1968

A preventive maintenance cost model application

Ted Chris Spiropoulos
Lehigh University

A PREVENTIVE MAINTENANCE COST MODEL APPLICATION

by

Ted Chris Spiropoulos

A Thesis

Presented to the Graduate Committee

of Lehigh University

in Candidacy for the Degree of

Master of Science

in Industrial Engineering

Lehigh University

1968

CERTIFICATE OF APPROVAL

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

May 21, 1968
Date

Wallace J. Richardson
Professor in Charge

Head of the Department of
Industrial Engineering

ACKNOWLEDGMENTS

The author wishes to thank Professor W. J. Richardson of the Industrial Engineering Department, Lehigh University, for his generous devotion of time, encouragement and excellent guidance during the counseling of this thesis.

The assistance received from Dr. N. D. Singpurwalla of the Western Electric Co., Inc. and former professor, Dr. B. Rhodes is also appreciated. The author also extends thanks to the Western Electric Co., Inc. whose sponsorship has made this study possible.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

This thesis has developed a proposed model for the solution of the problem of determining, on the basis of cost, how to maintain industrial equipment subject to wearout failure. Its features are the consideration of equipment as a system of parts and a process as a system of pieces of equipment. It includes use of a versatile probability distribution, solvable by using tables for derivatives of the normal distribution, as a potential means of characterizing failure distributions. Bayesian statistics are used to determine the parts or pieces of equipment which contribute the most to system failure. Renewal theory is used to predict the frequency of failures and preventive maintenance actions under a given maintenance policy; and the emphasis is placed on the maintenance engineer and practitioner as the man with the practical experience to conduct the preliminary analyses of equipment which form the foundation of and provide justification for a more extensive analysis, using the techniques suggested by the author.

The technique was applied to a sample of data obtained from a pharmaceutical firm. These data were typical of industry and satisfactory in general. The model proved to be a good predictor of actual experience. It should be emphasized that limitations of the use of the model probably will lie in the area of the completeness and accuracy of the historical records of maintenance experience.

I. INTRODUCTION

The idea of exercising maintenance action on a piece of equipment before it fails, what is called now Preventive Maintenance, is not new. L. C. Morrow, a maintenance consultant writing for Factory Magazine, claims to have a document on preventive maintenance dated 1925. At that time maintenance was considered as merely something to be tolerated. Very few records were kept with very little, if any, cost accounting.

In 1930 and 1931, Management, Maintenance and Materials Handling forums were held, which allowed approximately 200 to 300 maintenance people to express an interest in improving maintenance management. While the depression of the 1930's discouraged any further forums, World War II encouraged many developments in maintenance due to its key role in assuring high production levels. However, where money was readily available during the war, after the war cost cutting forced maintenance to backslide into its former subdominant role. This situation continued until 1950, when the first of a series of maintenance conferences was held in Cleveland, Ohio.(1)

Presently, such organizations as the American Management Association conduct regular seminars on maintenance topics. The boom of the space industries has contributed greatly to the increasing interest in new techniques of maintenance analysis. The field of reliability, or the study of the dependability of equipment, is growing with the space age. Before 1958, there were very few technical papers written on reliability. When the papers appeared, they were mainly concerned with such equipment as electron tubes and various other electronic components whose failure rates were exponentially distributed.

Since 1960, reliability analysis has expanded into the area of equipment subject to wearout failures, such as motors, bearings, and transmissions. Such authors as B. J. Flehinger, R. Barlow, F. Proschan, L. C. Hunter and others have written a number of articles covering a variety of possible preventive maintenance policies. Due to the nature of the mathematics of these articles, it is doubtful that many companies have attempted to apply these models. There seems to be a great deal of work to be done to translate the newly developed and applied mathematics into courses of action to guide maintenance managers.

II. BACKGROUND

A. The Empirical Approach

From many interviews with key maintenance personnel during the past four years and reading the available literature on preventive maintenance one can easily be convinced that preventive maintenance can offer a greatly needed solution for plant maintenance problems. Many lists of the benefits derived from P/M (abbreviation for preventive maintenance to be used from now on) are available. Such benefits are usually:

1. Less production downtime, with the corresponding savings produced by fewer breakdowns, i.e., higher employee morale, better quality product, lower production cost through greater efficiency.
2. Less overtime pay due to fewer emergency breakdown repairs compared to planned or scheduled maintenance.
3. Fewer large-scale repairs or overhauls and fewer repetitive repairs due to better control of the state of equipment. The result of these benefits should especially be better maintenance manpower and equipment loading.
4. Lower repair costs for small repairs due to more efficient spare parts management, fewer skills needed

for repairs, and fewer parts used on planned repairs.

5. Fewer product rejects and less spoilage due to more efficiently operated equipment.

6. Prolongment of equipment life resulting in minimum cash outlay for capital equipment.

7. Less standby equipment normally used to minimuze down-time produced by breakdowns.

8. Better control of misapplication of equipment, equipment operator abuse, and obsolescence.

9. Better control of maintenance spare parts inventory resulting from more predictable demand for repair parts.

10. Greater employee safety resulting from the better operating condition of equipment.

11. Lower unit manufacturing cost due to decreased down-time of production equipment.(2)

The fact that maintenance represents an appreciable cost to a manufacturing plant is demonstrated by published U.S. Government and Securities Exchange Commission statistics. According to the files of the Securities Exchange Commission, 687 companies spent an aggregate of 7.51 billion dollars for maintenance in 1965. Since these companies represented 53.8% of total sales for manufacturing companies, a linear projection of the total maintenance cost for all manufacturing concerns amounts to over 14 billion dollars. A U.S. Census Bureau appraisal of maintenance costs for 50,000 manufacturing plants in 1965 yielded a figure of approximately 9 billion dollars in annual cost. The breakdown of these costs is 50% for payrolls and 50% for materials, supplies, and contracted maintenance.[3] Although the two cost figures have different bases, they are large enough to cause great concern.

Many managers, trade magazine editors, and experienced maintenance personnel have realized both the great cost of maintenance and the great contribution preventive maintenance can make to reducing this cost. However, the literature contains little concerning the mechanics of reducing and evaluating maintenance expenditures, but emphasizes all the benefits to be obtained from the preventive maintenance approach. A few examples of the preventive maintenance programs instituted by some of the large industries in the country will be cited. This is not intended to be a complete survey of the field but merely a sample.

Union Carbide Company produces more than 60 products at its 318 acre, Texas plant. The equipment used can, in some cases, produce more than one product. Many products and intermediates are interrelated with each other in the production cycle. Demand varies independently for many of the products. This situation is a complex one, demanding excellent performance of equipment and complicated scheduling of required maintenance downtime.

A summary of the solution to this particular problem is as follows:

1. A formalized planning and scheduling program developed these six objectives:

   a. Obtain optimum maintenance at least cost. This requires optimum maintenance calls with best utilization of major maintenance equipment.

   b. Reduce downtime due to breakdown. Revamped schedules and methods have often cut downtime 15% - 20%. Planned shutdowns have in some cases been reduced from 10 to 3 shifts.

   c. Reduce maintenance overtime. Here the article simply states the difficulty of deciding the quantity of overtime to justify. Scheduled start-ups are given priority, which dictates to a great extent overtime hours. Still, overtime has been held to a three year average of 3.3%.

   d. Lower material and tool expenditures. No approach or method is cited in this case. Total tool cost, however, remained constant for 2 years.

e. Reduce the maintenance function of total manufacturing cost. Labor cost, a big factor, has been reduced due to 100% better worker activity resulting from improved scheduling. Fewer non-essential jobs are being done. The total result is $230,000 of labor savings over a three-year period.

f. Improve operating efficiency through reduction of conflicting shutdowns which lowered production output.

2. Better organization of maintenance information, including new repair forms and better communications throughout the plant due to meetings and improved handling of paperwork.(4)

Nowhere in the solution were the hard facts, the mechanics of the system cited. How were individual equipment characteristics incorporated in the scheduling of shutdowns? How were failure rates used to decide how much maintenance to do and with what frequency? To what extent does an extra dollar spent on maintenance of a piece of equipment reduce operating cost? How do you determine the pieces of equipment which deserve thorough scheduled maintenance and those which require a relatively small degree?

Monsanto Chemical Company has another approach to the problem of reducing high maintenance costs. They call it preventive engineering.

The claimed benefits are:

1. Reduced maintenance and production costs. Although no change in the maintenance craft operations were made, maintenance's portion of total manufacturing cost was reduced 12.6% over a two year period.
2. Improvement of maintenance downtime. Downtime was reduced from 5% of scheduled hours to 3½%.
3. Solution of costly industrial relations problems. Turnover of engineers is lower and relations of production supervision with plant engineering are improved.

Preventive engineering is defined as "having enough trained engineers in your maintenance organization to enable each one to use his talents to best advantage and do his job right."

The probram was set up by adding more graduate engineers to the maintenance engineering staff. The engineers were instructed continually to reduce the need for maintenance, decrease mechanical downtime, and improve equipment performance.

An empirical ratio of one engineer per $800,000 to $1,500,000 worth of process equipment was established. Each engineer was to perform engineering required for breakdown maintenance, execute the preventive maintenance program, supervise the spare parts program, engineer minor new installations and construction work, and still have time for preventive (maintenance) engineering.[5]

Again, although this approach also claims many benefits, nothing of substance is offered other than the fact that more technical people lead to a better maintenance program. No mention is made of what execution of the preventive engineering program entails.

Buick Motors Division of General Motors Corporation has instituted an extensive preventive maintenance program. The account of this activity is more detailed than average and gives some insight into the decisions required in the management of such a program. To determine the equipment which deserves P/M attention, the following checklist is used by plant supervisors:

1. Will the equipment's failure shut down integrated units?
2. Is it the only equipment available for the job?
3. Must it be continuously available for an indirect activity?
4. Do the equipment characteristics or environment demand P/M ?
5. Is the value of the equipment high enough to require

good maintenance protection?

6. Does its malfunctioning threaten product quality?

7. Is it an important property or building service installation?

After the proper equipment has been selected, the inspection frequency is established by considering manufacturer recommendations. However, such information is greatly amended by knowledge and experience with the inspected equipment. This type of analysis has led to 25% coverage of all facilities. Buick realizes that coverage shouldn't be 100%, but honestly admits that the proper coverage is an unknown quantity.[6] This problem seems to be universal.

L. C. Morrow is considered by some to be the pioneer of modern maintenance. His opinion on the state of managed maintenance was that there is a shocking contrast in American industry between progress in scientific management of production and the slow advances made in maintenance. Mr. Morrow's estimate of the state of the art was that half of America's manufacturers are less than 1/3 of the way along the road from the crudest management methods to the newest techniques. His conclusions were based on polls taken on a rate-yourself basis of companies participating in various management seminars. Mr. Morrow also concluded that management must place an emphasis on maintenance progress if advances are to be made.[1] Although these views seem to contain a good amount of truth, there also seems to be a general scarcity of good tools for evaluation of the maintenance function. This point was hinted at in the conclusion of the preceding paragraph.

To emphasize and clarify this statement a few maintenance articles will be cited to demonstrate some of the available information on measurement of maintenance performance, specifically the measurement of preventive maintenance operation.

A relatively recent article which was supposed to update the present approaches to managing P/M offered the advice not to go overboard on P/M and that it should be realized that there is an optimum approach. To give strength to these statements, it was mentioned that too little or too much maintenance can increase manufacturing costs. It is stated that the enlightened view allows P/M to be applied only where it will pay. Equipment which is essential for continued production is designated as deserving P/M treatment. As a further guidepost, the article concludes that cost of P/M should not be the only guide, since the effects of deferred maintenance may not be detected for a while.[7]

In other words, the article admits that the preventive maintenance function, although capable of being optimized, it is complex to the extent that concrete techniques and decision rules are not available to promote scientific management.

Another attempt to quantify the evaluation of preventive maintenance, states the problem as determining whether equipment is being over or under maintained. The solution offered is an individual analysis of actual results. One cost factor is that of inspections and services. The counterbalancing cost factors are over-all costs of repairs and breakdowns. The proposed decision rules are, if there are no repairs, there are chances of over-maintaining. If there are too many repairs, the inspections are inadequate. It is, therefore, concluded that a dollar-wise appraisal helps to arrive at the proper balance.(8)

Such empirical management tools as presented in all the previously cited articles are actually no more than sales pitches for preventive maintenance management. Any manager or practitioner searching for concrete solutions to his maintenance problems would have no concept of how to institute and gain the claimed benefits of these approaches.

B. The Theoretical Approach

In recent years a field called Reliability has evolved to meet the strict demands placed on the performing excellence of America's space program. Reliability has been defined as "the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered." (9)

The three essential aspects of Reliability are satisfactory operation of equipment, operation within specific performance limits, and operation for a definite time period. Reliability is integrally related with maintainability and availability. Maintainability (M) is defined as "the probability that a device will be restored to operational effectiveness within a given period of time when the maintenance action is performed in accordance with prescribed procedures."(9) (In short, the ease of repairing a piece of equipment.) Availability is the fraction of total on-stream time that a piece of equipment is operable. All these equalities are quantifiable in a manner which allows arithmetic evaluation of equipment operation parameters. Reliability is quantified as the mean time between failures. This mean value is the mean of a distribution of failure probabilities. Two types of failures contribute to the distribution of time between failures. One type is wearout failure, which is exemplified by piston rings, bearings, or any part whose reliability depends on equipment age. The average wearout failure distribution is Gaussian. The second type of failure is due to strictly random occurrences. A tire puncture or failure of a capacitor is an example of such a failure. The failure distribution for such parts is usually exponential, where reliability depends on operating time rather than age of equipment.(10)

Attempts have been made to apply Reliability Theory to chemical equipment. Since this is an area of interest of this paper due to the material available and the background of the writer, a few of the more recent contributions to the chemical field will be discussed.

An article by Edward J. Gibbons of Colgate-Palmolive Company claims that application of probability theory to chemical equipment configurations can help to make the following decisions:

1. Whether a given layout or design can offer the required on-stream time.
2. Whether too many pieces of equipment are arranged in series.
3. Whether standby equipment or alternate processing paths can be justified.
4. Whether upgrading the quality of equipment will sufficiently increase on-stream time.
5. Whether equipment should be grouped so that failure of one unit does not effect the balance of plant operations.(11)

With the same approach, Mr. Gibbons could have included the decision of what degree of maintenance will improve equipment on-stream time. The approach taken was that a parameter termed "Mechanical Availability" could be used to measure the degree of operating time offered by various equipment configurations. Mechanical Availability is identical with the parameter Availability already defined.

Figure 1 is a reproduction of a graph showing the relationship between repair costs, composed of breakdown and preventive maintenance, and percent Mechanical Availability. This curve represents the average relationship between the two plotted variables. The importance of the curve's shape is that throughout the range of Mechanical Availability an incremental change in repair cost will produce a corresponding change in Availability until an Availability of approximately 96% is approached. Beyond this point, an infinite maintenance expenditure will not offer 100% Availability, since

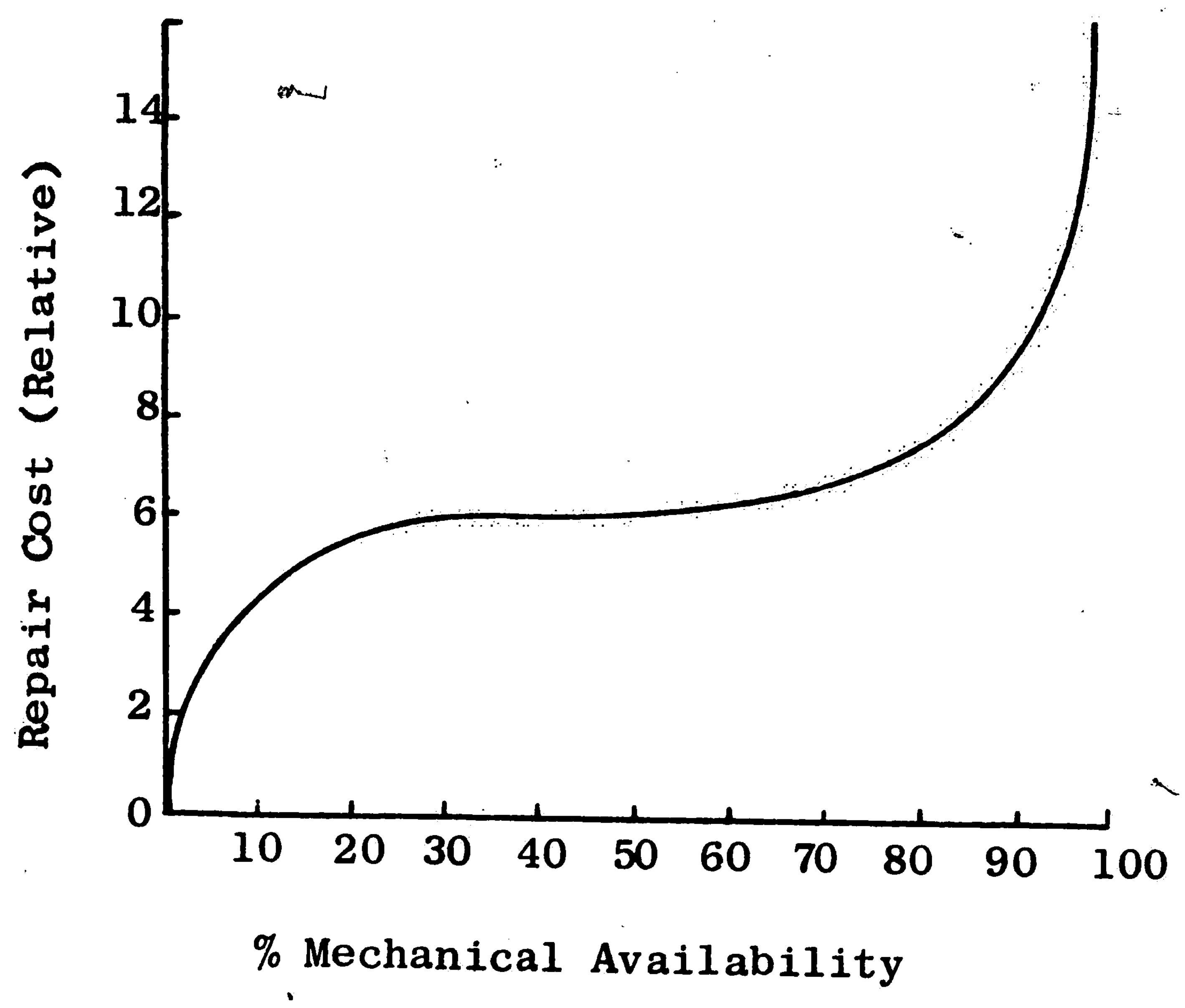


Figure 1. Maintenance Costs vs. Mechanical Availability

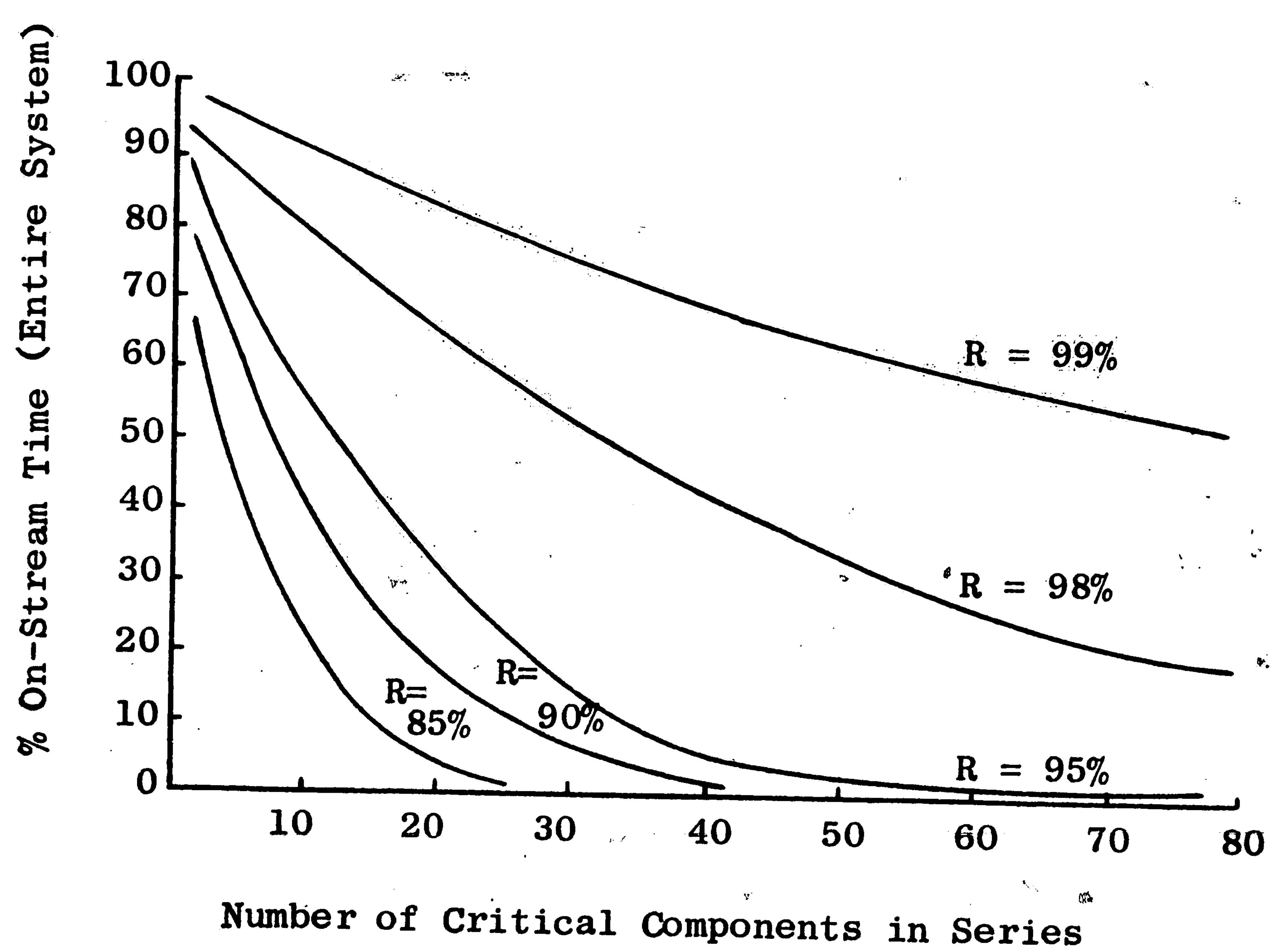


Figure 2. Percent On-Stream Time vs. Number of Components in Series

Availability approaches the 100% value asymptotically. Although the author does not detail the procedure, he states that at a point on this curve the maintenance cost will be high enough to exceed the advantage of added on-stream time.

The curves of Figure 2 show the relationship between Mechanical Availability, expressed as percent on-stream time, and the number of critical components in series. A critical component is defined as a piece of equipment whose failure would shut down the system of which it is a part. The one restricting assumption which permits use of this graph is that every critical component in series has identical Mechanical Availabilities. However, the graph is useful in that it indicates how too many series components can drastically reduce over-all system efficiency, even with the help of infinite maintenance expenditures.

The interesting aspect of the article is demonstrated in Figure 3. The author shows a particular configuration of equipment which could occur in many parts of a given chemical plant and especially in the chemical industry. Such a configuration could be used for batch mixing, effecting mild temperature changes, or even a crystallization.

The parts of the system are assigned constant probability values for Mechanical Availability. With this information the system availability can be determined. Decision 1 and 2 cited at the beginning of the discussion of the article can now be made multiplying the Availabilities of the systems in series in the following manner:

$$P_o = P_1 \cdot P_1 \cdot P_3 \cdots P_n$$

where

P_o = over-all system Availability

P_i = Availability of each unit i in the system, i = 1, ...n

n = number of units in system

Decision 3 can be calculated for parallel configurations of units in the over-all system. Since all the units in a parallel hook-up must fail for the system to fail, the P_o for this case is calculated as follows:

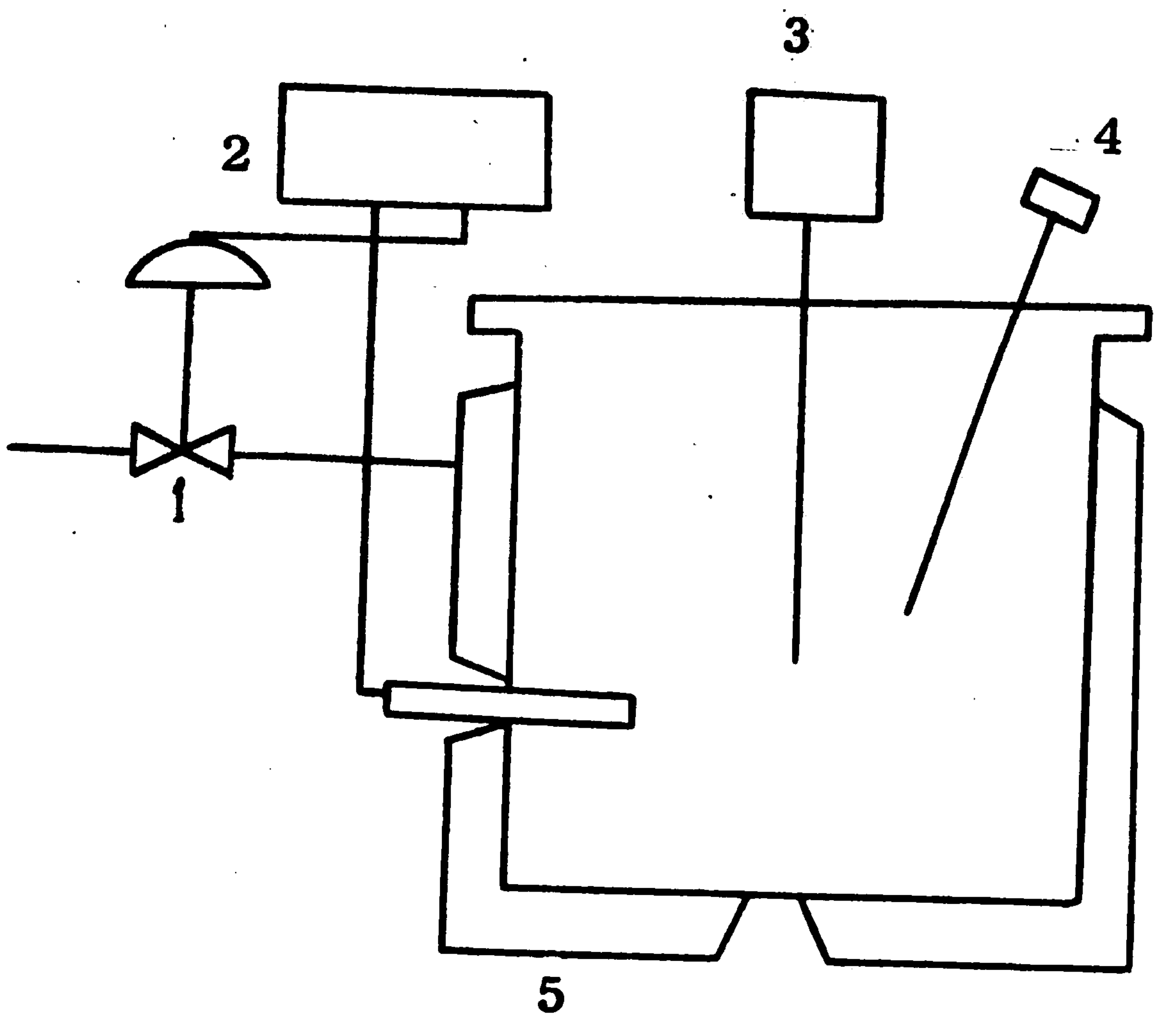
$$P_o = (1-P_1)\ (1-P_2)\ \ldots.(1-P_n),$$

where

$(1-P_1)$, $(1-P_2)$ = the probability of unit failure, i.e., the opposite of Mechanical Availability.

Decision 4 can be calculated by substituting the new Mechanical Availability values, caused by better equipment, into the preceding formulas. Decision 5 is calculated by using the formula for the parallel case until all parallel units in a configuration can be represented in series by substitution of the parallel unit P_o's in the series formula.[11]

The approach taken by the author is limited by a few practical drawbacks. The author assumes a constant probability value for the various units in a system. He also assumes that each member of a series of identical equipment configurations will have the same Mechanical Availability in each configuration. Although these simplifying assumptions enable crude comparisons to be made, they also



Number and Item

1. Control valve
2. Control instrument
3. Motor and mixer
4. Thermometer
5. Mixing Vessel

Figure 3. Diagram of Mixing Tank with Five Components in Series.

drastically fall short of representing reality. Rarely are even two similar pieces of equipment alike enough in maintenance or usage history to enable the assignment of fixed and equal Mechanical Availability values to each of them. More than two pieces of equipment complicates the situation to an even greater extent. Mechanical Availability is a time-dependent quantity which varies stochastically at a low level during initial start-up to a higher value, independent of equipment age, during normal life, and degenerates to a lower value in the period where parts are wearing out. Since this occurs, in many cases, in each part of a piece of equipment the over-all Availability is the combined result of many probability distributions. Only if these factors can be included in a model of maintenance characteristics, will the model be meaningful and broadly applicable.

An article by Dale F. Rudd of the University of Wisconsin takes the approach of introducing redundancies at various steps in a process design to improve over-all system performance. He takes the basic reliability approach and incorporates in his solution the failure characteristics of initial start-up, random failure, and wearout failures inherent in such equipment.

The author realizes that his proposed solution requires an economic means of evaluating alternatives in design. He uses dynamic programming to decide what level of redundancy is needed to offer the desired equipment performance. This application, however, applies only to a series hook-up of process equipment. For more complex designs combining parallel and series components, Baye's theorem is applied and demonstrated. The admitted drawbacks are that probabilities of equipment must be accurately determined before either analysis will yield good results for desicsion making. For Baye's theorem all the possible events which could occur in a design must be determined, then all probabilities for these events must be known. Only after this demanding analysis can the system reliability be known. The author admits that no current rational method is known for determining the optimal design of the complex systems consisting of both parallel and series elements.(12)

Summary

The problem which has evolved in the attempt to operate process equipment at optimum cost is the lack of practical quantitative methods of measuring the type and degree of maintenance to perform on this equipment. Specific experience in industry has only yielded specific rules of thumb, which are not capable of general application without causing a great deal of error. In some cases maintenance managers admit they have no idea and know no solution method for determining which level of maintenance is optimal. Several approaches toward evaluation and improvement of equipment operating efficiency were discussed; however, they were either over-simplified or required constant probability values, which are neither realistic nor easily obtainable.

The need is plainly for an equipment maintenance cost model that can be broadly applicable and rigorous enough to represent reality.

III. STATEMENT OF THE PROBLEM

It is known by experience that the greatest cost of downtime is associated with the greatest complexity of equipment layout and the greatest lack of thorough preventive maintenance analysis.

There seems to be a pattern in the nature of maintenance difficulties. The most serious problem seems to be in the inconsistent quality of the repair work done on plant equipment. Too many preventive maintenance inspections have not (most of the time) improved the frequency of failure. In fact, inspections in many cases do not investigate the most crucial parts of a piece of equipment. This is not a fault of the inspector, but can be attributed to a general lack of thorough inspection instructions.

For these reasons the department engineers, who are responsible for their particular pieces of equipment, rely on themselves to provide adequate maintenance protection and records. They are kept so busy that they have no time for a thorough preventive maintenance analysis in their areas. Even if they did, they did have the time, they would be interested in their own specific needs rather than an analysis with broad application throughout the plant. Another limitation is the fact that an engineer would not have at his command the proper statistical tools to develop the required solution.

The problem, therefore, seems to have evolved into a general need for the development of a method for balancing the cost of maintenance with the cost of product loss and/or production downtime caused by too little maintenance.

This thesis will provide the means for a decision on the basis of cost as to whether or not institute a P/M program on equipment subject to wearout failure, through:

1. Analysis of the important parts of equipment using a building block approach.
2. Developing relationships which exist between parts, regarding their contribution to a system failure and characterizing the failure distribution of the parts.
3. Use the preventive and failure maintenance, cost models developed for comparison in decision making.

IV. EXPERIMENTAL PROCEDURE

A. Bayesian Approach

In order to generate a preventive maintenance model which will have the greatest possible general applications, a basic equipment array is proposed as illustrated in Figure 5. Any piece of equipment should fall into this generalized array, which is divided into individual parts, part subsystems and an equipment system. There are:

n part groups

m subsystems

p systems

where n, m, and p can be any integers.

Each i^{th}, j^{th}, and k^{th} component of the over-all system can possibly cause a system failure. Therefore, for ease of symbolic representation an event, E, representing a failure will be subscripted E_{ijk} to identify its relation with the components of the system. Failure of a part is then related to the over-all system by identification of its particular subsystem and system group.

If the probability of a failure, B, depends on the occurrence of a number of events within its equipment system, it can be expressed as a function of certain X time depended functions, i.e. $P(B) = (X_{111}, \ldots., X_{ijk}, \ldots X_{nmp})$, where ijk are any integers within the configuration of n parts, m subsystems, and p systems. However, since the nature of this problem is such that good maintenance data are not available to allow development of probability values via the classical statistics

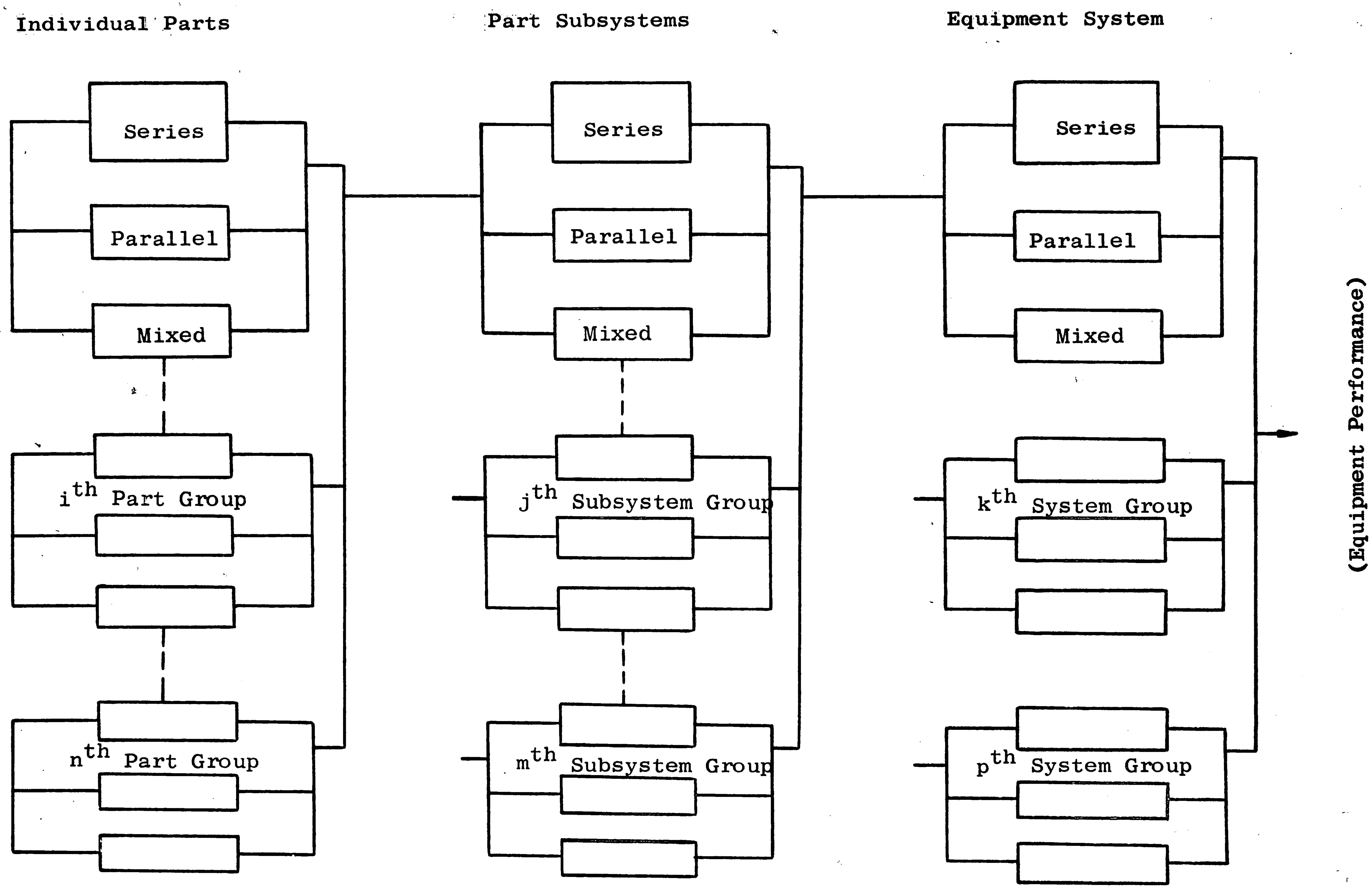


Figure 4 - Generalized Basic Structure of an Equipment Configuration

approach, the Bayesian approach to statistical decisions will be employed. Some advantages of this approach over the classical statistics approach, are the departure from the classical testing of a null hypothesis and attempting to protect within practical limits against Type I and Type II errors. Instead, the Bayesian approach emphasizes the risks of error of the decision rules being developed. The importance of this is that the risks of error for each decision rule are considered as a function of the possible values of the population being investigated. Rather than selecting the proper α and β values for a given sampling or data gathering scheme and then testing a hypothesis, the Bayesian approach attempts to select a decision rule which minimizes risks of error.[13] Such utilization of the risks of error allows assignment of a cost function to the error function, which appears to offer a good solution to the problem of incorporating the costs of various maintenance actions into the model of failure probability distributions. The technique for incorporating these costs will be treated later in the development of the over-all problem solution.

Another extremely practical aspect of the Bayesian approach is its use of a priori knowledge of a probability distribution in analyzing the probabilities of the occurrence of various events which depend on interactions with other events. This method easily lends itself to the calculation of the effect of various part failure probabilities on the failure characteristics of a part system. If the effect of a system's parts on the over-all system performance can be calculated and integrated with the cost alternatives involved, then the problem of developing cost decision rules for preventive maintenance will be solved. The solution would be an optimal allocation of equipment operating money to provide the greatest probability of equipment system performance at the lowest cost possible.

To initiate the solution, the Bayesian probability expressions for standard series, standard parallel, and mixed series-parallel systems will be developed.

Initially, Bayes' general formula is presented as follows: where there are Q mutually exclusive causes of failure E_{111}, E_{ijk}, ... E_{mnp}, E_{mnp} being the Q^{th} possible cause for an equipment system failure represented symbolically by B. The probability that a given part failure caused a system failure is given by:

$$P\left[E_{ijk}/B\right] = \frac{P\left[E_{ijk}B\right]}{P\left[B\right]} \qquad (1)$$

This formula is derived from the general expression for conditional probability, as developed by Hoel.[14] If the order of the conditional events in formula (1) is changed and the result solved for $P[E_{ijk}\,B]$, it becomes

$$P\left[E_{ijk}B\right] = P\left[E_{ijk}\right] P\left[B/E_{ijk}\right] \qquad (2)$$

Since the event B can occur only if one or more of the possible failures

E_{111}, E_{ijk}, ...E_{mnp} occurs, Then the probability of B occurring as a result of these failures is

$$P[B] = P[E_{111}B] + P[E_{ijk}B] + \ldots + P[E_{mnp}B] \quad (3)$$

where $E_{111}B$, $E_{ijk}B, \ldots E_{mnp}B$ are mutually exclusive events.

Now substituting formula (2) on the right of (3) we have:

$$P[B] = \sum_{i,j,k=1}^{mnp} P[E_{ijk}]\, P[B/E_{ijk}] \quad (4)$$

Substituting (4) and (2) in (1) we have:

$$P[E_{ijk}/B] = \frac{P[E_{ijk}]\, P[B/E_{ijk}]}{\sum_{i,j,k=1}^{mnp} P[E_{ijk}] P[B/E_{ijk}]} \quad (5)$$

which is Bayes' general formula for calculating the probabilities of causes expressed in terms of consistent notation used in the solution of the problem posed by this thesis.

If this formula is applied to the calculation of reliability for simple parallel networks as shown in Figure 5, the following result is obtained:

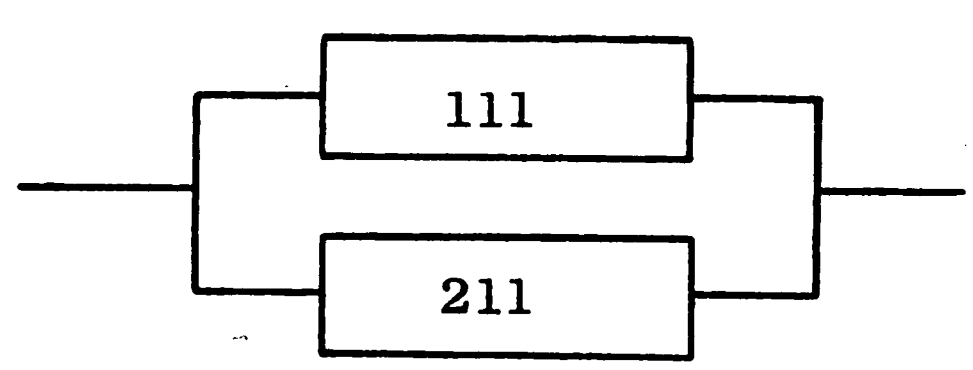


Figure 5. A Basic Parallel Network

If R_{ijk} = reliability of a given part = 1 - P (failure) = $1 - P[E_{ijk}]$ and $Q_{ijk} = P(E_{ijk})$ = probability of a part failure then

$$R_{ijk} = 1 - Q_{ijk}$$

Also, if Q_s = total system failure, then $Q_s = P[B]$. Such notation will keep the notation of this development consistent with the acknowledged texts in the field.[9,15]. Applying formula (4) to the mutually exclusive events that a system's reliability depends on the success or failure of a given component gives:

$$Q_s = P[B = R_{ijk} \cdot P[B/R_{ijk}] + Q_{ijk} \cdot P[B/Q_{ijk}] \qquad (6)$$

or

$$Q_s = R_{ijk} \cdot Q_s \left[\text{If part ijk operates}\right] + Q_{ijk} \cdot Q_s \left[\text{If part ijk fails}\right] \qquad (7)$$

Applying this formula to parallel case of Figure 6

$$Q_s = R_{111} \cdot P\left[B/R_{111}\right] + Q_{111} \cdot P\left[B/Q_{111}\right]$$

But, $P\left[B/R_{111}\right] = 0$, since the system can't fail if part 111 operates.

Thus, $Q_s = Q_{111} \cdot P\left[B/Q_{111}\right]$

Substituting,

$$Q_s = (1 - R_{111})(1 - R_{211}) = 1 - R_{111} - R_{211} + R_{111} \cdot R_{211}$$

Since

$$R_s = 1 - Q_s$$

$$= 1 - (1 - R_{111} - R_{211} + R_{111} \cdot R_{211}) = R_{111} + R_{211} - R_{111} \cdot R_{211}$$

which is the standard reliability expression for two components in parallel.

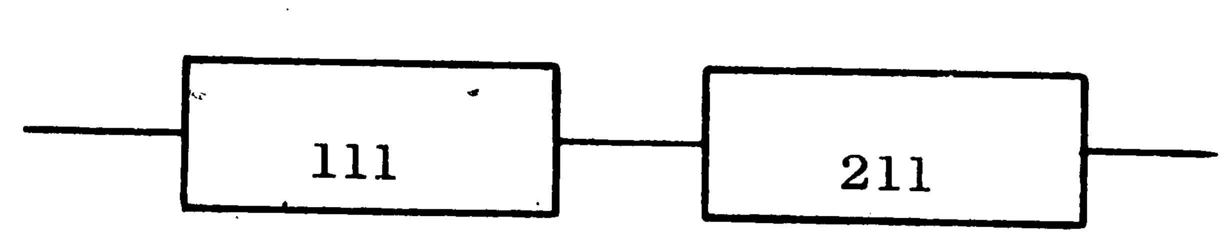


Figure 6 - Basic Series Network

For the series case illustrated in Figure 6, the following development results:

Part 111 will represent part ijk.

$$Q_s = R_{111} \cdot P[B/R_{111}] + Q_{111} \cdot P[B/Q_{111}]$$

$$Q_s = R_{111} \cdot (1 - R_{211}) + (1 - R_{111}) \cdot 1$$

$$Q_s = R_{111} - R_{111} \cdot R_{211} + 1 - R_{111} = 1 - R_{111} \cdot R_{211}$$

since

$$R_s = 1 - Q_s$$

$$R_s = 1 - (1 - R_{111} \cdot R_{211}) = R_{111} \cdot R_{211}$$

This result also is a standard expression for the reliability of two comp. in series.

Now that a more general tool of analysis has been examined and demonstrated in terms of the notation of this solution, the exact problem will be more fully developed and characterized.

Figure 7, consisting of Figures 7a, b, c, and d, represents graphically the general effect of a preventive maintenance overhaul on a piece of equipment consisting of eight parts.

Each part has a different failure distribution, which occurs at a different period in time. This is represented graphically in Figure 7a.

Figure 7b indicates the composite effect of the individual part failure characteristics on the failure of the over-all system. If a complete overhaul is performed on the system, which repairs or replaces the parts of Figure 7a, the failure pattern of the over-all

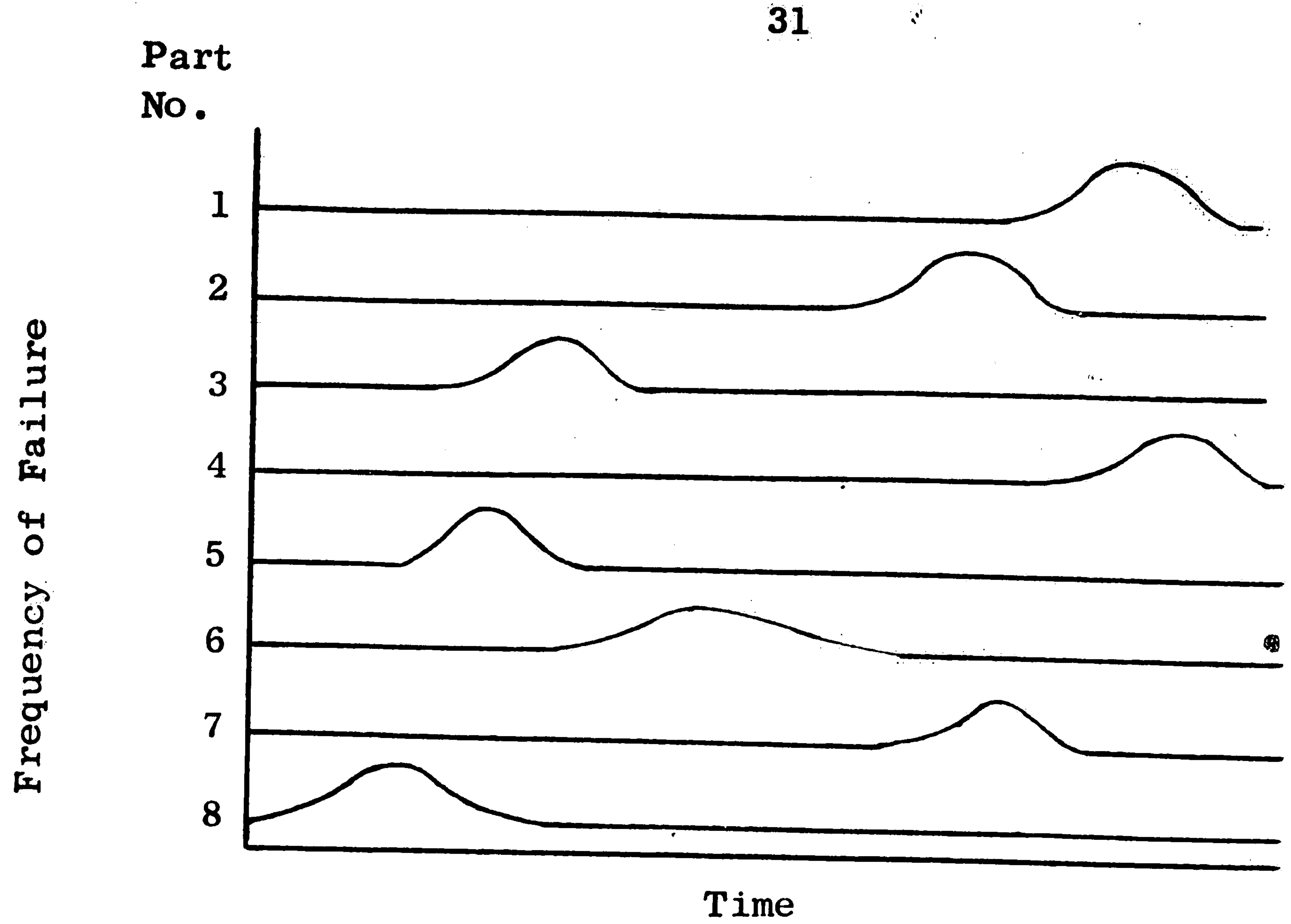


Figure 7a - Part Failure Frequency Distribution vs. Time

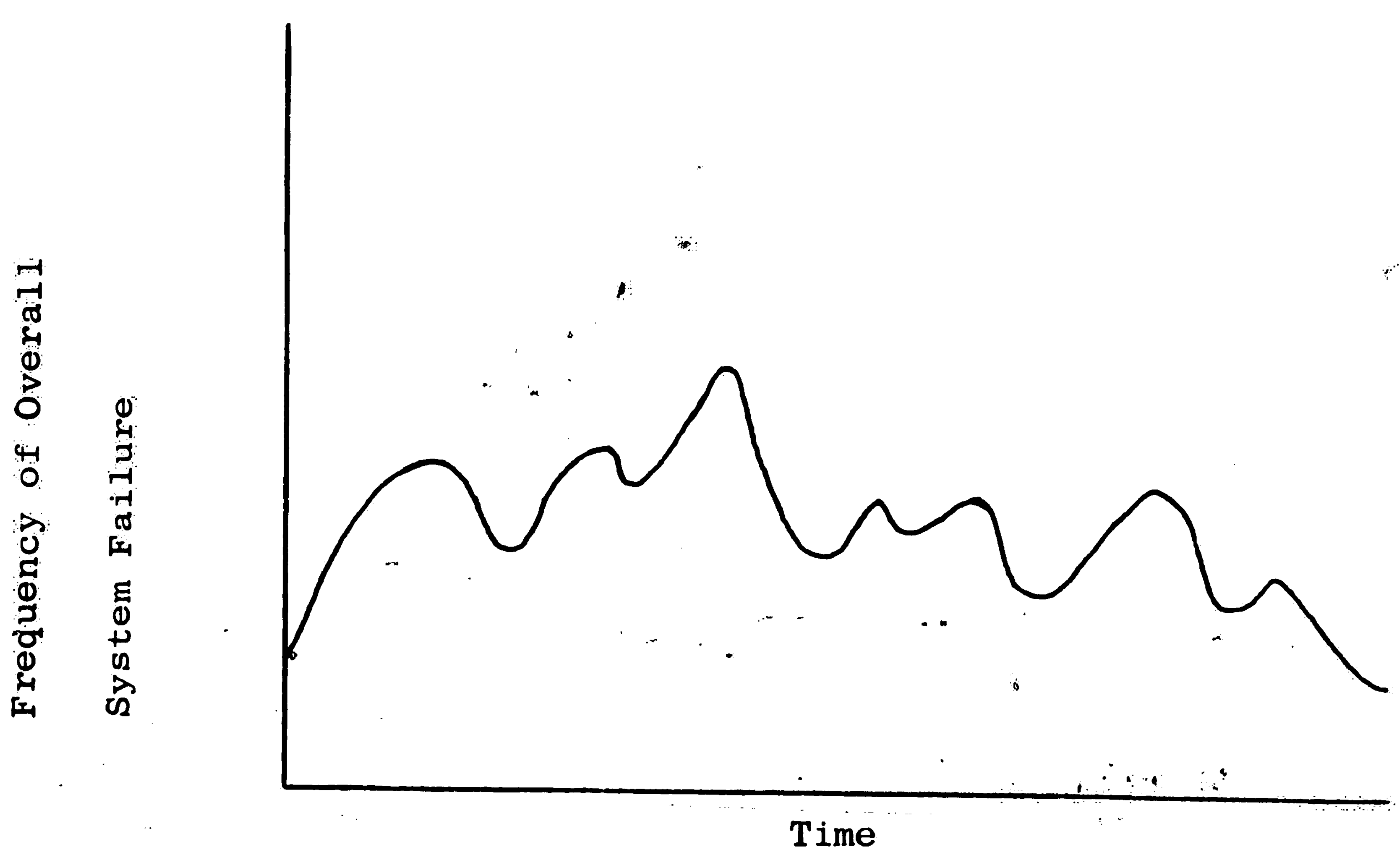


Figure 7b - Overall System Failure Distribution vs. Time

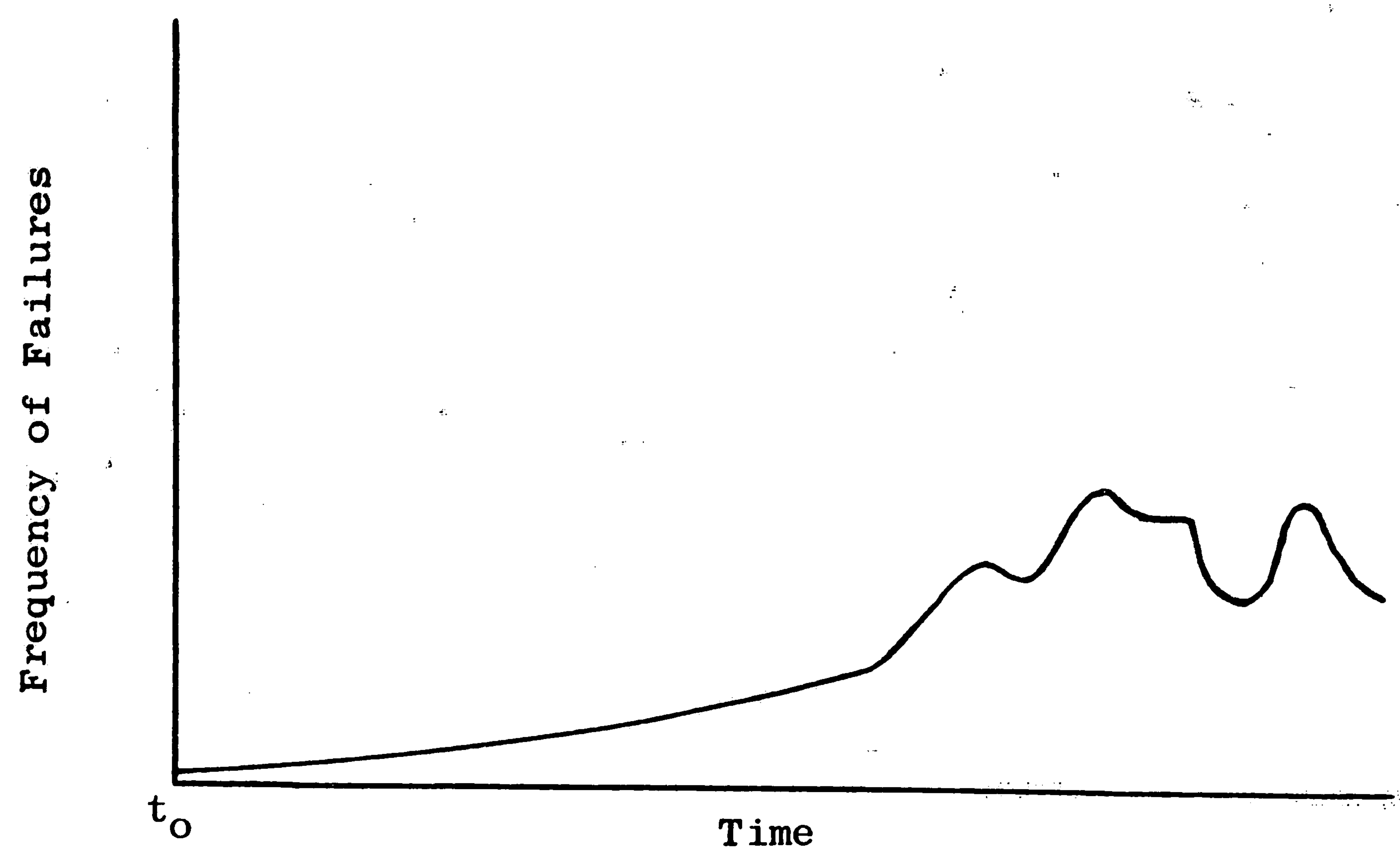


Figure 7c - Overall Failure Distribution of Equipment After A Major Overhaul at t_o

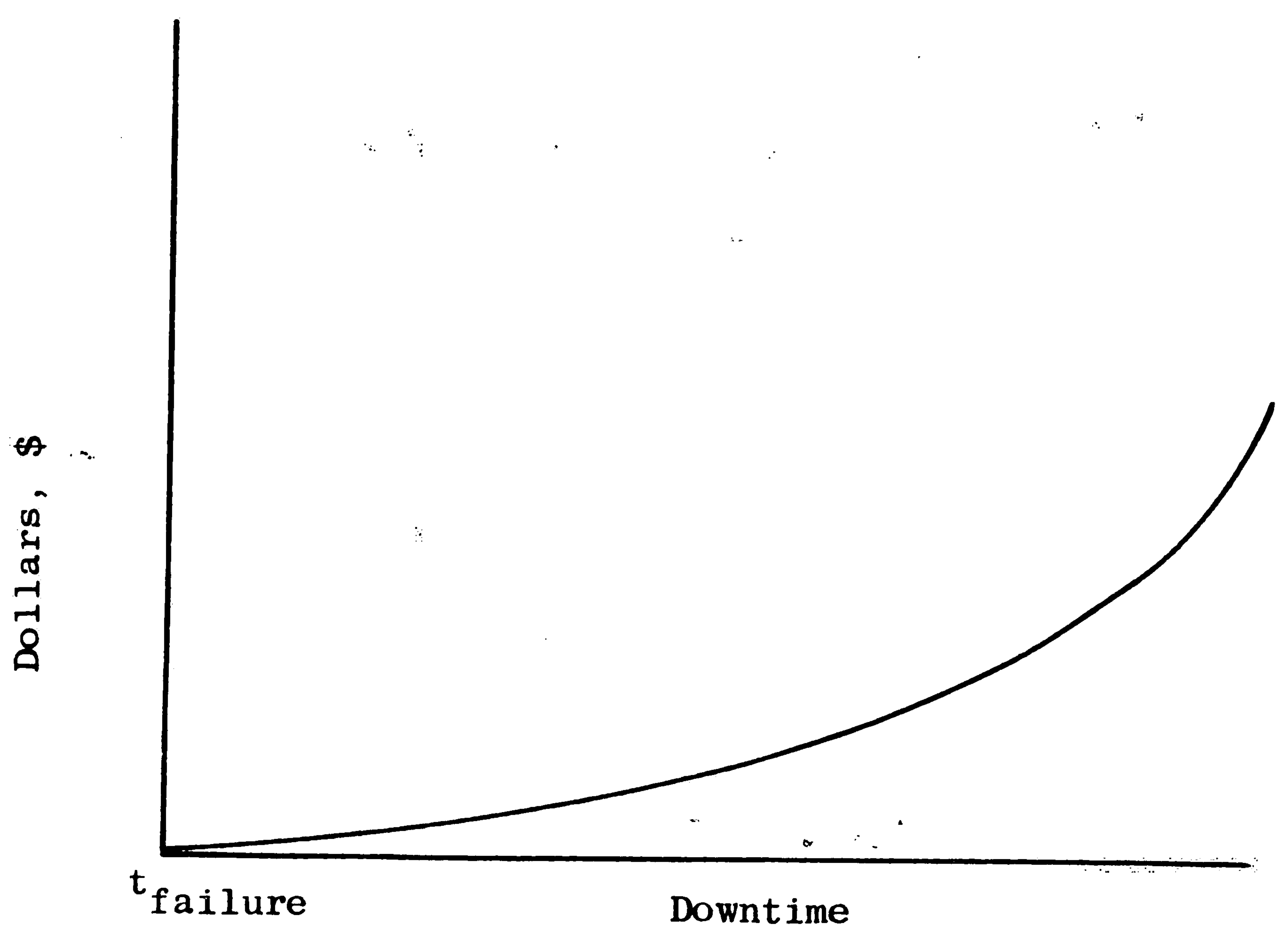


Figure 7d - Hypothetical Cost of Downtime For An Equipment Failure system is represented by Figure c. In effect, the overhaul has shifted the various part failure density functions further into time, so that the probability of a system failure has been decreased almost to zero for the near future.

After a certain time, however, the frequency of failure greatly increases to the point where possibly another overhaul or maintenance action of some type is warranted. Figure d represents a hypothetical equipment shutdown cost function.

Elimination of this cost of failure is the goal of preventive maintenance and any maintenance policy should be justified by cost comparison with the cost of failure, which should be the penalty of not performing the maintenance action.

The mathematics which characterize such an analysis of equipment operation will now be developed.

Let h = number of parts which cause a system failure. Let these parts be represented by h-tuple, $(X_1, X_2, \ldots X_h)$. Now if we also let the X's be the components of a vector $\bar{X}$ we have:

$$\bar{X} = \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_h \end{bmatrix}$$

Let $G(\bar{X})$ be the probability density function of $\bar{X}$.

Then,

$$G(\bar{X}) = \begin{Bmatrix} P(X_1) \\ P(X_2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ P(X_h) \end{Bmatrix}$$

The probability function P(X), to be realistically applicable to many different failure patterns, must possess a great amount of flexibility.

Here a probability function is presented which has a great amount of flexibility. It has the derivatives of the normal distribution as its terms, thereby readily available normal distribution tables can be used for solutions.

I wish to thank Professor Richardson for instigating the idea and Drs. Hamming and Rhodes for their constructive help to the solution.

Let P(X) be a probability density function which is expanded in a series of Hermite polynomials.(16)

$$P(X) = A_0H_0(X)e^{-X^2} + A_1H_1(X)e^{-X^2} + A_2H_2(X)e^{-X^2} \ldots \tag{8}$$

The Hermite polynomial, when integrated, gives the solution

$$\int_{-\infty}^{\infty} e^{-X^2} H_m(X)H_n(X)dx = \begin{cases} 0 & m \neq n \\ 2^n n! \sqrt{\pi} & m = n \end{cases} \tag{9}$$

To solve for the A terms, both sides of (8) are multiplied by $H_o(X)$ and integrated from $-\infty$ to ∞.

Thus,

$$\int_{-\infty}^{\infty} H_o(X)P(X)dx = A_o \int_{-\infty}^{\infty} H_o^2(X)e^{-X^2}dx + A_1 \int_{-\infty}^{\infty} H_o(X)H_1(X)e^{-X^2}dx + A_2 \int_{-\infty}^{\infty} H_o(X)H_2(X)e^{-X^2}dx + \ldots$$

$$\int_{-\infty}^{\infty} H_o(X)P(X)dx = A_o 2^o 0! \sqrt{\pi} = A_o\sqrt{\pi}$$

This implies:

$$A_o = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_o(X)P(X)dx$$

It is interesting that if,

$$H_o(X) = 1,\ A_o = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} P(X)dx = \frac{1}{\sqrt{\pi}}$$

or

$$H_o(X) = X,\ A_o = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} XP(X)dx = \frac{\mu}{\sqrt{\pi}}$$

or

$$H_o(X) = X^2,\ A_o = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} X^2P(X)dx = \frac{\sigma^2}{\sqrt{\pi}}$$

Continuing the solution of A_n's, if (8) is multiplied by $H_1(X)$ and integrated from $-\infty$ to ∞, we obtain

$$\int_{-\infty}^{\infty} H_1(X)P(X)dx = A_o \int_{-\infty}^{\infty} H_o(X)H_1(X)e^{-X^2}dx + A_1 \int_{-\infty}^{\infty} H_1{}^2(X)e^{-X^2}dx + A_2 \int_{-\infty}^{\infty} H_1(X)H_2(X)e^{-X^2}dx + \ldots$$

which reduces to

$$\int_{-\infty}^{\infty} H_1(X)P(X)dx = A_1 2^1\ 1! \sqrt{\pi} = 2A_1\sqrt{\pi}$$

Therefore,

$$A_1 = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} H_1(X)P(X)dx$$

Expressing the solution of the A_n terms in general form gives,

$$P(X) = \sum_{j=o}^{\infty} A_j H_j(X) e^{-X^2}$$

$$= \sum_{j=o}^{\infty} \left[\frac{1}{2^j j! \sqrt{\pi}} \int_{-\infty}^{\infty} H_j(X) P(X) dx \right] \cdot H_j(X) e^{-X^2} \quad (10)$$

Equation (9) may be altered to make its terms amenable to look up in normal distribution tables. If we let $X = \frac{Y}{\sqrt{2}}$, then $\frac{e^{-X^2}}{\sqrt{2\pi}}$ becomes

$$\frac{e^{-\frac{Y^2}{2}}}{\sqrt{2\pi}} = \Phi(Y)$$

Now that the probability distribution has been presented, the next step is to develop an expression for relating the individual part failure distributions with the effect on cost of maintenance actions.

B. The Cost Model

1. Assumptions

The basic assumptions which allow the development of the cost model are stated as follows:

1. The components being evaluated suffer wear-out type failure. Mathematically, this means the components experience increasing hazard with time, where hazard is defined as the ratio of failure probability to non-failure probability.

2. A component replacement due to failure results in a rescheduling of the next preventive maintenance action. Since the preventive maintenance replacements are based on component age, this assumption is straight forward.

A great deal of money is wasted on preventive maintenance inspections which vainly attempt to measure the state of well-being of a piece of equipment.

3. The basic failure distribution of components is known A distribution model was presented in previous pages that will answer the need for a general probability distribution to represent either failure data or empirical knowledge of a component failure pattern. The empirical approach can be used to bridge the gap from no failure data to the point where good quality data have been collected. The distribution model was presented since it can closely approximate the failure distributions experienced with wear-out failures.(17)

4. All component replacements are statistically identical with a failure distribution which adheres to the following mathematical definition:

   $F(t)$ = probability that a component will fail in an interval $[0,t]$ when it is new at $t = 0$.

   T_F = a non-negative random variable representing "time-to-failure" for each individual component.

   Therefore,

$$F(t) = Pr\left[T_F \leq t\right] \quad , \quad t \geq 0$$

   and $F(t)$ is a non-decreasing function, continuous to the right having $F(t) = 0$ for $t < 0$, $F(0) = 0$ and $\lim_{t\to\infty} F(t) = 1$. The mean time to failure is assumed to exist for the distribution F(t).

5. The basic distribution, which characterizes the scheduling of the preventive maintenance replacements, is known or can be determined. If a fixed interval of time occurs between preventive maintenance replacements, the exchange distribution is a step function. This is the situation which most easily conforms to scheduling operations and is commonly applicable.

Mathematically, G(t) is the probability that an exchange is performed in the interval $[0, t]$. This distribution describes the variation in time for replacement. T_G is defined as a non-negative random variable representing "time-to-exchange."

Therefore, $G(t) = \Pr[T_G \leq t]$, $t \geq 0$, where G(t) is a non-decreasing function, continuous to the right, with $G(t) = 0$ for $t < 0$ and $\lim_{t \to \infty} G(t) = 1$. A quantity $\overline{G(T)}$ is defined as $\overline{G(t)} = 1 - G(t) = \Pr[T_G > t]$, $t \geq 0$. It is assumed that the mean of the exchange distribution, T_G, exists for the distribution G(t). If the density function of the exchange distribution exists, it is defined as

$$g(t) = \frac{dG(t)}{dt} \quad , \; t > 0$$

6. It is assumed that the first generation component is new at time zero and that a component is new when it is installed.

This assumption is valid as long as spare parts do not deteriorate as they are stored, waiting for installation. Also, a repair of a part rather than new replacement requires characterizing the failure distribution of the repair unit, since it will invariably be different from the new component. This exception would not allow straightforward application of the standard renewal equations presented in the cost model.

7. It is assumed that the time required for a component replacement is negligible compared to the component's mean life and that components do not wear during the replacement interval. It is inherently assumed, therefore, that a failure is immediately detectable.[18]

2 The Cost Model Development

Without Preventive Replacements

For a single independent component position:

$$C(t) = h_2 F'(t) \tag{11}$$

where

$C(t)$ = the expected maintenance cost for a component position in the interval $[0,t]$.

h_2 = the cost of replacement when a failure occurs.

$F'(t)$ = the expected number of failures in a component position for the interval $[0,t]$.

With Preventive Replacements

For a single independent component position:

$$C^*(t) = h_1 G'^*(t) + h_2 F'^*(t) \tag{12}$$

where:

h_1 = the entire cost of a preventive replacement

h_2 = the cost of replacement when a failure occurs

$C^*(t)$ = the expected cost of maintenance in the interval $[0,t]$, when a preventive maintenance policy is in effect.

$G'^*(t)$ = the expected number of preventive maintenance replacements in the interval $[0,t]$, when a preventive maintenance policy is in effect.

$F'^*(t)$ = the expected number of failures in the interval $[0,t]$, when a preventive maintenance policy is in effect.

Minimum $C^*(t)$ is obtained, with fixed h_1 and h_2, by selecting different values of parameters of the distributions determined by maintenance policy and computing $C^*(t)$. Optimum parameter values can be obtained by minimizing $C^*(t)/h_2$; therefore, the ratio h_1/h_2, rather than the absolue value of each factor, is required for parameter selection.

Comparison of $C^*(t)$ with $C(t)$

If $C(t)$ is less than $C^*(t)$, the following possibilities exist: (a) preventive maintenance is not advantageous from a standpoint of cost, (b) reliability improvement is not worth the extra cost, (c) better quality data may or may not be worthwhile as determined by substituting various values of h_2 in equations (11) and (12).

For Steady State Conditions

If, in equation (11), the derivative of F'(t) exists, then for a single component position,

$$c(t) = h_2q(t) \tag{13}$$

where:

$c(t) = \frac{dC(t)}{dt}$, the expected cost rate, is the change in the cumulative (expected) cost per unit time when a preventive maintenance policy is not in effect.

$q(t) = \frac{dF'(t)}{dt}$, the expected rate of failure, is the change in the cumulative (expected) number of failures per unit time when a preventive replacement policy is not in effect.

If the derivatives exist, equation (12) can be differentiated to yield,

$$c^*(t) = h_1g^*(t) + h_2q^*(t) \tag{14}$$

where:

$c^*(t) = \frac{dC^*(t)}{dt}$, the expected cost rate when a preventive replacement policy is in effect.

$g^*(t) = \frac{dG'^*(t)}{dt}$, the expected rate of preventive replacements.

$q^*(t) = \frac{dF'^*(t)}{dt}$, the expected rate of failure when a preventive replacement policy is in effect.

The functions $g^*(t)$ and $q^*(t)$ are renewal rates which are not independent of each other. Since they approach steady-state values, the cost rate $c^*(t)$ also approaches a constant.

The Renewal Process with no Preventive Maintenance

The behavior of the function $F'(t)$ can be determined from the failure distribution, $F(t)$ of the individual component.

$F(t)$ is the probability that a component which is new at the beginning of an interval of length t fails at or before the end of the interval.

If, T_F = a positive random variable, "time-to-failure", then

$$F(t) = \Pr\left[T_F \leq t \right], \qquad t \geq 0 \qquad (15a)$$

$$F(t) = 0, \qquad t < 0 \qquad (15b)$$

$F(t)$ is a nondecreasing function of t, continuous to the right at points of discontinuity, i.e. $F(x) = F(x^+)$, having the limit $\lim_{t \to \infty} F(t) = 1$ and for this treatment, $F(0) = 0$.

Let $F(k,t)$ be defined as the probability that at least k consecutive failures occur in the interval $[0,t]$, when no preventive maintenance is performed. This is the distribution function of k generations.

$$F(k,t) = \Pr\left[T_{F1} + T_{F2} + \ldots + T_{Fk-1} + T_{Fk} \leq t\right] \qquad (16)$$

$$k = 1, 2, \ldots, \qquad t \geq 0$$

The recurrence relation between $F(k,t)$ and $F(k-1,t)$ can be exhibited as a convolution by the following procedure.

In (16) let T_{Fk} be a positive fixed value τ. This is permissible since T_{Fk} is independent of the other k-1 random variables; then

$$Pr\left[T_{F1} + T_{F2} + \ldots + T_{Fk-1} \leq t-\tau\right] = F\left[k-1,\ t-\tau\right]$$

Since F(k-1, t-τ) is dependent on the value of τ, the dependency may be eliminated by averaging F(k-1, t-τ) over all possible values of τ. Since τ is distributed according to the function F(τ) the expected value of F(k-1, t-τ) yields

$$F(k,t) = \int_0^t F(k-1,\ t-\tau)dF(\tau) \qquad k = 2, 3, \ldots$$

$$F(1,t) = F(t) \qquad t \geq 0 \tag{17}$$

where the upper limit of integration is t rather than ∞, since F(k-1,x) = 0 for x < 0. F(k,t) is also a nondecreasing function of t, continuous to the right, having F(k,0) = 0 and $\lim_{t\to\infty} F(k,t) = 1$.

Since F(k,t) represents the distribution function of a sum of identically distributed random variables, its mean equals k times the mean of F(t) provided the mean value of F(t) exists. F'(t) may be expressed in terms of the sequence F(k,t) by the following procedure. Let N(t) be an integral-valued, non-negative random variable which represents the number of failures in the interval $[0,t]$. Then, F'(t), the expected number of failures in this interval $[0,t]$ is

$$F'(t) = E\left[N(t)\right]$$

If we define, p(k,t) as the probability of exactly k failures in the interval $[0,t]$, then since $\sum_{k=0}^{\infty} p(k,t) = 1$

$$F'(t) = \sum_{k=0}^{\infty} kp(k,t)$$

In terms of random variables,

$$p(k,t) = Pr\left[T_{F1}+\ldots+T_{Fk} \leq t,\ T_{F1}+\ldots+T_{Fk}+T_{Fk+1} > t\right]$$

$$k = 1, 2, \ldots$$

and taking T_{Fk+1} as a fixed value τ yields,

$$Pr\left[t-\tau < T_{F1}+\ldots+T_{Fk} \leq t\right] = F(k,t) - F(k,t-\tau)$$

Averaging over-all τ using the distribution $F(\tau)$ yields

$$p(k,t) = F(k,t) - F(k+1,t) \qquad k = 1, 2, \ldots.$$

Thus, after substitution we obtain the desired result

$$F'(t) = \sum_{k=1}^{\infty} F(k,t) \qquad , t \geq 0 \tag{18}$$

The familiar integral equation in renewal theory is obtained using equations (17) and (18)

$$F'(t) = F(t) + \sum_{k=2}^{\infty} \int_0^t F(k-1,t-\tau)dF(\tau)$$

$$F'(t) = F(t) + \int_0^t \sum_{k=1}^{\infty} F(k,t-\tau)dF(\tau)$$

Using equation (18) again, we have

$$F'(t) = F(t) + \int_0^t F'(t-\tau)dF(\tau) \qquad , t \geq 0 \tag{19}$$

where $F'(t)$ is a unique, nondecreasing function, continuous to the right at points of discontinuity, having limit $\lim_{t \to \infty} F'(t) = \infty$ and in this development, $F'(0) = 0$.

When the distribution function $F(t)$ is absolutely continuous, we can define a density function, $f(t)$, where

$$f(t) = \frac{dF(t)}{dt} \qquad t > 0$$

$$f(0) = f(0^+) \qquad t = 0$$

Absolute continuity of F(t) implies absolute continuity of F'(t) and F(k,t) so that F'(t) may be differentiated to yield a renewal rate,

$$q(t) = \frac{dF'(t)}{dt} \quad , \quad t \geq 0 \tag{20}$$

which is termed the rate of failure. When the density function f(t) is continuous and non-negative for t>0 and right continuous at t=0, q(t) is unique, non-negative, and continuous for $t \geq 0$.

The rate of failure, q(t), may also be expressed as a summation and as an integral equation. If we define the density function of F(k,t) as

$$f(k,t) = \frac{dF(k,t)}{dt} \quad , \quad t > 0$$

then we have a convergent series

$$q(t) = \sum_{k=1}^{\infty} f(k,t) \quad , \quad t \geq 0 \tag{21}$$

The integral equation can then be developed from the recurrence relation between density functions so that

$$f(k,t) = \int_0^t f(k-1, t-\tau)f(\tau)d\tau$$

Thus,

$$q(t) = f(t) + \int_0^t q(t-\tau)f(\tau)d\tau \quad ; \quad t \geq 0 \tag{22}$$

Consequently, the expected cost rate, c(t), is given by

$$c(t) = \frac{dC(t)}{dt} = h_2 q(t) \quad , \quad t \geq 0 \tag{23}$$

This cost rate approaches a steady-state value in the sense that q(t) approaches a constant. It has been shown that with f(t) continuous and bounded where

$$\int_0^\infty f(t)dt = 1, \qquad \int_0^\infty tf(t)dt = T_f$$

and a finite second moment,

$$\lim_{t \to \infty} q(t) = \frac{1}{T_f} \tag{24}$$

where T_f is the mean time to component failure. If we define $c = \lim_{t \to \infty} c(t)$ as a steady-state cost rate,

$$c = \frac{h_2}{T_f} \tag{25}$$

For a distribution function F(t), which is not absolutely continuous, the absence of a density function can be compensated for in long range cost estimates by noting that

$$\lim_{t \to \infty} \frac{F'(t)}{t} = \frac{1}{T_f}$$

for a general distribution having a finite first moment.

The Failure Pattern in Presence of P/M Replacements

The Probability of k Consecutive Exchanges

A(k,t) is defined as the probability of at least k consecutive exchanges in the interval (0,t). Then,

$$A(k,t) = Pr\left[T_{G1} + T_{G2} + \ldots + T_{Gk} \leq t, \; T_{G1} \leq T_{F1}, \; T_{G2} \leq T_{F2}, \ldots, T_{Gk} \leq T_{Fk} \right]$$

$$K = 1, 2, \ldots \qquad t \geq 0 \tag{26}$$

The inequalities, $T_{Gj} \leq T_{Fj}$, define an initial sequence of k generations in which exchanges occur consecutively without any failures. The presence of the same random variable, T_{Gj}, in two events in the above expression indicates that these events are dependent even though all the random variables are mutually independent. This dependency prohibits evaluating this expression by simply taking the products of the probabilities of the individual events. Equation (26) is evaluated by letting T_{Gk} be a fixed positive value τ. Since T_{Fk} appears only once, the event $\tau < T_{Fk}$, for fixed τ, is independent of all other events, therefore, a product of probabilities is permitted and yields

$$\Pr\left[T_{G1}+\dots+T_{Gk-1} \leq t-\tau, T_{G1} \leq T_{F1}, \dots, T_{Gk-1} \leq T_{Fk-1}\right]\Pr\left[\tau < T_{Fk}\right]$$
$$= A(k-1, t-\tau)R(\tau)$$

Averaging over-all τ using the distribution $G(\tau)$ yields

$$A(k,t) = \int_0^t A(k-1, t-\tau)R(\tau)dG(\tau) \qquad K = 2, 3, \dots \tag{27}$$

$$A(1,t) = \int_0^t R(\tau)dG(\tau) \qquad , t \geq 0$$

From equation (26), $A(k,0) = 0$, $\lim_{t\to\infty} A(k,t) = p^k$ where, by definition,

$$p = \Pr\left[T_G \leq T_F\right] = \int_0^\infty R(\tau)dG(\tau)$$

Further, it can be seen from equation (27) that $A(k,t)$ is a non-decreasing function, since the integrand is always positive and is continuous to the right at points of discontinuity. Also, $A(k,t)$ is absolutely continuous if $G(t)$ and $F(t)$, for $t > 0$, are absolutely continuous.

Expected Number of Consecutive Exchanges

Let $a(t)$ be defined as the expected number of consecutive exchanges in the interval $(0,t)$. Thus, as may be demonstrated by the method used to develop F'(t) in equation (18),

$$a(t) = \sum_{k=1}^{\infty} A(k,t) \qquad , \; t \geq 0 \tag{28}$$

Replacing A(k,t) by the recurrence relation, equation (27) yields the integral equation,

$$a(t) = \int_0^t R(\tau)dG(\tau) + \int_0^t a(t-\tau)R(\tau)dG(\tau) \qquad t \geq 0 \tag{29}$$

a(t) is a unique, nondecreasing function, continuous to the right, having a(0) = 0 and $\lim_{t \to \infty} a(t) = \frac{P}{1-P}$. Furthermore, a(t) is absolutely continuous if both F(t) and G(t), t > 0, are absolutely continuous. The function a(t) may be evaluated in specific cases using either transform or numerical methods from equations (28) and (29).

The Probability of Failure on the k^{th} Generation After k-1 Exchanges

Let b(k,t) be defined as the proability that at least the k^{th} generation component fails during the interval $[0,t]$ after k-1 consecutive exchanges.

$$b(k,t) = \Pr\Big[T_{G1} + \ldots\ldots + T_{Gk-1} + T_{Fk} \leq t,\; T_{G1} \leq T_{F1}, \; \ldots. \ldots,\; T_{Gk-1} \leq T_{Fk-1},\; T_{Gk} > T_{Fk}\Big] \tag{30}$$

Allowing T_{Fk} to be a fixed value τ yields

$$b(k,t) = \int_0^t A(k-1, t-\tau)\overline{G(\tau)}dF(\tau), \qquad k = 2, 3, \ldots$$

$$b(1,t) = \int_0^t \overline{G(\tau)}dF(\tau) \qquad , \; t \geq 0 \tag{31}$$

The Resultant Failure Distribution Function

$F^*(t)$ is defined as the probability that at least one component failure occurs in the interval $[0,t]$, when preventive maintenance is being performed. This distribution describes the time between actual failures. Let T_{F^*} be defined as the non-negative random variable representing "time-between-actual failures." Then $F^*(t) = \Pr\left[T_{F^*} \leq t\right]$ for $t \geq 0$. A failure which occurs in the interval $[0,t]$ must occur in one of the possible component generations. Since $b(k,t)$ was defined primarily for the events, "first failure on each generation," the addition rule for mutually exclusive events yields,

$$F^*(t) = \sum_{k=1}^{\infty} b(k,t) \qquad , \ t \geq 0 \tag{32}$$

Using the integral expression for $b(k,t)$ equation (31) yields an integral expression for the resultant failure distribution.

$$F^*(t) = \int_0^t \overline{G(\tau)}\,dF(\tau) + \int_0^t a(t-\tau)\overline{G(\tau)}\,dF(\tau) \qquad t \geq 0 \tag{33}$$

T_{f^*} is defined as the mean time between actual failures,

$$T_{f^*} = \int_0^{\infty} t\,dF^*(t)$$

For absolutely continuous $F(t)$ and $G(t)$ we have a density function associated with $F^*(t)$, namely, $f^*(t) = \frac{dF^*(t)}{dt}$.

The Expected Number of Actual Failures

$F'^*(t)$ is defined as the expected number of actual failures in the interval $[0,t]$.

Using the same method, as previously we have:

$$F^*(k,t) = Pr\left[T_{F*1}+T_{F*2}+\ldots+T_{F*k} \leq t\right] \quad , \; t \geq 0$$

$$F^*(k,t) = \int_0^t F^*(k-1,t-\tau)dF^*(\tau) \quad , \; k = 2, 3, \ldots$$

$$F^*(1,t) = F^*(t) \quad , \quad t \geq 0 \tag{34}$$

These convolution expressions assume that the failure distribution, which is valid in the interval after $t = 0$, is also valid in an interval after a failure at τ.

$$F'^*(t) = \sum_{k=1}^{\infty} F^*(k,t) \tag{35}$$

$$F'^*(t) = F^*(t)+\int_0^t F'^*(t-\tau)dF^*(\tau) \quad , \quad t \geq 0 \tag{36}$$

The general behavior of $F'^*(t)$ is similar to $F'(t)$. The particular behavior as related to $G(t)$ and $F(t)$ may be deduced from that of $F^*(t)$. For steady-state costs, one may use

$$\lim_{t \to \infty} F'^*(t)/t = \frac{1}{T_{f*}}$$

With absolutely continuous $F(t)$ and $G(t)$, the behavior of the rate of failure function, $q^*(t)$ may be observed as,

$$q^*(t) = \frac{dF'^*(t)}{dt} \quad , \quad t > 0$$

$$q^*(t) = \sum_{k=1}^{\infty} f^*(k,t) \quad , \quad t \geq 0 \tag{37}$$

where

$$f^*(k,t) = \frac{dF^*(k,t)}{dt} \quad , \quad t > 0$$

$$q^*(t) = f^*(t) + \int_0^t q^*(t-\tau)f^*(\tau)d(\tau) \tag{38}$$

$$\lim_{t \to \infty} q^*(t) = \frac{1}{T_{f*}} \tag{39}$$

Exchange Pattern in the Presence of Failures

The Probability of k Consecutive Failures

I(k,t) is defined as the probability of at least k consecutive failures in the interval $[0,t]$.

$$I(k,t) = Pr\left[T_{F1}+T_{F2}+\ldots+T_{Fk} \leq t,\ T_{F1} < T_{G1},\ T_{F2} < T_{G2},\ \ldots T_{Fk} < T_{Gk}\right], \quad k = 1, 2, \ldots, \ t \geq 0 \qquad (40)$$

$$I(k,t) = \int_0^t I(k-1, t-\tau)\overline{G(\tau)}dF(\tau)$$

$$I(1,t) = \int_0^t \overline{G(\tau)}dF(\tau), \quad k = 1,2,\ldots \ t \geq 0 \qquad (41)$$

The Expected Number of Consecutive Failures

J(t) is defined as the expected number of consecutive failures in the interval $[0,t]$.

$$J(t) = \sum_{k=1}^{\infty} I(k,t), \quad t \geq 0 \qquad (42)$$

$$J(t) = \int_0^t \overline{G(\tau)}dF(\tau) + \int_0^t J(t-\tau)\overline{G(\tau)}dF(\tau), \quad t \geq 0 \qquad (43)$$

The Probability of Exchange on the k^{th} Generation After k-1 Failures

Let j(k,t) be defined as the probability that at least the k^{th} generation component is exchanged during the interval $[0,t]$ after k-1 consecutive failures

$$j(k,t) = Pr\left[T_{F1} + \ldots + T_{Fk-1} + T_{Gk} \leq t,\ T_{F1} < T_{G1},\ \ldots\ T_{Fk-1} < T_{Gk-1},\ T_{Fk} \geq T_{Gk}\right] \quad k = 1,2,3\ldots \qquad (44)$$

$$j(k,t) = \int_0^t I(k-1,t-\tau)R(\tau)dG(\tau)$$

$$j(1,t) = \int_0^t R(\tau)dG(\tau) \quad , \quad k = 1,2,\ldots \quad t \geq 0 \qquad (45)$$

The Resultant Exchange Distribution Function

Let G*(t) be defined as the probability that at least one exchange occurs in the interval $[0,t]$, where failures may occur between these exchanges.

Let T_{G*} be the random variable representing "time between actual exchanges."

$$G^*(t) = Pr[T_{G*} \leq t] , \quad t \geq 0$$

$$G^*(t) = \sum_{k=1}^{\infty} j(k,t), \quad t \geq 0 \qquad (46)$$

$$G^*(t) = \int_0^t R(\tau)dG(\tau) + \int_0^t J(t-\tau)R(\tau)dG(\tau) \qquad (47)$$

T_{g*} is defined as the mean time between actual exchanges and under the proper conditions,

$$g'^*(t) = \frac{dG^*(t)}{dt}$$

The Expected Number of Actual Exchanges

$$G^*(k,t) = Pr[T_{G*1}+T_{G*2}+\ldots+T_{G*k} \leq t] , \quad t \geq 0 \qquad (48)$$

$$G^*(k,t) = \int_0^t G^*(k-1,t-\tau)dG^*(\tau) \qquad (49)$$

$$G^*(1,t) = G^*(t) , \quad k = 2,3,\ldots \quad t \geq 0$$

Let G'*(t) be the expected number of actual exchanges in the interval $[0,t]$.

$$G'^*(t) = \sum_{k=1}^{\infty} G^*(k,t) \quad , \quad t \geq 0 \qquad (50)$$

$$G'^*(t) = G^*(t) + \int_0^t G'^*(t-\tau)dG^*(\tau) \qquad (51)$$

$$\lim_{t \to \infty} \frac{G'^*(t)}{t} = \frac{1}{T_{g^*}} \quad (52)$$

When the derivative exists, the rate of exchange is

$$g^*(t) = \frac{dG'^*(t)}{dt}, \quad t > 0 \quad (53)$$

$$g^*(t) = \sum_{k=1}^{\infty} g^*(k,t), \quad t \geq 0 \quad (54)$$

where

$$g'^*(k,t) = \frac{dG^*(k,t)}{dt}, \quad t > 0$$

$$g^*(t) = g^*(t) + \int_0^t g^*(t-\tau)g^*(\tau)d\tau, \quad t \geq 0 \quad (55)$$

$$\lim_{t \to \infty} g^*(t) = \frac{1}{T_{g^*}} \quad (56)$$

Therefore, behavior of the expected cost function

$$C^*(t) = h_2F'^*(t)+h_1G'^*(t)$$

has been completely described.

<u>The Expected Number of Removals</u>

Let $W(t)$ be defined as the expected number of removals in the interval $[0,t]$, when a preventive maintenance policy is in effect.

Since removals are due to failures or exchanges,

$$W(t) = F'^*(t) + G'^*(t), \quad t \geq 0 \quad (57)$$

A preventive maintenance policy always results in $W(t) > F'^*(t)$. If derivatives are permitted, then the rate of removals at t,

$$w(t) = q^*(t) + g^*(t) = \frac{dW(t)}{dt} \quad (58)$$

$W(t)$ can also be evaluated in terms of $F(t)$ and $G(t)$. To evaluate $W(t)$, a distribution function must be defined which represents the probability of a removal in the interval $[0,t]$. The first moment of this distribution, the "mean time between removals," defined as T_r, can be obtained by taking the limit $\lim_{t \to \infty} \frac{W(t)}{t}$ in equation (57).

$$T_r = \frac{T_{f*}\ T_{g*}}{T_{f*}+T_{g*}} = \int_0^{\infty} R(t)\overline{G(t)}\,dt \tag{59}$$

T_r is the mean value of the operating life of the components.

Explicit Form for the Mean Time Between Actual Failures

The mean time between actual failures, T_{f*}, not only indicates the extent of reliability improvement, but is used in the steady state cost expression.

T_{f*} and T_{g*} will be obtained explicitly as functions of the basic distributions F(t) and G(t). In order to derive results which will be applicable to discontinuous functions, Laplace-Stieltjes rather than Laplace transforms will be used. This transform is defined as

$$F(s) = \int_0^{\infty} e^{-st}\ dF(t) \tag{60}$$

where we use the same functional notation for the transform, F(s), and the determining function, F(t). The theorem on convolutions, namely, if

$$M(t) = \int_0^{t} F(t-\tau)\ dG(\tau) \qquad \text{where}$$

$$F(t=0) = 0 \text{ and } G(t=0) = 0. \qquad \text{Then,}$$

$$M(s) = F(s)\ G(s) \tag{61}$$

when these integrals converge.

The mean is obtained by a procedure similar to the methods used with moment generating functions as follows:

$$\frac{dF(s)}{ds} = -\int_0^{\infty} e^{-st} \quad tdF(t)$$

$$T_f = \lim_{s \to 0} \left[-\frac{dF(s)}{ds} \right] = \int_0^{\infty} tdF(t) \qquad (62)$$

These procedures require F(s) to be analytic in a region about the origin, s = 0. Higher moments of the distribution may be obtained in a similar manner.

In order to transform equation (29), we define

$$A(t) = \int_0^t R(\tau)\, dG(\tau)$$

Then equation (29) reduces to the form

$$a(t) = A(t) + \int_0^t a(t-\tau)\, dA(t)$$

$$a(s) = \frac{A(s)}{1-A(s)}$$

To handle equation (33), we define

$$I(t) = \int_0^t \overline{G(\tau)}\, dF(\tau)$$

Hence,

$$F^*(s) = I(s)\left[1 + a(s) \right] = \frac{I(s)}{1-A(s)} \qquad (63)$$

and $\lim_{s \to 0} F^*(s) = 1$

The remainder of the derivation consists of taking the derivative of $F^*(s)$, $\frac{dF^*(s)}{ds}$, allowing $s \to 0$, and simplifying the result using the identity

$$\int_0^{\infty} \overline{G(t)} \quad dF(t) \equiv \int_0^{\infty} F(t)\, dG(t) = 1-p$$

and finally integrating by parts to obtain

$$T_{f*} = \frac{T_f - \int_0^\infty G(t)R(t)dt}{\int_0^\infty F(t)\, dG(t)} \equiv \frac{\int_0^\infty \overline{G(t)}\, R(t)dt}{\int_0^\infty F(t)\, dG(t)} \qquad (64)$$

Equations (43) and (47) may be reduced in a similar manner to

$$T_{g*} = \frac{T_g - \int_0^\infty \overline{G(t)}F(t)dt}{1 - \int_0^\infty F(t)dG(t)} \qquad (65)$$

Note that if the inverse transform of $F^*(s)$ (equation 63) can be determined, an immediate solution results for $F^*(t)$ (and similarly for $G^*(t)$).

<u>Reliability</u>

In order to measure the improvement in reliability that can be obtained by the use of this type of replacement procedure, we use

$R(t) = 1 - F(t)$	The basic component reliability.
$R^*(t) = 1 - F^*(t)$	Resultant (operating) reliability in a component position.
$R^*(t)$	The probability that a failure does not occur within the interval $[0,t]$ after a removal.

If however, no exchanges will occur in time t, then R(t) measures reliability.

V. RESULTS

In order to demonstrate an actual problem solution, a standard type of centrifuge used to separate two heterogeneous liquid streams and a solid will be analyzed. The centrifuge consists mainly of an electric motor which provides rotation of the centrifuge bowl by transmission, using a belt drive. The liquid feed stream is fed by pump or gravity into the bottom of the bowl through the drag assembly. As the feed is forced upward in the bowl, the stream components are separated by centrifugal force. The solids remain trapped in the bowl, while the two liquids travel out the top of the bowl, the more dense liquid passing into the lower cover, the less dense one passing into the upper cover.

Table 1 shows the individual part description and the parameters of its failure distribution. Since all the parts listed in Table 1 are critical to the centrifuge's operation, in that any part could cause a detectable failure independent of any other part, the event of a failure by any one part is considered to be mutually exclusive in relation to a failure caused by another part.

The parameters of the distributions of Table 1 are given for the frequency of a failure versus time of an infinite number of parts. The cumulative probability of component failure with time equals the probability of a single part failing in some time interval, t. This distribution is obtained by integrating the expression for the part frequency of failure distribution represented by equation (10), part IV A. The result is the distribution required for solution of

TABLE 1

CENTRIFUGE EQUIPMENT SYSTEM

Part Code	Description	Mean $\mu = \bar{T}_F$	Failure Distribution Parameters: Standard Deviation	Third Moment	Fourth Moment
011	Bearing Assembly	0.2	0.02	-0.5	0.0
111	Male Clutch	4.2	0.80	0.0	0.0
211	Female Clutch Assembly	4.5	0.90	0.0	0.0
311	Flexible Coupling	2.5	0.50	-5.0	0.0
411	Bearing Seal Ring	0.2	0.02	-0.5	0.0
511	Bearing Washer	2.0	0.20	-1.0	0.0
611	Top Ball Bearing	7.0	1.50	-10.0	0.0
711	Bottom Ball Bearing	4.5	1.00	-8.0	0.0
021	Idler Assembly	1.2	0.20	-3.0	0.0
121	Idler Spring	3.5	0.50	-6.0	0.0
221	Top Idler Arm Bearing	7.0	1.00	0.0	0.0
321	Bottom Idler Arm Bearing	7.0	1.00	0.0	0.0
421	Idler Pulley Gasket	1.5	0.20	-3.0	0.0
521	Top Pulley Ball Bearing	7.5	1.00	-10.0	0.0

TABLE 1 (cont.)

CENTRIFUGE EQUIPMENT SYSTEM

Part Code	Description	Mean $\mu = \bar{T}_F$	Failure Distribution Parameters Standard Deviation	Third Moment	Fourth Moment
621	Bottom Pulley Ball Bearing	7.5	1.00	-10.0	0.0
031	Drag Unit	0.2	0.04	-0.5	0.0
131	Drag Spring	3.0	0.50	-5.0	0.0
231	Drag Bushing	0.2	0.03	-0.4	0.0
331	Feed Nozzle Gasket	0.4	0.05	0.0	0.0
041	Brake Assembly	2.0	0.30	-3.0	0.0
141	Brake Band Lining	2.0	0.30	-4.0	0.0
241	Brake Spring	3.0	0.50	0.0	0.0
051	Motor Assembly	3.5	0.50	-7.0	0.0
151	Top Shaft Bearing	7.5	1.00	-10.0	0.0
251	Top Shaft Seal	2.0	0.30	-3.0	0.0
351	Bottom Shaft Bearing	7.5	1.00	-10.0	0.0
451	Bottom Shaft Seal	2.0	0.20	-3.0	0.0

TABLE 1 (cont.)

CENTRIFUGE EQUIPMENT SYSTEM

Part Code	Description	Mean $\mu = \overline{T}_F$	Failure Distribution Parameters: Standard Deviation	Third Moment	Fourth Moment
061	Start-Stop Switch	0.2	0.04	-0.4	0.0
161	Stop Switch	3.0	0.50	-3.0	0.0
261	Run Switch	3.0	0.50	0.0	0.0
361	Start Switch	4.0	0.50	0.0	0.0
461	Start Solenoid	0.2	0.02	0.0	0.0
561	Run Solenoid	0.2	0.02	0.0	0.0
661	No. 1 Heater	1.0	0.03	0.0	0.0
761	No. 2 Heater	1.0	0.03	0.0	0.0
861	Fuse	8.0	0.50	0.0	0.0
071	Centrifuge Bowl	0.3	0.05	0.0	0.0
171	Bowl Boss Sleeve	0.3	0.05	0.0	0.0
271	Spindle	3.0	0.50	0.0	0.0
081	Drive Belt	4.0	0.70	0.0	0.0

the cost model and is equivalent to $F(t) = P_r\left[T_F < t\right]$, $t > 0$, which is presented in part IV B.

The evaluation of the centrifuge will be carried out in the following manner: each component position will be evaluated individually to determine the type of maintenance, preventive versus failure maintenance. Such parameters as the probability of k consecutive failures, the expected number of failures, and the expected number of total exchanges will be evaluated for the preventive and non-preventive maintenance case during a specific time interval. The solution will be carried out with the use of tables of normal distribution derivatives and will serve to demonstrate the type of results to be obtained through the use of the cost models.

The first step in the calculations will be to develop the general equations of part IV into specific form required for the problem solution.

The probability distribution in part IV was given in the form.

$$P(X) = \sum_{n=0}^{\infty}\left[\frac{1}{2^n\, n!\sqrt{\pi}}\int_{-\infty}^{\infty} H_n(s)P(s)\,ds\right]\cdot H_n(X)e^{-x^2} \qquad (66)$$

by substituting $X = \frac{y}{\sqrt{2}}$ and $\Phi(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}}$ we obtain

$$P(X) = \sum_{n=0}^{\infty}\left[\frac{1}{2^n\, n!}\int_{-\infty}^{\infty} H_n(s)P(s)\,ds\right] H_n\left[\frac{X}{\sqrt{2}}\right]\Phi(X) \qquad (67)$$

We also have:

$$H_o\ (X/\sqrt{2}) = 1$$

$$H_1\ (X/\sqrt{2}) = \sqrt{2}\ X$$

$$H_2\ (X/\sqrt{2}) = 2(X^2 - 1)$$

$$H_3\ (X/\sqrt{2}) = \frac{4}{\sqrt{2}}(X^3 - 3X)$$

$$H_4\ (X/\sqrt{2}) = 4(X^4 - 6X^2 + 3)$$

By expanding equation (67) we have:

$$P(X) = \int_{-\infty}^{\infty} P(s)\,ds\ \Phi(X) + \int_{-\infty}^{\infty} sP(s)\,ds\Phi(X) +$$

$$1/2\int_{-\infty}^{\infty}(s^2-1)P(s)\,ds(X^2-1)\Phi(X) +$$

$$1/6\int_{-\infty}^{\infty}(s^3-3s)P(s)\,ds(X^3-3X)\Phi(X) +$$

$$1/24\int_{-\infty}^{\infty}(s^4-6s^2+3)P(s)\,ds(X^4-6X^2+3)\Phi(X) + \ldots$$

$$P(X) = \Phi(X) + \mu\ X\Phi(X) + \tfrac{1}{2}(\sigma^2-1)(X^2-1)\Phi(X) +$$

$$1/6(\mu_3-3\mu)(X^3-3X)\Phi(X) +$$

$$1/24(\mu_4-6_\sigma 2+3)(X^4-6X^2+3)\Phi(X) + \ldots \qquad (68)$$

Applying the relationships:

$$\left.\begin{aligned} \Phi'(X) &= -X\Phi(X) \\ \Phi''(X) &= (X^2-1)\Phi(X) \\ \Phi'''(X) &= (3X-X^3)\Phi(X) \\ \Phi^{IV}(X) &= (X^4-6X^2+3)\Phi(X) \end{aligned}\right\} \qquad (69)$$

Using equation (69), equation (68) can be written as:

$$P(X) = \Phi(X) + \mu\ \Phi'(X) + \frac{\sigma^2-1}{2}\ \Phi''(X) + \frac{\mu_3-3\mu}{\sigma}\ \Phi'''(X) +$$

$$\frac{\mu_4-6\sigma^2+3}{24}\ \Phi^{IV}(X) + \ldots \qquad (70)$$

Since the function $\Phi(X)$ and its derivatives are standardized variables, then $\mu = 0$, and $\sigma^2 = 1$. Therefore, equation (70) becomes:

$$P(X) = \Phi(X) + \frac{\mu_3}{3!} \Phi'''(X) + \frac{\mu_4 - 3}{4!} \Phi^{IV}(X) + \ldots \quad (71)$$

Since the form of the failure distribution required for the cost model is the cumulative form; equation (71) becomes:

$$\int_o^t P(X) = \int_o^t \Phi(X) + \int_o^t \frac{\mu_3}{3!} \Phi'''(X) + \int_o^t \frac{\mu_4 - 3}{4!} \Phi^{IV}(X) + \ldots \quad (72)$$

$$F(t) = \int_{-\infty}^{\infty} P(X) = \Phi(X) + \frac{\mu_3}{3!} \Phi''(X) + \frac{\mu_4 - 3}{4!} \Phi'''(X) + \ldots \quad (73)$$

where X = normal variable = $\frac{t-\mu}{\sigma}$

For Failure Maintenance

Substituting equation (73) into equation (18) and utilizing the relation of the mean of $F(k+1, t) = \frac{k+1}{k}$ times the mean of $F(k, t)$ we have:

$$F'(t) = \sum_{k=1}^{\infty} F(k,t) = \sum_{k=1}^{\infty} \left[\Phi_{k+1} \left[\frac{t - \mu_k (\frac{k+1}{k})}{\sigma} \right] + \frac{\mu_3}{3!} \Phi''_{k+1} \left[\frac{t - \mu_k (\frac{k+1}{k})}{\sigma} \right] \right] \quad (74)$$

Therefore the failure cost model, equation (11),

$C(t) = h_2 F'(t)$ is completely defined

For Preventive Maintenance

In order to facilitate comprehension of how this model (equation 12) functions and avoid confusion, the nature of the cost factors h_1 (the P/M cost) and h_2 (the cost of failure maintenance) will be examined in Table 2.

TABLE 2

COMPARISON OF h_1 vs. h_2

(h_1 = P/M cost, h_2 = Failure Maint. Cost)

Cost Breakdown	h_1, h_2 Relationship	Comments
Part Installation		
(a) Materials	$h_{1a} = h_{2a}$	
(b) Labor	$h_{1b} \leq h_{2b}$	P/M allows replacement of more than one part per disassembly.
Downtime Cost		
(c)	$h_{1c} < h_{2c}$	In general scheduled downtime costs, very much less than failure (for that type of eq.).

It can be seen that P/M is generally less costly than failure maintenance in terms of labor and downtime. In the case of the centrifuge being evaluated, the unscheduled downtime cost is prohibitive. This situation results from the centrifuge's role in a batch processing operation where each stage is in series with the next. The min.mal effect of a failure is the backing up of in-process inventory in the stages prior to the one that failed, resulting therefore, in shortages in stages following and over-all drop in process production. For these reasons P/M is considered much less expensive in this case.

To assure maintenance entirely on a preventive basis, the preventive replacements will be scheduled so that they occur at $\overline{T}_F - 4\sigma_F$, where (as defined in section IV) $\overline{T}_F$ is the mean time to failure of a given component whose failure distribution is F(t) and σ_F is the standard deviation of the component failure distribution. This approach allows calculation of the exact number of preventive replacements in an interval $[0, t]$.

The preventive maintenance cost model (equation 12) then becomes:

$$C(t) = h_1 \left(\frac{t}{\overline{T}_F - 4\sigma_F}\right)$$

Table 3 is a summary of the results obtained through the application of the data of Table 1 in the appropriate formulas.

TABLE 3

SAMPLE PROBLEM SOLUTION - FOR A FIVE YEAR TIME INTERVAL
(Using Data From Table 1)

Part Code	Predicted P/M Actions $\frac{t}{\bar{T}_F - 4\sigma_F}$	F(k,t) k=1 F(k,t) = F(t)	k=2	k=3	k=4	k=5	k=6	Actual Recorded Failures	Predicted Failures Using Model F'(t) = $\sum F(k,t)$	Predicted Failures $\frac{t}{\bar{T}_F}$
		(k=23)	(k=24)	(k=25)	(k=26)					
011	41.7	1.00	1.00	1.00	0.00			22*	25.00	25.00
111	5.0	.84	.01	0.00				0	.85	1.19
211	5.5	.13	0.00					0	.13	1.11
311	10.0	1.00	.33	0.00				0	1.33	2.00
		(k=23)	(k=24)	(k=25)	(k=26)					
411	41.7	1.00	1.00	.03	0.00			21*	24.03	25.00
511	4.2	1.00	1.00	0.00				0	2.00	2.50
611	5.0	0.00						0	0.00	.72
711	10.0	.54	0.00					0	.54	1.11
021	12.5	1.00	1.00	1.00	.84	.16	0.00	3	4.00	4.16
121	3.3	.96	0.00					0	.96	1.43
221	1.7	.03						0	.03	.72

TABLE 3 (cont.)

SAMPLE PROBLEM SOLUTION - FOR A FIVE YEAR TIME INTERVAL
(Using Data From Table 1)

Part Code	Predicted P/M Actions $\frac{t}{T_F - 4\sigma_F}$	F(k,t) k=1 $F(k,t) = F(t)$	k=2	k=3	k=4	k=5	k=6	Actual Recorded Failures	Predicted Failures Using Model $F'(t) = \sum F(k,t)$	Predicted Failures $\frac{t}{T_F}$
321	1.7	.03						0	.03	.72
421	7.1	1.00	1.00	.82	.16	0.00		No Data	2.98	3.33
521	1.4	0.00						No Data	0.00	.67
621	1.4	0.00						No Data	0.00	.67
031	125.0	(k=24) 1.00	(k=25) .50	(k=26) 0.00				21	24.50	25.00
131	5.0	1.00	0.00					0	1.00	1.67
231	62.5	(k=24) 1.00	(k=25) .48	(k=26) 0.00				21	24.48	25.00
331	25.0	(k=11) 1.00	(k=12) 1.00	(k=13) 0.00				10	12.00	12.50
041	6.3	1.00	.84	.16	0.00			0	2.00	2.50
141	6.3	1.00	.84	.16	0.00			0	2.00	2.50

TABLE 3 (cont.)

SAMPLE PROBLEM SOLUTION - FOR A FIVE YEAR TIME INTERVAL
(Using Data From Table 1)

Part Code	Predicted P/M Actions $\frac{t}{\bar{T}_F - 4\sigma_F}$	F(k,t) k=1 F(k,t) = F(t)	k=2	k=3	k=4	k=5	k=6	Actual Recorded Failures	Predicted Failures Using Model $F'(t) = \sum F(k,t)$	Predicted Failures $\frac{t}{\bar{T}_F}$
241	5.0	1.00	.16	0.00				0	1.16	1.67
051	3.3	1.00	0.00					1	1.00	1.43
151	1.4	0.00						1	0.00	.67
251	6.2	1.00	1.00	0.00				1	2.00	2.50
351	1.4	0.00						1	0.00	.67
451	4.2	1.00	1.00	0.00				1	2.00	2.50
061	125.0	(k=24) 1.00	(k=25) .02	(k=26) 0.00				21	24.02	25.00
161	5.0	1.00	0.00					1	1.00	1.67
261	5.0	1.00	0.00					0	1.00	1.67
361	2.5	.03	0.00					0	0.03	1.25

TABLE 3 (cont.)

SAMPLE PROBLEM SOLUTION - FOR A FIVE YEAR TIME INTERVAL
(Using Data From Table 1)

Part Code	Predicted P/M Actions $\frac{t}{\overline{T}_F - 4\sigma_F}$	F(k,t) k=1 $F(k,t) = F(t)$	k=2	k=3	k=4	k=5	k=6	Actual Recorded Failures	Predicted Failures Using Model $F'(t) = \sum F(k,t)$	Predicted Failures $\frac{t}{\overline{T}_F}$
461	41.8	(k=24) 1.00	(k=25) .50	(k=26) 0.00				28	24.50	25.00
561	41.8	(k=24) 1.00	(k=25) .50	(k=26) 0.00				28	24.50	25.00
661	5.7	1.00	1.00	1.00	1.00	.5		5	4.50	5.00
761	5.7	1.00	1.00	1.00	1.00	.5		5	4.50	5.00
861	0.8	0.00						6*	0.00	.63
071	50.0	(k=15) 1.00	(k=16) 1.00	(k=17) .03				13	16.03	16.70
171	50.0	(k=15) 1.00	(k=16) 1.00	(k=17) .03				13	16.03	16.70
271	5.0	1.00	.03					2	1.03	1.67
081	4.2	1.00	0.00					1	1.00	1.25

* Indicates Trouble Area

VI CONCLUSIONS

As it can be seen from the results in Table 3, the models have proven to be good predictors of actual experience. A comparison of the estimated number of failures calculated with the models and the actual recorded failures indicate a good level of agreement.

It should be noted, however, that the model resulting from application of eq. (74) (column heading F'(t)) is considered an "optimistic" model when compared to actual failure rates; on the other hand model $\frac{t}{T_F}$ is "pessimistic" in nature. In general both models tend to overstate the expected failures as compared to actual results except for part codes 461 to 861 where both models understate the fact (actuality). this is due to very erratic performance due to the nature of the parts involved.

Although the equipment discussed in the application is of the type used in the chemical industry, it is basically similar to that used by manufacturing concerns in general. The basic solution proposed is to consider pieces of equipment as building blocks of a system. The analysis is based on the most elementary building blocks of all, the individual components which comprises a piece of equipment.

By treating the problem in this manner and using the Bayesian statistical approach of utilizing a priori knowledge of the probability distribution a versatile model has resulted.

It should be emphasized, however, that limitations of the model lie in the area of completeness and accuracy of historical records of maintenance experience.

VII RECOMMENDATIONS FOR FURTHER STUDY

In evaluating alternatives of all failure maintenance versus all preventive maintenance by simply calculating the mean expected failures in a given time interval, $\frac{t}{\bar{T}_F}$, and the expected number of preventive replacements, $\frac{t}{\bar{T}_F - 4\sigma_F}$, the presence or lack of cost differential can be observed. If a considerable cost differential seems to exist, then a further, more extensive study can be justified. The most applicable technique to be used would be the random number or Monte Carlo simulation. This technique would allow generation of simulated failure data according to assumed failure and preventive maintenance scheduling distributions of any type, to known probability distributions, or simply to a cumulative histogram of whatever data are available.

If the failure and preventive maintenance scheduling distributions are as shown in Figure 8 , random numbers would be used to determine when a hypothetical failure and preventive maintenance action occurred.

The time of failure would be compared with the time of the scheduled preventive maintenance action. The incident with the smaller time value would naturally have occurred first, determining which type of maintenance action it was. By generating data in such a manner, the functions A(k,t) and I(k,t), basic to the development of the maintenance cost model, can be determined.

The one major drawback to the simulation technique would be the extensive use of computer time and the need to analyze and curve fit

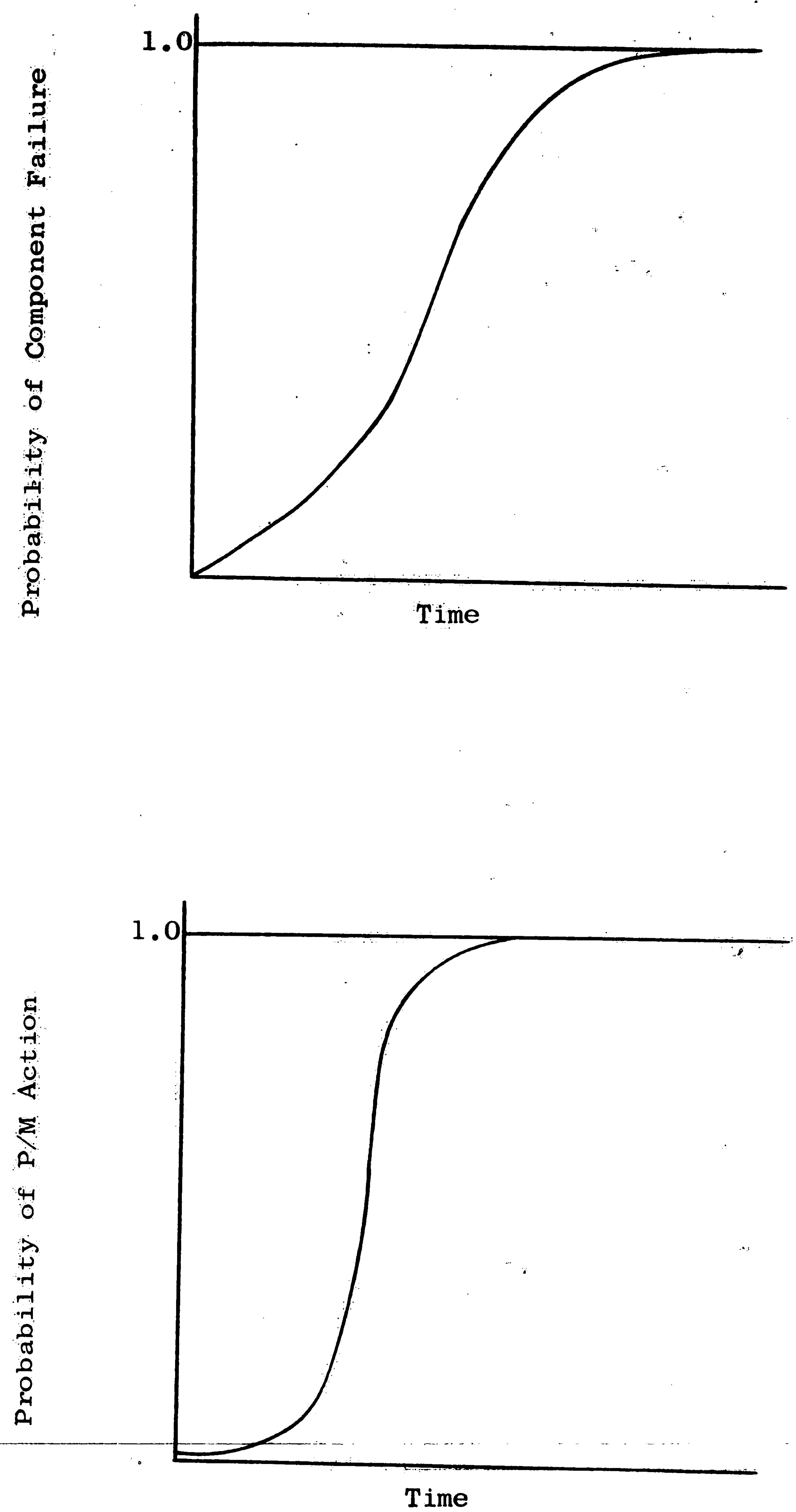


Figure 8 - Plots of Hypothetical Component Failure and Preventive Replacement Distributions vs. Time.

a large amount of data. An alternative solution method also could be suggested, which is termed the analytical approach.

This approach would incorporate the mathematical expressions for the component failure distribution and the preventive maintenance scheduling distribution into the renewal theory integrals of the cost model. As the expressions for the functions A(k,t) and I(k,t) would be developed for the various values of k, the general equations would be as follows:

$$F(k,t) = \int_0^t F(k-1, t-\tau)\, dF(\tau)$$

$$F(1,t) = F(t)$$

$$F(2,t) = \int_0^t F(1, t-\tau) dF(\tau) = \int_0^t F(t-\tau) dF(\tau)$$

$$F(3,t) = \int_0^t F(2, t-\tau) dF(\tau) =$$

$$\int_0^t \left[\int_0^t F((t-\tau) - \tau)\; dF(\tau) \right] dF(\tau) =$$

$$\int_0^t \left[\int_0^t F(t-2\tau) dF(\tau) \right] dF(\tau)$$

$$\vdots$$

$$F(k,t) = \int_0^t dF(\tau_1) \int_0^{\tau_1} dF(\tau_2) \ldots \int_0^{\tau_{n-1}} F(t-\tau_2-\tau_3 \ldots -\tau_n) dF(\tau_n)$$

By substitution of either the probability distribution suggested in this paper or another appropriate distribution function into this integral expression, a function might result which would either converge to a definite solution or might lend itself to approximation by a series expansion or Mill's ratio. According to the nature of the specific component an algebraic curve might be used to approximate the failure and preventive maintenance scheduling distributions.

BIBLIOGRAPHY

1. Morrow, L. C., "Managed Maintenance; Its Past, Its Present Status, Its Future Prospects," Factory, Vol. 118, January, 1960, pp. 64-71.

2. Wyder, C., "Preventive Maintenance; Special Report," Factory Management, Vol. 114, September, 1956, pp. 114-136.

3. Rice, R. S., "The High Cost of Maintenance," Factory, Vol. 117 May, 1966, p. 96.

4. Robinson, A. M., "Planned Repair Jobs Hold Down Costs," Factory, Vol. 119, July, 1961, pp. 132-136.

5. Knight, C. E., "Monsanto Says, Try Preventive Engineering if You Really Want to Cut Maintenance Costs," Factory Management, Vol. 112, November, 1951, pp. 104-105.

6. Bryrer, V. H., "How to Start a Preventive Maintenance Program," Factory, Vol. 119, October, 1961, pp. 236-240.

7. DeLong, A. F., "Thirteen Ways to Update Your Preventive Maintenance Programs," Plant, Vol. 23, January, 1961, pp. 43-45.

8. , "Preventive Maintenance; Heating, Piping, and Air Conditioning Equipment," Heating-Piping, Vol. 32, October, 1960, pp. 150-160.

9. Calabzo, S. R., Reliability Principles and Practices, New York: McGraw-Hill, Inc., 1962.

10. Editors, "Maintenance Prevention is Growing Up," Factory, Vol. 121, January, 1963, pp. 68-69.

11. Gibbons, E. J., "Probability, Downtime and Doomed Designs," Chemical Engineering, November 12, 1962, pp. 250-254.

12. Rudd, D. F., "Reliability Theory in Chemical System Design," Industrial and Engineering Chemistry, Fundamentals I, 1962, p. 138.

13. Hirshleifer, "The Bayesian Approach to Statistical Decision," Journal of Business (Univ. of Chicago, November, 1961, pp. 471-489.

14. Hoel, P. G., Introduction to Mathematical Statistics, New York: John Wiley & Sons, 1962.

BIBLIOGRAPHY (cont.)

15. Bazovsky, I., Reliability Theory and Practice, Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1961.

16. Hamming, R., Numerical Methods for Scientists and Engineers, New York: McGraw-Hill, Inc., 1960.

17. Brunner, L. K., L. J. Clevenger, and G. W. Dodson, "A Method of Identifying, Quantifying, and Specifying the Objective Predictors of Maintainability," School of Logistics, Institute of Technology Air University, Wright-Patterson Air Force Base, Ohio, June, 1960.

18. Brender, D. M., "The Statistical Dynamics of Preventive Replacements," Institute of Radio Engineers' Wescon Convention Record, 1959.

19. Sprinivasan, V. S., "The Effect of Standby Redundancy in Systems Failure with Repair Maintenance," Operations Research, December, 1966, Vol. 14, No. 6, p. 1024.

20. McCall, J. J., "Maintenance Policies for Stochastically Failing Equipment: A Survey," Management Science, Series A, Vol. 11, March, 1965, p. 493.

21. Vergin, R. C., "Scheduling Maintenance and Determining Crew Size for Stochastically Failing Equipment," Management Science, Vol. 13, No. 2, October, 1966, p. B-52.

22. Smith, W. L., "Renewal Theory and Its Ramifications," Journal of the Royal Statistical Society, Series B, Vol. 20, No. 2, pp. 243-302.

23. Bessler, S., H. Chernoff, and A. W. Marshall, "An Optimal Sequential Accelerated Life Test, Technometrics, Vol. 4, August, 1962, pp. 367-379.

24. Turban, Efraim., "The Use of Mathematical Methods in Plant Maintenance-Decision Making," Management Science, Vol. 13, No. 6, February, 1967.

25. Barlow, R. E. and L. C. Hunter, "System Efficiency and Reliability," Technometrics, Vol. 2, February, 1960, pp. 43-53.

26. ______, "Tests for the Validity of the Assumption that the Underlying Distribution of Life is Exponential - Part I," Technometrics, Vol. 2, February, 1960, pp. 83-101.

BIBLIOGRAPHY (cont.)

27. Kruse, M. E., "Problems of Preventive Maintenance Programs," Plant, Vol. 23, January, 1961, pp. 43-45.

28. Darin, A., "Preventive Maintenance Records Help Ford Maintain Equipment Reliability," Plant Engineer, Vol. 17, January, 1963, pp. 123-126.

29. Johnson, N. L., "Estimation of Sample Size," Technometrics, Vol. 4, February, 1962, pp. 59-67.

30. , "Maintenance Prevention Is Growing Up," Factory, Vol. 121, January, 1963, pp. 68-71.

31. Flehinger, B. J., "A General Model for the Reliability Analysis of Systems Under Various Preventive Maintenance Policies," Annals of Mathematical Statistics, Vol. 34, January, 1962, pp. 137-156.

32. Thorton, J. F., "Preventive Maintenance at Work; Abstract," Tool Engineer, Vol. 44, April, 1960, p. 337.

33. , "Clear Look at How Preventive Maintenance Pays Off Best," Factory, Vol. 118, April, 1960, p. 115.

34. Epstein, B., "Tests for the Validity of the Assumption that the Underlying Distribution of Life is Exponential - Part II," Technometrics, Vol. 2, May, 1960, pp. 167-183.

35. , "Statistical Life Test Acceptance Procedures," Technometrics, Vol. 2, November, 1960, pp. 435-446.

36. , "Maintenance List Fights Corrosion," Factory, Vol. 118, April, 1960, pp. 188-189.

37. Parkhill, S. M., "Periodic Preventive Maintenance; Pneumatic Tools," Compressed Air Magazine, Vol. 66, January, pp. 11-13; February, pp. 25-27; March, pp. 24-25; April, p. 27; May 1961, pp. 9-11.

38. Truelove, A. J., "Strategic Reliability and Preventive Maintenance," Operations Research, Vol. 9, January, 1961, pp. 22-29.

39. Krane, S. A., "Analysis of Survival Data by Regression Techniques," Technometrics, Vol. 5, May, 1963, pp. 161-174.

40. Gupta, S. S., "Life Test Sampling Plans for Normal and Lognormal Distributions," Technometrics, Vol. 4, May, 1962, pp. 151-175.

BIBLIOGRAPHY (cont.)

41. Chernoff, H., "Optimal Accelerated Life Designs for Estimation," Technometrics, Vol. 4, August, 1962, pp. 381-408.

42. Morrison, D. F., "The Optimum Allocation of Spare Components in Systems," Technometrics, Vol. 3, August, 1961, pp. 399-406.

43. , "System Reliability As a Function of System Age; Effects of Intermittent Component Usage and Periodic Maintenance," Operations Research, Vol. 8, January, 1966, pp. 30-44.

44. Mendenhall, W., E. H. Lehman, Jr., "An Approximation of the Negative Moments of the Positive Binomial Useful in Life Testing," Technometrics, Vol. 2, May, 1966, pp. 227-242.

45. Barlow, R. E. and L. C. Hunter, "Optimum Preventive Maintenance Policies," Operations Research, Vol. 8, January, 1960, pp. 90-100.

46. Garretson, R. C., "Preventive Maintenance Scheduling with Punched Cards," Lubrication Engineering, Vol. 19, January, 1963, pp. 10-20.

47. Gaver, D. P., Jr., "Random Hazard in Reliability Problems," Technometrics, Vol. 5, May, 1963, pp. 211-225.

48. Weiss, G. H., "The Reliability of Components Exhibiting Cumulative Damage Effects," Technometrics, Vol. 3, August, 1961, pp. 413-422.

49. Prairie, R. R., and B. Ostle, "An Analysis of Some Relay Failure Data from a Composite Exponential Population," Technometrics, Vol. 3, August, 1961, pp. 423-428.

50. Calenoff, R., "Stitch of Design Can Save Nine at Maintenance Time; A Comprehensive Survey and Probelist to Help You Short-Circuit Costs," Factory, Vol. 120, January, 1962, pp. 67-74.

51. "Control of Quality in Aircraft Maintenance as a Possible Means of Predicting Performance Reliability," Working Paper - WP 43, Operations Analysis Headquarters, United States Air Force, January, 1963.

52. Metaxas, T., "Preventive Maintenance, Sweet and Simple," Mill and Factory, Vol. 63, March, 1961, pp. 107-110.

53. Pritsker, A. A., "The Setting of Maintenance Tolerance Limits," Journal of Industrial Engineering, Vol. 14, March-April, 1963, pp. 80-86.

BIBLIOGRAPHY (cont.)

54. Winter, R. E., "Fix It Before It Breaks," Management Review, Vol. 50, May, 1961, pp. 77-79, Supervisory Management, Vol. 6, May, 1961, pp. 53-54.

55. Mott, S. C., "Help Yourself to Better Maintenance (Foreman's Responsibility)," Supervisory Management, Vol. 8, July, 1963, pp. 14-17.

56. Quinn, J. D., "Real Goal of Maintenance Engineering," Factory, Vol. 121, June, 1963, pp. 90-93.

57. Dyment, R., "Preventive Maintenance Program," Supervision, Vol. 25, March, 1963, pp. 10-12.

58. Zelen, M. and M. C. Dannemiller, "The Robustness of Life Testing Procedures Derived from the Exponential Distribution," Technometrics, Vol. 3, February, 1961, pp. 29-49.

59. Esary, J. D., and F. Proschan, "Relationship Between System Failure Rate and Component Failure Rates," Technometrics, Vol. 5, May, 1963, pp. 183-189.

VITA

PERSONAL HISTORY

Name:	Ted Chris Spiropoulos
Birth Place:	Patras, Greece
Birth Date:	May 16, 1931
Parents:	Chris and Mary Spiropoulos
Wife:	Peggy Spiropoulos
Children:	Chirs, Bill, Angele, Helen, Kathryn

EDUCATIONAL BACKGROUND

Prep. School of Athens Math and Physics Department	Attended	1949-1951
Washington University School of Engineering	Attended	1952-1954
Illinois Inst. of Technology BSME	Graduated	1962-1965
Lehigh University Candidate for Master of Science Degree in Industrial Engineering		1966-1968

PROFESSIONAL EXPERIENCE

Dema Engineering St. Louis, Missouri Position Held: Assistant Chief Engineer	1954-1957
Western Electric Company, Inc. Positions Held: Plant Engineer, Maintenance Engineer, Equipment Procurement, Cost Reduction Specialist, Research Engineer	1957-1968