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ABSTRACT

Column Curvature Curves are used to solve general inelastic beam-column problems in a manner similar to the use of Column Deflection Curves. Curvature curves are obtained analytically from differential equations while rotation and deflection are computed by numerical integration of curvature. Three cases of Column Curvature Curves cover all possible cases of elastic-plastic beam-columns. Complete elastic-plastic responses of beam-columns are investigated. Interaction relationships between thrust and end moment for ultimate strength of a beam-column are given. Numerical results are presented for beam-columns with a rectangular section and a wide-flange section.

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1. INTRODUCTION

Columns of buildings and framed piers of bridges are subject to bending moments as well as thrusts at their ends. The end moments are due either to eccentricities of the thrusts or to rotations of adjacent members. Elastic-plastic behavior of these members should be investigated as a beam-column problem. The solution of an inelastic beam-column problem is generally complicated. The purpose of this report is to show that the general response of elastic-plastic beam-columns may be obtained without laborious computations by using exact solutions with the aid of Column Curvature Curves.

2. PREVIOUS WORK

Many investigations have been carried out on elastic beam-column problems by Timoshenko and others [7]. In problems involving moment gradients Column Deflection Curves have been shown to be of great use by Chwalla [5]. Inelastic problems are difficult to solve because of the high nonlinearity of the basic differential equation. Most of the solutions obtained are by numerical computations [1,6]. Recently, Chen and Santathadaporn have succeeded in obtaining an analytical solution for the curvature of elastic-plastic columns [2]. Extending these solutions, Chen has solved an inelastic beam-column subjected to a concentrated lateral load at midspan [3]. Moment-curvature-thrust relations in the elastic-plastic range for rectangular, box, and wide flange sections with or without the influence of residual stresses have been reported [3,4].

3. MOMENT-CURVATURE-THRUST RELATIONSHIP

The initial yield quantities for a cross section are

$$M_y = \sigma_y S \quad \text{for moment } M$$

$$P_y = \sigma_y A \quad \text{for thrust } P \qquad (1)$$

$$\Phi_y = \frac{2\epsilon_y}{h} \quad \text{for curvature } \Phi$$

where

σ_y , ϵ_y are yield stress and yield strain of the material

h , A are depth and area of cross section

and (2)

S is elastic section modulus

Non-dimensional parameters are defined as

$$m = \frac{M}{M_y} \text{ , } p = \frac{P}{P_y} \text{ , } \varphi = \frac{\Phi}{\Phi_y} \qquad (3)$$

Following the work of Chen [3,4], the state of the generalized stress m in a cross section of a beam-column belongs to one of the following three regimes: Elastic regime, primary plastic regime, and secondary plastic regime (Fig. 1). The boundaries between the regimes are defined by initial yield curvature φ_1 and secondary yield curvature φ_2, respectively (Fig. 2). A general m-φ-p curve is assumed to be closely represented by the following expressions (Ref. 3):

In the elastic regime ($0 \leq \varphi \leq \varphi_1$)

$$m = a\,\varphi \tag{4}$$

In the primary plastic regime ($\varphi_1 \leq \varphi \leq \varphi_2$)

$$m = b - \frac{c}{\sqrt{\varphi}} \tag{5}$$

In the secondary plastic regime ($\varphi_2 \leq \varphi$)

$$m = m_{pc} - \frac{f}{\varphi^2} \tag{6}$$

where a, b, c, f, m_{pc}, φ_1, and φ_2 are functions of the thrust parameter p and the shape of the cross section. These functions have been obtained in Ref. 4 for several commonly used structural sections. As a simple example, for the case of a rectangular cross section

$$\begin{gathered} a = 1 \quad , \quad b = 3\ (1-p) \quad , \quad c = 2\ (1-p)^{3/2} \\ f = \frac{1}{2} \; , \; m_{pc} = \frac{3}{2}\ (1-p^2) \; , \; \varphi_1 = 1-p \; , \; \varphi_2 = \frac{1}{1-p} \end{gathered} \tag{7}$$

4. CONCEPT OF EQUIVALENT COLUMN

A beam-column AB of length L is subjected to axial compression P and bending moments M_A and M_B at its ends A and B, respectively [Fig. 3(a)]. The vertical reactions V at the ends are, from moment equilibrium,

$$V = (M_B - M_A)/L \tag{8}$$

If the resultant force of P and V is taken as

$$P^* = \sqrt{P^2 + V^2} \tag{9}$$

another equilibrium state is obtained as shown in Fig. 3(b). The two resultant forces have the same magnitude and are parallel to the slope of the beam-column at C where curvature is maximum (Appendix I). As shown here, the section where the maximum curvature occurs does not coincide with the section corresponding to the maximum deflection.

From this state of equilibrium it is possible to conclude that a simply supported column DE of length L^* can be chosen which is subjected to axial compression P^* only so that a portion $A^*C^*B^*$ may be in the same condition as the original beam-column ACB, Fig. 3(c). This is the "equivalent column" of the original beam-column. The length of the equivalent column L^* is unknown at this stage. The use of the equivalent column concept reduces ten cases of beam-column problem to three cases of equivalent column problem as shown in Fig. 4.

5. DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

The equation of equilibrium for a beam-column is

$$\frac{d^2m}{dx^2} + k^2 \varphi = 0 \tag{10}$$

where

$$k = \sqrt{P^*/EI}$$

and (11)

EI = bending rigidity of the elastic beam

Using the moment-curvature-thrust relations (Eqs. (4), (5), and (6)) the basic differential equations of the beam-column in terms of φ are obtained as follows:

In the elastic regime ($\varphi \leq \varphi_1$)

$$\frac{d^2\varphi}{dx^2} + \frac{k^2}{a}\varphi = 0 \tag{12}$$

In the primary plastic regime ($\varphi_1 \leq \varphi \leq \varphi_2$)

$$\varphi\frac{d^2\varphi}{dx^2} - \frac{3}{2}\left(\frac{d\varphi}{dx}\right)^2 + \frac{2k^2}{c}\varphi^{7/2} = 0 \tag{13}$$

In the secondary plastic regime ($\varphi_2 \leq \varphi$)

$$\varphi\frac{d^2\varphi}{dx^2} - 3\left(\frac{d\varphi}{dx}\right)^2 + \frac{k^2}{2f}\varphi^5 = 0 \tag{14}$$

The general solutions are:

In the elastic regime ($\varphi \leq \varphi_1$)

$$\varphi = A_1\cos\frac{kx}{\sqrt{a}} + B\sin\frac{kx}{\sqrt{a}} \tag{15}$$

In the primary plastic regime ($\varphi_1 \leq \varphi \leq \varphi_2$)

$$\frac{d\varphi}{dx} = \frac{2\sqrt{2}k}{\sqrt{c}} \left(D^{1/2} - \varphi^{1/2}\right)^{1/2} \varphi^{3/2} \tag{16}$$

$$x - x_p = -\frac{\sqrt{c}}{\sqrt{2}k} D^{-3/4} \left\{ \left(\frac{D}{\varphi}\right)^{1/2} \left[1 - \left(\frac{\varphi}{D}\right)^{1/2}\right]^{1/2} + \tanh^{-1} \left[1 - \left(\frac{\varphi}{D}\right)^{1/2}\right]^{1/2} \right\} \tag{17}$$

In the secondary plastic regime ($\varphi_2 \leq \varphi$)

$$\frac{d\varphi}{dx} = \frac{k}{\sqrt{f}} (1 + G\varphi)^{1/2} \varphi^{5/2} \tag{18}$$

$$x - x_s = \frac{2}{3} \frac{\sqrt{f}}{k} \left(G + \frac{1}{\varphi}\right)^{1/2} \left(2G - \frac{1}{\varphi}\right) \tag{19}$$

where A_1, B, D, G, x_p, and x_s are constants of integration.

6. BOUNDARY CONDITIONS AND DISCONTINUITY CONDITIONS

Consider the equivalent columns shown in Fig. 5. Curvatures corresponding to primary yielding (φ_1) and secondary yielding (φ_2) occur at sections $x = \rho_1$ and $x = \rho_2$, respectively. The maximum curvature occurs at midspan, $x = L^*/2$. The boundary conditions for curvature are, hence,

$$\begin{aligned} &\text{at } x = 0, && \varphi = 0 \\ &\text{at } x = \rho_1, && \varphi = \varphi_1 \\ &\text{at } x = \rho_2, && \varphi = \varphi_2 \\ &\text{at } x = L^*/2, && \varphi = \varphi_m \end{aligned} \tag{20}$$

Discontinuity conditions of $d\varphi/dx$ are obtained from continuity conditions of bending moment at the boundaries [3];

$$\text{at} \quad x = \rho_1, \qquad \frac{2a}{c}\varphi_1^{3/2}\left(\frac{d\varphi}{dx}\right)_E = \left(\frac{d\varphi}{dx}\right)_P$$

$$\text{at} \quad x = \rho_2, \qquad \frac{c}{4f}\varphi_2^{3/2}\left(\frac{d\varphi}{dx}\right)_P = \left(\frac{d\varphi}{dx}\right)_S \tag{21}$$

$$\text{at} \quad x = L^*/2, \qquad \frac{d\varphi}{dx} = 0$$

where subscripts, E, P, and S denote elastic, primary plastic and secondary plastic regimes, respectively.

7. COLUMN CURVATURE CURVES

From the general solutions and boundary conditions Column Curvature Curves (CCC-s) for the three different cases (Fig. 5) are obtained as follows:

Elastic Column ($0 \leq \varphi_m \leq \varphi_1$)

$$kx = \sqrt{a}\,\sin^{-1}\left(\frac{\varphi}{\varphi_m}\sin\frac{kL^*}{2\sqrt{a}}\right) \tag{22}$$

where

$$kL^* = \pi\sqrt{a} \tag{23}$$

One-Side Plastic Column ($\varphi_1 \leq \varphi_m \leq \varphi_2$)

In elastic regime ($0 \leq x \leq \rho_1$)

$$kx = \sqrt{a}\,\sin^{-1}\left(\frac{\varphi}{\varphi_1}\sin\frac{k\rho_1}{\sqrt{a}}\right) \tag{24}$$

In primary plastic regime $(\rho_1 \leq x \leq L^*/2)$

$$kx = \frac{kL^*}{2} - \frac{\sqrt{c}}{\sqrt{2}}\varphi_m^{-3/4}\left\{\left[\frac{\varphi_m}{\varphi} - \left(\frac{\varphi_m}{\varphi}\right)^{1/2}\right]^{1/2} + \tanh^{-1}\left[1 - \left(\frac{\varphi}{\varphi_m}\right)^{1/2}\right]^{1/2}\right\} \quad (25)$$

where

$$k\rho_1 = \sqrt{a}\,\cot^{-1}\left\{\sqrt{\frac{2c}{a}}\,\varphi_1^{-3/4}\left[\left(\frac{\varphi_m}{\varphi_1}\right)^{1/2} - 1\right]^{1/2}\right\} \quad (26)$$

$$\frac{kL^*}{2} = k\rho_1 + \frac{\sqrt{c}}{\sqrt{2}}\varphi_m^{-3/4}\left\{\left[\frac{\varphi_m}{\varphi_1} - \left(\frac{\varphi_m}{\varphi_1}\right)^{1/2}\right]^{1/2} + \tanh^{-1}\left[1 - \left(\frac{\varphi_1}{\varphi_m}\right)^{1/2}\right]^{1/2}\right\} \quad (27)$$

Two-Sides Plastic Column $(\varphi_2 \leq \varphi_m)$

In elastic regime $(0 \leq x \leq \rho_1)$

$$kx = \sqrt{a}\,\sin^{-1}\left(\frac{\varphi}{\varphi_1}\sin\frac{k\rho_1}{\sqrt{a}}\right) \quad (28)$$

In primary plastic regime $(\rho_1 \leq x \leq \rho_2)$

$$kx = kx_p - \frac{\sqrt{c}}{\sqrt{2}}D^{-3/4}\left\{\left(\frac{D}{\varphi}\right)^{1/2}\left[1 - \left(\frac{\varphi}{D}\right)^{1/2}\right]^{1/2} + \tanh^{-1}\left[1 - \left(\frac{\varphi}{D}\right)^{1/2}\right]^{1/2}\right\} \quad (29)$$

In secondary plastic regime $(\rho_1 \leq x \leq L^*/2)$

$$kx = \frac{kL^*}{2} - \frac{2}{3}\sqrt{f}\,\varphi_m^{-3/2}\left(2 + \frac{\varphi_m}{\varphi}\right)\left(\frac{\varphi_m}{\varphi} - 1\right)^{1/2} \quad (30)$$

where

$$D = \left[\sqrt{\varphi_2} + \frac{2f}{c}\left(\frac{1}{\varphi_2} - \frac{1}{\varphi_m}\right)\right]^2 \quad (31)$$

$$k\rho_1 = \sqrt{a}\,\tan^{-1}\left[\sqrt{\frac{a}{2c}}\,\varphi_1\Big/\left(D^{1/2} - \varphi_1^{1/2}\right)^{1/2}\right] \quad (32)$$

$$kx_p = k\rho_1 + \sqrt{\frac{c}{2}}\,D^{-3/4}\left\{\left(\frac{D}{\varphi_1}\right)^{1/2}\left[1-\left(\frac{\varphi_1}{D}\right)^{1/2}\right]^{1/2} \pm \tanh^{-1}\left[1-\left(\frac{\varphi_1}{D}\right)^{1/2}\right]^{1/2}\right\} \quad (33)$$

$$k\rho_2 = kx_p - \sqrt{\frac{c}{2}}\,D^{-3/4}\left\{\left(\frac{D}{\varphi_2}\right)^{1/2}\left[1-\left(\frac{\varphi_2}{D}\right)^{1/2}\right]^{1/2} + \tanh^{-1}\left[1-\left(\frac{\varphi_2}{D}\right)^{1/2}\right]^{1/2}\right\} \quad (34)$$

$$kL^* = 2k\rho_2 + \frac{4}{3}\sqrt{f}\left(\frac{1}{\varphi_2}-\frac{1}{\varphi_m}\right)^{1/2}\left(\frac{1}{\varphi_2}+\frac{2}{\varphi_m}\right) \quad (35)$$

The length x is expressed explicitly here as a function of curvature φ. The length of the equilvanet column kL^* is obtained as a function of φ_m and p^* from these equations and shown in Table 1.

Figure 6 is an example of CCC-s for a rectangular cross section when the axial thrust is constant ($p^* = 0.5$). In the secondary plastic regime ($2 \leq \varphi$) φ changes rapidly. The lowest curve ($\varphi_m^* = 0.5$) represents curvatures in the elastic regime. The half width of the abscissa for the elastic curve is equal to $\pi/2$ which corresponds to Euler's buckling.

8. SOLUTION OF THE BEAM-COLUMN PROBLEM

Column Curvature Curves (CCC-s) are used to solve various beam-column problems. Consider a beam-column as shown in Fig. 7(a). First, p^*, φ_A, φ_B, and kL are computed from M_A, M_B, P, L, and EI using Eqs. (4) to (11). For a fixed p^*-value CCC-s are drawn from Eqs. (22) to (35) as shown in Fig. 7(c). The curve, in which φ_A, φ_B, and kL match, gives the maximum curvature φ_m^*, the equivalent column length kL^*, and the curvature distribution φ corresponding to the original beam-column. It should be noted here that there can be two sets of matchings as shown in Fig. 8(a). This means, for a fixed thrust p^*, there can be two states of equilibrium which are represented by two maximum curvatures φ_{m1}^* and φ_{m2}^*. These two different states correspond to the two points on the loading and unloading portions of a load-curvature curve, respectively, as shown in Fig. 8(b).

As shown here, the maximum curvature φ_m of a beam-column does not necessarily equal φ_m^*, the maximum curvature of the equivalent column. Figure 9 shows a comparison between the maximum curvature φ_m of a beam-column and φ_m^* of the equivalent column for a rectangular section in the loading case of $m_A = 0.4$ (constant) and $m_B = 0.2$ (constant). When p is small, φ_m and φ_m^* are much different from each other, which means the maximum curvature φ_m^* occurs outside the beam-column, while φ_m occurs at one end of the beam-column. In the figure φ_m stays constant (= 0.4) while φ_m^* decreases as the thrust p increases. At some stage both φ_m and φ_m^* start to increase and become equal after a while. After this, maximum curvatures φ_m and φ_m^* remain the same and occur inside the beam-column.

Using Column Curvature Curves for single curvature beam-columns, beam-column problems with double curvature can be solved in exactly the same way as done using the Column Deflection Curve method. This implies drawing another set of CCC-s continuous with the original CCC-s at one end, the continuation being mirror images of the original CCC-s as shown in Fig. 10. Using this modified set of CCC-s, beam-columns with many types of end loading can be solved.

9. NUMERICAL RESULTS

Generally, the behavior of an elastic-plastic beam-column is determined by two independent non-dimensional variables,

$$\gamma = \frac{S}{AL} \quad \text{and} \quad \lambda = \frac{L}{r}\sqrt{\frac{\sigma_y}{E}} \tag{36}$$

where L/r is the slenderness ratio. Loading conditions for the left end A and the right end B are given by the following relations:

$$m_A = \alpha \cdot p + \bar{m}_A \ , \ m_B = \beta \cdot p + \bar{m}_B \tag{37}$$

where α and β represent non-dimensionally the eccentricities of the thrust at ends A and B, respectively. These are computed by the expressions,

$$\alpha = e_A \cdot \frac{A}{S} \quad \text{and} \quad \beta = e_B \frac{A}{S} \tag{38}$$

where e_A and e_B represent the actual eccentricities as shown in Fig. 11. Moments $\bar{m}_A$ and $\bar{m}_B$ are produced by rotation of adjacent members. In case of proportional loading Eq. (37) reduces to

$$m_A = \alpha \cdot p \quad , \quad m_B = \kappa \, m_A \tag{39}$$

As an example, numerical results for a beam-column (L = 600 in.) with a rectangular cross section (6" x 20") are considered. The moment-curvature-thrust relationship is determined by the parameters given in Eq. (7). Assuming A7 steel, and

$$E = 29{,}000 \text{ ksi} \quad , \quad \sigma_y = 33 \text{ ksi} \tag{40}$$

the non-dimensional variables of this beam-column are (Eq. (36))

$$\gamma = 5.56 \times 10^{-3} \quad , \quad \lambda = 3.51 \tag{41}$$

Figure 12 is a plot of the numerical results for maximum curvature φ_m vs. thrust p for the case with $\kappa = 0.5$ in Eq. (39). In the figure the (p, φ_m)-curves are divided into elastic regime, primary plastic regime, and secondary plastic regime by two dotted straight lines. Rotational angles at the ends, θ_A and θ_B, can also be computed and plotted as shown by curves in Fig. 13. Picking up the maximum points of these curves, interaction curves for the ultimate strength of this beam-column corresponding to loads m_A, m_B, and p can be obtained as shown in Fig. 14.

In Figs. 15, 16, and 17 numerical results for a 14W246 cross section neglecting the influence of residual stresses are shown, other parameters used being identical with those of the rectangular section.

10. FURTHER APPLICATIONS

As the end rotations of a beam-column can be obtained, the stiffness of a beam-column which is the ratio of end moment to end rotation can be computed. From the member stiffnesses, the stiffness matrix for a whole frame, such as the one shown in Fig. 18, can be constructed and, thus, the elastic-plastic behavior of the frame can be obtained. The application of the CCC-s to framed structures will be presented in a further report.

11. CONCLUSION

An approach using Column Curvature Curves (CCC-s) has been introduced to solve elastic-plastic beam-columns subjected to various end loadings. The basic idea is the same as in the method using Column Deflection Curves where deflections are obtained by numerical integration. The application of the CCC method is simple and three types of CCC-s cover all cases of elastic-plastic beam-columns. Complete elastic-plastic response of beam-columns is presented and ultimate strength and interaction relations are obtained. End rotation which gives member stiffness is also computed.

APPENDIX I - PROOF OF EQUIVALENT COLUMN

Consider a beam-column ACB subjected to axial thrust P, shear force V, and bending moments M_A and M_B as shown in Fig. 19(a). At section C curvature and, hence, moment are maximum.

$$\left.\frac{dM(x)}{dx}\right|_c = 0 \qquad (42)$$

Moment M(x) may be expressed in terms of deflection W(x):

$$M(x) = P\,W(x) + Vx + M_A \qquad (43)$$

Hence,

$$\frac{dM(x)}{dx} = P\,\frac{dW(x)}{dx} + V \qquad (44)$$

From Eqs. (42) and (43), at section C

$$\left.\frac{dM(x)}{dx}\right|_c = P\left.\frac{dW(x)}{dx}\right|_c + V = 0$$

$$\therefore \tan\theta_c = \left.\frac{dW(x)}{dx}\right|_c = -\frac{V}{P}$$

Thus, the slope at the section of maximum curvature ($\tan\theta_c$) is parallel to the resultant force of thrust P and shear V.

The part CB of the beam-column can therefore be replaced by a cantilever beam subjected to horizontal thrust $P^* = \sqrt{P^2 + V^2}$ and moment M_A at its free end as shown in Fig. 19(b). The length of this cantilever beam may be increased without altering the thrust, resulting in another cantilever beam D^*C^* which is subjected to horizontal thrust $P^* = \sqrt{P^2 + V^2}$ only and produces exactly the same state of stress over the original cantilever AC, Fig. 19(c). This proves that the beam-column AC is a part of the equivalent column D^*C^*. Similarly, the entire beam-column ACB is a part of the simple column $D^*C^*E^*$ subjected to thrust P^* only (Fig. 3).

The thrust on the equivalent column is the resultant of P and V

$$P^* = \sqrt{P^2 + V^2}$$
$$= \sqrt{P^2 + \left(\frac{M_B - M_A}{L}\right)^2} \tag{45}$$

For this equivalent column the unknown length L^* must be determined using Eqs. (22) to (35) which account for equilibrium and boundary conditions.

APPENDIX II - ROTATION AND DEFLECTION

From a knowledge of the curvature distribution, rotation and deflection of a beam-column at any section can be computed by integrations of the curvature diagram once and twice, respectively. Curvature distribution φ is not expressed explicitly as a function of x. Instead, kx is expressed as a function of φ (Eqs. (22) to (35)). Therefore, curvature φ needs to be obtained by numerical integration.

Consider half waves of CCC-s for a cantilever beam as shown in Fig. 20. First, one chooses n + 1 points on curvature axis:

$$\varphi_o \ (= 0), \ \varphi_1, \ \varphi_2, \ \ldots \ \varphi_N \ (= \varphi_m^*) \tag{46}$$

Corresponding kx values are computed using Eqs. (22) to (35):

$$kx_o \left(= \frac{kL^*}{2} \right), \ kx_1 \ kx_2, \ \ldots \ kx_N \ (= 0) \tag{47}$$

The Fourier Series for the curvature function is

$$\varphi = \sum_{j=1}^{N} C_j \cos \frac{(2j - 1)\pi}{2L^*} kx \tag{48}$$

Substituting for kx_i and φ_i $(i = 1, 2, \ldots, N)$, one gets n-simultaneous equations:

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_N \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ & & & \\ A_{N1} & \cdots\cdots & & A_{NN} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_N \end{bmatrix} \tag{49}$$

where

$$A_{ij} = \cos \frac{(2j - 1)\pi}{2L^*} kx_i \tag{50}$$

The coefficients C_j of the Fourier Series are obtained by pre-multiplying the φ-matrix by the inverse of A-matrix, that is

$$[C] = [A]^{-1} [\varphi] \tag{51}$$

Rotation angle at point x of the cantilever is, from Eq. (48),

$$\Theta^* = \int_o^x \Phi^* \, dx$$

$$= \Phi_y \int_o^x \varphi \, dx \tag{52}$$

$$= \Phi_y \sum_{j=1}^{N} C_j \frac{2L^*}{(2j-1)\pi k} \sin \frac{(2j-1)\pi}{2L^*} kx$$

Non-dimensional rotation is defined by

$$\theta^* = k\Theta^*/\Phi_y$$

$$= \sum_{j=1}^{N} C_j \frac{2L^*}{(2j-1)\pi} \sin \frac{(2j-1)\pi}{2L^*} kx \tag{53}$$

Deflection at point x is

$$W^* = \int_o^x = \Theta^* \, dx$$

$$= \frac{\Phi_y}{k} \sum_{j=1}^{N} C_j \frac{2L^*}{(2j-1)\pi} \int_o^x \sin \frac{(2j-1)\pi}{2L^*} kx \, dx \tag{54}$$

$$= \frac{\Phi_y}{k} \sum_{j=1}^{N} C_j \left[\frac{2L^*}{(2j-1)\pi}\right]^2 \left(1 - \cos \frac{(2j-1)\pi}{2L^*} kx\right)$$

Non-dimensional deflection is defined by

$$w^* = k^2 W^*/\Phi_y$$

$$= \sum_{j=1}^{N} C_j \left[\frac{2L^*}{(2j-1)\pi}\right]^2 \left(1 - \cos \frac{(2j-1)\pi}{2L^*} kx\right) \tag{55}$$

Now, consider rotations at ends A and B of a beam-column shown in Fig. 21(a). Maximum curvature occurs at section C with

$$\overline{AC} = \ell_A \quad , \quad \overline{CB} = \ell_B \tag{56}$$

Rotation angles at sections A and B with respect to the tangent at point C are

$$\Theta_A^* = \frac{\Phi_y}{k}\,\theta_A^* \quad , \quad \Theta_B^* = \frac{\Phi_y}{k}\,\theta_B^* \tag{57}$$

Deflections at sections A and B with respect to the tangent at point C are

$$W_A^* = \frac{\Phi_y}{k^2}\,w_A^* \quad , \quad W_B = \frac{\Phi_y}{k^2}\,w_B^* \tag{58}$$

Member rotation R is expressed by

$$R = \frac{W_B^* - W_A^*}{\ell_A + \ell_B} = \frac{\Phi_y}{k}\,\frac{w_B^* - w_A^*}{kL} \tag{59}$$

Then, rotation angles at A and B of the beam-column are

$$\Theta_A = \Theta_A^* + R = \frac{\Phi_y}{k}\left(\theta_A^* + \frac{w_B^* - w_A^*}{kL}\right) \tag{60}$$

$$\Theta_B = \Theta_B^* - R = \frac{\Phi_y}{k}\left(\theta_B^* - \frac{w_B^* - w_A^*}{kL}\right) \tag{61}$$

Non-dimensional rotations are obtained by

$$\theta_A = \frac{k}{\Phi_y}\,\Theta_A = \Theta_A^{\ *} + \frac{w_B^{\ *} - w_A^{\ *}}{kL} \qquad (62)$$

$$\theta_B = \frac{k}{\Phi_y}\,\Theta_B = \theta_B^{\ *} - \frac{w_B^{\ *} - w_A^{\ *}}{kL} \qquad (63)$$

Even if the maximum curvature does not occur inside the beam-column as shown in Fig. 21(b), the expression for end rotations are the same as given by Eqs. (62) and (63).

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APPENDIX IV - NOTATION

A = area of section

a, b, c, f = constants which define m-φ-p relationship

E = modulus of elasticity

e_A, e_B = eccentricities of the thrust at ends A and B

h = depth of section

I = moment of inertia of section

k^2 = P^*/EI

L = length of beam-column

L^* = length of equivalent column

M, (m) = bending moment, $(m = M/M_y)$

M_A, M_B, (m_A, m_B) = moment at ends A and B, $(m_A = M_A/M_y$, $m_B = M_B/M_y)$

M_{pc}, (m_{pc}) = full-plastic moment of section, $(m_{pc} = M_{pc}/M_y)$

m_1, m_2 = bending moment at initial and secondary yield

$\overline{M}_A$, $\overline{M}_B$, $(\overline{m}_A, \overline{m}_B)$ = constant bending moments at ends A and B, $(\overline{m}_A = \overline{M}_A/\overline{M}_y$, $\overline{m}_B = \overline{M}_B/\overline{M}_y)$

P, (p) = axial thrust of beam-column $(p = P/P_y)$

P^*, (p^*) = axial thrust of equivalent column, $(p^* = P^*/P_y)$

P_y = axial yield load

r = radius of gyration of section

S = elastic section modulus

V = shear force

W^*, (w^*) = deflection with respect to tangent at the point of maximum curvature (see Fig. 21), $(w^* = W^* k^2/\Phi_y)$

x = longitudinal coordinate axis

y = transverse coordinate axis

α, β	= coefficients of bending moment due to eccentric thrust at ends A and B (Eq. (38))
γ	= S/AL
ϵ_y	= strain at yield point
κ	= M_B/M_A
ρ_1	= location of initial yield point
ρ_2	= location of secondary yield point
λ	= $\frac{L}{r}\sqrt{\frac{\sigma_y}{E}}$
σ_y	= yield stress
Φ, (φ)	= curvature, ($\varphi = \Phi/\Phi_y$)
Φ_m, (φ_m)	= maximum curvature of beam-column, ($\varphi_m = \Phi_m/\Phi_y$)
Φ_m^*, (φ_m^*)	= maximum curvature of equivalent column, ($\varphi_m^* = \Phi_m^*/\Phi_y$)
Φ_y	= curvature at initial yielding for pure bending
φ_1	= curvature at initial yielding with bending and thrust
φ_2	= curvature at secondary yielding with bending and thrust

Table 1. Length of Equivalent Column kL^* for Rectangular Section

φ_m^* \ p^*	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.57	1.57	1.57	1.57	1.57	1.57	1.57	1.57	1.57
0.2	1.57	1.57	1.57	1.57	1.57	1.57	1.57	1.57	1.48
0.3	1.57	1.57	1.57	1.57	1.57	1.57	1.57	1.54	1.36
0.4	1.57	1.57	1.57	1.57	1.57	1.57	1.55	1.48	1.28
0.5	1.57	1.57	1.57	1.57	1.57	1.56	1.52	1.42	1.21
0.6	1.57	1.57	1.57	1.57	1.56	1.54	1.48	1.36	1.15
0.7	1.57	1.57	1.57	1.57	1.55	1.51	1.44	1.32	1.10
0.8	1.57	1.57	1.57	1.55	1.53	1.48	1.40	1.28	1.06
0.9	1.57	1.57	1.56	1.54	1.50	1.45	1.36	1.24	1.03
1.0	1.57	1.56	1.54	1.52	1.48	1.42	1.33	1.21	1.00
1.1	1.56	1.55	1.53	1.50	1.45	1.39	1.30	1.18	0.976
1.2	1.55	1.54	1.51	1.48	1.43	1.36	1.28	1.15	0.953
1.3	1.54	1.52	1.49	1.46	1.41	1.34	1.25	1.13	0.932
1.4	1.53	1.51	1.48	1.44	1.38	1.32	1.23	1.10	0.912
1.5	1.51	1.49	1.46	1.42	1.36	1.30	1.21	1.08	0.895
1.6	1.50	1.48	1.44	1.40	1.34	1.28	1.19	1.07	0.879
1.7	1.48	1.46	1.43	1.38	1.33	1.26	1.17	1.05	0.864
1.8	1.47	1.45	1.41	1.36	1.31	1.24	1.15	1.03	0.850
1.9	1.46	1.44	1.40	1.35	1.29	1.22	1.13	1.02	0.837
2.0	1.44	1.42	1.39	1.34	1.28	1.27	1.12	1.00	0.825
2.1	1.43	1.41	1.37	1.32	1.26	1.19	1.10	0.989	0.813
2.2	1.42	1.40	1.36	1.31	1.25	1.18	1.09	0.976	0.803

Elastic | Primary Plastic

← Secondary Plastic | Primary Plastic →

$$P^* = \left[P^2 + \left(\frac{M_B - M_A}{L} \right)^2 \right]^{1/2}$$

$$p^* = P^*/P_y$$

$$k = \sqrt{P^*/EI}$$

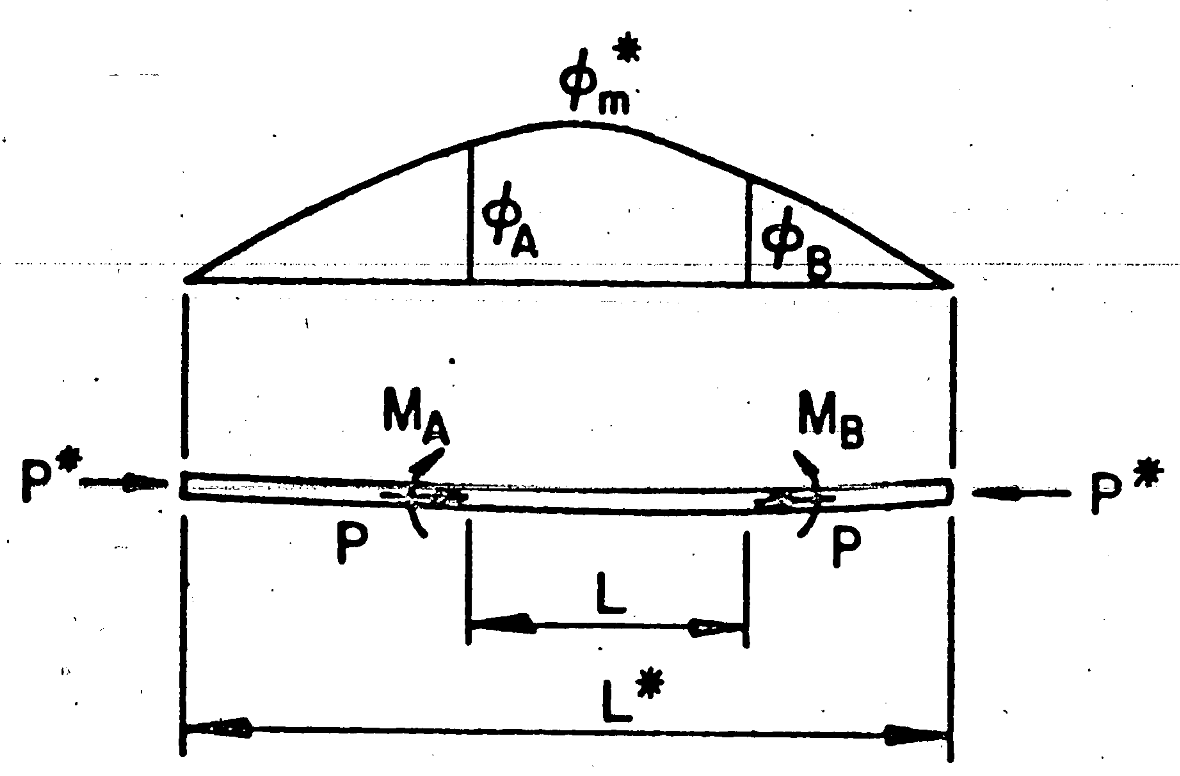
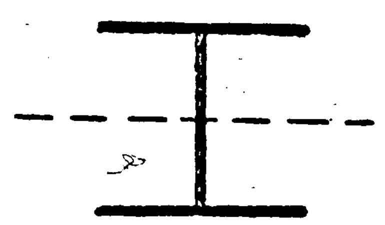
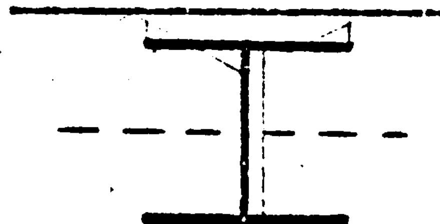
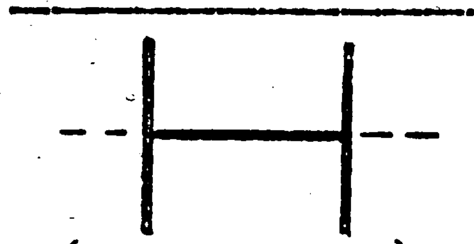
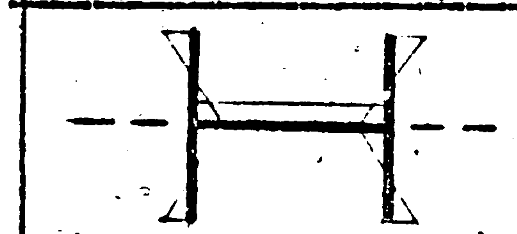
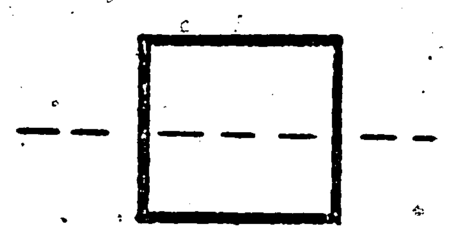
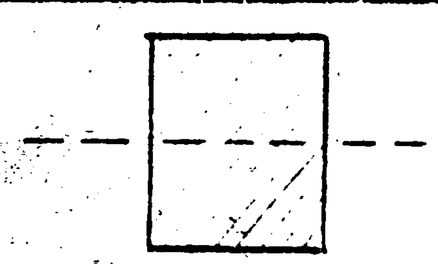


Table 2 Parameters for m-Ø-p Relationship (See Ref.(4))

$$\begin{cases} m = a\phi & (\phi \le \phi_1) \\ m = b - c/\sqrt{\phi} & (\phi_1 < \phi \le \phi_2) \\ m = m_{pc} - f/\phi^2 & (\phi_2 < \phi) \end{cases}$$

$$a = \frac{m_1}{\phi_1} \qquad b = \frac{m_2 - m_1\sqrt{\phi_1/\phi_2}}{1 - \sqrt{\phi_1/\phi_2}}$$

$$c = \frac{m_2 - m_1}{1/\sqrt{\phi_1} - 1/\sqrt{\phi_2}} \qquad f = (m_{pc} - m_2)\phi_2^2$$

SECTION	p	m_1	$\emptyset_1$	m_2	$\emptyset_2$	m_{pc}
 (no R.S.)	.225	1 − p	1 − p	$1+.778p-4.78p^2$	$(1-3.7p+8.4p^2)^{-1}$	$1.11-2.64p^2$
				$1.2(1-p)$	$2.2(1-p)$	$1.238-1.143p-.095p^2$
 (with R.S.)	.225	0.9 − p	0.9 − p	$.9+1.94p-9.4p^2$	$(1.11-7.35p+29.2p^2)^{-1}$	$1.11-2.64p^2$
	.8	$-1.1+3.1p-2p^2$	$3.3-8p+5p^2$	$1.1(1-p)$	1.3 − p	$1.238-1.143p-.095p^2$
 (no R.S.)	.252	1 − p	1 − p	$1+1.5p-2.5p^2$	$(1-1.57p+.785p^2)^{-1}$	$1.51(1-.185p^2)$
	.4			$.85+2.03p-2.88p^2$	$(.368+.645p-.862p^2)^{-1}$	$2.58(.52+p)x(1-p)$
 (with R.S.)	.252	0.9 − p	0.9 − p	$.9+p-2.5p^2$	$(1.11-2.11p+2.81p^2)^{-1}$	$1.51-.28p^2$
	.4	$.567+.1p-.667p^2$	0.5	$1+.25p-1.25p^2$	$(1.3-2.45p+2.45p^2)^{-1}$	$2.58(.52+p)x(1-p)$
	.467	1 − p	1 − p	$1-.9p-3.25p^2$	$(1-2.5p+4.17p^2)^{-1}$	$1.2-1.6p^2$
				$1.4(1-p)$	$2.5(1-p)$	$1.51-1.31p-.2p^2$
		1 − p	1 − p	$1+p-2p^2$	$(1-p)^{-1}$	$1.5(1-p^2)$

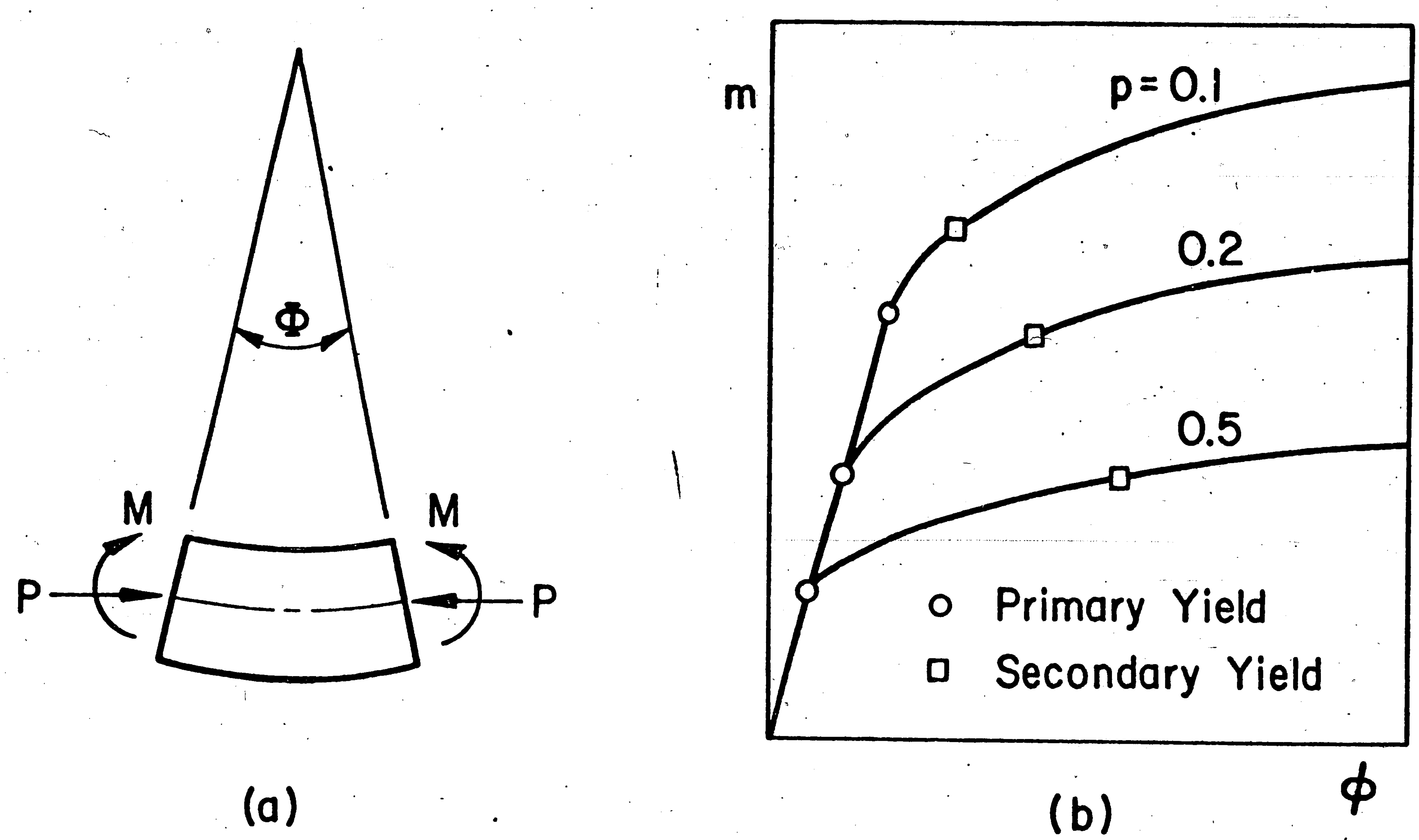


Fig. 1 Moment-Curvature-Thrust Relationship

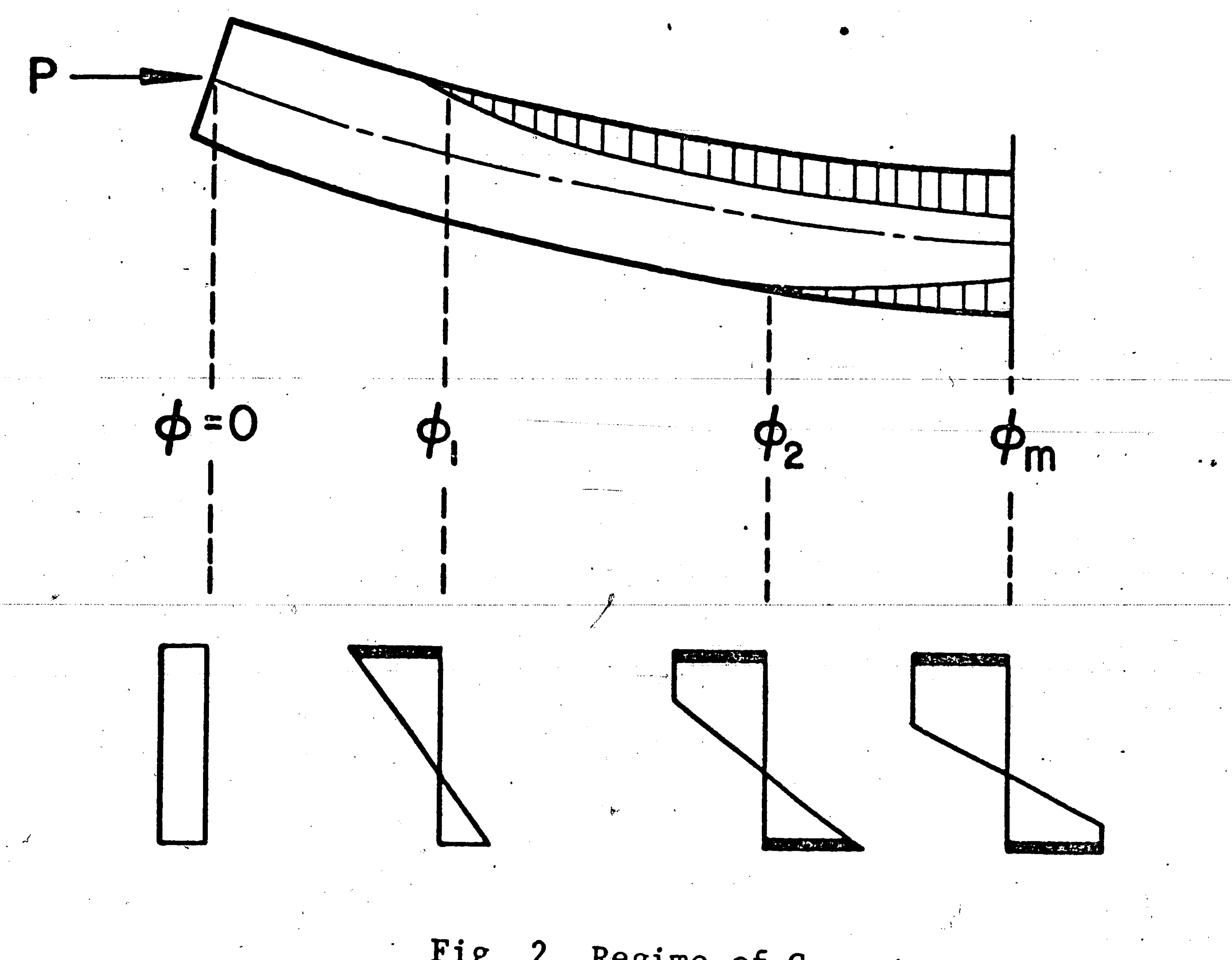


Fig. 2 Regime of Curvature

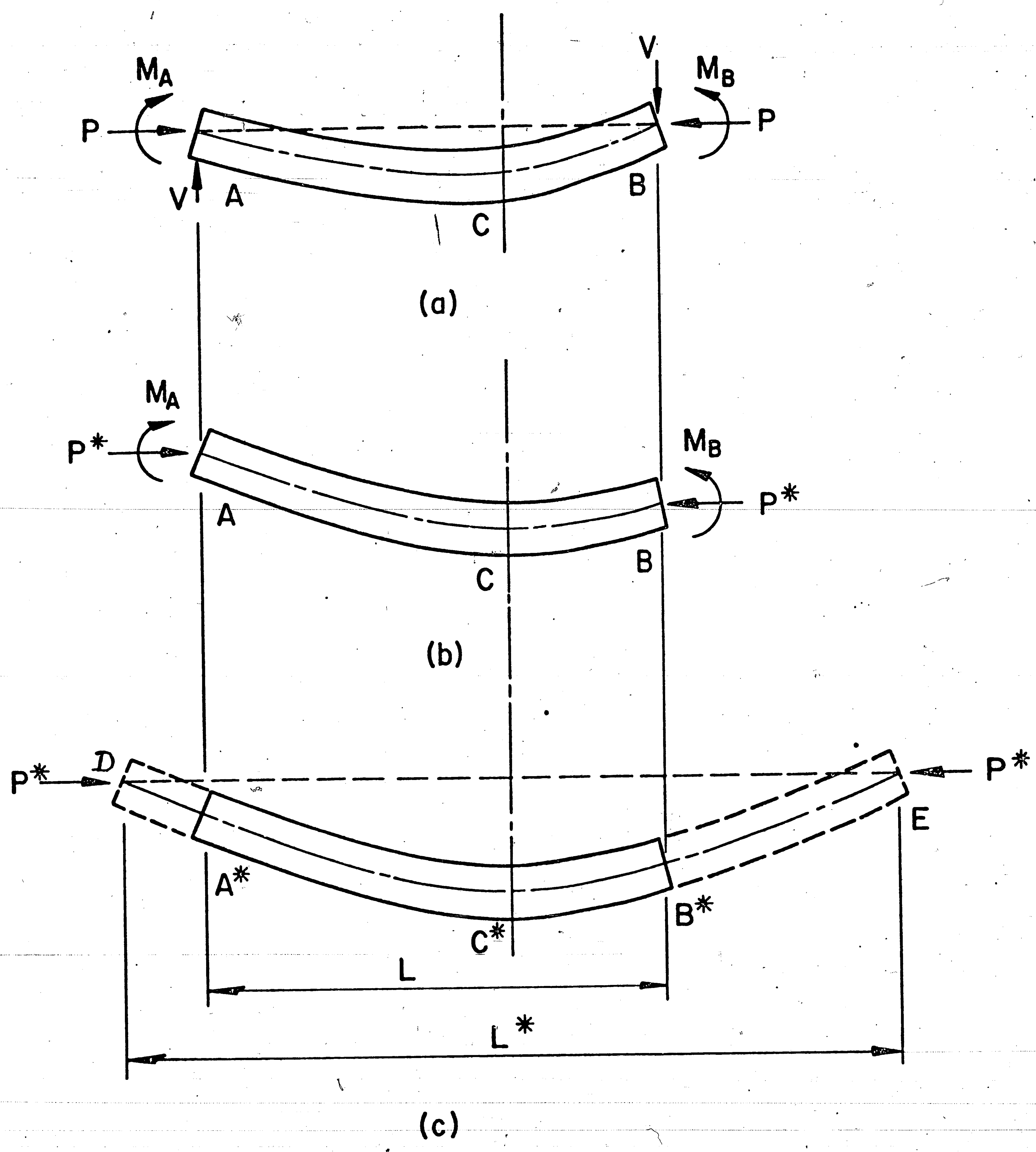


Fig. 3 Beam-Column and Its Equivalent Column

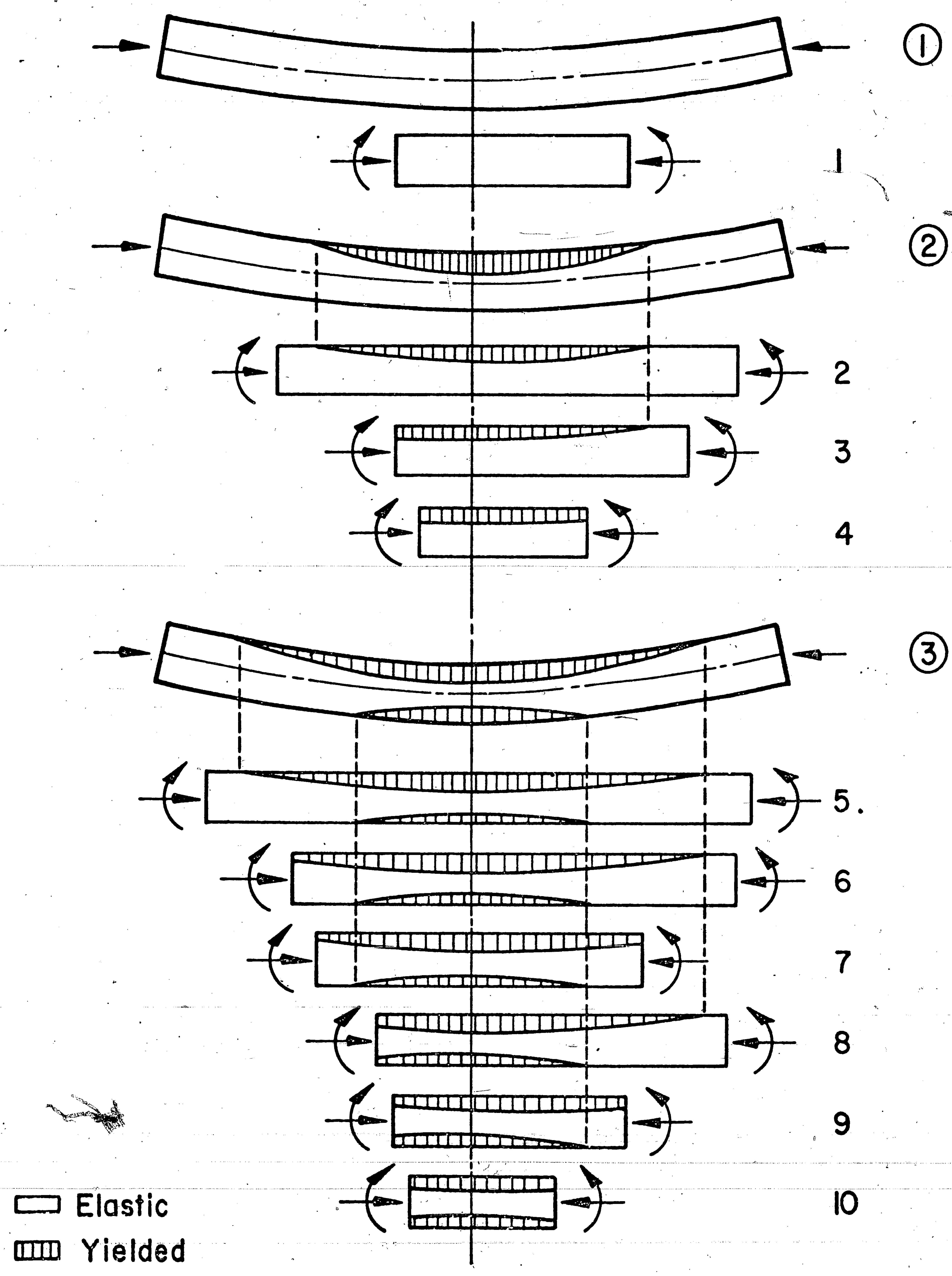


Fig. 4 Types of Elastic-Plastic Beam-Columns

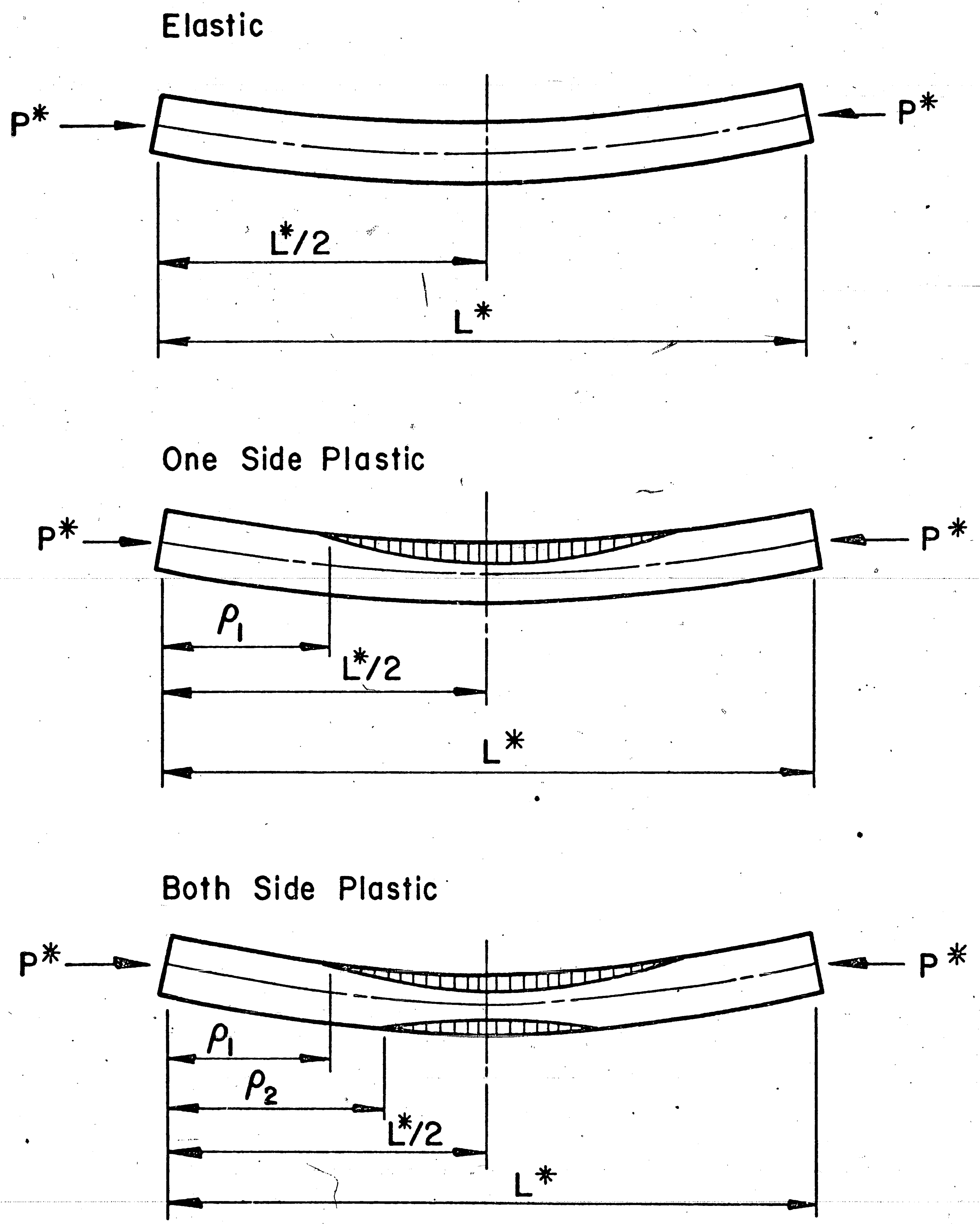


Fig. 5 Types of Equivalent Columns

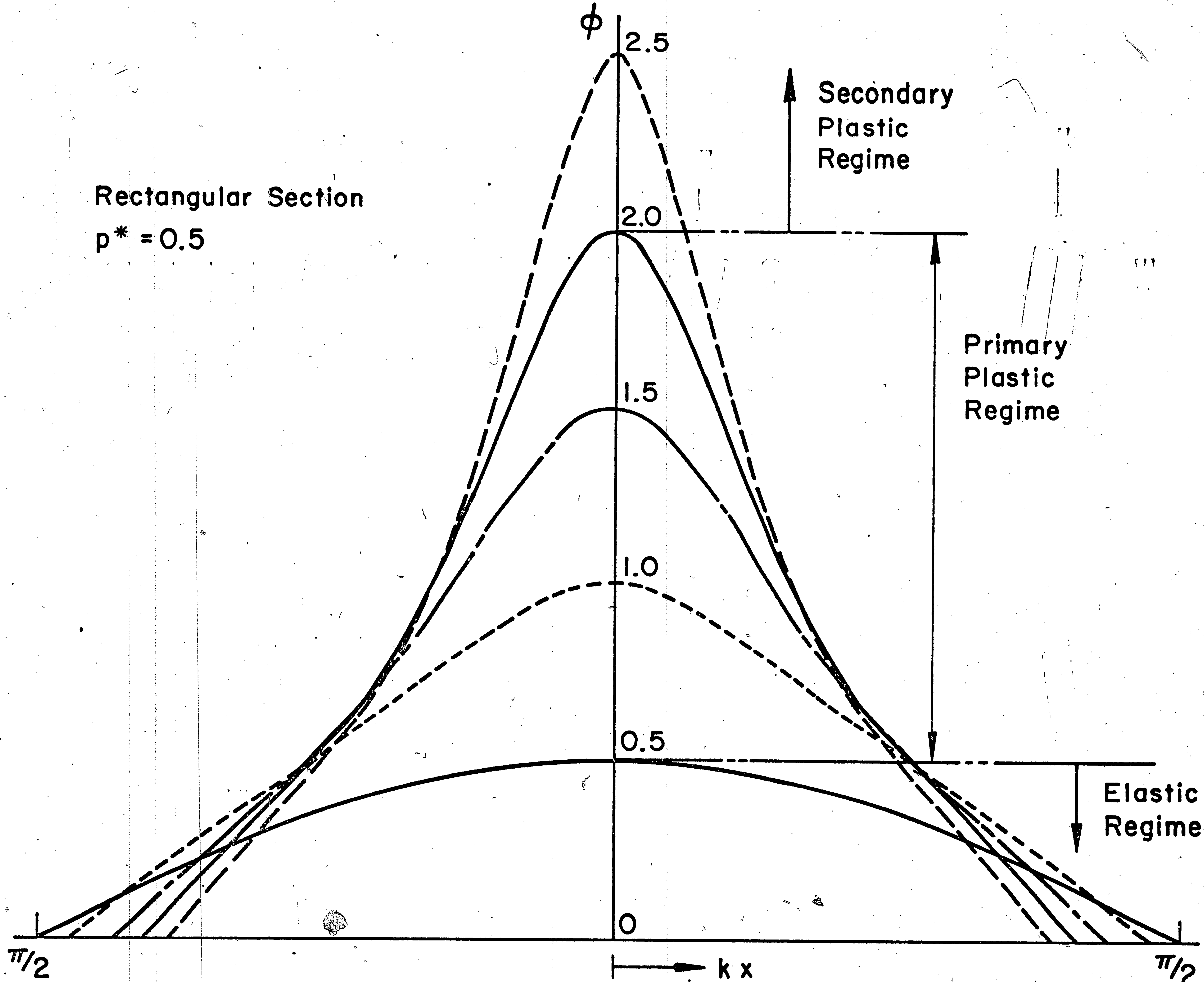


Fig. 6 An Example of Column Curvature Curves

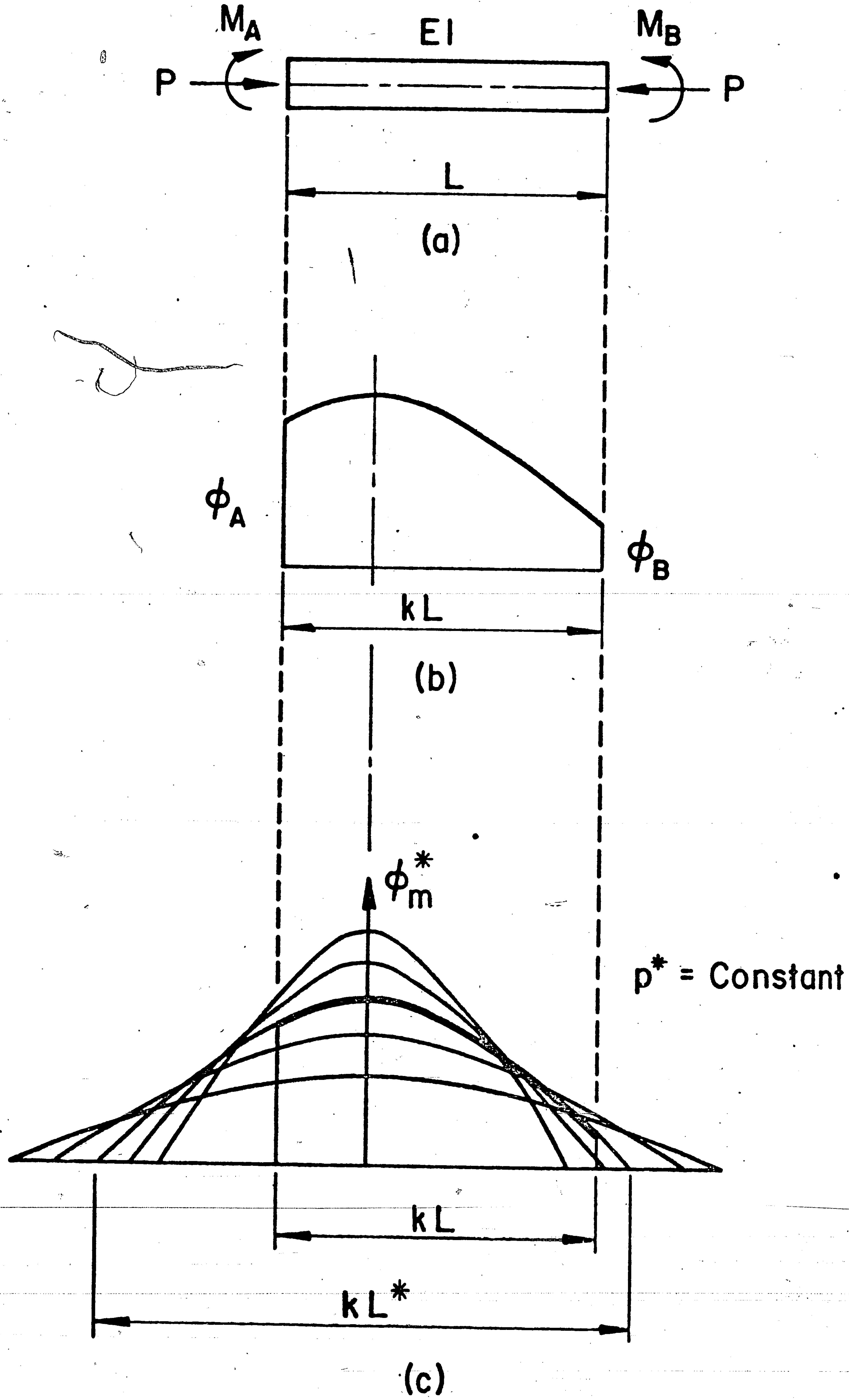


Fig. 7 Application of Column Curvature Curves

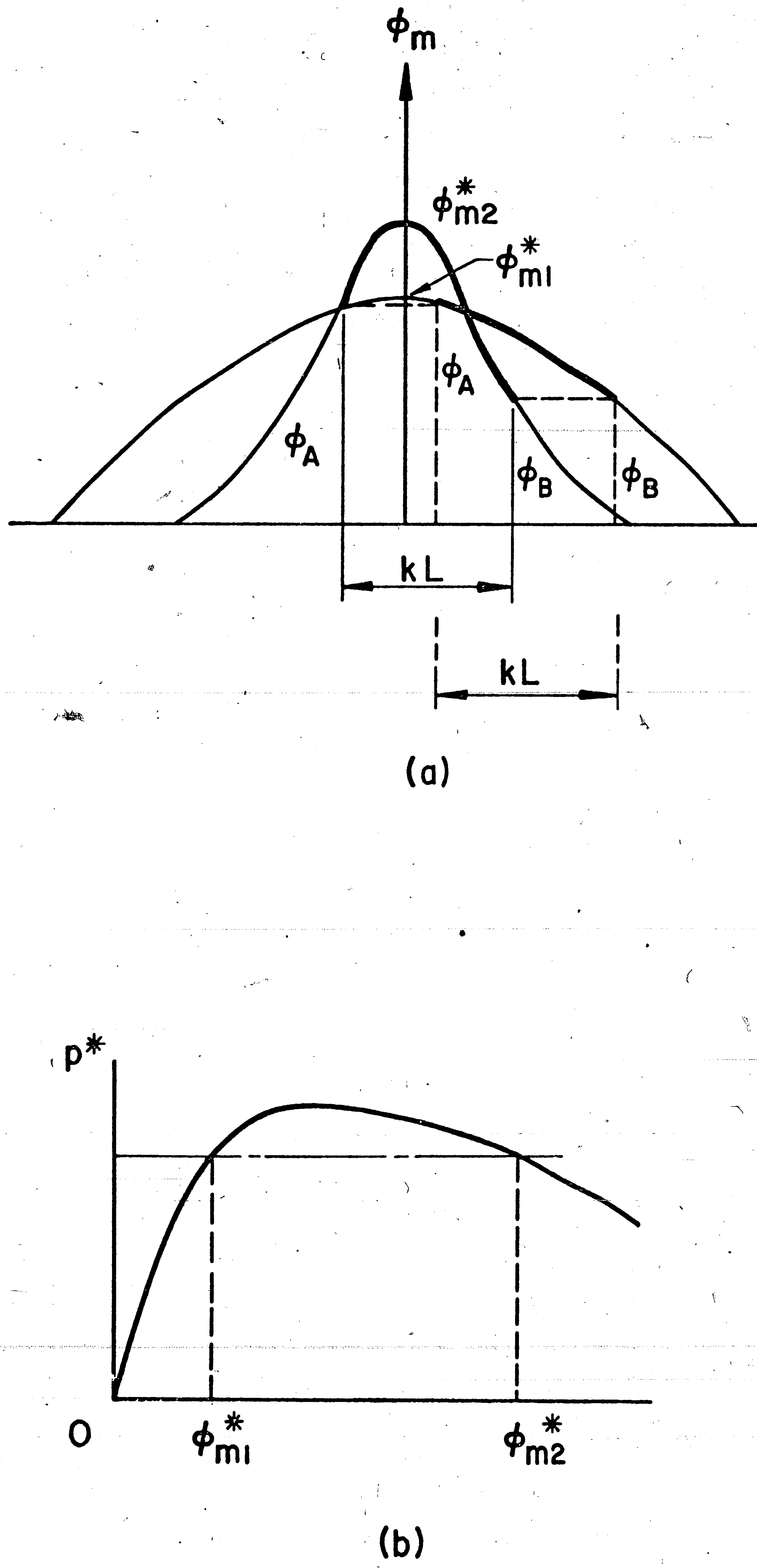


Fig. 8 Loading and Unloading

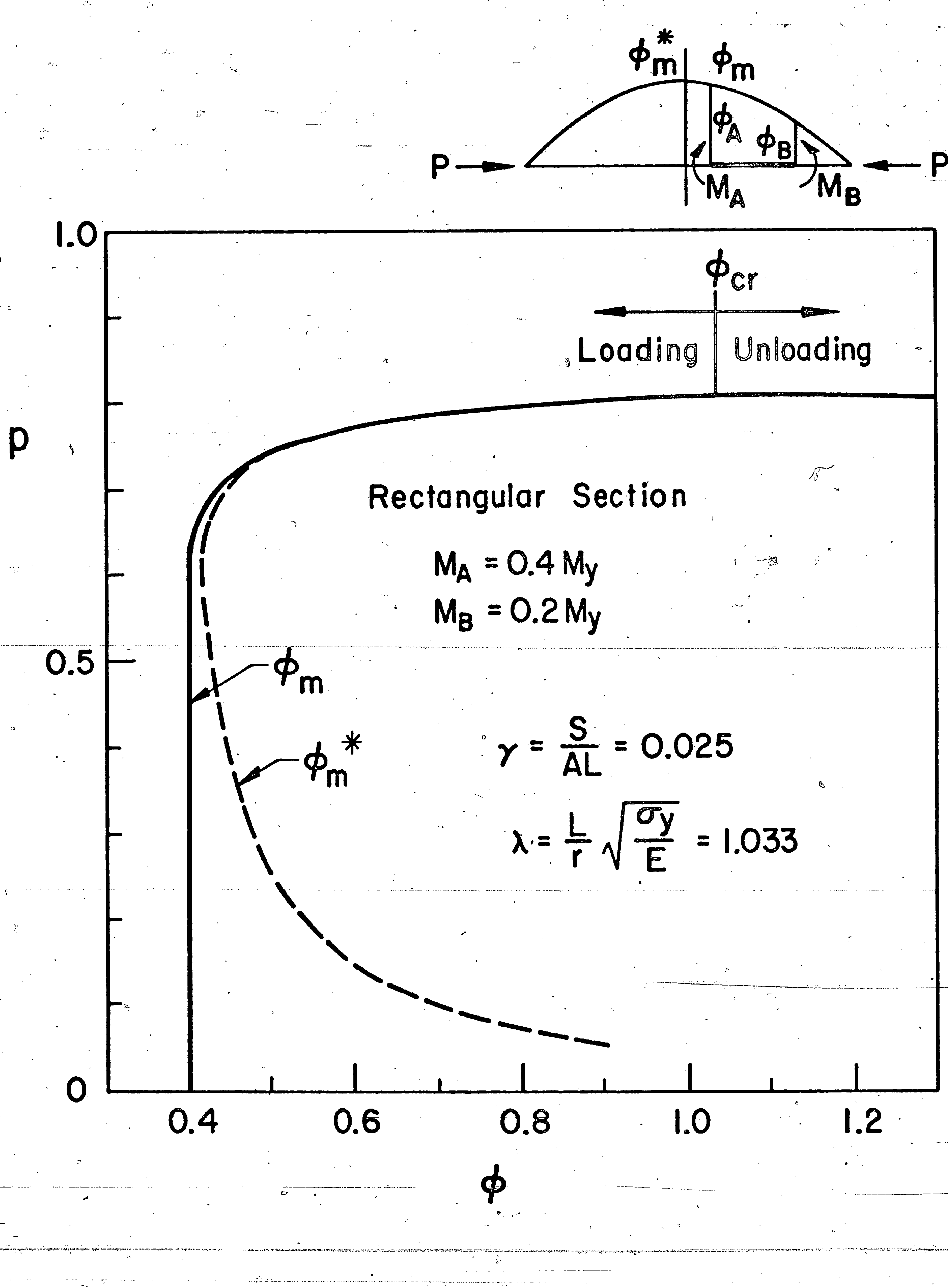


Fig. 9 Comparison of Maximum Curvatures φ_m and φ_m^*

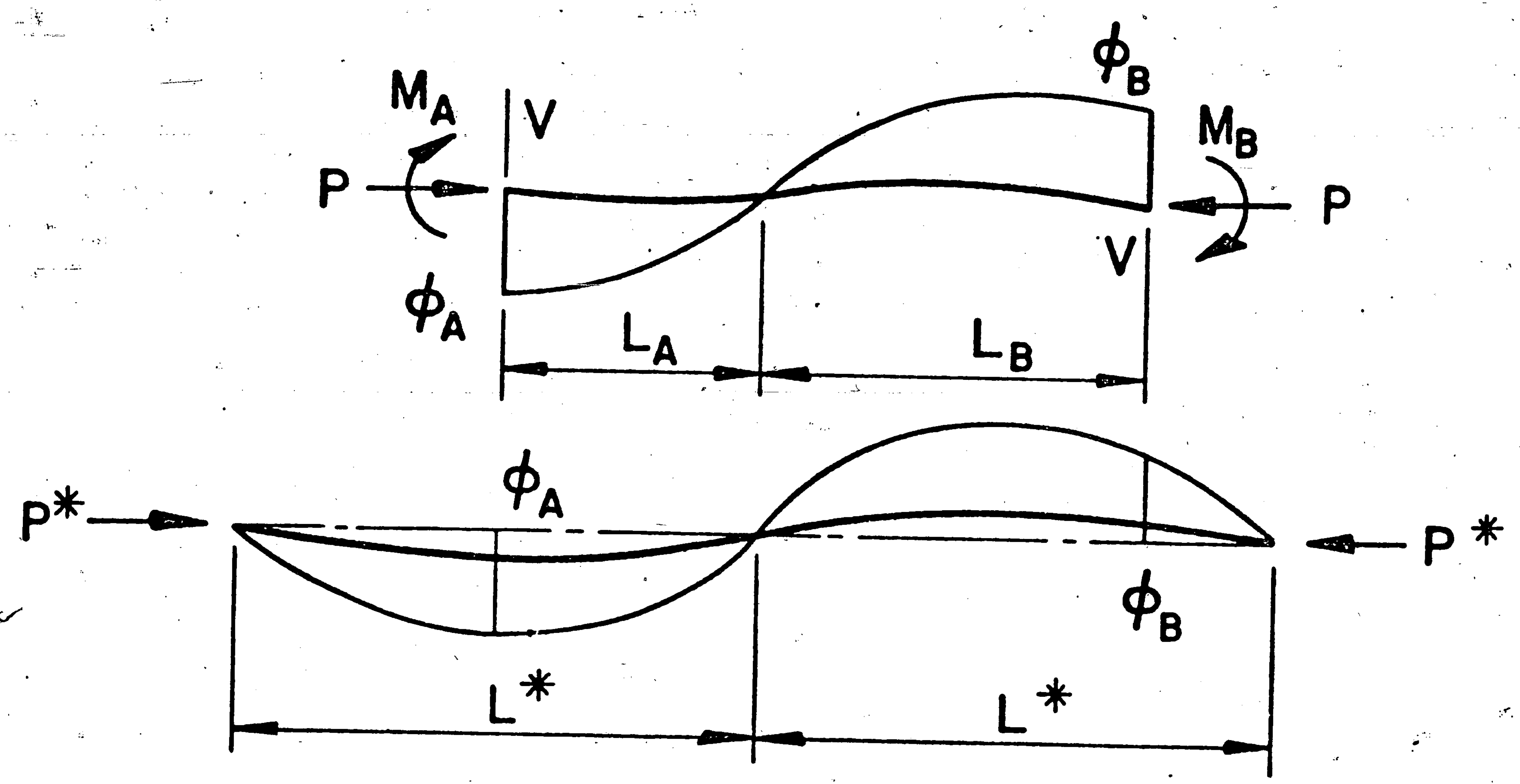


Fig. 10 Beam-Column with Double Curvature

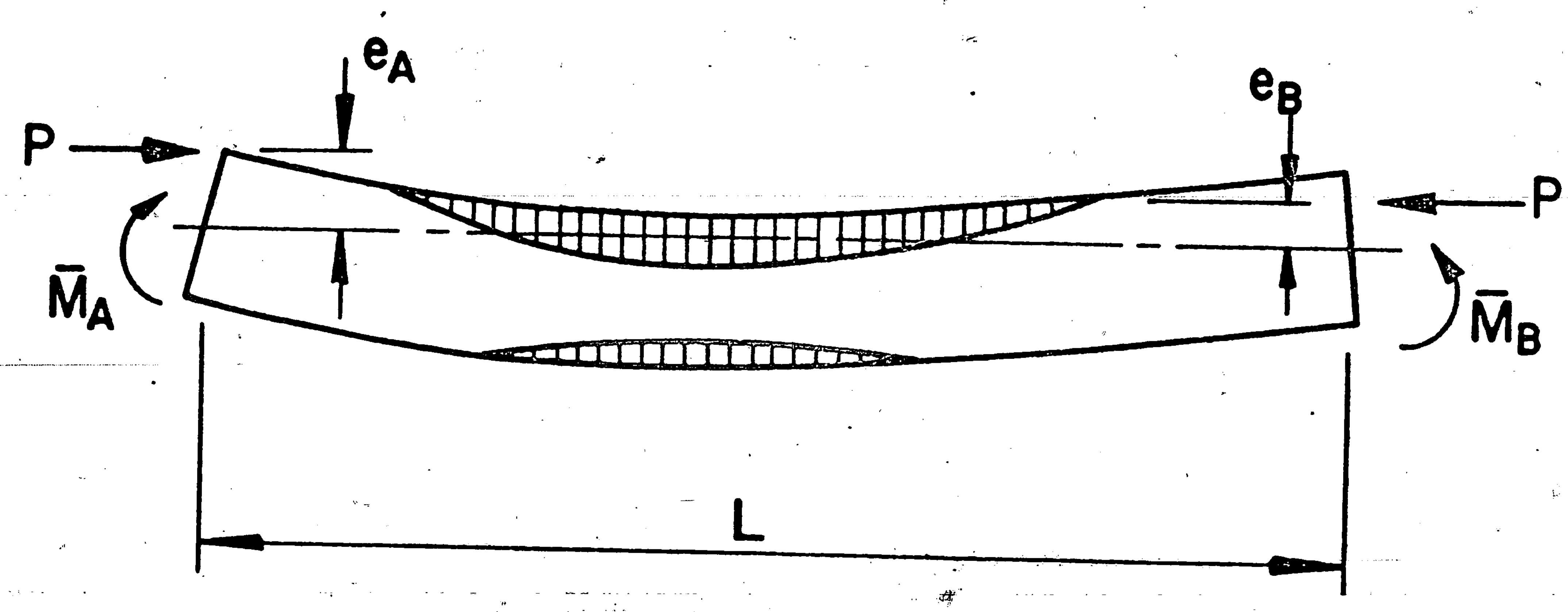


Fig. 11 Loading Condition of Beam-Column

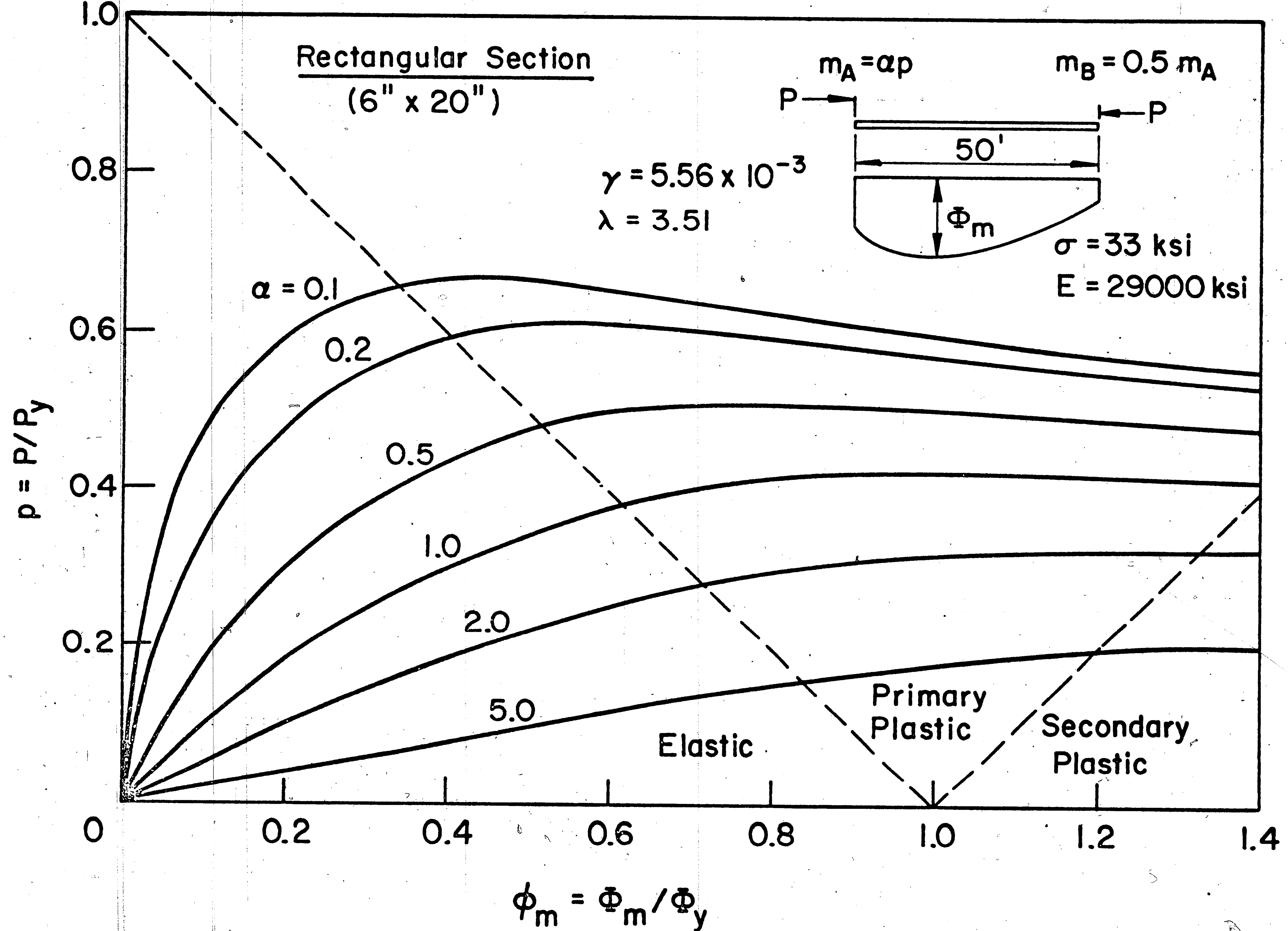


Fig. 12 Maximum Curvature

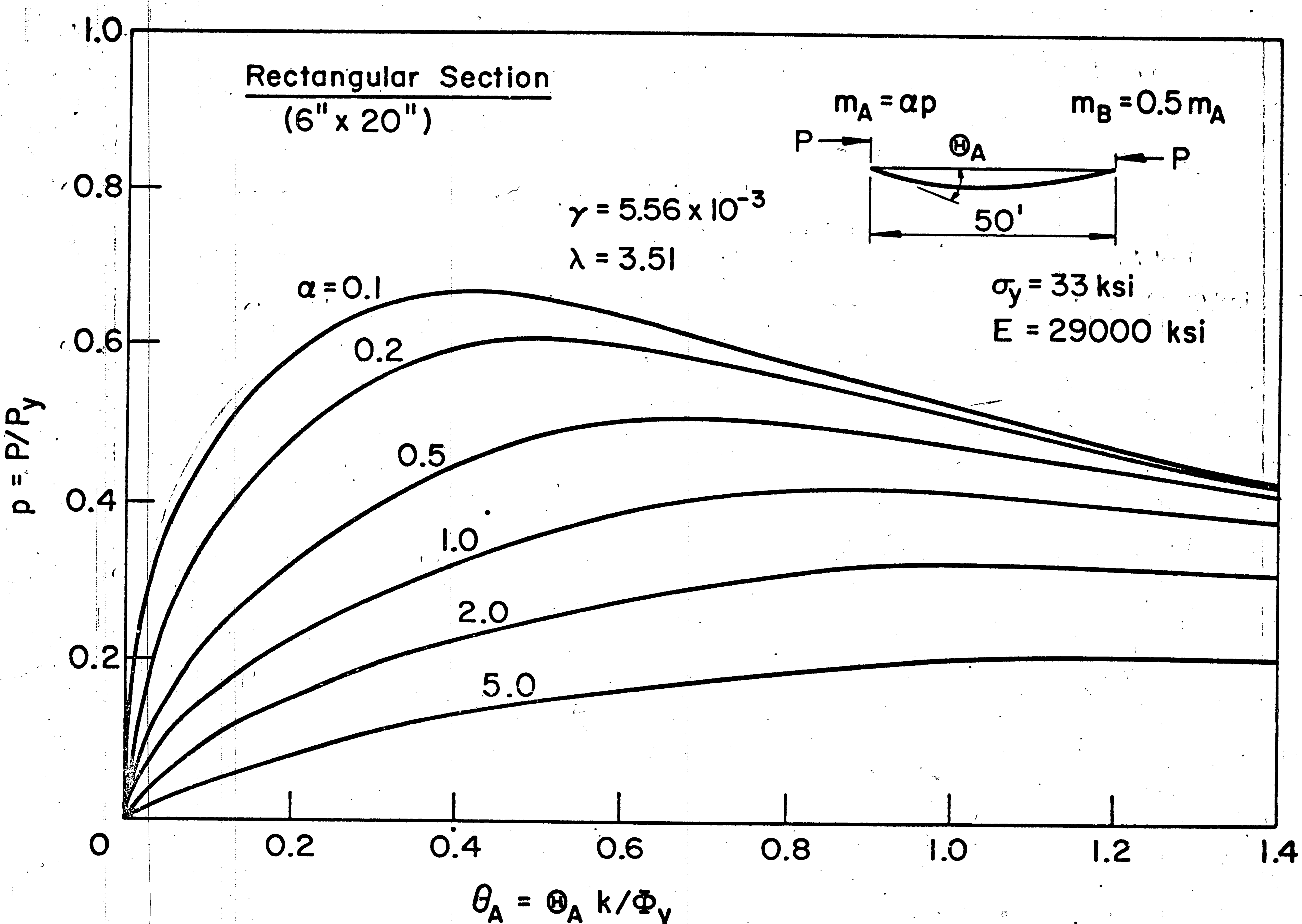


Fig. 13 End Rotation

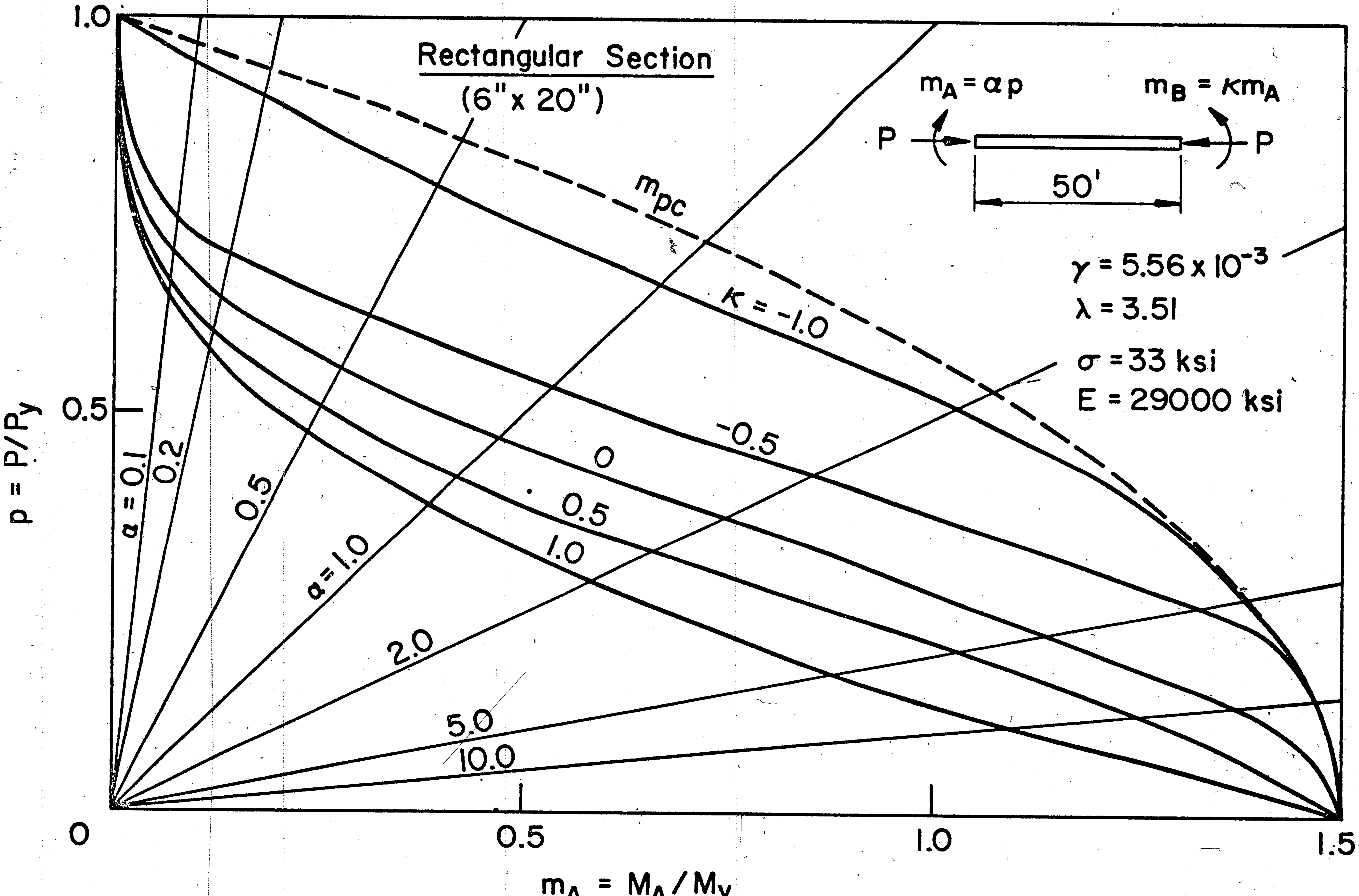


Fig. 14 Interaction Curves

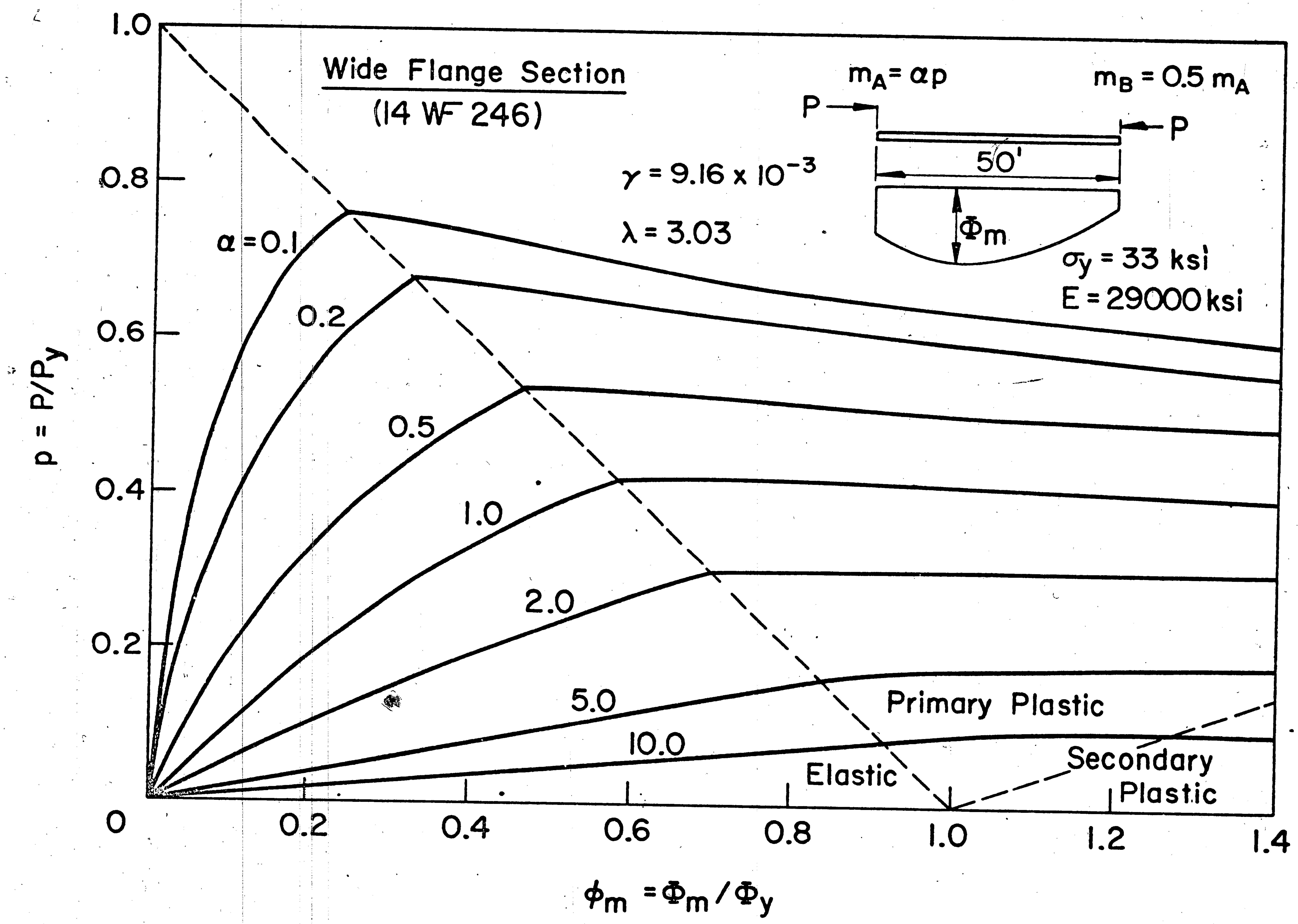


Fig. 15 Maximum Curvature

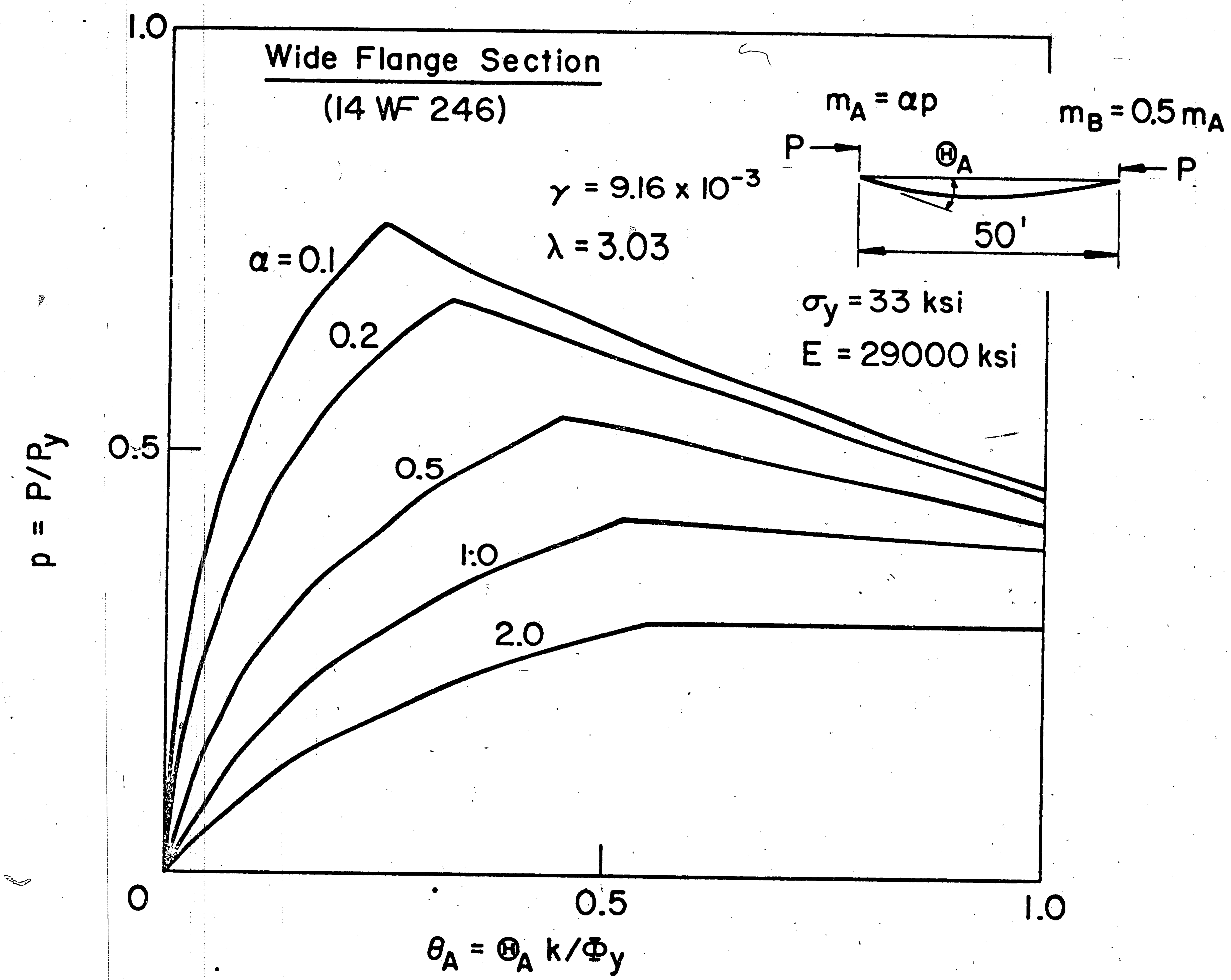


Fig. 16 End Rotation

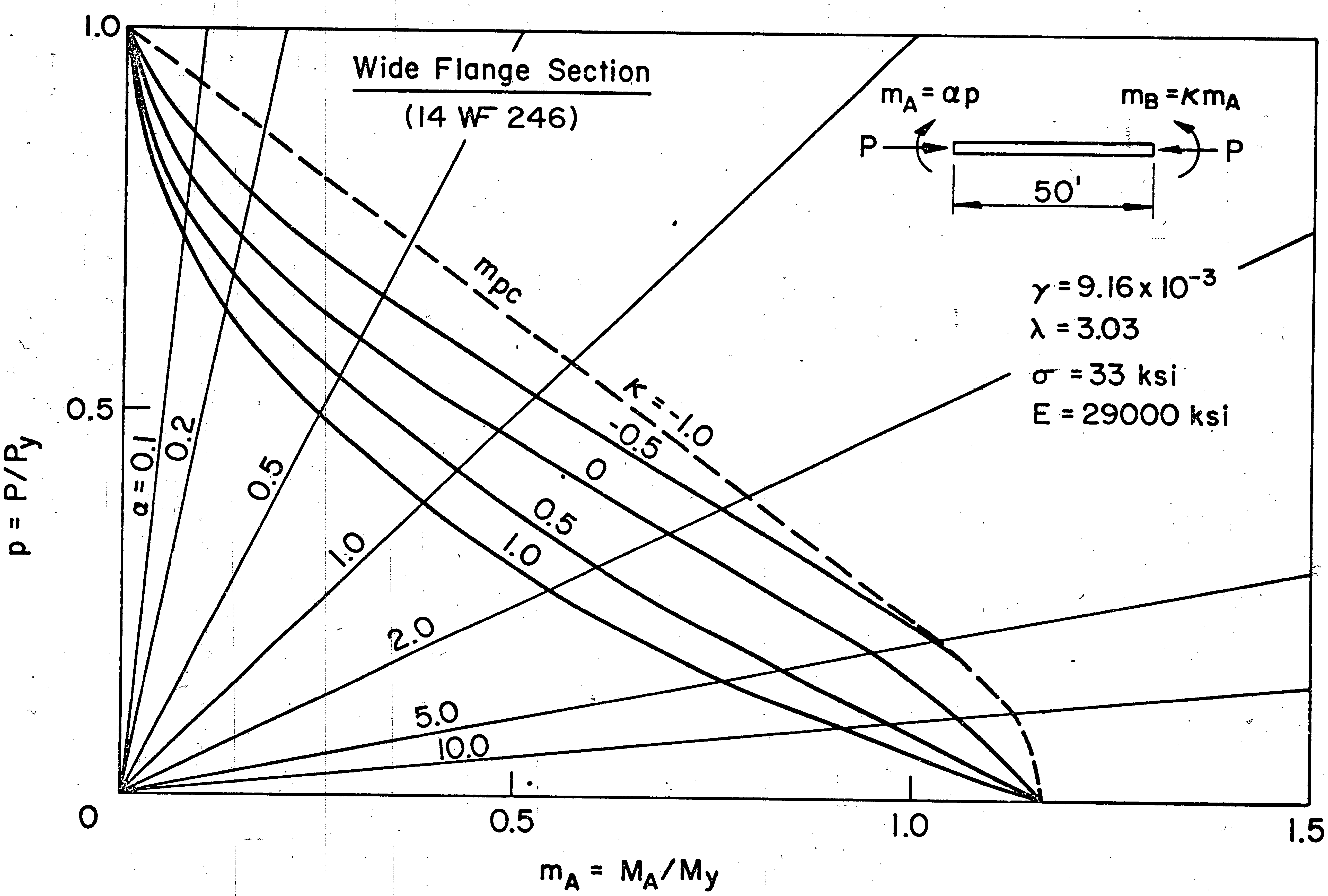


Fig. 17 Interaction Curves

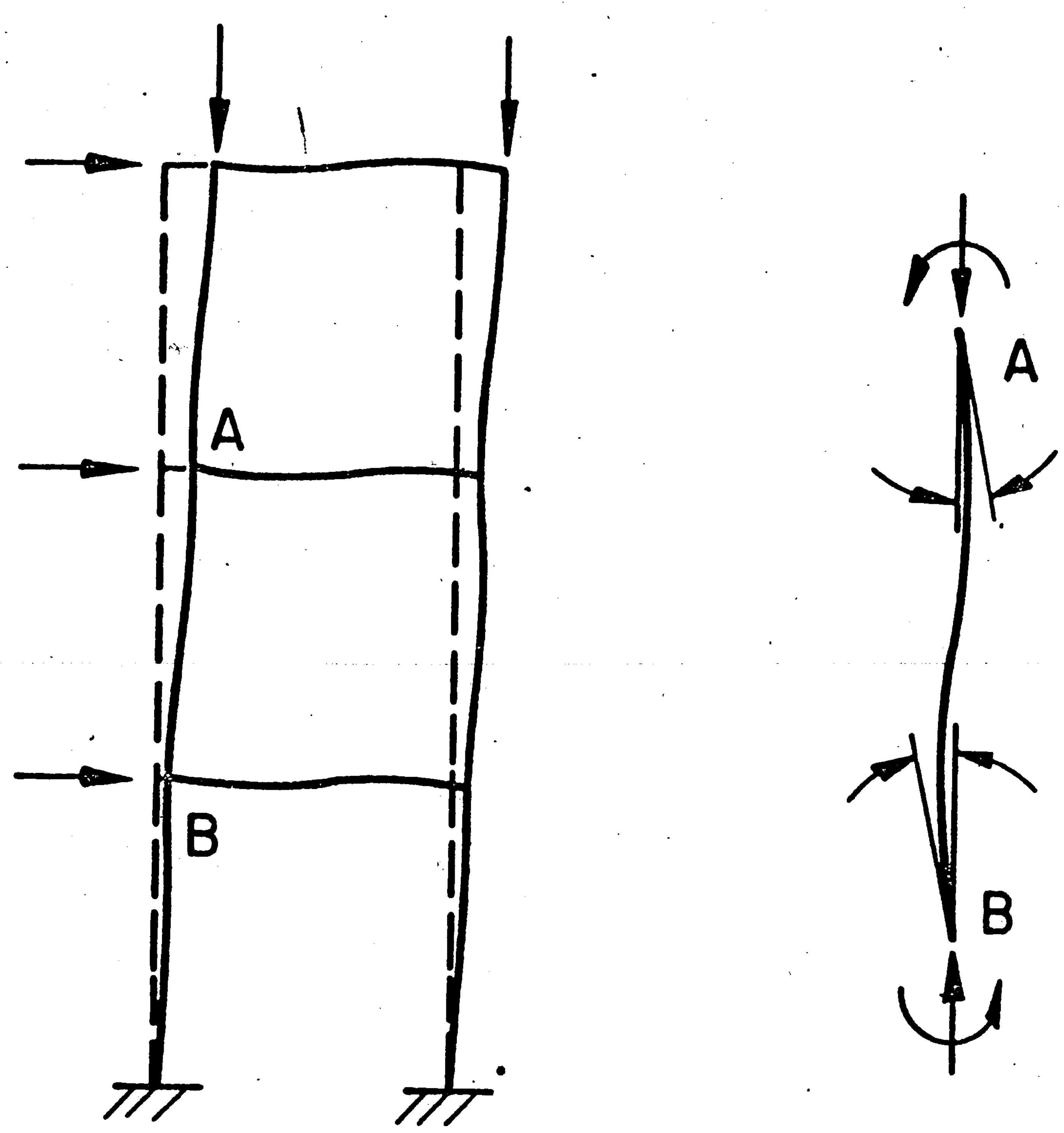


Fig. 18 Beam-Columns as Members of a Frame

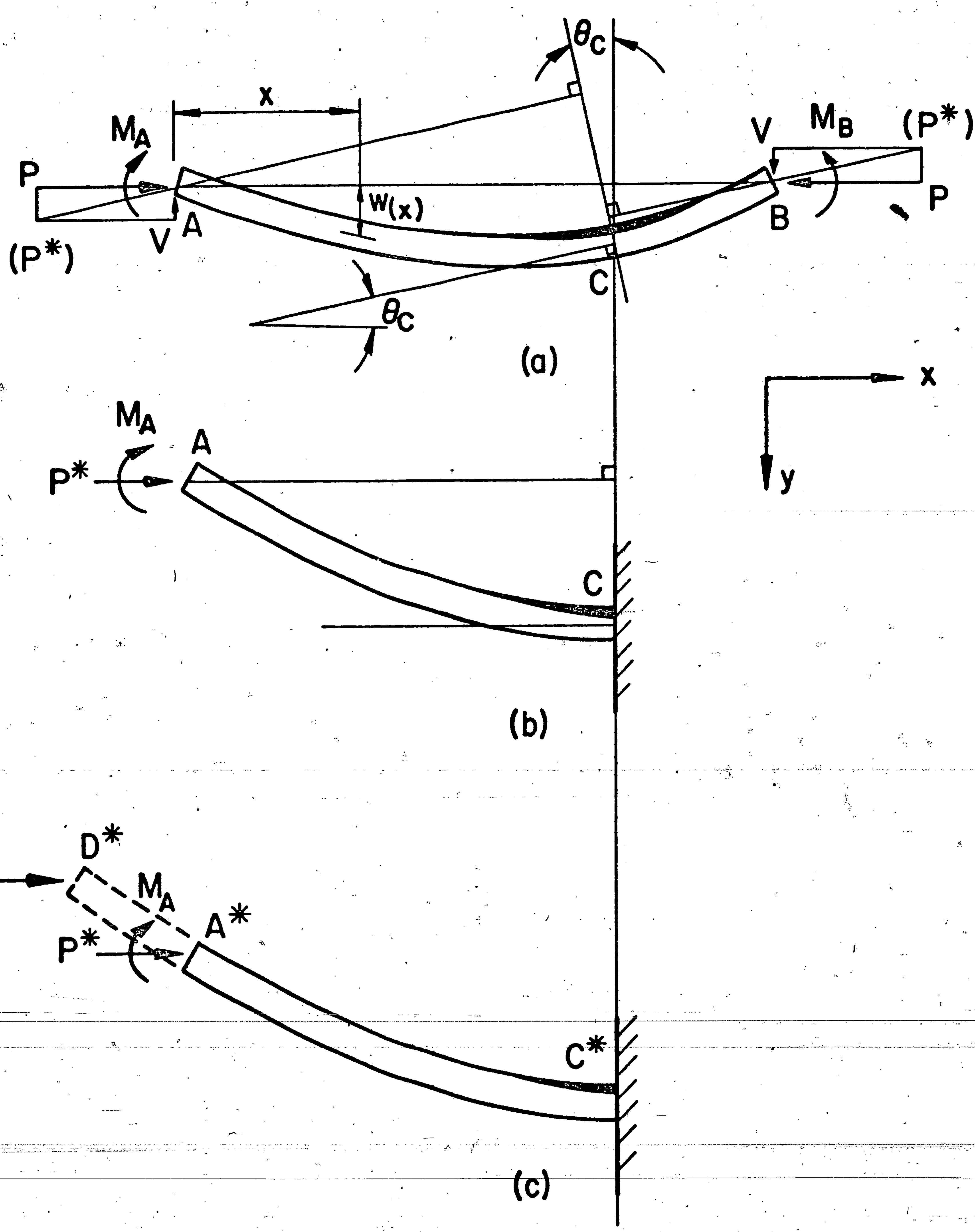


Fig. 19 Proof of Equivalent Column

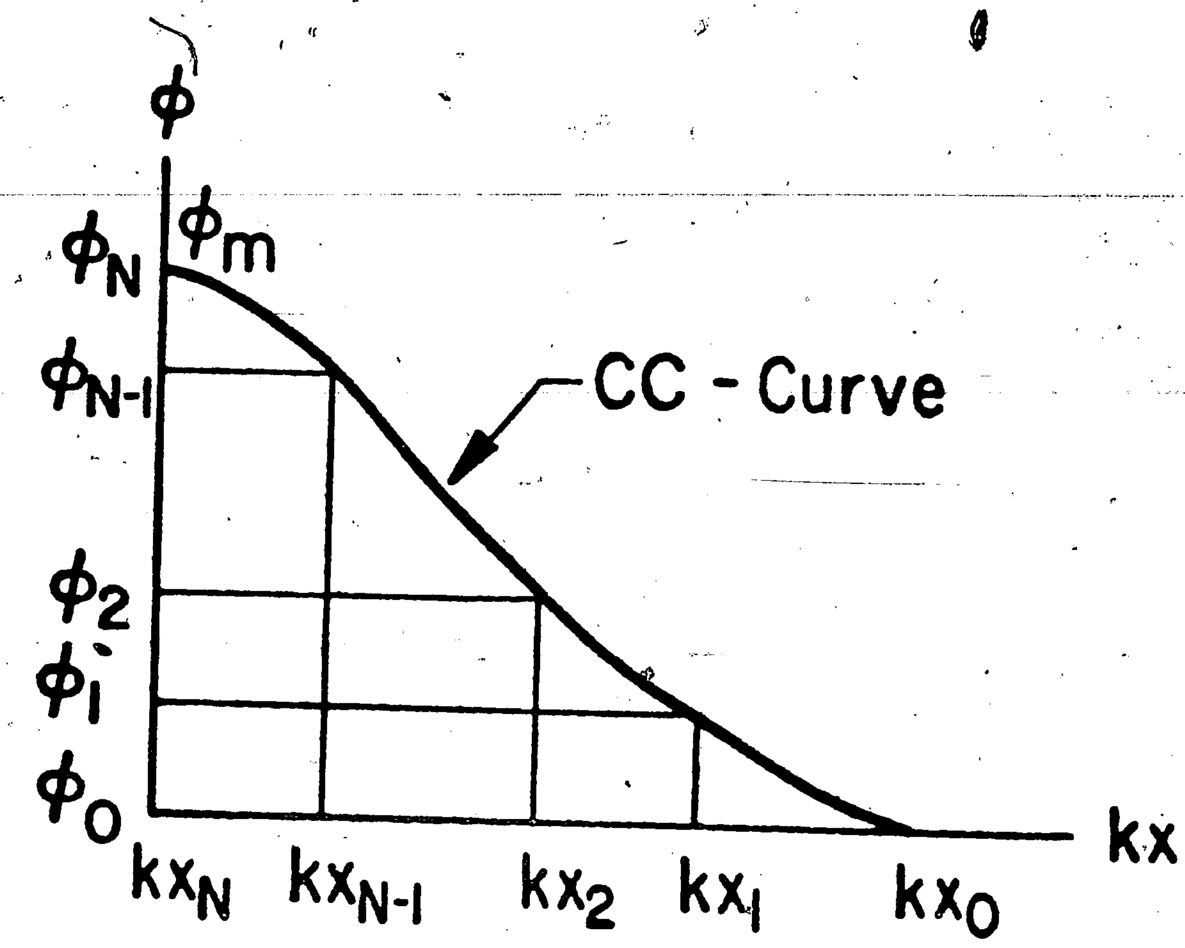


Fig. 20 Integration of Curvature Curve

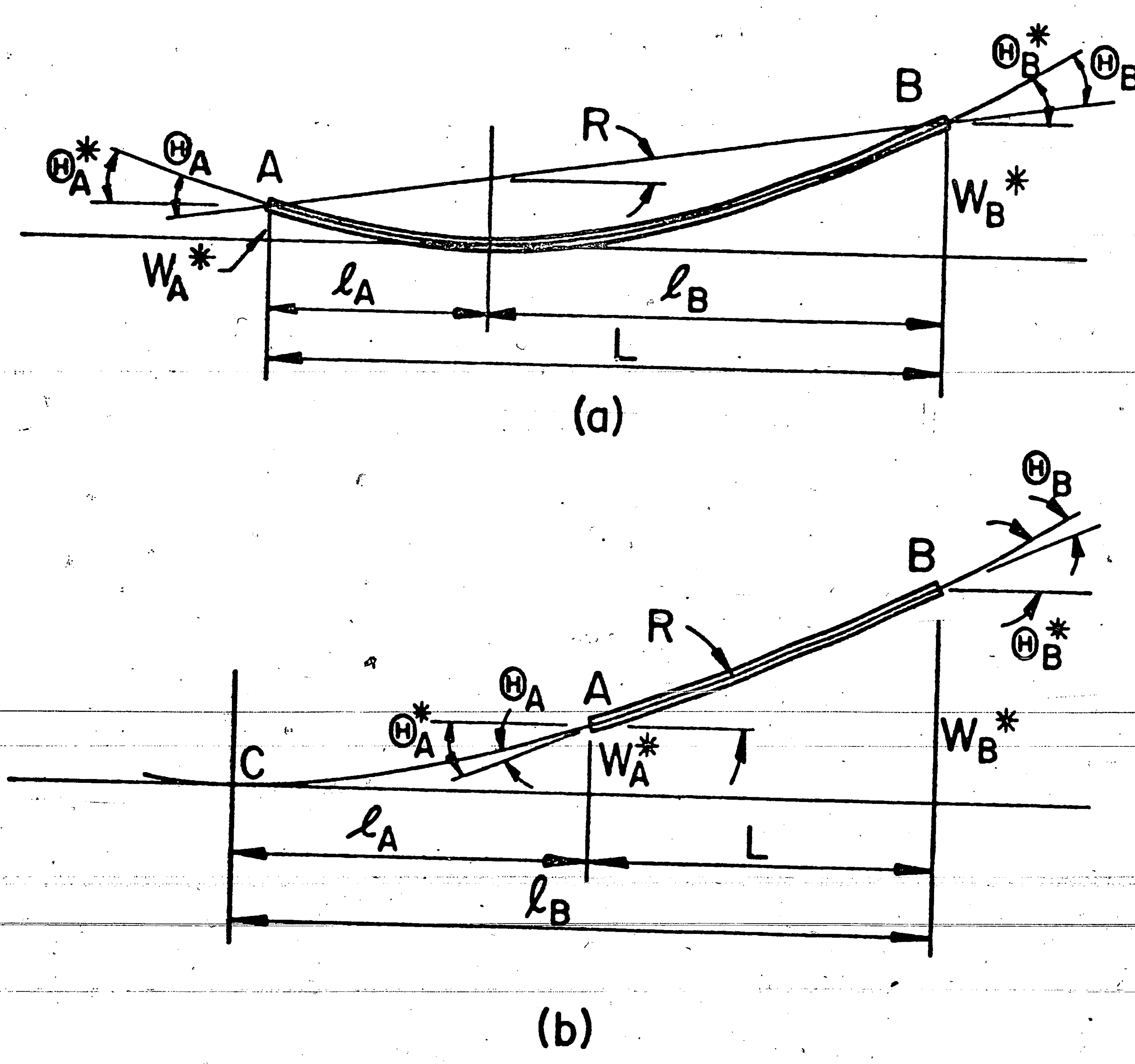
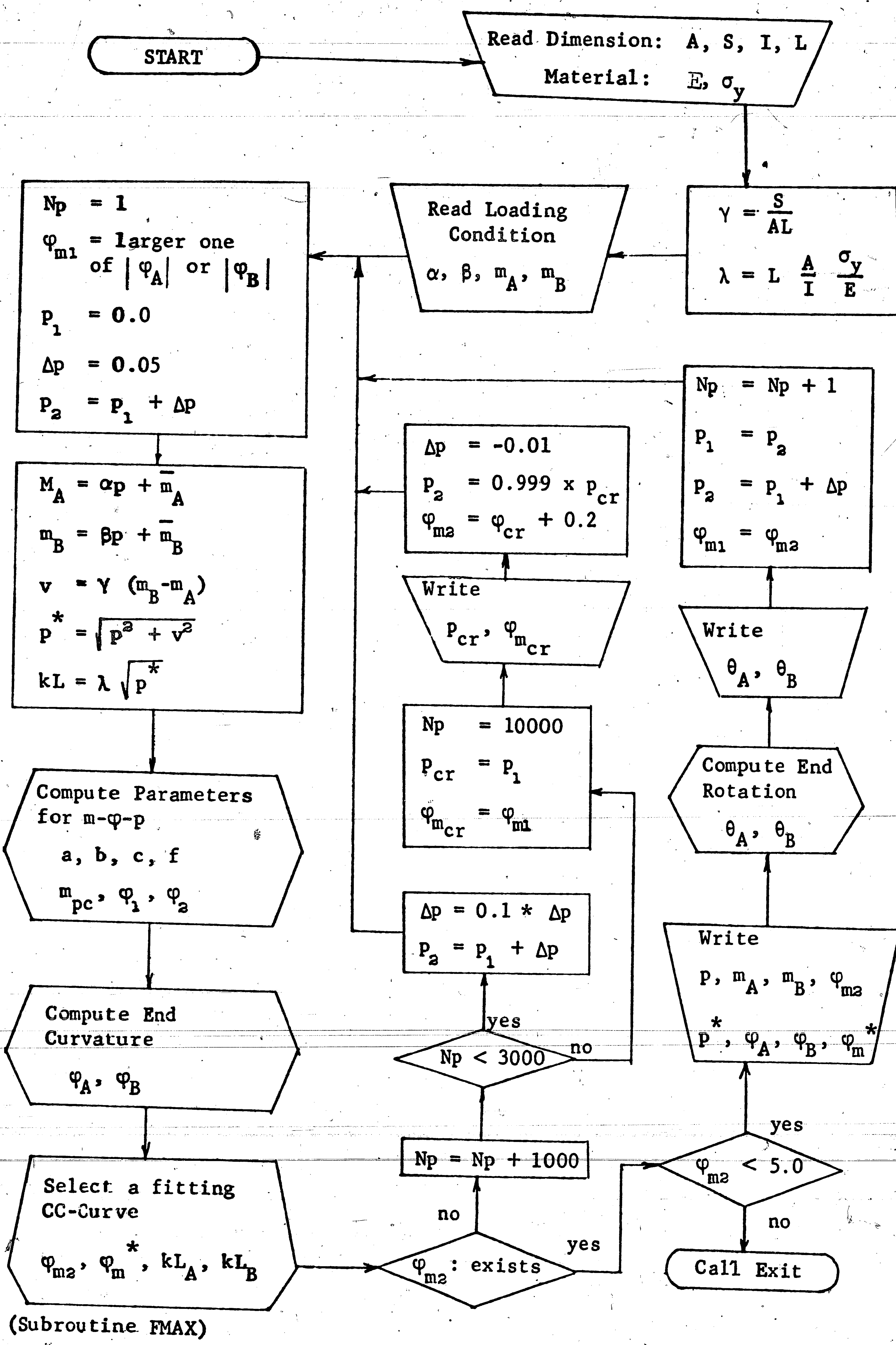
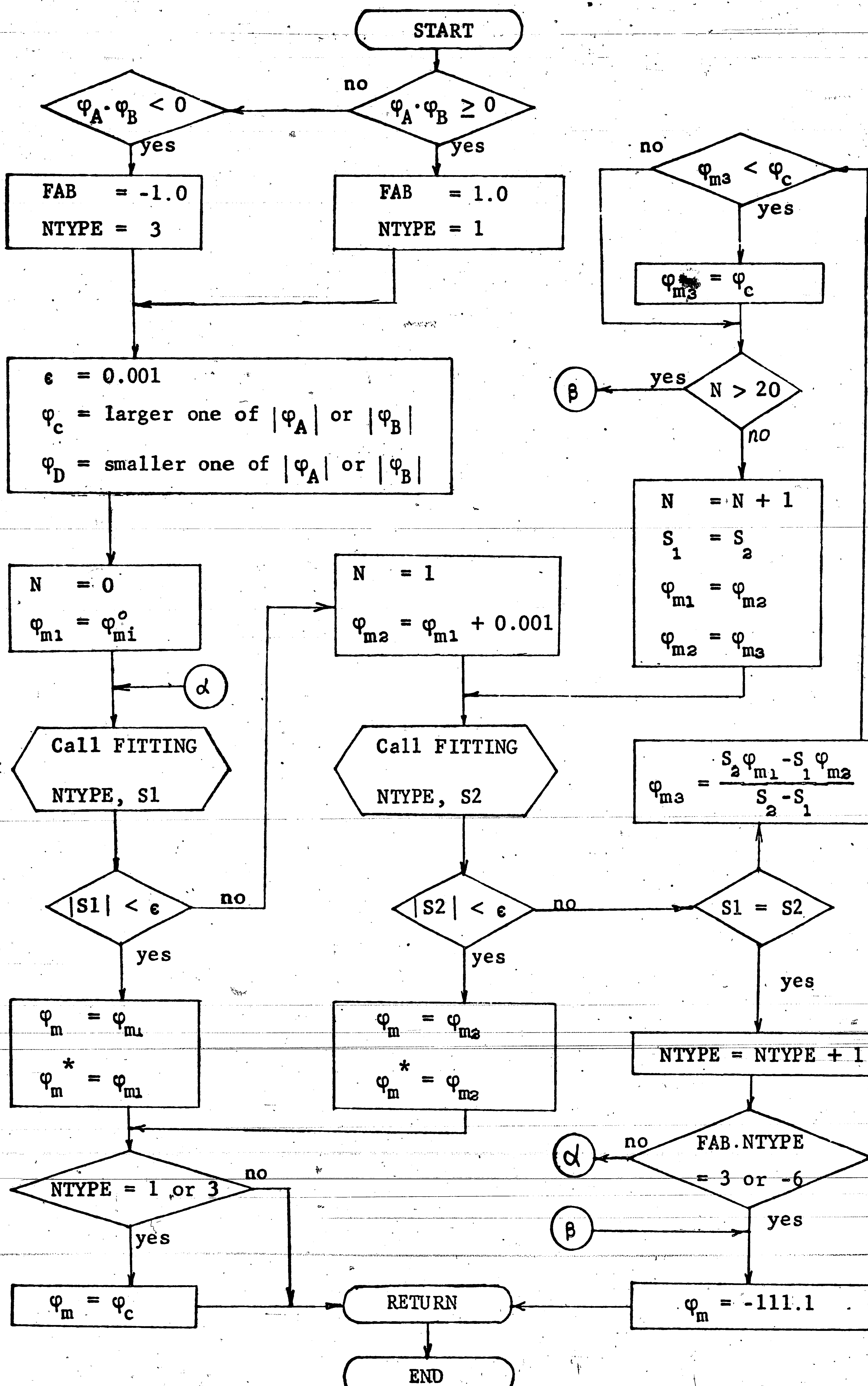


Fig. 21 End Rotation of Beam-Columns

Main Program

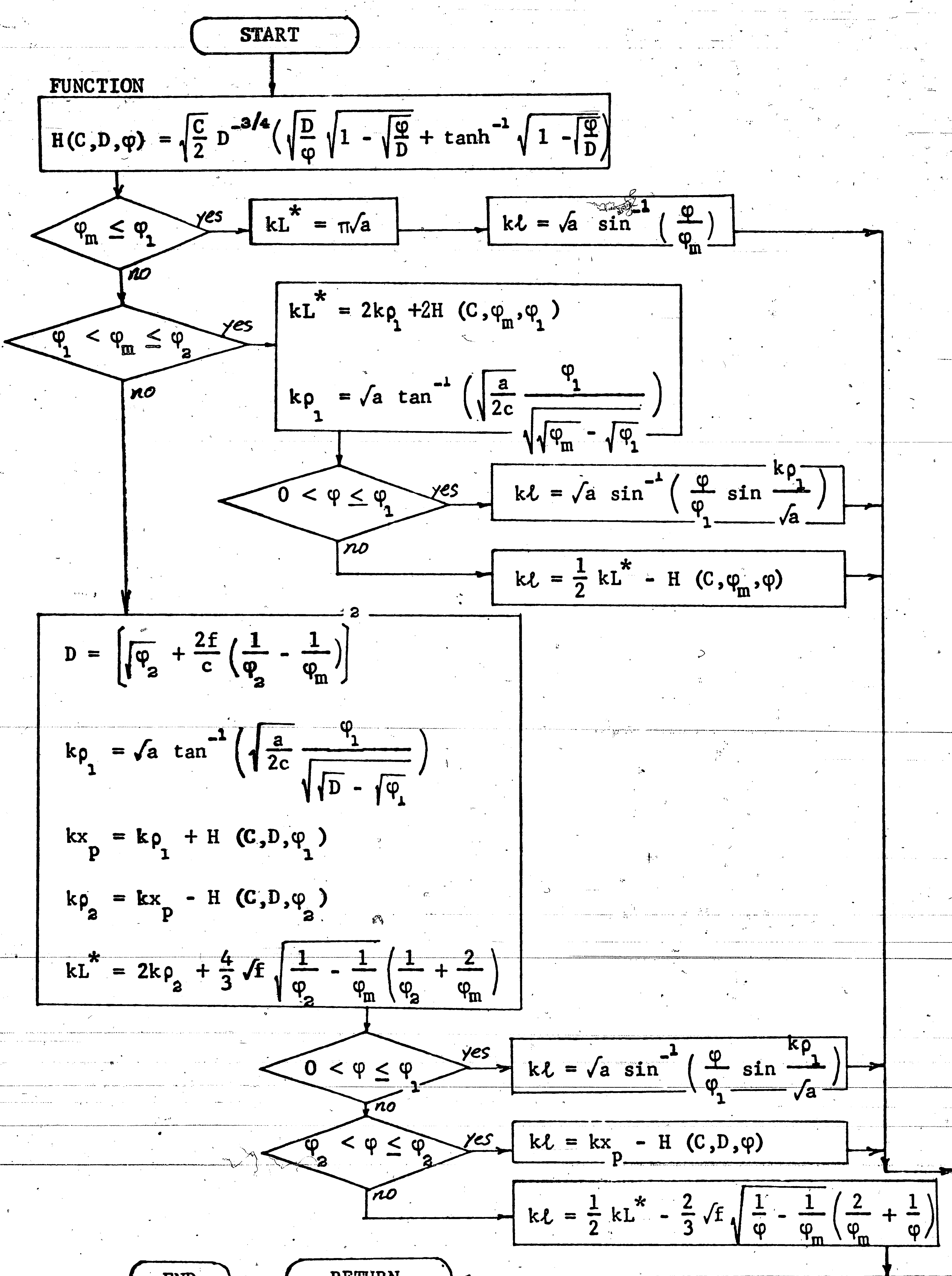


Subroutine FMAX



Subroutine SUBKL (kL^*, kL, φ_m, φ)

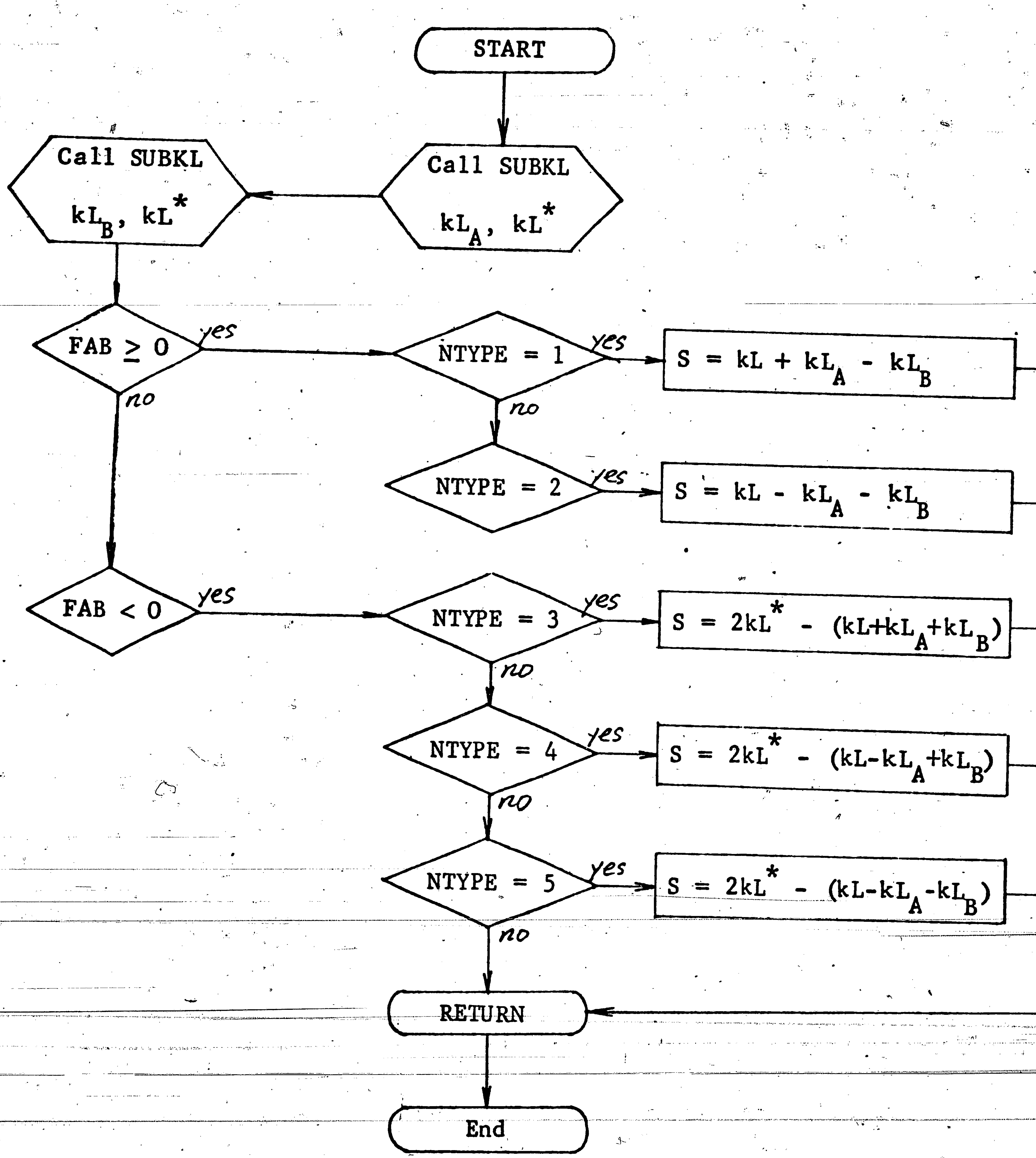
Common/a, b, c, f, m_{pc}, φ_1, φ_2



Subroutine FITTING(s)

Common/FAB, NTYPE, φ_c, φ_D-

/a, b, c, f, m_{pc}, φ_1, φ_2-



VITA

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