1}^{b} \mathcal{E}_{1,j_c}^{r+1}$.

Hence $L(G)$ is a regular language.

Def 1.2.11 Formal Grammar:

Let $A \neq \emptyset$ be a finite set.

Then a formal grammar G on A is a system consisting of:

1) Two subsets A_n, $A_t \subset A$ such that $A_n \cap A_t = \emptyset$, $A_n \cup A_t = A$, $A_t \neq \emptyset$ and $A_n \neq \emptyset$.

2) A finite set P of ordered pairs (α,β), $\alpha,\beta \in A^*$. We write $\alpha \dashrightarrow \beta$ for (α,β) and call the ordered pair a production.

3) A specific element $S \in A_n$ called the start symbol.

We write $G = (A_n, A_t, P, S)$ to denote a formal grammar.

Def 1.2.12 Derivation:

Let $G = (A_n, A_t, P, S)$ be a formal grammar,

$\gamma_1, \gamma_2 \in A^*$,

$\gamma_1 = \rho\alpha\theta$, $\gamma_2 = \rho\beta\theta$ and

$\alpha \dashrightarrow \beta \in P$.

Then we write $\gamma_1 \dashrightarrow \gamma_2$ call it a single step derivation.

If $\sigma_1, \sigma_2, \ldots, \sigma_n \in A^*$, $n \in N$ and

$\sigma_1 \dashrightarrow \sigma_2 \dashrightarrow \ldots \dashrightarrow \sigma_n$,

Then we can write $\sigma_1 \overset{*}{\dashrightarrow} \sigma_n$ and call $\sigma_1, \sigma_2, \ldots, \sigma_n$ a derivation of σ_n.

Note that for any $\gamma \in A^*$ we can write $\gamma \overset{*}{--->} \gamma$.

Def 1.2.13 Language Generated by a Formal Grammar:

Let $G = (A_n, A_t, P, S)$ be a formal grammar.

Then $L(G) = \{\alpha | \alpha \in A_t^*,\ S \overset{*}{--->} \alpha\}$ is defined to be the language generated by G.

Def 1.2.14 Right Linear and Normalized Right Linear Grammars:

Let $G = (a_n, A_t, P, S)$ be a formal grammar.

If all the productions in P are of one of the forms

$u --> \alpha v$ or

$u --> \alpha$ where $u, v \in A_n$ and $\alpha \in A_t^*$,

Then G is said to be a right linear grammar.

If all the productions in P are of one of the forms

$u --> \alpha v$ or

$u --> \Lambda$ where $u, v \in A_n$ and $\alpha \in A_t^*$

Then G is said to be a normalized right linear grammar.

Lemma 1.2.15:

Let G be a right linear grammar.

Then there exists a normalized right linear grammar G' such that

$L(G) = L(G')$.

Pf: by construction

Let $G = (A_n, A_t, P, S)$.

We construct $G' = (A_n', A_t', P', S')$ as follows:

Let $A_t' = A_t$.

Include in P' each production in P of the form

$x \dashrightarrow \alpha y$ or

$x \dashrightarrow \Lambda$ where $x, y \in A_n$ and $\alpha \in A_t^*$.

For each production in P of the form

$x \dashrightarrow \alpha$, $x \in A_n$, $\alpha \in A_t^+$,

create a new non-terminal $u_{x,\alpha} \in A_n$, and add the productions

$x \dashrightarrow \alpha u_{x,\alpha}$ and

$u_{x,\alpha} \dashrightarrow \Lambda$

to P'.

Let $A_n' = A_n \cup \{u_{x,\alpha} | x \dashrightarrow \alpha \in P, \alpha \in A_t^+\}$ and

$S' = S$.

Having constructed G', we must now show that

$L(G) = L(G')$.

Suppose $\beta \in L(G)$ and

$S = \sigma_0 \dashrightarrow \sigma_1 \dashrightarrow \ldots \dashrightarrow \sigma_k = \beta$, $k \in N$, is a derivation of β in G.

Suppose no production of the form $x \dashrightarrow \alpha$, $x \in A_n$, $\alpha \in A_t^+$, occurs in the derivation of β.

Then all the productions applied in the derivation of β are in P'.

Since $S = S'$, the derivation of β is in G'.

Thus $\beta \in L(G')$.

Now suppose that a production of the form $x \dashrightarrow \alpha$, $x \in A_n$, $\alpha \in A_t^+$, occurs in the derivation of β.

Note that such a production can only occur at the end of the derivation.

Thus $\sigma_{k-1} \text{-->} \sigma_k$ must be of the form

$$\gamma x \text{-->} \gamma\alpha = \beta,$$

where $x \in A_n$, $\alpha, \gamma \in A_t^*$ and $\alpha \neq \Lambda$.

Since the production $x \text{-->} \alpha$ is in P, the following productions must be in P':

$$x \text{-->} \alpha u_{x,\alpha} \quad \text{and}$$

$$u_{x,\alpha} \text{-->} \Lambda \quad \text{where } u_{x,\alpha} \in A'_n.$$

Thus we can replace the last single derivation in the derivation of β in G with the following:

$$\sigma_{k-1} = \gamma x \text{-->} \gamma\alpha u_{x,\alpha} \text{-->} \gamma\alpha = \beta.$$

The resulting derivation of β is in G', and hence

$$\beta \in L(G').$$

Thus $L(G) \subseteq L(G')$.

Now suppose $\beta \in L(G')$.

Then we can find a derivation

$$S' = \sigma_0 \text{-->} \sigma_1 \text{-->} \sigma_2 \text{-->} \ldots \text{-->} \sigma_k = \beta, \quad k \in N,$$

of β in G'.

Note that none of the new non-terminals in $A'_n \setminus A_n$ can appear in $\sigma_0, \ldots, \sigma_{k-2}$.

Hence all the single derivations in

$$S' = \sigma_0 \text{-->} \sigma_1 \text{-->} \sigma_2 \text{-->} \ldots \text{-->} \sigma_{k-2}$$

are single derivations in G.

Since G' is a normalized right linear grammar, the last two single derivations in the derivation of β in G' must be of form

$$\gamma x \dashrightarrow \gamma\alpha u_{x,\alpha} \dashrightarrow \gamma\alpha = \beta,$$

where $x \in A_n$, $x, u_{x,\alpha} \in A'_n$, $\alpha, \gamma \in A_t^*$ and $\alpha \neq \Lambda$.

If $u_{x,\alpha} \in A_n$,

Then both of the above single derivations are in G, and hence β L(G).

Suppose $u_{x,\alpha} \notin A_n$.

Then by construction of G', there must be a production

$$x \dashrightarrow \alpha$$

in P.

We can use this production to replace the last two single derivations in the derivation of β in G' with

$$\sigma_{k-2} = \gamma x \dashrightarrow \gamma\alpha = \beta.$$

and thereby obtain a derivation of β in G.

Hence $\beta \in L(G)$.

Therefore $L(G') \subseteq L(G)$.

Combining the above with the previous result, we obtain

$$L(G') = L(G).$$

Thm 1.2.16

Let G be a formal grammar.

Then G is a right linear grammar iff there exists a finite recognition automaton $\mathcal{Q}$ with an A-labeling such that

$L(\mathcal{A}) = L(G)$.

Pf: (==>) by construction

Suppose $G = (A_n, A_t, P, s)$ is a right linear grammar.

By Lemma 1.2.15, we can assume that G is a normalized right linear grammar, and thus that P contains only productions of the forms

$x \dashrightarrow \alpha y$ or

$x \dashrightarrow \Lambda$

where $x, y \in A_n$ and $\alpha \in A_t^*$.

We construct the finite recognition automaton

$\mathcal{A} = (D, A, l, S, F)$, $D = (V, E, \tau', \phi)$,

as follows:

Let $V = A_n$,

$A = A_t$ and

$S = \{s\}$.

For each production in P of the form

$x \dashrightarrow \alpha y$, $x, y \in A_n$, $\alpha \in A_t^*$,

include in E the edge $e_{x,\alpha,y}$ and define:

$\tau'(e_{x,\alpha,y}) = x$,

$\phi(e_{x,\alpha,y}) = y$ and

$l(e_{x,\alpha,y}) = \alpha$.

For each production in P of the form

$x \dashrightarrow \Lambda$, $x \in A_n$,

include x in F.

Having constructed $\mathcal{A}$, we must now show that

$L(\mathcal{A}) = L(G)$.

Suppose $\alpha \in L(G)$.

Then there exists a derivation

$$s \dashrightarrow \sigma_1 \dashrightarrow \sigma_2 \dashrightarrow \ldots \dashrightarrow \sigma_k \dashrightarrow \alpha, \quad k \in N.$$

Note that each σ_i, $i \in \{1,\ldots,k\}$, must be of the form

$$\alpha_1\alpha_2\ldots\alpha_i x_i, \quad \alpha_1,\alpha_2,\ldots,\alpha_i \in A_t^*, \quad x \in A_n,$$

and that for each σ_i, $i \in \{2,\ldots,k\}$, the production

$$x_{i-1} \dashrightarrow \alpha_i x_i \in P$$

is applied to obtain it from σ_{i-1}. Further σ_1 is obtained from s via the production

$$s \dashrightarrow \alpha_1 x_1 \in P$$

and α is obtained from σ_k though the production

$$x_k \dashrightarrow \Lambda \in P.$$

Thus, by construction of $\mathcal{A}$, we can construct the path

$$\Delta = s\ e_{s,\alpha_1,x_1}\ x_1\ e_{x_1,\alpha_2,x_2}\ x_2 \ldots x_{k-1}\ e_{x_{k-1},\alpha_k,x_k}\ x_k$$

in D.

By construction of ,

$$x_k \dashrightarrow \Lambda \in P \implies x_k \in F.$$

Also $S = \{s\}$.

Thus Δ is an admissable path in $\mathcal{A}$.

Since $l(\Delta) = l(e_{s,\alpha_1,x_1})l(e_{x_1,\alpha_2,x_2}) \ldots l(e_{x_{k-1},\alpha_k,x_k})$
$= \alpha_1\alpha_2\ldots\alpha_k = \alpha,$

we have that $\alpha \in L(\mathcal{A})$.

Therefore $L(G) \subseteq L(\mathcal{A})$.

Now suppose $\alpha \in L(\mathcal{A})$.

Then there exists an admissable path

$$\Delta = s\ e_{s,\alpha_1,x_1}\ x_1\ e_{x_1,\alpha_2,x_2}\ x_2 \cdots x_{k-1}\ e_{x_{k-1},\alpha_k,x_k}\ x_k,$$
$k \in N$,

in $\mathcal{A}$, such that

$$l(\Delta) = l(e_{s,\alpha_1,x_1})l(e_{x_1,\alpha_2,x_2}) \cdots l(e_{x_{k-1},\alpha_k,x_k})$$
$$= \alpha_1\alpha_2 \cdots \alpha_k = \alpha,$$

where $\alpha_1,\alpha_2,\ldots,\alpha_k \in A_t^*$.

By definition of E, for each e_{x_{i-1},α_i,x_i}, $i \in \{2,\ldots,K\}$, there exists a production

$$x_{i-1} \dashrightarrow \alpha_i x_i,\quad x_{i-1},x_i \in A_n,\quad \alpha_i \in A_t^*,$$

in P. Likewise, P contains the productions:

$s \dashrightarrow \alpha_1 x_1$ and

$x_k \dashrightarrow \Lambda$, $s,x_1,x_k \in A_n$, $\alpha_1 \in A_t^*$.

Thus we can construct the derivation

$$s \dashrightarrow \alpha_1 x_1 \dashrightarrow \alpha_1\alpha_2 x_2 \dashrightarrow \ldots \dashrightarrow \alpha_1\alpha_2\ldots\alpha_{k-1}x_{k-1} \dashrightarrow \alpha_1\alpha_2\ldots\alpha_k x_k$$
$$\dashrightarrow \alpha_1\alpha_2\ldots\alpha_k = \alpha.$$

Since this derivation is in G, it follows that

$\alpha \in L(G)$.

Therefore $L(\mathcal{A}) \subseteq L(G)$.

Combining the above with the previous result, we obtain

$L(\mathcal{A}) = L(G)$.

(==>) by construction

Let $\mathcal{A} = (D,A,l,S,F)$, $D = (V,E,\dot{r},\varphi)$ be a finite recognition automaton.

We construct $G = (A_n,A_t,P,s)$ as follows:

Let $A_t = A$,

$A_n = V$ and

$s = v_s$ where $\{v_s\} = S$.

For each $v \in F$, place the production

$v \dashrightarrow \Lambda$

in P.

For each $e \in E$, place the production

$\tau(e) \dashrightarrow l(e)\varphi(e)$

in P. Note that $\tau(e), \varphi(e) \in V = A_n$ and $l(e) \in A^* = A_t^*$.

Having constructed G, we must now show that

$L(G) = L(\mathcal{a})$.

Suppose $\alpha \in L(\mathcal{a})$.

Then there exists an admissable path

$\Delta = v_0 e_1 v_1 \ldots v_{k-1} e_k v_k, \quad k \in N,$

in $\mathcal{a}$ where $l(\Delta) = l(e_1)l(e_2)\ldots l(e_k)$

$= \alpha_1 \alpha_2 \ldots \alpha_k = \alpha.$

By construction of G, for each e_i, $i \in \{1,\ldots,k\}$, the production

$\tau(e_i) \dashrightarrow l(e_i)\varphi(e_i)$,

which can also be written

$v_{i-1} \dashrightarrow \alpha_i v_i, \quad v_{i-1}, v_i \in A_n, \quad \alpha_i \in A_t^*,$

is in P. Likewise, since $v_k \in F$, the production

$v_k \dashrightarrow \Lambda$

is also in P.

Thus we can construct the derivation:

$$v_0 \to \alpha_1 v_1 \to \alpha_1\alpha_2 v_2 \to \ldots \to \alpha_1\ldots\alpha_{k-1}v_{k-1} \to \alpha_1\ldots\alpha_k v_k$$
$$\to \alpha_1\ldots\alpha_k = \alpha.$$

Since Δ is an admissable path in $\mathcal{a}$, $v_0 = s$.

Thus we have constructed a derivation of α in G.

Hence $\alpha \in L(G)$.

Therefore $L(\mathcal{a}) \subseteq L(G)$.

Now suppose $\alpha \in L(G)$.

Then we can find a derivation of α in G as follows:

$$s = v_0 \to \alpha_1 v_1 \to \alpha_1\alpha_2 v_2 \to \ldots \to \alpha_1\ldots\alpha_{k-1}v_{k-1} \to \alpha_1\ldots\alpha_k v_k$$
$$\to \alpha_1\ldots\alpha_k = \alpha, \quad k \in \mathbb{N}.$$

By construction of G, for each production of the form

$$v_{i-1} \to \alpha_i v_i, \quad i \in \{1,\ldots,k\}, \quad v_{i-1}, v_i \in A_n, \ \alpha_i \in A_t^*,$$

in P, we can find an edge $e_i \in E$ such that

$\tau(e_i) = v_{i-1}$,

$l(e_i) = {}_i$ and

$\varphi(e_i) = v_i$.

Thus we can construct the path Δ in $\mathcal{a}$ such that

$$\Delta = v_0 e_1 v_1 \ldots v_{k-1} e_k v_k.$$

Since $v_0 = s$, $\{v_0\} = S$.

Since the production

$$v_k \to \Lambda$$

is in P, $v_k \in F$.

Thus Δ is an admissable path in $\mathcal{a}$.

Note that $l(\Delta) = l(e_1)l(e_2)\ldots l(e_k)$

$$= \alpha_1\alpha_2\ldots\alpha_k = \alpha.$$

Thus $\alpha \in L(\mathcal{G})$.

Therefore $L(G) \subseteq L(\mathcal{G})$.

Combining the above with the previous result, we obtain

$$L(G) = L(\mathcal{G}).$$

<u>Thm 1.2.17</u>:

Let $G = (A_n, A_t, P, S)$ be a formal grammar.

Then G is a right linear iff there exists a regular expression r on A_t such that

$$L(G) = L(r).$$

Pf:

Follows directly from Thms 1.2.10 & 1.2.16.

<u>Def 1.2.18 Pref( )</u>:

Let A be a set,

$L \subseteq A^*$ be a language on A.

Then $\mathrm{Pref}(L) = \{\alpha \mid \alpha, \beta \in A^*, \alpha\beta = \gamma \in L\}$.

Note: If $\alpha, \beta \in A^*$, $\alpha\beta = \gamma \in L$, we can write $\alpha \subseteq \gamma$.

<u>Thm 1.2.19</u>:

If L is a regular language on a set A,

Then so is Pref(L).

Pf: by construction

Since L is a regular language, by Thm 1.2.10 there exists a finite recognition automaton

$\mathcal{A}_L = (D,A,1,S,F)$, $D = (V,E,\tau,\varphi)$,

such that $L(\mathcal{A}_L) = L$.

We construct the finite recognition automaton $\mathcal{A}_P$ in two stages as follows:

1) We examine D to find all vertices $v \in V \setminus F$ such that there exists no edge $e \in E$ such that $\tau(e) = v$. Since D is finite, we can do this.

Since any path including such a v must end with it and thus not be an admissable path, we can remove v from V and all edges $e \in E$ such that $\varphi(e) = v$ without changing the language recognized by the automaton.

We do so, and repeat the process until there are no such vertices remaining. Call the result $\mathcal{A}_L'$,

$$\mathcal{A}_L' = (D',A,1',S,F), \quad D' = (V',E',\tau',\varphi).$$

Note that $L = L(\mathcal{A}_L) = L(\mathcal{A}_L')$. More importantly, note that given any path Δ_1 in D' with initial vertex in S, we can find a second path Δ_2 in D' such that $\Delta_1\Delta_2$ is defined and is an admissable path in $\mathcal{A}_L'$.

2) We form $\mathcal{A}_P$ from $\mathcal{A}_L'$ by setting F equal to the set of all vertices in D', thus making every vertex a final vertex. Hence

$$\mathcal{A}_P = (D',A,1',S,V'), \quad D' = (V',E',\tau',\varphi).$$

Having constructed $\mathcal{A}_P$, we must now show that

$$\text{Pref}(L) = L(\mathcal{A}_P).$$

Suppose $\alpha \in \text{Pref}(L)$.

Then there exists a $\beta \in A^*$ and a $\gamma \in L$ such that

$$\alpha\beta = \gamma.$$

Hence there exists an admissable path Δ in $\mathcal{A}_L'$ such that $l(\Delta) = \gamma$, and further, there exist two paths Δ_1 and Δ_2 in $\mathcal{A}_L'$ such that

$\Delta_1\Delta_2 = \Delta$,

$l(\Delta_1) = \alpha$ and

$l(\Delta_2) = \beta$.

Since Δ_1 starts in S and since each vertex in V' is a final vertex in $\mathcal{A}_P$, Δ_1 is an admissable path in $\mathcal{A}_P$.

Thus $\alpha \in L(\mathcal{A}_P)$.

Therefore $Pref(L) \subseteq L(\mathcal{A}_P)$.

Now suppose $\alpha \in L(\ _P)$.

Then there exists an admissable path Δ_1 in $\mathcal{A}_P$ such that

$\Delta_1 = v_0e_1v_1 \ldots v_{k-1}e_kv_k$, $k \in N$,

where $v_0 \in S$ and

$l(\Delta_1) = \alpha$.

As a result of the pruning process we used to obtain $\mathcal{A}_L'$, we can find a path Δ_2 in D' such that

$\Delta_2 = v_ke_{k+1}v_{k+1} \ldots v_{k+j-1}e_{k+j}v_{k+j}$, $j \in N$,

$v_{k+j} \in F$ and

$l(\Delta_2) = \beta$.

Note that $\Delta_1\Delta_2$ is defined in D'

Recall that both $\mathcal{A}_P$ and $\mathcal{A}_L'$ use D'.

Since $v_0 \in S$ and $v_{k+j} \in F$, $\Delta_1\Delta_2$ is an admissable path in $\mathcal{A}_L'$.

Hence $\gamma = \alpha\beta = 1(\delta_1\delta_2) \subseteq L(a_L') = L(a_L) = L$.

Thus $\alpha \in \mathrm{Pref}(L)$.

Therefore $L(a_p) \subseteq \mathrm{Pref}(L)$.

Combining the above with the previous result, we obtain

$$L(a_p) = \mathrm{Pref}(L).$$

Hence, if L is a regular language, so is Pref(L).

Segment 1.3 - Zorn's Lemma:

Zorn's Lemma and the three definitions given in this segment are used repeatedly in section two.

Def 1.3.1 N_∞:

Define $N_\infty = N \cup \{\infty\}$.

Def 1.3.2 Inequality and Incomparable:

Let $X \neq \emptyset$ be a finite set.

$Q, Q' \in N_\infty^{|X|}$.

Define $Q \leq Q' \iff Q(x) \leq Q'(x) \quad \forall x \in X$,

$Q < Q' \iff (Q \leq Q') \wedge (\exists x \in X \ni Q(x) < Q'(x))$,

$Q = Q' \iff (Q \leq Q') \wedge (Q' \leq Q) \iff Q(x) = Q'(x) \ \forall x \in X$.

We say that Q and Q' are incomparable iff neither $Q \leq Q'$ nor $Q' \leq Q$ holds. For example, in N_∞^2, (1,2) and (2,1) are incomparable.

Def 1.2.3 Pairwise Incomparable:

Let $X \neq \emptyset$ be a finite set,

$A \quad N_{\infty}^{|X|}$ be a set.

Then we say that A is a set of pairwise incomparable elements iff

for all $a, a' \in A$, a and a' are incomparable.

Thm 1.3.4 Zorn's Lemma:

Let $X \neq \emptyset$ be a finite set,

$A \subseteq N_{\infty}^{|X|}$ be a set of pairwise incomparable elements.

Then A is finite.

Pf: by contradiction

Suppose A is infinite.

Let $S = \{s \mid s \subsetneq X\}$.

Since X is finite, so is S.

For each $s \in S$, define

$$A_s = \{a \mid a \in A, (a(x) = \infty) \Longleftrightarrow (x \in s)\}.$$

Note that $A = \bigcup_{s \in S} A_s$.

Thus, there exists $s' \in S$ such that $A_{s'}$ is infinite.

Consider the following induction:

Base step:

Let $B_0 = A_{s'}$ and

$e_0 = s'$.

Note that $b_0(x) = b_0'(x) \; \forall \; b_0, b_0' \in B_0, \quad x \in e_0$,

$b_0 < \infty, \quad \forall \; b_0 \in B_0, \quad x \in X \setminus e_0$ and

$B_0 \subseteq N_{\infty}^{|X|}$ is an infinite set of pairwise incomparable elements.

Induction step:

Suppose that for $i \in N$, $i \geq 0$, we have shown that $B_i \in N_\infty^{|X|}$ is an infinite set of pairwise incomparable elements such that

$$b_i(x) = b_i'(x) \quad \forall\, b_i, b_i' \in B_i, \quad x \in e_i \quad \text{and}$$

$$b_i(x) < \infty \quad \forall b_i \in B_i, \quad x \in X \setminus e_i.$$

We construct B_{i+1} and e_{i+1} as follows:

Consider $b_i \in B_i$.

Since B_i is a set of pairwise incomparable elements,

$$\forall\, b_i' \in B_i,\ b_i' \neq b_i,\ \exists\, x \in X \setminus e_i \ni b_i'(x) < b_i(x).$$

Thus, for each $x \in X \setminus e_i$, we can define

$$C_x = \{b_i' \mid b_i' \in B_i,\ b_i'(x) < b_i(x)\},$$

where $B_i = (\bigcup_{x \in X \setminus e_i} C_x) \cup \{b_i\}$.

Since B_i is infinite and $X \setminus e_i$ is not, there exists $x' \in X \setminus e_i$ such that $C_{x'}$ is infinite.

Since $b_i(x') < \infty$ and $|C_{x'}| = \infty$, $\exists\, j \in N$, $0 \leq j < b_i(x')$, such that there are an infinite number of $c \in C_{x'}$ such that $c(x') = j$.

Define $B_{i+1} = \{b_{i+1} \mid b_{i+1} \in C_{x'},\ b_{i+1}(x') = j\}$ and

$$e_{i+1} = e_i \cup \{x'\}.$$

By definition of $C_{x'}$ & j, B_{i+1} is an infinite set of pairwise incomparable elements.

Further, since $B_{i+1} \subseteq B_i$ and $b_{i+1}(x') = j \ \forall\, b_{i+1} \in B_{i+1}$,

$$b_{i+1}(x) = b_{i+1}'(x) \quad \forall\, b_{i+1}, b_{i+1}' \in B_{i+1}, \quad x \in e_{i+1} \quad \text{and}$$

$$b_{i+1}(x) < \infty \ \forall \ b_{i+1} \in B_{i+1}, \quad x \in X \setminus e_{i+1}.$$

Now consider B_k, $k = |X| - |e_0|$.

Since each e_i has one more element than e_{i-1},

$$|e_k| = |X|.$$

Thus $e_k = X$.

Hence $b_k = b_k' \ \forall \ b_k, b_k' \in B_k$.

Since B_k is a set of pairwise incomparable elements,

$$|B_k| = 1.$$

But by the above induction, B_k must be infinite - a contradiction.

Hence A must be finite.

Section 2 - Petri Net Theory:

Segment 2.0 - Introduction:

Having completed the preliminaries, we now proceed with our developement of petri net theory. This section is devided into two segments. The first covers the basic definitions and results concerning petri nets and their related constructs. The second deals with those petri nets whose firing languges are regular.

While the basic thrust of this paper remains theoretical, examples have been included both for clarity and to indicate possible applications.

Segment 2.1 - Basic Definitions and Results:

Before beginning our development of petri net theory, we pause briefly to consider the goal towards which our effort is directed. Generally stated we wish to find some convenient and reasonably intuitive method of describing asynchronous, concurrent processes. In addition, we would like to be able to use this representation to answer such such questions as "Can these two processes deadlock?" or "Is the work space of process A safe from being overwritten by process B?". Finally, we would like to be able to automate much of the above so that we can deal with large systems of interrelated, concurrent processes such as operating systems or the more modern design natural language translators. What follows is an effort in that direction, which makes no assumptions about the hardware in question except for the existence of a hardware arbiter which prevents the simultaneous access of a single memory location by two or more processes. Since this feature is standard, the assumption is not unreasonable.

In our developement of petri net theory, the following two problems will be used repeatedly as examples:

Problem #1 - Dijkstra's Dining Philosophers:

Five philosophers live together. They spend their time either eating or thinking. They eat at a round table with five places. Each philosopher has his own place and will eat at no other. At each place there is a plate of food. Between each plate there is a fork. Each philosopher requires two forks to eat with, and will use only those forks on either side of his plate. A philosopher can only pick up one fork at a time, and once a philosopher picks up a fork, he will not put it down until he has finished eating, at which point he will put each fork back where he found it. No philosopher will eat forever. Design a scheduling algorithm for the dining philosophers such that no philosopher starves.

Problem #2 The Mutual Exclusion Problem:

Suppose an arbitrarily large number of processes share some resource (i.e. a printer). Assume that once a process obtains control of the resource, it will release it eventually. Design a scheduling algorithm with the following properties:

1) At most one process will have control of the resource at any one time.

2) Any process which requests control of the resource will obtain it eventually.

Unfortunately, most of the problems in concurrency require petri nets that are too large for us to deal with in this paper. Thus the above two examples have been chosen as much for brevity as for any other quality.

We now offer the definitions leading up to our definition of the petri net.

Def 2.1.1 Place Transition Graph (P/T Graph):

Let $P = \{p_1, p_2, \ldots, p_m\}$, $m \in \mathbf{N}$, be a set whose elements are called places,

$T = \{t_1, t_2, \ldots, t_n\}$, $n \in \mathbf{N}$, be a set whose elements are called transitions,

$P \cap T = \emptyset$,

$V = P \cup T$ be a set of vertices and

$E \subseteq (P \times T) \cup (T \times V) \subset (V \times V)$ be a set of edges.

Then the resulting bipartite directed graph, written (P,T,E), is said to be a place transition graph.

Def 2.1.2 Edge Multiplicity Function:

Let (P,T,E) be a p/t graph.

Define $W:E \dashrightarrow N$ to be the edge multiplicity function. Unless otherwise stated,

$W(e) = 1 \quad \forall\, e \in E.$

Def 2.1.3 Adjacency Functions:

Let $D = (V,E)$ be a dg,

$v, v' \in V$

$$\eta((v,v')) = \begin{cases} 1 & \text{if } (v,v') \in E, \\ 0 & \text{otherwise,} \end{cases}$$

$W:E \dashrightarrow N$ be defined as in Def 2.1.2.

Then $A:E \dashrightarrow N$, $A((v,v')) = \eta((v,v'))W((v,v'))$ is said to be the adjacency function for D.

Further, if $D = (P,T,E)$ is a p/t graph, then two functions $B,F:(P \times T) \dashrightarrow N$, called the backward adjacency function and the forwards adjacency function respectively, are defined as follows:

$$B((p,t)) = \begin{cases} A((p,t)) & \text{if } (p,t) \in E \\ 0 & \text{otherwise,} \end{cases}$$

$$F((p,t)) = \begin{cases} A((t,v)) & \text{if } (t,v) \in E \\ 0 & \text{otherwise,} \end{cases}$$

where $p \in P$ and $t \in T$.

Note that A,B & F can all be expressed as matrices.

Def 2.1.4 Incidence Function:

Let (P,T,E) be a p/t graph and

B & F be defined.

Then the incidence function $D:(P \times T) \text{-->} N$ is defined as follows:

$$D(p,t) = F(p,t) - B(p,t), \forall p \in P, t \in T.$$

Note that D can also be expressed as a matrix, and in this form, D is called the incidence matrix.

Def 2.1.5 Marking and Token:

Let (P,T,E) be a p/t graph.

Then a marking is a function $M:P \text{-->} N$. Note that M can be written as a column vector.

If for some $p \in P$, $M(p) = n$, $n \in N$, then p is said to contain n tokens in the marking M.

Def 2.1.6 Capacity Function:

Let (P,T,E) be a p/t graph.

Define the capacity function, $K:P \text{-->} N_{\infty}$, to represent the maximum number of tokens which may reside in any given place at any given time.

Thus for any marking M on P and any place $p \in P$, $M(p) \leq K(p)$.

Unless otherwise stated, assume $K(p) = \infty \ \forall p \in P$.

Def 2.1.7 P/T Net:

Let (P,T,E) be a p/t graph,

$B:(P \times T) \dashrightarrow N$ be the backward adjacency function associated with (P,T,E),

$F:(P \times T) \dashrightarrow N$ be the forward adjacency function associated with (P,T,E),

$K:P \dashrightarrow N_\infty$ be a capacity function on P and

$W:E \dashrightarrow N$ be an edge multiplicity function on E.

Then (P,T,B,F,K,W) is said to be a p/t net.

Note that B and F together uniquely define E.

If K is ommitted, assume $K(p) = \infty \ \forall p \in P$.

If W is ommitted, assume $W(e) = 1 \ \forall e \in E$.

Def 2.1.8 Petri Net:

Let (P,T,B,F,K,W) be a p/t net and

M_0 be a marking on P such that $M(p) \leq K(p) \ \forall p \in P$.

Then $N = (P,T,B,F,K,W,M_0)$ is said to be a petri net, and M_0 is called the initial marking of N.

Def 2.1.9 Strict Transition Rule:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$M,M':P \dashrightarrow N$ be markings on P,

$u \subseteq T$, $u \neq \emptyset$ be a set of transitions and

$c:T \dashrightarrow \{0,1\}$ be the characteristic function of u, i.e.

$$c(t) = \begin{cases} 1 & \text{if } t \in u \\ 0 & \text{otherwise.} \end{cases}$$

Then we write M[u>M' iff

1) $M(p) \geq \sum_{t \in u} B((p,t)) \quad \forall\, p \in P$,

2) $M'(p) = M(p) + \sum_{t \in u} D((p,t)) \quad \forall\, p \in P$,

3) $M'(p) \leq K(p) \quad \forall\, p \in P$ and

4) $\forall\, t,t' \in u \ni t \neq t'$, $({}^{\bullet}t \cup t^{\bullet}) \cap ({}^{\bullet}t' \cup t'^{\bullet}) = \emptyset$. i.e. No two transitions in u can involve the same place.

Note that if we view M,M',K and c as column vectors, B and D as matrices and the relations and operations X + Y, X - Y, X = Y and $X \leq Y$ componentwise, we can rewrite 1), 2) and 3) above as follows:

1) $M \geq B \cdot c$,

2) $M' = M + D \cdot c$ and

3) $M' \leq K$.

Unless otherwise stated, this notation will be used hence forth.

If M[u>M' holds, then we say that u is a set of <u>concurrently fireable transitions</u> with respect to M according to the <u>strict transition rule</u>.

Note that M[Λ>M always holds.

We say M[->M' iff there exists $u \in T$ such that M[u>M'.

Define [=> to denote the reflexive, transitive closure of the relation [->.

Let $M_0[u_1>M_1$, $M_1[u_2>M_2, \ldots, M_{n-1}[u_n>M_n$, $n \in \mathbf{N}$, all hold.

Then we may write $M_0[u_1,u_2,\ldots,u_n>M_n$, and if c_i is the characteristic function of $u_i \; \forall\, i \in \{1,\ldots,n\}$, then

$M_n = M_0 + D \cdot \sum_{i=1}^{n} c_i$.

Further, if $u_i = \{t_i\}$ $\forall i \in \{1,\ldots,n\}$, we ommit the brackets and write

$$M_0[t_1t_2\ldots t_n>M_n$$

for short. The word $w = t_1t_2\ldots t_n \in T^*$ is said to be a <u>firing sequence</u> which leads from M_0 to M_n.

For all $w \in T^*$, $M:P\text{-->}\mathbb{N}$, we write $M[w>$ iff there exists an $M':P\text{-->}\mathbb{N}$ such that $M[w>M'$.

Finally, if $w = t_1t_2\ldots t_n \in T^*$, $n \in \mathbb{N}$, is a firing sequence, we define

$$(D\cdot w)(p) = \sum_{i=1}^{n} D(p,t_i) \quad \forall\, p \in P.$$

The reader should note that the pairs of transitions shown in Fig. 2-1 cannot be fired concurrently under the strict transition rule. Recall that if R is a relation on a set S, $a,b,c \in S$, R is said to be reflexive iff aRa holds for all $a \in S$. R is said to be symmetric iff aRb <==> bRa for all $a,b \in S$. Also R is said to be transitive iff aRb and bRc ==> aRc. Finally the closure of R in S is defined to be the set C defined as follows:

$$C = \{a \mid C \subseteq A,\ a,a' \in C \implies ((aRa') \text{ or } (a'Ra))\}.$$

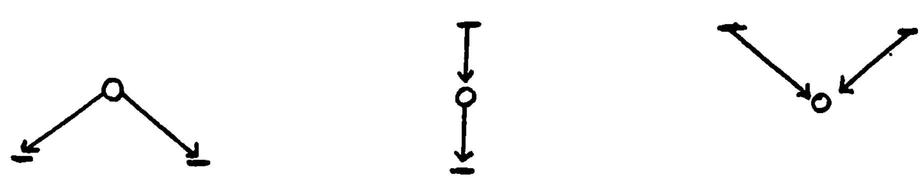


Fig 2-1 Pairs of transitions which cannot be fired concurrently under the strict transition rule:

In the above figure, and all others which follow, we represent places with circles and transitions with bars or lines.

Def 2.1.10 Weak Transition Rule:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$M,M':P \to N$ be markings,

$u \subseteq T$, $u \neq \emptyset$ be a set of transitions and

c be the characterisic function of u, i.e.

$$c(t) = \begin{cases} 1 & \text{if } t \in u \\ 0 & \text{otherwise.} \end{cases}$$

Then we write M(u>M' iff

1) $M \geq B \cdot c$,

2) $M' = M + D \cdot c$.

Note that the matrix notation defined in Def 2.1.9 is use here.

If M(u>M' holds, we say that u is a set of concurrently fireable transitions with respect to M according to the weak transition rule.

Note that M(Λ>M always holds.

We say that M(->M' iff there exists $u \in T$ such that M(u>M' holds.

Define (=> to denote the reflexive, transitive closure of the relation (->.

Let $M_0(u_1>M_1$, $M_1(u_2>M_2, \ldots, M_{n-1}(u_n>M_n$, $n \in N$, all hold.

Then we write $M_0(u_1,u_2,\ldots,u_n>M_n$, and if c_i is the characteristic function of u_i $\forall$ $i \in \{1,\ldots,n\}$, then

$$M_n = M_0 + D \sum_{i=1}^{n} c_i.$$

Further, if $u_i = \{t_i\}$ $\forall i \in \{1,\ldots,n\}$, we may ommit the brackets and write

$M_0(t_1 t_2 \ldots t_n > M_n$

for short. The word $w = t_1 t_2 \ldots t_n \in T^*$ is said to be a <u>firing sequence</u> which leads from M_0 to M_n.

For all $w \in T^*$, $M:P \to N$, we write $M(w>$ iff there exists an $M':P \to N$ such that $M(w>M'$. Further, if $w = t_1 t_2 \ldots t_n$, $n \in N$, we define:

$$(D \cdot w)(p) = \sum_{i=1}^{n} D(p,t_i) \ \forall p \in P.$$

Hence forth we refer only to firing sequences since any set of concurrently fireable transitions can be represented as a firing sequence but not vice versa.

<u>Def 2.1.11 Enabled</u>:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$t \in T$ and

$M,M':P \to N$ be markings on P.

If $M[t>$, we say that the transition t is <u>enabled</u> on the marking M under the strict transition rule.

If $M(t>$, we say that the transition t is <u>enabled</u> on the marking M under the weak transition rule.

Note that $M[\to M' \implies M(\to M'$. However the converse need not be true even if $K(p) = \infty$ for all $p \in P$. If $w \in T^*$ and $K(p) = \infty$ for all $p \in P$ then $M[w>M' \iff M(w>M'$.

Def 2.1.12 Reachable:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net and

$M,M':P\text{-->}N$ be markings on P.

Then we say that M' is <u>reachable</u> from M according to the weak transition rule iff there exists $w \in T^*$ such that

M(w>M'.

Further, if there exists $w \in T^*$ such that

M[w>M',

Then M' is <u>reachable</u> from M according to the strict transition rule.

Def 2.1.13 Marking Sets:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net and

$M:P\text{-->}N$ be a marking on P.

Then we define the strict and weak <u>forward reachable marking sets</u> of M as follows:

[M> = {M'|M[=>M'},

(M> = {M'|M(=>M'}.

We define the strict and weak <u>full marking sets</u> to be:

$[M] = \{M' | M \tilde{s} M'\}$,

$(M) = \{M' | M \tilde{w} M'\}$.

where $\tilde{s}$ is defined to be the transitive and semetric closure of the relation [=> and $\tilde{w}$ is defined to be the transitive and semetric closure of the relation (=>.

Finally, we define

$$R(N) = (M_0>$$

to be the <u>reachability set</u> of the petri net N.

Having defined the petri net and the strict and weak transition rules, we now apply these definitions to our two problems. In both cases, we use the weak transition rule.

Consider the following solution to Problem #1. Place the philosophers dining table in a dining room with a narrow entrance so that only one philosopher can enter the dining room at any one time. When a philosopher feels hungry and comes to the dining room, he looks in before he enters. If either of the philosophers who sit on either side of him are in the dining room, he goes away and comes back later. If neither are present, he enters the room, sits down and eats. Upon finishing, he leaves the dining room.

We can represent the above solution to problem #1 with the petri net N_1 in Fig. 2-2. For $i \in \{1,\ldots,5\}$, a token in c_i implies that philosopher i is thinking, a token in f_i implies that

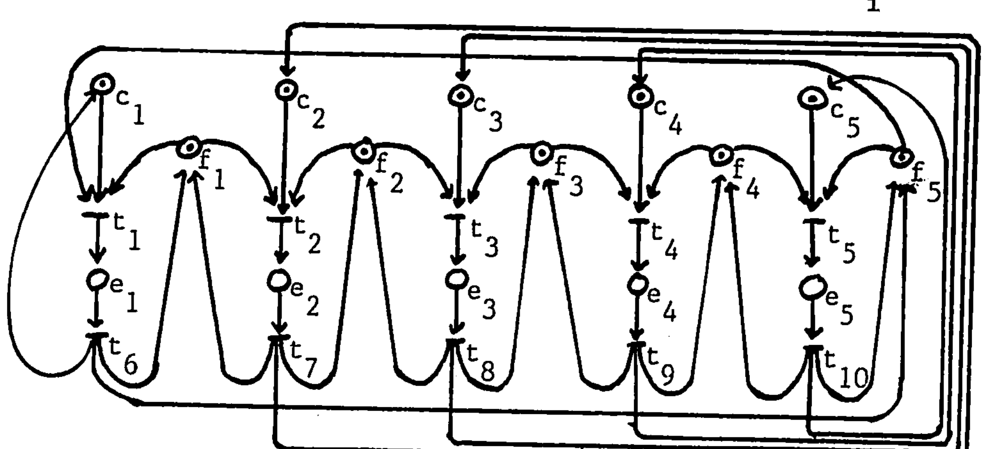


Fig 2-2 A Graphic Representation of N_1:

fork i is not in use and a token in e_i implies that philosopher i is eating. The representation of N_1 in Fig 2-2 is graphic. We can also represent $N_1 = (P,T,B,F,K,W,M_0)$ mathematically as follows:

$P = \{c_1,c_2,c_3,c_4,c_5,f_1,f_2,f_3,f_4,f_5,e_1,e_2,e_3,e_4,e_5\}$,

$T = \{t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8,t_9,t_{10}\}$,

$$B = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix},$$

$K(p) = \infty \quad \forall\ p \in P$,

$W(e) = 1 \quad \forall\ e \in E$,

(Note: Recall that B & F together uniquely define E.)

$$F = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \qquad M_0 = \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}.$$

At this point, we also include D for later reference:

$$D = B - F = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} .$$

Note that we represent B, F & D as $|P| \times |T|$ matrices and M_0 as a $|P| \times 1$ matrix. The places are represented top to bottom in the order in which they appear in P. Likewise, the transitions are represented left to right in the order they are listed in T. Thus the third column of B represents the tokens removed from c_3, f_2 & f_3 by transition t_3 with "1"s in rows 3,7 & 8. This notation will be used hence forth.

Now consider the following solution to Problem #2. When a process A requests control of the resource, check to see if there exists some process B which already has control. If there is no such process, give A control of the resource. If there is, wait until B yields control, and then give control to A. If more than one process is awaiting control at a given time, place them on a queue and deal with them on a first come/first served basis.

We can represent the above solution to problem #2 with the petri net in Fig. 2-3. Each token in p_1 represents a process doing what ever it is that processes do when they don't want control of the resource. Tokens in p_2 represent processes which have requested control of the resource but have not yet recieved it. (Note that since we can not tell one token from another in a petri net and since we require our petri nets to be finite, we can not represent an arbitrarily large first come/first served queue explicitly.) A (hopefully single) token in p_3 represents a process which has control of the resource.

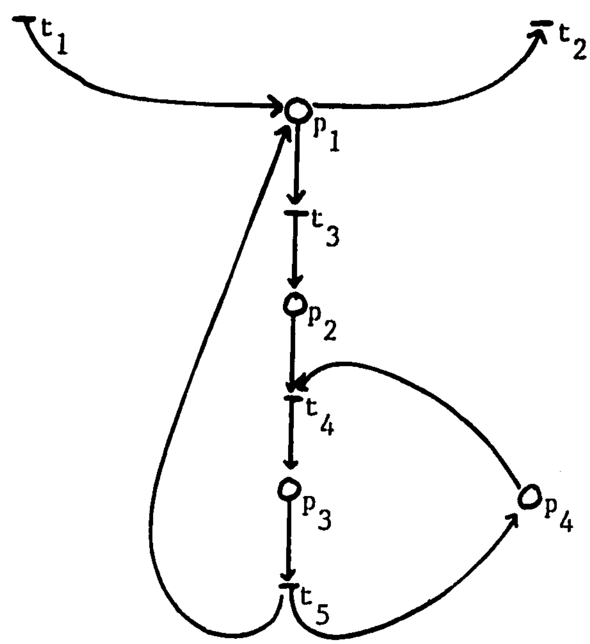


Fig 2-3 A Graphic Representation of N_2:

Again, we can represent N_2 mathematically, and do so as follows:

$P = \{p_1, p_2, p_3, p_4\}$,

$T = \{t_1, t_2, t_3, t_4, t_5\}$,

$$B = \begin{vmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix}, \quad F = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}, \quad M_0 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix},$$

$K(p) = \infty \;\; \forall \; p \in P$,

$W(e) = 1 \;\; \forall \; p \in P$.

Again, we include D for our later convience:

$$D = B - B = \begin{vmatrix} 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$$

In both of the above examples, we can verify by inspection that N_1 and N_2 are correct representations of our solutions to problems #1 & #2. (Since our initial statements of the solutions are written in English, they are, perforce, somewhat inexact.) However, that is all we have achieved.

The remainder of this segment is devoted to developing constructs which can be used to determine whether or not our solutions are correct.

Def 2.1.14 Strict Marking Graph:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net.

Then the strict marking graph of N is a system consisting of a directed graph

$G = (Z,E)$

and a labeling function 1:E-->P defined as follows:

$Z = [M_0 > \subseteq N^{|P|}$ is the set of vertices,

$E \subseteq Z \times Z$,

$E = \{(z,z') | z,z' \in Z, \exists\, t \in T \ni z[t>z'\}$ is the set of edges,

and for all (z,z') E,

$$l(z,z') = t$$

where $t \in T$ and $z[t>z'$.

We write

$$SMG(N) = (G,l), \quad G = (Z,E)$$

to denote the strong marking graph of N.

<u>Def 2.1.15 Weak Marking Graph</u>:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net.

Then the <u>weak marking graph</u> of N is a system consisting of a directed graph

$$G = (Z,E)$$

and a labeling function l:E-->P defined as follows:

$Z = (M_0 > \subseteq N^{|P|}$ is the set of vertices,

$E \subseteq Z \times Z$,

$E = \{(z,z') | z,z' \in Z, \exists\, t \in T \ni z(t>z\}$ is the set of edges,

and for all (z,z') E,

$$l(z,z') = t$$

where $t \in T$ and $z(t>z'$.

We write

$$WMG(N) = (G,l), \quad G = (Z,E)$$

to denote the weak marking graph of N.

When we represent either of the marking graphs graphically, instead of writing the vertices as column vectors (recall that the vertices are markings), we use the following notation: Let M be a marking/vertex in a marking graph. Then for each p P, we write p^i, $M(p) = i$, if $i > 0$ and ommit p entirely if $i = 0$. Thus if $P = \{p_1, p_2, p_3\}$, $M(p_1) = 1$, $M(p_2) = 0$ and $M(p_3) = 4$, we write $p_1^1 p_3^4$.

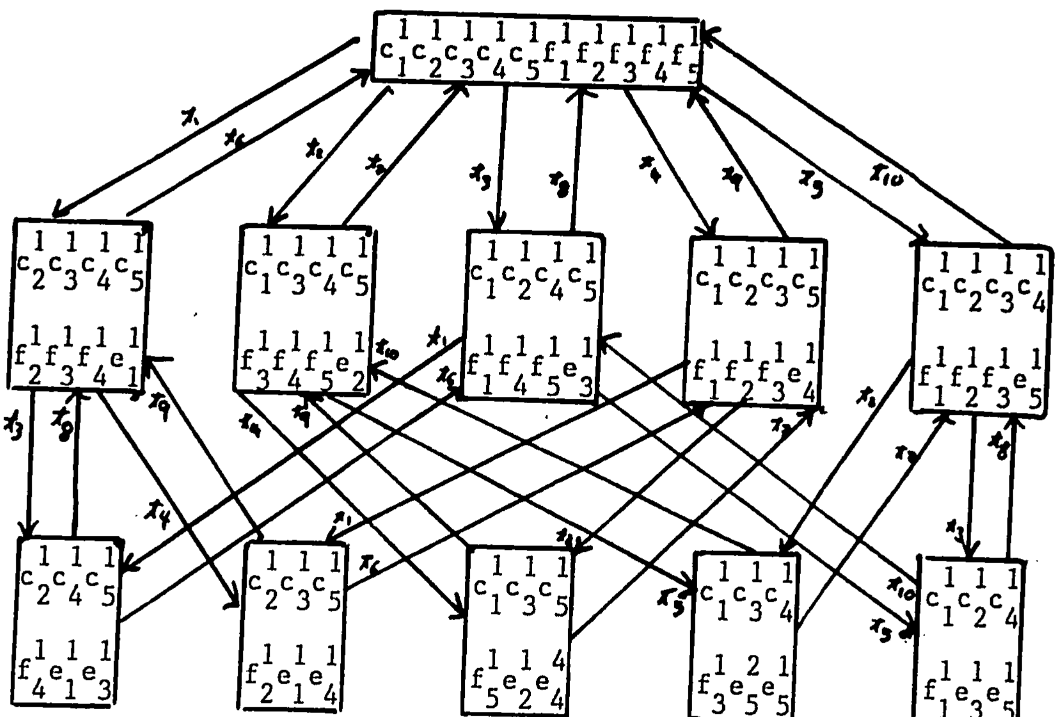


Fig 2-4 The Weak Marking Graph of N_1 WMG(N_1):

Consider WMG(N_1) in Fig 2-4. Note that since WMG(N_1) is finite, we can determine by inspection that N_1 is a correct solution to problem #1, since for all i {1,...,5} and for each vertex/marking M in WMG(N_1), we can find a directed path to a vertex/marking M' such that $M'(e_1) = 1$.

Now consider N_2. The reader can verify that $WMG(N_2)$ is infinite. Thus we cannot use $WMG(N_2)$ to determine whether or not N_2 is a correct solution to problem #2.

<u>Def 2.1.16 Weak Coverability Tree CT(N)</u>:

Let $N = (P,T,B,F,K,W.M_0)$ be a petri net.

Then the <u>weak coverabilty tree</u> of N is a system consisting of the tree

$T' = (V,E,\Upsilon,\Phi)$

and two labeling functions $l_V:V \dashrightarrow N_\infty^{|P|}$ and $l_E:E \dashrightarrow T$ which are defined via the following induction:

Base step - Depth 0:

Introduce the root vertex $r \in V$ such that

$l_V(r) = M_0$.

Induction step - Depth n + 1:

Assume that all vertices of depth $\leq n$, $n \in N$, $n \geq 0$, have been defined.

Let $s \in V$ be a vertex such that

Depth(s) = n

and

$l_V(s) = Q$

where $Q \in N_\infty^{|P|}$.

If one of the following hold, then s is a leaf:

1) On the path from r to s there exists a vertex $s' \in V$,

Depth(s') < n, such that

$$l_V(s') = Q.$$

2) There exists no $t \in T$ such that $Q(t>$.

If s is not a leaf, then there exists at least one $t \in T$ such that

$$Q(t>Q'$$

for some $Q' \in N_{\infty}^{|P|}$.

For each such t, introduce a new vertex s_t to V and a new edge e_t to E such that

$\tau(e_t) = s,$

$l_E(e_t) = t$ and

$\varphi(e_t) = s_t.$

We define $l_V(s_t)$ as follows:

Let $P(Q') = \{Q'' | Q''$ labels a vertex on the directed path from r to s_t, $Q'' \leq Q'\} \cup \{Q'\}$,

$$\tilde{Q} = Q' + \infty \cdot \sum_{Q'' \in P(Q')} \mathrm{Max}(0, (Q' - Q'')).$$

Recall that by convention, $0 \cdot \infty = 0$.

Define $l_V(s_t) = \tilde{Q}$.

Note that for all $p \in P$ such that

$$Q''(p) < Q'(p) \quad \forall Q'' \in P(Q'),$$

we have that

$$\tilde{Q}(p) = \infty,$$

and for all $p \in P$ such that

$$Q''(p) \geq Q'(p) \quad \forall Q'' \in P(Q'),$$

we have that

$$\tilde{Q}(p) = Q'(p) = Q(p) + D(p,t).$$

Note also that $(Q(p) = \infty \Longrightarrow \tilde{Q}(p) = \infty) \quad \forall p \in P$

Repeat the above process for all vertices of depth n. This defines all vertices of depth n + 1, or stops if there are no such vertices.

We write

$$CT(N) = (T', l_V, l_E), \quad T' = (V, E, \tau', \phi)$$

to denote the weak coverability tree of N.

Note that since T is finite, for all $v \in V$,

$$|\{e | e \in E, \tau'(e) = v\}| \leq |T| < \infty.$$

Thus CT(N), or more correctly T', is finitely branching.

Def 2.1.17 Strict Coverability Tree:

The strict coverability tree is constructed in the same fashon as the weak coverability tree, save that the strict transition rule is used in place of the weak transition rule.

We provide $CT(N_2)$ in Fig 2-5 as an example of a weak coverability tree. Note that we use the same notation for labeling vertices as we did for the weak marking graph of N_1. The following four theorems give us the information we require to interpret the weak coverability tree.

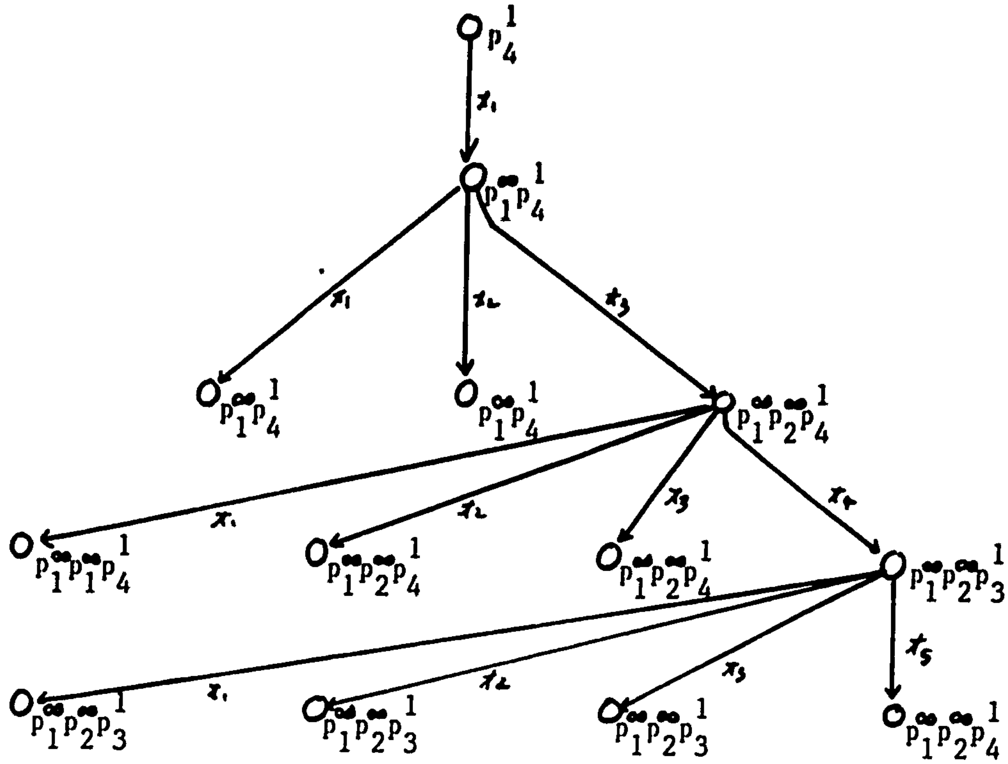


Fig 2-5 The Coverability Tree of N_2 $CT(N_2)$:

<u>Thm 2.1.18</u>:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net and

$CT(N) = (T',l_V,l_E)$, $T' = (V,E,\tau,\phi)$ be the coverability tree associated with N.

Then T' is finite.

Pf: by contradiction

Suppose that N is a petri net such that T' is infinite.

Since T' is finitely branching, by Thm 1.2.23, Konig's lemma, T' must have at least one infinite path.

Call this path

$$\Delta = v_0 e_1 v_1 e_2 v_2 \ldots$$

where v_0 is the root vertex of T'.

By definition of CT(N), associated with Δ is a sequence of labelings

$$L = l_0, l_1, l_2, \ldots$$

where $l_V(v_i) = l_i$ for all $i \in N$.

By the construction rules for CT(N), L has the following properties:

1) $l_i \neq l_j$ for all $i,j \in N$, $i \neq j$,

2) For all $i \in N$, there exists $t_i \in T$ such that $l_i(t_i>$.

The above must be true, for were they not, Δ would terminate and hence be finite.

Consider $l_0 = M_0$, the initial marking of N.

Since M_0 is a marking, it must, by definition, be finite. i.e.

$$M_0(p) < \infty \ \forall \ p \in P.$$

Hence there is at most a finite number of labelings $l \in N_\infty^{|P|}$ such that

$$l \leq l_0.$$

Further, since P, the set of places in N, is finite, by Thm 1.3.4, Zorn's Lemma, the set of pairwise incomparable labelings $l' \in N_\infty^{|P|}$, which are also incomparable with l_0 and any labeling $l'' \leq l_0$ which may occur in L, must also be finite.

Thus there must exist some finite $i \in N$ such that $l_i > l_0$.

By the construction rules for CT(N), there exists $p \in P$ such that

$l_i(p) = \infty$.

Since the definition of the edge multiplicity function requires a finite multiplicity on each edge in N, for all $j \in N$, $j > i$ and $p \in P$,

$$l_i(p) = \infty \implies l_j(p) = \infty.$$

Thus there are only a finite number of possible labelings $l \in N_\infty^{|P|}$ such that

$$l < l_i.$$

Again by Zorn's Lemma, the set of pairwise incomparable labelings $l' \in N_\infty^{|P|}$ which are also incomparable to l_i and any labeling $l'' \leq l_i$ which may occur in L, must be finite.

Thus there must exist some finite $k \in N$, $k > i$, such that

$$l_k > l_i.$$

By the construction rules for CT(N), there exists $p \in P$ such that

$$l_i(p) < \infty$$

and

$$l_k(p) = \infty.$$

The above argument can be repeated indefinitely.

However, since P, the set of places in N is finite, we must eventually reach some labeling $l_m \in N_\infty^{|P|}$, $m \in N$, such that

$$l_m(p) = \infty \ \forall p \in P.$$

But then, since the edge multiplicity function is defined to be finite for all edges in N,

$$l_{m+1}(p) = \infty \ \forall p \in P$$

Thus $l_m = l_{m+1}$.

But this is precisely one of the conditions for the termination of a path given in the construction rules for CT(N).

Hence Δ is finite and T' cannot contain an infinite path.

Therefore T', and hence CT(N), is finite.

Lemma 2.1.19:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$CT(N) = (T', l_V, l_E)$, $T' = (V,E,\iota,\varphi)$ be the coverability tree associated with N,

$\Delta = v_0 e_1 v_1 \ldots v_{n-1} e_n v_n$, $n \in N$, be a directed path in T' such that v_0 is the root vertex of T' and v_n is a leaf.

$l_V(v_i) = Q_i \in N_\infty^{|P|} \quad \forall\, i \in \{0,\ldots,n\}$,

$P_i^\infty \subseteq P$ be the set of places such that $Q_i(p) = \infty \iff p \in P_i^\infty$ for all $i \in \{0,\ldots,n\}$ and

$l_e(e_i) = t_i \in T \quad \forall\, i \in \{1,\ldots,n\}$.

Then for all $k \in N$, $0 \le k \le n$, such that $P_{k-1}^\infty \subset P_k$, there exists a $b_k \in N$, $0 \le b_k < k$ such that

$$Q_k(p) = Q_{b_k}(p) \quad \forall\; p \in P \setminus P_k^\infty$$

and

$$(D \cdot \textstyle\sum_{h=b_k+1}^{k} t_h)(p) > 0 \quad \forall\; p \in P_k^\infty \setminus P_{k-1}^\infty .$$

Further for each such k, we can construct a firing sequence $w_k \in T^*$ with the properties:

1) $(D \cdot w_k)(p) = 0 \quad \forall\; p \in P \setminus P_k^\infty$,

2) $(D \cdot w_k)(p) > 0 \quad \forall\; p \in P_k^\infty \setminus P_{b_k}^\infty$,

3) w_k is enabled on any marking M on P such that

a) $M(p) \geq Q_{b_k}(p) \;\; \forall \;\; p \in P \setminus P^{\infty}_{b_k}$,

b) $M(p) \geq (B \cdot w_k)(p) \;\; \forall \;\; p \in P^{\infty}_{b_k}$.

Pf: by construction

The existence and properties of b_k follow directly from the definition of CT(N).

We demonstrate that w_k can be constructed via the following recursive procedure:

Initially let $w_k = t_{b_k+1} t_{b_k+2} \cdots t_k$

By the construction rules for CT(N), the following must be true:

1') $(D \cdot w_k)(p) = 0 \;\; \forall \;\; p \in P \setminus P^{\infty}_{k}$,

2') $(D \cdot w_k)(p) > 0 \;\; \forall \;\; p \in P^{\infty}_{k} \setminus P^{\infty}_{k-1}$,

3') w_k is enabled on any marking M on P such that

a') $M(p) \geq Q_{b_k}(p) \;\; \forall \;\; p \in P \setminus P^{\infty}_{k-1}$,

b') $M(p) \geq (B \cdot w_k)(p) \;\; \forall \;\; p \in P^{\infty}_{k-1}$.

Note that 1') is identical to 1).

Further, if $P^{\infty}_{k-1} = P^{\infty}_{b_k}$, 2') and 3') are identical to 2) and 3) respectively.

Thus, if $P^{\infty}_{k-1} = P^{\infty}_{b_k}$, w_k as initially defined satisfies properties 1), 2) & 3) and we are done.

Suppose $P^{\infty}_{b_k} \subset P^{\infty}_{k-1}$.

Then we modify w_k as follows:

Initially, let $r = k - 1$.

Note that by the construction rules for CT(N), the following are true:

1") $(D \cdot w_k)(p) = 0 \quad \forall\, p \in P \setminus P_k^{\infty}$,

2") $(D \cdot w_k)(p) > 0 \quad \forall\, p \in P_k^{\infty} \setminus P_r^{\infty}$,

3") w_k is enabled on any marking M on P such that:

a") $M(p) \geq Q_{b_k}(p) \quad \forall\, p \in P \setminus P_r^{\infty}$,

b") $M(p) \geq (B \cdot w_k)(p) \quad \forall\, p \in P_r^{\infty}$.

We proceed via the following cycle:

i) Decrement r by 1.

ii) Modify w_k as indicated below.

iii) Demonstrate that 1"), 2") & 3") hold for the new value of r.

iv) If $r > b_k$, we return to i) and start over. If $r = b_k$, we are done, since 1"), 2") & 3") have become equivalent to 1), 2) & 3) respectively.

Our modifications to w_k in ii) and our argument in iii) depend upon whether $P_r^{\infty} = P_{r+1}^{\infty}$ or $P_r^{\infty} \subset P_{r+1}^{\infty}$. We deal with the former in Case 1 and the latter in Case 2.

Case 1 - ($P_r^{\infty} = P_{r+1}^{\infty}$):

No additions are required to w_k.

Since $P_r^{\infty} = P_{r+1}^{\infty}$, 1"), 2") & 3") still hold.

Case 2 - ($P_r^{\infty} \subset P_{r+1}^{\infty}$):

Define $s = r + 1$.

Construct w_s via recursive application of this procedure.

Then w_s has the properties:

1') $(D \cdot w_s)(p) = 0 \quad \forall\ p \in P \setminus P_s^{\infty}$,

2') $(D \cdot w_s)(p) > 0 \quad \forall\ p \in P_s^{\infty} \setminus P_{b_s}^{\infty}$,

3') w_s is enabled on any marking M on P such that:

a') $M(p) \geq Q_{b_s}(p) \quad \forall\ p \in P \setminus P_{b_s}^{\infty}$,

b') $M(p) \geq (B \cdot w_s)(p) \quad \forall\ p \in P_{b_s}^{\infty}$.

where $b_s \in N$, $0 \leq b_s < s$,

$$Q_{b_s}(p) = Q_s(p) \quad \forall\ p \in P \setminus P_s^{\infty}$$

and

$$\left(D \cdot \sum_{h=b_s+1}^{k} t_h\right)(p) > 0 \quad \forall\ p \in P_s^{\infty} \setminus P_{s-1}^{\infty}.$$

Choose $K \in N$ such that

$$K > 2((B \cdot w_k)(p)) \quad \forall\ p \in P_s^{\infty} \setminus P_r^{\infty}.$$

Choose $m \in N$ such that

$$m\ ((D \cdot w_s)(p)) > K \quad \forall\ p \in P_s^{\infty} \setminus P_r^{\infty}.$$

By 2'), such an m must exist since

$$(P_s^{\infty} \setminus P_r^{\infty}) \subseteq (P_s^{\infty} \setminus P_{b_s}^{\infty}).$$

Let I be the firing sequence formed by concatinating w_s with itself m times.

Note that I has the following properties:

$\bar{1}$) $(D \cdot I)(p) = 0 \quad \forall\ p \in P \setminus P_s^{\infty}$,

$\bar{2}$) $(D \cdot I)(p) > 0 \quad \forall\ p \in P_s^{\infty} \setminus P_{b_s}^{\infty}$,

$\bar{3}$) I is enabled on any marking M on P such that

$\bar{a}$) $M(p) \geq Q_{b_s}(p) \quad \forall\ p \in P \setminus P_{b_s}^{\infty}$,

$\bar{b}$) $M(p) \geq (B \cdot I)(p) \quad \forall\ p \in P_{b_s}^{\infty}$.

Let $a = t_{b_k+1} t \ldots t_s \in T^*$ and

$b = t_{s+1} \ldots t_k \in T^*$ be firing sequences such that

$w_k = ab.$

Define $w_k' = aIb.$

We must now show that w_k' has the properties:

$1^{\#}$) $(D \cdot w_k')(p) = 0 \quad \forall\, p \in P \setminus P_k^{\infty}$,

$2^{\#}$) $(D \cdot w_k')(p) > 0 \quad \forall\, p \in P_k^{\infty} \setminus P_r^{\infty}$,

$3^{\#}$) w_k' is enabled on any marking M on P such that:

$a^{\#}$) $M(p) \geq Q_{b_k}(p) \quad \forall\, p \in P \setminus P_r^{\infty}$,

$b^{\#}$) $M(p) \geq (B \cdot w_k')(p) \quad \forall\, p \in P_r^{\infty}$.

By 1"), $(D \cdot w_k)(p) = 0 \quad \forall\, p \in P \setminus P_k^{\infty}$.

Further, by $\bar{I}$), $(D \cdot I)(p) = 0 \quad \forall\, p \in P \setminus P_s^{\infty}$.

Since $P_s^{\infty} \subseteq P_k^{\infty}$, we obtain $1^{\#}$).

By definition of I,

$$(D \cdot I)(p) > (B \cdot w_k)(p) \quad \forall\, p \in P_s^{\infty} \setminus P_r^{\infty}$$

which implies

$$(D \cdot w_k')(p) > 0 \quad \forall\, p \in P_k^{\infty} \setminus P_s^{\infty}.$$

By $\bar{I}$), $(D \cdot I)(p) = 0 \quad \forall\, p \in P \setminus P_s^{\infty}$.

Thus we obtain $2^{\#}$).

To obtain $3^{\#}$), it is sufficient to show that w_k' is enabled on the the marking M on P, where:

$$M(p) = \begin{cases} Q_{b_k}(p) & \forall\, p \in P \setminus P_r^{\infty} \\ (B \cdot w_k')(p) & \forall\, p \in P_r^{\infty}. \end{cases}$$

Since for all $m \in N$, $b_k \leq m < s$,

$$Q_m(p) < \infty \quad \forall\, p \in P \setminus P_r^{\infty},$$

by the construction rules of CT(N) and the weak transition rule, a must be enabled on M.

Thus there exists a marking M' on P such that

$$M(a>M'$$

where

$$M'(p) = Q_s(p) = Q_{b_s}(p) \quad \forall \ p \in P \setminus P_s^{\infty}$$

and

$$M'(p) \geq (B \cdot (Ib))(p) \quad \forall \ p \in P_r^{\infty}.$$

By 3), to show that I is enabled on M', it suffices to show that

$$M'(p) \geq Q_{b_s}(p) \quad \forall \ p \in P_s^{\infty} \setminus P_r^{\infty}.$$

Consider the firing sequence

$$c = t_{b_s+1} t_{b_s+2} \cdots t_s.$$

Either $b_s < b_k$ or $b_s \geq b_k$.

Suppose $b_s < b_k$.

Let $d = t_{b_s+1} \cdots t_{b_k}$.

Then c = da. This is true since we have not yet modified w_k' for $i < r$.

Let $\tilde{M}$ be a marking on P such that

$$\tilde{M}(p) = \begin{cases} Q_{b_s}(p) & \forall \ p \in P \setminus P_r^{\infty} \\ (D \cdot d)(p) + (B \cdot w_k')(p) & \forall \ p \in P_r^{\infty}. \end{cases}$$

Since for all $m \in N$, $b_s \leq m < s$,

$$Q_m(p) < \infty \quad \forall \ p \in P \setminus P_r^{\infty},$$

by the construction rules for CT(N), and the weak transition rule, c is enabled on $\tilde{M}$, as is d.

Hence there exists a marking $\tilde{M}'$ on P such that

$$\tilde{M}(d>\tilde{M}'.$$

Further, by the weak transition rule,

$$\tilde{M}' = M$$

Thus we have

$$\tilde{M}(d>\tilde{M}'$$

and

$$M(a>M'$$

where $\tilde{M}' = M$.

But, by def of CT(N):

$$(D \cdot c)(p) > 0 \quad \forall\, p \in P_s^{\infty} \setminus P_r^{\infty}.$$

Hence

$$M'(p) \geq Q_{b_s}(p) \quad \forall\, p \in P_s^{\infty} \setminus P_r^{\infty}$$

and thus I is enabled on M'.

On the other hand, suppose $b_s \geq b_k$.

Now let $d = t_{b_k+1} \ldots t_{b_s}$.

Then a = dc.

Since a is enabled on M, so is d.

Thus there exists a marking $\tilde{M}$ on P such that

$$M(d>\tilde{M}.$$

Since for all $m \in N$, $b_k \leq m < s$,

$$Q_m(p) < \infty \quad \forall\, p \in P \setminus P_r^{\infty},$$

by the contruction rules for CT(N) and the weak transition rule, we have that:

$$\tilde{M}(p) = Q_{b_s}(p) \quad \forall\, p \in P \setminus P_r^{\infty}.$$

Since c is enabled on $\tilde{M}$, and since, by the construction rules for CT(N)

$$(D\cdot c)(p) > 0 \quad \forall\, p \in P_s^\infty \setminus P_r^\infty,$$

we have that

$$M(d>\bar{M}(c>M'$$

and

$$M'(p) > Q_{b_s}(p) \quad \forall\, p \in P_s^\infty \setminus P_r^\infty.$$

Hence, in this case as well, we have shown that I is enabled on M'.

Thus there exists a marking M" on P such that

$$M'(I>M".$$

It remains to be shown that b is enabled on M".

By $\bar{I}$), $(D\cdot I)(p) = 0 \quad \forall\, p \in P \setminus P_s^\infty$.

Thus $M''(p) = M'(p) = Q_s(p) \quad \forall\, p \in P_s^\infty$.

By definition of I and the weak transition rule,

$$M''(p) \geq (B\cdot b)(p) \quad \forall\, p \in P_s^\infty \setminus P_r^\infty.$$

By definition of M and the weak transition rule,

$$M''(p) \geq (B\cdot b)(p) \quad \forall\, p \in P_r^\infty.$$

To summarize:

$$M''(p) = Q_s(p) \quad \forall\, p \in P \setminus P_s^\infty,$$
$$M''(p) \geq (B\cdot b)(p) \quad \forall\, p \in P_s^\infty.$$

By 3"), w_k, and hence a, is enabled on the marking $\hat{M}$ on P, where:

$$\hat{M}(p) = \begin{cases} Q_{b_k}(p) & \forall\, p \in P \setminus P_s^\infty \\ (B\cdot w_k)(p) & \forall\, p \in P_s^\infty. \end{cases}$$

Thus, by the weak transition rule, b is enabled on the marking $\hat{M}'$ on P, where

$\hat{M}(a>\hat{M}'$,

$\hat{M}'(p) = Q_s(p) \; \forall \, p \in P \setminus P_s^{\infty}$

and

$\hat{M}'(p) \geq (B \cdot b)(p) \;\; \forall \; p \in P_s^{\infty}$.

Since $M''(p) = \hat{M}'(p) \;\; \forall \; p \in P \setminus P_s^{\infty}$, and both $M''(p)$ and $\hat{M}'(p)$ are greater than or equal to $(B \cdot b)(p)$ for all $p \in P_s^{\infty}$, by the weak transition rule, b is enabled on M" iff b is enabled on $\hat{M}'$.

Since we have shown that b is enabled on $\hat{M}'$, we have obtained $3^{\#}$).

We now redefine w_k to equal w_k', and note that $1^{\#}$), $2^{\#}$) & $3^{\#}$) are equivalent to 1"), 2") & 3").

This concludes our handling of Case 2.

One point remains to be delt with in our argument for our recursive construction procedure for w_k. We must show that the recursion is not infinite.

We do so by observing that if the construction of w_k requires the construction of w_s, then $P_s^{\infty} \subset P_k^{\infty}$.

Since P, and hence P_k^{∞}, is finite, the recursion must also be finite.

<u>Thm 2.1.20</u>:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$CT(N) = (T', l_V, l_E)$, $T' = (V,E,\epsilon,\phi)$ be the coverability tree asscoiated with N,

$v \in V$ and

$l_V(v) = Q \in N_{\infty}^{|P|}$.

Then i) If $Q(p) < \infty \ \forall p \in P$,

Then $Q \in R(N)$.

ii) If there exists $P^{\infty} \subseteq P$, $P^{\infty} \neq \emptyset$, such that

$$Q(p) = \infty \Longleftrightarrow p \in P^{\infty},$$

Then there exists an infinite sequence of markings

$$M_1, M_2, \ldots, M_i, \ldots$$

such that

a) $M_i(p) = Q(p) \quad \forall p \in P \setminus P^{\infty}$,

b) $M_1(p) < M_2(p) < \ldots \quad \forall p \in P^{\infty}$ and

c) $M_i \in R(N) \quad \forall i \in \{1,2,\ldots\}$.

Pf: by construction

Let $\Delta = v_0 e_1 v_1 \ldots v_{n-1} e_n v_n$, $n \in N$ be a directed path in T'

such that v_0 is the root vertex of T' and v_n is a leaf.

Recall that by Thm 2.1.18, T' must be finite. Thus every

vertex in V must lie along some such directed path,

$l_V(v_i) = Q_i \quad \forall i \in \{0,\ldots,n\}$,

$l_E(e_i) = t_i \quad \forall i \in \{1,\ldots,n\}$ and

$P_i^{\infty} \subseteq P$ be the set of places such that

$$[Q_i(p) = \infty \Longleftrightarrow p \in P_i^{\infty}] \quad \forall i \in \{0,\ldots,n\}.$$

We proceed by induction on i:

Base step - (i = 0):

By definition of CT(N),

$$Q_0 = M_0,$$

the initial marking of N.

Hence $Q_0 = M_0 \in R(N)$.

Induction step - ($i \geq 0$):

Suppose that the Thm holds for all Q_j, $0 \leq j \leq i$.

We demonstrate that the Thm holds for $i + 1$ as follows - three cases:

Case 1 - ($P_{i+1} = \emptyset$):

By def of CT(N), $P_i = \emptyset$.

Hence $Q_i \in R(N)$.

By def of CT(N), t_{i+1} is enabled on Q_i and

$$Q_i(t_{i+1} > Q_{i+1}.$$

Thus $Q_{i+1} \in R(N)$.

Case 2 - ($P_i = P_{i+1} \neq \emptyset$):

By the induction hypothesis, there exists an infinite sequence of markings

$$M_1, M_2, \ldots, M_h, \ldots \in R(N)$$

such that

$$M_h(p) = Q_i(p) \quad \forall\, p \in P \setminus P_i^{\infty},\ h \in \{1,2,\ldots\} \quad \text{and}$$

$$M_1(p) < M_2(p) < \ldots < M_h(p) < \ldots \quad \forall\, p \in P_i^{\infty},$$
$$h \in \{1,2,\ldots\}.$$

By def of CT(N) and the weak transition rule,

$$(B \cdot t_{i+1})(p) \leq Q_i(p) = M_h(p) \quad \forall\, p \in P \setminus P_i^{\infty},$$
$$h \in \{1,2,\ldots\}.$$

Further, since the edge multiplicity function is defined to be finite:

$$(B \cdot t_{i+1})(p) < \infty \quad \forall\, p \in P_i^{\infty}.$$

Thus there exists some $j \in N$ such that

$$(B \cdot t_{i+1})(p) \leq M_h(p) \quad \forall\, h \in \{j, j+1, \ldots\}, \quad p \in P_i^{\infty}.$$

Therefore, t_{i+1} is enabled on all M_h such that $h \in \{j, j+1, \ldots\}$.

Hence we can define a sequence of markings

$$M'_k = M_{k+j} + D \cdot t_{i+1} \quad \forall\, k \in N,\ k > 0.$$

Since $M_h(p) = Q_i(p) \quad \forall\, h \in \{1, 2, \ldots\},\ p \in P \setminus P_i^{\infty}$, by definitions of CT(N) and the weak transition rule, we have:

$$\text{a)}\ M'_k(p) = Q_{i+1}(p) \quad \forall\, p \in P \setminus P_{i+1}^{\infty},\ k \in N,\ k > 0.$$

Further, since

$$M_1(p) < M_2(p) < \ldots \quad \forall\, p \in P_i^{\infty},$$

we have that

$$\text{b)}\ M'_1(p) < M'_2(p) < \ldots \quad \forall\, p \in P_{i+1}^{\infty}.$$

Finally, since $M_h \in R(N)$ for all $h \in \{1, 2, \ldots\}$, and since t_{i+1} is enabled on all $M_h \ni h \in N,\ h \geq j$, we have that:

$$\text{c)}\ M'_k \in R(N) \quad \forall\, k \in \{1, 2, \ldots\}.$$

Thus, if $P_i^{\infty} = P_{i+1}^{\infty} \neq \emptyset$, we have shown that Q_{i+1} satisfies the Thm.

Case 3 - ($P_i^{\infty} \subset P_{i+1}^{\infty} \neq \emptyset$):

By the construction rules for CT(N), there exists $b \in N$, $0 \leq b \leq i$, such that

$$Q_b(p) = Q_{i+1}(p) \quad \forall\, p \in P \setminus P_{i+1}^{\infty}$$

and

$$(D \cdot \sum_{h=b+1}^{i+1} t_h)(p) > 0 \quad \forall\, p \in P^{\infty}_{i+1} \setminus P^{\infty}_{i}.$$

By Lemma 2.1.19, we can construct a finite firing sequence $w \in T^*$ such that:

1) $(D \cdot w)(p) = 0 \quad \forall\, p \in P \setminus P^{\infty}_{i+1}$,

2) $(D \cdot w)(p) > 0 \quad \forall\, p \in P^{\infty}_{i+1} \setminus P^{\infty}_{b}$,

3) w is enabled on any making M on P such that

a') $M(p) \geq Q_b(p) \quad \forall\, p \in P \setminus P^{\infty}_{b}$,

b') $M(p) \geq (B \cdot w)(p) \quad \forall\, p \in P^{\infty}_{b}$.

Suppose $P_b = \emptyset$.

Then, by the induction hypothesis, Q_b $R(N)$.

Since $(D \cdot w)(p) \geq 0 \quad \forall\, p \in P$, we can define the sequence of markings

$$M_h = Q_b + h \cdot D \cdot w \quad \forall\, h \in \{1,2,\ldots\}$$

where

c) $M_h \in R(N) \quad \forall\, h \in \{1,2,\ldots\}$

holds by construction.

Since $(D \cdot w)(p) = 0 \quad \forall\, p \in P \setminus P^{\infty}_{i+1}$, we have

a) $M_h(p) = Q_b(p) = Q_{i+1}(p) \quad \forall\, p \in P \setminus P^{\infty}_{i+1}$,
$h \in \{1,2,\ldots\}$.

Since $(D \cdot w)(p) > 0 \quad \forall\, p \in P^{\infty}_{i+1} \setminus P^{\infty}_{b} = P^{\infty}_{i+1}$, we have

b) $M_1(p) < M_2(p) < \ldots \quad \forall\, p \in P^{\infty}_{i+1}$.

On the other hand, suppose $P_b \neq \emptyset$.

Then, by the induction hypothesis, there exists an infinite sequence of markings

$$M_1, M_2, \ldots, M_h, \ldots$$

where

a") $M_h(p) = Q_b(p) \quad \forall\, p \in P \setminus P_b^\infty$,

b") $M_1(p) < M_2(p) < \ldots \quad \forall\, p \in P_b^\infty$ and

c") $M_h \in R(N) \quad \forall\, h \in \{1,2,\ldots\}$.

Since the arc weighting function is defined to be finite, and since w is finite, there exists $K \in N$ such that

$$K - (B\cdot w)(p) > 0 \quad \forall\, p \in P_b^\infty.$$

Define the function $f: N \dashrightarrow N$ such that $f(r)$ is equal to the least integer h such that

$$M_h(p) \geq 2\cdot r\cdot K \quad \forall\, p \in P_b^\infty.$$

Note that by b"), f must be defined for all $r \in N$.

Then we can define a sequence of markings

$$M'_r = M_{f(r)} \quad \forall\, r \in \{1,2,\ldots\}.$$

Since for all $r', r'' \in N$, $r' < r''$, there exists $h', h'' \in N$, $h' < h''$, such that

$$M'_{r'} = M_{h'} \quad \text{and} \quad M'_{r''} = M_{h''},$$

we have

$\bar{a}$) $M'_r(p) = Q_b(p) \quad \forall\, p \in P \setminus P_b^\infty$,

$\bar{b}$) $M'_1(p) < M'_2(p) < \ldots \quad \forall\, p \in P_b^\infty$ and

$\bar{c}$) $M'_r \in R(N) \quad \forall\, r \in \{1,2,\ldots\}$.

By 3), w concatinated with itself 2r times is enabled on M'_r for all $r \in \{1,2,\ldots\}$.

Thus we can define the infinite sequence of markings on P:

$$M''_r = M'_r + r(D\cdot w)$$

where

c) $M''_r \in R(N) \quad \forall\, r \in \{1,2,\ldots\}$

holds by construction.

Since $(D\cdot w)(p) = 0$ for all $p \in P \setminus P^{\infty}_{i+1}$ we have

a) $M''_r(p) = Q_b(p) = Q_{i+1}(p) \quad \forall\, p \in P \setminus P^{\infty}_{i+1}$.

Since $(D\cdot w)(p) > 0$ for all $p \in P^{\infty}_{i+1} \setminus P^{\infty}_b$ and

$M'_r(p) \geq 2r(B\cdot w)(p) \quad \forall\, p \in P^{\infty}_b,\ r \in \{1,2,\ldots\}$,

we have

b) $M''_1(p) < M''_2(p) < \ldots \quad \forall\, p \in P^{\infty}_{i+1}$.

Thus if $P_i \subset P_{i+1} \neq \emptyset$, we have shown that Q_{i+1} satisfies the Thm.

By the construction rules for CT(N), cases 1), 2) & 3) are the only possible cases in our induction.

Thus our induction is complete.

Thm 2.1.21:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$CT(N) = (T', l_V, l_E)$, $T' = (V,E,\tau,\varphi)$, be the coverability tree associated with N and

$M: P \rightarrow N$ be a marking on P.

Then $M \in R(N) \iff$ there exists a vertex $v \in V$ such that $l_V(v) = Q$ and a set $P^{\infty} \subseteq P$ such that

$Q(p) = \infty \iff p \in P^{\infty}$

and

$M(p) = Q(p) \quad \forall\, p \in P \setminus P^{\infty}$.

Pf: (==>) by construction

Suppose $M \in R(N)$.

Then there exists a firing sequence

$$w = t_1 t_2 \ldots t_n \in T^*, \quad n \in N$$

such that

$$M_0(t_1>M_1(t_2>M_2 \ldots M_{n-1}(t_n>M_n = M$$

or, more simply,

$$M_0(w>M.$$

Proof follows by induction on M_i, $0 \leq i \leq n$.

Base step:

By definition of CT(N), the root node r is labeled by M_0, the initial marking of N.

Therefore the Thm holds for M_0.

Induction step:

Suppose that we have proved the result for M_i, $i \in N$, $i \geq 0$.

If i = n, then we are done.

If i < n, we show that the Thm holds for i + 1 as follows:

Since the Thm holds for M_i, there exists a vertex $v_i \in V$ such that $l_V(v_i) = Q_i$ and a set $P_i^\infty \subseteq P$ such that:

$$Q_i(p) = \infty \iff p \in P_i^\infty$$

and

$$M_i(p) = Q_i(p) \quad \forall \; p \in P \setminus P_i^\infty$$

We show that the same holds for M_{i+1}. We do so in the following cases:

Case 1 - (v_i is not a leaf):

Since $Q_i \geq M_i$, t_{i+1} is enabled on Q_i.

Thus, by the construction rules for CT(N), there exists an edge $e_{i+1} \in E$, a virtex $v_{i+1} \in V$ and a set $P^{\infty}_{i+1} \subseteq P$ such that:

$$\iota(e_{i+1}) = v_i,$$

$$l_E(e_{i+1}) = t_{i+1},$$

$$\varphi(e_{i+1}) = v_{i+1}$$

$$l_V(v_{i+1}) = Q_{i+1}$$

$$Q_{i+1}(p) = Q_i(p) + (D \cdot t_{i+1})(p) \quad \forall\, p \in P \setminus P^{\infty}_{i+1}$$

where $p \in P^{\infty}_{i+1} \iff Q_{i+1}(p) = \infty$. Note also that by definition of CT(N), $P^{\infty}_i \subseteq P^{\infty}_{i+1}$.

Since the Thm holds for M_i, since $M_{i+1} = M_i + D \cdot t_{i+1}$ and since $P^{\infty}_i \subseteq P^{\infty}_{i+1} \subseteq P$, we have that

$$M_{i+1}(p) = Q_{i+1}(p) \quad \forall\, p \in P \setminus P^{\infty}_{i+1}.$$

Case 2 - (v_i is a leaf):

By the construction rules for CT(N), either

1) There exists no $t \in T$ such that t is enabled on Q_i

or

2) There exists a virtex $v' \in V$ on the directed path from r to v_i such that $l_V(v') = Q_i$.

Since t_{i+1} is enabled on M_i and $Q_i \geq M_i$, 2) must hold. Thus we can set v_i equal to v' without changing Q_i. We do so and proceed as in Case 1.

(<==)

Follows directly from Thm 2.1.20.

Def 2.1.22 Bounded and Unbounded on N:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$p \in P$.

Then p is said to be bounded on N iff there exists a $K \in \mathbb{N}$ such that

$M(p) \leq K \quad \forall M \in R(N)$.

If there is no such K, p is said to be unbounded on N.

Thm 2.1.23:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$CT(N) = (T', l_V, l_E)$, $T' = (V,E,\epsilon,\varphi)$ be the coverability tree associated with N and

$p \in P$.

Then p is unbounded on N iff there exists a vertex $v \in V$ such that $l_V(v) = Q$ where $Q(p) = \infty$.

Pf: (==>) by contradiction

Suppose that:

p is unbounded on N and

There exists no vertex $v \in V$ such that $l_V(v) = Q$ and $Q(p) = \infty$.

Since p is unbounded on N, there exists an infinite sequence markings $M_i \in R(N)$, $i \in \{1,2,\ldots\}$, such that

$M_1(p) < M_2(p) < \ldots < M_i(p) < \ldots$.

By Thm 2.1.21, for all such M_i, there exists $v_i \in V$ such that $l_V(v_i) = Q_i$ and $M_i \leq Q_i$.

By hypothesis, $Q(p) < \infty \;\forall\, i \in \{1,2,\ldots\}$.

Thus, again by Thm 2.1.21, $Q_i(p) = M_i(p) \;\forall\, i \in \{1,2,\ldots\}$.

Let $P_i^{\infty} \subseteq P$ be a subset of P such that

$$Q_i(p') = \infty \iff p' \in P_i^{\infty}.$$

Since $M_1(p) < M_2(p) < \ldots < M_i(p) < \ldots$

we have that

$$\{Q_i \mid \exists\, v \in V \ni ((l_V(v) = Q_i) \wedge (Q_i(p') = M_i(p') \quad p' \quad P\ P_i) \wedge (p \quad P_i))\}$$

is an infinite set.

Hence T', and thus CT(N), must be infinite.

But this contradicts Thm 2.1.18.

(<==)

Follows directly from Thm 2.1.20.

Return now to $CT(N_2)$ in Fig 2-5. By Thm 2.1.23, p_3 is bounded on N_2. Further, by Thm 2.1.21, $M(p_3) \leq 1$ for all M $R(N_2)$. Thus our solution to problem #2 meets the first requirement.

We demonstrate that N_2 meets the second requirement as follows: By the weak transition rule, t_5 is enabled on any marking M $R(N_2)$ such that $M(p_3) > 0$. Since it is given that any process which obtains control of the resource will relinquish it eventually, t_5 must fire eventually and yield some marking M' on P such that $M'(p_4) = 1$ and $M'(p_3) = 0$. Since $M' \geq M_0$ and since there exists $w \in T^*$ such that $M_0(w>M$, we have $M'(w>$. Thus another process can obtain control of the resource. Since the above argument can be repeated indefinitely, we have obtained the second condition.

The development of the coverability graph which follows, will be of use in segment 2.2.

Def 2.1.24 Weak Coverability Graph:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net and

$CT(N) = (T',l_V,l_E)$, $T' = (V',E',\iota',\varphi')$ be the weak coverability tree associated with N.

Then the weak coverability graph is a system consisting of the directed graph

$D' = (V,E,\iota,\varphi)$

and a labeling function $l:E \to T$ defined as follows.

Let $V = \{Q \mid Q \in N_{\infty}^{|P|},\ \exists\, v' \in V' \ni l_V(v') = Q\}$.

For each $e' \in E'$ such that $l_V(\iota'(e')) = Q$, $l_E(e') = t \in T$, $l_V(\varphi'(e')) = Q'$ and $Q,Q' \in V$, introduce a new edge $e \in E$ such that:

$\iota(e) = Q$,

$l(e) = t$ and

$\varphi(e) = Q'$.

Note that the labeling function $l:E \to T$ need not be distinct - i.e. for all $e_1,e_2 \in E$, $l(e_1) = l(e_2) \not\Rightarrow e_1 = e_2$. However, $(\iota(e_1) = \iota(e_2)) \wedge (\varphi(e_1) = \varphi(e_2)) \wedge (l(e_1) = l(e_2)) \Rightarrow e_1 = e_2$.

We write

$CG(N) = (D',l)$, $D = (V,E,\iota,\varphi)$

to denote the weak coverability graph associated with N. Note that we could also define the strong coverability graph by substituting the strong coverability tree of N for CT(N).

$CG(N_2)$ in Fig. 2-6 is offered as an example of a coverability graph. As will be shown in the following theorems, we can obtain much the same information from the coverability graph as we can from the coverability tree.

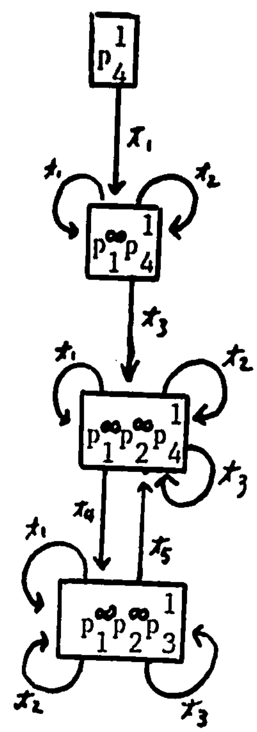


Fig 2-6 The Coverability Graph of N_2 $CG(N_2)$:

Thm 2.1.25:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$CG(N) = (D',l)$, $D' = (V,E,\tau,\varphi)$ be the weak coverability graph associated with N,

$Q_i, Q_j \in V$

$P_i^\infty, P_j^\infty \subseteq P$ be subsets of P such that

$$Q_i(p) = \infty \iff p \in P_i^\infty$$

and

$$Q_j(p) = \infty \iff p \in P_j^\infty,$$

Then 1) If $e \in E$, $\tau(e) = Q_i$, $\varphi(e) = Q_j$ and $l(e) = t \in T$,

Then a) $Q_i(p) \geq B\ t(p) \quad \forall\, p \in P$,

b) $P_i^\infty \subseteq P_j^\infty$ and

c)
$$Q_j(p) = \begin{cases} Q_i(p) + D\ t(p) & \forall\ p \in P \setminus P_j^\infty \\ \infty & p \in P_j^\infty, \end{cases}$$

2) $Q_i \in V \implies$ for all $k \in N$ there exists $M_k \in R(N)$ such that

$M_k(p) = Q_i(p) \quad \forall\ p \in P \setminus P_i^\infty$ and

$M_k(p) \geq k \quad \forall\ p \in P_i^\infty$,

3) A place $p \in P$ is unbounded on N iff there exists a $Q \in V$ such that $Q(p) = \infty$.

Pf:

1) Follows directly from the definitions of CT(N) and CG(N).

2) Follows directly from Thms 2.1.20 & 2.1.21.

3) Follows directly from Thm 2.1.23.

Thm 2.1.26:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$CG(N) = (D',l)$, $D' = (V,E,\tau,\varphi)$ be the weak coverability graph associated with N and

$w \in T^*$ label a loop in D'.

Then there exists a marking $M \in R(N)$ such that $M(w>$.

Pf:

For each $Q_i \in V$, define $P_i^\infty \subseteq P$ such that

$$Q_i(p) = \infty \iff p \in P_i^\infty.$$

Let $L = Q_0 e_1 Q_1 \ldots Q_{n-1} e_n Q_n$ be a loop in D' where

$Q_0 = Q_n$,

$l(e_j) = t_j \quad \forall\ j \in \{1,2,\ldots,n\}$ and

$w = t_1 t_2 \ldots t_n$.

By Thm 2.1.25 - 2), for all $k \in N$ there exists $M_k \in R(N)$ such that

$M_k(p) = Q_0(p) \quad \forall\ p \in P \setminus P_0^\infty$ and

$M_k(p) \geq k \quad \forall\ p \in P_0^\infty$.

By Thm 2.1.25 - 1), $P_0 = P_1 = \ldots = P_n$.

Suppose $P_0 = \emptyset$, then by definitions of CT(N) and CG(N),

$Q_i \in R(N) \quad \forall\ i \in \{0,1,\ldots,n\}$ and

$Q_{i-1}(t_i > Q_i \quad \forall\ i \in \{1,2,\ldots,n\}$.

Hence $Q_0 = M(w>M = Q_n$, and we are done.

Now Suppose $P_0 \neq \emptyset$.

Choose $K \in N$ such that

$K \geq (B \cdot w)(p) \quad \forall\ p \in P_0^\infty$.

By Thm 2.1.25 - 2) there exists $M = \tilde{M}_0 = \tilde{M}_n$ such that

$M(p) = Q_0(p) \quad \forall\ p \in P \setminus P_0^\infty$,

$M(p) \geq K \geq (B \cdot w)(p) \quad \forall\ p \in P_0^\infty$ and

$M \in R(N)$.

Define $\bar{M}_i = \bar{M}_0 + D(t_1 \ldots t_i) \;\; \forall\, i \in \{1,2,\ldots,n\}$.

By definition of CT(N) and CG(N) and the weak transition rule,

$$\bar{M}_i(p) = Q_i(p) \;\; \forall\, p \in P \cdot P_0^{\infty},$$

$$\bar{M}_i(p) \geq (B\cdot(t_{i+1} \ldots t_n))(p) \;\; \forall\, p \in P_0^{\infty} \text{ and}$$

$$\bar{M}_i \in R(N)$$

for all $i \in \{0,1,\ldots,n\}$.

Thus t_{i+1} is enabled on $\bar{M}_i$ for all $i \in \{0,1,\ldots,n\}$.

Therefore $\bar{M}_0 = \bar{M}_n = M(w\rangle$.

Segment 2.2 - Petri Nets with Regular Firing Languages:

In this segment, we show that petri nets with regular languages exist, and that for a given petri net N, it is decidable whether or not the firing language of N is regular. Since any actual problem in concurrency would be too unwieldy, we restrict ourselves to small examples chosen to illustrate specific points.

This section begins with some definitions of boundedness conditions for sets of places and results concerning them and their relationship with the weak coverability tree. This relationship is used to demonstrate the decidability of the regularity of the the firing language of a petri net. Thus if we seem to go far afield at first, the reader is asked to persevere as all is tied together in Thm 2.2.1. This said, we begin.

Def 2.2.1 Characteristic Function of a Subset:

Let P be a set,

$P' \subseteq P$ be a subset of P.

Then we define the characteristic function of P', written $U_{P'}$, as follows:

$$U_{P'}(p) = \begin{cases} 1 & \forall p \in P' \\ 0 & \forall p \in P \setminus P'. \end{cases}$$

Note that $U_{P'}$ can also be thought of as the characteristic vector of P'.

Def 2.2.2 Boundedness for Sets:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$M: P \rightarrow \mathbf{N}$ be a marking on P,

$P' \subseteq P$ be a set of places and

$U_{P'}: P \rightarrow \mathbf{N}$ be the characteristic function of P'.

Then a) P' is bounded for M iff there exists $k \in \mathbf{N}$ such that for each $M' \in (M>$,

$M'(p) \leq k$ for some $p \in P'$.

b) P' is uniformly bounded for M iff there exists $k \in \mathbf{N}$ such that for each $M' \in (M>$,

$M'(p) \leq k \quad \forall p \in P'$.

c) P' is bounded below for M iff there exists $k \in \mathbf{N}$ such that for each $M' \in (M + n \cdot U_{P'}>$, $n \in \mathbf{N}$,

$M'(p) \geq M(p) + n - k$

for some $p \in P'$.

d) P' is <u>uniformly bounded below for M</u> iff there exists $k \in \mathbb{N}$ such that for each $M' \in (M + n \cdot U_{P'}>$, $n \in \mathbb{N}$,

$$M'(p) \geq M(p) + n - k \quad \forall p \in P'.$$

<u>Thm 2.2.3</u>:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$M:P \dashrightarrow \mathbb{N}$ be a marking on P and

$P' \subseteq P$ be a set of places.

Then a) P' uniformly bounded for M ==> P' is bounded for M.

b) P' uniformly bounded below for M ==> P' is bounded below for M.

c) P' uniformly bounded for M <==> for all $p \in P'$, {p} is bounded for M.

d) P' uniformly bounded below for M ==> for all $p \in P'$, {p} is bounded below for M.

Pf: Follows directly from Def 2.2.2.

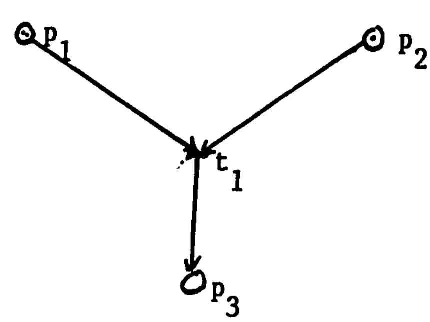


Fig 2-7 The petri net N':

The assymitry of d) may bother the reader at first, however a glance at the petri net N' in Fig 2-7 that while both $\{p_1\}$ and $\{p_2\}$ are bounded below for the initial marking, $\{p_1,p_2\}$ is not even bounded below for the initial marking, much less uniformly bounded below.

<u>Def 2.2.4 Unbounded With Context M</u>:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$M':P \dashrightarrow \mathbb{N}$ be a marking on P and

$P' \subseteq P$ be a non-empty set of places.

Then P' is said to be unbounded with context M' iff

a) $M'(p) = 0 \quad \forall\, p \in P'$.

b) For all $k \in \mathbb{N}$ there exists $M'' \in R(N)$ such that

1) $M''(p) = M'(p) \quad \forall\, p \in P \setminus P'$ and

2) $M''(p) \geq k \quad \forall\, p \in P'$.

<u>Def 2.2.5 Maximal</u>:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$\mathcal{Y}$ be the set of couples (P',M'), $P' \subseteq P$, $M':P \dashrightarrow \mathbb{N}$ a marking on P, such that P' is unbounded with context M' and

$(P',M'), (P'',M'') \in \mathcal{Y}$.

Define the partial ordering relationship $\preceq$ on $\mathcal{Y}$ by

$(P',M') \preceq (P'',M'')$

iff

1) $P' \subseteq P''$ and

2) $M'(p) \leq M''(p) \quad \forall p \in P - P''$.

Let $(P',M')^{\preccurlyeq} = \{(P'',M'') \mid (P'',M'') \in \mathcal{T}, (P',M') \preccurlyeq (P'',M'')\}$.

We say that (P',M') is __maximal__ or __maximally unbounded with context M'__ iff

$$(P',M')^{\preccurlyeq} = \{(P',M')\}$$

(i.e. $((P',M'),(P'',M'') \in \mathcal{T}) \wedge ((P',M') \preccurlyeq (P'',M'')) \Longrightarrow$
$P' = P''$ and $M' = M''$.).

__Thm 2.2.6__:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$\mathcal{T}$ be defined as in Def 2.2.5 and

$(P_1,M_1) \in \mathcal{T}$.

Then there exists $(P_m,M_m) \in (P_1,M_1)^{\preccurlyeq}$ such that (P_m,M_m) is maximal.

Pf: by contradiction

Suppose that $(P_1,M_1)^{\preccurlyeq}$ contains no maximal element.

Let $(P',M') \in (P_1,M_1)^{\preccurlyeq}$.

Since $P_1 \subseteq P' \subseteq P$ and $|P| < \infty$, the set

$$\bar{P} = \{P' \mid (P',M') \in (P_1,M_1)^{\preccurlyeq}\}$$

must be finite. Further, there must exist at least one $P^* \in \bar{P}$ such that

$$\{P' \mid P' \in \bar{P}, P^* \subseteq P'\} = \{P^*\}.$$

Consider the subset of $(P_1,M_1)^{\preccurlyeq}$ defined as follows:

$$S = \{(P',M') \mid (P',M') \in (P_1,M_1)^{\preccurlyeq}, P' = P^*\}.$$

Note that by our choice of P^*, if

$(P^*,M') \in S$

is maximal in S, it is also maximal in $(P_1,M_1)^{\preccurlyeq}$.

By hypothesis, $(P_1,M_1)^{\preccurlyeq}$ has no maximal element.

Thus S has no maximal element.

Hence for all $(P^*,M') \in S$, there exists $(P^*,M'') \in S$, $M' < M''$ such that

$$(P^*,M') \preccurlyeq (P^*,M'').$$

Therefore S is infinite, and we can define an infinite sequence of couples

$$(P^*,M_i)_{i \in \mathbf{N}} \in S \quad \forall i \in \mathbf{N}$$

with the property

$$M_i < M_{i+1} \quad \forall i \in \mathbf{N}.$$

Since $|P| < \infty$, there exists $\beta \in P \setminus P^*$ such that

$$|\{i | i \in \mathbf{N},\ M_i(\beta) < M_{i+1}(\beta)\}| = \infty.$$

Define $\hat{P} = P^* \cup \{\beta | \beta \in P \setminus P^*,\ |\{i | i \in \mathbf{N},\ M_i(\beta) < M_{i+1}(\beta)\}| = \infty\}$.

Since

$$|\{i | i \in \mathbf{N},\ M_i(p) < M_{i+1}(p),\ p \in P \setminus \hat{P}\}| < \infty,$$

there exists $j \in \mathbf{N}$ such that

$$M_i(p) = M_{i+1}(p) \quad \forall\ p \in P \setminus \hat{P},\ i \in \mathbf{N},\ i \geq j.$$

Define
$$\hat{M}(p) = \begin{cases} 0 & \forall\ p \in \hat{P} \\ M_j(p) & \forall\ p \in P \setminus \hat{P}. \end{cases}$$

Consider the pair $(\hat{P},\hat{M})$.

We now show that $\hat{P}$ is unbounded with context $\hat{M}$, and hence $(\hat{P},\hat{M}) \in \mathcal{Y}$.

By construction, $\hat{M}(p) = 0 \quad \forall\ p \in \hat{P}$.

Let $k \in \mathbf{N}$.

By construction of $\hat{P} \setminus P^*$, there exists $h \in N$, $h > j$ such that

$$M_h(p) \geq k \quad \forall\, p \in \hat{P} \setminus P^*.$$

Since $(P^*, M_h) \in S \subseteq (P_1, M_1)^{\preccurlyeq} \in \mathcal{T}$, there exists $\bar{M} \in R(N)$ such that

$$\bar{M}(p) = M_h(p) \quad \forall\, p \in P \setminus P^* \quad \text{and}$$

$$\bar{M}(p) \geq k \quad \forall\, p \in P^*.$$

But $\bar{M}(p) = M_h(p) = \hat{M}(p) \quad \forall\, p \in P \setminus \hat{P}$,

$$\bar{M}(p) = M_h(p) \geq k \quad \forall\, p \in \hat{P} \setminus P^* \quad \text{and}$$

$$\bar{M}(p) \geq k \quad \forall\, p \in P^*.$$

Thus $\hat{P}$ is unbounded with context $\hat{M}$ and $(\hat{P}, \hat{M}) \in \mathcal{T}$.

Since $(P_1, M_1) \preccurlyeq (\hat{P}, \hat{M})$,

$$(\hat{P}, \hat{M}) \in (P_1, M_1)^{\preccurlyeq}.$$

But $P^* \subset \hat{P}$, a contradiction.

Hence $(P_1, M_1)^{\preccurlyeq}$ contains a maximal element.

<u>Def 2.2.7 Maximal Vertex of a Coverability Graph</u>:

Let $N = (P,T,B,F,K,W.M_0)$ be a petri net,

$CG(N) = (D',1)$, $D' = (V,E,\tau,\varphi)$, be the weak coverability graph associated with N and

$Q \in V$.

Then Q is said to be a <u>Maximal vertex</u> of CG(N) iff for all $Q' \in V$,

$$Q' \geq Q \implies Q' = Q.$$

<u>Thm 2.2.8</u>:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net,

$CG(N) = (D',1)$, $D' = (V,E,\tau,\varphi)$ be the weak coverability graph associated with N,

$P_1 \subseteq P$ be a non-empty set of places and

$M_1 : P \dashrightarrow N$ be a marking on P.

Then P_1 is maximally unbounded with context M_1 iff there exists a maximal vertex $Q \in V$ in CG(N) such that

$$Q(p) = \begin{cases} \infty & \forall\, p \in P_1 \\ M_1(p) & \forall\, p \in P \setminus P_1. \end{cases}$$

Pf: (==>)

Suppose P_1 is unbounded with context M_1 and (P_1, M_1) is maximal.

Then for all $k \in N$ there exists $M' \in R(N)$ such that

a) $M'(p) = M_1(p) \quad \forall\, p \in P \setminus P_1$ and

b) $M'(p) \geq k \quad \forall\, p \in P_1$.

Since CG(N) is finite, we can find a constant $h \in N$ such that

$Q(p) \geq h \iff Q(p) = \infty \quad \forall\, Q \in V,\ p \in P$.

Choose $k > h$ and let $M' \in R(N)$ be defined as above.

Let $CT(N) = (T' 1_{V'}, 1_{E'})$, $T' = (V', E', r', \varphi)$ be the weak coverability tree associated with N.

By Thm 2.1.21, $M' \in R(N) \implies$ there exists a vertex $v \in V'$ such that $1_V(v) = \bar{Q}$, and a set $\bar{P} \subseteq P$ such that

$\bar{Q}(p) = M'(p) \quad \forall\, p \in P \setminus \bar{P}$ and

$\bar{Q}(p) = \infty \quad \forall\, p \in \bar{P}$.

By the construction of CG(N), $\bar{Q} \in V$.

By our choice of k, $P_1 \subseteq \bar{P}$.

We must now show that $P_1 = \bar{P}$:

Suppose $P_1 \neq \bar{P}$.

Then by part two of Thm 2.1.25, for all $j \in N$ there exists

$M_j \in R(N)$ such that:

$$M_j(p) = \bar{Q}(p) \quad \forall\, p \in P \setminus \bar{P}$$

$$M_j(p) \geq j \quad \forall\, p \in \bar{P}$$

or, to put it more simply, $\bar{P}$ is unbounded with context $\bar{M}$,

where $$\bar{M}(p) = \begin{cases} \bar{Q}(p) & \forall\, p \in P \setminus \bar{P} \\ 0 & \forall\, p \in \bar{P}. \end{cases}$$

Note that $(P_1, M_1) \precsim (\bar{P}, \bar{M})$.

But, by hypothesis, (P_1, M_1) is maximal.

Hence $\bar{P} = P_1$.

It remains to be shown that $\bar{Q}$ is a maximal vertex in CG(N). We do so by contradiction:

Suppose there exists $\hat{Q} \in V$ such that $\hat{Q} > \bar{Q}$.

Define $\hat{P}$ such that

$$\hat{Q}(p) = \infty \iff p \in \hat{P}.$$

Define: $$\hat{M}(p) = \begin{cases} \hat{Q}(p) & \forall\, p \in P \setminus \hat{P} \\ 0 & \forall\, p \in \hat{P}. \end{cases}$$

Again, by part two of Thm 2.1.15, for all $i \in N$ there exists

$M_i \in R(N)$ such that

$$M_i(p) = \hat{M}(p) \quad \forall\, p \in P \setminus \hat{P}$$

$$M_i(p) \geq i \quad \forall\, p \in \hat{P}$$

which is to say that $\hat{P}$ is unbounded with context $\hat{M}$.

By hypothesis, either

$$P_1 = \bar{P} \subset \hat{P}$$

or

There exists $p \in P \setminus \hat{P}$ such that $\hat{M}(p) > M_1(p)$.

In either case, $(P_1, M_1) \preccurlyeq (\hat{P}, \hat{M})$.

But this contradicts (P_1, M_1) maximal.

Hence $\bar{Q}$ is a maximal vertex of CG(N).

($\Longleftarrow$)

Suppose Q is a maximal vertex in CG(N).

Let $P_1 \subseteq P$ be a set of places such that

$$Q(p) = \infty \Longleftrightarrow p \in P_1.$$

Define:
$$M_1(p) = \begin{cases} Q(p) & \forall p \in P \setminus P_1 \\ 0 & \forall p \in P_1. \end{cases}$$

Again, by part two of Thm 2.1.25, P_1 is unbounded with context M_1.

It remains to be shown that (P_1, M_1) is maximal.

By Thm 2.2.6, $(P_1, M_1)^{\preccurlyeq}$ contains a maximal couple (P_m, M_m).

By the first half of this Thm, there exists $Q_m \in V$ such that

$$Q_m(p) = M_m(p) \quad \forall\ p \in P \setminus P_m,$$

$$Q_m(p) = \infty \quad \forall\ p \in P_m$$

and Q_m is a maximal vertex in CG(N).

Since $P_1 \subseteq P_m$ and $M_m(p) \geq M_1(p) \quad \forall p \in P \setminus P_m$, we have $Q_m \geq Q$.

By hypothesis, Q is a maximal vertex in CG(N), hence

$$Q = Q_m.$$

Thus $P_1 = P_m$ and $M_1 = M_m$.

Therefore (P_1, M_1) is maximal.

Thm 2.2.9:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net and

$CG(N) = (D',1)$, $D' = (V,E,\tau,\varphi)$ be the weak coverability graph associated with N.

Then there exists $P_1 \subseteq P$, $P_1 \neq \emptyset$ and $M_1 \in \mathbf{N}^{|P|}$ such that

1) P_1 is maximally unbounded with context M_1 and

2) P_1 is not uniformly bounded below for M_1

iff there exists a maximal vertex $Q \in V$, a loop Δ in CG(N) such that $1(\Delta) = w$ T*, and $\beta \in P_1$ such that

3) Δ has initial and final vertex Q,

4) $D(\beta,w) < 0$ and

5) $Q(\beta) = \infty$.

Pf: (==>)

Suppose (P_1,M_1) is maximal and P_1 is not uniformly bounded below for M_1.

By Thm 2.2.8, there exists a maximal vertex $Q \in V$ such that:

$$Q(p) = \begin{cases} \infty & \forall p \in P_1 \\ M_1(p) & \forall p \in P \setminus P_1. \end{cases}$$

Since P_1 is not uniformly bounded below for M_1, for all $k \in \mathbf{N}$ there exists $n_k \in \mathbf{N}$, $M'_k \in (M_1 + U_{P_1} \cdot n_k\rangle$, $w_k \in T^*$ and $p_k \in P_1$ such that

$$(M_1 + U_{P_1} \cdot n_k)(w_k\rangle M'_k \quad \text{and}$$
$$M'_k(p_k) < M_1(p_k) + n_k - k.$$

Thus for all $k \in \mathbf{N}$, $D(p_k,w_k) < -k$.

Since P_1 is finite, there exists $\beta \in P_1$ such that the set

$$\Lambda = \{k \mid k \in N,\ p_k = \beta\}$$

is infinite.

Let $d = -\,\mathrm{Min}\{D(p,t) \mid p \in P,\ t \in T\}$.

Note that the number of tokens that can be removed from any one place by the firing of any one transition is bounded by d.

Choose $\hat{k} \in N$ such that $\hat{k} > d \cdot |V|$.

By our choice of $\hat{k}$, we can devide $w_{\hat{k}}$ into $|V|$ firing sequences

$$w_{\hat{k}} = v_1 v_2 \ldots v_{|V|}$$

such that $D(\beta, v_i) < 0$ for all $i \in N$, $1 \leq i \leq |V|$.

Since

$$Q(p) = \begin{cases} \infty & \forall\, p \in P_1 \\ M_1(p) & \forall\, p \in P \setminus P_1, \end{cases}$$

CG(N) must contain a path Δ starting at Q and labeled by $w_{\hat{k}}$.

Since $l(\Delta) = w_{\hat{k}} = v_1 v_2 \ldots v_{|V|}$, we can devide Δ into $|V|$ segments such that

$$\Delta = \Delta_1 \Delta_2 \ldots \Delta_{|V|}$$

and

$$l(\Delta_i) = v_i \quad \forall\, i \in N,\ 1 \leq i \leq |V|.$$

For all $i \in N$, $1 \leq i \leq |V|$ let

$\tilde{Q}_i$ be the initial vertex of Δ_i and

$\bar{Q}_i$ be the final vertex of Δ_i.

By our choice of $\hat{k}$, there exists $j, j' \in N$, $1 \leq j \leq j' \leq |V|$ such that

$$\tilde{Q}_j = \bar{Q}_{j'}, \quad \text{and}$$

$$D(\beta,\ v_j v_{j+1} \ldots v_{j'}) < 0.$$

Define $\hat{Q} = \tilde{Q}_j = \bar{Q}_{j'}$,

$v = v_j v_{j+1} \dots v_{j'}$.

If $\hat{Q}$ is maximal, we are done.

If not, there exists a maximal $\hat{Q}' \in V$ such that $\hat{Q}' \geq \hat{Q}$.

Since $\hat{Q} \geq \hat{Q}$, by the construction of CG(N), there must exist a circuit in CG(N), labeled by v, which starts and ends in $\hat{Q}'$.

Since $D(\beta, v) < 0$, we have proved the first half of the Thm.

(<==)

Suppose there exists a maximal vertex $Q \in V$, a loop Δ in CG(N) such that $l(\Delta) = w \in T^*$ and $\beta \in P$ such that

1) Δ has initial and final vertex Q,

2) $D(\beta, w) < 0$ and

3) $Q(\beta) = \infty$.

Let $P_1 = \{p \mid p \in P,\ Q(p) = \infty\}$ and

$$M_1(p) = \begin{cases} 0 & \forall p \in P_1 \\ Q(p) & \forall p \in P \setminus P_1. \end{cases}$$

By Thm 2.2.8, P_1 is maximally unbounded with context M_1

By the construction of CG(N), $D(\beta, w) < 0 \implies Q(\beta) = \infty \implies \beta \in P_1$, as otherwise Δ could not be a loop.

It remains to be shown that P_1 is not uniformly bounded below for M_1.

For all $k \in N$, let

$n_k = (k + 1) \cdot \mathrm{Max}\{B(p,w) \mid p \in P_1\}$ and

$w_k = w^{k+1}$ = w concatinated with itself k + 1 times.

Then $(M_1 + U_{P_1} \cdot n_k)(w_k>$.

Hence for all $k \in N$ there exists $M_k' \in (M_1 + U_{P_1} \cdot n_k>$ such that

$$(M_1 + U_{P_1} \cdot n_k)(w_k > M'_k.$$

Since $D(\beta,w) < 0$,

$$D(\beta,w_k) < -k \quad \forall k \in \mathbf{N}.$$

Thus for all $k \in \mathbf{N}$

$$M'_k(\beta) < M_1(\beta) + n_k - k.$$

Hence P_1 is not uniformly bounded below for M_1.

Def 2.2.10 Language of Firing Sequences:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net and

$F(N) = \{w \mid w \in T^*, M_0(w>\}$.

Then $F(N)$ is said to be the language of firing sequences of N, or the firing language of N.

Def 2.2.11 Regular Petri Net:

A petri net N is said to be regular iff $F(N)$ is regular.

Thm 2.2.12:

A Petri Net $N = (P,T,B,F,K,W,M_0)$ is regular iff there exists $k \in \mathbf{N}$ such that for all M $R(N)$, $M' \in (M>$ and $p \in P$,

$M'(p) \geq M(p) - k$.

Pf: (<==) by construction

Suppose $N = (P,T,B,F,K,W,M_0)$ is a petri net such that there exists $k \in \mathbf{N}$ such that for all $M \in R(N)$, $M' \in (M>$ and $p \in P$,

$M'(p) \geq M(p) - k$.

We must show that $F(N)$ is a regular language. We shall do so by constructing a finite recognition automaton $\mathcal{A}'$ which recognizes $F(N)$.

Let $c = k + Max\{M_0(p) | p \in P\} + Max\{B(p,t) | p \in P,\ t \in T\}$.

We define $\mathcal{A}' = (D', A', 1', S', F')$, $D' = (V', E', \iota', \varphi')$ as follows:

Let $V' = \{M | M \in N^{|P|},\ M(p) \leq c \ \ \forall\, p \in P\} \cup \{v_g\}$

where v_g is a garbage vertex,

$S' = \{M_0\}$,

$F' = V' \setminus \{v_g\}$ and

$A' = T$.

For each $M, M' \in F'$, $t \in T$ such that

$$M(p) \geq B(p,t) \quad \forall\, p \in P$$

and

$$M'(p) = Min\{c,\ M(p) + D(p,t)\} \quad \forall\, p \in P,$$

include an edge e in E' such that

$\iota'(e) = M$,

$1'(e) = t$ and

$\varphi'(e) = M'$.

For all $M \in F'$ and $t \in T$ such that there exists $p \in P$ such that

$$M(p) < B(p,t),$$

include an edge e in E' such that

$\iota'(e) = M$,

$1'(e) = t$ and

$\varphi'(e) = v_g$.

Having defined $\mathcal{A}'$, we must now show that

$$w \in F(N) \iff w \in L(\mathcal{A}').$$

Suppose that $w \in F(N)$.

Let n equal the number of transitions in w.

Then we can write

$$w = t_1 t_2 \ldots t_n.$$

Further, for all $i \in \mathbb{N}$, $i \leq n$, there exists $M_i \in R(N)$ such that

$$M_0(t_1>M_1(t_2>M_2 \ldots M_{n-1}(t_n>M_n.$$

We now construct inductively an admissable path Δ in $\mathcal{A}'$ such that

$$l'(\Delta) = w.$$

Base step:

Since $S' = \{M_0\}$ F, the directed path of length zero

$$\Delta_0 = M_0$$

is admissable in $\mathcal{A}'$.

Since

$$l'(\Delta_0) = \Lambda$$

the nul firing sequence is accepted by $\mathcal{A}'$.

Induction step:

Suppose that for $i \in N$, $i \leq n$, there exists an admissable path Δ_i in $\mathcal{A}'$ such that

$$\Delta_i = M'_0 e_1 M'_1 \ldots M'_{i-1} e_i M'_i$$

where $l'(\Delta_i) = l'(e_1) l'(e_2) \ldots l'(e_i)$.

If $i = n$, $l'(\Delta_i) = w$ and we are done.

If $i < n$, we construct Δ_{i+1} via one of the following two cases:

Case 1 - ($M'_j = M_j \quad \forall j \in N,\ j \leq i$):

Since $M_i(t_{i+1}>M_{i+1}$, by the construction of $\mathcal{A}'$,

there exists $e_{i+1} \in E'$ and $M'_{i+1} \in F'$ such that

$$\tau'(e_{i+1}) = M'_i,$$

$$l'(e_{i+1}) = t_{i+1},$$

$$\varphi'(e_{i+1}) = M'_{i+1} \quad \text{and}$$

$$M'_{i+1}(p) = \mathrm{Min}\{c,\ M'_i(p) + D(p,t_{i+1})\} \quad \forall p \in P$$

$$= \mathrm{Min}\{c,\ M_{i+1}(p)\} \quad \forall p \in P.$$

Hence

$$\Delta_{i+1} = M'_0 e_1 M'_1 e_2 M'_2 \ldots M'_i e_{i+1} M'_{i+1}$$

is an admissable path in $\mathcal{A}'$ such that

$$l'(\Delta_{i+1}) = l'(e_1)l'(e_2)\ldots l'(e_{i+1})$$

$$= t_1 t_2 \ldots t_{i+1}.$$

Case 2 - (there exists $j \in N$, $1 \leq j \leq i$ and $\beta \in P$ such that $M_j(\beta) > M'_j(p) = c$):

If there is more than one such j, choose the least.

Let $v = t_{j+1} t_{j+2} \ldots t_i$.

Since $M_i \in (M_j>$, by hypothesis,

$$M_i(\beta) \geq M_j(\beta) - k.$$

Hence $D(v,\beta) \geq -k$.

Since $M'_j e_{j+1} \ldots e_i M'_i$ is a path in $\mathcal{A}'$ and

$l'(e_{j+1})\ldots l'(e_i) = v$, by construction of $\mathcal{A}'$ we have that

$$M'_i(\beta) \geq M'_j(\beta) - k = c - k.$$

Thus $M'_i(p) \geq c - k$

$$\geq Max\{B(t,p) | p \in P,\ t \in T\} + Max\{M_0(p) | p \in P\} > 0.$$

Hence t_{i+1} is enabled on M'_i for all $p \in P$ such that there exists $j \in N$, $0 \leq j \leq i$ such that

$$M_j(p) > M'_j(p) = c.$$

For all $p \in P$ such that for all $j \in N$, $0 \leq j \leq i$,

$M_j(p) = M'_j(p)$ we have that

$$M_i(p) = M'_i(p).$$

Since $M_i(t_{i+1} > M_{i+1}$, t_{i+1} is enabled on M'_i for all such p.

Thus t_{i+1} is enabled on M'_i.

Therefore, by the construction of $\mathcal{A}'$, there exists an edge $e_{i+1} \in E'$ and $M'_{i+1} \in F'$ such that

$$\tau'(e_{i+1}) = M'_i,$$

$$l'(e_{i+1}) = t_{i+1},$$

$$\varphi'(e_{i+1}) = M'_{i+1} \quad \text{and}$$

$$M'_{i+1}(p) = Min\{c,\ M'_i(p) + D(p,t_{i+1})\}\ \forall p \in P.$$

Thus we can construct

$$\Delta_{i+1} = \Delta_i e_{i+1} M'_{i+1}$$

$$= M'_0 e_1 m'_1 \ldots M'_i e_{i+1} M'_{i+1}$$

where $l'(\Delta_{i+1}) = l'(\Delta_i) l'(e_{i+1})$

$$= l'(e_1) l'(e_2) \ldots l'(e_i) l'(e_{i+1})$$

$$= t_1 t_2 \ldots t_i t_{i+1}$$

and Δ_{i+1} is an admissable path in $\mathcal{A}'$.

Therefore $w \in F(N) \Longrightarrow w \in L(\mathcal{A}')$.

Now suppose that $w \in L(\mathcal{A}')$.

Let n equal the number of transitions in w.

Then we can write

$$w = t_1 t_2 \ldots t_n.$$

Since $w \in L(\mathcal{A}')$, there exists an admissable path

$$\Delta = M_0' e_1 M_1' e_2 M_2' \ldots M_{n-1}' e_n M_n'$$

such that

$$1'(e_i) = t_i \quad \forall i \in \mathbb{N},\ 1 \leq i \leq n,$$

$$M_0' \in S' \quad \text{and}$$

$$M_0' = M_0.$$

We now show inductively that w is enabled on M_0.

Base step:

Since $M_0' = M_0$, by the construction of $\mathcal{A}'$, the firing sequence

$$w_1 = t_1$$

is enabled on M_0, and thus there exists an $M_1 \in R(N)$ such that

$$M_0(w_1>M_1.$$

Induction step:

Suppose that for $i \in \mathbb{N}$, $0 \leq i \leq n$, the firing sequence

$$w_i = t_1 t_2 \ldots t_i$$

is enabled on M_0, and thus for all $j \in \mathbb{N}$, $0 \leq i \leq n$ there exists $M_j \in R(N)$ such that

$$M_0(t_1>M_1(t_2>M_2 \ldots M_{i-1}(t_i>M_i.$$

If i = n, we are done.

If $i < n$, we must show that there exists $M_{i+1} \in R(N)$ such that

$$M_i(t_{i+1}>M_{i+1}.$$

By construction of $\mathcal{A}'$,

$$M'_{j+1}(p) = Min\{c, M'_j(p) + D(p,t_{j+1})\}$$

for all $p \in P$, $j \in \mathbb{N}$, $0 \leq j < n$.

Thus $M_j \geq M'_j$ for all $j \in \mathbb{N}$, $0 \leq j < n$.

By the weak transition rule,

$$((M_i \geq M'_i) \wedge (M'_i(t_{i+1}>)) \Longrightarrow M_i(t_{i+1}>.$$

Hence there exists $M_{i+1} \in R(N)$ such that

$$M_i(t_{i+1}>M_{i+1}$$

which concludes our induction.

Hence $w \in L(\mathcal{A}') \Longrightarrow w \in F(N)$.

Combining the above with the previous result, we obtain

$$w \in F(N) \Longleftrightarrow w \in L(\mathcal{A}').$$

Since F(N) is recognized by $\mathcal{A}'$ and $\mathcal{A}'$ is a finite recognition automaton, by Thm 1.2.10, F(N) is a regular language.

(==>) by contradiction

Suppose that $N = (P,T,B,F,K,W,M_0)$ is a petri net such that F(N) is a regular language. We must show that there exists $k \in \mathbb{N}$ such that for all $M \in R(N)$, $M' \in (M>$ and $p \in P$,

$$M'(p) \geq M(p) - k.$$

Proof follows by contradiction.

Suppose that for all $k \in \mathbb{N}$ there exists $M \in R(N)$, $M' \in (M>$ and $\hat{p} \in P$ such that

$M'(\beta) < M(\beta) - k.$

Since F(N) is regular, by Thm 1.2.10, there exists a finite recognition automaton

$\mathcal{A}' = (D', A', l', S', F'), \quad D' = (V', E', \alpha', \varphi)$

such that $F(N) = L(\mathcal{A}')$.

Let $k = |V'| \cdot (-\mathrm{Min}\{D(p,t) \mid p \in P,\ t \in T\})$.

Then, by hypothesis, there exist firing sequences $v, w \in T^*$, markings $M, M' \in R(N)$ and $\beta \in P$ such that

$M_0(v>M(w>M',$

$M(\beta) > k$ and

$M'(\beta) < M(\beta) - k.$

Further there must exist two paths Δ_v and Δ_w in $\mathcal{A}'$ such that

$l'(\Delta_v) = v,$

$l'(\Delta_w) = w$

and $\Delta_v \Delta_w$ exists and is an admissable path in $\mathcal{A}'$. i.e.

$l'(\Delta_v \Delta_w) = vw \in L(\mathcal{A}') = F(N).$

By our choice of k, w can be devided into at least $|V'|$ shorter firing sequences such that

$w = w_1 w_2 \ldots w_{|V'|}$

where $D(\beta, w_i) < 0$ for all $i \in \mathbb{N},\ 1 \leq i \leq |V'|$.

Similarly, we can devide Δ_w into $|V'|$ subpaths such that

$\Delta_w = \Delta_{w_1} \Delta_{w_2} \ldots \Delta_{w_{|V'|}}$

where $l'(\Delta_{w_i}) = w_i \quad \forall\, i \in \mathbb{N},\ 1 \leq i \leq |V'|$.

Let $M_i', M_i'' \in V,\ i \in \mathbb{N},\ 1 \leq i \leq |V'|$ be respectively the initial and final vertices of Δ_{w_i}.

Thus $M''_i = M'_{i+1}$ for all $i \in N$, $1 \leq i < |V'|$,

M'_1 is the initial vertex of Δ_w and

$M''_{|V'|}$ is the final vertex of Δ_w.

Since we have defined $|V'| + 1$ vertices as initial and/or final vertices of the Δ_{w_i}, there must exist $j, j' \in N$, $1 \leq j \leq j' \leq |V'|$ such that

$M'_j = M'_{j'}$.

Let $r = vw_1w_2 \ldots w_{j-1} \in T^*$,

$s = w_jw_{j+1} \ldots w_{j'} \in T^*$,

$\Delta_r = \Delta_v \Delta_{w_1} \Delta_{w_2} \ldots \Delta_{w_{j-1}}$ and

$\Delta_s = \Delta_{w_j} \Delta_{w_{j+1}} \ldots \Delta_{w_{j'}}$.

Thus $1'(\Delta_r) = r$ and

$1'(\Delta_s) = s$.

Since $Pref(F(N)) \subseteq F(N)$ and $vw \in F(N)$, it follows that $rs \in F(N)$ and thus $\Delta_r \Delta_s$ must be an admissable path in $\mathcal{A}'$.

Hence $M''_{j'} \in F'$.

Note that by our construction of w_i, $i \in N$, $1 \leq i \leq |V'|$,

$D(\beta, s) < 0$.

Since Δ_s is a loop, Δ_r followed by Δ_s n times, written $\Delta_r \Delta_s^n$, must also be an admissable path in $\mathcal{A}'$.

Let $\Delta_u = \Delta_r \Delta_s^{M(\beta)+1}$ and

$u = 1'(\Delta_u) = rs^{M(\beta)+1}$.

Then $u \in L(\mathcal{A}')$.

But $u \notin F(N)$, since firing u would leave a negative number of tokens in β.

Hence $L(\alpha') \neq F(N)$, which contradicts the hypothesis that $F(N)$ is regular.

Therefore there exists $k \in \mathbf{N}$ such that for all $M \in R(N)$, $M' \in (M\rangle$ and $p \in P$,

$$M'(p) \geq M(p) - k.$$

<u>Thm 2.2.13</u>:

Let $N = (P,T,B,F,K,W,M_0)$ be a petri net.

Then N is not regular iff there exists a marking $M_1 \in \mathbf{N}_{\infty}^{|P|}$ and a set $P_1 \subseteq P$ such that

1) P_1 is maximally unbounded with context M_1 and
2) P_1 is not uniformly bounded from below for M_1.

Pf: (==>)

Suppose N is not regular.

Then by Thm 2.2.12, for all $k \in \mathbf{N}$ there exists $M_k \in R(N)$, $M_k' \in (M_k\rangle$ and $p_k \in P$ such that

$$M_k'(p_k) < M_k(p_k) - k.$$

Since $|P| < \infty$, there exists $\beta \in P$ such that

$$\alpha = \{k \mid k \quad \mathbf{N},\ p_k = \beta\}$$

and $|\alpha| = \infty$.

Further, by Thm 1.3.4, Zorn's Lemma and since $|P| < \infty$, we can define the infinite set

$$\mathcal{B} = \{k \mid k \in \alpha, ((k,k' \in \mathcal{B},\ k < k') \Longrightarrow (M_k < M_{k'}))\}$$

which in turn defines an infinite increasing sequence of markings

$$(M_k)_{k\in\mathcal{B}}$$

Let $P' = \{p \mid p\in P,\ |\{M_k(p) \mid (k,k'\in\mathcal{B},\ k > k') \Longrightarrow$

$$(M_k(p) > M_{k'}(p))\}| = \infty\}.$$

Note that $\beta\in P'$.

By our choice of P', we can find an infinite subset $\mathcal{C}\subseteq\mathcal{B}$ such that

$$\mathcal{C} = \{k \mid k\in\mathcal{B},\ ((k,k'\in\mathcal{C},\ k < k',\ p\in P') \Longrightarrow$$

$$(M_k(p) < M_{k'}(p)))\}.$$

By definition of $\mathcal{C}$ and P',

$$|\{M_k(p) \mid p\in P\setminus P',\ k\in\mathcal{C},\ ((k'\in\mathcal{C},\ k > k') \Longrightarrow$$

$$(M_k(p) > M_{k'}(p)))\}| = \infty.$$

Thus there must exist some infinite subset $\mathcal{D}\subseteq\mathcal{C}$ such that

$$\mathcal{D} = \{k \mid k\in\mathcal{C},\ (k,k'\in\mathcal{D}) \Longrightarrow (M_k(p) = M_{k'}(p)\ \ \forall\ p\in P\setminus P')\}.$$

Since $\mathcal{D}\subseteq\mathbf{N}$, $\mathcal{D}$ is well ordered and thus contains a least element $\bar{k}\in\mathcal{D}$.

Define $\mathcal{E} = \mathcal{D}\setminus\{\bar{k}\}$.

Since $\mathcal{E}$ is also well ordered, we can assign an index $i\in\mathbf{N}$ to each $k\in\mathcal{E}$ such that

$$k_i < k_{i+1}\quad \forall\ k_i,k_{i+1}\in\mathcal{E}.$$

Further, by our choice of $\mathcal{E}$,

$$i < k_i\quad \forall\ i\in\mathbf{N},\ k_i\in\mathcal{E}.$$

Also, by our choice of $\mathcal{C}$ and $\mathcal{E}$,

$$M_{k_i}(p) > i\quad \forall\ p\in P',\ i\in\mathbf{N}.$$

To recapitulate, we have defined an infinite sequence of markings $(M_{k_i})_{i \in \mathbb{N}}$, where $k_i \in \mathscr{E}$ $\forall$ $i \in \mathbb{N}$ with the following properties:

1) For all $i \in \mathbb{N}$ there exists $k_i \in \mathscr{E} \subseteq \mathbb{N}$, $M_{k_i} \in R(N)$ and $M'_{k_i} \in (M_{k_i}>$ such that

$$0 \leq M'_{k_i}(\beta) < M_{k_i}(\beta) - k_i.$$

2) $M_{k_i} \leq M_{k_{i+1}}$ $\forall$ $i \in \mathbb{N}$.

3) $M_{k_i}(p) > i$ $\forall$ $p \in P'$, $i \in \mathbb{N}$.

4) $M_{k_i}(p) = M_{k_j}(p)$ $\forall$ $i,j \in \mathbb{N}$, $p \in P \setminus P'$.

5) $i < k_i$ $\forall$ $i \in \mathbb{N}$.

Define:

$$M'(p) = \begin{cases} 0 & \forall\ p \in P' \\ M_{k_1}(p) & \forall\ p \in P \setminus P'. \end{cases}$$

By definition of M' and properties 3) & 4) above, P' is unbounded with context M'.

By Thm 2.2.6, $(P',M')^{\preccurlyeq}$ contains a maximal element (P_1,M_1).

We must now show that P_1 is not uniformly bounded from below for M_1.

For all $i \in \mathbb{N}$, define $w_{k_i} \in T^*$ to be the firing sequence such that

$$M_{k_i}(w_{k_i}>M'_{k_i}$$

For $i \in \mathbb{N}$, define $n_{k_i} \in \mathbb{N}$ such that

$$n_{k_i} = \mathrm{Max}\{B(p,w_{k_i}) \mid p \in P_1\}.$$

Since $M_1(p) \geq M'(p) = M_{k_i}(p)$ $\forall$ $p \in P \setminus P_1$, $i \in \mathbb{N}$, by the weak transition rule,

$$(M_1 + U_{P_1} \cdot n_{k_i})(w_{k_i}> \quad \forall\, i \in \mathbb{N}$$

where U_{P_1} is the characteristic function of P_1.

Hence for all $i \in \mathbb{N}$ there exists $M''_{k_i} \in \mathbb{N}^{|P|}$ such that

$$(M_1 + U_{P_1} \cdot n_{k_i})(w_{k_i} > M''_{k_i}.$$

By properties 1) & 5) above and since $\beta \in P' \setminus P_1$,

$$-D(\beta, w_{k_i}) > k_i > i.$$

Thus for all $i \in \mathbb{N}$ there exists $M''_{k_i} \in (M_1 + U_{P_1} \cdot n_{k_i}>$, $n_{k_i} \in \mathbb{N}$ such that

$$M''_{k_i}(\beta) < M_1(\beta) + n_{k_i} - i.$$

Therefore P_1 is not uniformly bounded from below for M_1.

(<==)

Suppose that P_1 is maximally unbounded with context M_1 and not uniformly bounded from below for M_1.

We wish to show that for all $k \in \mathbb{N}$ there exists $M_k \in R(N)$, $M_k' \in (M_k>$ and $p_k \in P$ such that

$$M_k'(p_k) < M_k(p_k) - k$$

which will yield the desired result via Thm 2.2.12.

Since P_1 is not uniformly bounded from below for M_1, for all $k \in \mathbb{N}$ there exists $p_k \in P_1$, $n_k \in \mathbb{N}$, $n_k > 0$, $w_k \in T^*$ and $M_k \in \mathbb{N}^{|P|}$ such that

$$(M_1 + n_k \cdot U_{P_1})(w_k > M_k$$

where U_{P_1} is the characteristic function of P_1, and

$$M_k(p_k) < M_1(p_k) + n_k - k.$$

Since (P_1, M_1) is maximal, by Thm 2.2.8, CG(N) contains a maximal vertex Q such that

$$Q(p) = \begin{cases} M_1(p) & \forall p \in P \setminus P_1 \\ \infty & \forall p \in P_1. \end{cases}$$

By part two of Thm 2.1.25, for all $k \in \mathbb{N}$ there exists $\hat{M}_k \in R(N)$ such that

$$\hat{M}_k(p) = M_1(p) \quad \forall\ p \in P \setminus P_1 \quad \text{and}$$
$$\hat{M}_k(p) > n_k \quad \forall\ p \in P_1.$$

Thus, by the weak transition rule, for all $k \in \mathbb{N}$ there exists $\hat{M}'_k \in R(N)$ such that

$$\hat{M}_k(w_k>\hat{M}'_k$$

where

$$\hat{M}'_k(p_k) < \hat{M}_k(p_k) - k.$$

Therefore, by Thm 2.2.12, N is not regular.

<u>Thm 2.2.14</u>:

The regularity of a petri net $N = (P,T,B,F,K,W,M_0)$ is decidable.

Pf:

Let $CG(N) = (D',l)$, $D' = (V,E,\tau,\varphi)$ be the weak coverability graph associated with N.

By Thms 2.2.8 & 2.2.13, we have that N is not regular iff the following condition *) holds:

*) There exists a maximal $Q \in V$, $p' \in P$ such that $Q(p') = \infty$, and a loop Δ in CG(N) with $l(\Delta) = w \in T^*$ such that

1) Δ has initial and final vertex Q and
2) $D(p',w) < 0$.

If we provide an effective procedure for testing the truth of *), we will have shown that the regularity of N is decidable. We begin by showing that it is sufficient to test *) for simple loops only.

Suppose there exists $Q \in V$, $p' \in P$ and a loop Δ in CG(N) where $l(\Delta) = w \in T^*$ satisfying *). Further suppose that Δ is not a simple loop.

Then there exists $Q_1 \in V$ such that Δ can be devided into three segments

$$\Delta = \Delta_1 \Delta_2 \Delta_3$$

such that Δ_1 has initial vertex Q and final vertex Q_1, Δ_2 is a simple loop with initial and final vertex Q_1 and Δ_3 has initial vertex Q_1 and final vertex Q.

By part one of Thm 2.1.25,

$$Q(p) = \omega \iff Q_1(p) = \omega \quad \forall \; p \in P.$$

Define: $l(\Delta_1) = v_1$,

$l(\Delta_2) = v_2$ and

$l(\Delta_3) = V_3$.

Two cases:

Case 1 - ($D(p',v_2) \geq 0$):

Then $D(p',v_1 v_2) \leq D(p',w) < 0$ and $\Delta_1 \Delta_3$ is a loop satisfying *).

If $\Delta_1 \Delta_3$ is simple, then we are done.

If $\Delta_1 \Delta_3$ is not a simple loop, then it can be devided into three parts as before, at which point either case 1 or case 2 applies.

Case 2 - ($D(p',v_2) < 0$):

By construction of CG(N), there exists a maximal vertex

$\bar{Q}$ V such that $\bar{Q} \geq Q_1$.

Again by construction of CG(N), there exists a simple

loop Δ' in CG(N) with initial and final vertex $\bar{Q}$

such that $l(\Delta') = v_2$.

Thus we have shown that N is not regular iff there exists

$Q \in V$, $p' \in P$, $w \in T^*$ and a loop Δ such that

1) Q, p', w and Δ satisfy *) and

2) Δ is simple.

Since V is finite, the set of all simple loops Δ in CG(N) which start and end in a maximal vertex is also finite and hence can be enumerated.

For each such Δ let $l(\Delta) = w_\Delta \in T^*$.

Since $|P| < \infty$, we can calculate $D(p, w_\Delta)$ for all $p \in P$ and for each w_Δ.

Hence the regularity of N is decidable.

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