1969

The hornich topology for schlicht functions

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THE HORNICH TOPOLOGY FOR SCHLICHT FUNCTIONS

by

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A THESIS

Presented to the Graduate Committee

of Lehigh University

in Candidacy for the Degree of

Master of Science

in

Mathematics

Lehigh University

1969

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

28 Feb. 1969
(date)

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ACKNOWLEDGMENT

The author wishes to express his gratitude to Professor J. P. King for his help in the preparation of this thesis.

CONTENTS

ABSTRACT

A function f, analytic in the unit disk is called schlicht if and only if f is one-to-one. The class S of all schlicht functions has been the object of much research. Most of this research has involved classical complex function theory methods. Functional analytic and linear topological space methods have not been of great value in this research primarily because S is not a linear space.

Hans Hornich considered the linear space H of all functions f analytic in the unit disk and normalized by $f(0) = 0$. Hornich defined a metric for H and was then able to use non-classical methods to deduce certain properties of the class S by considering it as a subset of H. More recently Cima and Pfaltzgraff have continued the study of H and have initiated the study of certain similar metric spaces of meromorphic functions.

In this work the results of Hornich and some of the results of Cima and Pfaltzgraff are presented. These results are primarily concerned with the characterization of the interior of the closed set S, with connectivity of schlicht and of non-schlicht functions, and with the local structure of the space H.

INTRODUCTION

A function f, analytic in the unit disk, is called schlicht, or univalent, if and only if f is one-to-one. The class S of all schlicht functions f which are normalized by $f(0) = 0$, $f'(0) = 1$ has been the object of much research [2], [5], [8]. Most of this research has involved classical complex function theory methods. Functional analytic and linear topological space methods have not been of great value in this research primarily because S is not a linear space.

Hans Hornich [7] considered the linear space H of all functions f analytic in the unit disk and normalized by $f(0) = 0$. Hornich defined the functional $\|\cdot\|$ on H by $\|f\| = \sup_n |a_n|^{1/n}$, where $f(z) = \sum_{n=1}^{\infty} a_n z^n$, and and showed that this functional is a metric for H. He was then able to use non-classical methods to deduce certain properties of the class S by considering it as a subset of H. More recently Cima and Pfaltzgraff [1] have continued the study of H and, with a similar metric, initiated the study of certain spaces of meromorphic functions.

In this work the results of Hornich and some of the results of Cima and Pfaltzgraff are presented.

In Section 2, for each f in H there is developed an associated function of two variables, the non-vanishing of which is equivalent to the property that f is schlicht.

In Section 3, it is shown that in the metric space H the set S of schlicht functions is closed. A schlicht function f will be in the interior of S and called, therefore, stable-schlicht if and only if its derivative f' is bounded away from zero in the unit disk. Furthermore, the space H cannot be normed, is not locally compact, is not locally convex, but is complete.

In Section 4, several results concerning connectivity in the space H are given. Any two functions of a connected set must differ by an entire function; the set of all entire functions is connected. Each component of H is convex; no component of H is open. Every two components of H are a positive distance apart; the number of components of H is the cardinal number of the continuum. The space H has no countable basis; the space H is at no point locally connected.

In Section 5, connectivity results which are related to the schlicht functions are given. The set of all schlicht polynomials in the unit disk is connected in H. The set of all stable-schlict polynomials is connected; the set of all stable-schlicht entire functions is connected. There are components of H which contain no schlicht functions, and every component of H contains non-schlicht functions.

In every component of $\dot{H}$ the set of non-schlicht functions is connected.

In Section 6, examples of non-schlicht, of schlicht, and of stable-schlicht functions are given. A schlicht function which maps the unit disk onto a convex domain is stable-schlicht. However, as other examples show, other restrictions as to the type of curve which bounds the image of the unit disk do not, in general, assure that a schlicht function will be also stable-schlicht.

An appendix is provided in which some of the pertinent definitions and results for schlicht functions are given. Reference to the appendix is indicated by (*).

2. THE ASSOCIATED FUNCTION

Let H be the set of functions f which are analytic in $|z| < 1$, and let f be not constant and given by

$$f(z) = a_1 z + a_2 z^2 + \ldots$$

If f is schlicht on a circle $|z| = r < 1$, then f is schlicht in $|z| < r$(*). Therefore, if f is not schlict in the unit disk, then there are two points, z_1 and z_2, in the unit disk such that $f(z_1) = f(z_2)$, $z_1 \neq z_2$, but $|z_1| = |z_2|$. So for two complex numbers $re^{i\varphi_1}$ and $re^{i\varphi_2}$, $0 < r < 1$, we form the difference

$$f(re^{i\varphi_1}) - f(re^{i\varphi_2}) = \sum_1^\infty a_n r^n (e^{i\varphi_1 n} - e^{i\varphi_2 n})$$

$$= (e^{i\varphi_1} - e^{i\varphi_2}) \sum_1^\infty r^n a_n e^{i\varphi_2(n-1)} [1 + e^{i(\varphi_1-\varphi_2)} + \ldots + e^{i(n-1)(\varphi_1-\varphi_2)}]$$

$$= (e^{i\varphi_1} - e^{i\varphi_2}) \sum_1^\infty r^n a_n e^{i\varphi_2(n-1)} \left[\frac{\cos \frac{n-1}{2}(\varphi_1-\varphi_2) \sin \frac{n}{2}(\varphi_1-\varphi_2)}{\sin \frac{1}{2}(\varphi_1-\varphi_2)} + i \frac{\sin \frac{n-1}{2}(\varphi_1-\varphi_2) \sin \frac{n}{2}(\varphi_1-\varphi_2)}{\sin \frac{1}{2}(\varphi_1-\varphi_2)} \right]$$

$$= (e^{i\varphi_1} - e^{i\varphi_2}) \sum_1^\infty r^n a_n \frac{\sin \frac{n}{2}(\varphi_1-\varphi_2)}{\sin \frac{1}{2}(\varphi_1-\varphi_2)} e^{i \frac{n-1}{2}(\varphi_1+\varphi_2)} .$$

We set $re^{i\frac{1}{2}(\varphi_1+\varphi_2)} = z$ and $\frac{\varphi_1-\varphi_2}{2} = \varphi$; then

$$f(re^{i\varphi_1})-f(re^{i\varphi_2}) = (e^{i\varphi_1}-e^{i\varphi_2})\, r \sum_1^\infty a_n z^{n-1} \frac{\sin n\varphi}{\sin \varphi} .$$

We, therefore, designate the function

$$f(\varphi,z) = \sum_1^\infty a_n z^{n-1} \frac{\sin n\varphi}{\sin \varphi}$$

as the "associated function" of f.

Since $\frac{\sin n\varphi}{\sin \varphi} \to n$ as $\varphi \to 0$, we set $\frac{\sin n\varphi}{\sin \varphi} = n$ if $\varphi = 0$. Thus,

$$f(0,z) = \sum_1^\infty n a_n z^{n-1} = f'(z) .$$

So that $f(0,z) \neq 0$ in $|z| < 1$ is necessary for f to be schlicht in the unit disk (*). The non-vanishing of $f(0,z)$ for a z in the disk then means that there do not exist distinct points z_1 and z_2 which are arbitrarily near z with $f(z_1) = f(z_2)$(*).

Clearly, from the development and definition of $f(\varphi,z)$, f is schlicht in $|z| < 1$ if and only if the associated function $f(\varphi,z)$ is never zero for $|z| < 1$. The non-vanishing of $f(\varphi,z)$ in the unit disk and with $\varphi \neq n\pi$ indicates that there do not exist two points $z_1 = re^{i(\alpha+\varphi)}$ and $z_2 = re^{i(\alpha-\varphi)}$ such that $f(z_1) = f(z_2)$.

3. THE METRIC SPACE H

We define a functional $\|\cdot\|$ on H by

$$\|f\| = \sup_n \sqrt[n]{|a_n|} ,$$

where $f(z) = \sum_1^\infty a_n z^n$. Since the power series for f has radius of convergence not less than one, $\overline{\lim} \sqrt[n]{|a_n|} \leq 1$; so that the functional is finite for all $f \in H$. Clearly $\|f\| \geq 0$; and $\|f\| = 0$ if and only if $f = 0$. For any two functions f and g in H given by $f(z) = \sum_1^\infty a_n z^n$ and $g(z) = \sum_1^\infty b_n z^n$, we have for all n,

$$\sqrt[n]{|a_n + b_n|} \leq \sqrt[n]{|a_n| + |b_n|} \leq \sqrt[n]{|a_n|} + \sqrt[n]{|b_n|} .$$

The triangular inequality, $\|f + g\| \leq \|f\| + \|g\|$, is therefore satisfied. H is then a metric space (*) where we define the distance between f and g as $\|f - g\|$.

The functional is not a norm (*) since $\|\lambda f\| \neq |\lambda| \cdot \|f\|$ for all complex numbers λ and all $f \in H$.

Observe two inequalities involving the functional $\|\cdot\|$. If $\|f\| < \epsilon$ and if $0 < \epsilon < 1$, then $|a_n| \leq \epsilon^n$; we have

$$|f(z)| \leq \epsilon|z| + \epsilon^2|z|^2 + \ldots = \frac{\epsilon|z|}{1-\epsilon|z|} < \frac{\epsilon}{1-\epsilon} ,$$

and

$$|f'(z)| \leq \epsilon + 2\epsilon^2|z| + 3\epsilon^3|z|^2 + \ldots < \epsilon + 2\epsilon^2 + \ldots$$

$$+ \ldots = \frac{\epsilon}{(1-\epsilon)^2} .$$

Let the complement of the set S of all schlicht functions be designated as T. That is, $H = S \cup T$. Since there are convergent sequences of schlicht functions which have the identically vanishing function as their limit (for example, $f_n(z) = \frac{z}{n}$ gives $\|f_n - 0\| \to 0$ as $n \to \infty$), we must, in order to have the next result, take $f = 0$ to be in S.

THEOREM. The set S is closed; T is open.

Proof. We show that T is open. Let f be an arbitrary function in T; we show that an open nbd of f is contained in T. Let $f(\varphi_o, z_o) = 0$ where $|z_o| < 1$ since f is not schlicht. We assume $f(\varphi_o, z)$ is not identically zero. For, if it is, $a_1 = 0$; then $f(\varphi, o) = 0$ for all φ and we change the choices of φ_o and z_o: pick any φ_o' and pick $z_o' = 0$. Then $f(\varphi_o', z)$ is not identically zero; for, otherwise $f(\varphi, z)$ is identically zero which is impossible since $f = 0$ is in S.

Let C be the circle

$$C = \left\{\xi : |\xi - z_o| = \beta\right\}, \ \beta > 0 ,$$

inside the unit disk such that z_o is the only zero of $f(\varphi_o, z)$ inside and on $C(\xi^*)$. For all ξ on C we take

$$\text{Min } |f(\varphi_o, \xi)| = \delta > 0 ,$$

where δ exists because $f(\varphi_o, \xi)$ is continuous and the circle is compact. Further, for all ξ on C we let

$$\text{Max } \left\{ |f(\varphi_o, \xi)| , |f'(\varphi_o, \xi)| \right\} = M < \infty .$$

We choose an ϵ in $(0,1)$ such that

$$\frac{\epsilon}{(1-\epsilon)^2} < \delta ,$$

and such that

$$\frac{2M\epsilon\beta}{\delta(1-\epsilon)[\delta(1-\epsilon)^2 - \epsilon]} < 1 .$$

Now let g be an arbitrary function in H such that $\|g\| < \epsilon$. Then $f + g$ is an arbitrary function in the ϵ-nbd or ϵ-ball of f since $\|f + g - f\| < \epsilon$. We have

$$|g(\varphi_o, z)| \le |a_1| + 2|a_z| |z| + 3|a_3| |z|^2 + \ldots$$

$$< \epsilon + 2\epsilon^2 + 3\epsilon^3 + \ldots = \frac{\epsilon}{(1-\epsilon)^2} ,$$

and similarly

$$|g'(\varphi_o, z)| < \frac{2\epsilon^2}{(1-\epsilon)^3} .$$

On C, $f(\varphi_0,\xi) + g(\varphi_0,\xi) \neq 0$, by the choice of ϵ and

$$\left|\frac{f'(\varphi_0,\xi)}{f(\varphi_0,\xi)} - \frac{f'(\varphi_0,\xi) + g'(\varphi_0,\xi)}{f(\varphi_0,\xi) + g(\varphi_0,\xi)}\right|$$

$$= \left|\frac{f'(\varphi_0,\xi)g(\varphi_0,\xi) - f(\varphi_0,\xi)g'(\varphi_0,\xi)}{f(\varphi_0,\xi)[f(\varphi_0,\xi) + g(\varphi_0,\xi)]}\right|$$

$$\leq \frac{M\frac{\epsilon}{(1-\epsilon)^2} + M\frac{2\epsilon^2}{(1-\epsilon)^3}}{\delta(\delta - \frac{\epsilon}{(1-\epsilon)^2})} = \frac{M\,\epsilon\,(1-\epsilon) + 2\,M\,\epsilon^2}{\delta(1-\epsilon)[\delta(1-\epsilon)^2-\epsilon]}$$

$$= \frac{M\,\epsilon + M\,\epsilon^2}{\delta(1-\epsilon)[\delta(1-\epsilon)^2-\epsilon]} < \frac{2\,M\,\epsilon}{\delta(1-\epsilon)[\delta(1-\epsilon)^2-\epsilon]} < \frac{1}{\rho} \;.$$

Therefore,

$$\frac{1}{2\pi i}\int_C \left[\frac{f'(\varphi_0,\xi)}{f(\varphi_0,\xi)} - \frac{f'(\varphi_0,\xi) + g'(\varphi_0,\xi)}{f(\varphi_0,\xi) + g(\varphi_0,\xi)}\right] d\xi$$

$$< \frac{1}{2\pi}(2\pi\rho)\frac{1}{\rho} = 1.$$

Thus, where 1 is the number of zeros of $f(\varphi_0,\xi)$ inside C and N is the number of zeros of $f(\varphi_0,\xi) + g(\varphi_0,\xi)$ inside C, we have $|1-N| < 0$ or $N = 1(*)$. So $f + g$ is non-schlicht since its associated function has a zero. Therefore, T is the union of open nbds and is open; hence S is closed.

The interior of the closed set S is not empty as examples will later show. We designate the interior of S or the open kernel of S as the set of functions G, and define a term to indicate that a function is in G.

DEFINITION. A schlicht function f is called stable-schlicht if there is an $\epsilon > 0$ such for all g with $\|g\| < \epsilon$, the function $f + g$ is schlicht. (Thus, all the functions f_1 in the ϵ-nbd of f are schlicht since, if $\|f_1 - f\| < \epsilon$, then $f + f_1 - f = f_1$ is schlicht.) If there exists no such $\epsilon > 0$ for a schlicht function f, it is called not stable-schlicht. ($f = 0$ is not stable-schlicht since $\|(\frac{\epsilon}{2})^2 z^2 - 0\| < \epsilon$.)

THEOREM. A function f is stable-schlicht if and only if for the associated function $f(\varphi, z)$

$$\inf |f(\varphi,z)| > 0 ,$$

where the infimum is taken over all φ and all z in the unit disk.

Proof. (a) Suppose $\inf |f(\varphi,z)| = \delta > 0$. We choose an ϵ in $(0,1)$ such that $\frac{\epsilon}{(1-\epsilon)^2} < \delta$. Then, for every g with $\|g\| < \epsilon$ we have

$$|g(\varphi,z)| \leq \left|\sum_1^\infty n\,\epsilon^n z^{n-1}\right| < \sum_1^\infty n\,\epsilon^n = \frac{\epsilon}{(1-\epsilon)^2} < \delta .$$

Therefore,

$$\inf |f(\varphi,z) + g(\varphi,z)| \geq |\delta - |g(\varphi,z)|| > \delta - \frac{\epsilon}{(1-\epsilon)^2} > 0;$$

thus $f + g$ is schlicht since its associated function has no zeros.

(b) We prove equivalently that if f is schlicht and $\inf |f(\varphi,z)| = 0$ then f is not stable-schlicht. Let $\inf |f(\varphi,z)| = 0$, $\epsilon > 0$ be given, and let f be schlicht and not identically zero since the null function is already not stable-schlicht. Then there is, since $f(\varphi,z)$ is continuous, a φ_o and a z_o with $|z_o| < 1$ such that $0 < |f(\varphi_o,z_o)| < \epsilon$.

Take $g(z) = -zf(\varphi_o,z_o)$; then we have $\|g\| = 1 - f(\varphi_o,z)|$ $< \epsilon$. Then, however, $f + g$ is not schlicht since its associated function, which is $f(\varphi,z) - f(\varphi_o,z_o)$, vanishes for $\varphi = \varphi_o$ and $z = z_o$. Thus, since ϵ was arbitrary, no neighborhood of f contains only schlicht functions. So f is not stable-schlicht.

THEOREM. <u>A function</u> f <u>is stable-schlicht if and only if there is an</u> $h > 0$ <u>such that</u>

$$\left|\frac{f(z_1) - f(z_2)}{z_1 - z_2}\right| \geq h$$

<u>for all</u> z_1 <u>and</u> z_2 <u>in the unit disk such that</u> $z_1 \neq z_2$.

Proof. (a) Suppose $|f(z_1)-f(z_2)| \geq h|z_1-z_2|$ for all z_1 and z_2 in the unit disk, $z_1 \neq z_2$. Choose an ϵ in $(0,1)$ such that

$$\frac{\epsilon}{(1-\epsilon)^2} < \frac{h}{2}$$

Then for any g with $\|g\| < \epsilon$ we have

$$|g(z_1)-g(z_2)| = |\sum_1^\infty a_n(z_1^n-z_2^n)|$$

$$= |z_1-z_2| \; |\sum_1^\infty a_n(z_1^{n-1}+z_1^{n-2}z_2+\ldots+z_2^{n-1})|$$

$$< |z_1-z_2|(\epsilon + 2\epsilon^2 + 3\epsilon^3 +\ldots) < \frac{h}{2} |z_1-z_2| .$$

Therefore,

$$|(f+g)(z_1) - (f+g)(z_2)| \geq |f(z_1) - f(z_2)|$$

$$- |g(z_1) - g(z_2)|$$

$$> h|z_1-z_2| - \frac{h}{2} |z_1-z_2| = \frac{h}{2} |z_1-z_2| > 0.$$

Thus $(f+g)(z_1) \neq (f+g)(z_2)$ so that $f+g$ is schlicht and f stable-schlicht.

(b) Suppose f is stable-schlicht, so that $\inf|f(\varphi,z)| = \delta > 0$. Now, for any r in $(0,1)$ consider the infimum of the quotient

$$\left|\frac{f(z_1)-f(z_2)}{z_1 - z_2}\right|$$

for all $z_1 \neq z_2$, $|z_1| \leq r$, and $|z_2| \leq r$. Since f is schlicht, the quotient is never zero and we apply the Minimum Modulus Principle. (*). Allowing only z_1 to very, that is, fixing z_2 anywhere in $|z| \leq r$, we see that for variation of z_1 the minimum of the quotient cannot occurr unless $|z_1| = r$. Now fixing z_1 on $|z| = r$ and allowing z_2 to vary gives that the infimum of the quotient for z_1 and z_2 in $|z| \leq r$ will equal the infimum of the quotient when both z_1 and z_2 vary on $|z| = r$. Then

$$f(z_1) - f(z_2) = (z_1 - z_2)f(\varphi, z)$$

for some z such that $|z| = r$. Hence

$$\left|\frac{f(z_1)-f(z_2)}{z_1 - z_2}\right| \geq \inf|f(\varphi, z)| = \delta > 0$$

for all distinct z_1 and z_2 in the unit disk, since r was arbitrary in $(0,1)$.

THEOREM. A schlicht function f is stable-schlicht if and only if there is a $\delta > 0$ such that for z in $|z| < 1$

$$|f'(z)| \geq \delta$$

(i.e, if and only if the derivative of f is bounded away from zero in the unit disk).

Proof. (a) If f is stable-schlicht then, where $f(\varphi, z)$ is the associated function,

$$\inf |f(\varphi, z)| = \delta > 0.$$

But $f'(z) = f(0,z)$ so that

$$|f'(z)| \geq \inf |f(0,z)| \geq \inf |f(\varphi,z)| > 0.$$

(b) Suppose $|f'(z)| \geq \delta > 0$. Since $f'(z) \neq 0$ in the unit disk and since f is schlicht, there is a unique analytic inverse f^{-1} of f which maps the image by f of the unit disk one-to-one onto the unit disk (*). Where $f(z) = w$, we have $f^{-1}(w) = g(w) = z$ and $g'(w) = 1/f'(z)$. Then, for any z_1 and z_2 in the unit disk and $z_1 \neq z_2$ we have $f(z_1) \neq f(z_2)$. Let C be a simple rectifiable curve in the domain of f^{-1} from $f(z_2)$ to $f(z_1)$. The length of C equals $k|f(z_1) - f(z_2)|$ where $0 < k < \infty$. Then

$$z_1 - z_2 = \int_{f(z_2)}^{f(z_1)} g'(w)\,dw \quad ,$$

$$|z_1 - z_2| \leq k|f(z_1) - f(z_2)|\left(\frac{1}{\delta}\right)$$

since $|g'(w)| = 1/f'(z)| \leq 1/\delta$. So that

$$\left|\frac{f(z_1)-f(z_2)}{z_1 - z_2}\right| \geq \frac{\delta}{k} > 0$$

and f is stable-schlicht.

The topology on H cannot be induced by a norm.

THEOREM. <u>The space H cannot be normed.</u>

<u>Proof</u>. It suffices to show tha no open set in H can be brought into (contained by) any neighborhood of the origin, f = 0, under multiplication by a scalar [3]. It is clear, by a consideration of the $n^{\underline{th}}$ root of the absolute value of the $n^{\underline{th}}$ power series coefficient as $n \to \infty$, that every neighborhood of every function contains non-entire functions. Further, no scalar multiplication of a non-entire function can bring it closer to the origin than the reciprocal of its radius of convergence. In particular, we need only show that the ϵ-nbd of the origin cannot be broughtwithin the $\epsilon/4$ nbd of the origin. Let a be any scalar, then $f(z) = (\epsilon/2^m z^m$ where m is chosen so that $2^m > 1/|a|$. Then f is in the ϵ-nbd of the origin, but $\|af\| > \frac{1}{4}$ so af is not in the $\epsilon/4$-nbd of the origin.

THEOREM. H <u>is not locally compact</u>.

Proof. No ϵ-nbd of the origin is relatively compact. For, the f_n given by $f_n(z) = (\epsilon/2)^n z^n$ are all in the (closure of the) ϵ-nbd of the origin, but the f_n do not converge since $\|f_n - f_m\| = \frac{\epsilon}{2}$.

THEOREM. *The space H is not locally convex* [3], [4].

THEOREM. *The space H is complete* [3], [4].

4. GENERAL RESULTS CONCERNING CONNECTIVITY IN THE SPACE H

We investigate the structure of the metric space H and the sets S and T in reference to connectivity.

THEOREM. *A set M of H is connected only if for each two functions f and g in M given by* $f(z) = \sum_1^\infty a_n z^n$ *and* $g(z) = \sum_1^\infty b_n z^n$

$$\overline{\lim}\ \sqrt[n]{|a_n - b_n|} = 0.$$

That is, each two functions of a connected set differ by an entire function.

Proof. The two functions will differ by an entire function since the limit superior is the reciprocal of a radius of convergence.

For each two such functions f and g in M and each $\epsilon > 0$ there must be, since M is connected, a finite chain of functions f_K in M,

$$f_K(z) = \sum_{n=1}^\infty a_n^{(K)} z^n \ , \quad K = 0, 1, \ldots, r \ ,$$

where $f = f_o$, $g = f_r$ and such that

$$\|f_K - f_{K-1}\| < \epsilon \quad , \quad K = 1,2,\ldots,r.$$

For, suppose on the contrary that there is no such chain between f and g for some $\epsilon > 0$. Then consider the two sets, which will clearly be disjoint and non-empty, of functions for which there is an ϵ-chain to f and of all other functions in M. Then the union of the $\epsilon/4$-nbds of each function in one of the sets and the union of the $\epsilon/4$-nbds of functions in the other set are disjoint. So M is the union of two disjoint open sets. This is impossible.

Now for each $n \geq 1$

$$|a_n - b_n| \leq |a_n^{(o)} - a_n^{(1)}| + |a_n^{(1)} - a_n^{(2)}| + \ldots + |a_n^{(r-1)} - a_n^{(r)}| ,$$

so that for each $n \geq 1$ there is at least one ℓ such that

$$|a_n - b_n| \leq r|a_n^{(\ell)} - a_n^{(\ell+1)}| ,$$

$$|a_n^{(\ell)} - a_n^{(\ell+1)}| \geq \frac{1}{r} |a_n - b_n| ;$$

$$\sqrt[n]{|a_n^{(\ell)} - a_n^{(\ell+1)}|} \geq \frac{1}{\sqrt[n]{r}} \sqrt[n]{|a_n - b_n|}$$

Thus

$$\sup_n \sup_\ell \sqrt[n]{|a_n^{(\ell)} - a_n^{(\ell+1)}|} \geq \overline{\lim_n} \frac{1}{\sqrt[n]{r}} \sqrt[n]{|a_n - b_n|} = \overline{\lim_n} \sqrt[n]{|a_n - b_n|}.$$

Also $\epsilon > \sup_{\ell} \|f_{\ell} - f_{\ell+1}\| \geq \sup_{n} \sup_{\ell} \sqrt[n]{|a_n^{(\ell)} - a_n^{(\ell+1)}|}$

Hence, since ϵ was arbitrary,

$$\overline{\lim} \sqrt[n]{|a_n - b_n|} = 0.$$

THEOREM. The set of all entire functions in the space H is connected. The set all polynomials of degree $\leq N$ is connected; the set of all polynomials is connected.

Proof. There are several ways to show that the set of all polynomials of degree $\leq N$ is connected. We do so either (1) by induction, since any az and bz of degree 1 are path-connected, and since there is a path, p, given by $p(t) = t\, a_{n+1} z^{n+1} + \sum_{1}^{n} a_k z^k$, $0 \leq t \leq 1$, connecting any polynomial of degree $n + 1$ to a polynomial of degree n; or (2), where two polynomials are given by $P_1(z) = \sum_{1}^{n_1} a_k z^k$ and $P_2(z) = \sum_{1}^{n_2} b_k z^k$, by considering the two paths $P_1(tz)$ and $P_2(tz)$, $0 \leq t \leq 1$, which have the null function in common; or (3) by considering the path $tP_1 + (1-t)P_2$, $0 \leq t \leq 1$, connecting any two polynomials P_1 and P_2.

Therefore, the set of all polynomials is connected.

Finally, the set of all entire functions is connected since it is contained in the closure of the set of all polynomials. For, let g be any entire function given by $g(z) = \sum_{1}^{\infty} a_n z^n$. Then the partial sums P_N of g, $P_N(z) = \sum_{1}^{N} a_n z^n$, converge to g,

$$\|g-P_N\| = \|\sum_{N+1}^{\infty} a_n z^n\| \to 0 \quad \text{as} \quad N \to \infty$$

since $\overline{\lim} \sqrt[n]{|a_n|} = 0$.

THEOREM. <u>Each component</u> K <u>of</u> H <u>consists of functions</u> f + g, <u>where</u> f <u>may be arbitrarily chosen from</u> K <u>but then held fixed, and</u> g <u>varies over the set of all entire functions</u>.

<u>Proof</u>. Clearly a component can contain no more functions than this or some two functions in the component will differ by more than an entire function.

The function F from H x H to H given by $F(h_1,h_2) = h_1 + h_2$ is continuous since H is a linear topological space (*). The set of all functions of the form f + g where f is fixed and g is an entire function is then the continuous image of a connected set and, therefore, connected since f is connected and since the entire functions are connected.

Alternately, we may show that the $f + g$ are path-connected by segment paths. That is, we have the

THEOREM. <u>Each component</u> K <u>of</u> H <u>is convex</u>.

<u>Proof</u>. Let $f + g_1$ and $f + g_2$ be any two functions in K where f is in K and g_1 and g_2 are entire. Clearly the functions

$$f + g_1 + t(f + g_2 - f - g_1)\ , \quad 0 \leq t \leq 1\ ,$$

of the segment joining $f + g_1$ and $f + g_2$ are in K. Furthermore, this path of functions is (uniformly) continuous since g_1 and g_2 are entire. For, let $\epsilon > 0$ be given. Then $g_2 - g_1$, given by $(g_2-g_1)(z) = \sum_1^\infty c_n z^n$, is entire so that there exists an N such that for each $n > N$, $\sqrt[n]{|c_n|} < \epsilon$. Also $\sup_{n \leq N} \sqrt[n]{|c_n|} = \delta < \infty$. Therefore, where $f_{t_1} = f + g_1 + t_1(g_2-g_1)$ and $f_{t_2} = f + g_1 + t_2(g_2-g_1)$ are two functions of the segment, we have $\|f_{t_1} - f_{t_2}\| < \epsilon$ if $|t_1-t_2| < (\frac{\epsilon}{\delta})^N$.

$$\|f_{t_1} - f_{t_2}\| = \|(t_1-t_2)(g_2-g_1)\| = \mathrm{SUP}_{n<\infty} \sqrt[n]{|(t_1-t_2)c_n|}$$

$$\leq \max\left\{\epsilon\ ,\ \mathrm{SUP}_{n\leq N} \left(\sqrt[n]{|t_1-t_2|}\ \sqrt[n]{|c_n|}\right)\right\}$$

$$< \max \left\{\epsilon, \sqrt[N]{\left(\frac{\epsilon}{\delta}\right)^N \left(\frac{1}{\delta}\right)}\right\} = \epsilon \quad .$$

THEOREM. Every two components K and K' of H are a positive distance apart.

Proof. Let f_1 and f_2, given by $f_1(z) = \sum_1^\infty a_n z^n$ and $f_2(z) = \sum_1^\infty b_n z^n$, be two functions in K and K', respectively. Then any function $f_1 + g_1$ in K given by $f_1(z) + g_1(z) = \sum_1^\infty (a_n + \alpha_n) z^n$ with $\overline{\lim} \sqrt[n]{|\alpha_n|} = 0$, and any function $f_2 + g_2$ in K' given by $f_2(z) + g_2(z) = \sum_1^\infty (b_n + \beta_n) z^n$ with $\overline{\lim} \sqrt[n]{|\beta_n|} = 0$ have a distance

$$\|f_1 + g_1 - f_2 - g_2\| = \sup_n \sqrt[n]{|a_n + \alpha_n - b_n - \beta_n|}$$

$$\geq \overline{\lim} \sqrt[n]{|a_n + \alpha_n - b_n - \beta_n|} = \overline{\lim} \sqrt[n]{|a_n - b_n|} = \delta > 0 ,$$

since $f_1 - f_2$ is not entire. Therefore, the distance between K and K' is positive since δ is fixed for different choices of g_1 and g_2.

THEOREM. The number of components of H is the cardinal number of the continuum.

Proof. Let f, given by $f(z) = \sum_1^\infty a_n z^n$, be a function in H such that $\overline{\lim} \sqrt[n]{|a_n|} = \epsilon > 0$. So that if $s \neq t$ are two numbers, then the functions sf and tf belong to different components of H because (s-t)f is not an entire function. This proves the theorem.

Furthermore, every two components in the proof have a distance greater than or equal to (by the previous proof) $\overline{\lim} \sqrt[n]{|s-t|\,|a_n|} = \overline{\lim} \sqrt[n]{|a_n|} = \epsilon > 0$. Consequently, the e-nbds of functions from different components are disjoint and we have the

THEOREM. The space H has no countable basis.

Every component of any space is closed in that space; a component may also be open. For the space H we have the

THEOREM. No component of H is open. Each function in a component K of H is the limit of functions in H none of which are in K and no two of which are in the same component.

Proof. No component will be open since its complement will not be closed.

For each function f, given by $f(z) = \sum_{1}^{\infty} a_n z^n$, which is in some component K there is an arbitrarily near function f_ϵ,

$$f_\epsilon(z) = \sum_{1}^{\infty} (a_n + \epsilon^n)z^n \ , \ 0 < \epsilon < 1 \ ,$$

where $\|f_\epsilon - f\| = \epsilon$. The functions f_ϵ are not in K since the $(f_\epsilon - f)(z) = \sum_{1}^{\infty} \epsilon^n z^n$ are not entire. Also, for $\epsilon_1 \neq \epsilon_2$, f_{ϵ_1} and f_{ϵ_2} are, similarly, in different components.

Since an ϵ-nbd of any f in H contains $f_{\epsilon/2}$ as above, no neighborhood of f is connected. So that we have the

THEOREM. H <u>is at no point locally connected</u>.

5. CONNECTIVITY RESULTS FOR SCHLICHT FUNCTIONS

THEOREM. The set P_N of all schlicht polynomials in the unit disk of degree $\leq N$ is connected in H.

Proof. If $f(z) = \sum_1^N a_n z^n$ is schlicht for $|z| < 1$, then so are the functions given by $f(tz) = \sum_1^N a_n t^n z^n$ for $0 \leq t \leq 1$(*). Finally, the functions given by

$$\frac{1}{t} f(tz) = a_1 z + a_2 t z^2 + \ldots + a_N t^{N-1} z^n \,, \qquad 0 \leq t \leq 1 \,,$$

will be schlicht. These last functions provide a path of schlicht polynomials from f to $a_1 z$. Any other schlicht polynomial $g(z) = \sum_1^M b_n z^n$, $M \leq N$, will similarly be path-connected to $b_1 z$. The segment $a_1 z + t(b_1 - a_1)z$, $0 \leq t \leq 1$, connects $a_1 z$ to $b_1 z$.

THEOREM. The set of all stable-schlicht polynomials in the unit disk of degree $\leq N$ is connected.

Proof. The proof is similar to the previous proof. It is clear that the functions $\frac{1}{t} f(tz)$ will be stable-schlicht if f is a stable-schlicht polynomial. In particular, where $f(\varphi, z)$ is the associated function of f the associated function of $\frac{1}{t} f(tz)$ will be (by observing its expansion coefficients) $f(\varphi, tz)$, and will therefore, be bounded away from zero for each t if $f(\varphi, z)$ is so bounded.

The connected set of this last theorem may be enlarged by the following

THEOREM. The set of all entire functions which are stable-schlicht in the unit disk is connected.

Proof. Let f be any entire function which is stable-schlicht in $|z| < 1$. Where $f(z) = \sum_{1}^{\infty} a_n z^n$ we have

$$\inf \left| \sum_{1}^{\infty} a_n \frac{\sin n\varphi}{\sin \varphi} z^{n-1} \right| = \delta > 0.$$

For a fixed n in this infimum put $t \cdot a_n$ in place of a_n and let t vary from 1 to 0 in $0 \leq t \leq 1$. In this variation of t the infimum will experience at most a change of $n|a_n|$ since $\left|\frac{\sin n\varphi}{\sin \varphi} z^{n-1}\right| \leq n$.

Since f is entire, $f'(z) = \sum_{1}^{\infty} n a_n z^{n-1}$ is entire and absolutely convergent for $z = 1$. That is, $\sum_{1}^{\infty} n|a_n| < \infty$ and the sequence of partial sums converges. So that $\sum_{n=n_1}^{\infty} n|a_n| \to 0$ as $n_1 \to \infty$. We choose an N such that

$$\sum_{N+1}^{\infty} n|a_n| < \delta .$$

Now, if f we successively let all the a_n for $n > N$ go to zero by the described variation of $t \cdot a_n$ and obtain a sequence of functions

$$f = f_o \, , \; f_k = f_{k-1} - a_{N+k} z^{N+k} \, , \; k = 1,2,\ldots \, .$$

for which

$$\lim_{k\to\infty} f_k(z) = \sum_1^N a_n z^n = P_N(z).$$

This sequence also converges in H to P_N,

$$\|f_k - P_N\| = \| \sum_{n=N+k+1}^{\infty} a_n z^n \|$$

$$\|f_k - P_N\| \to 0 \quad \text{as} \quad k \to \infty \, .$$

since $\overline{\lim} \sqrt[n]{|a_n|} = 0$.

P_N as well as each f_k is stable-schlicht because as all $t \cdot a_n$ for $n > N$ go to zero, the total change in $\inf |f(\varphi,z)|$ is less than or equal to $\sum_{N+1}^{\infty} n|a_n|$ which is less than $\delta = \inf |f(\varphi,z)|$, that is

$$\inf |P_N(\varphi,z)| > 0 \, .$$

Therefore, we have a path of stable-schlicht entire functions from each f_k to f_{k+1} for all k; the f_k are connected and in the set of stable-schlicht entire functions. Finally, since the closure of this path includes the stable-schlicht polynomial P_N and since the stable-schlicht polynomials form a connected set, any two stable-schlicht entire functions may be connected. The theorem is proved.

Alternately, in this proof we may let the $t \cdot a_n$ go to zero simultaneously to obtain a (continuous) path from f to P_N. That is, where N is chosen as above, it can be shown as in the proof of the theorem thate every component of K is convex, that

$$f_t(z) = \sum_{1}^{N} a_n z^n + t \sum_{N+1}^{\infty} a_n z^n \ , \ 0 \leq t \leq 1 \ ,$$

provides a path from from f to $\sum_{1}^{N} a_n z^n = P_N(z)$.

THEOREM. <u>There are components of</u> H <u>which contain no no schlicht functions</u>.

<u>Proof</u>. Consider $f(z) = \dfrac{z}{(1-z)^n}$ for $n \geq 3$. Then, each of the functions is in H but each is non-schlicht (the derivative of each has a zero in the unit disk).

An elementary growth theorem (*) for schlicht functions is: if f is schlicht in the unit disk, then for $|z| = r$, $0 < r < 1$,

$$|f'(0)| \frac{r}{(1+r)^2} \leq |f(z)| \leq |f'(0)| \frac{r}{(1-r)^2} \ .$$

We see that $f(z) = \frac{z}{(1-z)^3}$, for instance, does not satisfy the inequality and hence f cannot be schlicht. Furthermore, for no entire function g will f + g satisfy the inequality as z approaches 1 in (0,1) because the entire function g must be bounded in $|z| \leq 1$. Therefore, the component containing f contains only non-schlicht functions.

LEMMA. <u>Every component of H contains non-schlicht functions.</u>

<u>Proof</u>. If a component contains a function f, we can choose an entire function g such that $g'(0) = -f'(0)$. That is, $f' + g'$ will vanish when $z = 0$. Then f + g is non-schlicht since the derivative of a schlicht function can have no zeros (*).

Observe that, for any two fixed functions f in H and g entire, there exists a complex t such that $f'(0) + t\,g'(0) = 0$; that is, the set of functions f + tg for variable t contains non-schlicht functions.

THEOREM. <u>In every component of H the set of non-schlicht functions is connected.</u>

<u>Proof</u>. Suppose f and f + g are functions in a component K where g is entire. We form with arbitrary

complex numbers t all functions $f + tg$, each of which will be in K. An $f + tg$ will be non-schlicht if and only if there are $z_1 \neq z_2$, $|z_1| < 1$, $|z_2| < 1$, such that $f(z_1) + tg(z_1) = f(z_2) + tg(z_2)$ or

$$t = - \frac{f(z_1) - f(z_2)}{g(z_1) - g(z_2)} .$$

To find all values of t which give a non-schlicht $f + tg$ we have only to vary z_1 and z_2 in the above calculation. The totality of such values for t, with fixed f and g and the variation of z_1 and z_2, forms in the t-plane a non-empty, open, and connected set R. This is so because the mapping defined by f and g in the calculation of t is open and is continuous for all but a finite number of pairs (z_1, z_2), and because the set of possible (z_1, z_2) is open and connected in the product topology. For all t in R and only for each of these is $f + tg$. non-schlicht. For all t in R the corresponding functions are connected in K since each two points of R may be connected by a (polygon) path; and, therefore, since g is entire, the corresponding two functions are path-connected.

In order to obtain in the form $f + tg$ all non-schlicht functions in the component K it is sufficient to consider a fixed f in K and for g to take all entire functions in H.

Thus, let $f + t_0 g_0$ and $f + t_1 g_1$ be any two non-schlicht functions in K, where for no two constants

α, β, not both zero, does $\alpha g_o = \beta g_1$. Otherwise, at least one of g_1 and g_2 can be expressed linearly in terms of the other and the connectivity of the two non-schlict functions is reduced to the case previously considered (where only the t values are different). We form the linear function-set

$$L = \left\{ g_\lambda ; g_\lambda = g_o + \lambda(g_1 - g_o) \, , \; 0 \leq \lambda \leq 1 \right\} .$$

For each function g_λ we form the set R_λ of all t-values for which $f + tg_\lambda$ is non-schlicht.

We choose two fixed points z_1 and z_2, $z_1 \neq z_2$, in the unit disk such that $g_\lambda(z_1) - g_\lambda(z_2) \neq 0$ for any $0 \leq \lambda \leq 1$. That is, the line segment between $g_o(z_1) - g_o(z_2)$ and $g_1(z_1) - g_1(z_2)$ does not touch the origin. This choice is possible because

$$\frac{g_1(z_1) - g_1(z_2)}{g_o(z_1) - g_o(z_2)}$$

is neither constant nor always infinite by our assumption $\alpha g_o \neq \beta g_1$. (If the quotient is constant, α, then let $z_2 = 0$: $g_1(z)/g_2(z) = \alpha$. If the quotient α were always zero or always infinite one of g_o or g_1 would equal zero which is impossible since then, assuming $g_o = 0$ $2g_o = 0g_1$, e.g.) So then, as previously, the values of $\frac{g_1(z_1) - g_1(z_2)}{g_o(z_1) - g_o(z_2)}$ must form an open set. So that any z_1

and z_2 such that the argument of the quotient is then not an odd multiple of π will suffice. Now

$$-\frac{f(z_1) - f(z_2)}{g_\lambda(z_1) - g_\lambda(z_2)} = t_{p_\lambda} = p_\lambda$$

is uniformly continuous in λ, since $[0,1]$ is compact, and the p_λ form a compact set. Each p_λ corresponds to a non-schlicht function, and each p_λ is in R_λ.

For each p_λ there is a circle in R_λ with center p_λ and radius r_λ. We choose the r_λ so that $r_\lambda \geq r > 0$. This is possible because the distance of p_λ to the frontier of R_λ is a continuous function of p_λ or of λ. So that since the p_λ or the λ are compact and each distance is positive (R_λ is open), the infimum of the distances is positive.

We now choose, as p_λ is uniformly continuous in λ, a $\delta > 0$ such that, for all λ' and λ'' in $[0,1]$, if $|\lambda'-\lambda''| < \delta$, then $|P'_\lambda - P''_\lambda| < r$.

For each λ in $[0,1]$ the intersection

$$\cap \{R_\ell : \ell \text{ in } (\lambda-\delta, \lambda+\delta) \cap [0,1]\}$$

is then non-empty; the intersection contains p_λ.

Let the sequence

$$0 = \lambda_o < \lambda_1 < \lambda_2 < \ldots < \lambda_N = 1$$

be chosen so that for each k with $0 \leq k \leq N$

$$\lambda_{k+1} < \lambda_k + \delta \ ; \ \lambda_{k+1} - \lambda_k < \delta \ .$$

The union of the intervals $(\lambda_k-\delta, \lambda_k+\delta)$ covers the interval $[0,1]$.

The points P_{λ_k} we designate as τ_k. Each τ_k lies in the intersection

$$\cap\left\{R_\ell : \ell \text{ in } (\lambda_k-\delta, \lambda_k+\delta)\right\} .$$

Furthermore, R_{λ_k} contains τ_k, τ_{k+1}, τ_{k-1} since the circle in R_{λ_k} around τ_k has a radius greater than or equal to r and $|\tau_{k+1}-\tau_k| < r$, etc.

Finally we have a path of non-schlicht functions in K from $f + t_o g_o$ to $f + t_1 g_1$. From $f + t_o g_o = f + t_o g_{\lambda_o}$ to $f + \tau_o g_{\lambda_o}$ there is a path of non-schlicht functions corresponding to a path in R_o connecting t_o and τ_o. For each k, $0 \leq k < N$, the path from $f + \tau_k g_{\lambda_k}$ to $f + \tau_k g_{\lambda_{k+1}}$, through the linear subset of L indicated, will contain only non-schlicht functions since τ_k is in each R_λ from $\lambda = \lambda_k$ to $\lambda = \lambda_{k+1}$. For each k, $0 \leq k < N$, there is a path of non-schlicht functions from $f + \tau_k g_{\lambda_{k+1}}$ to $f + \tau_{k+1} g_{\lambda_{k+1}}$ since there is a path from τ_k to τ_{k+1} inside $R_{\lambda_{k+1}}$. At last, there is a suitable path from $f + \tau_N g_{\lambda_N}$ to $f + t_1 g_{\lambda_N} = f + t_1 g_1$ since R_{λ_N} contains both τ_N and t_1. This proves the theorem.

6. EXAMPLES

We give examples of schlicht and of non-schlicht functions.

EXAMPLE 1. THEOREM. Let f in H be given by $f(z) = \sum_{1}^{\infty} a_n z^n$. Then:

(1). If $|a_1| > \sum_{2}^{\infty} n|a_n|$, then f is stable-schlicht,

(2). If $|a_1| \geq \sum_{2}^{\infty} n|a_n|$, then f is schlicht, and

(3). If $|a_1| < \sum_{2}^{\infty} n|a_n|$, then f may be non-schlicht

Proof. (1). Let $|a_1| - \sum_{2}^{\infty} n|a_n| = \delta > 0$. For any two z_1 and z_2, $z_1 \neq z_2$, in the unit disk

$$f(z_1) - f(z_2)$$

$$= (z_1 - z_2)(a_1 + \sum_{2}^{\infty} a_n(z_1^{n-1} + z_1^{n-2} z_2 + \ldots + z_2^{n-1})).$$

Then

$$\left| \frac{f(z_1) - f(z_2)}{(z_1 - z_2)} \right| > \delta > 0.$$

so that f is stable-schlicht.

(2). Similarly, if $|a_1| \geq \sum_2^\infty n|a_n|$, we have

$$f(z_1)-f(z_2) = (z_1-z_2)(a_1 + \sum_2^\infty a_n(z_1^{n-1} + z_1^{n-2}z_2 + \ldots + z_2^{n-1})) \neq 0$$

for any $z_1 \neq z_2$ since

$$|\sum_2^\infty a_n(z_1^{n-1} + z_1^{n-2}z_2 + \ldots + z_2^{n-1})| < \sum_2^\infty n|a_n| < |a_1| ,$$

so that f is schlicht.

(3). If $|a_1| < \sum_2^\infty n|a_n|$ and if $f(z) = \sum_1^\infty a_n z^n$ converges in $|z| < 1$, then $f'(z) = \sum_1^\infty n a_n z^{n-1}$ and converges absolutely for each z in $|z| < 1$. That is, $|a_1| + \sum_2^\infty n|a_n|\rho^{n-1}$ converges and is continuous for all ρ such that $0 \leq \rho < 1$; so $\sum_2^\infty n|a_n|\rho^{n-1} \to 0$ as $\rho \to 0$. Since $|a_1| < \sum_2^\infty n|a_n|$, $\lim_{\rho \to 1} \sum_2^\infty n|a_n|\rho^{n-1} > |a_1|$. Therefore there is a ρ_o in $(0,1)$ such that

$$|a_1| = \sum_2^\infty n|a_n|\rho_o^{n-1} .$$

Now, the arguments of the a_n in f may be such that f is non-schlicht; in particular

$$f(z) = |a_1|z - |a_2|^2 - |a_3|z^3 - \ldots$$

is non-schlicht since $f'(\rho_o) = 0$.

EXAMPLE 2. The functions f_ρ in H given by $f_\rho(z) = \frac{z}{1-\rho z}$, $|\rho| \leq 1$, are stable-schlicht since

$$|f_\rho(z_1)-f_\rho(z_2)| = \left|\frac{z_1 - z_2}{(1-\rho z_1)(1-\rho z_2)}\right| \geq |z_1-z_2| \frac{1}{(1+|\rho|)^2} .$$

In the following examples of schlicht functions, whether each function is also stable-schlicht is considered in relation to the boundary of the image, under the function, of the unit disk. We call the image by f of the unit disk $f(\mathbb{D})$.

EXAMPLE 3. The convex schlicht functions, schlicht functions which map the unit disk onto a covex domain, are stable-schlicht. This follows from the known result (*) that for any convex schlicht function f

$$|f'(z)| \geq \frac{1}{4} |f'(0)| > 0 , \quad |z| < 1.$$

Apparently, as further examples show, this may be the only elementary characterization of the boundary of the image of the disk which assures that the function is stable-schlicht.

EXAMPLE 4. The function $f(z) = \frac{(1-z)^2}{2} - \frac{1}{2}$ is in H and is schlicht. Furthermore, $f(D)$ is bounded by a rectifiable Jordan curve which is analytic except at $z = 1$, where the curve has a cusp. The boundary of $f(D)$ is also a C' curve. However f is not stable-schlicht since f' is not bounded away from zero.

EXAMPLE 5. The functions

$$f_a(z) = \left(\frac{\sqrt{i\left(\frac{1+z}{1-z}\right)} - 1}{\sqrt{i\left(\frac{1+z}{1-z}\right)} + 1} \right) z^a \ , \ 0 < a < 2 \ ,$$

show that even if the image of the disk is bounded by a rectifiable Jordan curve without a cusp, the schlicht function may not be stable-schlicht. Where $f_a(z) = w$,

$$f_a(D) = \left\{ w : |w| < 1 \ , \ 0 < \arg w < a \right\} .$$

Since f_a' is bounded away from zero in the unit disk if and only if $0 < a \leq 1$, $f_a(z) - f_a(0)$ will be schlicht but not stable-schlicht if $1 < a < 2$.

Piranian [9] has given an example of a schlicht function whose distance to every other schlicht function is at least one.

APPENDIX

DEFINITION. A linear topological space is a vector space equipped with a Hausdorff topology such that the operations of addition and of scalar multiplication are continuous.

DEFINITION. A linear space L is normed if there exists a function $p: L \to E^1$ such that

(1). $p(a) > 0$ for all $a \in L$

(2). $p(a) = 0$ if and only if $a = 0$

(3). $p(a+b) \leq p(a) + p(b)$ for all $a, b \in L$

(4). $p(\lambda a) = |\lambda| p(a)$ for all $a \in L$ and all $\lambda \in E'$.

We write $p(a) = \|a\|$ and call $\|a\|$ the norm of $a \in L$. We then have the distance $\|a-b\|$ as a metric for the space L.

DEFINITION. A function f which is analytic in a domain D is said to be schlicht (univalent, simple) in D if $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$. This holds if and only if the mapping defined by f is one-to-one.

THEOREM. A schlicht function of a schlicht function is schlicht. If f is schlicht from a domain D onto a domain D', and if F is schlicht in D' then $F \circ f$ is schlicht in D.

Proof. $F(f(z_1)) = f(f(a_2)$ implies $f(z_1) = f(z_2)$, and this implies $z_1 = z_2$.

THEOREM. If f is schlicht in D, then the derivative f' is never zero in D.

Proof. Suppose that $f'(z_o) = 0$. Then the power series expansion of f about z_o shows that $f - f(z_o)$ has a zero of order two or greater at z_o. Since $f - f(z_o)$ is not constant, the zeros of $f - f(z_o)$ as well as the zeros of f' must be isolated points. Hence we can find a circle C on which $f - f(z_o)$ does not vanish and inside of which f' has no zeros except at z_o. Therefore, let $m > 0$ be the minimum of $|f - f(z_o)|$ on this circle, where m exists because $|f - f(z_o)|$ is continuous and the circle is compact. Then, by Rouche's theorem, if $0 < |a| < m$, $f - f(z_o) - a$ has two or more zeros in the circle. It does not have a double zero since if it did, consideration of the power series expansion of f about this new point in the circle would show that f' would be zero there, which has already been excluded. Therefore, f takes the value $f(z_o) + a$ more than once in D. This is a contradiction.

Furthermore, if $f'(z_o)$ were zero, we see that, by varying the choice of "a", f would take on all values sufficiently near $f(z_o)$ at least twice.

THEOREM. Let f be analytic at some point $z = z_0$ and let $f'(z_0) \neq 0$. Then f is schlicht in the immediate neighborhood of $z = z_0$.

Proof. We may assume w.l.o.g. that $z = 0$. Let f be given by $f(z) = \sum_0^\infty a_n z^n$ where $a_1 \neq 0$. We show that if z_1 and z_2 are close enough to $z = 0$ then $f(z_1) = f(z_2)$ implies $z_1 = z_2$. If $f(z_1) = f(z_2)$ then

$$\sum_1^\infty a_n(z_1^n - z_2^n) = 0 \quad ,$$

$$(z_1 - z_2)[a_1 + \sum_2^\infty a_n(z_1^{n-1} + z_1^{n-2} z_2 + \ldots + z_2^{n-1})] = 0.$$

Now if $|z_1| < r$ and $|z_2| < r$, then the modulus of the second factor is greater than $|a_1| - \sum_2^\infty n|a_n|r^{n-1}$. Thus, since f' given by $f'(z) = \sum_1^\infty n\, a_n z^{n-1}$ is absolutely convergent in some nbd of zero, the modulus of the second factor is positive if r is small enough. Hence $z_1 = z_2$ if r is small enough.

THEOREM. Let f be analytic in a domain D whose boundary is a simple closed curve C, and let f be schlicht, i.e. one-to-one, on C. Then f is schlicht in D. See Titchmarsh, The Theory of Functions, p. 201.

THEOREM. Let f be schlicht in the unit disk and $f(0) = 0$. Then we have for $|z| = r$, $0 < r < 1$,

$$|f'(0)| \frac{r}{(1+r)^2} \leq |f(z)| \leq |f'(0)| \frac{r}{(1-r)^2} .$$

See [6, p. 4].

THEOREM. Let f be schlicht in the unit disk and $f(0) = 0$. Then if f maps the unit disk onto a convex domain, we have for $|z| = r$, $0 < r < 1$,

$$|f'(0)| \frac{1}{(1+r)^2} \leq |f'(z)| \leq |f'(0)| \frac{1}{(1-r)^2} .$$

See [6, p. 13]. These classical results and others appear in [6] and elsewhere in standard literature of complex functions.

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