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Variational methods and nonlinear differential equations

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ABSTRACT

This paper is a survey of recent applications of the Direct Methods of the Calculus of Variations to nonlinear equations. R. Courant has extensively treated the linear Sturm-Liouville Eigenvalue problem in terms of a quadratic functional called the Rayleigh Quotient. More recently, Z. Nehari and others have been able to generalize this technique for nonlinear second-order differential equations, both singular and nonsingular.

This variational approach has been most useful in establishing the existence of specific types of solutions for a given differential equation. For instance, it is shown that the equation

$$y'' + yF(y^2,x) = 0,$$

where $F(t,x)$ is subject to conditions guaranteeing positiveness and continuity, has solutions vanishing at both endpoints of any finite interval $[a,b]$. Moreover, for each integer k, there is a solution which vanishes $(k-1)$ times in the same interval (a,b). On the other hand, the equation

$$y'' - yF(y,x) = 0,$$

is shown to have a positive solution through any point (a,A), $A > 0$, which decreases monotonically in $[a,\infty)$.

The solution in each case is, except for a constant multiple, a function minimizing a functional $J(y)$, which is related to the given differential equation through Euler-

Lagrange conditions. An interesting feature of this variational approach is that once a positive lower bound for the functional and a minimizing sequence have been shown to exist, the differential equation is used to generate another minimizing sequence converging to a solution having the desired properties.

Finally, a singular problem is also discussed, namely,

$$u'' - u \quad x^{1-k} u^{k} = 0, \quad 1 < k < 5, \quad x \in [0,\infty).$$

This equation is shown to have a nonnegative continuous solution u(x) for which $u(0) = u(\infty) = 0$. Moreover, if $1 < k < 4$, it is shown that $u'(0)$ is finite.

I. INTRODUCTION

The simplest problem of the Calculus of Variations is that of minimizing

$$(1.1)\quad J(y) = \int_a^b f(x,y,y')\,dx$$

over the class C of all continuous functions with piecewise continuous derivative on $[a,b]$, passing through the two points (a,A) and (b,B). It is well known that a necessary condition for (1.1) to have a minimum y, is that y satisfy the Euler-Lagrange equation

$$(1.2)\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

If the minimum problem has a solution then that solution will necessarily be the solution to the corresponding boundary value problem for the differential equation (1.2).

However, sufficient conditions for the existence of an extremal are not as easily obtained. It is even possible to construct examples of variational problems for which there is no solution among the class of admissible functions. For instance, the functional

$$J(y) = \int_{-1}^{1} x^4 y'^2\,dx$$

has a greatest lower bound of zero over the class of all continuous functions having piecewise continuous derivative and satisfying $y(-1) = -1$, $y(1) = 1$. This can be seen by taking for arbitrarily small positive ϵ the function $y = -1$ for $x < \epsilon$, $y = x/\epsilon$ for $|x| \leq \epsilon$, $y = 1$ for $x > \epsilon$. However, the integral assumes the value zero only for the functions $y = \text{constant}$. Since no such function is admissible, the minimum problem has no solution. For a variational problem to have a solution it is often necessary to impose additional restraints on the class of admissible functions.

One of the most powerful methods devised for variational problems is the so-called Direct Method and is best illustrated by the Sturm-Liouville problem as treated by R. Courant [1]. The technique employed here will lead us to generalizations for nonlinear equations developed more recently by Z. Nehari [6], [7], and [8]. The results we present are most directly related to the oscillation questions of differential equations, although results concerning boundedness are also obtainable.

Consider the equation

$$(1.3) \quad (ry')' + (\lambda p - q)y = 0$$

with the homogeneous boundary conditions

$$(1.4a) \quad r(a)y'(a)\cos\alpha - y(a)\sin\alpha = 0$$

$$(1.4b) \quad r(b)y'(b)\cos\beta - y(b)\sin\beta = 0,$$

where $0 \leq \alpha < \pi$ $0 < \beta \leq \pi$, $r, p, q \in C[a,b]$. We define the functional $J(y)/\|y\|^2$ by

$$(1.5) \quad J(y) = r(x)y'(x)\Big|_a^b + \int_a^b (ry'^2 + qy^2)dx$$

$$(1.6) \quad \|y\|^2 = \int_a^b py^2 dx,$$

This quadratic functional is often referred to as Rayleigh's Quotient.

Theorem 1.1 The eigenvalue problem (1.3) + (1.4) is solved by a solution to the minimum problem

$$\min_{C} J(y)/\|y\|^2 ,$$

where C is the class of all continuous functions with piecewise continuous derivative on [a,b], and satisfying (1.4).

Proof: For $y \in C$, we can always find a normalization constant ρ such that $\|\rho y\|^2 = 1$. Moreover, since the integral of J(y) is positive definite, J(y) is bounded below. The minimum problem, therefore, makes sense and

$$\inf_{C} J(y)/\|y\|^2 = \lambda$$

exists. Given any set of boundary conditions (1.4), we can always construct a non-trivial function $\bar{y}$ for which J(y) is finite, so that C is non-empty. For instance, sin x is an admissible function for $y(0) = y(\pi) = 0$. Since this is true in general, the Rayleigh Quotient must be smaller than or equal to that given by this particular function. We may therefore restrict ourselves to the subclass $C^* \subset C$ for which $y \in C^*$ implies $\|\rho y\|^2 = 1$ and $J(y) \leq K$, where K is some finite constant. Since a solution to the minimal problem exists, there is a sequence $\{y_n\} \subset C^*$ such that $J(y_n) \to \lambda$ for $n \to \infty$. To show C* is a compact family, recall that r and p are positive and continuous so that there exist positive constants R, P, such that, $r(x) \geq R$, $p(x) \geq P$ on $[a,b]$.

If $u \in C^*$, then

$$|u(x_2) - u(x_1)|^2 = \left(\int_{x_1}^{x_2} u'dt\right)^2$$

$$\leq |x_2 - x_1| \int_{x_1}^{x_2} u'^2 dt$$

$$\leq R^{-1} \; x_2 - x_1 \int_a^b ru'^2 dt$$

$$\leq (K/R)\,|x_2 - x_1|$$

so that C^* is an equicontinuous family. Moreover, by the mean value theorem and the fact that $\|u\|^2 = 1$, we have for some $t \in (a,b)$

$$1 = \int_a^b pu^2 dx = u^2(t)\int_a^b pdx = u^2(t)A^2.$$

Therefore, for each $u \in C^*$, $u(t) = A^{-1}$ for some $t \in (a,b)$. Upon setting $x_2 = x$ and $x_1 = t$, we have in the above $|u(x) - A^{-1}|^2 = (K/R)(b-a)$, thereby demonstrating that C^* is uniformly bounded. By Ascoli's Lemma, there exists a sequence, which we shall also designate b y_n , contained in C^* and converging uniformly to a continuous function y. We must now show that y is the minimizing function, i.e., $J(y) = \lambda$.

Let $v \in D^1[a,b]$ for which $v(a) = v(b) = 0$. Then $y_n \pm \epsilon v \in C$ and we have $J(y_n \pm \epsilon v) \geq \|y_n \pm \epsilon v\|$

for arbitrary $\epsilon > 0$. This implies

(1.8) $A_n \pm 2\epsilon\, B_n + \epsilon^2 C \geq 0,$

where

$$A_n = J(y_n) - \|y_n\|^2,$$

$$B_n = B(y_n, v) - (y_n, v),$$

$$C = J(v) - \|v\|^2,$$

$$B(y_n, v) = r(x) y_n'(x) v(x) \Big|_a^b + \int_a^b (r y_n' v' + q y_n v)\, dx,$$

$$(y_n, v) = \int_a^b p y_n v\, dx.$$

Since $\lim_{n \to \infty} A_n = 0$, for n large enough we can have $A_n < \epsilon^2$,

Thus from (1.7) we see that

$$B_n \leq \frac{1}{2}(1 + C)\epsilon$$

and since ϵ can be made arbitrarily small we must conclude that

(1.8) $\lim_{n \to \infty} B(y_n, v) - (y_n, v) = 0$

In particular, let v be the Green's function defined by,

$$(b-a) g(x,t) = \begin{cases} (x-a)(b-t) & a \leq x \leq t \\ (t-a)(b-x) & t \leq x \leq b \end{cases}$$

Passing to the limit in (1.8) we obtain

$$B(y, g) - (y, g) = 0.$$

When this is written in integral form we see that

$$0 = \int_a^b [ry'g' + (q - \lambda p)yg]dx$$

$$= ryg'\Big|_a^t + ryg'\Big|_t^b - \int_a^b y[(rg')' + (\lambda p - q)g]dx,$$

which can be simplified to

$$(b-a)r(t)y(t) = (b-t)r(a)y(a) + (t-a)r(b)y(b)$$

$$+ \int_a^b y(x)[r'(x)(b-t) + (\lambda p-q)(x-a)(b-t)]dx$$

$$+ \int_a^b y(x)[r'(x)(a-t) + (\lambda p-q)(t-a)(b-x)]dx.$$

By differentiating the left side with respect to t, solving for $r(t)y'(t)$ and differentiating again, we find that y(t) is indeed a solution to the equation

$$\frac{d}{dt}[r\frac{dy}{dt}] + [\lambda p(t) - q(t)] = 0.$$

Since this is true for any $t \in [a,b]$, it follows that y is a C^2 solution of the system (1.3) + (1.4). Multiplying this equation by y and integrating once by parts, we see that $J(y) = \lambda\|y\|^2$ so that y does in fact minimize the Rayleigh Quotient. If there exists a $\lambda^* < \lambda$ and a corresponding function y^* satisfying the system (1.3) +(1.4), then multiplying (1.3) by y^* and integrating by parts, we see that $J(y^*) = \lambda^*\|y^*\|^2$ which contradicts the minimal property of λ. This completes the proof of the theorem.

This technique establishes the existence of the first eigenvalue and eigenfunction for the Sturm-Liouville eigenvalue problem. We remark that this problem may also be solved by first inverting the system (1.3) + (1.4) into its equivalent integral equation and then maximizing a suitable quadratic functional.

II: THE NONLINEAR EQUATION $y'' + p(x)y^{2n+1} = 0$

We consider in this section a particular nonlinear problem

$$(2.1) \quad y'' + p(x)y^{2n+1} = 0, \quad n > 0$$

with the boundary conditions

$$(2.2) \quad y(a) = y(b) = 0,$$

where p is a nonnegative continuous function on [a,b]. This equation is considered first because it is a direct generalization of the linear equation, (n = 0), and the method of discussing this problem depends to a great extent upon known results for the linear Sturm-Liouville eigenvalue problem. What we shall prove is the following:

THEOREM 2.1. For each pair of positive numbers a and b with $a < b < \infty$, equation (2.1) has a positive solution y satisfying (2.2).

Proof: Consider the functional

$$(2.3) \quad J(y) = \left(\int_a^b y'^2 dx\right)^{n+1} \Big/ \int_a^b p y^{2n+2} dx$$

which we shall call the generalized Rayleigh Quotient. What we shall attempt to do is to minimize J(y) over the class C of all continuous functions with piecewise continuous derivatives on [a,b], satisfying (2.2). First we show that J(y) has a positive lower bound. Let μ be the lowest eigenvalue of the system

$$(x^{-n}u')' + \mu p(x)u = 0, \quad u(a) = u(b) = 0.$$

Then by Wirtinger's inequality, for any $v \in D^1[a,b]$ such that $v(a) = 0$, we have

$$\mu\int_a^b p(x)v^2dx \leq \int_a^b x^{-n}v'^2dx.$$

Setting $v = y^{n+1}$, where $y \in C$, yields

$$\mu\int_a^b p(x)y^{2n+2}dx \leq (n+1)^2\int_a^b x^{-n}y^{2n}y'^2dx.$$

On the other hand, Schwarz's inequality shows that

$$y^2(x) = (\int_a^x y'dt)^2 \leq (x - a)\int_a^x y'^2dt \leq x\int_a^x y'^2dt$$

so that

$$\mu\int_a^b py^{2n+2}dx \leq (n+1)^2\int_a^b y'^2dt(\int_a^x y'^2dt)^n dx$$

$$= (n+1)(\int_a^b y'^2dx)^{n+1},$$

from which we obtain the inequality

$$0 < \mu \leq (n+1)J(y).$$

This shows that $J(y)$ has a positive lower bound. Let

$$\lambda = \min_C J(y),$$

where we already know that $\lambda > 0$. Hence there exists a minimizing sequence $\{y_n\}$ of functions such that

$$\lim_{n \to \infty} J(y_n) = \lambda.$$

We now proceed to show that the sequence $\{y_n\}$ may be replaced by another sequence $\{u_n\}$ where $u_n \in C^2[a,b]$, and satisfy the boundary conditions $u(a) = u'(b) = 0$.

LEMMA 2.2 Let α and $u(x)$ be the first eigenvalue and eigenfunction of the linear system

$$(2.5) \quad u'' + \alpha[p(x)y^{2n}]u = 0, \; u(a) = u(b) = 0.$$

If y in C is further normalized by the condition

$$(2.6) \quad \int_a^b y'^2 dx = 1,$$

then

$$(2.7) \quad J(u) \leq \alpha \leq J(y).$$

Proof: By Wirtinger's inequality we have

$$\alpha\int_a^b py^{2n}v^2 dx \leq \int_a^b v'^2 dx$$

for all admissible functions v. In particular y is an admissible function so that

$$\alpha\int_a^b py^{2n+2} dx \leq \int_a^b y'^2 dx,$$

which establishes the right half, namely, $\alpha \leq J(y)$.

By Holder's integral inequality for nonnegative functions f, g,

$$\int_a^b f^r g^s dx \leq \left(\int_a^b f dx\right)^r \left(\int_a^b g dx\right)^s, \; 0 < r < 1, \; r + s = 1,$$

and equality holds only if f and g are proportional.

By setting

$$f = p^{n/n+1}y^{2n}, \qquad g = p^{1/n+1}u^2$$

we see that

$$(2.8) \quad \left(\int_a^b py^{2n}u^2dx\right)^{n+1} \leq \left(\int_a^b py^{2n+2}dx\right)^n\left(\int_a^b pu^{2n+2}dx\right).$$

Now if u is a solution of (2.5) we have

$$\int_a^b u'^2dx = \alpha\int_a^b py^{2n}u^2dx.$$

However, by (2.8) we obtain

$$\left(\int_a^b u'^2dx\right)^{n+1} \leq \alpha\left(\alpha\int_a^b py^{2n+2}dx\right)^n\left(\int_a^b pu^{2n+2}dx\right).$$

Using the normalization (2.6) and the first half of inequality (2.7) we see that

$$\left(\int_a^b u'^2dx\right)^{n+1} \leq \alpha\int_a^b py^{2n+2}dx,$$

which completes the proof of (2.7).

By the homogeneity of $J(y)$, if $y \in C$, so is ky for any non-zero constant k so that we may suppose the minimizing sequence $\{y_n\}$ to be normalized by (2.6). Thus if we replace each $\{y_n\}$ by the corresponding u_n defined by (2.5) then Lemma 2.2 shows that $J(u_n) \leq \alpha_n \leq J(y_n)$, where α_n is the corresponding least eigenvalue of the linear system. Since each $u_n \in C$ we may also have $\lambda \leq J(u_n)$ so that in the limit

$$(2.9) \quad \lambda = \lim_{n\to\infty} J(u_n) = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} J(y_n).$$

The minimizing sequence $\{y_n\}$ may be replaced by another minimizing sequence $\{u_n\}$ which is of the class $C^2[a,b]$. These functions u_n are the first eigenfunctions of the systems

$$(2.5i) \quad u_i'' + \alpha_i[p(x)y_i^{2n}]u_i = 0, \quad u_i(a) = u_i'(b) = 0.$$

Each of these may be replaced by its equivalent Fredholm integral equation

$$(2.10) \quad u(t) = \alpha \int_a^b g(x,t)p(x)y^{2n}u\,dx$$

where $g(x,t) = (x - a)$ for $a \leq x \leq t$ and $g(x,t) = (t - a)$ for $t \leq x \leq b$. The subscript has been dropped for convenience. We remark here that if the boundary condition (2.2) is replaced by $y(a) = y(b) = 0$, we need only make an appropriate change in the definition of the Green's function to establish an alternative form of Theorem 2.1.

In either case

$$|g(x,t_2) - g(x,t_1)| \leq |t_2 - t_1|$$

so that

$$(2.11) \quad |u(t_2) - u(t_1)|^{2n+2} \leq \alpha^{2n+2}|t_2 - t_1| \left(\int_a^b py^{2n}u\,dx\right)^{2n+2}.$$

With the aid of the Schwarz inequality, and (2.8) we find that

$$(2.12) \quad \left(\int_a^b py^{2n}dx\right)^{2n+2} \leq \left(\int_a^b py^{2n}u^2dx\right)^{n+1}\left(\int_a^b py^{2n}dx\right)^{n+1}$$

$$\leq \left(\int_a^b py^{2n}u^2dx\right)^{n+1}\left(\int_a^b py^{2n+2}dx\right)\left(\int_a^b pdx\right).$$

Now if u is normalized by (2.6) or what amounts to the same thing,

$$\alpha\int_a^b py^{2n}u^2dx = \int_a^b u'^2dx = 1,$$

then

$$J(y) = \left(\int_a^b py^{2n+2}dx\right)^{-1} \geq \alpha$$

so that (2.12) reduces to

$$\left(\int_a^b py^{2n}udx\right)^{2n+2} \leq \alpha^{-(2n+1)}\int_a^b pdx.$$

Consequently, (2.11) becomes

$$(2.13) \quad |u(t_2) - u(t_1)| \leq M|t_2 - t_1|$$

where

$$M = \left(\alpha\int_a^b pdx\right)^{1/2n+2}.$$

Using the sequence $\{u_i\}$, define a double sequence $\{u_{ij}\}$ by

$$(2.14) \quad u_{i1} = u_i$$

$$u''_{ij} + \alpha_{ij}[p(x)u^{2n}_{ij-1}]u_{ij} = 0, \quad u_{ij}(a) = u'_{ij}(b) = 0$$

$$\int_a^b u'^2_{ij}dx = 1$$

where α_{ij} is the lowest eigenvalue of the system. By applying Lemma 2.2 twice we find that $J(u_{ij}) \leq \alpha_{ij} \leq J(u_{ij-1}) \leq \alpha_{ij-1}$ so that $\alpha_{ij} \leq \alpha_{ij-1}$, $\alpha_{ii} \leq \alpha_i$. Since $u_{ij} \in C$ we must have that $\lambda \leq \alpha_{ii} \leq \alpha_i$ and from (2.9) we see that

$$\lim_{n \to \infty} \alpha_i = \lambda.$$

From (2.13) and the fact that the α_{ii} are uniformly bounded we have

$$|u_{ii}(t_2) - u_{ii}(t_1)| \leq M|t_2 - t_1|;$$

where M is independent of t_2, t_1, and i. Consequently, the sequence $\{u_{ii}\}$ is equicontinuous on $[a,b]$. We denote this sequence by $\{v_i\}$ for simplicity and observe that

$$v_i^2(x) = (\int v_i' \, dt)^2 \leq (x - a)\int v_i'^2 dt \leq (b - a)$$

so that $\{v_i\}$ is also uniformly bounded. It follows by Ascoli's lemma that there exists a subsequence $\{v_i^*\}$ which converges uniformly to a continuous function y.

By the same arguement $\{u_{ii-1}\}$ is also compact and we may take, if necessary a subsequence which converges uniformly to y_1. We then must have by the normalization requirement for $n \to \infty$

$$\lambda\int_a^b p(x)y^{2n}y_1^2 dx = 1.$$

Thus, neither y nor y_1 can be identically zero on $[a,b]$. Since the convergence is uniform, both limit functions must satisfy the boundary conditions (2.2) Now, $y = y_1$ for by (2.8)

$$= \left(\int_a^b py^{2n}y^2dx\right)^{n+1} \leq \left(\int_a^b py_1^{2n+2}dx\right)^n\left(\int_a^b py^{2n+2}dx\right).$$

Since y and y_1 are concave functions in C, it follows from the normalization condition for both that

$$\lambda^{-n-1} \leq [J(y_1)]^{-n}[J(y)]^{-1} \leq \lambda^{-n-1}.$$

Thus the inequality (2.8) must in fact be equality from which we conclude that y and y_1 are proportional. In view of the boundary conditions (2.2) and the normalization (2.6), they are identical.

Using the representation (2.10) we see that in the limit the function y must be a solution of the nonlinear Fredholm integral equation

$$y(t) = \lambda \int_a^b g(x,t)p(x)y^{2n+1}dx.$$

The function y is then a C^2 solution of the equation $y'' + \lambda p(x)y^{2n+1} = 0$ with the boundary conditions (2.2). Upon setting $Y(x) = \lambda^{1/2n}y(x)$ we have the desired solution to (2.1).

We remark that we could establish the existence of a solution of (2.1) vanishing n-1 times in the open interval. However, we prefer to forego this discussion until Section IV where it will be treated for a more general class of equations which include (2.1) as a special case.

III: A GENERAL CLASS OF STRICTLY NON-LINEAR EQUATIONS

In this section we shall consider the more general non-linear equation

$$(3.1)\quad y'' + yF(y^2,x) = 0,$$

where $F(t,x)$ is assumed to satisfy the following conditions:

(3.2a) $F(t,x)$ is continuous in t and x for t in $[0,\infty)$ and x in $[a,b]$;

(3.2b) $F(t,x) > 0$ for $t > 0$ and x in $[a,b]$;

(3.2c) $t^{-\epsilon}F(t,x)$ is a non-decreasing function of t for t in $[0,\infty)$ and some positive ϵ.

It should be noted that because of (3.2c) the strictly linear equation (1.3) is excluded from this discussion. Another direct consequence of (3.2c) is that $F(t,x)$ is a monotonely increasing function of t. Thus, (2.1) is included as a special case of (3.1). However, the treatment of (3.1) is somewhat different in that the functional used to study solutions of (3.1) is of an entirely different nature from the generalized Rayleigh Quotient.

No Lipschitz condition has been imposed to guarantee uniqueness, however the trivial solution $y(x) = 0$ is unique as we shall now prove. Assume there exists a solution of (3.1) such that $y(a) = y'(a) = 0$, but $y(x) \neq 0$ on a small interval $[a, a+\epsilon]$. If $y(x) = 0$ on this interval, we may replace a by a', where a' is the largest value such that

$y(x) = 0$ on $[a,a']$. There are two cases to consider, namely whether of not $y(x)$ has a zero in $[a,a+\epsilon]$. In the latter case we may write by Taylor's formula with remainder

$$y(x) = y(a) + y'(a)(x - a) - \int_a^x (x - s)y(s)F(y^2,s)ds.$$

Since $y(a) = y'(a) = 0$ we obtain

$$y(x) + \int_a^x (x - s)y(s)\, F(y^2,s)ds = 0,$$

which is impossible as $y(x)$ does not change sign in $[a,a+\epsilon]$ and $F(y^2,x) > 0$. In the former case there exists a sequence of points $\{x_n\}$, such that $x_i \neq x_j$ for $i \neq j$, and $\lim x_n = a$, for which $y(x_n) = 0$. In the interval $[a,x_n]$ (3.1) may be replaced by the integral equation

$$y(x) = \int_a^{x_n} g(x,s)y(s)F(y^2,s)ds,$$

where the Green's function $g(x,s)$ is defined by $g(x,s) = (x_n - a)^{-1}(x - a)(s - x_n)$ and $g(x,s) = (x - a)^{-1})(s - a)\cdot(x_n - s)$ in the intervals $[a,s]$ and $[s,x_n]$ respectively. Let x be the point in $[a,x_n]$ at which $y(x)$ attains its maximum M_n. Therefore,

$$M_n \leq M_n \int_a^{x_n} g(x,s)F(y^2,s)ds.$$

However, $F(t,x)$ is increasing for $t > a$, $M_n \leq M_1$, and by observing that $4g(x,s) \leq (x_n - a)$ we have

$$4 \leq (x_n - a)\int_a^{x_n} F(M_1^2,s)ds.$$

Again we have a contradiction, since as $n \to \infty$, $(x_n - a) \to 0$. The uniqueness of the trivial solution is thus established. It has also been deduced by Moroney [4] that if $F(t,x)$ is a non-decreasing function of x for each $t > 0$, then the solution satisfying $y(a) = y'(b) = 0$ is unique.

For consideration of the oscillatory properties of (3.1) we consider the functional

$$(3.3)\quad J(y) = \int_a^b [y'^2 - G(y^2,x)]dx,$$

where

$$(3.4)\quad G(t,x) = \int_0^t F(s,x)ds.$$

Even though equation (3.1) is a necessary condition for the existence of a minimal for J(y), one further restraint is needed to insure the existence of a lower bound for J(y). The restriction in question is

$$(3.5)\quad \int_a^b y'^2dx = \int_a^b y^2F(y^2,x)dx,$$

which is satisfied by any solution of (3.1) having homogeneous boundary conditions.

If we take C to be the class of all non-null continuous functions with piecewise continuous derivative on [a,b], which satisfy (3.5) and the initial condition $y(a) = 0$, then J(y) will have a positive minimum. Furthermore, the minimizing function will be a solution of (3.1) for which $y(a) = y(b) = 0$. Observe that (3.5) is actually a normalization condition in the sense that if u is any non-trivial function in $D^1[a,b]$ for which $u(a) = 0$, we can always find a positive constant α such that the function $y = \alpha u$ will satisfy (3.5). This is equivalent to finding a number α such that

$$\int_a^b u'^2 dx = \int_a^b u^2 F(\alpha^2 u^2, x)\, dx.$$

However, since the left side is positive and the right side is a continuous function of α which, by (3.1c), tends to 0 for $\alpha \to 0$ and to ∞ for $\alpha \to \infty$, a normalization constant can be found.

First we show that J(y) is bounded from below by a positive number. Indeed, the assumption that $y = 0$ is essential in that $y = 0$ corresponds to the trivial solution with $J(y) = 0$. By (3.1c),

$$G(t,x) = \int_0^t s^{\epsilon}[s^{-\epsilon}F(s,x)]ds \leq t^{-\epsilon}F(t,x)\int_0^t s^{\epsilon} ds,$$

i.e.,

$$G(t,x) \leq (1 + \epsilon)^{-1} tF(t,x).$$

Therefore,

$$tF(t,x) - G(t,x) \geq \epsilon(1 + \epsilon)^{-1} tF(t,x),$$

and, by (3.3) and (3.5), we have

$$J(y) \geq \epsilon(1+\epsilon)^{-1}\int_a^b y'^2 dx.$$

Since the integral must exist and be non-zero for the problem to make sense, $J(y)$ is bounded from below.

We may further assume that

$$\int_a^b y'^2 dx < M < \infty,$$

where M is some fixed constant. Consequently, by elementary considerations, C is a compact family; hence, by Ascoli's lemma, C contains a subsequence $\{y_n\}$ which converges uniformly to a continuous function $y_o(x)$, i.e.,

$$\lim_{n\to\infty} J(y_n) = \lambda(a,b) = \inf_C J(y), \text{ and } \lim_{n\to\infty} y_n(x) = y_o(x).$$

In order to show that $J(y_o) = \lambda(a,b)$ it remains to be shown that $y_o \epsilon D'[a,b]$ and that

$$\lim_{n\to\infty}\int_a^b y_n'^2(x)dx = \int y_o'^2(x)dx.$$

By the uniform donvergence of $\{y_n\}$, $y_o(a) = y_o(b) = 0$.

For $y \epsilon \{y_n\}$, we define another function u as a solution to the linear differential system

(3.6) $u'' = -\alpha y(x)F(y^2,x)$, $u(a) = \dot{u}(b) = 0$,

where the positive constant α is to be determined by the normalization condition

$$(3.7)\quad \int_a^b u'^2 dx = \int_a^b u^2 F(u^2,x)\,dx.$$

As previously indicated, such a constant can always be determined.

By (3.5), (3.6), and (3.7), we have

$$\alpha^2\left(\int_a^b y^2 F(y^2,x)\,dx\right)^2 = \left(\int_a^b yu''dx\right)^2 = \left(\int_a^b y'u'dx\right)^2$$

$$\leq \left(\int_a^b u'^2 dx\right)\left(\int_a^b y'^2 dx\right)$$

$$= \left(\int_a^b y^2 F(y^2,x)\,dx\right)\left(\int_a^b u^2 F(u^2,x)\,dx\right)$$

i.e.,

$$\alpha^2\int_a^b y^2 F(y^2,x)\,dx \leq \int_a^b u^2 F(u^2,x)\,dx,$$

Moreover, by (3.6) and (3.7), we obtain

$$\left(\int_a^b u^2 F(u^2,x)\,dx\right)^2 = \left(\int_a^b u'^2 dx\right)^2 = \left(\int_a^b uu''dx\right)^2$$

$$= \left(\int_a^b uyF(y^2,x)\,dx\right)^2$$

$$\leq \alpha^2\left(\int_a^b u^2 F(y^2,x)\,dx\right)\left(\int_a^b y^2 F(y^2,x)\,dx\right).$$

These last two inequalities simplify to

$$(3.8)\quad \int_a^b u^2 F(u^2,x)\,dx \leq \int_a^b u^2 F(y^2,x)\,dx.$$

Since $F(t,x)$ is an increasing function of t, it follows that $G(t,x)$ is convex and we can then write

$$(3.9)\quad G(t,x) \geq G(t_o,x) + (t - t_o)F(t_o,x).$$

Using this we obtain

$$\int_a^b G(u^2,x)dx \geq \int_a^b G(y^2,x)dx + \int_a^b (u^2 - y^2)F(y^2,x)dx.$$

Combining this with (3.8) we have

$$\int_a^b [u^2F(u^2,x) - G(u^2,x)]dx \leq \int_a^b [y^2F(y^2,x) - G(y^2,x)]dx,$$

which implies in view of (3.3), (3.5) and (3.7),

$$(3.10)\qquad J(u) \leq J(y).$$

This shows that the sequence $\{y_n\}$ may be replaced by a minimal sequence $\{u_n\}$, where each u_n is obtained from y_n by means of (3.6) and (3.7). In addition, since y_n converges uniformly to a continuous function, we see from (3.6) that u_n'' also converges uniformly to a continuous function. Consequently. u_n' and u_n tend to continuous limits. If we write $\lim u_n = u_o$, then $u_o \in C^2[a,b]$ and is itself a solution to the minimal problem, i.e.,

$$\lim_{n \to \infty} J(u_n) = J(u_o) = \lambda(a,b).$$

By writing

$$\beta = \int_a^b u'^2 dx,$$

we find that if $u = 0$ then

$$u^2(x) = \left(\int_a^x u' dx\right)^2 \leq \beta(x - a).$$

In view of (3.7) and the fact that $\beta>0$,

$$(3.11)\quad 1 \leq \int_a^b (x - a)F(\beta(x - a), x)dx,$$

which implies that β has a positive lower limit β_o.
Hence

$$\int u_o'^2 dx \geq \beta_o$$

and u_o is indeed a non-trivial function. Furthermore, $u_o(x)$ must be a positive concave function of x in [a,b] if u_o is further normalized by the condition $u_o'(a) > 0$. This is a direct consequence of the fact that (3.3) and (3.5) remain unchanged if y(x) is replaced by -y(x). As a result, we only consider the subclass of C consisting of nonnegative functions. Therefore, by (3.6), $u''(x) \leq 0$ and

$$u'(x) = \int_x^{x_o} yF(y^2, x)dx > 0,$$

Where x_o is a point in [a,b] such that $u'(x_o) = 0$.
Moreover,

$$u(x) = \int_a^x u'(x)dx > 0$$

which proves our assertion.

Now in (3.10) equality is attained only if $\dot{y}$ and u are identical because of the manner in which the Schwarz inequality was used. As a result, (3.6) shows that y is a solution of $u'' + uF(u^2,x) = 0$. Since $u(a) = u(b) = 0$ and

$$\int_a^b u'^2 dx = \alpha^2 \int_a^b u^2 F(u^2,x)\,dx,$$

a comparison with (3.7) shows that $\alpha = 1$. Hence equality is attained in (3.9) only if y is a solution of (3.1) such that $y(a) = y(b) = 0$. Now if the transformation (3.6) is applied to $y = u_o$, then $\lambda < J(u)$ since u_o is an admissible function, and $J(u) \leq J(u_o) = \lambda$ because of the minimal property of u_o. Consequently, we must have equality in (3.9), which proves that $u_o(x)$ is a solution of (3.1). We now state this result as a theorem.

THEOREM 3.1 Let C be the family of non-null piecewise differentiable functions on [a,b] which satisfy the normalization condition (3.5) and the initial condtion $y(a) = 0$. If $J(y)$ is the functional defined by (3.3), then the problem

$$\min_C J(y) = \lambda(a,b)$$

is solved by a solution $y(x)$ of (3.1) such that $y(a) = y(b) = 0$ and $y(x) > 0$ in (a,b) and the minimal value $\lambda(a,b)$ is positive.

We define $\lambda(a,b)$ to be the characteristic value of

(3.1) and the boundary conditions $y(a) = y(b) = 0$. The following properties will characterize $\lambda(a,b)$.

THEOREM 3.2.

(i) If $(a',b') \subset (a,b)$, then $\lambda(a,b) \leq \lambda(a',b')$;

(ii) $\lambda(a,b) \to \infty$ for $(b - a) \to 0$;

(iii) $\lambda(a,b)$ is a continuous function of both a and b.

Proof: To verify (i), let u be the minimizing function for $J(y)$ on the smaller interval $[a',b']$. Define v an admissible function as follows: $v = u$ for $x \in [a',b']$, $v = 0$ for $x \in [a,a') \cup (b',b]$. Thus, $\lambda(a',b') = J(u) = J(v)$, but $J(v) \geq \lambda(a,b)$.

For part (ii), let $\delta = (b - a)$ and use the inequality (3.11) to obtain

$$1 < \int_a^b (x - a)F(\beta(x - a),x)dx < \delta \int_a^b F(\beta\delta,x)dx$$

where β is defined as before and y is the minimizing function for $J(y)$ on $[a,b]$. If β were bounded from above, then there would exist a constant M such that $\beta \leq M$ for all $\delta \in (0,\delta_0)$ for some $\delta_0 > 0$. We would then have

$$1 < \delta \int_a^{a+\epsilon} F(M\delta_0,x)dx$$

for all $\delta \in (0,\delta_0)$ which is absurd. Therefore, $\beta \to \infty$ for $\delta \to 0$ and because $J(y)$ is bounded from below by

$$\epsilon(1 + \epsilon)^{-1}\int_a^b y'^2dx = \epsilon(1 + \epsilon)^{-1}\beta,$$

the result (ii) is established.

For (iii) it is sufficient to show that $\lambda(a,b)$ is a continuous function of b, because the argument will be the same for the lower limit a. For simplicity set $a = 0$ and let y denote the minimizing function for the interval $[0,b]$. Let $0 < b' < b$, and write $t = bb'^{-1}$. On the interval $[0,b']$ define the function $u(x)$ by $u(x) = y(tx)$. Now let $w = \alpha u$ and determine the constant α such that

$$\int_0^{b'} u'^2 dx = \int_0^{b'} u^2 F(u^2,x)\, dx.$$

This is equivalent to

$$t^2 \int_0^{b} y'^2 dx = \int_0^{b} y^2 F(\alpha^2 y^2, t^{-1}x)\, dx.$$

Since F is monotonic in its first argument, and continuous in both, this shows that α is a continuous function of t for $t \geq 1$. The normalization condition (3.5) with $a = 0$ shows that $\alpha \to 1$ for $t \to 1$. Therefore $|\alpha - 1|$ can be made arbitrarily small by making $(t - 1)$ small, i.e., by taking b' close to b.

By assumption $y(0) = y(b) = 0$, and thus by definition of w, $w(0) = w(b') = 0$. Hence, w is an admissible function. By Theorem 3.1 we have

$$\lambda(0,b') \leq J(w) = \int_0^{b'} [w'^2 - G(w^2,x)]dx.$$

By changing the variable of integration from x to tx and observing that $w(x) = y(tx)$ we obtain

$$\lambda(0,b') \leq t^{-1}\int_0^b [\alpha^2 t^2 y'^2 - G(\alpha^2 y^2, xt^{-1})]dx$$

where $y = y(x)$. Since $G(t,x)$ is continuous in both variables, and $t \to 1$ implies $\alpha \to 1$, the right side of the above can be made smaller than

$$\int_0^b [y'^2 - G(y^2,x)]dx + \epsilon$$

for an arbitrarily small ϵ by taking b' close enough to b. But the last expression is just $\lambda(0,b) + \epsilon$. Property (i) shows that $\lambda(0,b) \leq \lambda(0,b')$ and thus we have shown that $\lambda(0,b)$ is a continuous function of b.

THEOREM 3.3 Consider the equations

(3.12) $y'' + yF_i(y^2,x) = 0, \quad i = 1,2$

where $F_i(t,x)$ satisfy the conditions (3.2). Let $\lambda_i(a,b)$ denote the corresponding characteristic values. Then if

$$F_1(t,x) \leq F_2(t,x)$$

for all positive t and $x \in [a,b]$, then

$$\lambda_1(a,b) \leq \lambda_2(a,b).$$

Proof: Let u denote the minimizing function for $J_1(y)$ and let the normalization constant α be determined by

$$\int_a^b u'^2 dx = \int_a^b u^2 F_2(\alpha^2 u^2, x)\, dx.$$

Then the function $w = \alpha u$ is subject to the normalization

(3.5). Since w is an admissible function we must have

$$\lambda_2(a,b) \leq \int_a^b [w'^2 - G_2(w^2,x)]dx.$$

Since $F_1 \leq F_2$, definition (3.4) shows that $G_1 \leq G_2$. Moreover, the convexity condition (3.9) yields

$$\int_a^b G_1(\alpha^2u^2,x)dx \geq \int_a^b G_1(u^2,x)dx + (\alpha^2 - 1)\int_a^b u^2F_1(u^2,x)dx.$$

Putting these results together leads us to the conclusion

$$\begin{aligned}
\lambda_2(a,b) &\leq \alpha^2\int_a^b u'^2dx - \int_a^b G_2(\alpha^2u^2,x)dx \\
&\leq \alpha^2\int_a^b u'^2dx - \int_a^b G_1(\alpha^2u^2,x)dx \\
&\leq \alpha^2\int_a^b u'^2dx - \int_a^b G_1(u^2,x)dx - (\alpha^2 - 1)\int_a^b u^2F_1(u^2,x)dx \\
&\leq \int_a^b u^2F_1(u^2,x)dx - \int_a^b G_1(u^2,x)dx = \lambda_1(a,b)
\end{aligned}$$

where the last simplification is a result of the normalization of u.

This result, it should be noted is in the nature of a comparison theorem similar to that for the eigenvalues of a linear differential system.

IV: THE CHARACTERISTIC VALUES λ_n

In this section we continue our investigation of the equation

$$(4.1) \quad y'' + yF(y^2,x) = 0,$$

where as before $F(t,x)$ satisfies the conditions (3.2). Our discussion will demonstrate a significant distinction between solutions of (4.1) and the linear equation (1.3). We have already mentioned in Section II that on any interval $[a,b]$ and for each positive integer n, there exists a solution of (2.1) such that it vanishes at the endpoints and n-1 times in the open interval (a,b). The same result is true for (4.1) as well.

We shall proceed to formulate the problem in the following way. Let $\pi(a,b)$ denote any partition of $[a,b]$ such that $\pi(a,b)$ contains n+1 distinct points x_i, $i = 0,\ldots,n$ and $x_0 = a$, $x_n = b$. Let y_i be called an admissible function if y_i is in $D^1[x_{i-1},x_i]$, $y_i(x_{i-1}) = y_i(x) = 0$, but $y_i = 0$, and y_i satisfies the normalization condition

$$(4.2) \quad \int_{x_{i-1}}^{x_i} y'^2 dx = \int_{x_{i-1}}^{x_i} y^2 F(y^2,x)dx.$$

If $x \in [x_{i-1},x_i]$, write $y_i = y$ and define the nth characteristic value λ_n as

$$(4.3) \quad \lambda_n = \min \int_a^b [y'^2 - G(y^2,x)]dx,$$

where for each partition π, y ranges over the class of all admissible functions.

By Theorem 3.1 we know that for any partition $\pi(a,b)$ a minimizing function for (4.2) must coincide with the soultion of (3.1) in each interval $[x_{i-1},x_i]$ such that y vanishes at x_{i-1} and x_i. What we shall demonstrate is that the integral has a positive lower bound and that the partition for which (4.3) attains its minimum value is such that the corresponding solutions of (3.1) combine to form a C^2 solution of (3.1) in tha interval [a,b]. This solution will then vanish n-1 times in (a,b).

By the previous remarks it is sufficient to show that there exists a partition $\pi_0(a,b)$ for which (4.3) or equivalently the expression

$$\lambda^* = \sum_1^n \lambda(x_{i-1},x_i)$$

attains its minimum. By Theorem 3.2, part (ii), λ^* is a continuous function of the variables $x_1,\ldots,x_{n-1}$. Also by the same theorem the variables x_i must be bounded away from each other for any sequence of partitions for which λ^* approaches its minimum. Clearly, just as J(y) is bounded below by a positive constant, so is λ^*. Thus the minimum problem has a solution, call it y_n. This function, as already pointed out, coincides with a solution of (3.1) on each interval and vanishing at the endpoints. Hence, the solution y_n has n-1 zeros in (a,b).

We now proceed to verify that the solution to the problem is a C^2 solution of (3.1) on [a,b]. Since the sign of the minimizing function y does not alter the admissibility conditions nor the value of the functional J(y), we may therefore assume that y changes its sign at each of the points x_i (i = 1,..,n-1). Thus the extremal function will be a solution of (3.1) if, and only if,

$$(4.4) \quad \lim_{x \to x_i^-} y'(x) = \lim_{x \to x_i^+} y'(x), \qquad i = 1,..,n-1.$$

To this end we shall show that if (4.4) fails to hold at some point, y could not be a solution for the minimal problem (4.3).

To this end we suppose that $y'(x_i^-) = y'(x_i^+)$ and set $x_{i-1} = P$, $x_i = c$, and $x_{i+1} = q$ to simplify the discussion. We may also assume without loss of generality that $y > 0$ in (p,c) and $y < 0$ in (c,q). Let δ denote a small positive quantity and define u in the following manner: let u = y for $x \in [p,c-\delta) \cup (c+\delta,q]$ and $u = y(c-\delta) + (2\delta)^{-1}(x - c + \delta)\cdot[y(c+\delta) - y(c-\delta)]$ for $x \in [c-\delta,c+\delta]$. Obviously u is a continuous function in [p,q] and the linear segment vanishes for x = c' where c' is given as a solution of

$$2\delta y(c-\delta) + (c' - c + \delta)[y(c+\delta) - y(c-\delta)] = 0.$$

In order to obtain a function that is subject to the Normalization (4.2) we multiply u by constants σ and ρ in the intervals [p,c'] and [c',q] respectively , so that

$$(4.5) \quad \int_p^{c'} u'^2 dx = \int_p^{c'} u^2 F(\sigma^2 u^2,x)\,dx,$$

$$\int_{c'}^{q} u'^2 dx = \int_{c'}^{q} u^2 F(\rho^2 u^2, x)\, dx.$$

Thus the function v obtained by writing $v = \sigma u$ and $v = \rho v$ in the intervals $[p,c']$ and $[c',q]$ respectively will be normalized by the condition (4.2). Furthermore, the function y_1 obtained from y by substituting v for y in the interval $[p,q]$ is an admissible function for the problem (4.3) on $[a,b]$.

Since $G(t,x)$ is convex, we have by (3.9) that

$$\int_{p}^{q}[v'^2 - G(v^2,x)]dx \leq \int_{p}^{q}[v'^2 - G(y^2,x) - (v^2 - y^2)F(y^2,x)]dx$$

$$= \int_{p}^{q}[y^2F(y^2,x) - G(y^2,x)]dx + \int_{p}^{q}[v'^2 - v^2F(y^2,x)]dx.$$

Thus, by (4.2) and the definition of $J(y)$,

$$J(y_1) \leq J(y) + \int_{p}^{q}[v'^2 - v^2F(y^2,x)]dx,$$

or equivalently

$$\text{(4.6)} \quad J(y_1) \leq J(y) + \sigma^2\int_{p}^{c'}[u'^2 - u^2F(y^2,x)]dx$$

$$+ \rho^2\int_{c'}^{q}[u'^2 - u^2F(y^2,x)]dx.$$

We intend to show that the sum of the last two terms in the above can be made negative by taking δ sufficiently small. By omitting the negative term of the integrand in the interval $(c-\delta,c+\delta)$ and observing that $y = u$ in $[p,c-\delta]$ and $[c+\delta,q]$ we obtain

$$(4.7)\quad J(y_1) \leq J(y) + \sigma^2\int_p^{c-\delta}[u'^2 - u^2F(u^2,x)]dx + \sigma^2\int_{c-\delta}^{c}u'^2dx$$

$$+ \rho^2\int_{c+\delta}^{q}[u'^2 - u^2F(u^2,x)]dx + \rho^2\int_{c}^{c+\delta}u'^2dx.$$

Since y is a solution of (4.1) in each interval for which $y(p) = y(q) = 0$, we have

$$\int_p^{c-\delta}[u'^2 - u^2F(u^2,x)]dx = y(c-\delta)y'(c-\delta)$$

and

$$\int_{c+\delta}^{q}[u'^2 - u^2F(u^2,x)]dx = -y(c+\delta)y'(c+\delta).$$

From these identities and the fact that

$$\sigma^2\int_{c-\delta}^{c}u'^2dx \leq \int_{c-\delta}^{c}u'^2dx + |\sigma^2 - 1|\int_{c-\delta}^{c}u'^2dx,$$

$$\rho^2\int_{c}^{c+\delta}u'^2dx \leq \int_{c}^{c+\delta}u'^2dx + |\rho^2 - 1|\int_{c}^{c+\delta}u'^2dx,$$

(4.5) becomes

$$(4.8)\quad J(y_1) \leq J(y) + y(c-\delta)y'(c-\delta) - y(c+\delta)y'(c+\delta)$$

$$+ (\sigma^2-1)y(c-\delta)y'(c-\delta) - (\rho^2-1)y(c+\delta)y'(c+\delta)$$

$$+ \int_{c-\delta}^{c+\delta}u'^2dx + [|\rho^2-1| + |\sigma^2-1|]\int_{c-\delta}^{c+\delta}u'^2dx$$

From the differential equation (4.1), $y(c) = 0$ implies $y''(c) = 0$. Thus by Taylor's formula , $y(c+\delta) = y'(c^+) + O(\delta^3)$, $y(c-\delta) = -\delta y'(c^-) + O(\delta^3)$, $y'(c+\delta) = y'(c^+) + O(\delta^2)$ and $y'(c-\delta) = y'(c^-) + O(\delta^2)$. Since F is a continuous function of both of its arguments and since $c' \to c$ for $\delta \to 0$, (4.5) shows that both σ^2 and $\rho^2 \to 1$ for $\delta \to 0$. Thus $(\sigma^2 - 1)y(c-\delta)$ $y'(c - \delta)$ and $(\rho^2 - 1)y(c + \delta)y'(c + \delta)$ are $o(\delta)$. From the definition of the linear segment $u(x)$ in $[c - \delta, c + \delta]$ we have

$$\int_{c-\delta}^{c+\delta} u'^2 dx = (2\delta)^{-1}[y(c + \delta) - y(c - \delta)]^2 = \delta[y'(c^+) + y'(c^-)]^2/2$$

$$+ O(\delta^3).$$

Therefore, the last term in (4.8) is $o(\delta)$. Since

$$y(c-\delta)y'(c-\delta) - y(c+\delta)y'(c+\delta) = -\delta[y'^2(c) - y'^2(c)] + O(\delta^3)$$

(4.8) reduces to

$$J(y_1) \leq J(y) - (\delta/2)[y'(c^+) - y'(c^-)]^2 + o(\delta).$$

Now the assumption that $y'(c^+) = y'(c^-)$ implies that $-(\delta/2)[y'(c^+) - y'(c^-)]^2 + o(\delta)$ can be made negative by choosing δ sufficiently small. Thus the corresponding function $y_1(x)$ will satisfy the strict inequality $J(y_1) <$ $J(y)$. But y_1 is an admissible function for the extremal problem (4.2) and since y is a minimizing function for (4.2) we arrive at a contradiction. Thus $y'(c^+) = y'(c^-)$, where c may be identified with any of the partition points x_i $(i=1,\ldots n-1)$, we have therefore proved the following result.

THEOREM 4.1 Let C_n be the class of all piecewise continuous functions y vanishing at a and b and at least n-1 times in the interval (a,b) and y subject to the normalization (4.2), then

$$\min\int_a^b [y'^2 - G(y^2,x)]\,dx = \lambda_n, \quad y \in C_n$$

has a solution y_n with continuous derivative throughout [a,b]. The function y_n has precisely n-1 zeros in (a,b) and is a solution of (4.1) for whixh y(a) = y(b) = 0.

The following corollaries follow quite easily.

COROLLARY 4.2 The characteristic values λ_n are strictly increasing with n.

Proof: Let y be the minimizing function for (4.2) and define a function u(x) = y(x) for $x \in [a,x_{n-1}]$ and u(x) = 0 for $x \in [x_{n-1},b]$, where $x_1,..,x_{n-1}$ are the zeros of y(x) in (a,b). Now u(x) is an admissible function for the extremal problem (4.3) corresponding to the index n-1. Thus it follows that

$$\lambda_{n-1} \leq \int_a^b [u'^2 - G(u^2,x)]dx = \int_a^{x_{n-1}} [y'^2 - G(y^2,x)]dx$$

$$= \lambda_n - \int_{x_{n-1}}^b [y'^2 - G(y^2,x)]dx.$$

By the estimate for the lower bound of J(y) applied to the interval $[x_{n-1},b]$ we have

$$\int_{x_{n-1}}^{b}[y'^2 - G(y^2,x)]dx \geq \epsilon(1+\epsilon)^{-1}\int_{x_{n-1}}^{b} y'^2 dx > 0,$$

and thus conclude that $\lambda_{n-1} < \lambda_n$.

COROLLARY 4.3 The characteristic value $\lambda_n(a,b)$ has the following properties:

(i) $\lambda_n(a,b) \leq \lambda_n(a',b')$ whenever $(a,b) \subset (a',b')$

(ii) $\lambda_n(a,b) \to \infty$ for $b-a \to 0$;

(iii) $\lambda_n(a,b)$ is a continuous function of both a and b.

This result is a trivial consequence of Theorem 3.2.

It is well known [9] that if λ_n is the nth eigenvalue of the linear Sturum-Liouville system, then

$$\lim_{n \to \infty} \frac{\lambda_n/n^2}{n^2} = \text{const}.$$

And, a general comparison theorem for the eigenvalues of a linear system exists. These familiar results should be compared with the following theorems which describe the asympotic behaviour of the characteristic values λ_n.

THEOREM 4.4 If λ_n is the nth characteristic value of the problem (4.3), then

$$\lim_{n \to \infty} \lambda_n/n^2 = \infty,$$

and the exponent 2 is sharp.

Proof: Let y be the minimizing function and $x_1,\ldots,x_{n-1}$ the zeros of y on (a,b). Then for $x \in [x_{i-1},x_i]$,

$$y^2(x) = \left(\int_{x_{i-1}}^{x_i} y' dx\right)^2 \le (x_i - x_{i-1}) \int_{x_{i-1}}^{x_i} y'^2 dx.$$

By the normalization condition (4.2), we find that

$$1 \le (x_i - x_{i-1}) \int_{x_{i-1}}^{x_i} F(y^2,x)\,dx,$$

and consequently

$$\int_{x_{i-1}}^{x_i} F(y^2,x)\,dx \ge (x_i - x_{i-1}).$$

By Holder's inequality, this reduces to

$$(4.9) \quad \int_a^b F(y^2,x)\,dx \ge n^2(b - a)^{-1}.$$

For a positive constant ρ, the convexity condition (3.9) becomes

$$\int_a^b G(\rho^2,x)\,dx \ge \int_a^b G(y^2,x)\,dx + \int_a^b (\rho^2 - y^2)F(y^2,x)\,dx,$$

which in view of (4.2) becomes

$$\int_a^b [y'^2 - G(y^2,x)]\,dx \ge \rho^2 \int_a^b F(y^2,x)\,dx - \int_a^b G(k^2,x)\,dx.$$

Since y is the minimizing function, the last inequality and (4.9) imply that

$$\lambda_n \geq \rho^2 n^2 (b - a)^{-1} - \int_a^b G(\rho^2, x)\,dx.$$

Hence, we conclude that lim inf $\lambda_n/n^2 \geq \rho^2(b - a)$. Since ρ is an arbitrary constant, it may be taken as large as we wish, thus proving our assertion.

To show that the exponent 2 is sharp, we merely calculate λ_n for the system

$$(4.10) \quad y'' + y^{2m+1} = 0, \; y(0) = y(b) = 0.$$

Our result will show that

$$(4.11) \quad \lambda_n = [m(m+1)^{1/m}][4\beta^2 n^2 b^{-1}]^{(m+1)/m}(m+2)^{-1},$$

where

$$\beta = \int_0^1 (1 - t^{2m+2})\,dt.$$

Since m may be taken arbitrarily large, the exponent 2 is the largest possible.

To derive (4.11) let y be a solution of (4.10) for which $y(0) = 0$, $y'(0) = a > 0$, then

$$(4.12) \quad y'^2 + (m+1)^{-1} y^{2m+2} = a^2,$$

which implies that y is a periodic function of x, oscillating between the limits $\pm M$, the maximum and minimum of y respectively, determined by the equation

$$M^{2m+2} = (m+1)a^2.$$

Let $x = T$ be the lowest positive value for which $y^2(T) = M^2$, then the zeros in $(0,\infty)$ are at 2T, 4T.... Since y is a continuous function of the parameter a, the zeros move to the left continuously as a^2 increases. Thus there is a unique solution of the system (4.10) with n-1 zeros in (0,b). By Theorem 4.1 this solution is identical with the minimizing function of (4.3) and we obtain

$$\lambda_n = \int_0^b [y'^2 - (n+1)^{-1} y^{2n+2}]dx$$

if y is the solution in question.

Since the equation in (4.10) is unchanged by a translation we find that y satisfies the identities $y(y+T) = y(x+2T) = -y(x)$, which simplifies the expression for λ_n to

$$\lambda_n = (2n) \int_0^T [y'^2 - (m+1)^{-1} y^{2m+2}]dx, \quad T = (b/2n).$$

Using the normalization condition

$$(4.13) \quad \int_0^T y'^2 dx = \int_0^T y^{2n+2} dx,$$

we can further simplify the above expression to

$$(4.14) \quad \lambda_n = 2nm(m+1)^{-1} \int_0^T y'^2 dx, \quad T = (b/2n).$$

Upon integrating (4.12) from 0 to T, we find that

$$\int_0^T y'^2 dx + (m+1)^{-1} \int_0^T y^{2m+2} dx = a^2 T,$$

which by (4.13) reduces to

(4.15) $\int_0^T y'^2 dx = (m+2)^{-1}(m+1)a^2T.$

To compute a, we observe that y is increasing in (0,T) and thus, by using (4.12), we have

$$T = \int_0^M [a^2 - (m+1)^{-1}y^{2m+2}]^{-1/2}dy.$$

Since $m^{2m+2} = (m+1)a^2$, we have by a change of variable

$$T = (m+1)^{1/2}M^{-m}\int_0^1 [1 - t^{2m+2}]^{-1/2}dt = (m+1)^{1/2}M^{-m} .$$

Eliminating M between these expressions we have

$$a^2 = (m+1)^{-1}[(m+1)^{1/2}\,T^{-1}]^{2(m+1)/m}.$$

Combining this with (4.14) and (4.15) we obtain the indicated value for λ_n.

THEOREM: 4.5 Let $F_i(t,x)$, i=1,2, satisfy conditions (3.2) and let λ_n', λ_n'' denote the respective nth characteristic values of the systems,

(4.16) $y'' + yF(y^2,x) = 0,\ y(a) = y(b) = 0,\ i=1,2.$

If,

(4.17) $\lim_{t \to \infty} F_1(t,x)/F_2(t,x) = 1$

uniformly in x, then

(4.18) $\lim_{n \to \infty} \lambda_n'/\lambda_n'' = 1.$

Proof: Let u, v be solutions to the system (4.16) corresponding to i = 1,2, respectively. Consider in addition the system

$$(4.19) \quad w'' + (1+\delta)wF(w^2,x) = 0, \quad w(a) = w(b) = 0,$$

where δ is a small positive number and λ_n is the corresponding characteristic value. Let $x_o, x_1, \ldots, x_n$ be the zeros of the characteristic function u. If the constants a_i are determined by the condition

$$\int_{x_{i-1}}^{x_i} u'^2 dx = (1+\delta)\int_{x_{i-1}}^{x_i} u^2 F(a_i^2 u^2, x)\, dx,$$

then by theorem 4.3

$$\lambda_n \leq \sum_1^n \int_{x_{i-1}}^{x_i} [a_i^2 u'^2 - (1+\delta)G_2(a_i^2u^2,x)]dx$$

$$= \sum_1^n \int_{x_{i-1}}^{x_i} [a_i^2 u'^2 - G_1(a_i^2u^2,x)]dx$$

$$+ \sum_1^n \int_{x_{i-1}}^{x_i} [G_1(a_i^2u^2,x) - (1+\delta)G_2(a_i^2u^2,x)]dx.$$

In view of the normalization (4.2) for u and the convexity condition (3.9) for G_1, we have

$$\sum_1^n \int_{x_{i-1}}^{x_i} [a_i^2 u'^2 - G_1(a_i^2u^2,x)]dx \leq$$

$$\sum_{1}^{n} \int_{x_{i-1}}^{x_i} [a_i^2 u'^2 - G_1(u^2,x) - (a_i^2-1)u^2 F_1(u^2,x)]dx$$

$$= \int_a^b [u^2 F_1(u^2,x) - G_1(u^2,x)]dx = \lambda_n',$$

and thus

$$(4.20) \quad \lambda_n \leq \lambda_n' + \sum_{1}^{n} \int_{x_{i-1}}^{x_i} [G_1(a_i^2u^2,x) - (1+\delta)G_2(a_i^2u^2,x)]dx.$$

By definition (3.4), we have

$$G_1(a_i^2u^2,x) - (1+\delta)G_2(a_i^2u^2,x) = \int_0^{a_i^2u^2} [F_1(t,x) - (1+\delta)F_2(t,x)]dx.$$

Moreover, from (4.17) we see that the integrand becomes negative for $t > M^2$, where M^2 is a function of δ, hence

$$G_1(a_i^2u^2,x) - (1+\delta)G_2(a_i^2u^2,x) \leq \int_0^{M^2} F_1(t,x)dt = G_1(M^2,x).$$

Using this in (4.20) we get the estimate

$$(4.21) \quad \lambda_n \leq \lambda_n' + \int_a^b G_1(m^2,x)dx.$$

Now let $a = x_0', \ldots, x_n' = b$ be the zeros of the nth characteristic function w of (4.19) and let the constants $b_1, \ldots, b_n$ be determined by the conditions

$$(4.22) \quad \int_{x_{i-1}'}^{x_i'} w'^2 dx = \int_{x_{i-1}'}^{x_i'} w^2 F_2(b_i^2 w^2,x)dx.$$

According to (4.19), we must have

$$\int_{x'_{i-1}}^{x'_i} w'^2 dx = (1+\delta)\int_{x'_{i-1}}^{x'_i} w^2 F_2(w^2,x)\,dx,$$

and therefore

$$\int_{x'_{i-1}}^{x'_i} w^2[F_2(b_i^2 w^2,x) - F_2(w^2,x)]\,dx > 0,$$

which in turn implies $F_2(b_i^2w^2,x) > F_2(w^2,x)$. Hence we conclude that $b_i^2 > 1$. Now by (3.2c),

$$F_2(b_i^2 w^2,x) > b_i^{2\epsilon} F_2(w^2,x),$$

and consequently, from (4.22) we have

$$(1+\delta)\int_{x'_{i-1}}^{x'_i} F_2(w^2,x)\,dx = \int_{x'_{i-1}}^{x'_i} w^2 F_2(b_i^2 w^2,x)\,dx$$

$$\geq b_i^{2\epsilon}\int_{x'_{i-1}}^{x'_i} w^2 F_2(w^2,x)\,dx.$$

Therefore, $b_i^{2\epsilon} \leq (1+\delta) < k$, where k is a constant independent of i.

The function w_1 defined by $w_1 = b_i w$ for $x \in [x'_{i-1}, x'_i]$ has the normalization (4.2) for $y = w_1$. As a result, we can estimate λ''_n by

$$\lambda''_n \leq \int_a^b [w_1'^2 - G_2(w_1^2,x)]\,dx = \sum_1^n \int_{x_{i-1}}^{x_i} [b_i^2 w'^2 - G_2(w^2,x)]\,dx$$

$$= \sum_{1}^{n} \int_{x'_{i-1}}^{x'_i} [b_i^2 w'^2 - (1+\delta) G_2(b_i^2 w^2, x)] dx + \int G_2(b_i^2 w^2, x) dx..$$

By the convexity of G_2 and (4.22), it follows that

$$\sum_{1}^{n} \int_{x'_{i-1}}^{x'_i} [b_i^2 w'^2 - (1+\delta) G_2(b_i^2 w^2, x)] dx$$

$$\le \sum_{1}^{n} \int_{x'_{i-1}}^{x'_i} [b_i^2 w'^2 - (1+\delta) G_2(w^2, x) - (1+\delta)(b_i^2 - 1) w^2 F_2(w^2, x)] dx$$

$$= \int_a^b [w'^2 - (1+\delta) G_2(w^2, x)] dx = \lambda_n .$$

Hence, we have

$$(4.24) \qquad \lambda_n'' \le \lambda_n + \delta \sum_{1}^{n} \int_{x'_{i-1}}^{x'_i} G_2(b_i^2 w^2, x) dx.$$

The convexity condition (3.9) for G_2 yields, upon setting $t = 0$ and $t_o = b_i^2 w^2$,

$$G_2(b_i^2 w^2, x) \le b_i^2 w^2 F_2(b_i^2 w^2, x).$$

Using (4.21) and $b_i^2 < k$ we obtain

$$\int_{x_{i-1}}^{x_i} G_2(b_i^2 w^2, x) dx \le k \int_{x_{i-1}}^{x_i} w'^2 dx.$$

Finally, the estimate (4.24) becomes

$$\lambda_n'' \le \lambda_n + \delta k \int_a^b w'^2 dx.$$

Now

$$J(y) \geq \epsilon(1+\epsilon)^{-1}\int_a^b y'^2 dx$$

so if we identify y with the minimizing function w for (4.19) we find that

$$\int_a^b w'^2 dx \leq (1+\epsilon)\epsilon^{-1}\lambda_n,$$

or finally,

$$\lambda_n'' \leq \lambda_n[1 + \delta(1+\epsilon)\epsilon^{-1}].$$

Combining this with (4.21) we find that

$$\limsup_{n \to \infty} \lambda_n''/\lambda_n' \leq 1 + \delta(1+\epsilon)\epsilon^{-1},$$

and since δ can be made arbitrarily small and $\epsilon^{-1}(1+\epsilon)$ does not depend on n we must conclude that

$$\limsup_{n \to \infty} \lambda_n''/\lambda_n' \leq 1.$$

By interchanging the roles of F_1 and F_2, the same arguement will yield

$$\limsup_{n \to \infty} \lambda_n'/\lambda_n'' \leq 1,$$

which completes the proof of the theorem.

This last result shows quite easily that if $p_k(x)$ are continuous and nonnegative on $[a,b]$ for $k = 1,..,m$ and if $p_m(x)$ is strictly positive, then the characteristic value of the system

$$y'' + \sum_{1}^{m} p_k(x) y^{2k+1} = 0, \quad y(a) = y(b) = 0,$$

is completely determined by the function $p_m(x)$. One can also establish that for large n

$$\lambda_n = An^{2(1+1/m)}[1 + O(n^{-2})]$$

where A is a constant and λ_n is the nth characteristic value of the system

$$y'' + p(x)y^{2m+1} = 0, \quad y(a) = y(b) = 0.$$

V. A SINGULAR PROBLEM

The problems considered thus far have all been of the nonsingular type. In this section we shall present a singular problem for which the variational methods have also proved useful. The problem in question is the existence of a continuous solution to the equation

$$(5.1)\quad y'' - y + x^{1-k}y^k = 0, \qquad 1 < k < 5$$

on $[0,\infty)$ which vanishes at the origin, $x = 0$, and as $x \to \infty$. The basic idea is again to minimize an appropriate functional under a suitable normalization condition. However, one difficulty which arises is that (5.1) is singular at both ends of the interval. Consequently, the results of Section III do not apply and the problem must be dealt with separately.

The interest in this equation arises from the fact that spherically symmetric solutions of the 3-dimensional partial differential equation $\Delta u = u + u^2$ satisfy the ordinary differential equation $u'' + (2/x)u' - u - u^2 = 0$ which is a special case of (5.1) for $k = 2$ under the transformation $y(x) = -xu(x)$.

Consider the problem of minimizing the functional

$$(5.2)\quad J(y) = \int_0^\infty (y^2 + y'^2)\,dx$$

over the class C of all nonnegative continuous functions on $[0,\infty)$ which vanish for $x = 0$, and are normalized by the condition

$$(5.3) \quad \int_0^\infty y^{k+1}x^{1-k}dx = 1.$$

We shall show that this variational problem nas a nonnegative solution y which vanishes at both ends of the interval and that except for a multiplicative constant, y is a solution of (5.1).

We note that a necessary condition that the variational problem have a solution is that y satisfy the Euler-Lagrange equation, which is (5.1) except for an undetermined multiplier.

Before proceeding with the existence proof, we also note that unlike the classical Dirichlet problem, the existence of the integral (5.3) need not be assumed as we shall now show.

LEMMA 5.1. For $k \leq 5$ and y(x) in C, the existence of (5.2) implies the existence of (5.3).

Proof: Let

$$(5.4) \quad f^2(X) = \int_0^X (y^2 + y'^2)dx, \qquad (f^2(X) > 0)$$

Since y(0) = 0 we have

$$(5.5) \quad y^2(X) = (\int_0^X y'dx)^2 \leq X \int_0^X y'^2dx \leq Xf^2(X)$$

$$(5.6) \quad y^2(X) = 2\int_0^X yy'dx \leq \int_0^X (y^2 + y'^2)dx \leq f^2(X).$$

For $X > 0$, y in C and y' square integrable on $[0,\infty)$

$$0 \le \int_0^X [y' - y/(2x)]^2 dx = \int_0^X [y'^2 + (y/2x)^2]dx - \int_0^X yy'/x\, dx$$

$$= \int_0^X [y'^2 - (y/2x)^2]dx - y^2(X)/X$$

$$\le \int_0^X [y'^2 - (1/4x^2)y^2]dx$$

from which we conclude that

$$(5.7) \quad \int_0^X y^2/x^2\, dx \le 4\int_0^X y'^2 dx.$$

Furthermore for $X \ge 1$ and $k \le 5$ so that

$$\int_0^X x^{1-k}y^{k+1}dx \le f^{k-1}(X)\int_0^X x^{(k-1)/2}\, y^2 dx$$

$$\le f^{k-1}(X)\, [\int_0^1 (y^2/x^2)x^{(k-1)/2}\, dx + \int_1^X y^2 dx\,]$$

$$\le f^{k-1}(X)\int_0^X (y^2/x^2)dx + f^{k+1}(X)$$

$$\le 5f^{k+1}(X).$$

Hence,

$$(5.8) \quad [\int_0^X x^{1-k}y^{k+1}dx]^2 \le 25\, [\int_0^X (y'^2 + y^2)dx]^{k+1}.$$

Since the right side of (5.8) exists for all X in $[0,\infty)$, the proof of Lemma 5.1 is complete.

In view of the inequality (5.8) and the normalization condition (5.3), the functional $J(y)$ necessarily has a positive lower bound. If we set

$$(5.9) \quad \lambda = \inf_{C} J(y), \qquad (\lambda > 0),$$

then there will exist a sequence of admissible functions $\{y_n\}$ such that

$$(5.10) \quad \lim_{n \to \infty} J(y_n) = \lambda.$$

From (5.10) we may assume that the sequence of numbers $J(y_n)$ is uniformly bounded by a constant b^2. By (5.6), $y_n^2 \leq J(y_n) \leq b^2$. Furthermore, for $0 \leq x_1 \leq x_2 < \infty$,

$$|y_n(x_2) - y_n(x_1)|^2 = \left(\int_{x_1}^{x_2} y'dx\right)^2 \leq (x_2 - x_1)\int_{x_1}^{x_2} y'^2 dx$$

$$\leq b^2(x_2 - x_1).$$

Hence $\{y_n\}$ is a compact family and by Ascoli's lemma, there exists a sequence, also to be designated by $\{y_n\}$, which converges uniformly to a continuous function $y(x)$ in any finite interval $[0,X]$. Moreover, this sequence has property (5.10) also.

We shall now proceed to show that for some positive constant μ, $\mu y(x)$ is a solution of (5.1) for $1 < k < 5$ such that $y(0) = y(\infty) = 0$, and that for $1 < k \leq 4$, $\lim_{x \to 0} y(x)/x$ is finite. To this end consider the linear differential system

$$(5.11) \quad v_n'' - v_n + a_n x^{1-k} y_n^k = 0, \quad v_n(0) = v_n(\infty) = 0;$$

where the positive constants a_n will be determined later. If (5.11) has a solution, we can write it in the form

$$(5.12)\quad v_n(x) = a_n \int_0^\infty g(x,t)t^{1-k}y_n^k(t)dt,$$

where $g(x,t)$ is the Green's function for the differential operator $L(u) = u'' - u$ with the boundary conditions $u(0) = u(\ ) = 0$. Thus, $g(x,t)$ is the function defined as follows:

$$(5.13)\quad \begin{aligned} g(x,t) &= e^{-x}\sinh t & 0 \le t \le x \\ g(x,t) &= e^{-t}\sinh x & t \ge x. \end{aligned}$$

We first show that under the conditions imposed on y_n, v_n defined by (5.12) is in fact a solution of (5.11). For simplicity write $a_n = 1$ and (5.12) as follows:

$$(5.14)\quad v_n(x) = e^{-x}p(x) + q(x)\sinh x,$$

where

$$(5.15)\quad p(x) = \int_0^x \sinh t\, y_n^k(t)t^{1-k}dt$$

$$(5.16)\quad q(x) = \int_x^\infty e^{-t}y_n^k(t)t^{1-k}dt.$$

The following estimates for $p(x)$ and $q(x)$ will now be established.

$$(5.17)\quad q(x) \le b^k x^{1-k} e^{-x}$$

$$(5.18)\quad p(x) \le 4b^k x^{3-k/2}\cosh x, \qquad k < 5$$

$$(5.19)\quad p(x) \leq M + b^k x^{1-k} e^x,\ M \text{ a constant},\ k < 5,\ t > 2(k-1)$$

$$(5.20)\quad q(x) \leq 5b^k,\quad k \leq 4$$

$$(5.21)\quad q(x) \leq b^k(1 + 4\, x^{2-k/2}),\quad k > 4.$$

Since $y_n^k(t) \leq b^k$ and $t^{1-k} \leq x^{1-k}$ for $k > 1$, (5.17) follows. Now $y_n^2(t) \leq tb^2$ and $\sinh t \leq t \cosh t$, so for $k < 5$

$$p(x) \leq b^{k-2} \cosh x \int_0^x (y_n^2/t^2)\, t^{3-k/2} dt$$

$$\leq b^{3-k/2} x^{3-k/2} \cosh x \int_0^x (y_n^2/t^2)\, dt,$$

and thus (5.18) follows from (5.7). From the identity

$$p(x) = p(x_o) + \int_{x_o}^x t^{1-k} \sinh t\, y_n^k\, dt,\ 0 < x_o < x$$

we obtain

$$p(x) \leq p(x_o) + b^k/2 \int_{x_o}^x t^{1-k} e^t dt.$$

If we set $x_o = 2(k-1)$, $t^{1-k} e^{t/2}$ is an increasing function for $t > 2(k-1)$ and we have

$$p(x) \leq p(x_o) + (b^k/2) x^{1-k} e^{x/2} \int_{x_o}^x e^{t/2} dt$$

$$< p(x_o) + b^k x^{1-k} e^x.$$

Using (5.18) to estimate $p(x_o)$, (5.19) results. Writing

$$q(x) = \int_x^1 t^{1-k} e^{-t} y_n^k \, dt + q(1)$$

we find that since $q(1) \leq b^k$ by (5.17)

$$q(x) \leq b^k + b^{k-2} \int_x^1 (y_n^2/t^2) t^{2-k/2} \, dt.$$

Since $t^{2-k/2} \leq 1$ for $k \leq 4$, (5.20) is established as a result of (5.7). However, if $k > 4$, $t^{2-k/2} \leq x^{2-k/2}$ and thus (5.21) is verified.

By using these identities we will show that $v_n(x)$ defined by (5.14) approaches zero as $x \to 0$ and $x \to \infty$. By the estimates (5.18) and (5.20), the estmate for $v_n(x)$ from (5.14) becomes

$$v_n(x) \leq 4b^k x^{3-k/2} \cosh x + 5b^k \sinh x, \quad 1 \leq k \leq 4.$$

If (5.20) is replaced by (5.21), the corresponding estimate is

$$v_n(x) \leq 4b^k x^{3-k/2} \cosh x + b^k(1 + 4x^{2-k/2}) \sinh x, \quad k > 4.$$

These inequalities imply that $v_n(x) \to 0$ for $x \to 0$, provided of course $1 < k < 5$. Now if $p(x)$ and $q(x)$ in (5.14) are estimated by (5.17) and (5.19) we have

$$v_n(x) \leq M e^{-x} + 2b^k x^{1-k}.$$

Since M does not depent on n and $k > 1$, $v_n(x) \to 0$ for $x \to \infty$.

From (5.14) we calculate the difference quotient $[v_n(t+h) - v_n(t)]/h$, take the limit as $h \to 0$ and find that $v_n'(x)$ exists and is given by

$$\int_0^X x^{1-k} y_n^k v_n \, dx \leq 5[\int_0^X (v_n'^2 + v_n^2) dx]^{1/2} = 5f(X).$$

(5.24) therefore implies

$$[f^2(X) - 5a_n/2] \leq 25a_n^2 / 4 + v_n(X)v_n'(X).$$

Since $v_n(X)v_n'(X) \to 0$ for $X \to 0$, we have established the existence of J(y). Moreover, by Lemma 5.1

$$\int_0^\infty x^{1-k} v_n^{k+1} \, dx$$

exists so that we may choose a_n in such a way that

$$(5.26) \quad \int_0^\infty x^{1-k} v_n^{k+1} \, dx = 1.$$

The right side of (5.24) will therefore be bounded by unity, from which we derive the inequality

$$(5.27) \quad J(v_n) \leq \int_0^\infty (v_n'^2 + v_n^2) dx \leq a_n.$$

Multiplying (5.11) by $y_n(x)$ and integrating from 0 to ∞ we have

$$\int_0^\infty (v_n' y_n' + v_n y_n) dx = a_n \int_0^\infty x^{1-k} y_n^{k+1} \, dx$$

because $v_n' y_n'$ tends to zero for $x \to 0$ and $x \to \infty$ (in the former case we use the fact that $v_n' = O(x^{3-k/2})$, $y_n = O(x^{1/2})$). In view of the normalization of y_n and the definition (5.2) of J(y) we may write

$$(5.28) \quad J(v_n) + J(y_n) - J(v_n - y_n) = 2a_n.$$

$$v_n'(x) = -p(x)e^{-x} + q(x)\cosh x.$$

The estimates (5.17) and (5.19) show that $v_n'(x) \to 0$ for $x \to \infty$. Similarly (5.18), (5.20) and (5.21) show that $v_n'(x)$ is bounded near $x = 0$ if $1 < k \leq 4$ and $v_n'(x) = 0(x^{2-k/2})$ if $k > 4$. The estimates for $v_n(x)$ near zero imply that

$$(5.22) \quad \lim_{x \to 0} v_n(x)v_n'(x) = 0, \quad 1 < k < 5.$$

A similar computation shows that $v_n''(x)$ exists and in fact $v_n(x)$ is a solution of (5.11) for $a_n = 1$. Since $y_n(x)$ is nonnegative in $[0,\infty)$, (5.12) makes it clear that $v_n(x)$ is likewise nonnegative.

We now establish the inequality

$$(5.23) \quad J(v_n) \leq J(y_n).$$

Multiplying (5.11) by $v_n(x)$, integrating from 0 to X, and using (5.22), we have

$$(5.24) \quad F^2(X) = \int_0^X (v_n'^2 + v_n^2)dx$$

$$= a_n \int_0^X x^{1-k} y_n^k v_n dx + v_n(X)v_n'(X);$$

By Holder's inequality we have

$$(5.25) \quad \int_0^X x^{1-k} y_n^k v_n dx \leq \left(\int_0^X x^{1-k} v_n^{k+1} dx\right)^{1/k+1} \left(\int_0^X x^{1-k} y_n^{k+1} dx\right)^{k/k+1}.$$

Thus by (5.8) and the normalization condition (5.3) we obtain

Since $J(v_n) \leq a_n$ we have by (5.27) that

$$J(v_n) \leq J(y_n) - J(v_n - y_n).$$

Since $J(v_n - y_n) \geq 0$, we have established (5.23), and in case $v_n = y_n$, then by (5.11) y_n is a constant multiple of a solution to (5.1).

In view of (5.27) and (5.28), we obtain the estimate $J(y_n) \geq a_n + J(v_n - y_n) \geq a_n$, and we thus have

$$J(v_n) \leq a_n \leq J(y_n).$$

Since $v_n(x)$ is an admissible function for the variational problem, $J(v_n) \geq \lambda$. Thus by (5.10), we obtain the limit relations

$$(5.29) \quad \lim_{n \to \infty} J(v_n) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} J(y_n) = \lambda.$$

On account of (5.28), $J(v_n - y_n) \to 0$ for $n \to \infty$. If we set $J(v_n - y_n) = \epsilon_n^2$ and use (5.6) we arrive at the estimate

$$(5.30) \quad |v_n(x) - y_n(x)| \leq \epsilon_n, \quad \lim_{n \to \infty} \epsilon_n = 0.$$

In the definition (5.12), we split the integration into the intervals $[0,X]$ and $[X, \infty)$, and use the estimate (5.17) for $q(x)$ and obtain

$$\left|y_n(x) - a_n \int_0^X t^{1-k} g(x,t) y_n^k \, dt\right|$$

$$= \left|y_n(x) - v_n(x) + a_n \int_X^\infty t^{1-k} g(x,t) y_n^k \, dt\right|$$

$$\leq |y_n(x) - v_n(x)| + a_n \sinh X\, b^k e^{-X} X^{1-k}$$

$$\leq \epsilon_n + a_n b^k X^{1-k}.$$

As n tends to infinity, $y_n(x)$ uniformly approaches its limit y(x) in [0,X]. At the same time $\epsilon_n \to 0$ and $a_n \to \lambda$. Consequently,

$$|y(x) - \lambda \int t^{1-k} g(x,t) y^k \, dt| \leq b^k X^{1-k}.$$

Moreover, in view of the fact that $k > 1$, we obtain for $X \to \infty$

$$(5.31) \quad y(x) = \lambda \int_0^\infty t^{1-k} g(x,t) y^k \, dt.$$

The function

$$y(x) = \lim_{n \to \infty} y_n(x) = \lim_{n \to \infty} v_n(x)$$

is then a solution of the integral equation (5.31). Since the convergence is uniform for $v_n(x)$, $y(x) \to 0$ for $x \to 0$ and $x \to \infty$.

To show that y(x) is a non-trivial solution of (5.31), multiply (5.11) by $v_n(x)$ and integrate from X to ∞ to obtain

$$(5.32) \quad \int_X^\infty (v_n'^2 + v_n^2)\, dx = a_n \int_X^\infty x^{1-k} v_n y_n^k \, dx - v_n'(X) v_n(X)$$

since $\lim_{X \to \infty} v_n(X) = \lim_{X \to \infty} v_n'(X) = 0$. Now

$$\int_X^\infty (v_n'^2 + v_n^2)\, dx = \int_0^\infty (v_n'^2 + v_n^2)\, dx - \int_0^X (v_n'^2 + v_n^2)\, dx$$

$$\geq \lambda - \int_0^X (v_n'^2 + v_n^2)\, dx,$$

and by (5.6)

$$\int_X^\infty x^{1-k} v_n y_n^k \, dx \le [J(y_n)]^{k-1/2} X^{1-k} \int_X^\infty v_n y_n \, dx$$

$$\le [J(y_n)]^{(k-1)/2} X^{1-k} [J(v_n) J(y_n)]^{1/2}$$

$$\le [J(y_n)]^{(k+1)/2} X^{1-k},$$

Where the last simplification is a result of $J(v_n) \le J(y_n)$. (5.32) leads to the inequality

$$\lambda \le \int_0^X (v_n'^2 + v_n^2) dx + a_n X^{1-k} [J(y_n)]^{(k+1)/2} - v_n(X) v_n'(X).$$

With the aid of (5.24) we can write this as

$$\lambda \le a_n \int_0^X x^{1-k} v_n y_v^k \, dx + a_n X^{1-k} [J(y_n)]^{(k+1)/2}.$$

If it were true that $0 = y(x) = \lim v_n(x) \neq \lim y_n(x)$, then in the limit the above would become

$$\lambda \le \lambda^{(k+3)/2} X^{1-k},$$

since $\lambda = \lim J(y_n)$. But this is absurd because $\lambda > 0$ and X^{1-k} can be made arbitrarily small for X large enough. Thus y(x) is indeed a non-trivial solution of (5.31).

Since g(x,t) is nonnegative in $(0,\infty)$, y(x) is also. By differentiating y(x) from (5.31) twice with respect to x we find that y(x) is a solution to the differential system

$$y'' - y + x^{1-k} y^k = 0, \qquad y(0) = y(\infty) = 0.$$

By the transformation $u = \lambda^{k-1} y$, we see that u satisfies the system

$$u'' - u + x^{1-k}u^k = 0, \qquad u(0) = u(\infty) = 0:$$

We then have the following theorem.

THEOREM 5.1. There exists a nonnegative solution of (5.1) such that $u(0) = u(\infty) = 0$ for $1 < k < 5$. Furthermore, if $1 < k \leq 4$, $u'(0)$ is finite.

As a result of the technique used in the proof, Nehari [8] has also established a method of constructing approximating solutions to (5.1). In fact he has shown that $k = 5$ is critical in that one cannot have a continuous solution of (5.1) for which $\lim_{x \to \infty} u(x)/x = 0$.

VI. CONCLUSION

We note that in the proof of Theorem (3.1), only slight modifications are needed to yield a more general result, In fact, the conclusion of this Theorem is true for a general set of homogeneous boundary conditions. This is achieved by an appropriate change in the class of admissible functions C, and requiring the approximating u defined by (3.6) to satisfy the same set of homogeneous boundary constions.

This paper, except for Section V, has been primarily related to the question of oscillation for the second order nonlinear equation. This is to be expected since for each equation we have investigated, $yy'' \leq 0$ so that every solution is concave. However, this variational type of argument has also been employed successfully by Wong [10] to obtain the existence of a unique convex solution y to the second order nonlinear equation

$$(6.1) \quad y'' - yF(y,x) = 0,$$

where $F(t,x)$ satisfies condtions similar to (3.2). In fact, he has shown that there is only one solution of the class $C^2[a,\infty)$ passing through any point (a,A), $A > 0$, and decreasing monotonically in $[a,\infty)$. The argument is quite interesting in that he first establishes the result for finite subintervals $[c,b]$ of $[a,\infty)$ and then shows that a suitable limiting process allows one to pass toethe limit as the right endpoint $b \to \infty$.

The extension of the results analagous to Theorem (3.1) for a class of even order nonlinear equations can also be obtained by obvious modifications of its proof. In fact, the following is a statement of what one can expect.

THEOREM 6.1 Let C be the family of all non-null functions defined on [a,b] which satisfy the normalization condition

$$(6.2)\quad \int_a^b y(n)^2 dx = \int_a^b y^2 F(y^2,x)\,dx,$$

the initial conditions

$$y(a) = ,\ldots, = y^{(2n-2)}(a) = 0,$$

and for which the nth derivative is piecewise continuous. Then for each positive interger n, if J(y) is the functional defined by

$$(6.3)\quad J(y) = \int_a^b [y^{(n)2} - G(y^2,x)]\,dx$$

the minimal problem

$$\min_C J(y) = \lambda$$

has a solution y. This minimizing function is in fact a solution of the system

$$(6.4)\quad y^{(2n)} - (-1)^n yF(y^2,x) = 0$$

$$y(a) = ,\ldots, = y^{(2n-2)}(a) = 0,$$

$$y'(b) = ,\ldots, = y^{(2n-1)}(b) = 0,$$

where $F(t,x)$ satisfies conditions (3.2) and $G(t,x)$ is defined by (3.4). However, the existence of a solution vanishing $(k-1)$ times in the interval (a,b) for each positive integer k, and the succeeding results of Section IV may not be so easily obtainable.

Observe that the term $(-1)^n$ appears in the equation (6.4) because of the form of Euler-Lagrange equation for functionals containing derivatives of order n. One should then also expect to obtain a result for the equation

$$(6.5) \quad y^{(2n)} - (-1)^n F(y,x) \quad 0, n \quad 1,2,\dots,$$

which is a generalization of the result obtained by Wong [10] for $n = 1$. However, the obvious modifications do not yield useful results, and more subtle conditions must be found for the class of admissible functions C or for the function $F(t,x)$, or both. The problem does indeed become more complicated with increasing n becuase the number of degrees of freedom increase accordingly.

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