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Economic lot sizes in a multi-stage production-inventory system

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ECONOMIC LOT SIZES IN A MULTI-STAGE PRODUCTION-INVENTORY SYSTEM

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ABSTRACT

Economic lot sizes for multi-stage production-inventory systems treated in the literature generally do not consider the ordering, inter-departmental transfer costs or the holding costs at the consuming stage. The classical economic lot size models consider either the producer or the consumer, but not both. If the producer and the consumer are considered as a system, a common, producer-consumer economic lot size can be derived in closed form if raw material inventories at the producing stage are not considered. The addition of raw material inventories to a system where the production rate is greater than the demand rate at the consuming stage destroys all of the convex-concave properties of the system cost function. This pheonmenon is due to the extra inventory holding costs incurred during the production idle time. One model found in the literature restricts the raw material inventory lot sizes to integer multiples of the system lot size to circumvent the non-convexity property of the cost function.

A simulation model was programmed to investigate the behaviour of the system cost function for a simple two-stage system with four raw material inventories; static, deterministic demand; and various production rates. It was found that the optimum system cost is obtained when the lot sizes for the raw material inventories are an integer multiple of the system lot size if the Wilson EOQ for those inventories is greater than one-half the system lot size. If the EOQ is less than one-half the system lot size, the cost performance cannot be predicted.

Therefore, in any mathematical formulation of this type of problem, the raw material inventories can be restricted to be integer multiples of the system lot size. If any inventory has an EOQ which is less than one-half the derived system lot size, this inventory should be investigated to see if further reduction in the lot size would improve the system cost.

CHAPTER I

INTRODUCTION

A. General

Inventories, in general, are a means of maintaining smooth production flow, obtaining maximum utilization of capital and facilities, and providing reasonable service to customers. Inherent in any production-inventory system are two sets of costs:

1. Those fixed costs associated with manufacturing set-up and ordering.
2. The variable costs associated with manufacturing, inventory storage, and any associated transportation costs.

The objective of a production-inventory control model is to provide a policy that will effect a tradeoff in the two sets of costs that will minimize the total system costs over a planning horizon while maintaining a reasonable level of service to the customer.

Production-inventory systems can be classified into four general categories as follows:

1. Single-facility system
2. Parallel-facility system
3. Series-facility system
4. Series-parallel facility system

Inventory control models for the single facility system are well documented in the technical literature. In the single-stage (single-facility) system, each location manufactures or orders separately and is concerned only with its own financial welfare. The most widely used model for the single stage system is the economic order quantity model. This model is used for systems with deterministic demand to determine the economic order quantity (EOQ) of items to be purchased. Analogous to this model is the economic manufacturing quantity (EMQ) model which is used to determine the manufacturing lot size for items that are produced and consumed internally by an organization. The introduction of stochastic demand functions complicates the computations for EOQ's and EMQ's but the problem is not intractable.

Since the time of the development of the single-stage models, a myriad of models has been documented that deal with multi-stage production-inventory systems. This paper will concern itself with a particular case of the series production-inventory system.

There are many industries that exhibit a production-inventory system as shown in Figure 1. In this system, there is a series of production facilities with a customer inventory for finished product. Each production facility has a supply of raw materials and subassemblies for use in product manufacture. The semi-finished product is then shipped to the succeeding manufacturing stage in predetermined lot sizes. This process continues until the finished product is delivered to the consumer's warehouse.

Most of the models that deal with multi-echelon production-inventory system. If the various facilities are under different managerial control, it is possible that each facility will determine its own EOQ/ EMQ and operate as efficiently as it can. Another way to operate a system such as this is to use the technique of batch processing wherein

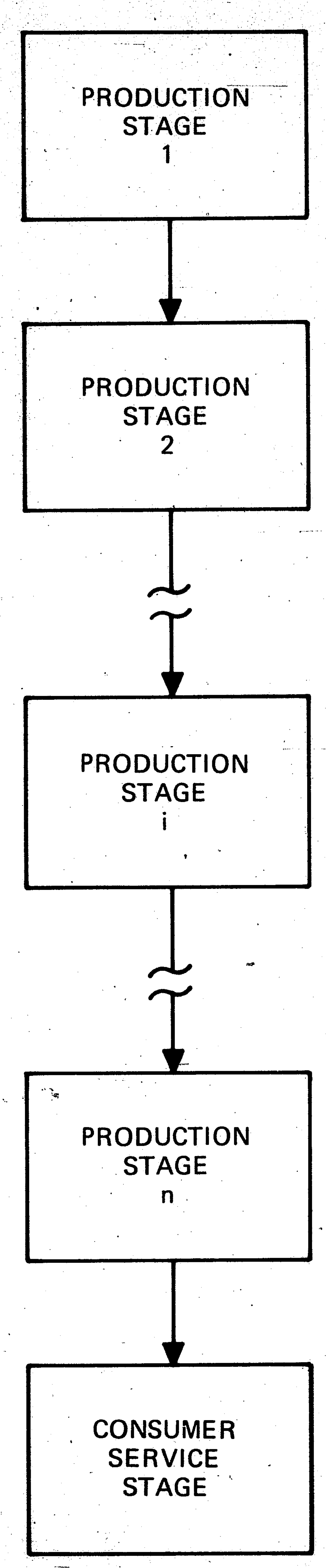

Figure 1. Typical Series Production-Inventory System.

a common lot size is determined and the same size lot is processed through all stages. A variation of this method is the case where the consumer orders a lot size that is equivalent to the economic manufacturing lot size, regardless of the cost structure at the consuming stage.

B. Objectives

The interaction between stages and the existence of buffer inventories require that the parameters of the entire production inventory system be considered in the development of economic lot size between stages. The purpose of this paper is to investigate the simple series production-inventory system and its cost structure and to develop the methodology for determining the economic interstage lot sizes that will minimize the total system cost.

CHAPTER II

BACKGROUND

A. General

The development of an economic lot size model for a production-inventory system involves delicate tradeoffs of the various cost elements of the system. These costs can be generally classified into the following categories.

1. Preparation or setup costs
2. Production costs
3. Handling and storage costs
4. Shortage costs
5. Capital costs

In order to make the mathematics more manageable, many of the economic lot size models in the literature make simplifying assumptions about one or more of the various cost elements above. The application of a particular model to a particular situation, therefore, may lead to disastrous results. To quote Starr and Miller [2]:

> "If an important cost or set of such costs is overlooked or purposely ignored, the analysis (of the production-inventory system) produces incorrect conclusions. Many times the evidence of such errors of omission or judgement is lost . . . hopelessly entangled and absorbed in the amalgamated overhead of non-specific costs which are felt but not recognized by the company."

B. Classical Models

Prior to the discussion of the various pertinent lot size models, certain factors must be defined:

Define:

TIC = total incremental cost

TIC* = total incremental cost of the optimal solution

Q = lot size

Q* = optimum lot size

D = annual demand in units

C_H = inventory holding cost per unit per year

C_p = set up and preparation costs per order

p = production rate

The original model for economic lot sizes is the classical "Wilson" model. This model is a single station inventory model with static deterministic demand. The inventory fluctuation for this model is presented in figure 2(a). The inventory is depleted at a rate D during the consumption period t_c, where $t_c = Q/D$. The total incremental cost equation is:

$$TIC = \frac{Q}{2} C_H + \frac{D}{Q} C_p$$

Taking the derivative with respect to Q and equating the result to zero gives:

$$Q^* = \sqrt{2C_pD/C_H}$$

Substituting Q* into the TIC equation gives:

$$TIC^* = \sqrt{2C_pC_HD}$$

One extension of the Wilson model is the inventory model for production runs where the production lot is received in inventory over a period of time as production progresses. The inventory fluctuation

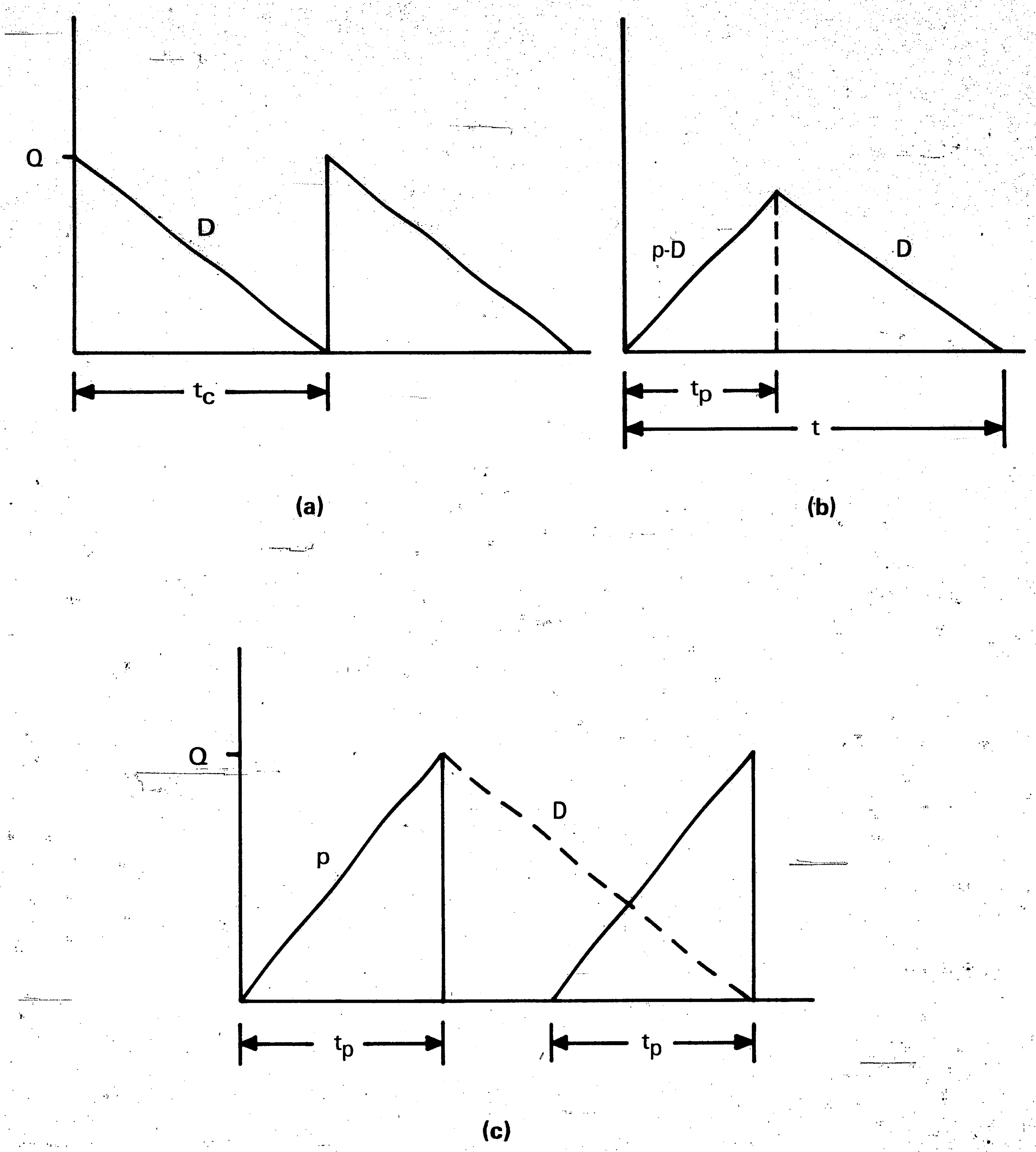

Figure 2. Inventory Cycle Times for Three Systems.

per cycle is presented in figure 2(b). The peak inventory per cycle is $(p - D)t_p$, where $t_p = Q/p$. The average inventory per cycle is:

$$\bar{I} = \frac{(p - D)Q^2}{2pD}$$

The total incremental cost equation is:

$$TIC = C_p \frac{D}{Q} + \frac{D}{Q}\frac{(p - D)Q^2}{2pD} C_H$$

$$= C_p \frac{D}{Q} + \frac{Q}{2}(1 - D/p)C_H$$

Taking the derivative with respect to Q and equating the result to zero gives:

$$Q^* = \sqrt{2C_pD/C_H(1 - D/p)}$$

with a resulting optimum cost of:

$$TIC^* = \sqrt{2D\ C_p\ C_H(1 - D/p)}$$

A common practice in manufacturing is not to release the product until a lot of predetermined size is completely produced. During the production phase, inventory is accumulated at the rate p during the production phase, t_p, at which time the lot size, Q, is released for consumption. During the consumption phase, t_c, the inventory is depleted at a rate D. This inventory variation is depicted in figure 2(c). The peak inventory is Q. The average inventory per cycle is:

$$\bar{I} = 1/2Qt_p = Q^2/2p$$

The number of cycles is D/Q and the total incremental cost is:

$$TIC = \frac{D}{Q} C_p + \frac{D}{Q}\frac{Q^2}{2p} C_H$$

$$= \frac{D}{Q} C_p + \frac{D\ Q\ C_H}{2p}$$

Taking the derivative with respect to Q and equating the result to zero gives:

$$Q^* = \sqrt{2pC_p/C_H}$$

with an optimum TIC of:

$$TIC^* = D\sqrt{2C_pC_H/p}$$

Wagner and Whitin [3] developed a different fundamental approach to economic lot size models. Their approach was to divide time into discrete time periods with known deterministic demand. A single product was considered and backlogging of unsatisfied demand was not allowed. The method of solution was dynamic programming.

C. Multi-stage Models

The fundamental shortcoming of the formulations presented above is that the models are designed to optimize the operating costs of single-stations (stages). Very few real-life production-inventory systems operate as independent entities. In general, production-inventory systems are comprised of various production and inventory stages, tied together in a multi-level or multi-echelon fashion. Many models have been developed which deal with multi-echelon systems. Most of the models in the literature deal with pure inventory systems without any consideration given to the relevant costs at the consuming stage or do not consider the production facility(ies). The models presented by Gross [4] and Hadley and Whitin [5] are typical. These two models deal with an inventory system where a single product type is stored at various locations with a central inventory control point. This modeling approach is common in view of the fact that, many times, the production facilities and the inventory facilities are under different managerial control.

Lele [6] has developed an economic lot size model for a two stage production-inventory system. This simple model contains one production stage and one consuming stage. Raw material inventories at the production stage are not considered. The model is a single product model without competition for facilities. The production facility produces the product, one at a time, until a batch of size Q is completed. The completed product is then shipped to the consuming stage where it is depleted at a rate D.

The total incremental cost equation for this model is:

$$TIC = N(C_p + C_c) + N(\overline{I}_p h_p + \overline{I}_c h_c)$$

where:

N = the number of cycles per time period (normally a year) = D/Q.

D = demand per time period

Q = lot size

C_p = setup and preparation costs at the production stage

C_c = setup, preparation, ordering, and/or interdepartmental transfer costs at the consuming stage.

$\overline{I}_p$ = average inventory per cycle at the production stage = $Q^2/2p$

p = production rate per time period

$\overline{I}_c$ = average inventory per cycle at the consuming stage = $Q^2/2D$.

h_p = inventory holding costs at the production stage

h_c = inventory holding costs at the consuming stage

Taking the derivative of the TIC equation with respect to Q and equating the result to zero, the optimum value of Q is found to be:

$$Q^* = \sqrt{\frac{2(C_p + C_c)}{(h_p/p) + (h_c/D)}}$$

Substituting the value of Q* into the TIC equation, the optimum system cost is found to be:

$$TIC^* = D\sqrt{2(C_p + C_c)\left(\frac{h_p}{p} + \frac{h_c}{D}\right)}$$

The problem with the results shown is that the raw material inventories are not considered in the formulation. These inventories are not normally considered the assumption is made that either they are under different managerial control or the materials are purchased from an outside supplier and the costs are carried on the company books in an account separate from the manufacturing account.

D. Definition of the Problem

The problem under consideration concerns a multistage production-inventory system where a single product is manufactured and moves serially through the system in lots of predetermined size, Q_i. The system under consideration is depicted in figure 3. This system contains the production facilities, the associated raw material inventories, and the inventory for the consuming stage. The characteristics of the system are:

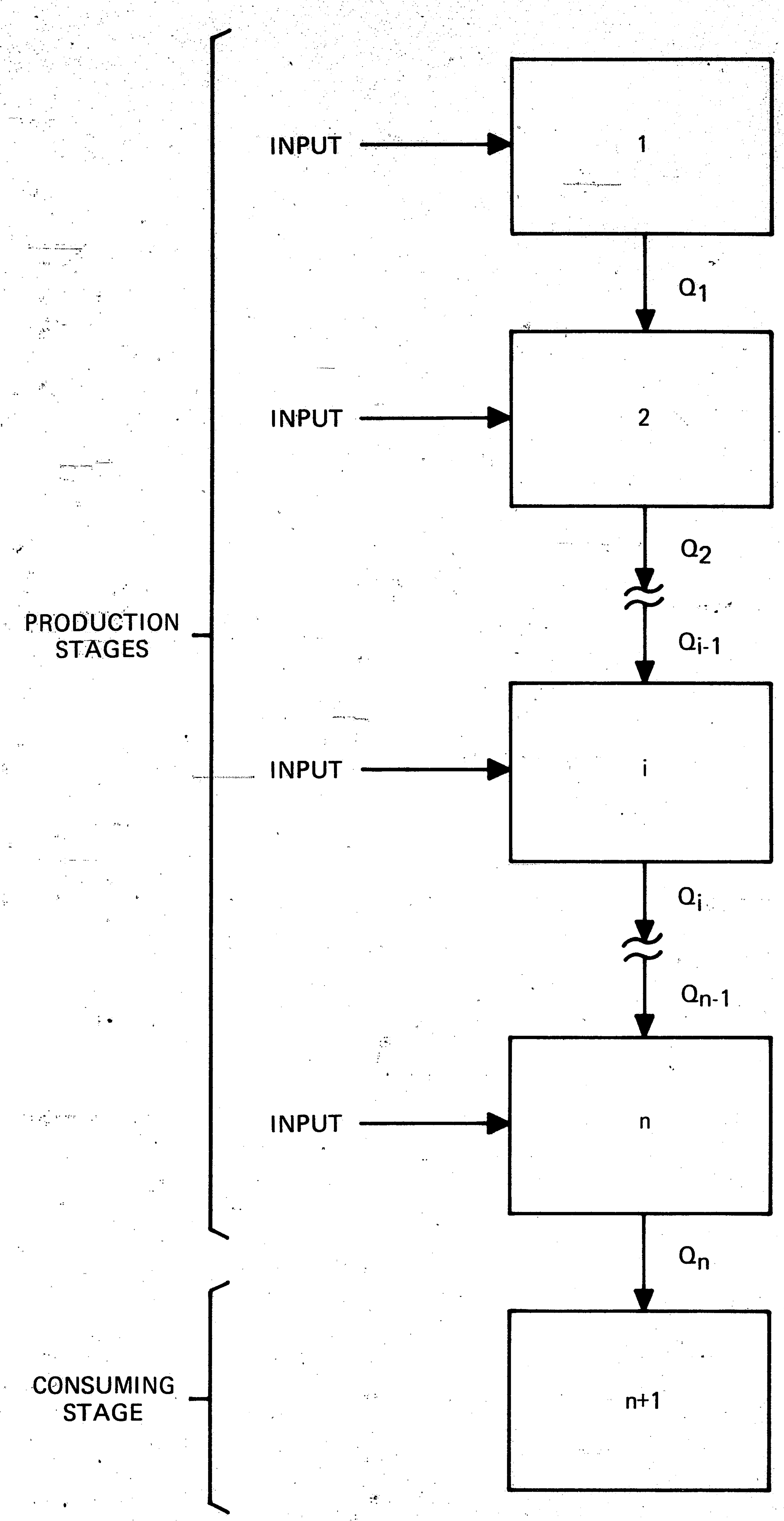

Figure 3. The n-Stage Production-Inventory System and Associated Consuming Stage.

1. Demand at the consuming stage is deterministic and assumed constant over time.
2. The replenishment lot size at any stage i, $i \neq 1$, is constant, the lot size Q_{i-1}.
3. The scheduling period is constant.
4. The replenishment rate is infinite.
5. The lead time between stages is zero.
6. The various setup, preparation and order costs are constant over time.
7. The setup time at the various manufacturing stages is negligible.
8. The inventory holding costs are constant.
9. Shortages at any stage are not allowed.
10. If a raw material inventory reaches the zero level at the end of a production run, this inventory will not be replenished until the start of the next production run. Otherwise, it will be replenished immediately.

The production scheme is such that at each stage i, the product is worked upon (and its value enhanced) until a lot size of Q_i is completed. At this point, the lot is shipped to stage i+1. Raw materials or semi-finished products can be added to the product at any stage i, $i \neq n+1$ (stage n+1 is the consuming stage).

The interactions between stages necessitates that the cost structure of the entire system be considered in the development of the economic lot sizes, Q_i^*. The pertinent inventory costs for the raw material inventories should also be considered since they do represent a capital investment. The problem, then, is to develop a method for determining the size of the production lots at each stage so as to minimize the total cost of the system.

E. Functional Relationships

The complete production-inventory system is shown in figure 4. The various parameters of the system are defined below.

n = the number of production stages

$n+1$ = the consuming stage

Q_i = the lot size shipped from stage i, i=i, 2, ..., n

p_i = the production rate at stage i

D = the demand at the consuming stage

J_i = the number of raw materials required at stage i

q_i^j = the lot size of the jth raw material at stage i; $i = 1, 2, \ldots, n; j = 1, 2, \ldots, J_i$.

R_i^j = the number of units of the jth raw material required for each unit of production at stage i

h_i^j = the holding costs for one unit of the jth raw material at stage i

H_i = the holding costs for one unit of product at production stage i

H_c = the holding costs for one unit of product at the consuming stage

s_i^j = the ordering cost for the jth raw material at stage i

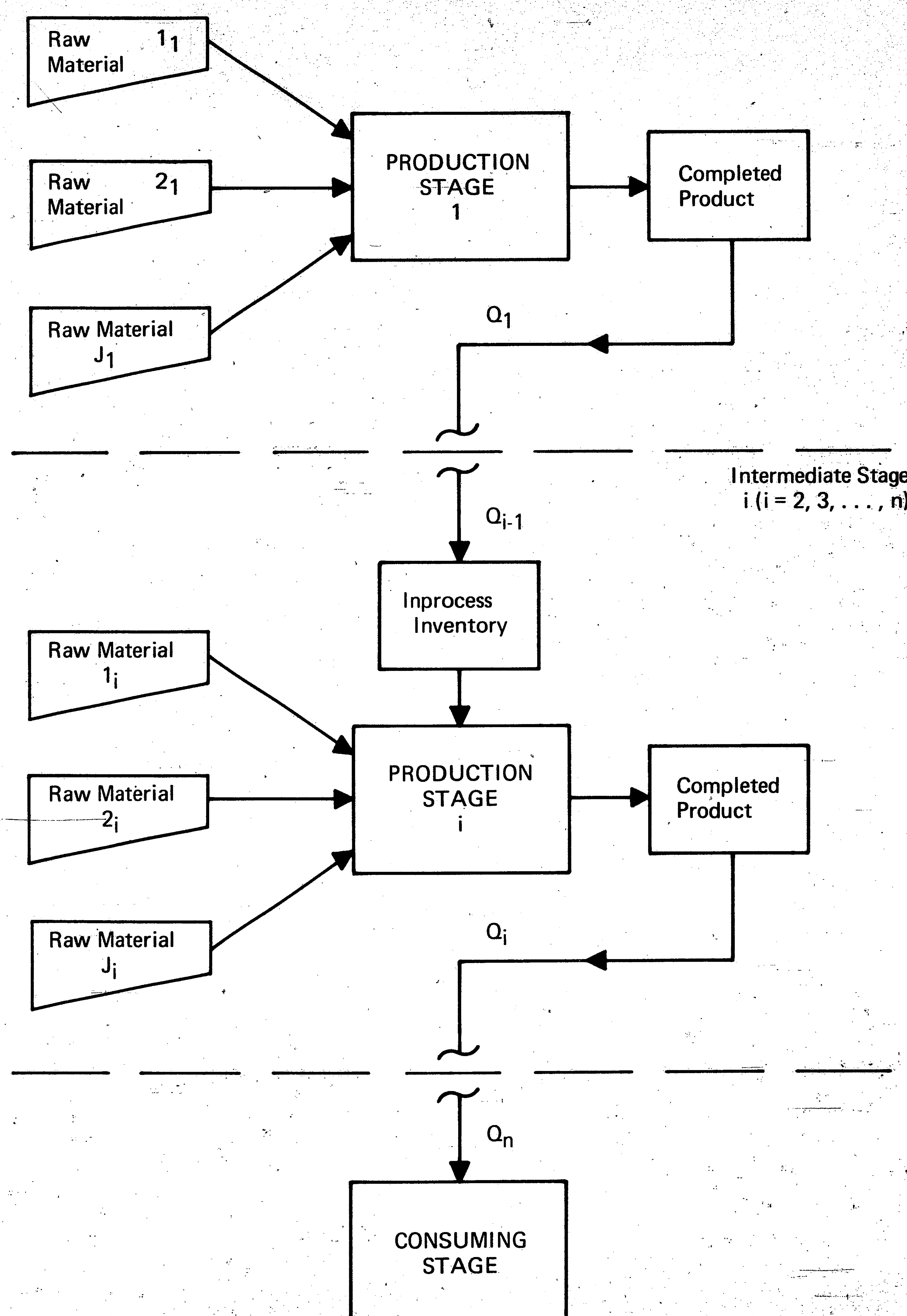

Figure 4. The Complete, n-Stage Production Inventory System

S_i = the setup and preparation costs at production stage i

S_c = the ordering and preparation costs at the consuming stage

The model described above, without the cost structure at the consuming stage, has been formulated by Taha and Skeith [7]. Their model also consists of n production-inventory stages with static deterministic demand. In the development of their model, it was assumed that any unfilled demand at stage n was backlogged; shortages are not allowed at the intermediate stages i; and the lot size produced at each stage, Q_i, is an integer multiple of the final lot size Q_n. The lot sizes for the raw material inventories are also assumed to be integer multiples of the final lot size. The procedure developed by Taha and Skeith could be used to solve the problem under consideration (including the cost structure at the consuming stage) but the integer multiple aspect of the solution creates some question as to the optimality of the solution. With this in mind, it was decided to investigate the formulation of the cost structure for a two stage system (i.e. one production stage and one consuming stage).

F. The Two Stage System with Raw Material Inventories

The two stage system is shown in figure 5. The demand at the consuming stage must be satisfied, therefore it is assumed that the production rate (p) is equal to or greater than the demand, D. If the production rate is equal to the demand rate, the production facility would operate continuously. Under this condition, the economic lot size for the raw material inventories is obtained from the "Wilson" equation. Therefore, it is further assumed that p is strictly

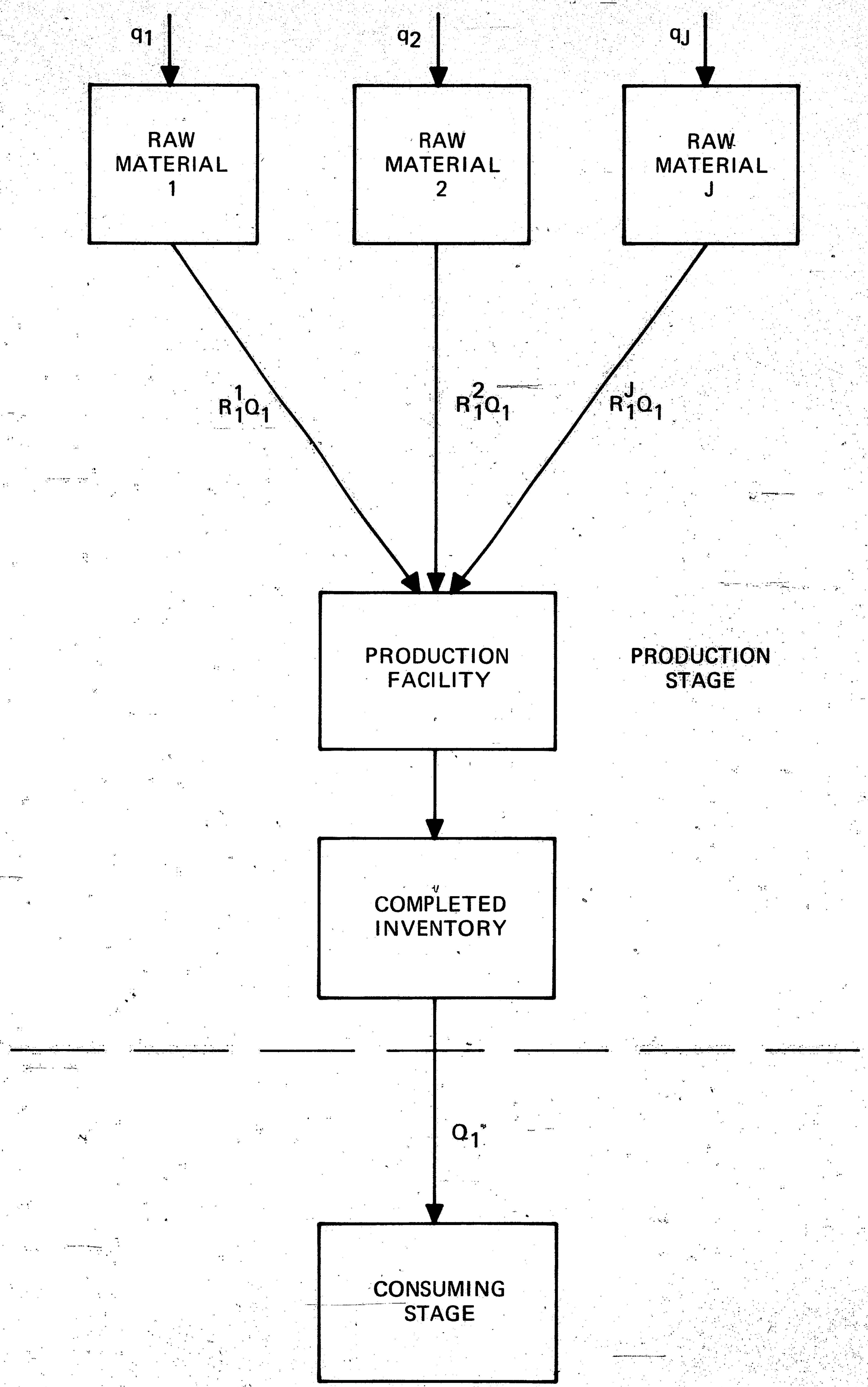

Figure 5. The Two-Stage System with Raw Material Inventories

greater than D.

The inventory position for the system for several cycles is presented in figure 6. The production time for the producer is $t_p (t_p = Q_1/p)$ and the average inventory per cycle is:

$$\overline{I}_p = 1/2 Q_1 t_p = (Q_1)^2/2p$$

The time required for the inventory at the consuming stage to reach the zero level is t_c, $(t_c = Q_1/D)$. The average inventory at the consuming stage is

$$\overline{I}_c = 1/2 Q_1 t_c = (Q_1)^2/2D$$

Both of these inventories will cycle Q_1/D times per year.

There are many possible combinations for the relationship between the raw material lot size, q_j, and the system lot size, Q_1. The raw material lot size, q_j, can be expressed as some multiple of the system lot size (i.e., $q_j = k_j R_j Q_1$).

a. Case 1. Assume that $k_j = 1$ (i.e., $q_j = R_j Q_1$). In this case, the cycle time for the raw material inventory is equal to the cycle time of the producer and consumer. The average inventory per cycle for the raw material inventory is

$$\overline{I}_j = 1/2 R_j Q_1 t_p = R_j (Q_1)^2/2p$$

and the number of cycles is D/Q_1.

b. Case 2. Assume that $q_j = 1.5 R_j Q_1$. The average inventory per cycle is

$$\overline{I}_j = \frac{R_j Q_1^2}{2p} + \frac{.5 R_j Q_1^2}{D} + \frac{.5 R_j Q_1^2}{2p}$$

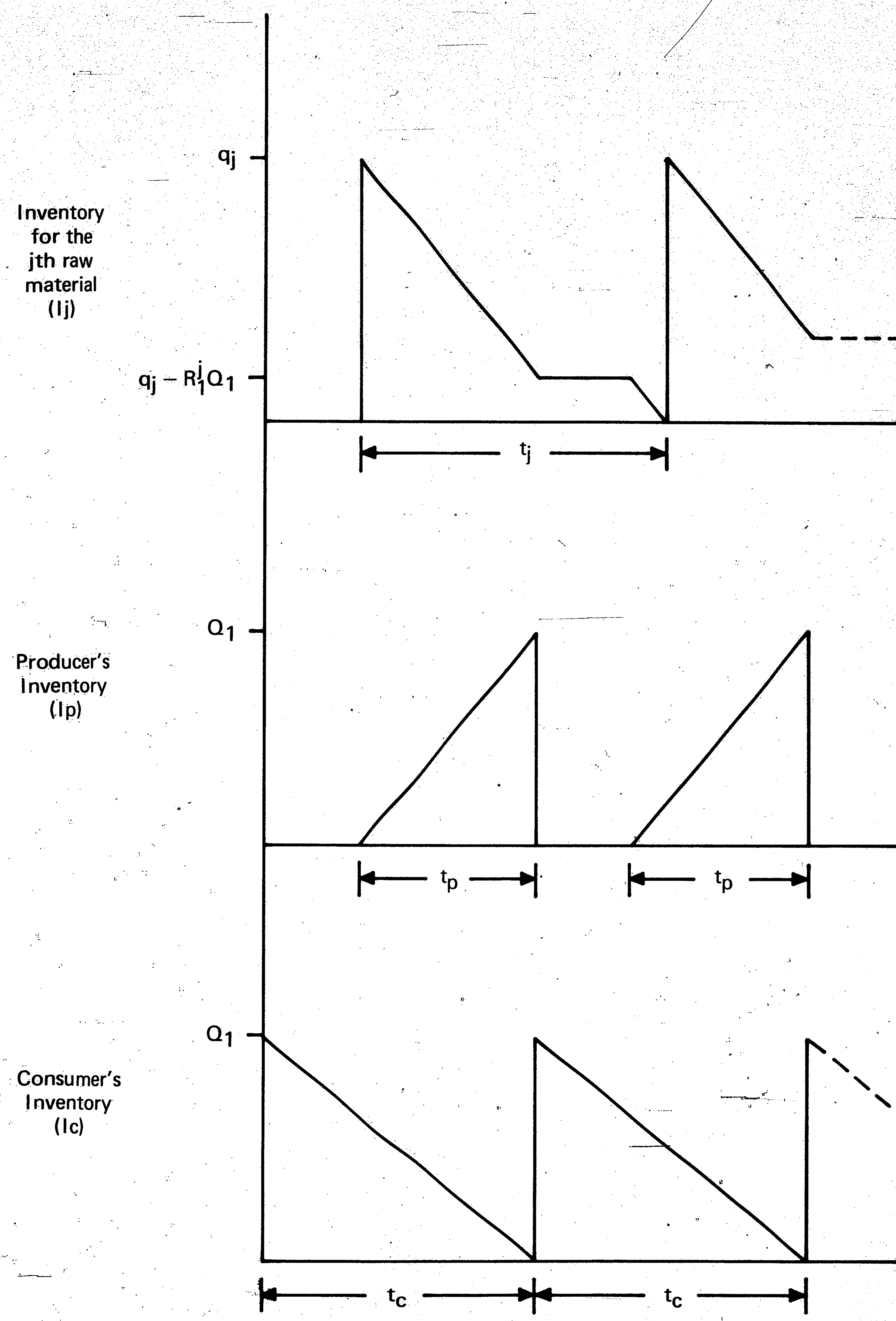

Figure 6. Inventory Positions for the Two-Stage System.

and the number of cycle is $D/1.5Q_1$.

c. Case 3. Assume $q_j = 2R_jQ_1$. The average inventory is

$$\overline{I}_j = \frac{2R_jQ_1^2}{2p} + \frac{R_jQ_1^2}{D}$$

d. The general two stage case: It can be shown that by following the procedures used for cases 1 through 3 above, the average inventory for the jth raw material, where $k_j \geq 1$, is

$$\overline{I}_j = \frac{\lambda R_jQ_1^2}{2p} + \frac{\lambda(\lambda-1)}{2}\frac{R_jQ_1^2}{D} + \beta R_jQ_1\left[\frac{\lambda Q_1}{D} + \frac{\beta R_jQ_1}{2p}\right]$$

and the number of cycles per year is D/k_jQ_1; where

λ = the largest integer $\leq k_j$

$\beta = k_j - \lambda$

The total incremental cost equation for the two-stage system with J raw materials is

$$TIC = N(\overline{I}_c h_c + S_c) + N(\overline{I}_p h_p + S_p) + N_j(\overline{I}_j h_j + S_j)$$

where

N = the number of cycles per time period for the producer consumer inventory

N_j = the number of cycles per time period for the jth raw material inventory

h_n = the inventory holding costs at the nth location

S_n = the fixed costs at the nth location associated with startup, ordering, etc.

Without the raw material inventories, the optimum value of Q_1 can be found using classical calculus. The addition of the raw material inventories makes the problem intractible. In order to evaluate the TIC equation, a common cycle time for the J+2 inventories must be found. If the raw material lot sizes are assumed to be integer multiples of the system lot size, then the equation can be solved. But the optimality of the results is still in question.

CHAPTER III

HYPOTHESIS AND EXPERIMENTAL PROCEDURE

A. Hypothesis

It has been shown that an economic producer-consumer lot size can be derived for the simple two-stage system without raw material inventories. It has also been shown that a solution in closed form can be obtained for the generalized n-stage system if it is assumed that the various interstage and raw material lot sizes are integer multiples of the output lot size, Q_n. The integer multiple aspect of the formulation for the n-stage system leads to the question of optimality of the solution. It can be hypothesized that there is an optimal solution for the system in question that does not involve integer multiples.

B. Experimental Procedure

It was shown in Chapter II that non-integer multiples of the various lot sizes do not yield neat and manageable mathematical expressions for the system cost. For this reason, it was decided to program an evaluator (deterministic simulator) on the computer that described the operation of a two-stage system with raw materails. The characteristics described in Chapter II for the generalized n-stage system are part of the model. Specific values for the various cost elements were used. For each evaluation, the common producer-consumer lot size was held at a fixed value and the lot size of the raw material inventory was incremented between a specific range of values. As a result of these various evaluations, the pertinent cost curves were plotted.

C. Evaluator Number 1

The evaluator that was programmed is a deterministic "next event" simulator. The objective of the evaluator is to obtain average system costs for the parameters under consideration. These costs are:

1. Total system costs
2. Average costs incurred at the production stage only
3. Average costs incurred at the consuming stage only
4. Average raw material inventory costs
   a. Average holding costs
   b. Average setup costs
   c. Average holding costs during the production cycle
   d. Average holding costs during the idel period in the production stage

This evaluator has the following characteristics:

1. The demand at the consuming stage is deterministic and is constant over time.
2. The production lot size and the replenishment lot size, Q, at the consuming stage are equal and constant for any evaluation.
3. The start of the production cycle is timed so that a lot of size Q is completed at precisely the time the consumer's inventory reaches the zero level.
4. The setup time at the production stage is zero.

5. The replenishment rate is infinite.
6. The various holding setup, preparation, and order costs are constant over time.
7. If the raw material inventory reaches the zero level at the end of a production run, it will not be replenished until the start of the next production run. Otherwise, the inventory will be replenished upon reaching the zero level.

For each evaluation the producer-consumer lot size was held constant; then the raw material inventory lot size was incremented in steps of 10 over a specific range. At each step the various system costs are evaluated over a 50-year period.

D. Evaluator Number 2

This evaluator was programmed to evaluate the two-stage system under the situation where the producer, the consumer, and the raw material inventory are operated independently, each using its own Wilson lot size. This evaluation is also deterministic in nature and has the following characteristics:

1. The demand at the consuming stage is deterministic and is constant over time.
2. The lot size that is shipped to the consuming stage can be either the manufacturer's lot size or the producer's lot size; either procedure tends to build up inventories.
3. The start of the production cycle is timed so that the appropriate lot size is completed at precisely the time the consumer's inventory reaches the zero level.

4. The setup time at the production stage is zero.
5. The replenishment rate is infinite.
6. The various pertinent costs are constant over time.
7. If the raw material inventory reaches the zero level at the end of a production run, it will not be replenished until the start of the next production run. Otherwise, the inventory will be replenished upon reaching the zero level.

CHAPTER IV

EXPERIMENTAL RESULTS AND DISCUSSION

A. General

The two evaluators discussed in Chapter III were used to evaluate the two-stage system whose parameters are shown in table 1. The system under consideration consists of a consuming stage and a production stage with four raw material inventories. The evaluation time used in each evaluator was 50 years.

The objective of evaluator number 2 is to obtain the average system cost when each inventory center (i.e., production, consuming, raw material) is operated as a separate entity, each center using its own Wilson economic lot size. The objective of evaluator number 1 is to evaluate the average system cost when a common producer-consumer lot size is used and to study the effects of varying raw material inventories upon the system cost.

B. Preliminary Results

Evaluator number 1 was used to evaluate the system described by the parameters in table 1. The common producer-consumer lot size was set arbitrarily at 200 units, between the consumers' EOQ and the producer's EMQ of 424. All raw material sizes were held constant with the exception of raw material 2. Raw material 2 has a Wilson EOQ of 317. An initial lot size of 190 units was specified for the model and the system was simulated for 50 years. The raw material inventory was then increased to 200 units and the simulation repeated. The process

Table 1

Parameters for the Two-Stage System

	Raw Material Inventories 1	2	3	4	Producer	Consumer
Rqmts/Unit	2	1	1	1	-	-
Holding Cost	.01	.02	.05	1.0	1.0	2.0
Setup/Ordering	10.	1.	1.	5.	60	30
Production Rate					1500	
Demand Rate						1000
EOQ/EMQ	2000	317	200	100	424	173

above was repeated, incrementing the raw material inventory by ten units until a value of 580 units was reached. At each point, the pertinent average system costs were obtained. These costs are plotted in figure 7.

The appearance of the total system cost curve is perplexing in that it does not exhibit any convex-concave properties which are necessary for solution in closed form using classical optimization techniques. In fact, the minimum cost for the system configuration under investigation occurs when the lot size for inventory 2 is set to 400 units, an integer multiple of the system lot size of 200. Other local minima appear at lot sizes of 200, 300, and 500.

The above experiment was repeated, this time incrementing inventory 2 between the range of 390 through 410 units in increments of one unit. The resultant cost curves are presented in figure 8. Again, the cost curves do not exhibit any convex-concave characteristics.

The system lot size was then set at 260. This is approximately the optimum producer-consumer lot size for a two-stage system without raw material inventories obtained from the equation derived by Lele [9] and described in Chapter II. (The optimum lot size is 259.) With the system lot size held constant at 260 units, the experiment was repeated by incrementing inventory 2 through a range of 120 units through 540 units in steps of 10. The resultant cost curves are presented in figure 9. In this instance, the optimum system cost is obtained when the raw material lot size is equal to the system lot size, 260 units. Other local minima appear at lot sizes of 130, 390, and 520 units.

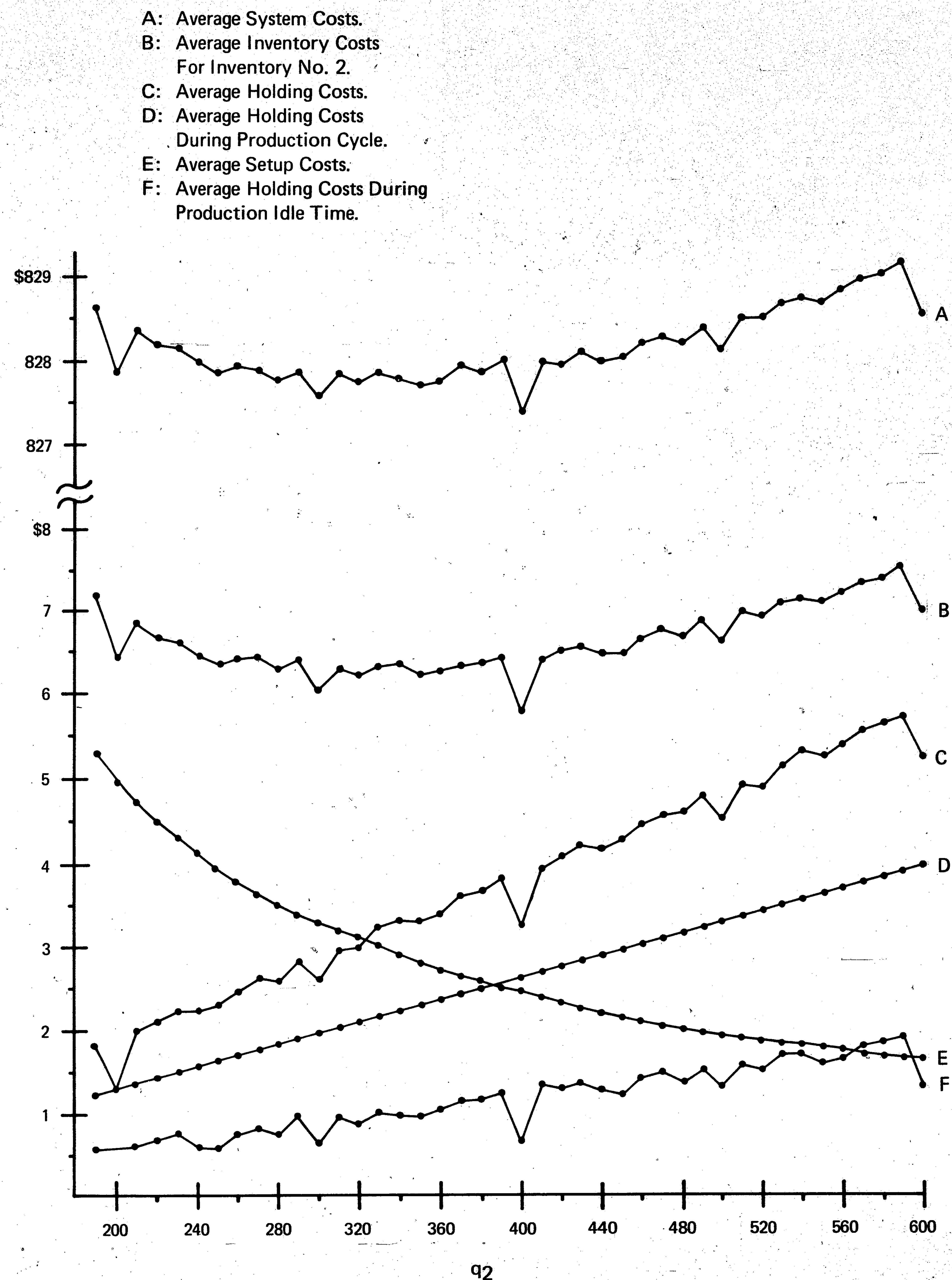

Figure 7. Cost Functions for System Q=200 and EOQ_2 = 317.

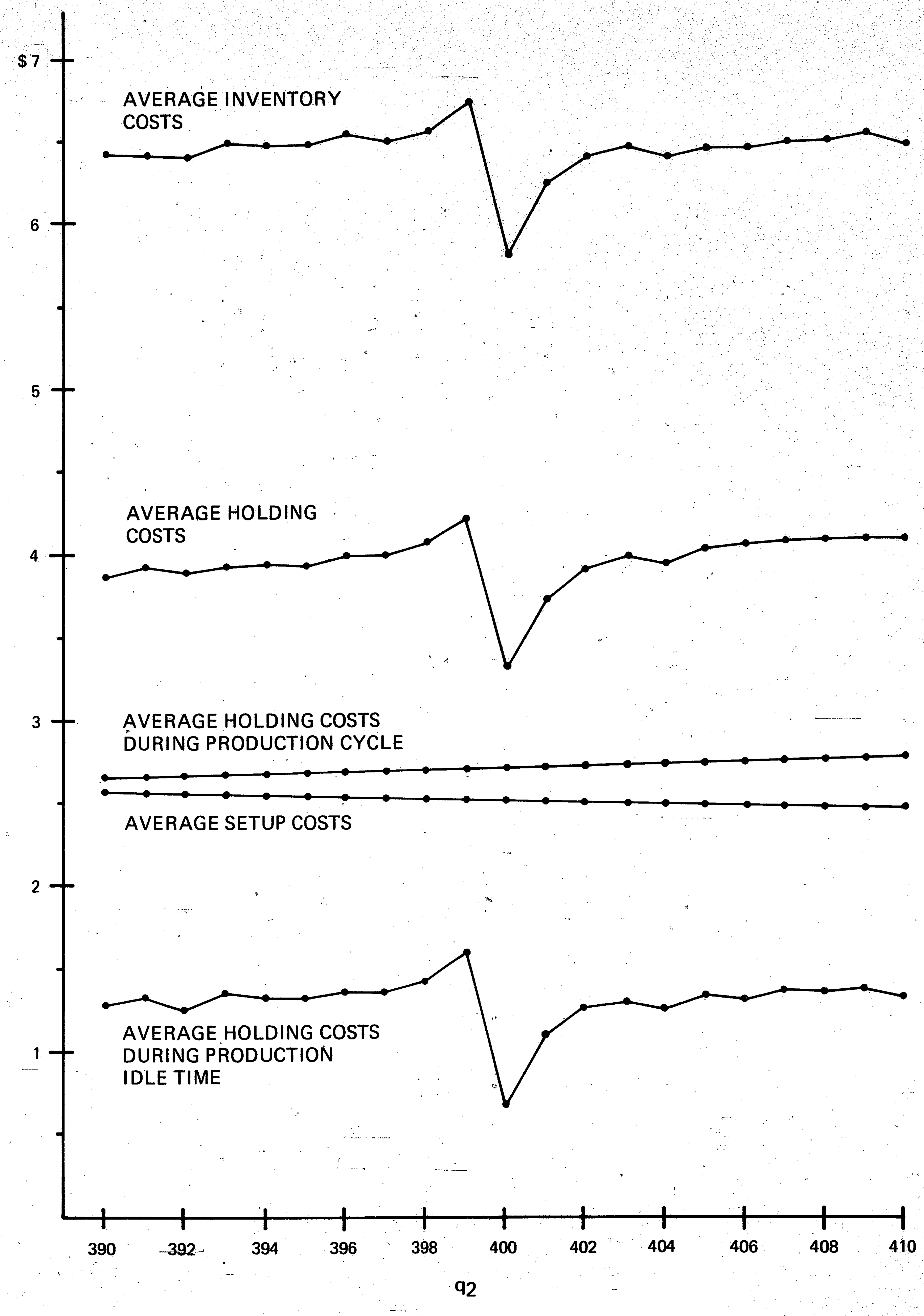

Figure 8. Cost Functions Around Optimum Point ($Q = 200$, $EOQ_2 = 317$)

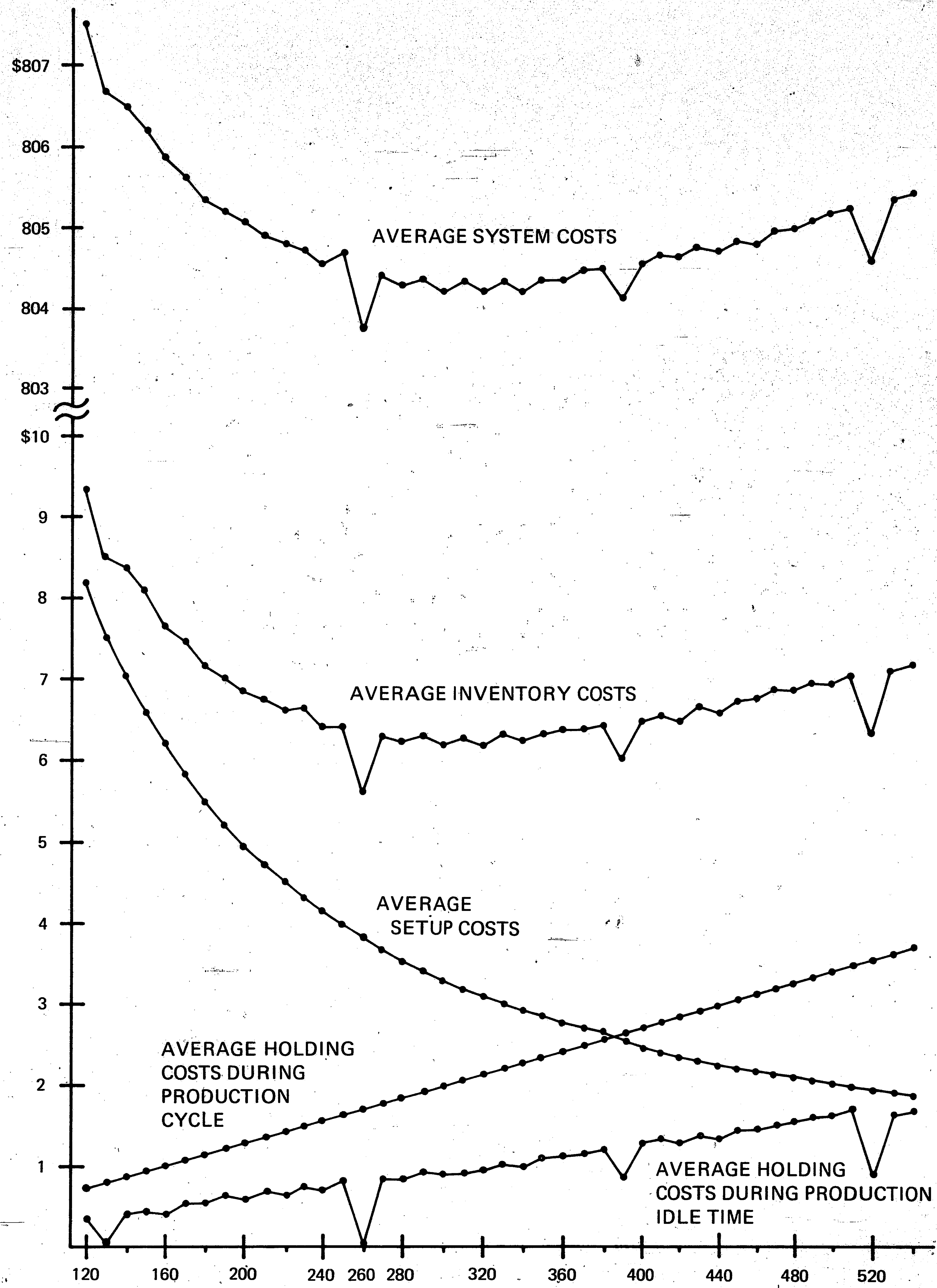

Figure 9. Cost Functions for System Q = 260 and EOQ_Z = 317

Inspection of the various cost curves shown in figures 7 through 9 reveals the following:

1. The average setup costs for inventory 2 are essentailly convex over the range and predictable.
2. The average holding costs during the production cycle increase linearly as the lot size is increased.
3. The cost element that is causing the non-convexity of the total cost function is the cost incurred during the production idle time.

The pseudo-random behaviour of the holding costs can be intuitively rationalized by use of examples.

1. Consider the case where the system lot size, Q, is 200 units and the raw material inventory lot size is also 200. This means that the raw material inventory will be depleted at the end of each production cycle. Since the raw material inventory is not replenished until the start of the next production run, holding costs will not be incurred during the production idel time. Therefore, regardless of the length of the simulation period, there will never be any holding costs incurred during the production idle time.

2. Suppose, though, that the raw material lot size, q, is increased to 201 units and the system lot size is held at 200. Then, at the end of the first production cycle, one unit will be held over during the production idle time; at the end of the second production cycle, two units will be held over; at the end of the third, three units will be held over, etc. This buildup of inventory continues until the end of

200th production cycle. At this time 200 units will be held over. Since the inventory will not be replenished during the 201st cycle, the raw material inventory will be depleted to zero and holding costs will not be incurred during the production idle time immediately after the 201st cycle. After the 202nd production cycle, the buildup cycle starts to repeat itself and zero cost intervals will appear at the end of production cycles numbered 402, 603, 804, 1005, etc. The number of zero cost intervals included in the cost function in a simulation is dependent upon the length of the simulation.

3. Suppose Q=200 and q=203. The raw material inventory will have three units left over at the end of the first production cycle; six at the end of the second production cycle; nine at the end of the third; etc. At the end of the 66th production cycle, the raw material inventory has built up to 198 units. During the 67th cycle, the inventory will be replenished, and at the end of the cycle there will be 201 units left in inventory. During the 68th cycle the inventory will not be replenished, and at the end of the cycle there will be one unit left in inventory. The inventory starts to build up in increments of three until the 135th cycle when there will be 202 units left over. At the end of the 136th cycle, there will be two units left in inventory and the buildup process starts again. At the end of the 202nd cycle there will be exactly 200 units left in inventory, and at the end of the 203 cycle the inventory will be depleted. Therefore, every 203 cycles there will be an idle interval where holding costs will not be incurred.

A counter was programmed into the evaluator which counts the number zero-cost intervals for each of the models presented in figures 7, 8, and 9. The pertinent data are presented in table 2.

Table 2(a) presents the data associated with the model in figure 7. It can be seen that the greatest number of zero-cost intervals appears at an inventory lot size of 200 units with a lesser number at a lot size of 400 units (2Q). The trade-offs between the holding costs and the setup costs force the minimum system cost to occur at a lot size of 400.

Table 2(b) presents the data for the costs around the optimum point of 400 units. The number of zero-cost intervals is small at every point except at 400. Table 2(c) presents the data for the model presented in figure 9. The maximum number of zero-cost intervals appears at lot sizes of 130 units (1/2Q) and 260 units. A lesser number of intervals appears at 520 (2Q). Again, the trade-offs between the holding costs and the setup costs force the optimum to appear at 260 units.

The number of zero-cost intervals observed within a simulation period is dependent upon the inventory cycle length (i.e., the time it takes for the inventory, starting at a level of q_i to be depleted to a level of exactly zero at the end of a production cycle.) This variable cycle length is the cause of the anomalies observed in figures 7 through 9. The length appears to be random from lot sizes to lot size but it can be calculated.

Table 2

Zero Cost Intervals for Various Lot Sizes

Q = System lot size.
q = Lot size for raw material number 2.
N = Number of zero cost intervals per 50 years simulation.

(a)
Figure 7
Q = 200

q	N	q	N
190	13	400	125
200	250	410	6
210	11	420	11
220	22	430	5
230	10	440	22
240	41	450	27
250	50	460	10
260	19	470	5
270	9	480	20
280	35	490	5
290	8	500	50
300	83	510	4
310	8	520	19
320	31	530	4
330	7	540	9
340	14	550	22
350	35	560	17
360	27	570	4
370	6	580	8
380	13	590	4
390	6		

(b)
Figure 8
Q = 200

q	N
390	6
391	0
392	5
392	0
394	1
395	3
396	2
397	0
398	1
399	0
400	125
401	0
402	1
403	0
404	2
405	3
406	1
407	0
408	4
409	0
410	6

(c)
Figure 9
Q = 260

q	N	q	N
120	31	330	5
130	185	340	11
140	26	350	5
150	12	360	10
160	23	370	5
170	11	380	10
180	20	390	61
190	10	400	9
200	18	410	4
210	9	420	9
220	17	430	4
230	8	440	8
240	16	450	4
250	7	460	8
260	185	470	4
270	7	480	8
280	13	490	3
290	6	500	7
300	12	510	3
310	6	520	91
320	12		

C. Calculation of the Raw Material Inventory Cycle Length for the Two-Stage System

The length of the raw material cycle is designated by L and will be in terms of production cycles. The common, producer-consumer lot size is Q and the raw material lot size (Wilson EOQ) is q. If q is equal to Q, the calculation of the inventory cycle length is trivial, e.g. $L=q/Q=1$, or one production cycle. At the end of each production cycle, there will be a period where holding costs will not be incurred.

The cycle length of the raw material inventory is

$L = nq/|(q-Q)|$ production cycles where n is the smallest integer that will result in an integer value for L; $1/2Q \leq q < Q$ and $q > Q$.

PROOF:

1. Let q be greater than Q. At the end of the first production cyle, the raw material inventory, I_j will have (q-Q) units left. These will be held over until the second production cycle. At the end of the second production cycle there will be 2(q-Q) units left; at the end of the third cycle there will be 3(q-Q) units; etc. We are looking for the inventory cycle time such that the inventory reaches zero at the end of a production cycle. In effect, we are searching for the number, L, of production cycles until the inventory reaches zero. This can only happen if at the (L-1)st cycle, the accumulated inventory I_j equals Q, the production lot size.

Define N = the greatest integer $\leq Q/(q-Q)$

a. If $N(q-Q)=Q$, then in the $(N+1)$th production cycle, the raw material inventory will drop to zero.

i.e. $N(q-Q)-Q=0$

Solve for N

$$N = \frac{Q}{q-Q}$$

Define L = the length of the raw material cycle time in terms of production cycles.

In this case $L = N+1$ Substitute

$$L = \frac{Q}{q-Q} + 1 = \frac{Q}{q-Q} + \frac{q-Q}{q-Q} = \frac{q}{q-Q}$$

b. If $I_j = N(q-Q) < Q$ (I_j cannot be greater than Q because of the definition for N).

Then

The I_j in the $(N+1)$th cycle is $I_j = N(q-Q)+q-Q=(N+1)(q-Q) > Q$.
Since I_j is $> Q$, replenishment will not take place during the $(N+2)$th cycle, and the inventory at the end of the production cycle is

$$I_j = [(N+1)(q-Q)-Q] << Q$$

In the $(N+2)$ cycle, I_j will be incremented by $(q-Q)$ and continue until the end of the $[(N+2)+N]$th cycle.

At the end of the $(2N+2)$th cycle, the accumulated inventory is:

$$\begin{aligned} I_j &= [(N+1)(q-Q)-Q] + N(q-Q) \\ &= N(q-Q)+q-2Q+N(q-Q) \\ &= 2N(q-Q)+q-2Q \end{aligned}$$

If $I_j=Q$, then in the $(2N+3)$th, the inventory will drop to 0; i.e.,
$I_j = 2N(q-Q) + q-3Q = 0$ and $L = 2N+3$

From I_j

$$N = \frac{3Q-q}{2(q-Q)}$$

Substituting into L

$$L = \frac{3Q-q}{q-Q} + \frac{3(q-Q)}{q-Q} = \frac{3Q-q + 3q-3Q}{q-Q} = \frac{2q}{q-Q}$$

c. If $I_j < Q$, then the inventory in the (2N+4)th cycle is

$$I_j = [2N(q-Q) + q-2Q] + q-Q > Q$$

$$= [(2N+1)(q-Q) + q-2Q] > Q$$

Since $I_j > Q$, the inventory is not replenished during the (2N+5)th cycle and the inventory at the end of this cycle is:

$$I_j = [(2N+1)(q-Q) + q-2Q] - Q << Q$$

$$= (2N+1)(q-Q) + q-3Q << Q$$

The process of adding the increment (q-Q) starts again and, at the end of the [(2N+4)+N]th cycle, the inventory is:

$$I_j = [(2N+1)(q-Q) + q-3Q] + N(q-Q)$$

$$= 3N(q-Q) + 2q-4Q$$

If $I_j = Q$, then at the end of the (3N+5)th production cycle

$$I_j = [3N(q-Q) + 2q-4Q]-Q = 0$$

$$= 3N(q-Q) + 2q-5Q = 0$$

Solving for N

$$N = \frac{5Q-2q}{3(q-Q)}$$

Substituting into L

$$L = \frac{5Q-2q}{(q-Q)} + \frac{5(q-Q)}{(q-Q)} = \frac{3q}{(q-Q)}$$

In case a above, the lot size, q, was such that the inventory-buildup cycle cycled once before the excess inventory built up to size Q and then depleted to zero.

$$I_j = N(q-Q) - Q = 0$$

$$N = \frac{Q}{q-Q}$$

$$L = N+1 = \frac{q}{q-Q}$$

In case b above, the lot size, q, was such that the inventory-buildup cycle cycled twice before the excess inventory built up to size Q and then depleted to zero.

$$I_j = 2N(q-Q) + q-3Q = 0$$

$$N = \frac{3Q-q}{2(q-Q)}$$

$$L = 2N+3 = \frac{2q}{q-Q}$$

In case c above, the cycle cycled three times before the excess inventory built up to size Q, then depleted to zero.

$$I_j = 3N(q-Q) + 2q-5Q = 0$$

$$N = \frac{5Q-2q}{3(q-Q)}$$

$$L = 3N+5 = \frac{3q}{(q-Q)}$$

By induction it can be seen that the buildup cycle has to cycle n times before the excess inventory builds up to Q and then depleted to zero, the length of the inventory cycle, as a function of production cycles is

$$L = \frac{nq}{(q-Q)}$$

If the buildup cycle has to cycle n+1 times, then

$$L = \frac{(n+1)q}{(q-Q)}$$

In general, the cycle time of a raw material inventory, as a function of production cycles with a constant lot size q from which lots of size Q are being withdrawn is given by

$$L = nq/(q - Q)$$

where n is the smallest integer that will result in an integer value for L.

Examples:

a. Let Q = 200 and q = 201

$q - Q = 1;\ L = 201/1 = 201$

b. Let Q = 200 and q = 202

$q - Q = 2;\ L = 202/2 = 101$

c. Let Q = 200 and q = 203

$q - Q = 3;\ L = (3)(203)/3 = 203$

d. Let Q = 200 and q = 205

$q - Q = 5;\ L = 205/5 = 41$

e. Let Q = 200 and q = 207

$q - Q = 7;\ L = (7)(207)/7 = 207$

f. Let Q = 200 and q = 210

$q - Q = 10;\ L = 210/10 = 21$

2. Let $.5Q < q < Q$. We are searching for the inventory cycle time such that the inventory reaches exactly a zero level at the end of a production cycle. In effect, we are searching for the number of production cycles, L, until the inventory reaches a zero level. This can only happen if, at the (L-1)st cycle, the accumulated

inventory, I_j, is equal to $Q - q$.

Define: N = the greatest integer $\leq Q/(Q-q)$.

Since $.5Q < q \leq Q$, at the end of the first production cycle, $I_j = 2q - Q$. At the end of the second production cycle, the amount left over will be $(2q-Q) - (Q-q)$. At the end of the (N-2)th cycle, the remaining inventory will be:

$$I_j = (2q-Q) - (N-3)(Q-q)$$
$$= 2q - Q - N(Q-q) + 3Q - 3q$$
$$= 2Q - q - N(Q-q)$$

a. If $I_j = Q - q$, then the raw material inventory has cycled once and at the end of the (N-1)th production cycle, the remaining inventory is:

$$I_j = [2Q - q - N(Q-q)] -Q + q = 0$$
$$= Q - N(Q-q) = 0;$$
$$N = Q/(Q-q);$$

and $L = N - 1 = Q/(Q-q) - 1 = q/(Q-q)$

b. If $I_j \neq Q -q$, then the inventory must cycle at least once more. At the end of the (N-1)th production cycle, the remaining inventory is

$$I_j = Q - N(Q-q) << Q$$

and the cycle starts repeating itself. At the end of the Nth production cycle, the inventory is

$$I_j = Q - N(Q-q) + 2q -Q = 2q - N(q-q).$$

At the end of the (2N-2)th production cycle, the inventory is

$$I_j = [2q - N(Q-q)] - (N-2)(Q-q)$$
$$= 2q - N(Q-q) - N(Q-q) + 2Q - 2q$$
$$= 2Q - 2N(Q-q).$$

If $I_j = (Q-q)$, then at the end of the $(2N-1)$th production cycle, the raw material inventory is

$$I_j = [2Q - 2N(Q-q)] + q - Q = Q + q - 2N(Q-q) = 0$$

$$N = \frac{Q+q}{2(Q-q)}$$

$$L = 2N - 1 = \frac{Q+q}{(Q-q)} - 1 = \frac{2q}{(Q-q)}$$

c. If $I_j \neq (Q-q)$, the inventory must cycle at least once more. At the end of the $(2N-1)$th production cycle, the remaining inventory is

$$I_j = Q + q - 2N(Q-q).$$

At the end of the $(2N)$th production cycle, the inventory is

$$I_j = Q + q - 2N(Q-q) + 2q - Q$$
$$= 3q - 2N(Q-q).$$

At the end of the $(3N-2)$th cycle, the inventory is

$$I_j = [3q - 2N(Q-q)] - (N-2)(Q-q)$$
$$= 3q - 2N(Q-q) - N(Q-q) + 2Q - 2q$$
$$= 2Q + q - 3N(Q-q).$$

If $I_j = (Q-q)$, then at the end of the $(3N-1)$th production cycle, the raw material inventory is

$$I_j = [2Q + q - 3N(Q-q)] + q - Q = Q + 2q - 3N(Q-q)$$

$$N = \frac{2q + Q}{3(Q-q)}$$

and

$$L = 3N - 1 = \frac{2q + Q}{(Q-q)} - \frac{Q-q}{Q-q} = \frac{3q}{(Q-q)}$$

By induction, then, if the depletion cycle has to cycle n times before the raw material inventory reaches the zero level, the length of the inventory cycle, as a function of production cycles is

$$L = \frac{nq}{Q-q}$$

If the inventory has to cycle n+1 times, then

$$L = \frac{(n+1)q}{(q - Q)}$$

In general, if $.5Q \leq q \leq Q$, the cycle time of a raw material inventory, as a function of production cycles, with a constant replenishment lot size q, from which lots of size Q are being withdrawn, is given by the expression

$$L = \frac{nq}{Q-q}$$

where n is the smallest integer that will result in an integer value for L.

3. If q is less than 1/2Q, a concise formulation for the length of the inventory cycle cannot be obtained. For any particular set of values for q and Q though, the cycle length can be calculated using the approach presented in 1 and 2 above.

D. The Integer Multiple Aspect

It can be seen in figures 7 through 9 that the economic lot sizes for the raw material inventories are some integer multiple of the common, producer-consumer lot size. The question remains as to whether the Wilson lot size for the raw material inventories are rounded up or down to the integer multiple of the system lot size. The rounding process has to be a function of the holding costs, the length of the production idle time, and the length of the raw material cycle time, L. It was shown in paragraphs C. 1 and 2 above, that the average inventory fluctuates randomly, depending upon the relationship between q and Q. Evaluator number 1 was used to investigate behaviour of the various inventories when different production rates (and thereby different system lot sizes) were used.

The demand rate for the system presented in table 1 was held constant at 1000 units. The production rate was varied between 1000 units and 5000 units. The system lot sizes and the pertinent times are presented in table 3. The optimum lot sizes for the four inventories are presented in table 4 for the various production rates.

When the production rate is equal to the demand rate, production is constant and the model for the raw material inventories is the classic Wilson model for constant deterministic demand. The economic lot sizes for the inventories are those derived from the Wilson equation. When the production rate is greater than the demand

Table 3

Production Rates and System Lot Sizes

D = 1000
p = production rate
Q = system lot size
t_c = Q/D
t_p = Q/p

p	Q	t_c	t_p	$t_c - t_p$
1000	245	.245	.245	0
1100	248	.248	.225	.023
1200	252	.252	.210	.042
1300	254	.254	.195	.059
1400	257	.257	.183	.074
1500	259	.259	.173	.086
1600	261	.261	.164	.097
1700	262	.263	.155	.108
1800	265	.265	.147	.118
1900	266	.266	.140	.126
2000	268	.268	.134	.134
2250	271	.271	.120	.151
2500	273	.273	.109	.164
2700	276	.276	.100	.176
3000	277	.277	.092	.185
3250	279	.279	.085	.194
3500	280	.280	.080	.200
3750	281	.281	.075	.206
4000	282	.282	.070	.212
4250	283	.283	.066	.217

Table 4

Optimum Inventory Lot Sizes vs. System Lot Sizes

$r = q/(R_j Q)$

		Raw Material Inventories							
		1		2		3		4	
	EOQ	2000		317		200		100	
	R_j	2		1		1		1	
P	Q	q	r	q	r	q	r	q	r
1000	245	2000	-	317	-	200	-	100	-
1100	248	1984	4	248	1	248	1	124	.5
1200	252	2016	4	252	1	252	1	126	.5
1300	254	2032	4	254	1	254	1	127	.5
1400	257	2056	4	257	1	257	1	100	-
1500	259	2072	4	259	1	259	1	100	-
1600	261	2088	4	261	1	261	1	87	.33
1700	263	2104	4	263	1	263	1	263	1
1800	265	2130	4	265	1	265	1	265	1
1900	266	2138	4	266	1	266	1	133	.5
2000	268	2144	4	268	1	268	1	134	.5
2250	271	2168	4	271	1	271	1	271	1
2500	273	2184	4	273	1	273	1	273	1
2750	275	2200	4	275	1	275	1	275	1
3000	277	2216	4	277	1	277	1	277	1
3250	279	2232	4	279	1	279	1	279	1
3500	280	2240	4	280	1	280	1	140	.5
3750	281	2248	4	281	1	281	1	281	1
4000	282	2256	4	282	1	282	1	282	1
4250	283	2264	4	283	1	283	1	283	1
4500	284	2272	4	284	1	284	1	284	1
4750	285	2280	4	285	1	285	1	285	1
5000	286	2288	4	286	1	286	1	286	1

rate, additional holding costs are incurred during the production idle time. The optimum lot size for the raw material inventories must be such that these extra holding costs are minimized in relation to any additional ordering costs incurred during the system cycle being evaluated.

Each unit produced at the production stage requires two units of inventory 1. Therefore, during each production cycle, 2Q units are removed from inventory 1. For the range of production rates which were investigated, the economic lot size for inventory 1 is four times the system lot size. The lot size will shift to a multiple of 3 when the production idle time is such that the extra holding costs are greater than the ordering cost.

Each unit produced at the production stage requires one unit each of raw materials 2,3, and 4. The Wilson EOQ's for raw materials 2 and 3 are both greater than 1/2Q. Both of these raw materials have an optimum lot size in the system that is equal to the system lot size. The optimum lot size of raw material 2 will remain an integer multiple of 1 until such time as the system lot size is 634. At this time, the optimum lot size for inventory 2 will be 317 or 1/2Q.

The optimum lot size for raw material 3 will remain an integer multiple of 1 until the system lot size is 400. At this time, the optimum lot size for raw material 3 will be 200 or 1/2Q.

Raw material 4 has a Wilson EOQ of 100, which is less than 1/2Q in every case investigated. The optimum lot size for this inventory under the production rates investigated does not behave in a consistent fashion. The optimum lot size for this inventory is very dependent upon the ordering costs, the holding costs, the production rate and system lot size, and the production idle time. In addition, the lot size is dependent upon whether the system lot size is an even odd, or prime number. Table 5 presents the average annual inventory costs for this inventory for various lot sizes and production rates.

The holding cost for inventory 4 is \$1/unit/year. The ordering cost is \$5. At a production rate of 1200 units, the system lot size is 252 units. The cycle time for production is .210 years (from table 3). If an inventory lot size of 252 is used, then the cost/production cycle is

$$C = C_s + (1/2)q(t_p)c_p$$
$$= \$5 + 1/2(252)(.210)(\$1)$$
$$= \$31.46$$

If an inventory lot size of 126, (1/2Q), is used, then the inventory has to be replenished twice during any production cycle and the cost is

$$C = 2C_s + 2(1/2Q)(1/2t_p)c_p$$
$$= \$10 + 2(113)(.105)(\$1)$$
$$= \$23.23;$$

which implies that an inventory lot size of 1/2Q should be used for inventory 4 for this production rate and system lot size. This analysis bears out the results shown in table 5. This same procedure can be used to confirm the results shown in the table except for the cases where the system lot size is a prime number (e.g.,

Table 5

Average Annual Inventory Costs for Inventory 4

Q = system lot size
q* = optimum inventory lot size
EOQ = 100

p	Q	Inventory Lot Size EOQ	1/3Q	1/2Q	Q	q*
1100	248	$99.80		$96.70	$133.04	(1/2)Q
1200	252	99.68	$94.48	92.20	124.95	(1/2)Q
1300	254	99.80		88.26	117.50	(1/2)Q
1400	257	99.86			111.31	EOQ
1500	259	99.82			105.61	EOQ
1600	261	99.82	84.55		100.74	(1/3)Q
1700	263	99.86			96.30	Q
1800	265	98.74			92.50	Q
1900	266	99.12		72.60	88.81	(1/2)Q
2000	268	99.00		70.74	85.58	(1/2)Q
2250	271	99.40			78.45	Q
2500	273	99.60	73.00		72.85	Q
2750	275	91.86			68.18	Q
3000	277	99.62			64.04	Q
3250	279	99.64	67.99		60.77	Q
3500	280	92.58		55.54	57.67	(1/2)Q
3750	281	99.62			55.27	Q
4000	282	99.29	64.83	52.99	52.89	Q
4250	283	99.19			50.76	Q
4500	284	99.22		50.97	49.14	Q
4750	285	97.70	62.47		47.43	Q
5000	285	98.98		49.15	46.00	Q

257 at p = 1400 or 263 at p = 1700).

The fractional aspects of the problem for this inventory and the relationship of q to Q prevent any further analysis.

E. Comparison of the Single-Lot-Size Model to the Wilson Model

The two evaluators were used to obtain the relative costs between the two stage model using a common, producer-consumer lot size and the two stage model which used the Wilson EOQ's at each inventory center. The lot sizes obtained in D above were programmed into evaluator 1 with a production rate of 1500 units and a demand rate of 1000 units. An evaluation of this model yielded an average annual system cost of $825.67.

Evaluator 2 has two options: ship a lot size which is equal to the producer's EMQ (424 units) or ship the consumer's EOQ (173 units). If the producer's EMQ is shipped and the EOQ's for the raw materials are used, the average annual system cost for a fifty year evaluation is $1,247.01. If the consumer's EOQ is shipped, then the average annual cost is $1,050.69. Both of these configurations yield annual costs which are greater than the costs from the model using the common, producer-consumer lot size.

CHAPTER V

CONCLUSIONS

The concepts of systems analysis can be and should be applied to the total production-inventory system within a manufacturing concern. Generally, a production-inventory system consists of production stages, inventory holding stages and consuming stages. If the manufacturing concern is large enough, it is possible that the three stages will be under different managerial control, with each manager trying to optimize his operating costs. The production-inventory system represents a flow of the firm's capital and, as such, the operations of the various stages should be synergistic.

It was shown in the previous chapter that, for a two-stage production-inventory system, a considerable improvment in the system costs was obtained when a common, producer-consumer lotsize was used. It should be noted though, that neither stage is operating at its optimum production level. In fact, the operating cost of one of the stages will be higher than the optimum cost if the stage were operating independently. The overall effect though, is a lower system cost.

Whenever raw material inventories have been considered in the literature, assumptions have been made about their cost structure that allow for solution of the economic order quantities in closed form using the classical optimization techniques available. These assumptions range from ignoring the cost of the inventories to assuming that the economic lot sizes of the raw material inventories are integer multiples of the production lot size. Obviously, to obtain a true system cost, the costs associated with the raw material inventories should be considered. It was shown in Chapter II that the solution of economic lot sizes for a generalized non-integer formulation of the model with raw material inventories is not possible. The problem arises in the definition of a common time base of such length that all of the inventories in the system fluctuate in precisely the same manner from time base to time base. If all of the raw material inventories are integer multiples of the production lot size, this common time base is easily found.

To overcome the mathematical difficulties, an empirical investigation was performed using a computer simulator to determine the characteristics of the cost functions of a two-stage system with raw material inventories under static, deterministic demand. The cost curves obtained by varying the various inventory lot sizes in relation to the production lot size were far from the much-assumed convex cost functions found in the literature for the case where the production rate is greater than the demand rate. The cost functions which were obtained were generally convex, with irregularities, and with sharply defined local minima at multiples of .5, 1, 1.5, 2, and 3 of the production lot size. The irregularities in the cost functions are related to the length of the raw material inventory cycle time (i.e. the time it takes for the inventory to drop to precisely zero at the end of a production run); the length of the production cycle; the production lot size; and the length of the production idle time.

Given a specific inventory lot size and production lot size, the length of the raw material inventory cycle time, L, in terms of production cycles can be computed by the equation:

$$L = \frac{nQ}{|Q-q|}$$

where

Q is the production lot size.

q is the raw material inventory lot size,

$1/2Q \leq q < Q$; or $q > Q$.

n is the smallest integer which will yield an integer value for L.

The resulting value for n is the number of times that the raw material inventory has to cycle before it reaches a value of precisely zero at the end of a production cycle.

If the raw material lot size is equal to the production lot size, then the inventory will be depleted to zero and inventory will not be held over during the production idle time. If the inventory lot size is larger than the production lot size, holding costs will be incurred during the production idle time until such time as the inventory reaches the zero level at the end of a production cycle. At this time there will be an idle period where holding costs are not incurred, then the cycle starts repeating itself. The objective of any optimizing technique is to minimize the total system costs by effecting a tradeoff between the ordering costs and the holding costs for the inventories. If the production rate is equal to the demand rate, then the raw material inventories are faced with static deterministic demand and the economic lot sizes will be those derived using the Wilson formulation.

The empirical investigation into the cost structure of a two-stage system with four raw material inventories revealed that the optimal system cost was obtained when the lot sizes for the raw material inventories were integer multiples of the system lot size as long as the Wilson lot size for these inventories was greater than one-half the system lot size. For inventories which have a Wilson EOQ of less than 1/2Q, the optimum lot size cannot be predicted.

With these observations in mind, it can be said that the procedure proposed by Taha and Skeith [7] is optimal if one further constraint is added. The Wilson EOQ for the raw material inventories should be calculated. If any lot size is less than one-half the system lot size, this inventory should be investigated to see if further reduction in the inventory lot size (to 1/2Q, 1/3Q, etc., or the EOQ) will further reduce the average inventory cost.

CHAPTER VI

RECOMMENDATION FOR FURTHER STUDY

One obvious extension of the work presented in this thesis is to develop the model for a two stage system under stochastic demand. Of interest here would be the safety stock level at the consuming stage. Since we are concerned with a total system, what is the effect on safety stocks if a forecasting model is added to the two-stage system? The level of the safety stock would be derived from the standard deviation of forecast error instead of the standard deviation of demand distribution. Since there are many forecasting models available, an investigation of several of the models could be performed to determine the relative level of safety stocks with various demand distributions for each model.

Another extension of the model presented in this thesis is the investigation of economic lot sizes if demand at the consuming stage is stochastic but the product is withdrawn in lots of varying sizes.

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