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On the determination of optimal price break procurement policies

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ON

THE DETERMINATION OF OPTIMAL PRICE BREAK PROCUREMENT POLICIES

by

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Abstract

Two problems which have the common characteristic of a concave cost functions are discussed in this paper. The basis problem is to determine the optimum allocation policy where the unit cost of the products being allocated is a function of the total quantity purchased from that supplier. In the first case under consideration any economies or diseconomies which result from changes in the total allocation to a supplier are reflected on the unit cost of all products being allocated to that supplier. The second case is the problem where the economies or diseconomies from a change in the total allocations to a supplier reflect only upon the incremental units. The exact formulation of these problems (as an integer and mixed integer Linear Programming Problem) are presented together with two methods which do not require integer restrictions to be placed on some of the variables.

These two simplifications were compared against the method of total enumeration of all feasible solutions and the results showed the simplifications would result in savings of up to 67% in the computational effort.

The largest problem considered was a problem of four suppliers, three products and three price breaks. It is expected that these methods could be applied to a problem of up to eight suppliers before it will become unwieldy and require too much time for solution. The number of products (within reasonable limits) should not significantly affect the time for a solution.

Description of the Problem

A problem which is faced in many situations when planning purchasing or manufacturing is the problem of allocating orders when the cost of the service or product is dependent upon the total quantity allocated to that source.

There are two general cases of this problem; the first , which will be referred to later as Case I can be described as follows:

The price per unit of all products purchased or manufactured is dependent upon the total quantity allocated to this source. Any economies or diseconomies which result from this policy in terms of unit price is applied against all units. An example which would demonstrate this would be the case where a supplier has one price schedule if the total number of items purchased were between 0 and 499 and he had a second price schedule for all products purchased if the total quantity purchased was in the range ≥ 500 units.

The second case, Case II is similar to Case I except that in this case the total cost curve for each product is continuous rather than discontinuous as in Case I. In this case the economies or diseconomies resulting from the incremental changes in quantity required are reflected only on the incremental production. The change in cost per unit does not apply to all the units produced but only to the additional units produced.

A problem similar to this could arise in manufacturing when it must be determined how much overtime should be worked at various locations (and possibly variable cost for this overtime production) to meet shipping schedules at the lowest overall cost.

For both of these cases there are three possible ways that the prices can vary. The price structure can be convex; the cost per unit is either increasing or is constant as the quantity increases (diseconomies of scale). The second possibility is the concave case where the cost per unit is decreasing as the quantity increases (economies of scale) and the third possibility is a combination of economies and diseconomies of scale within the price structure.

History of the Problem

Closely associated with the mathematics necessary for the exact formulation of the problems, defined previously as Cases I and II (for a concave cost function), are the following six problems which all have in common that their exact formulation as a linear programming problem requires that they have certain variables which are restricted to integer values.

These problems are:

1. Fixed Cost Transportation Problem [3,7,20].
2. Traveling Salesman Problem [7].
3. p - coloring a Map [10, p. 548].
4. Orthagonal Latin Squares [10, p. 547].
5. Scheduling Problem [7].
6. Stock Cutting Problem [11,13].

With certain of these problems the exact formulation of the problem with integer restrictions is a critical part of the formulation while in others the rounding to integers of the solution obtained from an ordinary Linear Programming formulation without the integer restrictions would be sufficiently accurate for most needs and the extra work required to obtain the integer solution is not warranted. The Fixed Cost Transportation problem is a case where the integer restriction is required while in the Stock Cutting Problem the integer restrictions can sometimes be relaxed if the quantities under consideration are large. The rounding in this case would represent only a small error.

For the cases where a concave cost function is being represented using separable programming techniques together with integer or mixed integer restrictions it is not possible to set up the problem or to solve the exact formulation without the integer restrictions since the resulting solution would bear no resemblence to the problem initially formulated.

Shetly [19] and Markowitz and Manne [17] both dealt with the problem of unit cost of a single product being a function of the quantity purchased and they laid the groundwork for the representation of these problems by the use of separable programming. Shetly dealt with the convex cost function while Markowitz and Manne dealt with both the convex and concave cases.

A problem which initially appears to be very similar to the problem defined in Cases I and II is determining the optimum order quantities when discounts are permitted in the price structure. Churchman, Ackoff and Arnoff [4, p. 235-54] discuss this problem for the case of a concave cost function. The procedure which they present to obtain the optimum solution is an iterative procedure as opposed to a closed solution.

The major difference between this problem and the problem in Cases I and II is that the solution for the Economic Order Quantity is based on only one cost schedule while in Cases I and II separate cost functions are required for each possible supplier-product combination.

No direct references were found in any of the literature dealing with the problem of setting up the two problems defined earlier as Cases I and II.

Objective of this Thesis

The concave case (the case of economies of scale) associated with both Cases I and II will be studied both from a mathematical point of view and from the standpoint of applying two computational algorithms to obtain the optimum solution.

The advantages of the computational procedures over the method of total enumeration will be demonstrated for several test problems by using a computer program to enumerate all feasible solutions and then to apply the simplifications called for by these algorithms to demonstrate what the savings would have been if these techniques had been applied.

Mathematical Development of the Problem

A. Separable Programming

Certain kinds of non-linerities - diseconomy of scale (i.e., convex cost function) may be incorporated directly within a linear programming model. For the case of "economies of scale" (i.e., concave cost function), an attempt to employ linear programming is likely to produce results which are entirely misleading.

Consider first the case of diseconomics. Suppose that the total cost for producing a quantity x is given by f(x) in Figure 1. The function f(x) can then be approximated by a piece-wise linear curve f'(x). Over the range $0 \leq x \leq q_K$, the function f'(x) can be described by the variables x_1, x_2, ...x_K as follows:

(1) $$x = \sum_{r=1}^{K} q_r$$

(2) $$f'(x) = \sum_{r=1}^{K} c_r \; q_r$$

(3) $$0 \leq q_r \leq (x_r - x_{r-1}) \qquad r = 1, 2, \ldots\ldots K$$

(4) $$q_r < (x_r - x_{r-1}) \text{ implies } q_{r+1} = 0 \; r = 1, 2, \ldots K-1$$

Where c_r represents the slope of f'(x) from x_{r-1} to x_r and in addition $c_r \geq c_{r-1}$.

Relations (1), (2), (3) may be incorporated within a linear

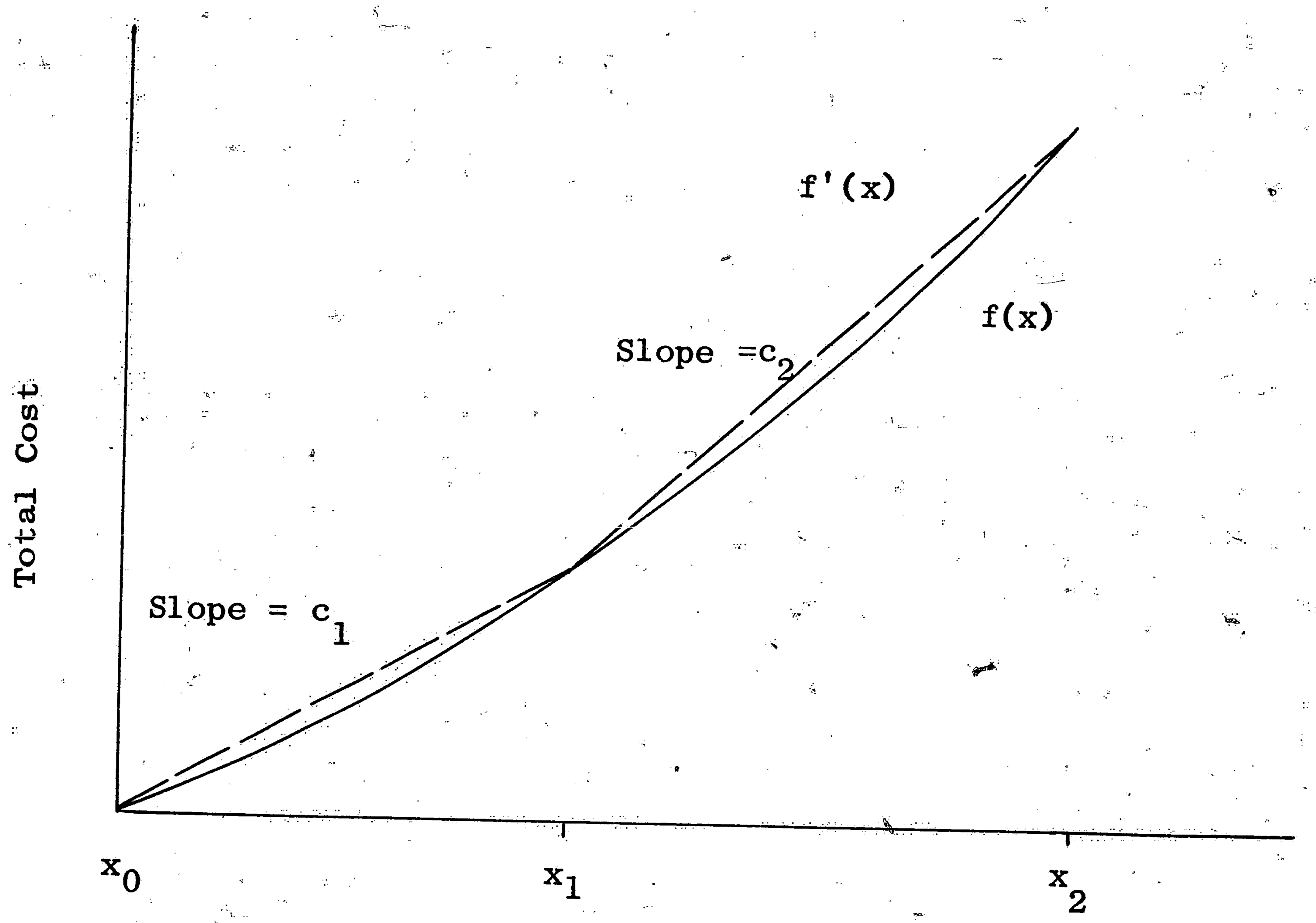


Figure 1: Convex Cost Function

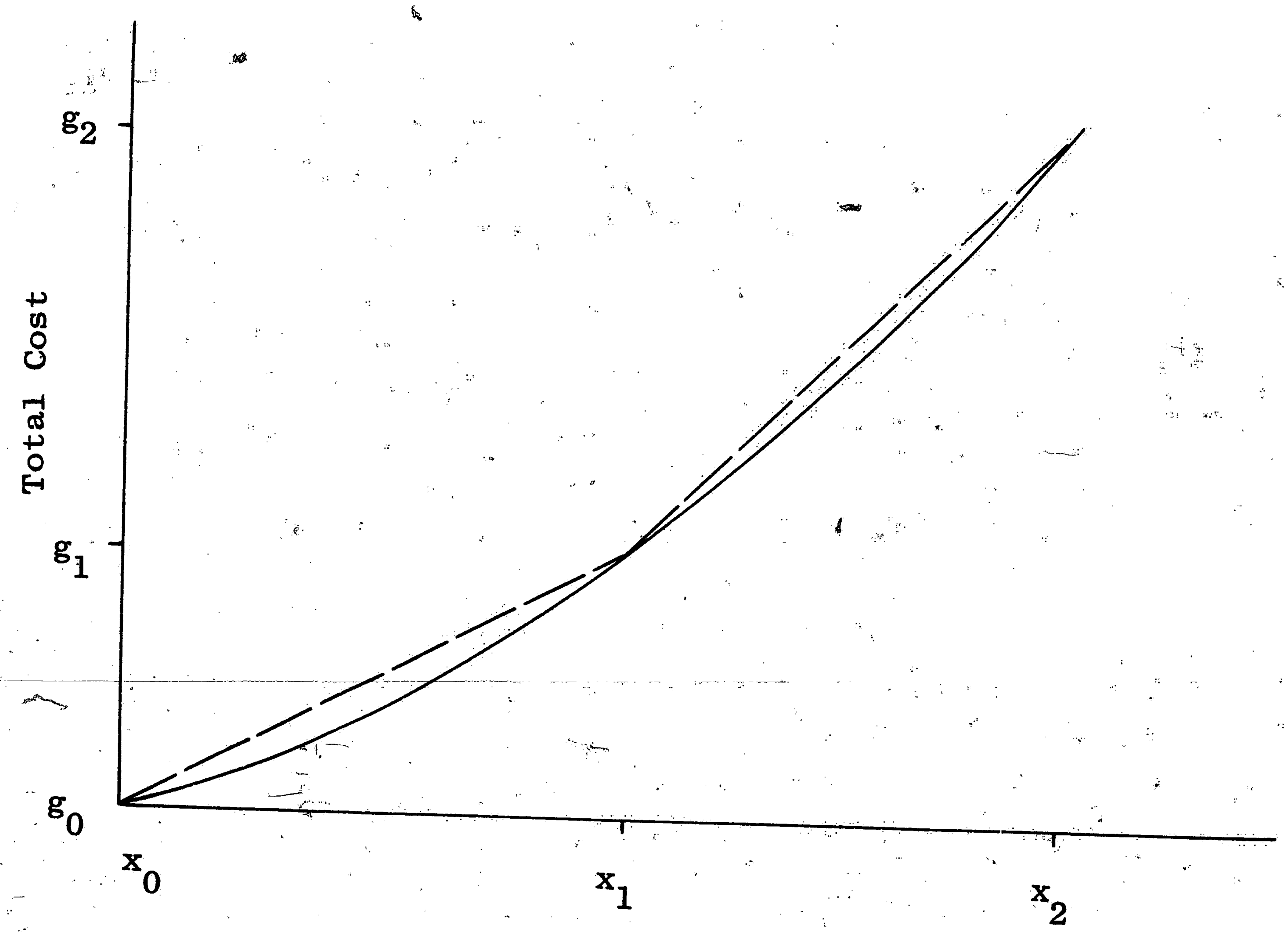


Figure 2: Convex Cost Function, Separable Programming

programming model without making condition (4) explicit. In the optimal solution since c_r is more economical than c_{r+1} condition (4) will hold automatically assuming that there is a place to dispose of the excess of x at no cost.

In addition, an additional representation can be made in the following manner: Any point on the curve f'(x) can be represented as the weighted average of two successive break points.

For each r $r=0, 1, 2, \ldots K$ let (f_r, g_r) be the coordinates of the break point $(x_r, f(x_r))$ of the function f(x) in Figure 2. Any x in the range $f_o \leq x_r \leq f_K$ may be represented by

(5) $x = \Phi_o f_o + \Phi_1 f_1 + \ldots\ldots \Phi_K f_K$

if f(x) is a convex function we have

(6) $f'(x) = \Phi_o g_o + \Phi_1 g_1 + \ldots\ldots \Phi_K f_K$

and in addition

(7) $1 = \Phi_o + \Phi_1 + \ldots\ldots \Phi_K$

Relations (5), (6) and (7) may also be incorporated directly into a linear programming model with the addition of the correct notation for the variables.

Non-linearities that correspond to economies of scale cannot be handled by either of the above representations. This is the concave curve f(x) shown in Figures3 and4. As in the previous cases a piece-wise linear approximation f'(x) may be

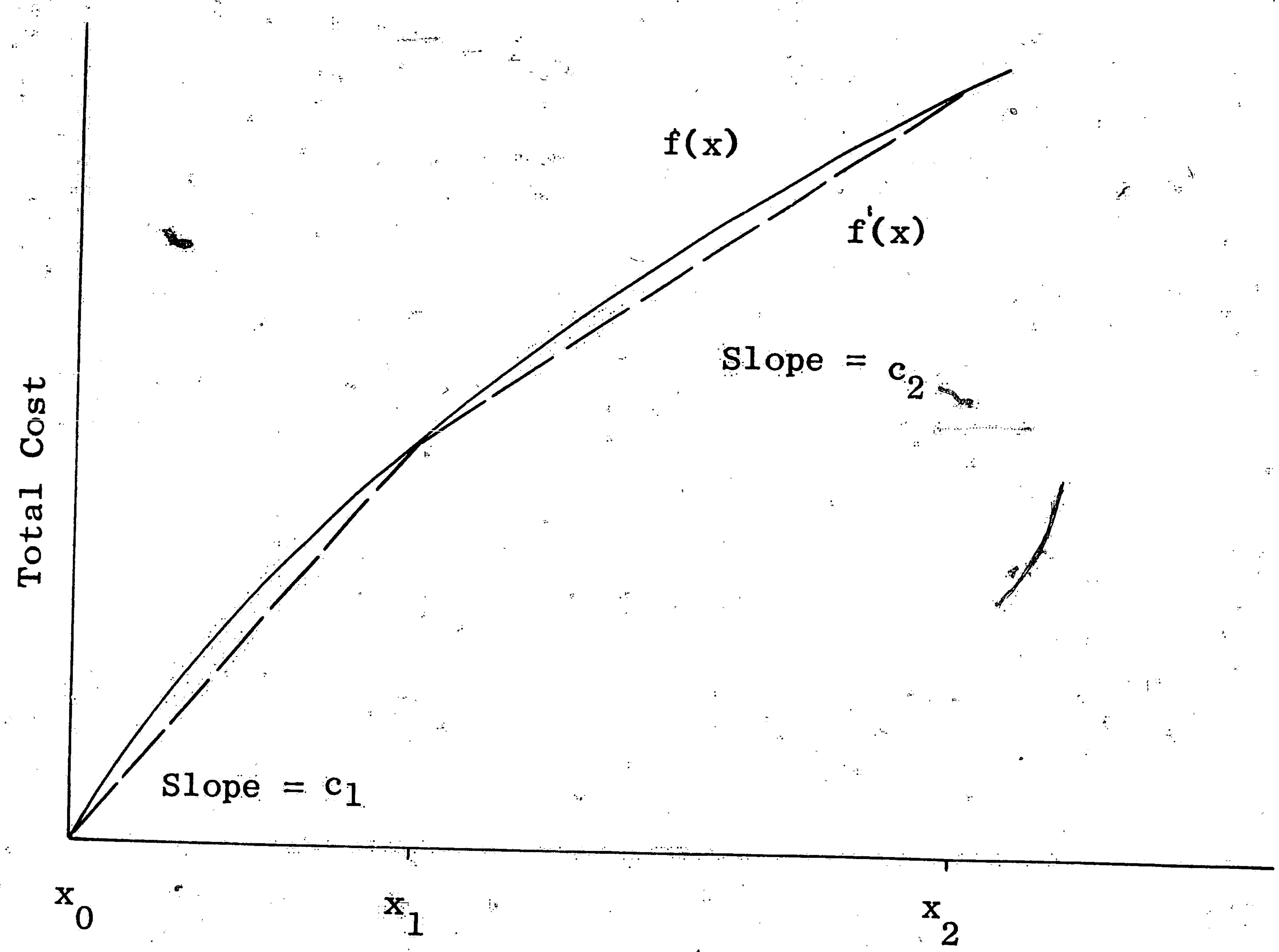


Figure 3: Concave Cost Function, Integer Representation

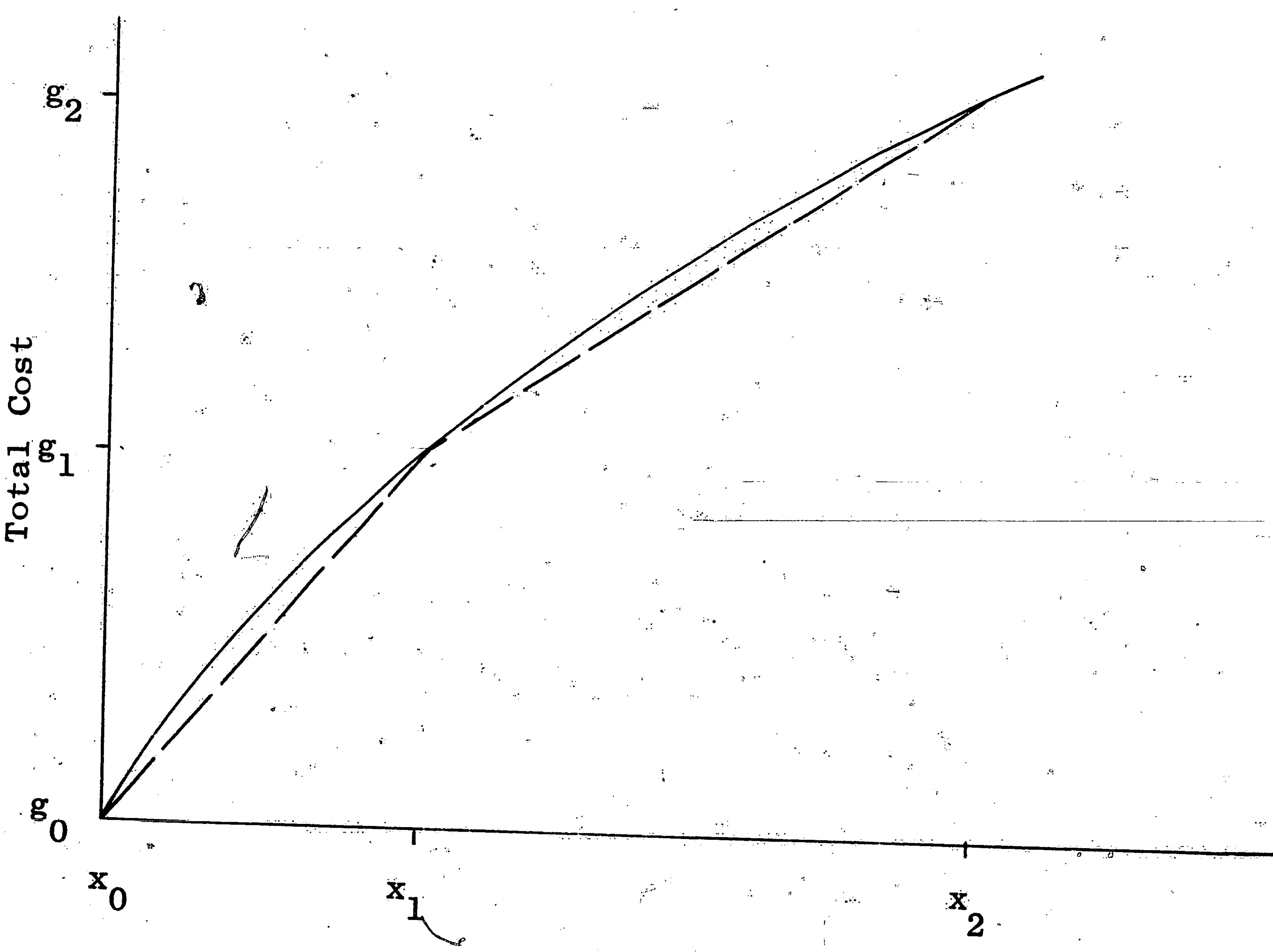


Figure 4: Concave Cost Function, Mixed Integer Representation

described by conditions (1), (2), (3) and (4). In this case $c_r \leq c_{r-1}$. If the constraints (1), (2) and (3) were again embedded in a linear programming model, it would be "optimal" to select the variables in a backward sequence.

$$q_r \leq (x_r - x_{r-1}) \text{ implies } q_{r-1} = 0 \quad r = 1, 2, \ldots . K$$

In this case the model would incorrectly substitute a case of diseconomy of scale for one of economy of scale.

In the second representation of the problem equation (5), (6), and (7) could be used to represent this case, see Figure 4, but in this case the solution would be a point on the line defined by $[x_o, f(x_o)]$, $[x_r, f(x_r)]$ and it would be below the curve defined by the curve $f'(x)$. In this case the model has substituted a linear function for the concave cost function. The inability of these two models to approximate the non-linear relationships shown in Figures 3 and 4 go deeper than the isolated failure of these two models. The points which satisfy the constraints of a linear programming model form a convex set. This is not true of the points on or above $f'(x)$ in Figures (3) and (4). These points form a concave set.

As long as a linear function is maximized over a convex set then we can be sure that a local optimum is also the absolute optimum. Once non-convex constraint sets are admitted, it is possible for a local optimum not to be an absolute optimum.

The economics of scale indicated by the function f'(x) in Figures 3 and 4 may be treated in a discrete programming formulation by using equations (1) and (2) and by defining a set of zero-one variables ($a_1 \ldots\ldots a_K$) related through linear constraints to the continuous variables ($q_1 \ldots\ldots q_K$) as follows:

(8) $$a_r \geq \left[\frac{1}{x_r - x_{r-1}}\right] q_r \qquad r = 1, \ldots\ldots K$$

and

(9) $$a_{r+1} \leq \left[\frac{1}{x_r - x_{r-1}}\right] q_r \qquad r = 1, \ldots\ldots K-1$$

(10) a_r is on integer o, 1 for all r

(11) $q_r \geq 0$ for all r

Condition (8) and (10) insure that if q_r is to be greater than zero, a_r will be forced up to the value of unity. Similarly (9) and (10) provide that unless the variable q_r has attained its maximum value, it is impossible for a_{r+1} to be greater than zero, i.e. impossible to assign a positive value to the variable q_{r+1}. This formulation insures that the variables q_r will be employed in the proper sequence rather than in reverse order. The restriction in (8), (9) and (10) can be rewritten as follows:

(12) $$-q_r + (x_{r+1} - x_r) \geq 0$$

(13) $$q_r - (x_{r+1} - x_r) + (x_{r+1} - x_r)\ a_r \geq 0$$

(14) $$q_{r+1} - (x_{r+2} - x_{r+1})\ \ (1-a_r) \geq 0$$

$$a_r = o \text{ or } 1$$

A second representation of this problem can be formulated in a manner similar to that used in equations (5), (6) and (7):

(15) $x = \Phi_o f_o + \Phi_1 f_1 + \ldots\ldots \Phi_K f_K \qquad \Phi_r \geq 0$

(16) $f'(x) = \Phi_o g_o + \Phi_1 g_1 \ldots\ldots \Phi_K g_K$

(17) $1 = \Phi_o + \Phi_1 + \ldots\ldots \Phi_K$

Then in addition we must impose the condition that all $\Phi_r = o$ except at most one pair Φ_r and Φ_{r+1}. This can be done by introducing the discrete variable a_r defined so that $a_r = 0$ or 1.

(18)
$$\begin{aligned} \Phi_o &\leq a_o \\ \Phi_1 &\leq a_o + a_1 \\ \Phi_2 &\leq a_1 + a_2 \\ \Phi_{K-1} &\leq a_{K-2} + a_{K-1} \\ \Phi_K &\leq a_{K-1} \end{aligned}$$

(19) $1 = a_o + a_1 + \ldots\ldots\ldots a_{K-1}$

An additional representation of this problem can be made in the following manner.

(20) $x = \Phi'_o f_o + \Phi_1 f_1 + \Phi'_1 f_1 + \Phi_2 f_2 + \Phi'_2 f_2 + \ldots\ldots$
$\ldots\ldots + \Phi'_{K-1} f_{K-1} + \Phi_K f_K$

(21) $f'(x) = \Phi'_o g_o + \Phi_1 g_1 + \Phi'_1 g_1 + \Phi_2 g_2 + \Phi'_2 g_2 + \ldots\ldots$
$\ldots\ldots + \Phi'_{K-1} g_{K-1} + \Phi_K g_K$

(22) $1 = \Phi'_o + \Phi_1 + \Phi'_1 + \Phi_2 + \Phi'_2 + \ldots\ldots \Phi'_{K-1} + \Phi_K$

To impose the restriction that all x_r and $x'_r = 0$ except for one pair the following restrictions are necessary:

(23) $\Phi'_o + \Phi_1 = a_o$

$\Phi_1 + \Phi'_1 = a_1$

$\Phi_2 + \Phi'_2 = a_2$

$\Phi_{K-1} + \Phi_K = a'_{K-1}$ $\qquad a_r = 0, 1$

Development of Case I

A representation similar to that used in equations (20), (21), (22) and (23) can be used to represent the case where the effect of allocating additional units, whether it is a net increase in total cost or a decrease, is reflected on all units rather than on only the incremental units. The cost function for this formulation is shown in Figure 5 for the case of diseconomies of scale and in Figure 6 for the case of economies of scale.

In all previous cases it was possible to represent the case of diseconomies of scale without the use of the zero-one variables , equations 23 , but in this case it was necessary to include these discrete variables. If the problem had been represented without the use of these variables the solution would fall on the line defined by (x_o, g_o), (x_1, g_1), (x_2, g_2) ....(x_K, g_K) in Figure 5.

Over the range $0 \leq x \leq q_r$ the problem can be represented as follows:

(24) $x = \Phi'_o x_o + \Phi_1 x_1 + \Phi'_1 x_1 + \ldots\ldots \Phi_K x_K \quad x_r = x'_r$

(25) $f(x) = \Phi'_o g_o + \Phi_1 g_1 + \Phi'_1 g_1 + \ldots\ldots \Phi_K g_K$

(26) $1 = \Phi'_o + \Phi_1 + \Phi'_1 + \ldots\ldots \Phi_K$

and in addition

$$\text{(27)} \quad \Phi_o' + \Phi_1 = 0, 1$$

$$\Phi_1' + \Phi_2 = 0, 1$$

$$\Phi_{K-1}' + \Phi_K = 0, 1$$

This representation differs from the representation used in equations (20) through (23) in that $g_r \neq g_r'$, i.e. the total cost curve is not continuous.

A similar representation could be made for the case of economies of scale shown in Figure 6.

This representation can be expanded to take into account all n products produced by the j^{th} supplier in a manner similar to that used for case II.

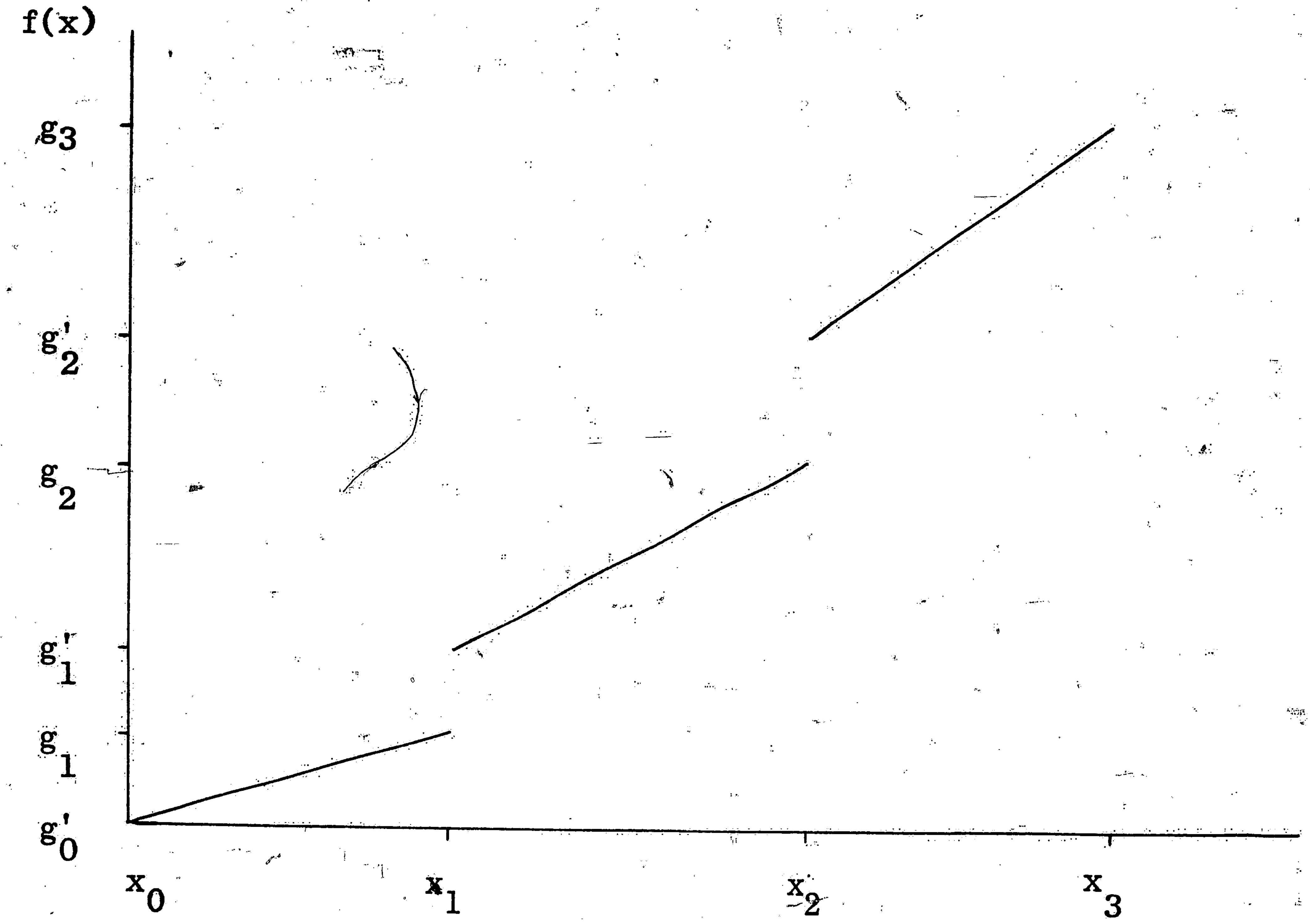


Figure 5: Cost Function for Case II, diseconomies of scale.

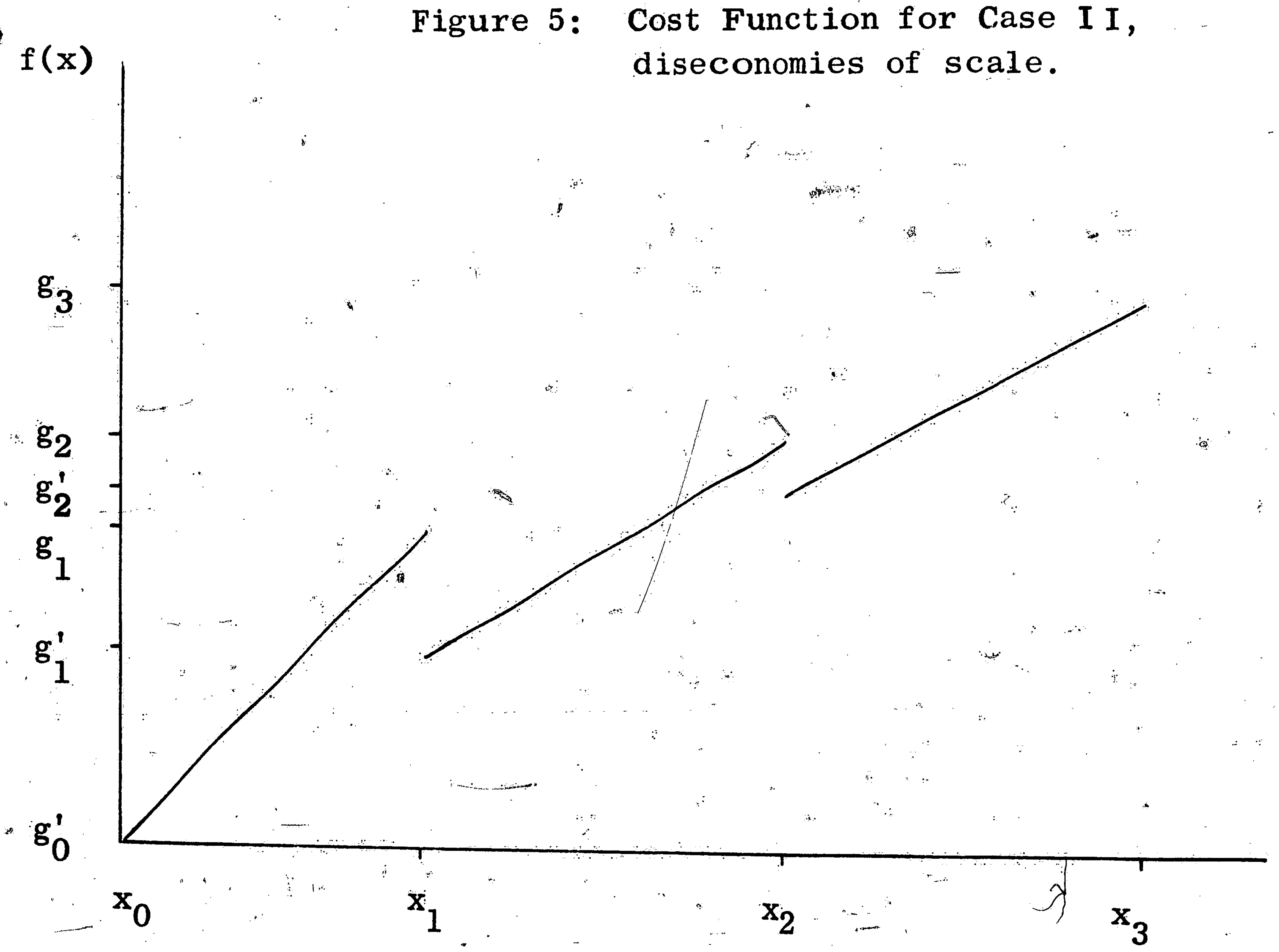


Figure 6: Cost Function for Case II economies of scale.

Development of Case II

A second problem area is that of representing the cost function for the case of where the cost per unit is a function of the total quantity purchased from the individual supplier rather than being a function only of the quantity of the individual product produced, for the problem defined as Case II.

Using the representation of a piece-wise convex cost function used in equation (1), (2), (3) and (4) but expanding the representation to include all the m products that the ith supplier produces we can write the following set of equations:

x_{ij} represents the quantity shipped from supplier j to destination i.

(28) $$x_{ij} = \sum_{r=1}^{K} q_{rij}$$

(29) $$b_j \leq \sum_{r=1}^{K} \sum_{i=1}^{m} q_{rij}$$

(30) $$Z_j = \sum_{r=1}^{K} \sum_{i=1}^{m} c_{rij} \quad q_{rij}$$

(31) $$\gamma_{jr} \leq \sum_{i=1}^{m} q_{rij}$$

b_j is the total amount of all products which could be supplied by supplier j.

γ_{jr} represents the maximum number of units that can be allocated from supplier j at the prices associated with schedule r .

If we have a convex cost function then for all i

$c_{rij} \leq c_{r+1ij}$

(32) $\sum_{i=1}^{m} q_{rij} < \gamma_{jr}$ implies that $q_{r+1,i,j} = 0$

For the case where the cost function is concave ($c_{rij} >$ c_{r+1ij}) it is necessary to implicitly state the relationship implied by equation (32) by means of a zero are variable a_{rj}.

(33) $x_{ij} = \sum_{r=1}^{K} ari\ q_{rij}$

(34) $b_j \leq \sum_{r=1}^{K} \sum_{i=1}^{m} a_{ri}\ q_{rij}$

(35) $\gamma_{rj} \leq \sum_{i=1}^{m} q_{rij}$

(36) $z_j = \sum_{r=1}^{K} \sum_{i=1}^{m} c_{rij}\ q_{rij}\ a_{ri}$

and in addition

(37) if $\gamma_{rj} - \sum_{i=1}^{m} q_{rij} = 0 \quad a_{r+1,i} = 1$

$\gamma_{rj} - \sum_{i=1}^{m} q_{rij} > 0 \quad a_{r+1,i} = 0$

Even though it is possible to solve this representation (see Appendix I) it appears that this method of formulation is not practical due to the size of the resulting problem and also because of the computational state of the art limits the wide-spread use of the present algorithms [9, 12, 15].

Computational Procedure

It would be desirable if both of these problems, Cases I and II, could be formulated in a manner which would not require the problem to be set up as an integer or mixed integer linear programming problem. To accomplish this it is first necessary to make some initial assumption about how the order will be allocated, how many units will be purchased from each supplier, before the problem can be set up as a linear programming problem. The setup of the linear programming problem for Cases I and II are shown respectively in Appendix II and III. For both cases after the initial assumption is made it must be determined if the assumed allocations will result in a feasible solution, i.e., the requirements for the various products can all be met, then the minimum cost solution under these assumptions would be determined. By proceeding in this manner the cost for each alternate feasible solution can be found and the least cost solution can be determined. Computationally, this procedure has many drawbacks, one of which is the necessity of setting up and solving many relatively large linear programming problems. For both cases a systematic approach has been developed to determine the optimum solution. For Case II an alternative approach has been developed and it is compared against the brute force solution (total enumeration), and also against the other computational technique.

The first technique which will be evaluated for both Case I and Case II can be described as follows:

For a case of m suppliers supplying n products which have K price breaks the first assumption to be made is that all m suppliers will participate in the order; all suppliers will be assumed to be supplying products at a cost associated with at least their lowest quantity range in their price structure (this range should be $0 \leq x_{ij1} \leq \gamma_{j1}$). After solving this set of feasible solutions it will be found that in many cases, even though it was assumed that an allocation would be made to a supplier (in the range $0 \leq x_{ij1} \leq \gamma_{j1}$) the resulting minimum cost solution says no allocation should be made.

For the cases where this occurred it will be possible to eliminate the need to solve certain other feasible solutions since the answer to these are already known. This can be shown in Table 1.

Table 1

Simplification 1

	Assumed Range	Optimum Solution	Ranges for Other Feasible Solutions which need not be considered.		
Supplier I	0-500	0	0	0-500	0
Supplier II	0-500	400	0-500	0-500	0-500
Supplier III	0-500	200	0-500	0-500	0-500
Supplier IV	0-500	0	0-500	0	0

The next step in the solution would be to consider all feasible solutions for the assumption that (m-1) suppliers will be included in the order. After a feasible solution has been obtained it should be checked against the list of feasible solutions which have been eliminated in the previous steps. If this solution has not been already eliminated then the problem would be solved and as in the previous step, the solution would be examined to see if other feasible solutions could be excluded.

This method would proceed until each supplier was assumed to fill the orders for all products.

As a last step in the procedure the list of feasible solutions would be examined to determine the lowest cost solution.

For Case II where the cost function is a continuous function there is an alternate procedure which could be used. This procedure would work from the other end of the family of solutions (C(m,1) to C(m,m) rather than C(m,m) to C(m,1)). This procedure can be described as follows:

This procedure begins with the assumption that the total order will be allocated to each of the various suppliers (one at a time) in turn (this will cause the allocation to be made at the lowest per unit cost for each supplier). After making this assumption the feasible solutions are determined (if there are any) and the linear program is set up and solved to determine the lowest cost solution (along with any alternate equivalent cost solutions).

The next step in the procedure assumes that the total order will be allocated to two suppliers (all the combinations of m suppliers taken two at a time will be considered). For each of these combinations of suppliers all feasible solutions are found, the Linear Programming Problem is set up, and then they are solved for the minimum cost solution. The lowest cost solution or solutions are then compared with the solutions for the assumption of one supplier filling all orders. If a lower cost solution was not found in step two or a different equivalent cost solution was not found in step two, then the solution in step one is optimum. If the solution in step one is not shown to be optimum then the procedure will be repeated for three suppliers splitting the order and the optimum solution will be compared to the optimum for two suppliers using the same criteria previously used. This method is continued until either an optimum is found or all suppliers are considered participating in the order. This procedure will work for Case II but cannot be applied to Case I because of the discontinous total cost curves.

Evaluation

A. Objectives

A series of test problems were run to demonstrate the computational procedure and to evaluate the effectiveness of these two procedures against the method of total enumeration.

The specific objectives of this evaulation were to:

1. Determine the savings in computational effort which would result from the application of the computational procedures to this problem.
2. Demonstrate that the solution obtained by both methods would be the optimal solution.
3. Demonstrate the approximate size of the problem which would be encountered while applying this technique to obtain solutions involving four or less suppliers.

B. Test Problems

Two test problems were designed to demonstrate the simplification techniques. The same test problems were used for both Cases I and II except in the data used for Case II the costs associated with supplier 4 were eliminated since the maximum number of suppliers which could be handled in the program for this case was three. This limitation of three suppliers for Case I and four suppliers for Case II was a restriction imposed by the size of the memory of the computer used (IBM 1620, 20K memory) and by the time necessary to solve these problems (all feasible solutions had to be evaluated to insure optimality while evaluating these procedures to solve these two cases). It was necessary to transfer the program to an IBM 1410 to take advantage of the faster internal speed, but the size of the problem was not increased since this would have required a major reprogramming effort, and it was felt that a program of the size originally programmed would demonstrate the techniques required for these two procedures adequately. The major considerations used when designing the test problems were that the unit cost of the product decreased as the quantity increased and that the cost of each product at each level was within reasonably close limits from all suppliers (± 10% for test problem I and ± 20% for test problem II). The cost matrices for these test problems are shown in Appendix IV.

C. Results

A sample calculation to demonstrate how the two computational techniques were applied to the data is shown in Appendix V. The results of the computations for price structures I and II at various levels of demand are shown in Tables 3 and 4.

Looking initially at Case I the following results were obtained:

1. The number of feasible solutions requiring a solution to obtain the optimum solution to the problem was reduced by between 51% and 67% for the range of requirements under consideration using the first simplification technique.

2. The resulting solution obtained was in all cases the optimal and in addition all unique solutions were also obtained. This gives the person planning the project alternate solutions which have higher costs from which he may want to choose due to some other consideration not implicit in the set up of the problem.

For Case II the following results were observed:

1a. The number of feasible solutions requiring a solution to obtain the optimum solution to the problem for Simplification I over the method of total enumeration resulted in a savings of between 14% and 53%.

1b. For Simplification II the number of feasible solutions which had to be solved to obtain the answer showed a reduction of up to 36% over total enumeration of all solutions.

1c. The solution obtained in all cases for Simplification I were superior with respect to the amount of computation required compared to Simplification II.

2. The answer obtained by both methods yielded the optimum solution for all cases tested.

It should be noted that these conclusions are based on two test problems and the total amount of product allocated was a small percentage of the total amount available (5 to 25 percent). Plotting the total number of feasible solutions against the ratio of the total allocated for all products to the total overall quantity available, results similar to those shown in Figure 7 would be expected for both Cases I and II. No generalization can be made about the general shape of a similar curve for unique solutions (the number of solutions necessary to be solved to obtain an optimum solution for Cases I and II with Simplification 1). Only two points on such a curve are known without testing. These are the extreme points (o, m), (1, 1). The results for Case I and Case II using Test Problem 1 and 2 are plotted respectively in Figures 8 and 9 as Percent Savings versus the ratio of the total quantity of all products allocated to the total quantity of product available. This ratio was not made $> .3$ because for larger values the price breaks in the cost function limits the number of feasible solutions which can be obtained. This limitation would not affect the computation in practice since this ratio would normally be a small number, i.e., the available capacity from which assignments must be made normally is much larger then the demand, especially when allocating orders to suppliers.

Simplification 2 did not yield as high a Percentage Savings as was expected it might have (see Table 3), because of the design of the test problem. Experience with this procedure has shown that it will work most efficiently if the number of suppliers is large. A problem with only three suppliers was not large enough to adequately test this procedure.

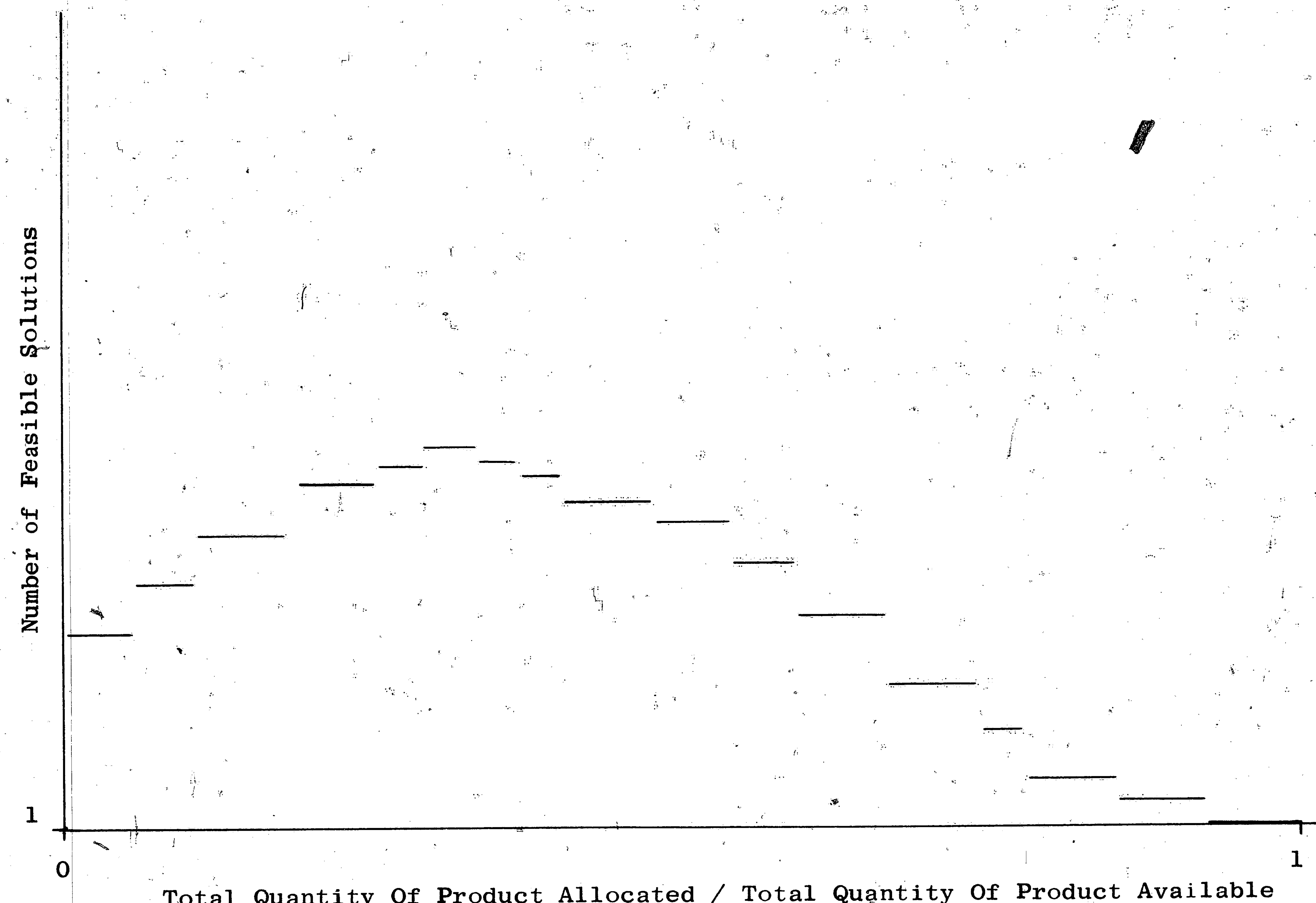
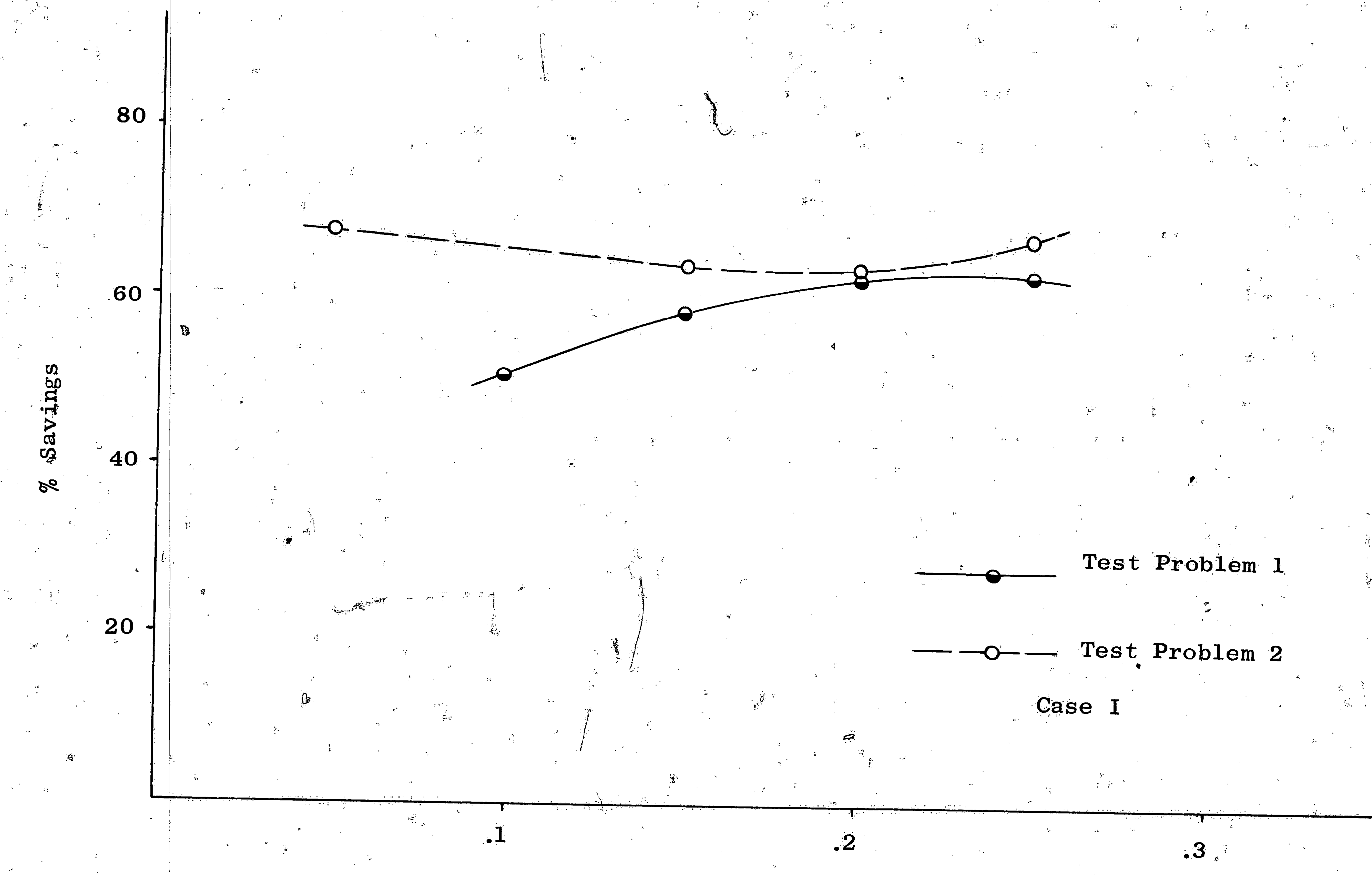


Figure 7: Distribution of Feasible Solutions



Total Quantity of Product Allocated/Total Quantity of Product Available

Figure 8: Results for Case I, Computational Method 1

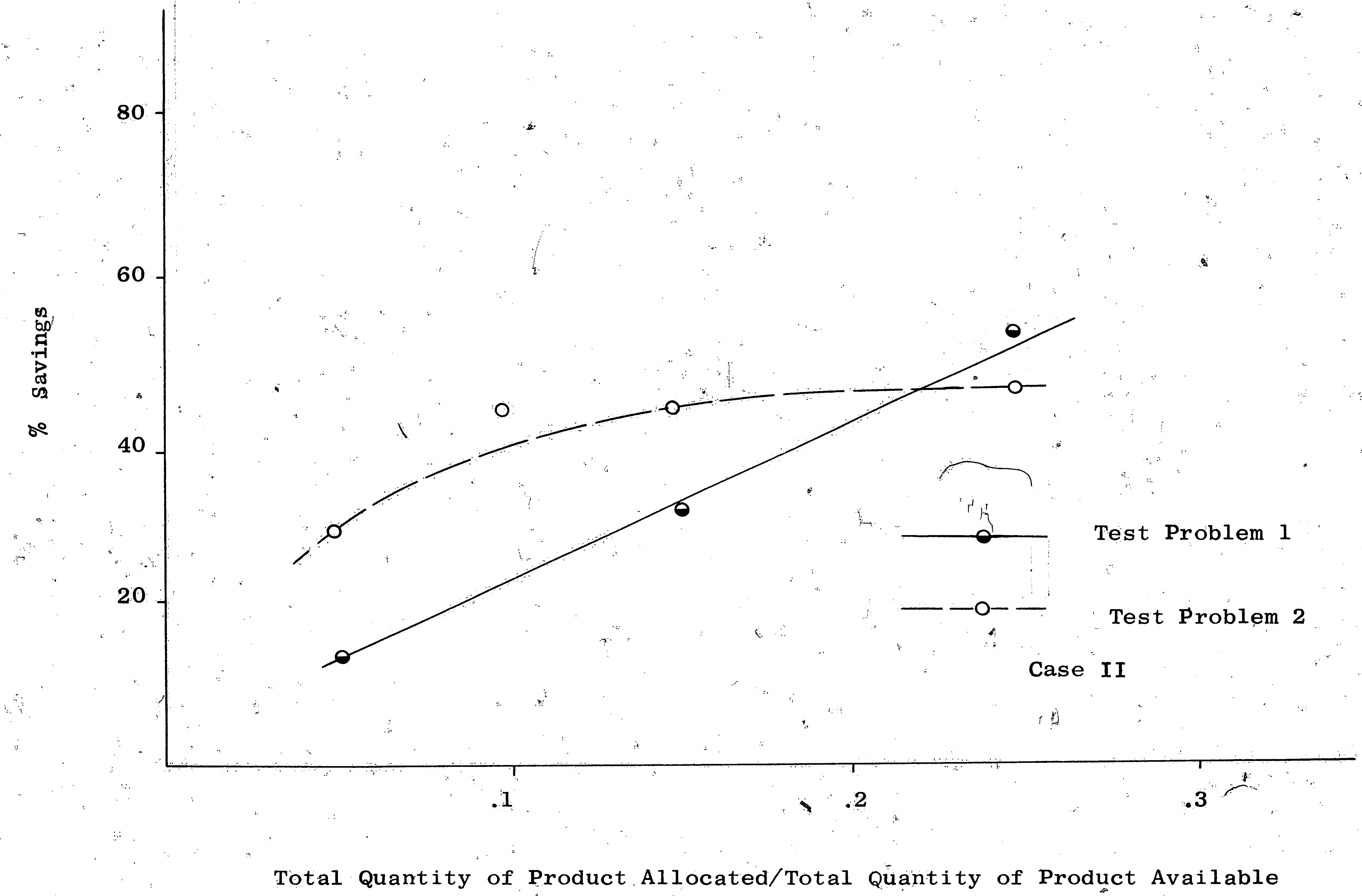


Figure 9: Results for Case II, Computational Method 1

Test Problem 1

Requirements Product 1	Product 2	Product 3	Number of Feasible Solutions by Total Enumeration	Number of Required Solutions for Simplification 1	% Savings
200	200	200	43	21	51%
300	300	300	43	18	58%
400	400	400	89	33	63%
500	500	500	89	33	63%

Test Problem 2

Requirements Product 1	Product 2	Product 3	Number of Feasible Solutions by Total Enumeration	Number of Required Solutions for Simplification 1	% Savings
100	100	100	15	5	67%
200	200	200	43	16	63%
300	300	300	43	16	63%
500	500	500	89	30	66%

Table 2: Summary Case I

Test Problem 1

Requirements Product 1	Requirements Product 2	Requirements Product 3	Number of Feasible Solutions by Total Enumeration	Number of Required Solutions for Simplification 1	% Savings	Number of Required Solutions for Simplification 2	% Savings
100	100	100	7	6	14.3%	7	0
300	300	300	16	9	31.0%	16	0
500	500	500	28	13	53.5%	18	35.8%

Test Problem 2

Requirements Product 1	Requirements Product 2	Requirements Product 3	Number of Feasible Solutions by Total Enumeration	Number of Required Solutions for Simplification 1	% Savings	Number of Required Solutions for Simplification 2	% Savings
100	100	100	7	5	29%	7	0
200	200	200	16	9	44%	16	0
300	300	300	16	9	44%	16	0
500	500	500	28	15	46%	18	36%

Table 3: Summary Case II

Conclusions

Both the procedures which were evaluated represented an improvement over the method of total enumeration of all possible feasible solutions to find the optimum solution. For Case II where two computational procedures were being compared, the first technique (looking at all solutions from C(m,m) to C(m,1)) showed a definite improvement over the second computational procedure. The first technique appears to be a practical alternative to solving the mixed integer linear programming formulation (concave case) required for Case II for small to moderate sized problems and it holds even more promise for Case I since the exact formulation to this problem is a more complicated Integer Linear Programming Problem or a Quadratic Programming Problem with Integer Restrictions.

One additional drawback to the exact formulation of this problem is the size of the resulting problem. What would appear to be a small problem turns out to be complicated and lengthy when using the integer and mixed integer algorithms. Although the size and time required to solve the problem by the two computational methods presented here increase as the size of the problem increases, for problems where there are less than 3 price breaks for each product it appears that these methods could be used in a practical problem for up to 6-8 suppliers (the limit is the memory size of the computer and the time necessary for the solution). The number of products involved should not affect the computations required significantly but the amount of the total available product and the break-points in the price structure significantly affect the computations.

Recommendations for Additional Work

The two problems under consideration both used the total quantity purchased from a supplier as the criteria for determining which price schedule would be used. A logical extension of this problem would be the case where the price schedule used is a function of the total value of the purchases from each supplier. It might be possible to modify the procedure presented to solve this case but at best it would show less savings than for the cases tested.

If it had been required to exclude certain supplier-product combinations from consideration, either because the supplier did not manufacture this product or for some other reason, it is necessary in the present set-up of the problem to associate a high cost with this supplier-product combination so that normally it will not enter into the solution. This will not work in every case because at some stage of the problem (especially Case II) the assumption will be made that each supplier will supply all of the requirements. If this supplier is capable of meeting all the requirements in terms of total quantity (a feasible solution) then the problem will be set-up and solved and this illegal supplier-product combination will be forced into the solution simply because there was no alternate source of supply.

A refinement in the method of determining feasible solutions and in the set up of the Linear Programming Problem would eliminate this situation. If there were only a few of these undesirable supplier-product combination the above method would work adequately since even though a solution was found, the total cost of this solution would be so high that it would not be selected as an optimal solution. If there were a large number of these undesirable supplier-product combinations, it is possible that for Simplification 2 when applied to Case II, the criteria for an optimum solution might be met and yet one or more of these undesirable supplier-product combinations might be in the solution. For this situation a refinement in the method of determing feasible solutions would be necessary.

When considering the exact formulation of the problem of the price of a product being only a function of the quantity purchased from that supplier two representations were presented, equation (15) through (19) and equation (20) through (23) which appear to be quite similar. A closer look will show that for the same problem equation (15) through (19) would have less variables (the same number of integer variables) but would have more equation that the representation in equation (20) through (23). It would be interesting to formulate the same problem in both representations and to compare the necessary computational work to solve both representations.

Appendix I

The mathematical representation of the problem in equation (32) through (37) is as follows:

(33) $$x_{ij} = \sum_{r=1}^{K} a_{ri}\, q_{rij}$$

(34) $$b_j \leq \sum_{r=1}^{K} \sum_{i=1}^{m} a_{ri}\, q_{rij}$$

(35) $$\gamma_{jr} \leq \sum_{i=1}^{m} a_{ri}\, q_{rij}$$

(36) $$z_j = \sum_{r=1}^{K} \sum_{i=1}^{m} c_{rij}\, q_{rij}$$

(37) $$\text{if } \gamma_{jr} - \sum_{i=1}^{m} q_{rij} = 0 \qquad a_{r+1\ i} = 1$$

$$\gamma_{jr} - \sum_{i=1}^{m} q_{rij} > 0 \qquad a_{r+1\ i} = 0$$

These are two basic problems associated with these representations, one the cost function has been expressed as the product of three terms - two of which are variables and the second problem is that any solution will be a mixed integer form since the a_{ri}'s have been defined as a zero-one variables.

To eliminate the product term it is possible to define two new variables U_{rij} and V_{rij} in the following manner:

(38) $$U_{rij} = (1/2)\,(q_{rij} - a_{ri})$$

(39) $$V_{rij} = (1/2)\,(q_{rij} + a_{ri})$$

The product term $a_{ri}\, q_{rij}$ can now be expressed as:

$$a_{ri} \cdot q_{rij} = V^2_{rij} - U^2_{rij}$$

With this simplification we can now represent the problem in the following way:

$$(40) \quad x_{ij} = \sum_{r=1}^{K} (V^2_{rij} - U^2_{rij})$$

$$(41) \quad b_j \leq \sum_{r=1}^{K} \sum_{i=1}^{m} (V^2_{rij} - U^2_{rij})$$

$$(42) \quad z_j = \sum_{r=1}^{K} \sum_{i=1}^{m} c_{rij} \; (V^2_{rij} - U^2_{rij})$$

$$(43) \quad \gamma_{jr} \leq \sum_{i=1}^{m} (V^2_{rij} - U^2_{rij})$$

At this point there are now two techniques which could be used to solve this representation of the problem. The first is to solve the problem as a Quadratic Programming problem or secondly the non-linearities can be approximated by a piece-wise linear function (the resulting function will be convex). Doing the latter will restrict the set of possible solutions to only certain values depending upon the number of linear elements used to approximate the quadratic function. This would increase the size of the problem excessively and therefore it would not be a practical approach to the solution to this problem.

Appendix II

Linear Programming Representation for Case I

In order to determine all feasible solutions and the lowest cost solution among the set of feasible solutions, a program was written in FORTRAN IV which would determine which solutions were feasible, set up the required Linear Programming Problem and solve it for the minimum cost solution. This program was written for a maximum of 3 products, 4 suppliers and 3 price breaks for each supplier. The input to this program consists of the following items of data:

1. Price Break Points for each Supplier.
2. Cost for each product between each Break Point.
3. Requirement for each of the Products.
4. Information as to which supplier will be supplying products towards meeting the total requirement.

Set-up of the Problem

The Linear Program which was set up by the computer to represent the problem in Case I is shown in Equation (II - 1) through (II - 4). All costs which are associated with allocations which are not allowed to enter the solution (because of the initial assumption on allocations) are set at a large positive cost so that no allocation will be made to these variables when minimizing the cost function z.

$$\text{(II - 1)} \quad A_i = \sum_{r=1}^{K} \sum_{j=1}^{n} x_{rij} \qquad i = 1, \ldots m,$$

$$\text{(II - 2)} \quad L_{r_{max}j} \geq \sum_{r=1}^{K} \sum_{i=1}^{m} x_{rij} \qquad j = 1, \ldots n$$

$$\text{(II - 3)} \quad L_{r_{max}-1j} \leq \sum_{r=1}^{K} \sum_{i=1}^{m} x_{rij} \qquad j = 1, \ldots n$$

$$\text{(II - 4)} \quad Z = \sum_{r=1}^{K} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{rij}\, x_{rij}$$

i = Product $\quad i = 1, \ldots m$

j = Supplier $\quad j = 1, \ldots n$

L_{rj} - Break point in price structure for supplier j in units. $0 \leq r < K$

x_{rij} - Quantity of product i purchased from supplier j at a cost associated with an allocation in the range between L_{r-1j} and L_{rj} $\quad 1 \leq r \leq K$

c_{rij} - Unit cost for product i from supplier j associated with an allocation in the range $L_{r-1j} \leq \sum_{r=1}^{K} \sum_{j=1}^{n} x_{rij} \leq L_{rj}$

A_i - Demand for product i

r_{max} - The break point in quantity below which the allocation has been assumed to be made (r_{max} is determined for each j from the assumption made about the allocation).

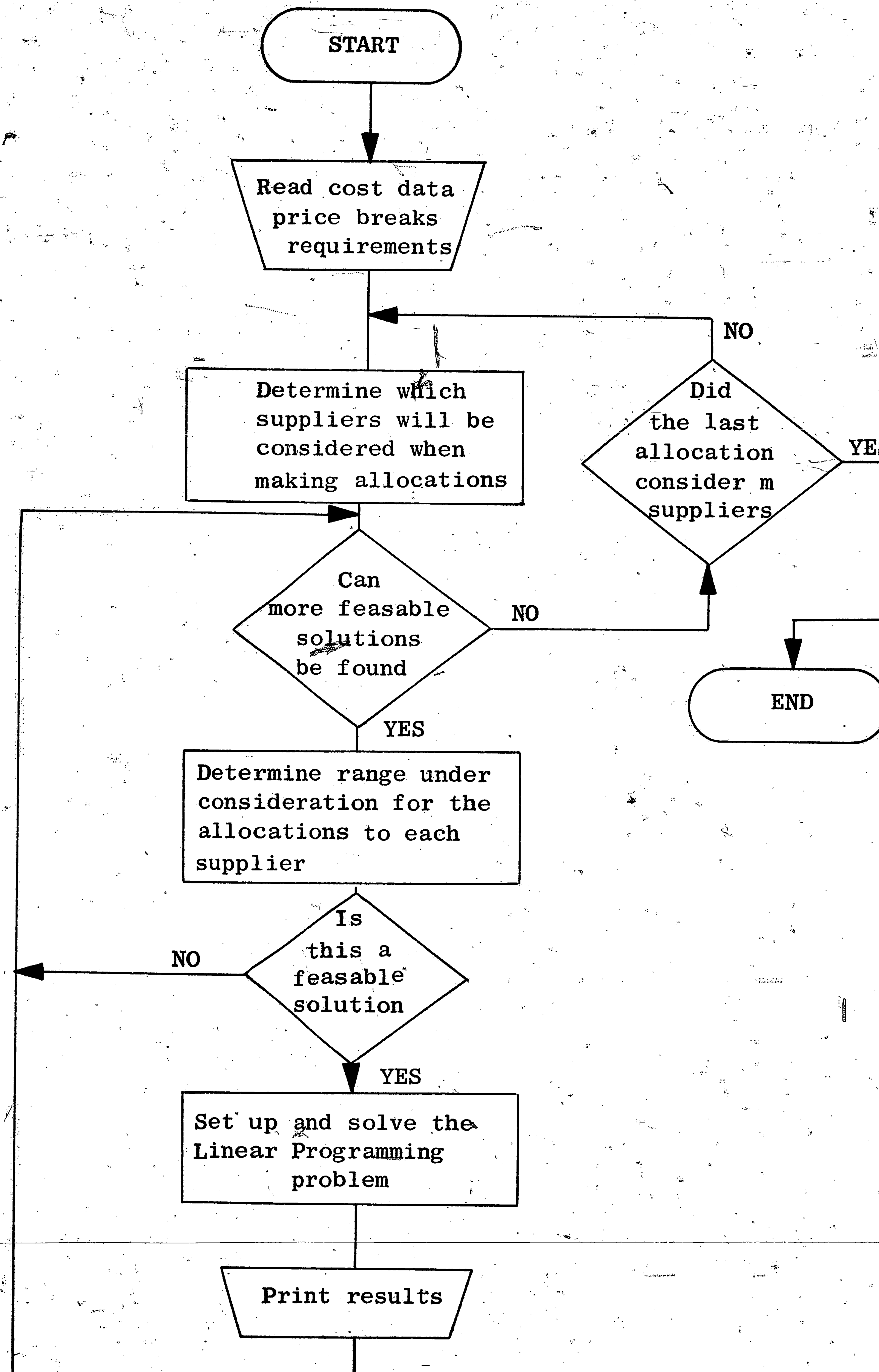


Figure 10: Flow chart for program used to enumerate all feasable solutions for Case I.

Appendix III

Linear Programming Representation for Case II

A program which has the same function as the program written for Case I was prepared for this case (see Appendix I). This program was designed for a maximum of three products, three suppliers and three price breaks. The inputs and outputs of this program are similar to those for the program for Case I.

Set-up of the Problem

The Linear Program which was set-up by the computer to represent the problem in Case II is shown in equations (III - 1) through (III - 5). All costs which are associated with allocations which are not allowed to enter the solution (because of the initial assumptions on allocations) are set at a large positive cost so that allocations will not be made to these when minimizing the cost function z.

(III - 1) $L_{Kj} \leq \sum_{r=1}^{K} \sum_{i=1}^{m} x_{rij} \quad j = 1, \ldots n$

(III - 2) $A_i = \sum_{r=1}^{K} \sum_{j=1}^{n} x_{rij} \quad i = 1, \ldots m$

(III - 3) $L_{rj} \leq \sum_{i=1}^{m} x_{rij} \quad r = 1, \ldots K$

(III - 4) $\sum_{i=1}^{m} A_i - \sum_{j=1}^{n} L_{r_{max}-1\,j} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{r_{max}ij}$

(III - 5) $z = \sum_{r=1}^{K} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{rij}\, x_{rij}$

These variables have been defined earlier in Appendix II.

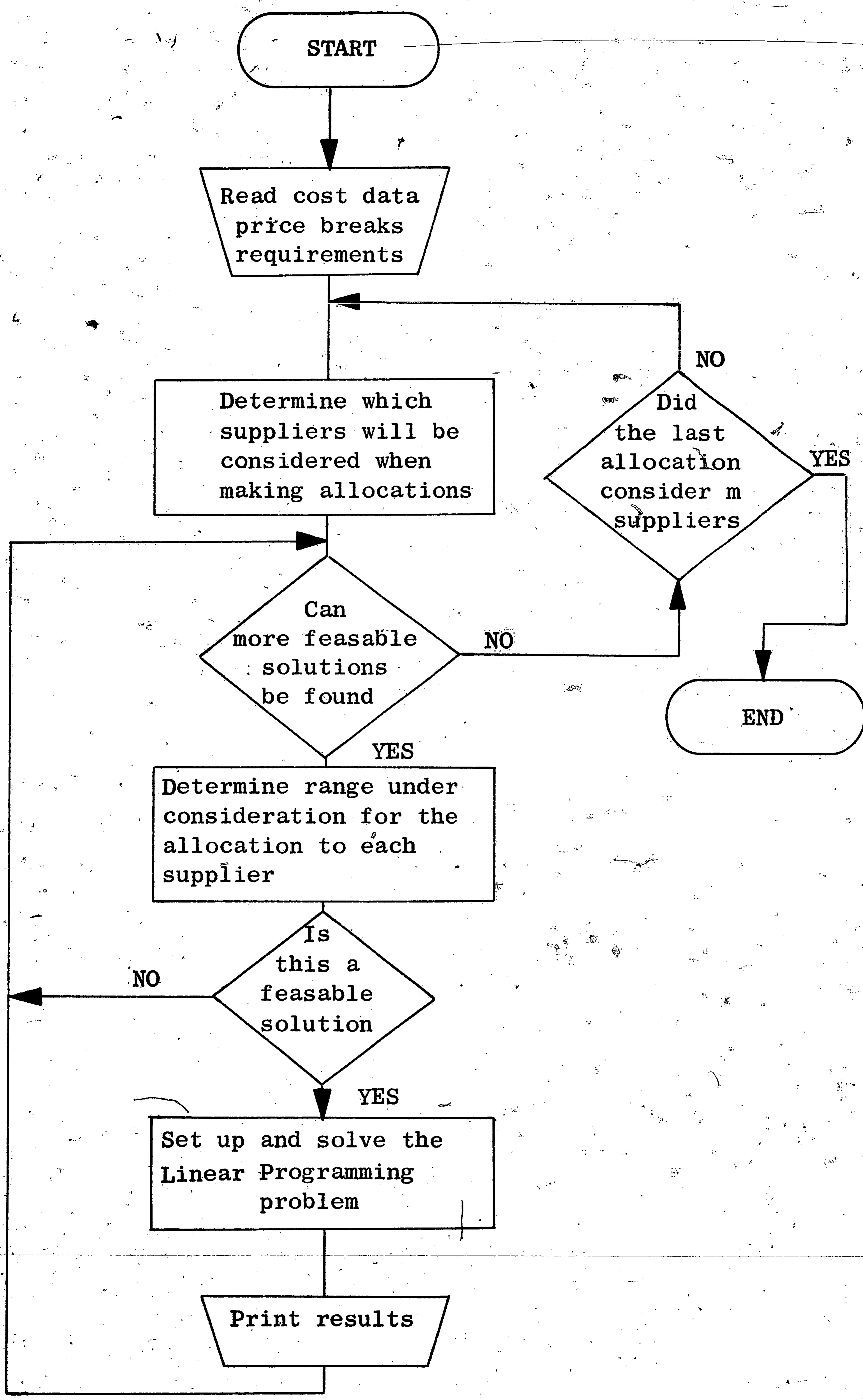


Figure 11: Flow chart for program used to enumerate all feasable solutions for Case II.

Appendix IV

Cost Matrix Test Problem I

	Supplier 1			Supplier 2			Supplier 3			Supplier 4		
Product 1	.10	.09	.08	.11	.10	.09	.09	.03	.07	.10	.09	.08
Product 2	.25	.20	.15	.24	.21	.14	.26	.20	.14	.27	.22	.16
Product 3	.09	.08	.07	.28	.07	.06	.09	.08	.07	.10	.09	.08
	0	501	1001	0	501	1001	0	501	1001	0	501	1001
	500	1000	2000	500	1000	2000	500	1000	2000	500	1000	2000

Cost Matrix Test Problem II

	Supplier 1			Supplier 2			Supplier 3			Supplier 4		
Product 1	.10	.08	.06	.09	.09	.08	.11	.10	.09	.10	.09	.07
Product 2	.50	.40	.30	.40	.40	.30	.60	.50	.40	.45	.40	.35
Product 3	.10	.09	.08	.10	.10	.08	.11	.10	.08	.09	.08	.07
	0	501	1001	0	501	1001	0	501	1001	0	501	1001
	500	1000	2000	500	1000	2000	500	1000	2000	500	1000	2000

Appendix V

Simplification #1 Cost Structure 2
Case II

Feasible Solution #	Assumed Range for Supplier 1	2	3	Actual Allocation Supplier 1	2	3	Cost	Equivalent Solutions that can be eliminated from considerations 1	2	3	Same Solution as Solution #
1	D	B	B	1500	0	0	165	D	B	A	
								D	A	B	
								D	A	A	
2	C	C	B	999	501	0	190	C	C	A	
3	C	B	C	999	0	501	190	C	A	C	
4	C	B	B	1000	500	0	190	C	B	A	
5	B	D	B	0	1500	0	160	A	D	B	
								B	D	A	
								A	D	A	
6	B	C	C	0	501	999	190	A	C	C	
7	B	C	B	0	1000	500	190	A	C	B	
8	B	B	D	0	0	1500	155	A	B	D	
								B	A	D	
								A	A	D	
9	B	B	C	0	500	1000	185	A	B	C	
10	B	B	B	500	500	500	190				

Appendix V (cont'd)

Feasible Solution #	Assumed Range for Supplier 1	2	3	Actual Allocation Supplier 1	2	3	Cost	Equivalent Solutions that can be eliminated from considerations 1	2	3	Same Solution as Solution #
	D	A	B								1
	C	A	C								3
11	C	A	B	1000	0	500	190				—
	B	A	D								8
12	B	A	C	500	0	1000	190				
	A	D	B								5
	A	C	C								6
	A	C	B								7
	A	B	D								8
	A	B	C								9
	D	B	A								1
	C	C	A								2
	C	B	A								4
	B	D	A								5
13	B	C	A	500	1000	0	195				
	A	A	D								8
	A	D	A								5
	D	A	A								

Appendix V (cont'd)

Range "A" No allocation is assumed

Range "B" 0-500 Units

Range "C" 501-1000 Units

Range "D" 1001-2000 Units

Simplification #2 Cost Structure 2

Case II

Feasible Solution #	Assumed Range for Supplier 1	2	3	Cost	List of (Variables) and allocation		
1	D	A	A	165	(2) 500	(12) 500	(19) 500
2	A	D	A	160	(5) 500	(15) 500	(22) 500
3	A	A	D	155*	(7) 500	(18) 500	(26) 500
4	B	C	A	195	(1) 500	(14) 500	(22) 500
5	B	D	A	160	(5) 500	(15) 500	(22) 500
6	C	B	A	190	(1) 500	(11) 500	(22) 500
7	C	C	A	190	(1) 500	(11) 500	(22) 500
8	D	B	A	165	(2) 500	(12) 500	(19) 500
9	A	B	C	185	(7) 500	(17) 500	(22) 500
10	A	B	D	155*	(7) 500	(18) 500	(26) 500
11	A	C	B	190	(7) 500	(14) 500	(22) 500
12	A	C	C	185	(7) 500	(17) 500	(22) 500
13	A	D	B	160	(5) 500	(15) 500	(22) 500
14	B	A	C	190	(7) 500	(17) 500	(19) 500
15	B	A	D	155*	(7) 500	(18) 500	(26) 500
16	C	A	B	190	(7) 500	(11) 500	(19) 500
17	C	A	C	190	(7) 500	(11) 500	(19) 500
18	D	A	B	165	(2) 500	(12) 500	(19) 500

Range "A" No allocation is assumed

Range "B" 0-500 Units

Range "C" 501-1000 Units

Range "D" 1001-2000 Units

*Optimum Solution in any group

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