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Affine connections and groups of holonomy

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AFFINE CONNECTIONS

AND GROUPS OF HOLONOMY

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INTRODUCTION

In recent years many interesting and important topics on differential geometry have been developed quite well. One of these topics is the affine connections and groups of holonomy on differentiable manifolds. The purpose of this thesis is to make a primary study on this topic by using the technique of principal bundles in algebraic topology.

In §1 we first define a manifold $\{M, F_k\}$ to be a separable Hausdorff space M with a family F_k ($1 \leqq k \leqq \infty$) of real-valued functions defined on open subsets of M and satisfying certain conditions. This definition is identical with the usual one by means of overlapping neighborhoods. It is shown that it is possible to determine the family F_k from the knowledge of a certain sub-family of F_k. Finally the product space of two differentiable manifolds is defined.

In §2 we define first the equivalence classes of functions of F_k at a point p on a differentiable manifold $\{M, F_k\}$ and then the spaces of tangent covectors and vectors at p. It is proved that every tangent space of a differentiable manifold of dimension n is a vector space of dimension n.

Exterior differentiation and multiplication on the differential forms, and the properties of the Grassman ring are defined and given in §3.

In §4 on a differentiable manifold M a certain structure called an affine connection is first defined. This introduces the covariant differentiation of tensor fields, the torsion tensor and the curvature tensor.

The principal bundle B on a differentiable manifold M is defined in §5. Equation of structure and Bianchi identities are derived. Necessary and sufficient conditions for a system of linear differential forms in B to define an affine connection on M are obtained, and a geometrical interpretation of an affine connection in terms of the relationship between M and B is given.

In §6 it is proved that an affine connection is locally flat, if and only if the torsion tensor and the curvature tensor vanish. A discussion on frames and principal bundles leads to the definition of the group of holonomy.

1. DIFFERENTIABLE MANIFOLDS

DEFINITION 1.1. *A differentiable manifold is a separable Hausdorff space* M *with a family* F_k $(1 \leq k \leq \infty)$ *of real-valued functions, defined on open subsets of* M *such that the following conditions are satisfied:*

1) *Every function* f, *whose domain* U *of definition on* M *is the union of a family of open sets* U_α, *belongs to* F_k, *if and only if its restriction* $f|U_\alpha$ *to each* U_α *belongs to* F_k.

2) *For each point* $p \in M$ *there is a neighborhood* U *of* p *and a homeomorphism* $h : U \rightarrow h(U) \subset E^n$, *where* E^n *is an* n-*dimen-*

sional Euclidean space, such that the family F_k of functions defined in an open subset V, $p \in V \subset U$, is identical with the family g o h, where g runs over all functions of class k in h(V). (A function in an open subset V' of E^n is said to be of class k, if it has partial derivatives of order $\leqq k$ at every point of the subset, and those of order k are continuous.)

The functions of the family F_k are called functions of class k on M, and are said to define a differentiable structure of class k on M. The differentiable manifold and the Hausdorff space M are said to be of dimension n. The space M is called the underlying topological space. The differentiable manifold will be denoted by $\{M, F_k\}$, and also by M when ever there is no danger of confusion. If we let g run only over the analytic functions in h(V), that is, all functions which at every point of their domains of definition can be expanded into convergent power series, then the manifold to be denoted by $\{M, F_\omega\}$ is called analytic.

Let h be defined by

(1.1) $$h(q) = (h'(q), \ldots, h^n(q)), \quad q \in V,$$

then we can write

(1.2) $$g \circ h(q) = g(h'(q), \ldots, h^n(q)), \quad q \in V.$$

The functions $h'(q), \ldots, h^n(q)$, which obviously belong to F_k, are called the local coordinates in U or the local coordinates at the point p.

Suppose that, instead of U, there exists a different neighborhood U' of p, with the homeomorphism $h' : U' \to h'(U')$, having the same properties. By restricting to a smaller neighborhood when necessary, we assume $U = U'$. Then $h' \circ h^{-1}$ is a homeomorphism of $h(U)$ onto $h'(U)$ and can be defined by the equations

$$h'^i = h'^i(h', \ldots, h^n), \qquad i = 1, \ldots, n, \tag{1.3}$$

where the functions in the right-hand side are of class k, since $h'^1(q), \ldots, h'^n(q)$ are functions of class k in U. A function of class k in V can now be written in one of the two forms:

$$f = g \circ h = g' \circ h', \tag{1.4}$$

and we have

$$g = g' \circ h' \circ h^{-1} \tag{1.5}$$

or

$$g(h', \ldots, h^n) = g'(h'^1(h', \ldots, h^n), \ldots, h'^n(h', \ldots, h^n)). \tag{1.6}$$

This formula gives the relation between two "representations" g, g' of a function of class k in V in terms of two systems of local coordinates.

From the family of functions F_k we can select a subfamily F'_k having the following property:

(k) To every point $p \in M$ there is a neighborhood U containing p in which there are n functions $h', \ldots, h^n$ of F'_k such that

1) The mapping $h : U \to h(U)$ of U onto an open subset $h(U)$ of E^n defined by

$$h(q) = (h'(q), \ldots, h^n(q)), \qquad q \in U$$

is a homeomorphism.

2) If $f \in F'_k$ is defined at p, there is an open subset V of U, $p \in V$, such that $f|V$ is identical with $g \circ h$, g being a function of class k in $h(V)$.

This shows that if we know F'_k we can determine F_k by the following theorem:

THEOREM 1.1. Let M be a separable Hausdorff space with a family F'_k of real-valued functions defined on open subsets of M such that property (k) holds. There exists one and only one minimal family of functions F_k defined on open subsets of M, which contains F'_k as a sub-family, such that $\{M, F_k\}$ is a differentiable manifold.

Let $f \in F'_k$ defined on an open subset $U' \subset M$. Let $p \in U'$ with U as its neighborhood having property (k). Suppose there is a neighborhood $V \subset U \cap U'$ such that $f|V$ is identical with $g \circ h$, where g is a function of class k in $h(V)$. If f has this property at every point $p \in U'$, then it belongs to any family F_k of functions, which contain F'_k, and defines a differentiable structure of class k on M.

Let F_k be the family of all these functions f. Then it is easy to verify conditions 1) and 2) of definition (1.1), so that $\{M, F_k\}$ is a differentiable manifold of class k.

COROLLARY 1.1. A differentiable structure of class k on M defines a (minimal) differentiable structure of class

$1<k$ by the condition that the functions of class k are also functions of class 1.

We can define the family of functions F_k in the topological sense, that there exists a countable open covering $\{U_\alpha\}$ of M such that to each α there are n functions $u_\alpha^1, \ldots, u_\alpha^n$ of a family H_k, which also determine the differentiable structure, with U_α as their domain of definition and the mapping $h_\alpha : U_\alpha \to E^n$ defined by

$$h_\alpha(q) = h_\alpha^1(q), \ldots, h_\alpha^n(q), \qquad q \in U_\alpha$$

is a homeomorphism of U_α onto the open subset $h_\alpha(U_\alpha)$ of E^n. Also we might add that if $f \in H_k$ is defined at p and $p \in U_\alpha$, there exists an open subset $V \subset U_\alpha$ containing p such that $f|V$ is identical with $g \circ h_\alpha$, where g is a function of class k in $h_\alpha(V)$. From the above discussion we have:

THEOREM 1.2. The underlying space M of a differentiable manifold $\{M, F_k\}$ has a countable open covering $\{U_\alpha\}$ with the following properties:

1) To each α there is a homeomorphism $h_\alpha : U_\alpha \to E^n$.

2) If $p \in U_\alpha \cap U_\beta$, there exists a neighborhood V of p such that each coordinate of the point $h_\beta h_\alpha^{-1}(q)$, $q \in h_\alpha(V)$ is a function of class k in $h_\alpha(V)$, and the functional determinant of these n coordinate functions is $\neq 0$.

Conversely, given on a separable Hausdorff space M a countable open covering $\{U_\alpha\}$ and a homeomorphism $h_\alpha : U_\alpha \to E^n$ for each α, such that condition 2) is satisfied, there exists a uniquely determined differentiable structure of class k on M which admits the coordinate functions in $h_\alpha(U_\alpha)$ as functions of class k.

When a differentiable manifold has a covering $\{U_\alpha\}$ with the properties 1), 2) of the above theorem, then we call U_α the coordinate neighborhoods, and the coordinate $h_\alpha(p)$ relative to U_α are called the local coordinates of p.

DEFINITION 1.2. Given two differentiable manifolds $\{M, F_k\}$, $\{M', F'_k\}$, and a map $\varphi: M \to M'$. If the function $f' \circ \varphi / \varphi^{-1}(U')$ defined in the open subset $\varphi^{-1}(U')$ corresponding to a function $f'|U'$ in F'_k belongs to F_k for every f' of F'_k, then φ is called differentiable.

DEFINITION 1.3. Let $\{M, F_k\}$ and $\{M', F'_k\}$ be two differentiable manifolds of the same class k. Let $\pi : M \mathrm{x} M' \to M$, $\pi' : M\mathrm{x}M' \to M'$ be the projections defined respectively by

$$\pi(p,p') = p, \qquad \pi'(p,p') = p'.$$

To a function $f|U$ in F_k corresponds a function $f \circ \pi|\pi^{-1}(U)$, and similarly, to a function $f'|U'$ in F'_k corresponds a function $f' \circ \pi'|\pi'^{-1}(U')$. These functions define a differentiable structure of class k with the underlying space $M\mathrm{x}M'$. It is called the product space of the two given manifolds.

2. TANGENT SPACES

Let $p \in M$ be a point on a differentiable manifold $\{M, F_k\}$.

By a function of class k at p we mean a function of F_k with a domain of definition which is a neighborhood of p. To such functions we introduce an equivalence relation: two functions f and g of class k at p belong to the same class if they are identical in a neighborhood of p. This is clearly an equivalence relation. Define addition and scalar multiplication of two classes of functions $\{f\}$ and $\{g\}$ by adding and multiplying their representatives respectively. With this definition of addition and scalar multiplication, all classes of functions of class k at p form an infinite-dimensional vector space denoted by S_p.

Let U be a neighborhood of p, and $u : U \rightarrow u(U)$ a homeomorphism of U onto an open subset of E^n, defined by

$$u(q) = (u'(q), \ldots, u^n(q)), \qquad q \in U,$$

where $u'(q), \ldots, u^n(q)$ are n functions of class k in U. Then a function f of class k at p has the form $f = g \circ u$, with g of class $k \geqq 1$; f is said to have zero differential at p, if all the first partial derivatives of g with respect to $u', \ldots, u^n$ vanish at u(p). This property is obviously a property of a class of functions $\{f\}$ at p, and is also independent of the choice of the local coordinates $u', \ldots, u^n$. For, if $u'^1, \ldots, u'^n$ form another local coordinate system at p, the two representations g, g' of f are related by the formula (1.6). Using the formula for the partial derivatives of a composite function, we get

$$\left(\frac{\partial g}{\partial u^i}\right)_{u(p)} = \sum_{j=1}^{n}\left(\frac{\partial g'}{\partial u'^j}\right)_{u'(p)}\left(\frac{\partial u'^j}{\partial u^i}\right)_{u(p)}, \quad i = 1,\ldots, n.$$

It follows that the vanishing of the partial derivatives $\left(\frac{\partial g'}{\partial u'^i}\right)_{u'(p)}$ implies the vanishing of $\left(\frac{\partial g}{\partial u^i}\right)_{u(p)}$.

Thus it is perfectly meaningful to speak of the classes of functions of zero differential at p. Clearly they form a linear subspace of S_p, which we shall denote by Z_p.

DEFINITION 2.1. _The quotient space_ $V^*_p = S_p/Z_p$ _is called the space of tangent covectors at_ p, _its elements being tangent covectors or covectors. The duel space_ V_p _of_ V^*_p _is called the space of tangent vectors or the tangent space at_ p, _its elements being tangent vectors or vectors._

A covector at p is therefore a residue class relative to Z_p of a class of functions $\{f\}$. It is uniquely determined by a representative f, and shall denote it by df(p) or df. A vector will be denoted by X(p) or X. The scalar product of X and df will be defined by (X, df) = Xf.

THEOREM 2.1. _Let_ $f, f',\ldots, f^m$ _be functions of class_ k _at_ p, _such that_

(2.1) $$f = F(f',\ldots, f^m),$$

where $F(f',\ldots, f^m)$ _is a function of class_ 1 _in a neighborhood of_ $(f'(p),\ldots, f^m(p))$. _Then_

$$df(p) = \sum_{j=1}^{m}\left(\frac{\partial F}{\partial f^j}\right)_{(f'(p),\ldots,\, f^m(p))} df^j(p).$$

For simplicity let us write

$$\alpha_j = \left(\frac{\partial F}{\partial f^j}\right)_{(f'(p),\dots,\ f^m(p))}, \qquad j = 1,\dots, m.$$

The theorem then asserts that the function

$$f - \sum_{j=1}^{m} \alpha_j f^j$$

has zero differential at p. In terms of a local coordinate system $h',\dots, h^n$ at p let $g, g',\dots, g^m$ be the representations of $f, f',\dots, f^m$ respectively, so that

$$f = g \circ h, \qquad f^j = g^j \circ h.$$

Then we have, by applying the homeomorphism h^{-1} to (2.1),

$$g(h',\dots, h^n) = F(g'(h',\dots, h^n),\dots, g^m(h',\dots, h^n)).$$

Hence the function

$$g - \sum_{j=1}^{m} \alpha_j g^j$$

has all first partial derivatives equal to 0 at h(p).

COROLLARY 2.1. *The covectors at* p *satisfy the identities*

(2.1) $$d(\alpha f + \beta g) = \alpha df + \beta dg,$$

(2.2) $$d(fg) = f(p)dg + g(p)df,$$

where f, g *are functions of class* k *at* p, *and* α, β *are real numbers*.

COROLLARY 2.2. *Every tangent space of a differentiable manifold of dimension* n *is a vector space of dimension* n.

PROOF. Since every function f of class k at p is represented by $F(u^1,\dots, u^n)$, df is a linear combination of $du',\dots, du^n$.

By Corollary 2.1, a relation of the form

$$\alpha_1 du^1 + \ldots + \alpha_n du^n = 0,$$

$\alpha_1, \ldots, \alpha_n$ being real numbers, implies

$$d(\alpha_1 u^1 + \ldots + \alpha_n u^n) = 0,$$

which means that $\alpha_1 u^1 + \ldots + \alpha_n u^n$ has zero differential at p. But this is true only when all the α's are zero. Hence we conclude that the tangent space of dimension n is a vector space of dimension n.

COROLLARY 2.3.

(2.3) $$X(\alpha f + \beta g) = \alpha Xf + \beta Xg,$$

(2.4) $$X(fg) = f(p)Xg + g(p)Xf.$$

3. EXTERIOR DIFFERENTIATION AND MULTIPLICATION

At first let us consider vector spaces V_r of anti-symmetric tensor of order (r, o), $r = 1, \ldots, n$, and define $V = V_o + V_1 + \ldots + V_n$, V_o being the one dimensional vector space isomorphic to the real field. Then V is a vector space of dimension 2^n.

V becomes a ring called Grassman ring, by introducing a multiplication $\wedge$ which has the following properties:

(3.1) $$f \wedge (g_1 + g_2) = f \wedge g_1 + f \wedge g_2, \quad f, g_1, g_2 \in V.$$

(3.2) $$(f_1 + f_2) \wedge g = f_1 \wedge g + f_2 \wedge g, \quad f_1, f_2, g \in V.$$

(3.3) If $f \in V_r$, $g \in V_s$, then

$$f \wedge g = (-1)^{rs} g \wedge f.$$

A differential polynomial is a mapping $\omega : M \to W$, where $W = \bigcup_{p \in M} G_p$ and G_p is the Grassman ring associated with the point p. This mapping ω is always to be locally differentiable of class $\geqq 2$. If $\omega(p)$, $p \in M$, is a form of degree r, ω is called a differentiable form of degree r.

We shall define an operation d, called <u>exterior differentiation</u> which carries differential polynomials into differential polynomials by the following properties:

(3.4) $$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2.$$

(3.5) $$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2,$$

where ω_1 is a differential form of degree r.

(3.6) If f is a scalar (that is, a differential form of degree zero), df is the covariant vector such that $d(df) = 0$.

Let us choose for V_1 a base with the differentials dx^i $(i = 1, \ldots, n)$. Then a base for V_r will be

$$dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_r}, \qquad i_1 < i_2 < \ldots < i_r.$$

Hence we can write a differential form ω of order r as

(3.7) $$\omega = \sum_{i_1 < \ldots < i_r} a_{i_1 \ldots i_r} \, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_r},$$

where the coefficients may be assumed to be anti-symmetric. It follows from (3.5), (3.6), (3.7),

$$d(dx^{i_1} \wedge \ldots \wedge dx^{i_r}) = 0,$$

$$d\omega = \sum_{i_1 < \ldots < i_r} da_{i_1 \ldots i_r} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_r},$$

$$d(d\omega) = 0.$$

A differential form ω is called <u>exact</u> if $d\omega = 0$, and is called derived if there exists a differential form θ such that $\omega = d\theta$. Then from $d(d\omega) = 0$ it follows that every derived form is exact.

Let $(x^1, x^2, \ldots, x^n)$ be the local coordinates of a point $x \in U$, where U is an open subset in M. Then we define the exterior multiplication $\wedge$ on the differentials dx^i by the following properties:

(3.8) $\quad (dx^i \wedge dx^j) \wedge (dx^k) = dx^i \wedge (dx^j \wedge dx^k),$

(3.9) $\quad (dx^i + dx^j) \wedge dx^k = dx^i \wedge dx^k + dx^j \wedge dx^k,$

(3.10) $\quad dx^i \wedge dx^j = -dx^j \wedge dx^i,$

(3.11) $\quad a \wedge dx^i = adx^i,$

(3.12) $\quad dx^i \wedge adx^j = adx^i \wedge dx^j,$

where a is a real function of $x^1, \ldots, x^n$. From (3.10) we thus have

$$dx^i \wedge dx^i = 0,$$
$$dx^i \wedge dx^j \neq 0 \qquad \text{for } i \neq j.$$

4. AFFINE CONNECTIONS

In order that differentiation of tensors be defined on a differentiable manifold M intrinsically, that is, independent of the choice of local coordinates, we shall need an additional structure called an affine connection.

DEFINITION 4.1. <u>An affine connection is defined, by giving</u>

in every coordinate neighborhood, a set of linear differential forms ω^i_k, such that in the intersection of two coordinate neighborhoods U, V, we have

(4.1) $$dp^i_j + \omega^{*i}_k p^k_j = p^i_k \omega^k_j,$$

where repeated indices imply summation, and

(4.2) $$p^i_j = \frac{\partial x^{*i}}{\partial x^j}$$

the x^i's and x^{*i}'s being the respective local coordinates.

To show the consistency of the above definition, consider in three local coordinate systems x^i, x^{*i}, x^{**i}

$$d\left(\frac{\partial x^{*i}}{\partial x^j}\right) + \omega^{*i}_k \frac{\partial x^{*k}}{\partial x^j} = \frac{\partial x^{*i}}{\partial x^k}\omega^k_j,$$

$$d\left(\frac{\partial x^{**i}}{\partial x^{*j}}\right) + \omega^{**i}_k \frac{\partial x^{**k}}{\partial x^{*j}} = \frac{\partial x^{**i}}{\partial x^{*k}}\omega^{*k}_j,$$

from which we get

$$d\left(\frac{\partial x^{**i}}{\partial x^j}\right) + \omega^{**i}_k \frac{\partial x^{**k}}{\partial x^j} = d\left(\frac{\partial x^{**i}}{\partial x^{*k}}\frac{\partial x^{*k}}{\partial x^j}\right) + \omega^{**i}_l \frac{\partial x^{**l}}{\partial x^{*k}}\frac{\partial x^{*k}}{\partial x^j}$$

$$= \frac{\partial x^{**}}{\partial x^j}\left[d\left(\frac{\partial x^{**i}}{\partial x^{*k}}\right) + \omega^{**i}_l \frac{\partial x^{**l}}{\partial x^{*k}}\right] + \frac{\partial x^{**i}}{\partial x^{*k}} d\left(\frac{\partial x^{*k}}{\partial x^j}\right)$$

$$= \frac{\partial x^{**i}}{\partial x^{*l}}\left[\omega^{*l}_k \frac{\partial x^{*k}}{\partial x^j} + d\left(\frac{\partial x^{*l}}{\partial x^j}\right)\right]$$

$$= \frac{\partial x^{**i}}{\partial x^{*l}}\frac{\partial x^{*l}}{\partial x^k}\omega^k_j = \frac{\partial x^{**i}}{\partial x^k}\omega^k_j,$$

which proves the consistency of the definition, that is, the relation (4.1) in one local coordinate system is a consequence of the relation in other two coordinate systems.

By introducing

(4.3) $$q^i_k = \frac{\partial x^i}{\partial x^{*k}},$$

we have

$$p^j_i q^k_j = q^j_i p^k_j = \delta^k_i, \tag{4.4}$$

that is, the matrices (p^j_i) and (q^j_i) are inverses to each other, and δ^k_i are the Kronecker deltas. Differentiating any one of these sets of equations (4.4), after simplification we obtain

$$dq^i_j + \omega^i_k q^k_j = q^i_k \omega^{*k}_{\ j}. \tag{4.5}$$

These conditions are clearly equivalent to (4.1).

Now let us assume an affine connection be given on a differentiable manifold M, and consider a contravariant vector field X, whose components in two local coordinate systems x^i and x^{*i} are X^i and X^{*i} respectively. Then we can write

$$X^{*i} = p^i_j X^j \ . \tag{4.6}$$

By differentiating (4.6) and using (4.1) we get

$$\begin{aligned} dX^{*i} &= p^i_j dX^j + X^j dp^i_j \\ &= p^i_j dX^j + (p^i_k \omega^k_j - \omega^{*i}_{\ k} p^k_j) X^j \ , \end{aligned} \tag{4.7}$$

or

$$dX^{*i} + \omega^{*i}_{\ k} X^{*k} = p^i_j (dX^j + \omega^j_k X^k) \ . \tag{4.8}$$

Define

$$DX^i = X^i_{,j} dx^j = dX^i + \omega^i_j X^j \ . \tag{4.9}$$

Then $X^i_{,j}$ are the components of a tensor field of type (1.1). Moreover, we can write (4.8) in the form

$$DX^{*i} = p^i_j DX^j \ , \tag{4.10}$$

so that DX^i are linear differential forms behaving like contravariant vector. They are said to define the covariant differential of the vector field X.

Now let us extend our investigation to a component of a tensor field $X^{ij}{}_k$ of the type (2, 1) relative to the local coordinates x^i, and show that by the existence of the affine connection on the manifold we can get a component of a tensor field of the type (2,2). Under a change of the local coordinates we have

$$X^{*ij}{}_k = p^i_l p^j_m q^r_k X^{lm}{}_r \ . \tag{4.11}$$

By differentiating (4.11), and using (4.1) and (4.5) we can easily obtain

$$dX^{*ij}{}_k + X^{*jh}{}_k \omega^{*i}_h + X^{*ih}{}_k \omega^{*j}_h - X^{*ij}{}_h \omega^{*h}_k$$
$$= p^i_l p^j_m q^r_k (dX^{lm}{}_r + \omega^l_h X^{hi}{}_r + \omega^m_h X^{lh}{}_r - \omega^h_r X^{lm}{}_h) \ .$$

Putting

$$\begin{aligned} DX^{ij}{}_k &= X^{ij}{}_{k,l}\, dx^l \\ &= dX^{ij}{}_k + \omega^i_l X^{lj}{}_k + \omega^j_l X^{il}{}_k - \omega^l_k X^{ij}{}_l \ , \end{aligned} \tag{4.12}$$

we then have

$$DX^{*ij}{}_k = p^i_l p^j_m q^r_k DX^{lm}{}_r \ . \tag{4.13}$$

From (4.12) it follows that $X^{ij}{}_{k,l}$ define a tensor field of type (2,2). Again we shall say that $DX^{ij}{}_k$ define the covariant differential of the tensor field $X^{ij}{}_k$.

Similarly, with the same procedure we can extend our definition to a tensor field of any type. In particular, for a tensor field of type (0,0) its covariant differential is the ordinary differential.

In the previous discussion, especially in the construction of (4.9), (4.13) from (4.7), (4.11) we notice that the covariant differentiation of a tensor leads to a new tensor field with one more covariant index, and this depends only on the affine connection.

Now we introduce the differential form

(4.14) $$\omega^i_k = \Gamma^i_{jk} dx^j .$$

To find Γ^i_{jk}, substituting (4.14) in (4.1) we obtain

(4.15) $$\frac{\partial^2 x^{*i}}{\partial x^j \partial x^k} dx^k + \Gamma^{*i}_{lk} \frac{\partial x^{*k}}{\partial x^j} dx^{*l} = \frac{\partial x^{*i}}{\partial x^k} \Gamma^k_{hj} dx^h .$$

Multiplication of (4.15) by $\frac{\partial x^p}{\partial x^{*i}}$ thus gives

(4.16) $$\Gamma^p_{hj} = \frac{\partial^2 x^{*i}}{\partial x^j \partial x^n} \frac{\partial x^p}{\partial x^{*i}} + \Gamma^{*i}_{lk} \frac{\partial x^{*k}}{\partial x^j} \frac{\partial x^{*l}}{\partial x^h} \frac{\partial x^p}{\partial x^{*i}} ,$$

which are the classical formulas for transformation of the components Γ^i_{jk} of an affine connection. Now let C be a parametrized curve in M with parameter t. The tangent vectors X(t) are said to be parallel along C, if $DX^i = 0$, or from (4.9), (4.14)

(4.17) $$\frac{dX^i}{dt} + \Gamma^i_{jk} \frac{dx^j}{dt} X^k = 0 .$$

This generalized notion of parallelism is called the parallelism of Levi-Civita. (4.17) are a system of ordinary differential equations of the first order. From the existence theorem of differential equations, this system has unique solutions $X^i(t)$, when the initial values $X^i(t_o)$ are given. In other words, every tangent vector can be displaced parallely along a curve C.

A parametrized curve is called an auto-parallel curve, or a path, or a geodesic, if its tangent vectors are parallel along it. If the arc length s is taken as the parameter of a curve C, then the components of the tangent vectors of C are given by

(4.18) $$X^i = dx^k/ds \ .$$

If C is a geodesic, then from (4.17) X^i satisfies

(4.19) $$\frac{dX^i}{ds} + \Gamma^i_{jk} X^j X^k = 0 \ .$$

Under a parallel displacement, the scalar product of two tangent vectors remain unchanged. Moreover, we have

(4.29) $$\frac{d}{dt}(X^1 X^2) = \frac{dX^1}{dt} X^2 + \frac{dX^2}{dt} X^1 \ .$$

Consider first a scalar function f. Relative to a local coordinate system x^i, by definition we obtain

(4.21) $$f,_i = \partial f/\partial x^i \ ,$$

(4.22) $$f,_{i},_{j} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ji} \frac{\partial f}{\partial x^k} \ ,$$

$$f,_{j},_{i} = \frac{\partial^2 f}{\partial x^j \partial x^i} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \ .$$

From (4.22) it follows

(4.23) $$f,_{i},_{j} - f,_{j},_{i} = (\Gamma^k_{ij} - \Gamma^k_{ji}) \frac{\partial f}{\partial x^k} \ .$$

Define

(4.24) $$T^k_{\ ij} = \Gamma^k_{ij} - \Gamma^k_{ji} \ ,$$

where $T^k_{\ ij}$ is anti-symmetric in i and j, and is called the <u>Torsion Tensor</u>.

Next consider a contravariant vector field with the components X^i. By definition we have

(4.25) $$X^i,_j = \frac{\partial X^i}{\partial x^j} + \Gamma^i_{jl} X^l \ .$$

Covariantly differentiating (4.25) with respect to x^k we get

(4.26) $$X^i{}_{,j;k} = \frac{\partial X^i{}_{,j}}{\partial x^k} + \Gamma^i_{kh}X^h{}_{,j} - \Gamma^h_{jk}X^i{}_{,h}$$

$$= \left(\frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{km}\Gamma^m_{jl}\right)X^l - \Gamma^l_{kj}X^i{}_{,l}$$

$$+ \left(\frac{\partial^2 X^i}{\partial x^k \partial x^j} + \Gamma^i_{jl}\frac{\partial X^l}{\partial x^k} + \Gamma^i_{kl}\frac{\partial X^l}{\partial x^j}\right),$$

where the term in the parentheses are symmetric in j,k.

Similarly,

(4.27) $$X^i{}_{,k;j} = \left(\frac{\partial \Gamma^i_{kl}}{\partial x^j} + \Gamma^i_{jm}\Gamma^m_{kl}\right)X^l - \Gamma^l_{jk}X^i{}_{,l}$$

$$+ \left(\frac{\partial^2 X^i}{\partial x^j \partial x^k} + \Gamma^i_{kl}\frac{\partial X^l}{\partial x^j} + \Gamma^i_{jl}\frac{\partial X^l}{\partial x^k}\right).$$

From (4.26), (4.27) it follows

(4.28) $$X^i{}_{,j;k} - X^i{}_{,k;j} = \left[\left(\frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{kl}}{\partial x^j} + (\Gamma^i_{km}\Gamma^m_{jl} - \Gamma^i_{jm}\Gamma^m_{kl}\right)\right]X^l$$

$$+ (\Gamma^l_{jk} - \Gamma^l_{kj})X^i{}_{,l}$$

$$= R^i{}_{lkj}X^l + T^l{}_{jk}X^i{}_{,l},$$

where

(4.29) $$R^i{}_{ljk} = \frac{\partial \Gamma^i_{kl}}{\partial x^j} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + (\Gamma^i_{jm}\Gamma^m_{kl} - \Gamma^i_{km}\Gamma^m_{jl}).$$

We notice that $R^i{}_{ljk}$ and $T^i{}_{jk}$ depend only on the affine connection, and are the components of tensor fields of types (1,3), (0,2) respectively. $R^i{}_{ljk}$ is called the <u>Curvature Tensor</u>.

From (4.29) it follows

(4.30) $$R^a{}_{ijk} = - R^a{}_{ikj}$$

for all affine connections, and

(4.31) $$R^a{}_{ijk} + R^a{}_{jki} + R^a{}_{kij} = 0$$

for all symmetric affine connections.

If we take the derivatives of similar formulas for general tensor fields, we do not get new tensors of the affine connection. We shall discuss this by simplifying the computation by using the principal bundle.

5. THE PRINCIPAL BUNDLES

DEFINITION 5.1. A frame is the object formed by a point p on a differentiable manifold M of dimension n and n linearly independent tangent vectors at p.

DEFINITION 5.2. The principal bundle B is the space of all frames over M; its dimension is $n^2 + n$.

To a local coordinate system x^i on M there corresponds a system of local coordinates x^i, X_i^k in B such that the n vectors of the frame are given by

(5.1) $$1_i = X_i^k \frac{\partial}{\partial x^k} .$$

Since these vectors are linearly independent,

(5.2) Let $$(X_i^k) \neq 0.$$

Let the matrix (Y_i^k) be inverse to the matrix (X_i^k) so that

(5.3) $$X_i^j Y_j^k = Y_i^j X_j^k = \delta_i^k .$$

Suppose we restrict our discussion to a neighborhood, in which the x^i and the second local coordinate system x^{*i} are valid. Let x^{*i}, X^{*k}_{i} be the system of local coordinates in B corresponding to x^{*i}. By introducing the Y^{*k}_{i} we then have

(5.4) $$X^{*k}_{i} = \frac{\partial x^{*k}}{\partial x^{j}} X^{j}_{i} ,$$

(5.5) $$Y^{*i}_{k} = Y^{i}_{j} \frac{\partial x^{j}}{\partial x^{*k}} ,$$

the last of which implies, in particular,

(5.6) $$Y^{*i}_{j} dx^{*j} = Y^{i}_{j} dx^{j} .$$

It follows that the differential form α^{i} having as representatives both sides of (5.6) is independent of the choice of local coordinates and is defined in B.

Now suppose an affine connection to be given in M. In the expression for DX^{i} in (4.9) we regard the X^{i} as independent variables and apply it to each of the vectors of our frame. Then

(5.7) $$DX^{j}_{i} = dX^{j}_{i} + \omega^{j}_{k} X^{k}_{i}$$

are linear differential forms in x^{i}, X^{k}_{i}. From (4.10) we obtain

(5.8) $$DX^{*j}_{i} = \frac{\partial x^{*j}}{\partial x^{k}} DX^{k}_{i} ,$$

which and (5.5) imply

(5.9) $$Y^{*k}_{j} DX^{*j}_{i} = Y^{k}_{j} DX^{j}_{i} ,$$

It follows that the two members of (5.9), denoted by α^{k}_{i}, are representatives in the coordinates x^{i} and x^{*i} respectively of differential forms in B. Notice that α^{i}, α^{k}_{i} are defined on B by means of the differential structure and the affine connection on M respectively. It is clear that these $n^{2} + n$ linear differential forms α^{i}, α^{k}_{i} are linearly independent.

Rewriting equations (5.6) and (5.9), we have

$$(5.10)\qquad \alpha^i = Y^i_j dx^j , \qquad \alpha^k_i = Y^k_j(dX^j_i + \omega^j_l X^l_i) .$$

or, by (5.3),

$$(5.11)\qquad dx^i = X^i_j \alpha^j , \qquad X^k_j \alpha^j_i = dX^k_i + \omega^k_l X^l_i .$$

Applying exterior differentiation to the first equation of (5.11) and using (5.11), (4.14) we obtain

$$X^i_k \alpha^k_j \wedge \alpha^j + X^i_j d\alpha^j = \omega^i_k \wedge dx^k = \Gamma^i_{kj} dx^k \wedge dx^j .$$

Since

$$\begin{aligned}\Gamma^i_{kj} dx^k \wedge dx^j &= -\Gamma^i_{kj} dx^j \wedge dx^k \\ &= \tfrac{1}{2}(\Gamma^i_{kj} - \Gamma^i_{jk}) dx^k \wedge dx^j \\ &= \tfrac{1}{2} T^i{}_{kj} dx^k \wedge dx^j ,\end{aligned}$$

we have

$$(5.12)\qquad \begin{aligned} X^i_j(d\alpha^j - \alpha^k \wedge \alpha^j_k) &= \omega^i_j \wedge dx^j \\ &= \tfrac{1}{2} T^i{}_{jk} dx^j \wedge dx^k .\end{aligned}$$

Similarly, applying exterior differentiation to the second equation of (5.11) and simplifying, we obtain

$$(5.13)\qquad \begin{aligned} X^k_j(d\alpha^j_i - \alpha^l_i \wedge \alpha^j_l) &= X^j_i(d\omega^k_j - \omega^l_j \wedge \omega^k_l) \\ &= \tfrac{1}{2} X^j_i R^k{}_{jlm} dx^l \wedge dx^m .\end{aligned}$$

From (5.10), (5.11), (5.12), (5.13) it follows

$$(5.14)\qquad \begin{aligned} d\alpha^j - \alpha^k \wedge \alpha^j_k &= \tfrac{1}{2} P^j{}_{lm} \alpha^l \wedge \alpha^m , \\ d\alpha^j_i - \alpha^l_i \wedge \alpha^j_l &= \tfrac{1}{2} S^j{}_{ilm} \alpha^l \wedge \alpha^m ,\end{aligned}$$

where

$$(5.15)\qquad \begin{aligned} P^j{}_{lm} &= Y^j_i X^p_l X^q_m T^i{}_{pq} , \\ S^j{}_{ilm} &= Y^j_k X^p_i X^q_l X^r_m R^k{}_{pqr} .\end{aligned}$$

Equations (5.14) play an extremely important role in the theory of affine connections, and we shall call them the equations of structure. Since both sides of (5.14) do not involve the local coordinates, they are defined on the whole space B.

In particular, if $X_i^k = Y_i^k = \delta_i^k$, then from (5.10) $\alpha^i = dx^i$, $\alpha_i^j = \omega_i^j$, and therefore (5.14) is reduced to

$$(5.16)\qquad \begin{aligned} - dx^k \wedge \omega_k^i &= T^j{}_{lm} dx^l \wedge dx^m , \\ d\omega_i^j - \omega_i^l \wedge \omega_l^j &= \tfrac{1}{2} R^j{}_{ilm} dx^l \wedge dx^m . \end{aligned}$$

Sometimes it is convenient to introduce the exterior quadratic differential forms

$$(5.17)\qquad \begin{aligned} H^j &= \tfrac{1}{2} P^j{}_{lm} \alpha^l \wedge \alpha^m , \\ H_i^j &= \tfrac{1}{2} S^j{}_{ilm} \alpha^l \wedge \alpha^m , \end{aligned}$$

so that (5.14) are written as

$$(5.18)\qquad \begin{aligned} d\alpha^j - \alpha^k \wedge \alpha_k^j &= H^j \\ d\alpha_i^j - \alpha_i^k \wedge \alpha_k^j &= H_i^j \end{aligned}$$

Exterior differentiation of (5.18) gives

$$(5.19)\qquad \begin{aligned} dH^j &= \alpha^k \wedge H_k^j - H^k \wedge \alpha_k^j , \\ dH_i^j &= \alpha_i^k \wedge H_k^j - H_i^k \wedge \alpha_k^j , \end{aligned}$$

which are called the Bianchi Identities.

So far we have been assuming that there is an affine connection in M; this in turn gives $n^2 + n$ linearly independent differential forms α^i, α_i^k in B, and the exterior derivatives of those differential forms are given in a simple form. Now we need to investigate the possibility of getting an affine connection from a set of differential forms α_i^k by showing the type of structure, which they should have, in order to define an affine connection.

THEOREM 5.1. In order that the linear differential forms α_i^j in B define an affine connection on M, it is necessary and sufficient that α_i^j together with α^i satisfy the equations of structure (5.14).

PROOF. Equations (5.14) are necessary condition. It remains only to prove that they are also sufficient. Since α^i are given in B by the differentiable structure of M, the problem is to determine α_i^j so that (5.14) are satisfied. Exterior differentiation of the first equations of (5.10) yields

$$d\alpha^i = dY_k^i \wedge dx^k = - Y_j^i dX_l^j \wedge Y_k^l dx^k$$
$$= - Y_j^i dX_l^j \wedge \alpha^l .$$

Substituting the above equation in the first set of equations (5.14) we obtain

$$\alpha^j \wedge (\alpha_j^i + \tfrac{1}{2}P^i{}_{jl}\alpha^l - Y_l^i dX_j^l) = 0 ,$$

which implies that $\alpha_j^i - Y_l^i dX_j^l$ are linear combinations of α^k, and guided by the second set of formulas in (5.11) we shall put

(5.20) $$X_j^k \alpha_i^j - dX_i^k = \omega_j^k X_i^j ,$$

where ω_j^k are linear combinations of dx^i. Now we need only to show that ω_j^k are linear differential forms in x^i only, or Γ_{ij}^k are independent of X_l^m, since

(5.21) $$\omega_j^k = \Gamma_{ij}^k dx^i .$$

To this purpose applying the exterior differentiation to (5.20) and using (5.20), (5.16) we obtain

(5.22) $$X^j_i(d\omega^k_j - \omega^l_j \wedge \omega^k_l) = \tfrac{1}{2}X^k_j S^j{}_{ilm}\alpha^l \wedge \alpha^m ,$$

which implies that $d\omega^k_j$ is an exterior quadratic differential form in dx^i. On the other hand, from (5.21) we have

$$d\omega^k_j = \frac{\partial \Gamma^k_{ij}}{\partial x^l} dx^l \wedge dx^i + \frac{\partial \Gamma^k_{ij}}{\partial X^m_l} dX^m_l \wedge dx^i ,$$

from which it follows that

$$\frac{\partial \Gamma^k_{ij}}{\partial X^m_l} = 0 ,$$

so that Γ^k_{ij} are independent of X^m_l. Thus Theorem (5.1) is proved.

We now study the notion of a tensor field from the point of view of the principal bundle. For definiteness, we consider a tensor field of type (2,1), whose components $X^{ij}{}_k$, $X^{*ij}{}_k$ in terms of two local coordinate systems x^i, x^{*i} are related by (4.11). From (4.11) (5.4), (5.5) it follows immediately

(5.23) $$Y^{*i}_q Y^{*j}_r X^{*s}_k X^{*qr}{}_s = Y^i_q Y^j_r X^s_k X^{qr}{}_s .$$

The common expression of this equation is therefore a function in B, independent of the choice of local coordinates. There are altogether n^3 such functions which we shall denote by $T^{ij}{}_k$. In general, for a tensor field of type (k,l) there will exist n^{k+l} functions in B.

Now we ask the question: When does a set of functions $T^{ij}{}_k$ arise from a tensor field? A necessary and sufficient condition is that they have the expression on the right side of (5.23) in the local coordinate system x^i. When there is an affine connection on M this condition can be given in a different form. To do this, we compute the differential $dT^{ij}{}_k$. From (5.11) we have

(5.24) $$dX^k_i = X^k_j \alpha^j_i - \omega^k_j X^j_i$$

Differentiation of (5.3) and use of (5.24) yield immediately

(5.25) $$dY^k_i = - Y^j_i \alpha^k_j + \omega^j_i Y^k_j \; .$$

From (5.23), (5.24), (5.25), (4.12) it follows that

(5.26) $$\begin{aligned} dT^{ij}{}_k &= d(Y^i_q Y^j_r X^s_k X^{qr}{}_s) \\ &= - T^{lj}{}_k \alpha^i_l - T^{il}{}_k \alpha^j_l \\ &\quad + T^{ij}{}_l \alpha^l_k + Y^i_q Y^j_r X^s_k DX^{qr}{}_s \; . \end{aligned}$$

THEOREM 5.2. <u>In order for the function $T^{ij}{}_k$ in B to be of the form (5.23) and, therefore, to arise from a tensor field M, it is necessary and sufficient that they should satisfy</u>

(5.27) $$dT^{ij}{}_k = - T^{lj}{}_k \alpha^i_l - T^{il}{}_k \alpha^j_l + T^{ij}{}_l \alpha^l_k + T^{ij}{}_{kl} \alpha^l \; .$$

<u>The functions $T^{ij}{}_{kl}$ are related to the covariant differential $DX^{qr}{}_s$ of the tensor field by the formula</u>

(5.28) $$T^{ij}{}_{kl} \alpha^l = Y^i_q Y^j_r X^s_k (DX^{qr}{}_s) \; .$$

<u>Similar relations exist between tensor fields of general types M and corresponding functions in B.</u>

The necessity of this condition has been established above. To prove the sufficiency we shall show that $X^q_i X^r_j Y^k_s T^{ij}{}_k$ are functions of x^i only. This follows from the formula

$$d(X^q_i X^r_j Y^k_s T^{ij}{}_k) = 0 \qquad (\text{mod. } dx^l) \; ,$$

which can be verified by direct differentiation and use of (5.24), (5.25), (5.27). Then formula (5.28) is a consequence of (5.26).

To find a geometrical interpretation of the affine connection in terms of the relationship between M and B, we notice that B has a group of homeomorphisms defined as follows: Let $pe_1 \ldots e_n$ be the frames over M. In B we call a translation T a homeomorphism

$$pe_1 \ldots e_n \to pe'_1 \ldots e'_n \tag{5.29}$$

defined by

$$e'_i = a_i^j e_j , \tag{5.30}$$

where a_i^j are constants such that $\det(a_i^j) \neq 0$. Clearly all the translations in B form a group. From the general discussions on differentiable manifolds it follows that a translation T induces an isomorphism of the tangent space V(b) at a point $b \in B$ onto the tangent space V(T(b)) at the image point T(b). This in turn induces a dual isomorphism of $V^*(T(b))$ onto $V^*(b)$.

THEOREM 5.3. _Let_ $\pi : B \to M$ _be the projection of_ B _onto_ M, _which assigns to a frame_ $b = pe_1 \ldots e_n$ _the point_ $p \in M$. _The definition of an affine connection_ M _is equivalent to that of a family of linear subspaces_ g(b) _supplementary to_ $\pi^* V^*(p)$ _in the space of covectors_ $V^*(p)$ _at_ b, _such that the family_ g(b) is invariant under the group of translations in B.

PROOF. We notice that g(b) is a vector space of dimension n^2. If B' denotes the bundle over B of the n-dimensional linear subspaces of covectors, then g is a cross-section of this bundle. The subspace g(b) can be defined by n^2 linear differential forms which, together with α^i, span $V^*(b)$. From (5.4), (5.5) it follows that the forms defined in a local coordinate system by $Y^k_j dX^j_i$ are determined mod α^1. Since these forms are clearly independent, g(b) defines and be defined by the n^2 linear differential forms

(5.31) $$\alpha^k_i = (dX^j_i + \varphi^j_i)Y^k_j \, ,$$

where φ^j_i are linear combinations of dx^k. Our problem is to study the forms φ^j_i, if g(b) is invariant under the translation T. Denote by x^i, X'^k_i the local coordinates at the image point b' of the point b under T, and denote the corresponding quantities at b' by the same symbols with dashes. Then we have

$$X'^k_i = a^j_i X^k_j \, ,$$
$$Y^i_k = a^i_j Y'^j_k \, .$$

From (5.31), (5.32) it follows

(5.33) $$T^*\alpha'^k_i = (a^l_i dX^j_l + \varphi'^j_i)Y'^k_j \, ,$$

where T^* is the dual mapping of T. The invariance of g(b) under T therefore implies that

(5.34) $$\varphi'^j_i = a^l_i \varphi^j_l \, .$$

From (5.34), (5.32) it follows that

(5.35) $$Y'^j_i \varphi'^k_j = Y^j_i {}^k_j \, .$$

The expression on the right side of (5.35) is invariant under the translation T, and is therefore independent of X^m_l. Hence we can put

(5.36) $$Y^j_i \varphi^k_j = \omega^k_i \, ,$$

or

(5.37) $$\varnothing_j^k = X_j^i \omega_i^k ,$$

where ω_i^k are linear differential forms in x^j only. Substituting (5.37) in (5.31) we can see that α_i^k here obtained are identical with those in (5.10). Hence the family of linear subspaces $g(b)$ defines an affine connection. Also from the discussion above it is shown that an affine connection defines a family of $g(b)$ with the properties given in the theorem. Thus the theorem is proved.

6. GROUPS OF HOLONOMY

It is known that for the ordinary affine space relative to the affine coordinates, all Γ_{ij}^k are zero. Therefore an affine connection is said to be locally flat, if there exists a local coordinate system with respect to which all Γ_{ij}^k are zero. In the case of a locally flat affine connection, the torsion tensor and curvature tensor must be zero, and the converse is also true.

THEOREM 6.1. <u>An affine connection is locally flat, if and only if both the torsion tensor and the curvature tensor vanish</u>.

We need only to prove the sufficiency. By (4.1), (4.2) it suffices to prove that a local coordinate system x^{*i} exists such that

(6.1) $$dx^{*i} = p_k^i dx^k ,$$

$$(6.2) \qquad dp^i_k = p^i_j \omega^i_k .$$

In this system we regard x^i, p^i_k, x^{*i} as variables.

Let F be a differential system of dimension r on a manifold M, so that for any point $p \in M$, F(p) is a linear subspace of dimension r of the tangent space V(p) of M at p. A submanifold (φ, N) is called an integral manifold of F if, for any $q \in N$, $\varphi(V(q))$ is contained in $F(\varphi(q))$. F is called completely integrable, if every point $p \in M$ has a coordinate neighborhood with the local coordinates $x^1, \ldots, x^n$ such that the coordinate slices $x^{r+1} = \text{const.}, \ldots, x^n = \text{const.}$, are integral manifolds of F. Let $p \in M$ and U a coordinate neighborhood containing p. If F is of class $r \geqq 1$ it defines and can be defined in U by a nonzero decomposable form Ω of degree n-r determined up to a nonzero factor. The system F is said to satisfy the condition (C) at p, if U can be so chosen that $d\Omega$ is a multiple of Ω by a linear differential form. For the use in this section we shall only state, without proof,

THEOREM 6.2. (Frobenius). <u>A necessary and sufficient condition that a differential system F of class $r \geqq 1$ be completely integrable is that the condition (C) be satisfied at all points</u> $p \in M$.

By taking exterior differentiation of (6.1), (6.2), we have

$$(6.3) \qquad dp^i_k \wedge dx^k \equiv p^i_j \omega^j_k \wedge dx^k = 0 ,$$

$$(6.4) \qquad dp^i_j \wedge \omega^j_k + p^i_j d\omega^j_k \equiv p^i_j (\omega^j_l \wedge \omega^l_k + d\omega^j_k) = 0 ,$$

so that the conditions in Frobenius' theorem are satisfied. It follows that in a neighborhood of a point (x_o^i) in the space of the variables x^i there are functions $x^{*i}(x^k)$, $p_k^i(x^j)$ satisfying (6.1), (6.2) such that the initial values $x^{*i}(x_o^k)$, $p_k^i(x_o^j)$ can be arbitrarily assigned. In particular, we can assign the initial values such that $\left|p_k^i(x_o^j)\right| \neq 0$, which means by (6.1) that the functional determinant $\left|\frac{\partial x^{*i}}{\partial x^k}(x_o^i)\right| \neq 0$. Therefore x^{*i} form a local coordinate system in the neighborhood of (x_o^i), and the theorem is proved.

Now let us consider the principal bundles B, B^* of two affinely connected manifolds M, M^* of dimension n. Then the affine connections determine $2(n^2 + n)$ linear differential forms α^i, α_i^j and α^{*i}, α^{*j}_i in B and B^* respectively. Let V, V^* be coordinate neighborhoods in B and B^* with (x^i, X_i^k), (x^{*i}, X^{*k}_i) as their local coordinates respectively, and let $f : V \to V^*$ be a local differentiable homeomorphism such that

$$(6.5) \qquad f^*\alpha^{*i} = \alpha^i, \quad f^*\alpha^{*j}_i = \alpha_i^j,$$

where the differential forms denote their restriction to the respective neighborhoods V and V^*. From the first equation of (6.5) it follows that $\alpha^{*i} = 0$ implies $\alpha^i = 0$. Thus frames with the same origin in $\sigma^*(V^*)$ are mapped into frames with the same origin in $\sigma(V)$, where σ, σ^* are the projections of B, B^* onto M, M^* respectively. Moreover, f induces a differentiable homeomorphism $f' : \sigma(V) \to \sigma^*(V^*)$. If by means of f' we take the local coordinates x^i of $p \in \sigma(V)$ to be those of $f'(p)$, then $q \in V$ and $f(g) \in V^*$ have the same local coordinates (x^i, X_i^k). From the formulas for α^i, α_i^j in terms of the local coordinates x^i, X_i^k it follows that the affine connection in M^* has, relative to x^i, the same components as the affine connection in M. These considerations justify the definition: The two affine connections are called locally equivalent (relatively to V, V^*), if there is a differentiable homeomorphism f of V onto V^* such that equations (6.5) hold.

With the above notion of local equivalence, we can identify the locally flat affinely connected manifolds with those which are locally equivalent to the ordinary affine space. Denote by A^n the ordinary affine space, and by B_o its principal bundle. By definition there exist local coordinates x^i in a neighborhood of M relatively to which all the components of the affine connection equal to zero. Then the mapping f, which maps the point (x^i, X_i^k) of B into the frame A^n whose origin has x^i as the local coordinates and whose vectors have the components $(x_i^1, \ldots, x_i^n)$, establishes the required local equivalence. Conversely, if the affine connection in M is locally equivalent to that of A^n, then we can take the coordinates of a point in A^n as the local coordinates of its corresponding point in M. From (6.5) it follows that relative to this local coordinate system the affine connection will have all its components equal to zero.

By Theorem 6.1 we can conclude that if both torsion and curvature tensors of M are zero, then the frames with origins in a neighborhood in M can be mapped by a homeomorphism onto the frames in a neighborhood in A^n so that (6.5) holds.

It is obviously not possible to do the same for a general affine connection. But to generalize the above geometrical situation, we have to be restricted to the frames whose origin lie on a parametrized curves C(u) in M, where u is the parameter. We shall denote the restrictions of the differential forms α^i, α_i^j to this submanifold of frames by the same symbols. To describe these frames we take a particular family $pa_1 \ldots a_n(u)$ along C(u), one at each point of C(u). Then a general frame of the family will be $p(u)e_1 \ldots e_n$, where

$$(6.6) \qquad e_i = x_i^k a_k(u) \ , \qquad \text{where } \left| x_i^k \right| \neq 0 \ .$$

We need to show that the frames $p(u)e_1 \ldots e_n$ can be mapped into frames in A^n such that α^i, α_i^j are the dual images of the corresponding differential forms in A^n. Geometrically this can be described by saying that we "develop" these frames into the affine space so that their relative position remain unchanged. If $p'e_1' \ldots e_n'$ denotes the image of $pe_1 \ldots e_n$, this means that the vectors p', $e_1', \ldots, e_n'$ satisfy the differential system

$$dp' = \alpha^i e_i' \ ,$$
$$de_i' = \alpha_i^k e_k' \ .$$

This is a differential system with variables, which are u, X_i^k and the components of the vectors $p', e_1', \ldots, e_n'$. Since (H)^i,

Θ^j_i are quadratic in the differentials of the local coordinates, they vanish along a parametrized curve. Just as in the locally flat case, it follows that (6.7) is completely integrable. Therefore, there is one and only one family of vectors $p', e'_1, \ldots, e'_n$ satisfying (6.7) and taking arbitrary initial positions for $u = u_o$ and $X^k_i = \delta^k_i$.

We shall first prove that $p'e'_1 \ldots e'_n$ is a frame, that is, that the vectors $e'_1, \ldots, e'_n$ are linearly independent, if the same is true of the initial position. In fact let

(6.8) $$\Delta = (e'_1, \ldots, e'_n)$$

denote the determinant whose i'th column consists of the components of e'_i. We find

$$\begin{aligned} d\Delta &= \sum_i (e'_1, \ldots, e'_{i-1}, \theta^i_i e'_i, e'_{i+1}, \ldots, e'_n) \\ &= (\sum_i \theta^i_i)\Delta . \end{aligned}$$

Integration gives

(6.9) $$\Delta = \Delta_o e^{\int \sum_i \theta^i_i} ,$$

where Δ_o denotes the initial value of the determinant. Hence Δ never vanishes if $\Delta_o \neq 0$.

Moreover since α^i are multiples of du, p' is a function of u only so that its locus is a parametrized curve C'(u) in A^n. Let $p'a'_1 \ldots a'_n$ be the image of $pa_1 \ldots a_n$. To determine α^j_i mod du we then write, in a local coordinate system x^i,

(6.10) $$a_i(u) = a^j_i(u)\frac{\partial}{\partial x^j} .$$

Let also y^j_i be the element of the matrix inverse to (x^k_i) so that

(6.11) $$x_i^j y_j^k = y_i^j x_j^k = \delta_i^k$$

From (6.7), (6.6) we have

$$de_i = \alpha_i^j e_j = \alpha_i^j x_j^k a_k ,$$
$$de_i = a_k dx_i^k + x_i^k da_k ,$$

which imply that

(6.11)' $$\alpha_i^j x_j^k = dx_i^k \qquad (\text{mod } u) .$$

From (5.10), (6.11), (6.11)' we thus have

(6.12) $$\alpha_i^m = Y_j^m dX_i^j = y_1^m dx_i^1 \qquad (\text{mod } du) .$$

Differentiation and a use of (6.7), (6.12) give

(6.13) $$d(y_i^j e_j') = dy_i^j e_j' + y_i^j \alpha_j^k e_k'$$
$$= (dy_i^j + y_i^k dx_k^1 y_1^j) e_j' = 0 \qquad (\text{mod } du) ,$$

which is zero by differentiating and using (6.11). Therefore $y_i^j e_j'$ are functions of u only, and

(6.14) $$y_i^j e_j' = a_i'(u) \quad \text{or} \quad e_i' = x_i^j a_j'(u) .$$

This shows that the vector e_i' satisfying (6.7) are of a very special form.

The above discussions help us to define an affine transformation of the tangent space $V(p(u_1))$ onto the tangent space $V(p(u_2))$, where $p(u_1)$, $p(u_2)$ are two points of $C(u)$ corresponding to the two parameters u_1, u_2 respectively. In fact, a point of the tangent space $V(p(u))$ can be defined by $x^i a_i(u)$. We map the point $x^i a_i(u_1)$ of $V(p(u_1))$ into the point $x^{*i} a_i(u_2)$ of $V(p(u_2))$, where x^{*i} are defined by the equation

(6.15) $$p'(u_1) + x^i a_i'(u) = p'(u_2) + x^{*i} a_i'(u_2) .$$

Let us denote the above affine transformation by $T_{u_2 u_1}$.

Equation (6.15) is the analytical expression of the operation of developing the tangent spaces along a curve into an affine space and comparing these tangent spaces by means of the development. We notice from (6.15) that $p(u_2)$ may not be the image of $p(u_1)$ or $x^i = 0$ does not have to imply that $x^{*i} = 0$, so that $T_{u_2u_1}$ in general is not a linear mapping of the tangent spaces. However, there is a linear mapping $T'_{u_2u_1}$ of the vector space $V(p(u_1))$ into $V(p(u_2))$, by which the vector $x^i a_i(u_1)$ goes into the vector $x^{*i}a_1(u_2)$, where x^{*i} are defined by

$$x^i a_i(u_1) = x^{*i} a'_i(u_2) \; . \tag{6.16}$$

In order for the mappings $T_{u_2u_1}$ and $T'_{u_2u_1}$ to be meaningful they must be independent of the following choices which have been made: 1) Choice of the initial frame $p'e'_1 \ldots e'_n$ for $u = u_o$ and $x^k_i = \delta^k_i$. 2) Choice of the family of frames $pa_1 \ldots a_n(u)$ along $C(u)$. To prove their independence from the choice 1) we denote by p'^k, e'^k_i the components of p', e'_i respectively. Then (6.7) can be written as

$$\begin{aligned} dp'^k &= \alpha^i e'^k_i \, , \\ de'^k_i &= \alpha^j_i e'^k_j \, , \end{aligned} \tag{6.17}$$

from which it follows that if p'^k, e'^k_i are solutions of (6.17), then the functions

$$\begin{aligned} p''^k &= h^k + g^k_1 p'^1 \, , \\ e''^k_i &= g^k_1 e'^1_i \, , \qquad \left| g^k_1 \right| \neq 0 \, , \end{aligned} \tag{6.18}$$

where g^k_1, h^k are constants and also solutions of (6.17). Moreover, these are the most general solutions, since by a proper choice of g_1^k, h^k, the solutions p''^k, e'^k_i can take arbitrary values for $u = u_o$ and $x_i^k = \delta_i^k$. The conditions (6.15), (6.16) remain unchanged, if p', e'_i are replaced by p'', e''_i. This shows that $T_{u_2u_1}$, $T'_{u_2u_1}$ are independent of the choice 1). Independence of the choice 2) follows immediately from (6.6), (6.14), when x_i^k are taken as functions of u.

From (6.15), (6.4) it is clear that the mappings $T_{u_2u_1}$ and $T'_{u_2u_1}$ have the following properties:

$$T_{u_3u_1} = T_{u_3u_2}T_{u_2u_1}, \quad T_{u_2u_1} = T^{-1}_{u_1u_2},$$
$$T'_{u_3u_1} = T'_{u_3u_2}T'_{u_2u_1}, \quad T'_{u_2u_1} = T'^{-1}_{u_1u_2},$$

where u_1, u_2, u_3 are three distinct values for u.

From the above discussions we can arrive at the definition of the group of holonomy. Let M be connected, and O a point on M. Let $\{\gamma\}$ be the family of closed parametrized curves in M with O as initial point. To each closed curve γ there corresponds an affine transformation T_γ of the tangent space V at O. All these affine transformations form a group, which we shall call the group of holonomy at O and shall denote by H_o. Similarly the linear mappings T'_γ corresponding to the curve γ also form a group, to be called the restricted group of holonomy at O and denoted by H'_o.

Suppose that, instead of O, we choose a point O' as initial point. Let β be a parametrized curve joining O to O'. By means of β we map a closed parametrized curve γ through O into

$\gamma' = \beta\gamma\beta^{-1}$. This induces a mapping of T_γ into $T_{\gamma'}$, and defines a homomorphism of H_0 into $H_{0'}$. It is easy to see that this homomorphism is an isomorphism onto. In particular, it follows that the group of holonomy, as an abstract group, is independent of the choice of 0. The same result holds for the restricted group of holonomy.

DEFINITION 6.1. Let Q be a positive definite quadratic form. If the restricted group of holonomy of an affine connection leaves Q invariant, then the affine connection is called a metrical connection. If the restricted group of holonomy of an affine connection leaves Q = 0 invariant, then it is called a Weyl connection.

The importance of the groups of holonomy lies in the fact that notions which are invariant under them can be defined. For example, in the case of a metrical connection, we can define the scalar product of two tangent vectors with the same origin. To show this, suppose the scalar product of two vectors at a point 0 be defined. Let v_1, v_2 be two vectors at a point $p \in M$. Join p and 0 by a parametrized curve α, and denote by T'_α the linear mapping of V(0) onto V(p). Then we define the scalar product as

(6.20) $$v_1 \cdot v_2 = (T'^{-1}_\alpha v_1) \cdot (T'^{-1}_\alpha v_2) .$$

To show that it is independent of the choice of α, let β be a second parametrized curve joining 0 to p, and T'_β the corresponding linear mapping of V(0) onto V(p). Then $T' = T'^{-1}_\beta T'_\alpha$

is an element of H_0', and leaves the scalar product of vectors at 0 invariant. Hence we have

$$(T_\alpha'^{-1}v_1).(T_\alpha'^{-1}v_2) = (T'T_\alpha'^{-1}v_1).(T'T_\alpha'^{-1}v_2)$$
$$= (T_\beta'^{-1}v_1).(T_\beta'^{-1}v_2) ,$$

which shows that the definition (6.20) is independent of the choice of α.

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