1966

Characterization of x in beta(x)

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CHARACTERIZATION OF X IN $\beta(X)$

by

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A THESIS

Presented to the Graduate Faculty

of Lehigh University

in Candidacy for the Degree of

Master of Science

Lehigh University

1966

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

May 26, 1966
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ACKNOWLEDGEMENT

The author wishes to thank Professor Stephen Willard of Lehigh University who provided the idea and whose advice and counsel made the completion of this thesis possible.

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ABSTRACT

This paper represents an attempt to unify some fairly recent results in the field of point-set topology. With the emphasis on clarification and, in some instances, simplification we have tried to characterize certain properties of X in terms of its Stone-Čech compactification, $\beta(X)$. For this reason we consider only completely regular spaces throughout the discussion. The concepts with which we shall be chiefly concerned are topological completeness, paracompactness, the Lindelöf property, real compactness, and complete metrizability.

Sections I and II contain preliminary terminology and a brief discussion of uniformities, pseudometric uniformities, and uniformly continuous and continuous pseudometrics. In Section III partitions of unity are introduced and an important theorem relating open coverings of a normal space to partitions of unity is proved. Section IV contains the standard result concerning pseudometrizable spaces and paracompactness, i.e. pseudometrizable implies paracompact. The theorem is proved without the usual restrictions of regularity or Hausdorff on the space. In Section V $\beta(X)$ is defined and its maximality among compactifications of X is demonstrated.

Section VI begins the main body of the thesis. Within this section the first major equivalence theorem is proved equating paracompactness to several other properties of X in

$\beta(X)$. Sections VII - IX are analogous to Section VI in that the method of proof of the theorems within is exactly the same. The interrelations between the essentially different concepts is thus made clear.

In Section X we prove Čech's theorem: A metric space is a G_δ in $\beta(X)$ if and only if it is completely metrizable. The proof is given in its original form with the exception of several clarifying comments. In conclusion these results are discussed and a still further question is formulated.

Introduction

$\beta(X)$ was first introduced and its properties studied by Stone and Čech. Their interests lay chiefly in metric spaces with the main result being Čech's well-known theorem: A metric space X is a G_δ in $\beta(X)$ if and only if it is completely metrizable. Thus a metric is imbedded in its Stone-Čech compactification in the same way in which it is imbedded in any compactification. The question naturally arose as to what could be said about X if its properties were less restricted, for example, if X were only completely regular. Some of the major contributions in this direction were made by Tamano and Michael, part of whose work is studied in this paper. Tamano, especially, has developed some very exact conditions under which a space X will be paracompact, topologically complete, and real compact. We shall study these results and attempt to make evident their correlations. The main point of interest will be the striking similarities in both statement and method of proof of these theorems. As a result one is quite easily able to list these properties of a space in a sort of hierarchy depending on the corresponding equivalent conditions that the space satisfies when imbedded in $\beta(X)$.

The first five sections of this paper contain the preliminary development of some ideas that we shall need later in our discussion and, although some interesting results are proved, their main purpose is to supply us with some machinery for proving more important theorems.

Sections VI-X constitute the main body of the thesis.

The equivalence theorems proved therein represent the focal point of our discussion. These theorems, which are the result of work done by Tamano, have been enlarged to a certain extent to provide a clearer understanding of the motivations behind various constructions and arguments.

With Čech's theorem in Section X we have a further characterization of X in $\beta(X)$ if X is metric. This was one of the first theorems to appear which related X to $\beta(X)$, in that it completely describes metric spaces which are G_δ's in $\beta(X)$.

It is hoped that the ordering and presentation of this material will provide a systematic approach to this type of study and will make clear the general mode of argument used to obtain the desired results.

I

In the following discussion the topological spaces of chief concern will be completely regular spaces so that from the outset it will be assumed that any arbitrary space X is completely regular unless otherwise stated. These are spaces for which the existence of uniformity is known and this property will prove useful in providing certain results.

For any topological space X and subset A denote the interior of A in X by $Int_X A$ and the closure of A in X by $Cl_X A$. X-A designates the complement of A in X. The diagonal of A in X is the set $\Delta_A=\{(x,x)\in X\times X \mid x\in A\}$. A collection of subsets of X is written $\{B_\alpha \mid \alpha\in\Gamma\}$ or $\{B_\alpha\}_{\alpha\in\Gamma}$. When the indexing set is of no particular importance we abbreviate this to $\{B_\alpha\}$.

If X is a topological space then C(X) designates the set of all continuous functions $f:X \to R$, where R is the real numbers. $C^*(X)$ is the set of all bounded continuous functions $f:X \to R$. For an arbitrary function $f:X \to R$ let $O(f)=\{x\in X \mid f(x)\neq 0\}$ and $Z(f)=\{x\in X \mid f(x)=0\}$. O(f) is called the support of f in X and Z(f), the zero set of f in X.

If X is a subspace of Y and U is open in X then an open set U^* of Y is an extension of U provided $U^*\cap X=U$. The largest such extension is called the proper extension of U and is denoted by U^ε. It is easily seen that $U^\varepsilon=Y-Cl_Y(X-U)$.

II

Let X be any arbitrary set. By a uniformity on X we mean a collection μ of subsets of $X\times X$ having the following properties:

i.) If $U \varepsilon \mu$ then $\Delta_X \subset U$.

ii.) If $U \varepsilon \mu$ then $U^{-1} \varepsilon \mu$. (If $U=U^{-1}$ then U is said to be symmetric.)

iii.) If $U \varepsilon \mu$ then there is a $V \varepsilon \mu$ such that $V \circ V \subset U$.

iv.) If $U, V \varepsilon \mu$ then $U \cap V \varepsilon \mu$.

v.) If $U \varepsilon \mu$ and $U \subset V$ then $V \varepsilon \mu$.

The resulting structure (X,μ) is a uniform space. In a uniform space (X,μ) property iii.) can actually be strengthened by saying that for any $U \varepsilon \mu$ there is a symmetric $W \varepsilon \mu$ such that $W \circ W \subset U$. For the V of property iii.) let $W=V \cap V^{-1}$. Then W is symmetric and $W \circ W \subset U$. Moreover, we may argue inductively that for any $U \varepsilon \mu$ and for any integer $n>0$ there is a symmetric $W \varepsilon \mu$ such that $W \circ W \circ \ldots \circ W$ (n-times) $= W^n \subset U$.

From the uniformity μ we may construct a topology τ_μ for the set X in the following manner. Let τ_μ consist of all the sets $O \subset X$ such that for each $x \varepsilon O$ there is a $U \varepsilon \mu$ with $U[x] \subset O$ where $U[x]=\{y \varepsilon X \mid (x,y) \varepsilon U\}$. In view of the definition of a uniformity it is not difficult to verify that τ_μ is indeed a topology for X. τ_μ is called the uniform topology. If (X,τ) is a topological space and μ is a uniformity for X then we say that μ is compatible with X if $\tau_\mu = \tau$.

If there exist uniformities μ, ζ for a set X and $\mu \subset \zeta$ then we say that μ is weaker than ζ and write $\mu \leq \zeta$. If ρ is a pseudometric on X then we may construct a uniformity μ_ρ for X by taking as a base, all subsets of $X \times X$ of the form $\{(x,y) \varepsilon X \times X \mid \rho(x,y) < r\}$ for $r>0$. It is easily proven that μ_ρ is a uniformity for X. We say that a uniform space (X,ζ) is pseudometrizable if and only if there exists a pseudometric ρ on X such that $\mu_\rho = \zeta$. A pseudometric ρ on the uniform space (X,ζ) is said to be uniformly continuous on $X \times X$ if and only if $\mu_\rho \leq \zeta$. If (X,τ) is a topological space and ρ is a pseudometric on X then ρ is said to be a topologically weaker pseudometric if $\tau_\rho \leq \tau$ where τ_ρ is the topology induced by ρ. A basic property of uniformities is given by the following:

<u>Theorem 2.1</u>: Let μ be a uniformity compatible with (X,τ) and $U \varepsilon \mu$. Then there exists a uniformly continuous pseudometric ρ on X such that

$$\ldots W_n \subset W_{n-1} \subset \ldots \subset W_0 \subset U \text{ where}$$
$$W_n = \{(x,y) \varepsilon X \times X \mid \rho(x,y) < 2^{-n}\}.$$

Proof: Let $U_0 = U$. Then there exists a symmetric $U_1 \varepsilon \mu$ such that $U \circ U \circ U \subset U_0$. For $n > 1$ suppose U_n has been defined. Let $U_{n+1} \varepsilon \mu$ be symmetric and have the property that $U^3_{n+1} \subset U_n$. We thus obtain the collection $\{U_n\}$ of symmetric members of μ. Now define a real valued function f on $X \times X$ as follows:

$$f(x,y) = \begin{cases} 3 & \text{if and only if } (x,y) \notin U_0 = U. \\ 2^{n-1} & \text{if and only if } (x,y) \varepsilon U_{n-1} - U_n, n>0 \\ 0 & \text{if and only if } (x,y) \varepsilon U_n,\ n=0,1,\ldots \end{cases}$$

For each $(x,y) \varepsilon X \times X$ set $\rho(x,y) = \min\{1, \inf(\sum_{i=0}^{n} f(x_i, x_{i+1}) \mid i=0,1,.n\}$ where the infimum is taken over all finite sequences $\{x_0, x_1, x_2, \ldots, x_{n+1}\}$ with $x_0 = x$ and $x_{n+1} = y$. From the definition of ρ it is clear that $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for any $x,y,z \varepsilon X$. Also since each U_n is symmetric $f(x,y) = f(y,x)$ for all $x,y \varepsilon X$

and hence $\rho(x,y)=\rho(y,x)$. If $x=y$ then $f(x,y)=0$ and thus $\rho(x,y)=0$. Therefore ρ is a pseudometric on X.

We proceed now by induction to show that

$$f(x_0,x_{n+1}) \leq 2\sum_{i=0}^{n} \{f(x_i,x_{i+1}) \mid i=0,1,\ldots,n\}.$$

This is clearly the case for $n=0$. Assume that the inequality is valid for all $k \leq n-1$. We agree to call $\sum_{i=r}^{s} f(x_i,x_{i+1})$ the length of the chain from r to s+1 and a the length of the entire chain $\sum_{i=0}^{n} f(x_i,x_{i+1})$. Let t be the largest integer such that the length of the chain from 0 to t is at most a/2. The chain from t+1 to n+1 is then also of length at most a/2. Hence by hypothesis $f(x_0,x_t) \leq 2(a/2)=a$ and $f(x_{t+1},x_{n+1}) \leq a$. It is clear that $f(x_t,x_{t+1}) \leq a$. Let m be the smallest integer such that $2^{-m} \leq a$. Then we have $f(x_0,x_t) < 2^{-m+1}$ so that $(x_0,x_t) \notin U_{m-1}-U_m$. Hence $(x_0,x_t) \varepsilon U_m$ and in the same way (x_{t+1},x_{n+1}), $(x_t,x_{t+1}) \varepsilon U_m$. Therefore $(x_0,x_{n+1}) \varepsilon U_{m-1}$ and $f(x_0,x_{n+1}) \leq 2^{-m+1} \leq 2a = 2\{\sum_{i=0}^{n} f(x_i,x_{i+1})\}$.

Now if $\rho(x,y)<1$ then $f(x,y)<2$ and $(x,y) \varepsilon U_0=U$. For $n=0,1,\ldots$ let $W_n=\{(x,y) \varepsilon X\times X \mid \rho(x,y)< 2^{-n}\}$, then we have:

$$\ldots W_n \subset W_{n-1} \subset \ldots \subset W_0 \subset U.$$

We have left to show only that $\mu_\rho \leq \mu$. Let $V \varepsilon \mu_\rho$. Then for some $r>0$ $\{(x,y) \varepsilon X\times X \mid \rho(x,y)<r\}$ V. Choose n so large that $2^{-n}<r$, then $W_n \subset V$. Now $\rho(x,y) \leq f(x,y)$ so that if $(x,y) \varepsilon U_{n+1}$, $f(x,y) \leq 2^{-n-1}$ and hence $\rho(x,y) \leq 2^{-n-1}$. Thus we have $U_{n+1} \subset W_n$. But $U_{n+1} \varepsilon \mu$ so $V \varepsilon \mu$ by the definition of uniformity.

A consequence of the previous theorem is the following:

Theorem 2.2: The uniform space (X,μ) is pseudometrizable if and only if μ has a countable base.

Proof: The necessity of the condition is clear. To prove the sufficiency let $\{U_n \mid n=0,1,\ldots\}$ be a countable base for μ. For each U_n we may apply Theorem 2.2 to obtain a uniformly continuous pseudometric ρ_n which satisfies the descending chain property. Let $\rho(x,y)=\sum_{n=0}^{\infty}\rho_n(x,y)/2^n$. ρ is clearly the desired pseudometric on X.

A uniformity of a very special nature is the fine uniformity μ^* which consists of all sets $U\subset X\times X$ such that there exists some continuous pseudometric ρ on $X\times X$ with $W_0^\rho\subset U$ where $W_0^\rho=\{(x,y)\in X\times X \mid \rho(x,y)<1\}$. μ^* is clearly a uniformity on X. Furthermore it is the largest uniformity compatible with X for by Theorem 2.2, every uniformity μ compatible with the space (X,τ) is contained in μ^* since $\mu_\rho\leq\mu$ implies $\tau_\rho\subset\tau$.

A subset $V\subset X\times X$ is said to be a surrounding for X if and only if V is a member of some uniformity compatible with X. In view of the maximality property of the fine uniformity μ^*, V is a surrounding for X if and only if $V\in\mu^*$. Thus V is a surrounding for X if and only if there exists a continuous pseudometric ρ on $X\times X$ such that $W_0^\rho\subset V$ where

$$W_0^\rho=\{(x,y)\in X\times X \mid \rho(x,y)<1\}.$$

Before introducing the concept of complete uniformity we need some terminology concerning nets. A net $\{S_n, n\in D\}$ is eventually in a subset A of X if there exists an $m\in D$ such that $S_n\in A$ for all $n\geq m$. With this in mind we say that a net $\{S_n, n\in D\}$ in the topological space (X,τ) converges to a point $s\in X$ if

$\{S_n, n \varepsilon D\}$ is eventually in each neighborhood of s. A net $\{S_n, n \varepsilon D\}$ in the uniform space (X,μ) is a Cauchy net if and only if for each $U \varepsilon \mu$ there is an $m \varepsilon D$ such that $(S_k, Sn) \varepsilon U$ for all $k,n \geq m$. Evidently this is equivalent to requiring that the net $\{(S_k, S_n), (k,n) \varepsilon D \times D\}$ is eventually in each member of some base for μ. If (X,μ) is a uniform space then we say that (X,μ) is complete if and only if every Cauchy net in X converges to a point in X.

III

A partition of unity on a space X is a family F of continuous functions $f:X \to R^+$, R^+ the non-negative real numbers, such that $\sum\{f(x) \mid f \varepsilon F\}=1$ for each $x \varepsilon X$ and all but a finite number of members of F vanish on some neighborhood of each point $x \varepsilon X$. This last condition is equivalent to requiring that the family of supports of F is a locally finite covering of X. In connection with locally finite coverings and partitions of unity, normal spaces have a characteristic property. If ζ is a covering of the space X and $\sum\{f \mid f \varepsilon F\}=1$ is a partition of unity on X then we say that F is subordinate to ζ if for each $f \varepsilon F$, $O(f) \subset U$ for some $U \varepsilon \zeta$. Then, for each locally finite covering ζ of a normal space X there is a partition of unity which is subordinate to ζ. Before proving this we require the following lemma:

<u>Lemma 3.1</u>: If ζ is a locally finite open covering of a normal space X then for each $U \varepsilon \zeta$ there is an open set G(U) such that $Cl_X G(U) \subset U$ and $\{G(U) \mid U \varepsilon \zeta\}$ is a covering of X.

Proof: We employ Zorn's Lemma in the following manner. Let F be a set function defined on a subfamily of ζ satisfying the properties:

i.) D_F=domain of F is a subfamily of ζ.

ii.) $F(U)$ is an open set such that $Cl_X\{F(U)\} \subset U$ for each $U \varepsilon D_F$.

iii.) $\bigcup\{F(U) \mid U \varepsilon D_F\} \cup \bigcup\{V \mid V \varepsilon \zeta,\ V \not\varepsilon D_F\} = X$.

Consider the set Ψ of all such set functions F. Ψ is a non-empty collection; for choose any $V \varepsilon \Psi$ and note that the set $A = X - \bigcup\{U \mid U \varepsilon \Psi,\ U \neq V\}$ is a closed set and $A \subset V$. If $A = \Phi$ then let F be the set function defined only on V such that $F(V) = \Phi$. Clearly F satisfies the above three properties. If $A \neq \Phi$ then there is a continuous function g such that $g=1$ on $X-V$ and $g=0$ on A. (The existence of g follows from the normality of X and Urysohn's Lemma.) Let $F(V) = \{x \varepsilon V \mid g(x) < 1/2\}$. Then $Cl_X\{F(V)\} \subset V$, $D_F \subset \zeta$, and property iii.) is also satisfied since $A \subset F(V)$.

The collection Ψ is partially ordered by the obvious relation: $F \leq G$ if and only if $D_F \subset D_G$ and $G|D_F = F$. Let Ω be a linearly ordered subset of Ψ. Define a new set function F^+ in the following way. Let $D_{F^+} = \{D_F \mid F \varepsilon \Omega\}$ and for $U \varepsilon D_{F^+}$ define $F^+(U) = \bigcup\{F(U) \mid F \varepsilon \Omega\}$. It is clear that F^+ satisfies properties i.) and iii.) above. In addition $F^+(U)$ is open and since is a locally finite collection $Cl_X\{F^+(U)\} = Cl_X \bigcup\{F(U) \mid F \varepsilon \Omega\} = \bigcup\{Cl_X F(U) \mid F \varepsilon \Omega\} \subset U$. Hence $F \leq F^+$ for each $F \varepsilon \Omega$. Applying Zorn's Lemma we denote the maximal element of Ψ by G, and we further assert that $D_G = \zeta$. Assume by way of contradiction that $\zeta - D_G \neq \Phi$ and let $V \varepsilon \zeta - D_G$. Let H be the set function whose domain

$D_H=D_G \cup \{V\}$, $H(U)=G(U)$ for $U \varepsilon D_G$ and define $H(V)$ in this way: Let $A=X-\{\bigcup\{G(U) \mid U\varepsilon D_G\} \cup \bigcup\{W \mid W\varepsilon\zeta,\ W\not\in D_G,\ W\neq V\}\}$. If $A=\Phi$ let $H(V)=\Phi$. We then have $H\geq G$ which is a contradiction. Now suppose $A\neq\Phi$. Then A is a closed set and $A\subset V$. Let h be the function which is 0 on A and 1 on X-V. The existence of h is again guaranteed by Urysohn's Lemma. Now we set $H(V)=\{x\varepsilon X \mid h(x)<1/2\}$ and note that $A\subset H(V)$. We also have $Cl_X\{H(V)\}\subset V$ so that only property iii) remains to be verified. To prove this we observe that:

$$\bigcup\{H(U) \mid U\varepsilon D_H\} \cup \bigcup\{W\varepsilon\zeta \mid W\not\in D_H\}=H(V) \cup \bigcup\{G(U) \mid U\varepsilon D_G\}\cup \bigcup\{W\varepsilon\zeta \mid W\not\in D_G,\ W\neq V\} \supset A \cup (X-A)=X.$$

Hence we have $H\varepsilon\Psi$ and $H\geq G$. But G is maximal, and therefore we must have $\zeta-D_G=\Phi$ and $\bigcup\{G(U) \mid U\varepsilon\zeta\}=X$.

Furnished with this lemma we are now prepared to prove:

Theorem 3.2: If ζ is a locally finite open covering of the normal space X then there is a partition of unity subordinate to ζ.

Proof: Applying Lemma 3.1 choose for each $U\varepsilon\zeta$ an open set $G(U)\subset U$ such that $Cl_X\{G(U)\}\subset U$ and $\{G(U) \mid U\varepsilon\zeta\}$ is a covering of X. For each $U\varepsilon\zeta$ there is a continuous function f_U on X such that $f_U=1$ on $Cl_X\{G(U)\}$ and $f_U=0$ on X-U. Furthermore, for each $x\varepsilon X$ there exists a neighborhood $V(x)$ such that for each $y\varepsilon V(x)$, $f_U(y)=0$ for all but a finite number of $U\varepsilon\zeta$. Hence $\sum\{f_U(x) \mid U\varepsilon\zeta\}$ is bounded for each $x\varepsilon X$. For $U\varepsilon\zeta$ let $g_U=\dfrac{f_U}{\sum\{f_U \mid U\varepsilon\zeta\}}$. Since f_U is continuous g_U is clearly continuous for each $U \epsilon \zeta$ and furthermore $\sum\{g_U(x) \mid U\epsilon\zeta\}=1$ for each $x\epsilon X$. $\{f_U \mid U\epsilon\zeta\}$ has the property that all but a finite number vanish on some neighborhood of each point of X and so g_U also has this same property. Thus $\{g_U \mid U\epsilon\zeta\}$ is the desired partition of unity.

From the proof of this theorem we observe that the partition of unity can actually be chosen so that the cardinality of its indexing set is the same as the cardinality of the covering ζ.

IV

If μ is an open covering of the space X then we say that a collection ζ of subsets of X is a locally finite refinement of μ if:

i.) For each $V\epsilon\zeta$ there is a $U\epsilon\mu$ such that $V\subset U$.

ii.) For each $x\epsilon X$ there is a neighborhood $O(x)$ such that $O(x)$ meets at most a finite number of $V\epsilon\zeta$.

iii.) $\bigcup\{V \mid V\epsilon\zeta\}=X$.

A space X is said to be paracompact if every open covering of X has an open locally finite refinement. Note that we do not require X to be Hausdorff or regular but the following result remains valid:

<u>Theorem 4.1</u>: Every pseudometrizable space X is paracompact.

Before proceeding to prove this theorem we need some preliminary definitions. If (X,ρ) is a pseudometric space and A,B are subsets of X then $\rho(A,B)=\inf\{\rho(x,y) \mid x\epsilon A,\ y\epsilon B\}$

is called the distance between the sets A and B. If one of the sets, for example A, consists of a single point $\{x\}$ then $\rho(A,B)=\rho(x,B)$ is called the distance between the point x and the set B. Note that if $x\varepsilon B$ then $\rho(x,B)=0$ and if B is closed and $x\not\varepsilon B$ then $\rho(x,B)>0$. For any set $E\subset X$ let $B_n(E)=\{x\varepsilon X|\ \rho(x,E)<1/2^n\}$. $B_n(E)$ is an open set and $E\subset B_n(E)$. If $E=\{x\}$ then $B_n(E)=B_n(x)$ is just the ball of radius $1/2^n$ about x. If we define $C_n(E)=\{x\varepsilon X|\ B_n(x)\subset E\}$ then it is easy to see that $C_n(E)=X-B_n(X-E)$. $C_n(E)$ is a closed set and $C_n(E)\subset E$. We also have $B_n\{C_n(E)\}\subset E$ and if $m>n$, $Cl_X\{B_m(E)\}\subset B_n(E)$. Let $x\varepsilon C_n(E)$ and $y\varepsilon X-E$. Then $\rho(x,y)>1/2^n$ from the definition of $C_n(E)$ and hence $\rho(C_n(E),X-E)\geq 1/2^n$. We now prove Theorem 4.1 in the following slightly altered form:

Theorem 4.2: For every open covering of a pseudometrizable space (X,τ) there is an open locally finite covering which refines it.

Proof: Let ρ be a pseudometric on X which gives the topology τ. Suppose $\{G_\alpha\}_{\alpha\varepsilon A}$ is any open covering of X and assume that A has been well-ordered. For each integer n define

$E^n_\alpha=C_n(G_\alpha-\bigcup_{\beta<\alpha}E^n_\beta)$ for each $\alpha\varepsilon A$. For each point $x\varepsilon X$ let $\xi=\min\{\alpha|\ x\varepsilon G_\alpha\}$. Since G_ξ is open there is an integer n such that $B_n(x)\subset G_\xi$. If $x\not\varepsilon E^n_\xi$, then $B_n(x)\not\subset G_\xi-\bigcup_{\beta<\xi}E^n_\beta$. Since $B_n(x)\subset G_\xi$, we have $B_n(x)\cap\bigcup_{\beta<\xi}E^n_\beta\neq\Phi$. Thus $B_n(x)\cap E^n_\alpha\neq\Phi$ for some $\alpha<\xi$ and we have for this α,

$$x\varepsilon B_n(E^n_\alpha)=B_n\{C_n(G_\alpha-\bigcup_{\beta<\alpha}E^n_\beta)\}\subset G_\alpha-\bigcup_{\beta<\alpha}E^n_\beta\subset G_\alpha.$$

But this is a contradiction since $\xi=\min\{\alpha|\ x\varepsilon G_\alpha\}$. Hence we must have $x \varepsilon E_{\xi}^{n}$ and $\{E_{\alpha}^{n} | \alpha \varepsilon A, n \varepsilon N^{+}\}$ is a covering of X.

For each $n \varepsilon N^{+}$ and $\alpha \varepsilon A$ we define $F_{\alpha}^{n} = Cl_X\{B_{n+3}(E_{\alpha}^{n})\}$ and $G_{\alpha}^{n} = B_{n+2}(E_{\alpha}^{n})$. It is clear that $F_{\alpha}^{n} \subset G_{\alpha}^{n}$. We wish to show first of all that $\rho(F_{\alpha}^{n}, F_{\beta}^{n}) \geq 1/2^{n+1}$ whenever $\alpha \neq \beta$. To do this it is sufficient to show that $\rho(E^{n}, E^{n}) \geq 1/2^{n}$ whenever $\alpha \neq \beta$. Now $B_n(E_{\beta}^{n}) = B_n\{C_n(G_{\beta} - \bigcup_{\alpha<\beta} E_{\alpha}^{n})\} \subset G_{\beta} - \bigcup_{\alpha \leq \beta} E_{\alpha}^{n} \subset X - E_{\alpha}^{n}$ for $\alpha < \beta$. Hence

$B_n(E_{\beta}^{n}) \cap E_{\alpha}^{n} = \emptyset$ for all β and for all $\alpha < \beta$. Thus we have

$\rho(E_{\alpha}^{n}, E_{\beta}^{n}) \geq 1/2^{n}$ whenever $\alpha \neq \beta$. Therefore $\rho(F^{n}, F^{n}) \geq 1/2^{n+1}$ and if we set $F^{n} = \bigcup_{\alpha} F^{n}$ then F^{n} is a closed set for each n.

We are now prepared to construct the refinement of $\{G_{\alpha} | \alpha \varepsilon A\}$. Let $V_{\alpha}^{n} = G_{\alpha}^{n} - \bigcup_{k<n} F^{k}$ for each $n \varepsilon N^{+}$ and for each $\alpha \varepsilon A$. V_{α}^{n} is an open set for each pair n, α.

Let $x \varepsilon X$. Then since $\{F_{\alpha}^{n} | n \varepsilon N^{+}, \alpha \varepsilon A\}$ is a covering of X there exists a least integer n such that $x \varepsilon F_{\alpha}^{n}$ for some $\alpha \varepsilon A$. We have

$$x \varepsilon F_{\alpha}^{n} - \bigcup_{k<n} \bigcup_{\beta} F_{\beta}^{k} = F_{\alpha}^{n} - \bigcup_{k<n} F^{k} \subset G_{\alpha}^{n} - \bigcup_{k<n} F^{k} = V_{\alpha}^{n}.$$

Hence $\{V_{\alpha}^{n} | n \varepsilon N^{+}, \alpha \varepsilon A\}$ is an open covering of X. It is clear from the construction of $\{V_{\alpha}^{n} | \alpha \varepsilon A, n \varepsilon N^{+}\}$ that this collection is a refinement of $\{G_{\alpha} | \alpha \varepsilon A\}$. Thus we have only to show that the collection $\{V_{\alpha}^{n} | n \varepsilon N^{+}, \alpha \varepsilon A\}$ is locally finite.

Let $x \varepsilon X$ and suppose $x \varepsilon E^{n}$. Then $B_{n+3}(x) \subset B_{n+3}(E_{\alpha}^{n}) \subset Cl_X\{B_{n+3}(E_{\alpha}^{n})\} = F_{\alpha}^{n} \subset F^{n}$. Thus $B_{n+3}(x) \cap V_{\alpha}^{k} = \Phi$ for all $k > n$ and for all $\alpha \varepsilon A$. Since $\rho(F_{\alpha}^{n}, F_{\beta}^{n}) \geq 1/2^{n+1}$, for a fixed $k \leq n$, $B_{n+3}(x)$ can intersect at most one G_{α}^{k} and consequently can intersect at most one V_{α}^{k}. We now have a neighborhood of x which intersects at most a finite number of V_{α}^{k}'s. Hence $\{V_{\alpha}^{n} | n \varepsilon N^{+}, \alpha \varepsilon A\}$ is locally finite and we are finished.

Now since every open covering of a pseudometrizable space X has an open locally finite refinement we have shown that X is paracompact.

V

A compactification of a space X is a pair (Y,f) where $f:X \to Y$ is an imbedding, Y is compact and $Cl_Y\{f(X)\}=Y$. By an imbedding $f:X \to Y$ we shall mean a continuous function such that if f(X) is given the relative topology of Y then $f:X \to f(X)$ is a homeomorphism. We are interested only in that largest compactification of a Tychonoff space X, the Stone-Čech compactification. Let Q=[0,1] and let F(X) be the collection of all continuous functions $f:X \to Q$. If we let A=cardinality of F(X) then $\pi(f_\alpha):X \to Q^A$ is a continuous mapping where π is the usual evaluation mapping. We note that Q^A is compact, Hausdorff and regular. The Stone-Čech compactification is $(\beta(X),\pi f_\alpha)$ where $\beta(X)=Cl(\pi f_\alpha[X])$. $\beta(X)$ is evidently compact, Hausdorff and, since regularity is hereditary, $\beta(X)$ is regular. We now state without proof a theorem that enables us to say that among all of the compactifications of X $\beta(X)$ is the largest in the following sense:

Theorem 5.1: If X is a Tychonoff space and $f:X \to Y$ is a continuous function where Y is a compact Hausdorff space, then there is a continuous function $f^*:\beta(X) \to Y$ such that $f^*|X=f$.

In particular if (Y,h) is any other compactification of X then there is a continuous function $h^*:\beta(X) \to Y$ such that $h^*|X=h$; that is, $\beta(X)$ is the largest compactification of X.

VI

We now begin the main body of our discussion with some results relating the paracompactness of a Tychonoff space X to $\beta(X)$.

<u>Theorem 6.1</u>: The topological space X is paracompact if and only if for each compact set $F \subset \beta(X)-X$ there exists a locally finite open covering $\{U_\lambda\}$ of X such that $Cl_{\beta(X)}U_\lambda \cap F=\Phi$ for each λ.

Proof: Suppose first of all that X is paracompact and let $F \subset \beta(X)-X$ be any compact set. Since $\beta(X)$ is regular there is for each $x \varepsilon X$ a neighborhood U_x^*, open in $\beta(X)$, such that $x \varepsilon U_x^*$ and $Cl_{\beta(X)}U_x^* \cap F=\Phi$. For $x \varepsilon X$ let $U_x=U_x^* \cap X$. $\{U_x\}$ is now an open covering of X and since X is paracompact there is an open locally finite refinement $\{U_\lambda\}$ of $\{U_x\}$. For each λ there is an $x \varepsilon X$ such that $Cl_{\beta(X)}U_\lambda \subset Cl_{\beta(X)}U_x \subset Cl_{\beta(X)}U_x^*$. Hence $Cl_{\beta(X)}U_\lambda \cap F=\Phi$ for each λ.

To prove the sufficiency let $\{U_\alpha\}$ be any open covering of X. For each α fix one open set U_α^* in $\beta(X)$ such that $U_\alpha^* \cap X=U_\alpha$. Let $F_\alpha=\beta(X)-U_\alpha^*$ for each α and let $F=\bigcap_\alpha F$. F is closed in $\beta(X)$ for each α and hence F is closed and compact; furthermore, $F \subset \beta(X)-X$. By hypothesis there is a locally finite open covering $\{O_\lambda\}$ of X such that $Cl_{\beta(X)}O_\lambda \cap F=\Phi$ for each λ. Therefore $Cl_{\beta(X)}O_\lambda \subset \bigcup_\alpha U_\alpha^*$ for each λ and, since

$Cl_{\beta(X)}O_\lambda$ is compact, there is a finite subcollection $\{U_k\}_1^m$ such that $Cl_{\beta(X)}O_\lambda \subset \bigcup_1^m U_k^*$ for each λ, hence we also have $O_\lambda \subset \bigcup_1^m U_k^*$. Since $U_k^* \cap X = U_k$ we have $O_\lambda \subset \bigcup_1^m U_k$ for each λ. Now let $H_{\lambda,k} = U_k \cap O_\lambda$, $k=1,\ldots,m$, for each λ. Then $O_\lambda = \bigcup_1^m H_{\lambda,k}$ and $H_{\lambda,k}$ is open for each λ and for each k. Furthermore $X = \bigcup_{\lambda,k} H_{\lambda,k}$ and by the construction $\{H_{\lambda,k}\}$ is a locally finite refinement of $\{U_\alpha\}$.

If for any set V XxX we define $\tilde{V} = Int_{\beta(X)x\beta(X)} Cl_{\beta(X)x\beta(X)} V$ we are led to the following:

<u>Theorem 6.2</u>: Let $F \subset \beta(X)-X$ be any compact set. If there is a surrounding V of X such that $\tilde{V} \cap \Delta_F = \Phi$, then there exists a locally finite open covering $\{U_\lambda\}$ of X such that $Cl_{\beta(X)} U_\lambda \cap F = \Phi$.

Proof: Since V is a surrounding for X there exists a (topologically) weaker pseudometric d on X such that

$$\ldots W_n \subset W_{n-1} \subset \ldots \subset W_0 \subset V \text{ where}$$
$$W_n = \{(x,y) \varepsilon XxX \mid d(x,y) < 1/2^n\}.$$

Let μ be the uniformity generated by d. Then τ_μ is the pseudometric topology on X generated by d, so (X,τ_μ) is paracompact. Consider the covering of X by $\{W_3[x] \mid x \varepsilon X\}$. There is a locally finite open covering $\{U_\lambda\}$ which refines $\{W_3[x]\}$ in (X,τ_μ) and $Cl_{\beta(X)} U_\lambda \cap F = \Phi$ for each λ. For suppose by way of contradiction that there is an element $p \varepsilon Cl_{\beta(X)} U_\lambda \cap F$ for some λ. Then $U_\lambda \subset W_3[x_0]$ for some $x_0 \varepsilon X$ since $\{U_\lambda\}$ refines $\{W_3[x]\}$. Hence p is an accumulation point of $W_3[x_0]$. Let

$d^*_{x_0} = d|x_0 x X$. $d^*_{x_0}(p) \leq 1/2^3 < 1/2^2$ and there is a neighborhood O^* of p open in $\beta(X)$ such that $d_{x_0}(y) < 1/2^2$ for each $y \in O = O^* \cap X$. Hence $O x O \subset W_1 = \{(x,y) \in X x X \mid d(x,y) < 1/2\}$. But $O \tilde{x} O \cap \Delta_F \subset \tilde{W}_1 \cap \Delta_F \subset \tilde{V} \cap \Delta_F = \Phi$ and this clearly cannot happen. Therefore we must have $Cl_{\beta(X)} U_\lambda \cap F = \Phi$ for each λ and $\{U_\lambda\}$ is the desired covering.

Corollary: Let $F \subset \beta(X) - X$ be any compact set. If there exists a surrounding V of X such that $\tilde{V} \cap \Delta_F = \Phi$ then X is paracompact.

Proof: If such a surrounding, V, exists then we may find a locally finite open cover $\{U_\lambda\}$ of X such that $Cl_{\beta(X)} U_\lambda \cap F = \Phi$ for each λ. X is paracompact by Theorem 6.1.

The following result supplies us with several conditions which are equivalent to paracompactness in Tychonoff spaces. In a normal space X we have shown that for any locally finite covering $\{U_\alpha\}$ there is a partition of unity $\Phi = \{\phi_\lambda \mid \sum_\lambda \phi_\lambda = 1\}$ subordinate to the covering $\{U_\alpha\}$.

Theorem 6.3: In a Tychonoff space X the following are equivalent:

i.) X is paracompact.

ii.) For each compact set $F \subset \beta(X) - X$, there is a locally finite covering $\{U_\lambda\}$ of X such that $Cl_{\beta(X)} U_\lambda \cap F = \Phi$ for each λ.

iii.) For each compact set $F \subset \beta(X) - X$, there is a partition of unity $\Phi = \{\phi_\lambda \mid \sum_\lambda \phi_\lambda = 1\}$ such that $Cl_{\beta(X)} \{O(\phi_\lambda)\} \cap F = \Phi$ for each λ.

iv.) For each compact set $F \subset \beta(X)-X$ there is a surrounding V such that $\tilde{V} \cap \Delta_F = \Phi$.

v.) $X \times \beta(X)$ is normal.

vi.) If $G = X \times C$ is a closed subset of $X \times \beta(X)$ such that $G \cap \Delta_X = \Phi$ then G and Δ_X are separated by some member of $C^*(X \times \beta(X))$.

Proof: i.) implies ii.). This is the necessary condition of Theorem 6.1.

ii.) implies iii.). Let $F \subset \beta(X)-X$ be any compact set and let $\{U_\alpha\}$ be a locally finite covering of X such that $Cl_{\beta(X)} U_\alpha \cap F = \Phi$ for each α. Since $\{U_\alpha\}$ is a locally finite covering there is a partition of unity $\Phi = \{\phi_\lambda \mid \sum_\lambda \phi_\lambda = 1\}$ on X subordinate to $\{U_\alpha\}$. Hence for each λ, $Cl_{\beta(X)}\{O(\phi_\lambda)\} \cap F \subset Cl_{\beta(X)} U_\alpha \cap F = \emptyset$ for some α.

iii.) implies iv.). Let $F \subset \beta(X)-X$ be any compact set and suppose that $\Phi = \{\phi_\lambda \mid \sum_\lambda \phi_\lambda = 1\}$ is a partition of unity on X such that $Cl_{\beta(X)}\{O(\phi_\lambda)\} \cap F = \Phi$ for each λ. Let $d(x,y) = \sum_\lambda |\phi_\lambda(x) - \phi_\lambda(y)|$ and set $V_n = \{(x,y) \varepsilon X \times X \mid d(x,y) < 1/2^n\}$. Then V_1 is a surrounding for X and we need only show that $\tilde{V}_1 \cap \Delta_F = \Phi$. Suppose on the contrary that $(p,p) \varepsilon \tilde{V}_1 \cap \Delta_F$. Then since V_1 is open in $\beta(X) \times \beta(X)$ there is a neighborhood U^* of p open in $\beta(X)$ such that $U^* \times U^* \subset \tilde{V}_1$. Choose $x \varepsilon U = U^* \cap X$. At most finitely many ϕ_λ, say $\phi_{\lambda_1}, \phi_{\lambda_2}, \ldots, \phi_{\lambda_n}$, do not vanish at x. For $k=1,2,\ldots,n$ let $H_k = \{y \varepsilon X \mid \phi_{\lambda_k}(y) > 0\}$. If $y \notin \bigcup_1^n H_k$ then we have the following:

$d(x,y) = \sum_\lambda |\phi_\lambda(x) - \phi_\lambda(y)| = \sum_1^n \phi_{\lambda_k}(x) + \sum_\lambda \phi_\lambda(y) > 1$ and $y \notin U$. Hence

$p\varepsilon Cl_{\beta(X)}\{\bigcup_1^n H_k\}=\bigcup_1^n Cl_{\beta(X)}\{O(\phi_{\lambda_k})\}$ but $p\notin Cl_{\beta(X)}\{O(\phi_\lambda)\}$ for any λ since $Cl_{\beta(X)}\{O(\phi_\lambda)\}\cap F=\Phi$ and we have a contradiction. Therefore we must have $\tilde{V}_1\cap\Delta_F=\Phi$ and V_1 is the desired surrounding.

iv.) implies i.). This is the content of the Corollary to Theorem 6.2.

i.) implies v.). If X is paracompact then $X\times\beta(X)$ is paracompact hence normal.

v.) implies vi.). Suppose $X\times\beta(X)$ is normal and let $G=X\times C$ be a closed subset of $X\times\beta(X)$ such that $G\cap\Delta_X=\Phi$. Since G and Δ_X are two closed disjoint subsets of $X\times\beta(X)$ by Uryssohn's Lemma there is an $f\varepsilon C^*(X\times\beta(X))$ such that $f(G)=0$ and $f(\Delta_X)=1$.

vi.) implies iii.). Let C be any compact set in $\beta(X)-X$ and let $F\varepsilon C^*(X\times\beta(X))$ be such that $F(X\times C)=1$ and $F(\Delta_X)=0$. Let F_x be the restriction of F to $\{x\}\times\beta(X)$ and put

$$d(x,y)=\|F_x(p)-F_y(p)\|=\sup_{p\varepsilon\beta(X)}|F_x(p)-F_y(p)|.$$ d is clearly a pseudometric on X.

Let τ be the induced topology of d. The topological space (X,τ) is paracompact. Let $U_x=\{y\varepsilon X|\ d(x,y)<1/2\}$ and consider the covering $\{U_x|\ x\varepsilon X\}$. Let $\{O_\lambda\}$ be a locally finite refinement of $\{U_x|\ x\varepsilon X\}$ and let $\Phi=\{\phi_\lambda|\ \sum_\lambda\phi_\lambda=1\}$ be a partition of unity on X subordinate to $\{O_\lambda\}$. If $d(x,y)<1/2$ then $|F_x(y)|=|F_x(y)-F_y(y)|<1/2$ and therefore $F_x(p)\leq 1/2$ for each $p\varepsilon Cl_{\beta(X)}U_x$. But $F_x(p)=F(x,p)=1$ for each $p\varepsilon C$ and hence $Cl_{\beta(X)}U_x\cap C=\Phi$ for each $x\varepsilon X$. Thus $Cl_{\beta(X)}\{O(\phi_\lambda)\}\cap C=\Phi$ for each

λ since Φ is subordinate to $\{O_\lambda\}$ and this covering is in turn a refinement of $\{U_x\}$.

VII

A topological space X is said to be topologically complete if there is a uniformity for X relative to which X is complete. If $\{V_\alpha\}$ is a uniformity for X in which X is complete then this is equivalent to $\Delta_X = \bigcap_\alpha \tilde{V}_\alpha$. Hence the uniform space $\{X, \{V_\alpha\}\}$ is complete if and only if $\Delta_X = \bigcap_\alpha V_\alpha^{\varepsilon(\beta(X) x \beta(X))}$.

<u>Theorem 7.1</u>: For a Tychonoff space X the following are equivalent:

i.) X is topologically complete.

ii.) For each point $p \varepsilon \beta(X) - X$ there is a locally finite covering $\{U_\lambda\}$ such that $p \not\varepsilon Cl_{\beta(X)} U_\lambda$ for each λ.

iii.) For each point $p \varepsilon \beta(X) - X$ there is a partition of unity $\Phi = \{\phi_\lambda \mid \sum_\lambda \phi_\lambda = 1\}$ such that $p \not\varepsilon Cl_{\beta(X)} \{O(\phi_\lambda)\}$ for each λ.

iv.) For each point $p \varepsilon \beta(X) - X$ there is a surrounding V such that $(p,p) \not\varepsilon \tilde{V}$.

v.) For each point $p \varepsilon \beta(X)$ if $(Xxp) \cap \Delta_X = \Phi$ then Xxp and Δ_X are separated by some member of $C(Xx\beta(X))$.

Proof: Let $\{V_\alpha\}$ be a uniformity on X relative to which X is complete; that is, $\Delta_X = \bigcap_\alpha V_\alpha^{\varepsilon(\beta(X) x \beta(X))}$. We wish to show first that i.) implies ii.). For each $p \varepsilon \beta(X) - X$ there is a V_α such that $(p,p) \not\varepsilon V_\alpha^{\varepsilon(\beta(X) x \beta(X))}$. Let d be a pseudometric on X such that $d(x,y) = 1$ whenever $(x,y) \not\varepsilon V_\alpha$ and let τ denote the induced topology. Then (X,τ) is paracompact. Consider the open covering of X by $\{U_x | x\varepsilon X\}$ where $U_x=\{y\varepsilon X| d(x,y)<1/2^2\}$ and let $\{O_\lambda\}$ be a locally finite open refinement of $\{U_x\}$. $p\notin Cl_{\beta(X)}U_x$ for each $x\varepsilon X$. For suppose $p\varepsilon Cl_{\beta(X)}U_x$ for some $x\varepsilon X$. Then for any neighborhood U^* of p open in $\beta(X)$, $d(x,y)<1/2^2$ for some $y\varepsilon U^* \cap X$. Let $d_x(y)=d(x,y)$ and let d_x^* be the extension of $d_x^*(y)$ over $\beta(X)$. Then $d_x^*(p)\leq 1/2^2<1/2$, hence there is a neighborhood W^* of p open in $\beta(X)$ such that

$$[W^* x W^*] \cap (XxX)=[W^* \cap X] \ x \ ]W^* \cap X] \subset V_\alpha \text{ for each } \alpha.$$

Thus $(p,p)\varepsilon V_\alpha^{\varepsilon(\beta(X)x\beta(X))}$ but this is a contradiction. Therefore we must have $p\notin Cl_{\beta(X)}O_\lambda$ for each λ.

ii.) implies iii.). Suppose $\{U_\lambda\}$ is a locally finite covering of X with the property that $p\notin Cl_{\beta(X)}U_\lambda$ for each λ and for any $p\varepsilon\beta(X)-X$. Then since X is normal we may find a partition of unity $\Phi=\{\phi_\lambda | \sum_\lambda \phi_\lambda=1\}$ which is subordinate to $\{U_\lambda\}$. That $p\notin Cl_{\beta(X)}\{O(\phi_\lambda)\}$ for each λ and for any $p\varepsilon\beta(X)-X$ follows immediately.

iii.) implies iv.). Let $p\varepsilon\beta(X)-X$ and suppose $\Phi=\{\phi_\lambda | \sum\phi_\lambda=1\}$ is a partition of unity on X such that $p\notin Cl_{\beta(X)}\{O(\phi_\lambda)\}$ for each λ. Let d be defined by $d(x,y)=\sum_\lambda|\phi_\lambda(x)-\phi_\lambda(y)|$ for $(x,y)\varepsilon XxX$ and let $V_n=\{(x,y)\varepsilon XxX| d(x,y)<1/2^n\}$. It is clear that V_1 is a surrounding. Furthermore, $(p,p)\notin\tilde{V}_1$.

iv.) implies i.). For each point $p\varepsilon\beta(X)-X$ there is a surrounding V such that $(p,p)\notin\tilde{V}$. Let $\{V_\alpha\}$ be the collection of all such surroundings. $\{X,\{V_\alpha\}\}$ is a uniform space and $\bigcap_\alpha \tilde{V}_\alpha=\Delta_X$.

Hence X is topologically complete.

iii.) implies v.). For each point $p\varepsilon\beta(X)-X$ assume there is a partition of unity $\Phi=\{\phi_\lambda \mid \sum_\lambda \phi_\lambda=1\}$ such that $p\notin Cl_{\beta(X)}\{O(\phi_\lambda)\}$ for each λ. Let $d(x,y)=\sum_\lambda |\phi_\lambda(x)-\phi_\lambda(y)|$ for $(x,y)\varepsilon X\times X$ and let d^* be the continuous extension of d over $X\times\beta(X)$. Let $p\varepsilon\beta(X)-X$ then $(X\times p)\cap\Delta_X=\Phi$. $d^*=1$ on $X\times p$ and $d^*=0$ on Δ_X, $d^*\varepsilon C(X\times\beta(X))$.

v.) implies iii.). If $p\varepsilon\beta(X)$ and $(X\times p)\cap\Delta_X=\Phi$ then $p\varepsilon\beta(X)-X$. Let $F\varepsilon C(X\times\beta(X))$ be such that F=1 on $X\times p$ and F=0 on Δ_X. Let $F_x=F|\ \{x\}\times\beta(X)$ and set $d(x,y)=\|F_x(p)-F_y(p)\|=$

$$=\sup_{p\varepsilon\beta(X)} |F_x(p)-F_y(p)|.$$

d is a pseudometric on X. Let τ_d be the topology induced by d. Then (X,τ_d) is paracompact. Consider a covering of X by $\{U_x \mid x\varepsilon X\}$ where $U_x=\{y\varepsilon X \mid d(x,y)<1/2\}$ and let $\{O_\alpha\}$ be a locally finite open refinement of $\{U_x \mid x\varepsilon X\}$. There is a partition of unity $\Phi=\{\phi_\lambda \mid \sum_\lambda\phi_\lambda=1\}$ on X subordinate to $\{O_\alpha\}$ and it is clear that $p\notin Cl_{\beta(X)}\{O(\phi_\lambda)\}$ for each λ.

VIII

We shall say that a space X is real compact if it is complete relative to the weakest uniformity for X with respect to which every continuous function on X is uniformly continuous. It is not difficult to show that this uniformity is generated by sets of the form $V_f=\{(x,y)\varepsilon X\times X \mid |f(x)-f(y)|<\}$ for $f\varepsilon C(X)$. (Cf. [7]). Hence X is real compact if and only if $\Delta_X=\bigcap_f V_f^{\varepsilon(\beta(X)\times\beta(X))}$, and we have the following theorem:

<u>Theorem 8.1</u>: Let X be a Tychonoff space. Then the following

are equivalent:

i.) X is real compact.

ii.) For each point $p\varepsilon\beta(X)-X$ there is a closed G_δ set, C, of $\beta(X)$ such that $p\varepsilon C\subset\beta(X)-X$.

iii.) For each point $p\varepsilon\beta(X)-X$ there is a countable star-finite partition of unity $\Phi=\{\phi_n | \sum_n \phi=1\}$ on X such that $p\notin Cl_{\beta(X)}O(\phi_n)$ for each n.

Proof: Let R^* be the one-point compactification of R, the real numbers. Then for any $f\varepsilon C(X)$ there is an $f^*\varepsilon C(\beta(X))$ such that $f^*|X=f$. Let $X_f=\{p\varepsilon\beta(X) | f^*(p)\varepsilon R\}$ and let the complementary set $C_f=\beta(X)-X_f$. For each $f\varepsilon C(X)$ let

$$V_f=\{(x,y)\varepsilon X\times X | \ |f(x)-f(y)|<1\}.$$

We wish to show that $\Delta_X=\bigcap_f V_f^{\varepsilon(\beta(X)\times\beta(X))}$. If $x\varepsilon X_f$ then there is a neighborhood U^* of x open in $\beta(X)$ such that $|f(x)-f(y)|<1$ for each $y\varepsilon U^*$. Hence $(x,x)\varepsilon V_f^\varepsilon$ for each $x\varepsilon X_f$. Conversely, if $p\notin X_f$ then for every neighborhood O^* of p open in $\beta(X)$, there are points $x,y\varepsilon O^*$ such that $|f(x)-f(y)|>1$ and hence $(p,p)\varepsilon Cl_{\beta(X)\times\beta(X)}(X\times X-V_f)$; that is, $(p,p)\notin V_f^\varepsilon$.

We have $\Delta_{X_f}=\Delta_{\beta(X)}\cap V_f^\varepsilon$. But $\cap V_f^\varepsilon\subset\Delta_{\beta(X)}$ and therefore X is real compact if and only if $X=\bigcap_f X_f$. C_f is a closed G_δ set of $\beta(X)$, $C_f\subset\beta(X)-X$ and every closed G_δ set in $\beta(X)-X$ is a C_f for some $f\varepsilon C(X)$. Hence we have proved that i.) and ii.) are equivalent.

ii.) implies iii.). If $f\varepsilon C(X)$ is such that $Z(f^*)=C\subset\beta(X)-X$

then let

$$h_n(x)=\begin{cases} n+1/n(x-1/n) + 1/n^2 & \text{if } 1/n+1\le x\le 1/n \\ -(n-1)/n(x-1/n) + 1/n^2 & \text{if } 1/n\le x\le 1/n-1 \\ 0 & \text{otherwise} \end{cases}$$

h_n is a continuous function on R, the real numbers. Let $g_n=h_n\circ f$; then g_n is continuous. Now set $\phi_n=\frac{g_n}{\sum_n g_n}$. Note that $\sum_n g_n\le\sum_n 1/n^2<\infty$. By this construction $\Phi=\{\phi_n \mid \sum_n \phi_n=1\}$ is a countable star-finite partition of unity and clearly $C\cap O(\phi_n)=\Phi$ for each n.

iii.) implies ii.). If $\Phi=\{\phi_n \mid \sum_n \phi_n=1\}$ is a countable star-finite partition of unity on X such that $p\notin Cl_{\beta(X)}\{O(\phi_n)\}$ for each n and for $p\in\beta(X)-X$ then $f=\sum_n \phi_n/2^n$ is a continuous function on X and $p\in Z(f^*)\subset\beta(X)-X$.

IX

It is known that in a regular space X the property of being Lindelöf is equivalent to the following: Every open covering of X has has a countable star-finite open refinement. (Cf. [4]) We will use this result in proving the following theorem:

Theorem 9.1: Let B(X) be any compactification of X. Then the following are equivalent:

i.) X is Lindelöf.

ii.) For each compact set $C \subset B(X)-X$ there is a countable star-finite partition of unity $\Phi=\{\phi_n \mid \sum_n \phi_n=1\}$ on X such that $C \cap Cl_{B(X)}\{O(\phi_n)\}=\Phi$ for each n.

iii.) For each compact set $C \subset B(X)-X$ there is a closed G_δ set F of B(X) such that $C \subset F \subset B(X)-X$.

iv.) For each compact set $C \subset B(X)-X$ there is a countable family $\{G_n\}$ of compact subsets of B(X) such that $G_n \cap C=\Phi$ for each n and $\bigcup_1^\infty G_n \supset X$.

Proof: i.) implies ii.). Suppose X is a Lindelöf space and let $C \subset B(X)-X$ be any compact set. For each $x \in X$ let U_x be a neighborhood of x such that $Cl_{B(X)}U_x \cap C=\Phi$ and consider an open covering of X by $\{U_x \mid x \in X\}$. Since X is Lindelöf there is a countable star-finite refinement $\{W_n\}$ of $\{U_x \mid x \in X\}$. Let $\Phi=\{\phi_n \mid \sum_n \phi_n=1\}$ be a partition of unity on X which is subordinate to $\{W_n\}$. We can do this since X is normal. Then clearly $Cl_{B(X)}\{O(\phi_n)\} \cap C=\emptyset$ for each n.

ii.) implies iii.). Let $f_n \in C(B(X))$ be such that $0 \le f_n \le 1$, $f_n=0$ on C, and $f_n=1$ on $Cl_{B(X)}\{O(\phi_n)\}$. Set $f=\sum_n f_n/2^n$. Z(f) is a closed G_δ set of B(X) and furthermore $C \subset Z(f) \subset B(X)-X$.

iii.) implies iv.). Note that the complement of a G_δ set in B(X) is an F_σ set. Now if for $C \subset B(X)-X$ there is a G_δ set G in B(X) such that $C\subset G\subset B(X)-X$, then $B(X)-G\supset X$ and $B(X)-G=\bigcup_1^\infty G_n$ where G_n is closed in B(X) and hence compact for each n.

iv.) implies i.). Let $\{U_\alpha\}$ be any open covering of X and let U_α^ε be the proper extension of U_α over B(X) for each α. Let $C=B(X)-\bigcup_\alpha U_\alpha^\varepsilon$. Then C is a closed subset of $B(X)-X$. If there exists a collection $\{G_n\}$ of compact subsets of B(X) such that $G_n\cap C=\Phi$ for each n and $\bigcup_1^\infty G_n\supset X$, then each G_n is covered by a finite number of U_α^ε's, say $\{U_{\alpha_n}^\varepsilon\}$, for each n. Hence $\bigcup_1^\infty G_n$ is covered by a countable subfamily of $\{U_\alpha^\varepsilon\}$. Therefore we have $X\subset\bigcup_1^\infty U_{\alpha_k}^\varepsilon$ and hence $X\subset\bigcup_1^\infty U_{\alpha_k}$ and X is Lindelöf.

X

A metric space X is called metrically complete if given any Cauchy sequence $\{x_n\}$ in X, there exists a point $x\varepsilon X$ such that $\lim_{n\to\infty} x_n=x$. A topological space is said to be completely metrizable if it is homeomorphic with a metrically complete space. We now prove the well-known result:

Theorem 10.1: A metric space (X,ρ) is a G_δ in $\beta(X)$ if and only if it is completely metrizable.

Proof: We may suppose first of all that $\rho(x,y)\leq 1$ for every pair $(x,y)\varepsilon X\text{x}X$ for if not then we may replace ρ by the equivalent metric $\rho_1=\min(1,\rho(x,y))$. For any $x_0\varepsilon X$, $\rho(x_0,y)$ is a bounded continuous function on X hence there exists a continuous function $\psi_{x_0} \varepsilon C(\beta(X))$ such that $\psi_{x_0}(x)=\rho(x_0,x)$ for each $x \varepsilon X$. It is clear that $\rho(x,y) \leq \psi_x(z) + \psi_y(z)$ for $x,y \varepsilon X$, and for any $z \varepsilon \beta(X)$.

For $x_0 \varepsilon X$ and $n=1,2,\ldots$ let $\Gamma(x_0,n)=\{x \varepsilon \beta(X) \mid \psi_{x_0}(x)<1/n\}$. $\Gamma(x_0,n)$ is an open subset of $\beta(X)$ for each $x_0 \varepsilon X$ and for every $n=1,2,\ldots$. Thus $G_n=\bigcup_{x_0 \varepsilon X} \Gamma(x_0,n)$ is open in $\beta(X)$. $X \subset G_n$ for each n and hence $X \subset \bigcup_1^\infty G_n$. Let $y \varepsilon \bigcup_1^\infty G_n$. Then there exist points $x_n, x_m \varepsilon X$ such that $\rho(x_n,x_m) \leq \psi_{x_n}(y) + \psi_{x_m}(y)<1/n+1/m$. Therefore $\{x_n \mid n=1,2,\ldots\}$ is a Cauchy sequence in X and there is some point $x \varepsilon X$ such that $\lim_{n\to\infty} x_n=x$. The claim is that $x=y$. Suppose on the contrary that $x \neq y$. Then there exist open sets U,V of $\beta(X)$ such that $x \varepsilon U$, $y \varepsilon V$ and $U \cap V=\Phi$. Now $x \varepsilon U \cap X$ hence there is an integer $n>0$ such that if $\rho(x,z)<2/n$ then $z \varepsilon U \cap X$. If we set $W=\{z \varepsilon \beta(X) \mid \psi_x(z)<2/n\}$ then W is open in $\beta(X)$ and $X \cap W \subset U$. Now since X is dense in $\beta(X)$ we have,

$$W \subset Cl_{\beta(X)}(X \cap W) \subset Cl_{\beta(X)} U \subset \beta(X)-V,$$

hence $W \cap V=\Phi$. For each $z \varepsilon V$ we have $\psi_x(z) \geq 2/n$. In particular $\psi_x(y) \geq 2/n$, hence for each $z \varepsilon X$ we have

$$\rho(x,z) \leq \rho(x,x_n) + \rho(x_n,z) \leq 1/n + \rho(x_n,z)$$

and therefore,

$\psi_x(z) \leq \psi_{x_n}(z) + 1/n$ for each $z \varepsilon \beta(X)$. But then we have,

$\psi_x(y) \leq \psi_{x_n}(y) + 1/n \leq 1/n + 1/n = 2/n$ and this is a contradiction. Thus we must have $x=y$ and $X=\bigcup_1^\infty G_n$. X is then a G_δ in $\beta(X)$. and is topolog

Now suppose X is a G_δ in $\beta(X)$. $\beta(X)-X=\bigcup_1^\infty F_n$ where F_n is closed in $\beta(X)$. If $\beta(X)-X=\Phi$ then $X=\beta(X)$ and X is a compact metric space and is metrically complete. Thus we may assume $X\neq\beta(X)$. For any $x\varepsilon X$, $\rho(x,y)$ is a bounded continuous function on X and hence has a continuous extension ψ_x to $\beta(X)$. If $x\neq y$ then there exist open sets U,V of $\beta(X)$ such that $x\varepsilon U$, $y\varepsilon V$ and $U\cap V=\Phi$. There exists an $\varepsilon>0$ such that $\rho(x,z)<\varepsilon$ for every $z\varepsilon U\cap X$. Then $\rho(x,z)\leq\varepsilon$ for every $z\varepsilon Cl_{\beta(X)}(U\cap X)$ and since $U\subset\beta(X)-V=Cl_{\beta(X)}(\beta(X)-V)$ we have $Cl_{\beta(X)}U\subset\beta(X)-V$. Thus Thus $y\varepsilon\beta(X)-Cl_{\beta(X)}U$ and $\psi_x(y)\geq\varepsilon$. Thus for every $y\varepsilon\beta(X)$ $\psi_x(y)>0$ if $x\neq y$. Since F_n is closed in $\beta(X)$, $\psi_x(y)$ attains a minimum value in F_n, say $\sigma(x,F_n)$. Since $x\varepsilon X$, $F_n\cap X=\Phi$ we have $\sigma(x,F_n)>0$.

If $x,y\varepsilon X$ then $\rho(x,z)\leq\rho(x,y)+\rho(y,z)$ for every $z\varepsilon X$ and hence $\psi_x(z)\leq\rho(x,y)+\psi_y(z)$ for every $z\varepsilon\beta(X)$. Thus,

$\sigma(x,F_n)\leq\rho(x,y)+\sigma(y,F_n)$ and similarly

$\sigma(y,F_n)\leq\rho(x,y)+\sigma(x,F_n)$. Therefore,

$|\sigma(x,F_n)-\sigma(y,F_n)|\leq\rho(x,y)$.

For $x,y\varepsilon X$ set $f_n(x,y)=\rho(x,y)+\sigma(x,F_n)+\sigma(y,F_n)>0$ and

$g_n(x,y)=\dfrac{\rho(x,y)}{f_n(x,y)}$, $\rho_0=\rho(x,y)+\sum_1^\infty 2^{-n}g_n(x,y)$. $0\leq g_n\leq 1$ so the series converges. It is clear, $\rho_0(x,y)=\rho(y,x)$ and $\rho_0(x,x)=0$. If $x\neq y$ then $\rho_0(x,y)>0$. From the definition of $g_n(x,z)$ it follows that

$$g_n(x,z)\leq\frac{\rho(x,y)+\rho(y,z)}{\rho(x,y)+\rho(y,z)+\sigma(x,F_n)+\sigma(z,F_n)}$$

Also since $\sigma(y,F_n) \leq \rho(x,y) + \sigma(x,F_n)$

and $\sigma(y,F_n) \leq \rho(y,z) + \sigma(z,F_n)$ we have

$$\rho(x,y) + \rho(y,z) + \sigma(x,F_n) + \sigma(z,F_n) \geq \begin{cases} \rho(x,y)+\sigma(x,F_n)+\sigma(y,F_n) \\ \rho(y,z)+\sigma(y,F_n)+\sigma(z,F_n) \end{cases}$$

Thus $g_n(x,z) \leq g_n(x,y) + g_n(y,z)$ and evidently

$$\rho_0(x,z) \leq \rho_0(x,y) + \rho_0(y,z).$$

We have thus shown that ρ_0 is a distance function for X. We wish to prove that ρ and ρ_0 are equivalent metrics. Let $\{x_n\}$ be a Cauchy sequence converging to x in the metric ρ_0. $\lim_{n\to\infty} \rho_0(x_n,x)=0$ implies $\lim_{n\to\infty} \rho(x_n,x)=0$ since $\rho(x_n,x) \leq \rho_0(x_n,x)$. Now let $\varepsilon>0$ and choose an integer $k>0$ such that $1/2^{k+1}<\varepsilon$. Then for all $n=1,2,\ldots$

$$\sum_{i=k+1}^{\infty} 2^{-i} g_i(x_n,x) \leq \sum_{i=k+1}^{\infty} 2^{-i} < \varepsilon/2 \quad \text{and hence}$$

$$\rho_0(x_n,x) < \rho(x_n,x) + \sum_{i=1}^{k} 2^{-i} \frac{\rho(x_n,x)}{\rho(x_n,x)+\sigma(x,F_i)+\sigma(x_n,F_i)} + \varepsilon/2$$

$$\leq \rho(x_n,x) + \sum_{i=1}^{k} 2^{-i} \frac{\rho(x_n,x)}{\rho(x_n,x)+\sigma(x,F_i)} + \varepsilon/2.$$

Since $\lim_{n\to\infty} \rho(x_n,x)=0$,

$$\lim_{n\to\infty} \sum_{i=1}^{k} 2^{-i} \frac{\rho(x_n,x)}{\rho(x_n,x)+\sigma(x,F_i)} = 0 \text{ so there}$$

exists an integer p such that for $n>p$,

$$0 \leq \sum_{i=1}^{k} 2^{-i} \frac{\rho(x_n,x)}{\rho(x_n,x)+\sigma(x,F_i)} \leq \varepsilon/2. \quad \text{We then have}$$

$\rho_0(x_n,x) < \rho(x_n,x)+\varepsilon$. If $\lim_{n\to\infty} \rho(x_n,x)=0$ then $\lim_{n\to\infty} \rho_0(x_n,x)=0$

since $\varepsilon>0$ is arbitrary. Thus we have shown that ρ, ρ_0 are equivalent metrics for X or that $(X,\rho), (X,\rho_0)$ are homeomorphic.

We wish to show now that (X,ρ_0) is metrically complete. Suppose $\{x_n\}$ is a Cauchy sequence in (X,ρ_0). We prove the existence of a point $x\varepsilon X$ such that $\lim_{n\to\infty} \rho_0(x_n,x)=0$ or equivalently $\lim_{n\to\infty} \rho(x_n,x)=0$.

Since $\beta(X)$ is compact there exists a point $x\varepsilon\beta(X)$ such that given any neighborhood U of x in $\beta(X)$, $x_n\varepsilon U$ for an infinite number of n. We will show that $x\varepsilon X$ which will complete the proof. Suppose on the contrary that $x\varepsilon\beta(X)-X=\bigcup_1^\infty F_n$. Then there exists an integer k such that $x\varepsilon F_k$. For $\varepsilon>0$ there exists an integer $p>0$ such that for $n,m>p$, $\rho(x_n,x_m)\leq\rho_0(x_n,x_m)<\varepsilon$ then $\sigma(x_n,F_k)$ is the minimum value of $\psi_{x_n}(y)$ on F_k, therefore $0<\sigma(x_n,F_k)\leq\psi_{x_n}(x)$. There exists a neighborhood V_n of x in $\beta(X)$ such that $|\psi_{x_n}(z)-\psi_{x_n}(x)|<\varepsilon$ for every $z\varepsilon V_n$ and there is an integer $m_n>p$ such that $x_{m_n}\varepsilon V_n$ hence $|\psi_{x_n}(x_{m_n})-\psi_{x_n}(x)|<\varepsilon$, or equivalently $|\rho(x_n,x_{m_n})-\psi_{x_n}(x)|<\varepsilon$. Since $n,m_n>p$, $\rho(x_n,x_{m_n})<\varepsilon$ hence $\psi_{x_n}(x)<2\varepsilon$. Therefore $\sigma(x_n,F_k)<2\varepsilon$ for $n>p$ so we have $\lim_{n\to\infty} \sigma(x_n,F_k)=0$. Since $\{x_n\}$ is a Cauchy sequence in (X,ρ_0) there exists an integer $p>0$ such that $\rho_0(x_n,x_p)<2^{-k-2}$ for every $n>p$. We then have $\rho_0(x_n,x_p)\geq 2^{-k}\, g_k(x_n,x_p)=$

$$2^{-k}\frac{\rho(x_n,x_p)}{\rho(x_n,x_p)+\sigma(x_n,F_k)+\sigma(x_p,F_k)}$$

and therefore,

$$\rho_0(x_n,x_p) \geq 2^{-k+1} \frac{\rho(x_n,x_p)}{\rho(x_n,x_p) + \sigma(x_n,F_k)} \geq 0$$

since $\sigma(x_p,F_k) \leq \rho(x_n,x_p) + \sigma(x_n,F_k)$.

Hence for each $n>p$,

$$0 \leq \frac{\rho(x_n,x_p)}{\rho(x_n,x_p) + \sigma(x_n,F_k)} < 2^{-k-2} \quad \text{and thus}$$

$\rho(x_n,x_p)<\sigma(x_n,F_k)$. But $\lim_{n\to\infty} \sigma(x_n,F_k)=0$ and therefore $\lim_{n\to\infty} \rho(x_n,x_p)=0$. There exists an integer $q>p$ such that for every $n>q$ $\rho(x_n,x_p)<1/2\ \psi_{x_p}(x)$. There is a neighborhood U of x in $\beta(X)$ such that $\psi_{x_p}(z)> 1/2\ \psi_{x_p}(x)$ for any $z\varepsilon U$. Hence there exists an integer $n>q$ such that $x_n \varepsilon U$ and we have $\rho(x_n,x_p)=\psi_{x_p}(x_n)> 1/2\ \psi_{x_p}(x)$. But this is a contradiction since $x_n \to x$ in $\beta(X)$. Hence we must have $x\varepsilon X$ and (X,ρ_0) is metrically complete.

X

As is evident, the properties which we have investigated yield some very concise results. At the risk of appearing repetitious let us review these conclusions in a different light with an eye to finding some connection between the seemingly different properties.

In Theorem 6.3 we showed that if a Tychonoff space X is paracompact then we can, in a sense, separate X from any compact subset of $\beta(X)-X$. One method whereby we can accomplish this is by means of a locally finite covering. Although this does not imply any definite measure concepts for the elements of the covering it does give us an idea of just what is happening. The same is true for the other methods of separation, i.e. by partitions of unity and by surroundings. Theorem 7.1 supplies the analogous conditions under which a Tychonoff space will be topologically complete. This occurs when we can separate X from points in $\beta(X)-X$ in exactly the same way as we separated compact subsets of $\beta(X)-X$ from X in Theorem 6.3. Thus we make the obvious inference that a paracompact Tychonoff space is topologically complete.

Again Theorem 8.1 repeats this same type of structure. It states that a Tychonoff space is real compact if we can separate points in $\beta(X)-X$ from X by means of closed G_δ's in $\beta(X)$. The parallel is Theorem 9.1 for the Lindelöf property. One of the conditions is that X can be separated from compact subsets of $B(X)-X$, where $B(X)$ is now any compactification of X, by means of closed G_δ's in $B(X)$. Hence if a space is Lindelöf it is real compact.

Čech's theorem completely characterizes spaces which are G_δ's in $\beta(X)$; they are completely metrizable spaces. The pattern is therefore clear except for one link which seems to be lacking. We have as yet no equivalence condition for X metric. It seems that there should be some analogue in the case where X is metric in $\beta(X)$. Tamano has studied this problem and given a solution which involves X in $X \times \beta(X)$ but this seems to be unsatisfactory. The question is: What property of X in $\beta(X)$ will insure that X is metric?

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