1962

Theory of charcteristics of second order partial differential equations

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THEORY OF CHARACTERISTICS

OF

SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

by

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A THESIS

Presented to the Graduate Faculty

of Lehigh University

in Candidacy for the Degree of

Master of Science

Lehigh University

June, 1962

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

May 19, 1962
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SECTION I

Canonical Forms

In the following material we shall be concerned with the second-order partial differential equation,

$$au_{xx} + 2bu_{xy} + cu_{yy} + f(x,y,u,u_x,u_y) = 0, \tag{1}$$

in which $u = u(x,y)$ is a function of the independent variables x and y. The coefficients a,b,c are continuous functions of x and y, having as many derivatives with respect to x and y as is necessary to the discussion.

1.1 To facilitate our study of characteristics, we shall, under certain conditions, be able to reduce the linear second-order partial differential equation,

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + hu + f = 0, \tag{2}$$

to one of three canonical forms. This reduction will be possible if we effect a change of variables, writing (2) in terms of $u_{\xi\xi}$, $u_{\xi\eta}$, $u_{\eta\eta}$, etc., where $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$, are particular functions of x,y. It is supposed that $\frac{d(\xi,\eta)}{d(x,y)} \neq 0$, so that these equations define x and y implicitly as functions of ξ and η.

Writing u_x, u_y in terms of the new variables ξ and η we have:

$$u_x = u_\xi \xi_x + u_\eta \eta_x, \qquad u_y = u_\xi \xi_y + u_\eta \eta_y.$$

Using these expressions to obtain u_{xx}, etc., in terms of ξ and η, we find

$$u_{xx} = (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + u_\xi\xi_{xx} + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)\eta_x + u_\eta\eta_{xx}$$

$$= u_{\xi\xi}(\xi_x)^2 + u_{\eta\eta}(\eta_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_\xi\xi_{xx} + u_\eta\eta_{xx},$$

with analogous expressions for u_{xy} and u_{yy}. On substituting these values in equation (1) we find

$$\alpha u_{\xi\xi} + 2\beta u_{\xi\eta} + \gamma u_{\eta\eta} + \varphi(\xi, \eta, u, u_\xi, u_\eta) = 0 \qquad (3)$$

where,

$$\alpha = a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2$$

$$\beta = \{a\xi_x\eta_y + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y\}$$

$$\gamma = a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2,$$

and equation (2) becomes

$$\alpha u_{\xi\xi} + 2\beta u_{\xi\eta} + \gamma u_{\eta\eta} + \delta u_\xi + \epsilon u_\eta + hu + f = 0. \qquad (4)$$

1.2 In order to find canonical forms for equation (3) we shall consider the sign of the discriminent of coefficients a,b,c in the expressions for α, β, γ. First we shall investigate the case when $ac - b^2 < 0$. In this case the quadratic equation,

$$az^2 + 2bz + c = 0, \qquad (5)$$

has two real roots, say λ_1 and λ_2, so that

$$az^2 + 2bz + c = a(z - \lambda_1)(z - \lambda_2). \qquad (6)$$

At this point let us choose ξ and η so that they satisfy the following two equations:

$$\xi_x = \lambda_1 \xi_y, \qquad \eta_x = \lambda_2 \eta_y. \tag{7}$$

Substituting these values of ξ_x and η_x into our expressions for α and γ we find:

$$\alpha = a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 = a(\lambda_1\xi_y)^2 + 2b(\lambda_1\xi_y)\xi_y + c(\xi_y)^2$$

$$= (a\lambda_1^2 + 2b\lambda_1 + c)(\xi_y)^2,$$

and

$$\gamma = a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2 = a(\lambda_2\eta_y)^2 + 2b(\lambda_2\eta_y)\eta_y + c(\eta_y)^2$$

$$= (a\lambda_2^2 + 2b\lambda_2 + c)(\eta_y)^2.$$

Hence by equation (6) $\alpha = \gamma = 0$, for all x and y.

That it is possible to choose ξ and η in this seemingly arbitrary manner we can justify by considering the following. Suppose

$$\frac{dy}{dx} + \lambda_1(x,y) = 0, \qquad \frac{dy}{dx} + \lambda_2(x,y) = 0. \tag{8}$$

These equations will have unique families of solutions if λ_1 and λ_2 have continuous first partial derivatives with respect to x and y. Since λ_1 and λ_2 can each be expressed as an algebraic combination of a,b,c and a,b,c are known to possess continuous partial derivatives of all necessary orders in x and y, we know λ_1 and λ_2 have continuous first partials with respect to x and y. Let the solutions of the two first order equations (8) be given respectively by

$$\xi(x,y) = \xi_o, \qquad \eta(x,y) = \eta_o.$$

Then

$$\xi_x + \xi_y \frac{dy}{dx} = 0, \qquad \eta_x + \eta_y \frac{dy}{dx} = 0,$$

or

$$\lambda_1 = -\frac{dy}{dx} = \frac{\xi_x}{\xi_y}, \qquad \lambda_2 = -\frac{dy}{dx} = \frac{\eta_x}{\eta_y},$$

the latter two equations satisfying equation (7). Note that this choice of ξ and η satisfies the condition that they be independent since if

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x = 0,$$

then $\dfrac{\xi_x}{\xi_y} = \dfrac{\eta_x}{\eta_y}$, or $\lambda_1 = \lambda_2$ which contradicts $ac - b^2 < 0$.

Next, we wish to show that $\beta \neq 0$. To do this we consider the equation

$$\alpha\gamma - \beta^2 = (ac - b^2)(\xi_x\eta_y - \xi_y\eta_x)^2. \tag{9}$$

We can justify this equality by expressing $\alpha\gamma - \beta^2$ in determinent notation:

$$\begin{vmatrix} \alpha & \beta \\ \beta & \gamma \end{vmatrix} =$$

$$\begin{vmatrix} a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 & a\xi_x\eta_x + b(\xi_x\eta_y + \eta_x\xi_y) + c\xi_y\eta_y \\ a\xi_x\eta_x + b(\xi_x\eta_y + \eta_x\xi_y) + c\xi_y\eta_y & a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (a\xi_x + b\xi_y)\xi_x + (b\xi_x + c\xi_y)\xi_y & (a\eta_x + b\eta_y)\xi_x + (b\eta_x + c\eta_y)\xi_y \\ (a\xi_x + b\xi_y)\eta_x + (b\xi_x + c\xi_y)\eta_y & (a\eta_x + b\eta_y)\eta_x + (b\eta_x + c\eta_y)\eta_y \end{vmatrix}$$

$$= \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \begin{vmatrix} a\xi_x + b\xi_y & a\eta_x + b\eta_y \\ b\xi_x + c\xi_y & b\eta_x + c\eta_y \end{vmatrix} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \begin{vmatrix} a & b \\ b & c \end{vmatrix} \begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix}$$

But

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix}, \text{ hence } \begin{vmatrix} \alpha & \beta \\ \beta & \gamma \end{vmatrix} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \begin{vmatrix} a & b \\ b & c \end{vmatrix} \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$$

and

$$\alpha\gamma - \beta^2 = (\xi_x\eta_y - \eta_x\xi_y)(ac - b^2)(\xi_x\eta_y - \xi_y\eta_x) = (ac - b^2)\cdot(\xi_x\eta_y - \eta_x\xi_y)^2.$$

Since $\alpha = \gamma = 0$, this shows that β is equal to the product of two non-zero factors and hence is different from zero. Hence our transformed equation takes the form,

$$u_{\xi\eta} = F(\xi, \eta, u, u_\xi, u_\eta), \tag{10}$$

which will be known as the canonical form for the case when $ac - b^2 < 0$. We shall also refer to it as the hyperbolic case.

1.3 Next let us consider $ac - b^2 = 0$. In this case equation (6) becomes

$$az^2 + 2bz + c = a(z - \lambda_1)(z - \lambda_2) = 0.$$

As in the previous case let us choose ξ in such a manner that $\xi_x = \lambda_1\xi_y$, and specify η as an arbitrary function independent of ξ. This choice of ξ makes $\alpha = 0$, and then equation (9) shows that $\beta = 0$, since $ac - b^2 = 0$. Now $\gamma \neq 0$, since if it were we would have

$$\frac{\eta_x}{\eta_y} = \lambda_1 \quad \text{and thus} \quad \frac{\eta_x}{\eta_y} = \frac{\xi_x}{\xi_y}, \quad \text{or} \quad \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = 0.$$

Then ξ and η would not be independent. We arrive at

$$u_{\eta\eta} = F(\xi, \eta, u, u_\xi, u_\eta) \tag{11}$$

as the canonical form for $ac - b^2 = 0$, and refer to it as the parabolic case.

1.4 Finally, suppose $ac - b^2 > 0$. If we allow complex values of ξ and η, equation (6) will have two distinct solutions λ_1 and λ_2 which are complex conjugates. Again choosing ξ and η satisfying $\xi_x = \lambda_1 \xi_y$ and $\eta_x = \lambda_2 \eta_y$, we obtain as before,

$$u_{\xi\eta} = F(\xi, \eta, u, u_\xi, u_\eta). \tag{12}$$

Let us change the independent variables in (12) by putting

$$\xi_1 = \frac{\xi + \eta}{2}, \qquad \eta_1 = \frac{\xi - \eta}{2i}. \tag{13}$$

We note that ξ_1 and η_1 are independent and real. The transformed equation is

$$u_{\xi_1\xi_1} + u_{\eta_1\eta_1} = G(\xi_1, \eta_1, u, u_{\xi_1}, u_{\eta_1}). \tag{14}$$

We refer to this case as being elliptic.

1.5 As an example to illustrate the elliptic case let us consider the equation

$$u_{xx} + yu_{yy} = 0 \quad (y>0). \qquad (15)$$

Note that for $y > 0$ the equation is not elliptic since $ac - b^2 > 0$ implies in this case $1 \cdot y + 0 > 0$.

Solving equation (6) we have

$$az^2 + 2bz + c = z^2 + y = (z + i\sqrt{y})(z - i\sqrt{y}) = 0$$

or $\lambda_1 = i\sqrt{y}$ and $\lambda_2 = -i\sqrt{y}$. Applying the transformation given by equation (13) we find

$$\xi_1 = \frac{4\sqrt{y}}{2} = 2\sqrt{y}, \qquad \eta_1 = \frac{2ix}{2i} = x.$$

Hence, $u_x = u_{\eta_1}$, $u_y = \frac{1}{\sqrt{y}} u_{\xi_1}$, $u_{xx} = u_{\eta_1 \eta_1}$, $u_{yy} = \frac{1}{y} u_{\xi_1 \xi_1} - \frac{1}{2\sqrt{y^3}} u_{\xi_1}$.

Substituting these values in equation (15) we have

$$u_{xx} + yu_{yy} = u_{\eta_1 \eta_1} + u_{\xi_1 \xi_1} - \frac{yu_{\xi_1}}{2\sqrt{y^3}} = u_{\xi_1 \xi_1} + u_{\eta_1 \eta_1} - \frac{u_{\xi_1}}{\xi_1} = 0$$

where

$$u_{\xi_1 \xi_1} + u_{\eta_1 \eta_1} - \frac{u_{\xi_1}}{\xi_1} = 0$$

expresses our canonical form.

SECTION II

Characteristics

2.1 When dealing with an ordinary differential equation of the first order, $\frac{dy}{dx} = f(x,y)$ defines an angle $\theta = \tan^{-1} f(x,y)$ which the tangent to an integral curve must make with the x-axis, and the solution of the equation consists of the set of curves having the given direction at any point in the xy-plane. Analogously, a linear partial differential equation of the first order,

$$Pu_x + Qu_y - R = 0, \qquad (16)$$

may be thought of as determining at any point of three-space a direction P,Q,R which by the differential equation is perpendicular to the direction $u_x, u_y, -1$. But this latter direction is normal to the integral surface $u(x,y,u) =$ constant. Thus P,Q,R is tangent to the integral surface. A curve having the direction P,Q,R at each point is called a characteristic curve of (16). Any surface built up from characteristic curves turns out to be an integral surface of the differential equation.

2.2 In a similar way we can investigate the problem of determining solutions of linear second-order partial differential equations by considering ordinary equations of second order. Concerning the latter, the initial conditions prescribe a point in the plane through which the integral curve is to pass and also prescribe the slope of the integral curve at that point. This is equivalent to prescribing the values of the dependent variable y and its derivative corresponding to a given value of the independent variable x. In the case of a partial differential equation, the analogous initial conditions prescribe a curve C in space which is to lie in the integral surface and also prescribe the orientation of the tangent planes to the integral surface along that curve. These conditions are equivalent to conditions which specify the values of u and its partial derivatives u_x and u_y along the projection C_o of the curve C onto the xy-plane. However, the values of u, u_x, u_y cannot be specified in a completely independent way if u is to be differentiable along C_o since if τ is a parameter prescribing position along C_o we have

$$\dot{u} = u_x \dot{x} + u_y \dot{y}, \tag{17}$$

where the dots indicate differentiation with respect to τ. A curve C in space, together with values of u_x and u_y specified in such a way that equation (17) is satisfied is called a strip. We may also define a strip as a curve endowed with a planar element at each point, provided the planar element passes through the tangent to the curve and the orientation of the planar element varies continously as we proceed along the curve. Should the planar elements be tangent to the integral surface, the strip is called an integral strip of the first order. Finding a solution of the given differential equation through the integral strip is known as the Cauchy problem.

Let us adopt for convenience the convention

$$u_x = p, \quad u_y = q, \quad u_{xx} = r, \quad u_{yy} = t, \quad u_{xy} = s.$$

Then equation (2) becomes

$$ar + 2bs + ct + d = 0. \tag{18}$$

As has been previously mentioned, p and q must satisfy equation (17). Hence, if we consider a space curve given in terms of the parameter τ:

$$x_o = x_o(\tau), \quad y_o = y_o(\tau), \quad u_o = u_o(\tau) \tag{19}$$

then it must be true that

$$\dot{u}_o = p_o\dot{x}_o + q_o\dot{y}_o. \tag{20}$$

We will now investigate whether it is possible to express the solution of the Cauchy problem in the form of a Taylor's series expansion about a point τ_o of the initial curve. At τ_o the coordinates x_o, y_o and the derivatives p_o, q_o are known. We now have to determine the values of r, s, t. By equation (18) we have

$$ar + 2bs + ct = -d. \tag{21}$$

If we differentiate $p_o = p_o(\tau)$ and $q_o(\tau)$ with respect to τ we have

$$\begin{aligned} \dot{p}_o &= p_x\dot{x} + p_y\dot{y} = r\dot{x} + s\dot{y} \\ \dot{q}_o &= q_x\dot{x} + q_y\dot{y} = s\dot{x} + t\dot{y} \end{aligned} \tag{22}$$

The three equations given by (21) and (22) will have a unique solution in r, s, t if and only if the determinant of their coefficients is not zero. But

$$\triangle = \begin{vmatrix} a & 2b & c \\ \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \end{vmatrix} = a\dot{y}^2 - 2b\dot{x}\dot{y} + c\dot{x}^2. \tag{23}$$

Hence if $\triangle \neq 0$ for the space curve given by equation (19), then r,s,t are uniquely determined along it. It turns out that the same condition guarantees unique values for all the higher partial derivatives of u. We may reasonably expect then, that there exists a unique integral surface of equation (18) which includes the strip consisting of equation (19) and u_x, u_y along it satisfying equation (17).

Now suppose that for all points on the given space curve,

$$a\dot{y}^2 - 2b\dot{x}\dot{y} + c\dot{x}^2 = 0. \tag{24}$$

Then we cannot solve equations (21) and (22) for s unless the numerator determinant in the solution by Cramer's rule is also zero, that is,

$$\begin{vmatrix} a & -d & c \\ \dot{x} & \dot{p} & 0 \\ 0 & \dot{q} & \dot{y} \end{vmatrix} = 0, \quad \text{or} \quad a\dot{p}\dot{y} + c\dot{q}\dot{x} + d\dot{x}\dot{y} = 0. \tag{25}$$

Hence if both equations (24) and (25) are satisfied we can find infinitely many values of r,s,t satisfying equations (21) and (22). If equation (24) is satisfied and equation (25) is not then no solution exists.

Let

$$\xi(x,y) = \xi_o \tag{26}$$

be the projection of equation (19) on the xy-plane. Differentiating yields $\xi_x \dot{x} + \xi_y \dot{y} = 0$. If we use this equation to eliminate $\dot{y}$ from (24) we get

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0. \tag{27}$$

In this case we call $\xi(x,y) = \xi_o$ a characteristic curve of equation (18).

Let us analyze equation (24) for the three cases discussed previously, that is, hyperbolic $(ac - b^2 < 0)$, parabolic $(ac - b^2 = 0)$, and elliptic $(ac - b^2 > 0)$. We can rewrite equation (24) in the form $a(\frac{dy}{dx})^2 - 2b(\frac{dy}{dx}) + c = 0$. Then

$$\frac{dy}{dx} = \frac{2b \pm \sqrt{4b^2 - 4ac}}{2a} = \frac{b \pm \sqrt{b^2 - ac}}{a}, \tag{28}$$

and for a real solution we must have $b^2 - ac$ non-negative. That is, characteristic curves exist for hyperbolic and parabolic linear equations of second order, but not for elliptic linear equations of second order. If our equation is hyperbolic, that is, $ac - b^2 < 0$ and $a \neq 0$, we have two families of curves in the xy-plane; if it is parabolic, $ac - b^2 = 0$, there will be one such family.

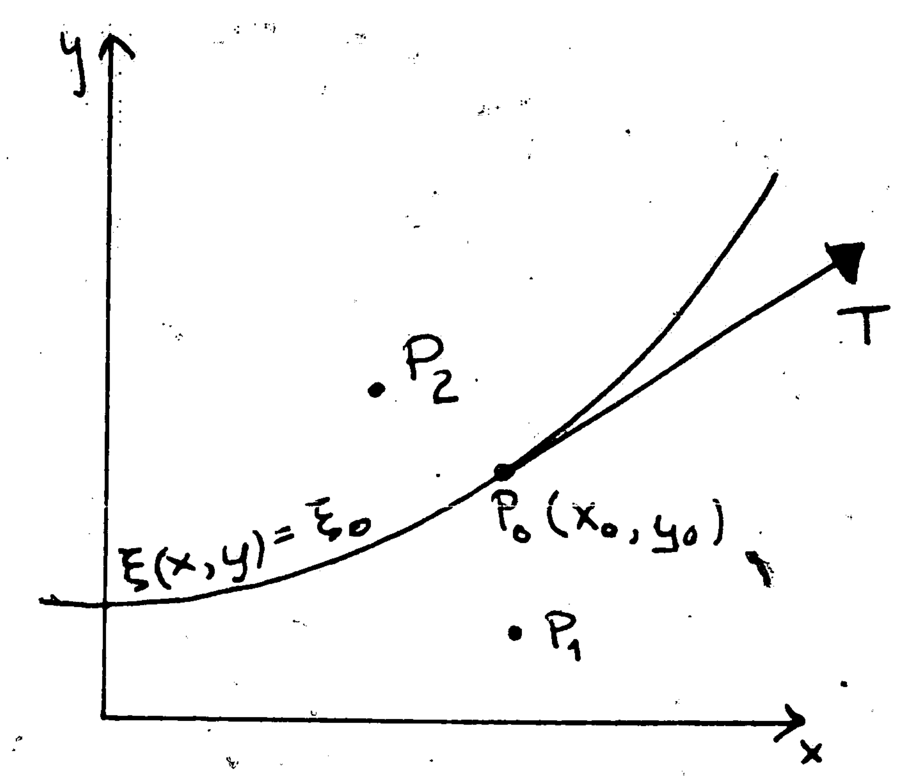
2.3 In order to investigate the relation of characteristic curves to second-order discontinuities of an integral surface we shall first make a few useful definitions. Let $f(x,y)$ be a function continuous at all points in the xy-plane with the exception of points on the curve $\xi(x,y) = \xi_o$, where $\xi(x,y)$ has derivatives of all necessary orders. Let P_1 and P_2 be variable points on either side of a point P_o lying on $\xi(x,y) = \xi_o$. Then we define the

$$\text{"jump of } f \text{ at } P_o\text{"} \equiv [f]_{P_o} = \lim_{P_1, P_2 \to P_o} [f(P_1) - f(P_2)] . \qquad (29)$$

We shall refer to the directional derivative of f at P_o along the tangent to $\xi(x,y) = \xi_o$ as the tangential derivative of f at P_o (assuming, of course, f has first partial derivatives).

Since ξ_x, ξ_y is a vector normal to $\xi(x,y) = \xi_o$, the vector $\xi_y, -\xi_x$, being perpendicular to this, lies along the tangent. Therefore, the tangential derivative of f with respect to P_o is

$$\frac{df}{dT} = \frac{f_x \xi_y(P_o) - f_y \xi_x(P_o)}{\xi_x^2(P_o) + \xi_y^2(P_o)} . \qquad (30)$$



If the right member of equation (30) is continuous for any value of P_o when P_o is on the curve $\xi(x,y) = \xi_o$, then we shall say the tangential derivative of f, $\frac{df}{dT}$, is continuous on $\xi(x,y) = \xi_o$.

It will now be possible to show that if a solution u of equation (3) has second order discontinuities along a curve $\xi(x,y) = \xi_o$ then this curve is necessarily a characteristic curve of equation (3).

Suppose $u(x,y) = 0$ is a solution of equation (18) with continuous derivatives of all orders except that at every point along $\xi(x,y) = \xi_o$ not all of its second derivatives are continuous. Further, suppose that at all points of $\xi(x,y) = \xi_o$ u_x and u_y have continuous tangential derivatives. The last assumption ($\frac{du_x}{dT}$ and $\frac{du_y}{dT}$ continuous) implies that for all points on $\xi(x,y) = \xi_o$ the following expressions are continuous:

$$\frac{du_x}{dT} = \frac{u_{xx}\xi_y(P_o) - u_{xy}\xi_x(P_o)}{\sqrt{\xi_x^2(P_o) + \xi_y^2(P_o)}}, \quad \frac{du_y}{dT} = \frac{u_{xy}\xi_y(P_o) - u_{yy}\xi_x(P_o)}{\sqrt{\xi_x^2(P_o) + \xi_y^2(P_o)}}.$$

But in each expression for a particular P_o the denominator is constant, hence for every P_o on $\xi(x,y) = \xi_o$, the numerators are continuous. Thus if we set $w_1 = u_{xx}\xi_y(P_o) - u_{xy}\xi_x(P_o)$, then

$$[w_1]_{P_o} = \lim_{P_1,P_2 \to P_o} [w_1(P_2) - w_1(P_1)] = 0$$

since w_1 is continuous, and hence

$$\begin{aligned}[w_1]_{P_o} &= \lim_{P_1,P_2 \to P_o} \left([u_{xx}\xi_y(P_2) - u_{xy}\xi_x(P_2)] - [u_{xx}\xi_y(P_1) - u_{xy}\xi_x(P_1)] \right) \\ &= \xi_y(P_o) \lim_{P_1,P_2 \to P_o} [u_{xx}(P_2) - u_{xx}(P_1)] - \xi_x(P_o) \lim_{P_1,P_2 \to P_o} [u_{xy}(P_2) - u_{xy}(P_1)] \\ &= \xi_y(P_o)[u_{xx}]_{P_o} - \xi_x(P_o)[u_{xy}]_{P_o} = 0.\end{aligned}$$

Setting $w_2 = u_{xy}\xi_y(P_o) - u_{yy}\xi_x(P_o)$, we have similarly

$$[w_2]_{P_o} = \xi_y(P_o)[u_{xy}]_{P_o} - \xi_x(P_o)[u_{yy}]_{P_o} = 0.$$

Thus (31)

$$0 = [u_{xx}]_{P_o}\xi_y(P_o) - [u_{xy}]_{P_o}\xi_x(P_o) = [u_{xy}]_{P_o}\xi_y(P_o) - [u_{yy}]_{P_o}\xi_x(P_o)$$

We can therefore write $\left[u_{xy}\right]_{P_o} = \frac{\xi_x(P_o)}{\xi_y(P_o)}\left[u_{yy}\right]_{P_o}$, and

$$\left[u_{xx}\right]_{P_o} = \frac{\xi_x(P_o)}{\xi_y(P_o)}\left[u_{xy}\right]_{P_o} = \frac{\xi_x^2(P_o)}{\xi_y^2(P_o)}\left[u_{yy}\right]_{P_o} .$$

If we set

$$\lambda = \frac{u_{yy}\ P_o}{\xi_y^2(P_o)}, \tag{32}$$

we can write (33)

$$\left[u_{xx}\right]_{P_o} = \lambda\xi_x^2(P_o)\ ,\ \left[u_{xy}\right]_{P_o} = \lambda\xi_x(P_o)\xi_y(P_o)\ ,\ \left[u_{yy}\right]_{P_o} = \lambda\xi_y^2(P_o).$$

Now consider ξ defined by $\xi(x,y) = \xi_o$ and a function η given by stipulating that $(f(\xi,\eta))_\eta = \frac{df}{dT}$. We shall show that λ as defined by equation (32) can be written as a function of η alone.

Earlier we found that

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\xi_y^2 + u_\xi\xi_{xx} + u_y\eta_{xx}$$

when we made a change of variables $\xi = \xi(x,y)$, $\eta = \eta(x,y)$. By hypothesis u_x and u_y are continuous. Hence the two equations $u_x = u_\xi\xi_x + u_\eta\eta_x$ and $u_y = u_\xi\xi_y + u_\eta\eta_y$ taken together imply u_ξ and u_η are also continuous. In addition we know that u_x and u_y have continuous tangential derivatives. That is, we know

$$\frac{du_x}{dT} = (u_\xi\xi_x + u_\eta\eta_x)_\eta = u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x ,$$

and

$$\frac{du_y}{dT} = (u_\xi\xi_y + u_\eta\eta_y)_\eta = u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y$$

where the right member of each equation represents a continuous function. Hence the functions $u_{\xi\eta}$ and $u_{\eta\eta}$ are continuous. We then consider

$$\left[u_{xx}\right]_{P_o} = \lim_{P_1, P_2 \to P_o} \left[u_{xx}(P_2) - u_{xx}(P_1)\right]$$

$$= \lim_{P_1, P_2 \to P_o} \Big\{ \left[u_{\xi\xi}(P_2)\xi_x^2(P_2) - u_{\xi\xi}(P_1)\xi_x^2(P_1)\right] +$$

$$\left[2u_{\xi\eta}(P_2)\xi_x(P_2)\xi_y(P_2) - 2u_{\xi\eta}(P_1)\xi_x(P_1)\xi_y(P_1)\right] +$$

$$\left[u_{\eta\eta}(P_2)\xi_y^2(P_2) - u_{\eta\eta}(P_1)\xi_y^2(P_1)\right] + \left[u_\xi(P_2)\xi_{xx}(P_2) -\right.$$

$$\left.u_\xi(P_1)\xi_{xx}(P_1)\right] + \left[u_\eta(P_2)\eta_{xx}(P_2) - u_\eta(P_1)\eta_{xx}(P_1)\right]$$

By the continuity of $u_{\xi\eta}$, $u_{\eta\eta}$, u_η and u_ξ the jump of each of these functions disappears and we are left with

$$\left[u_{xx}\right]_{P_o} = \lim_{P_1, P_2 \to P_o} \left[u_{\xi\eta}(P_2)\xi_x^2(P_2) - u_{\xi\xi}(P_1)\xi_x^2(P_1)\right]$$

$$= \xi_x^2(P_o) \lim_{P_1, P_2 \to P_o} \left[u_{\xi\xi}(P_2) - u_{\xi\xi}(P_1)\right] = \xi_x^2(P_o)\left[u_{\xi\xi}\right]_{P_o} .$$

Hence, from the definition of λ we find

$$\lambda = \left[u_{\xi\xi}\right]_{P_o} \qquad (34)$$

We originally assumed that at every point of the curve $\xi(x,y) = \xi_o$, at least one of the second derivatives of u was not continuous. Thus at every point of this curve, at least one of the quantities $[u_{xx}]$, $[u_{xy}]$, and $[u_{yy}]$ is not zero. This implies that $\lambda \neq 0$ since by equation (33) λ appears as a factor in each expression. Hence at every point of $\xi(x,y) = \xi_o$, $[u_{\xi\xi}]$ is non-zero.

In a similar way we can consider the equation previously derived:

$$\alpha u_{\xi\xi} + 2\beta u_{\xi\eta} + \gamma u_{\eta\eta} + \varphi(x,y,u,u_{\xi},u_{\eta}) = 0. \qquad (3)$$

Again the continuity of $u_{\xi\eta}$, $u_{\eta\eta}$, u_{ξ}, and u_{η} will reduce the jump of the left member at P_o to $\alpha\left[u_{\xi\xi}\right]_{P_o}$. Thus we have the equation:

$$\alpha\left[u_{\xi\xi}\right]_{P_o} = 0.$$

But we have just shown $\left[u_{\xi\xi}\right]$ to be non-zero at every point of $\xi(x,y) = \xi_o$, which implies $\alpha = 0$, at every point on the curve. By our previous evaluation for α:

$$\alpha = a(\xi_x)^2 + 2b\xi_x\eta_y + c(\xi_y)^2 = 0 \qquad (35)$$

which is nothing more than the definition of a characteristic curve.

If we then differentiate equation (3) with respect to ξ we obtain:

$$u_{\xi\xi\xi}\cdot\alpha + u_{\xi\xi}\alpha_{\xi} + 2u_{\xi\xi\eta}\cdot\beta + 2u_{\xi\eta}\beta_{\xi} + u_{\xi\eta\eta}\cdot\gamma + u_{\eta\eta}\gamma_{\xi} + u_{\xi\xi}\cdot\delta + u_{\xi}\delta_{\xi} + u_{\xi\eta}\cdot\epsilon + u_{\eta}\epsilon_{\xi} + u_{\xi}h + uh_{\xi} + f_{\xi} = 0. \qquad (36)$$

The term involving α equals zero by our previous work and the terms in $u_{\xi\eta}$, $u_{\eta\eta}$, u_{η}, u_{ξ}, u, f_{ξ} all vanish due to continuity when we evaluate the jump of the left member of equation (36). We know also that the term involving $u_{\xi\eta\eta}$ vanishes. Hence we can write:

$$\left[u_{\xi\xi}\right]_{P_o}(\alpha_\xi + \delta) + 2\beta\left[u_{\xi\xi\eta}\right]_{P_o} = 0. \qquad (37)$$

Remembering that $\lambda = \left[u_{\xi\xi}\right]_{P_o}$, we know

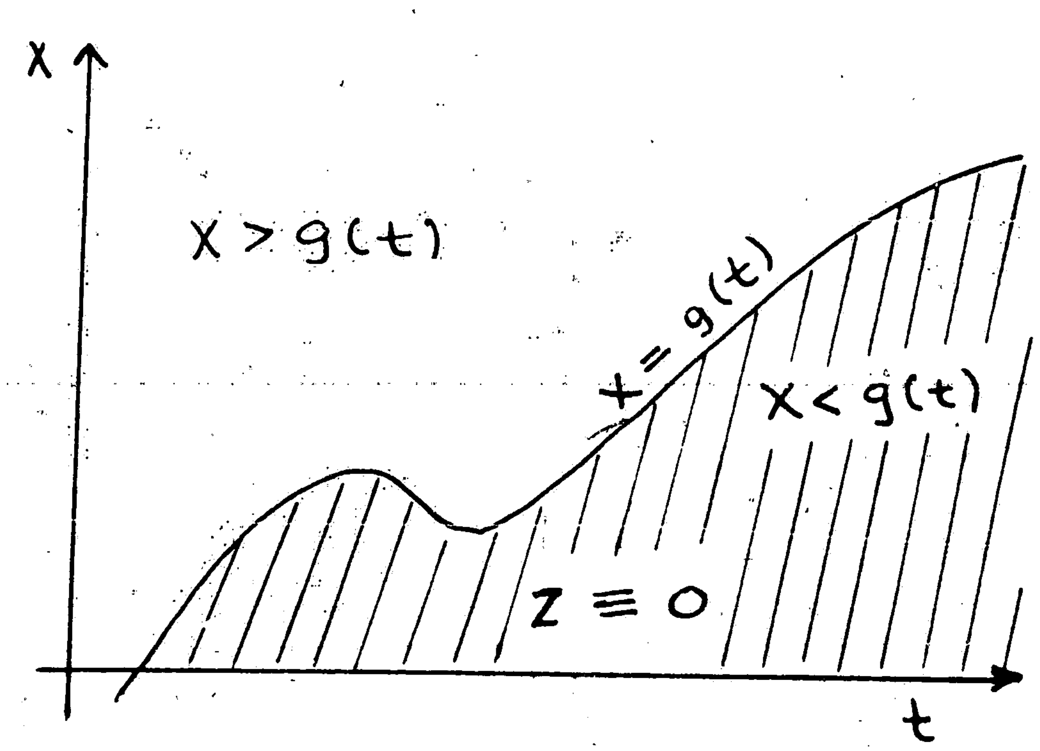
$$\frac{d\lambda}{d\eta} = \frac{d}{d\eta}\left[u_{\xi\xi}\right] = \left[u_{\xi\xi\eta}\right] \quad \text{or} \quad \lambda(\alpha_\xi + \delta) + 2\beta\frac{d\lambda}{d\eta} = 0$$

which can be written $\frac{d\lambda}{d\eta} = \lambda w(\eta)$. The general solution of this equation is given by:

$$\lambda(\eta) = \lambda(\eta_o)e^{\int_{\eta_o}^{\eta} w(\eta)d\eta}. \qquad (38)$$

Consequently, if on the curve $\xi(x,y) = \xi_o$ the function $\lambda(\eta_o) = 0$ for some η_o, we know $\lambda(\eta) \equiv 0$. ~~But~~ $\lambda(\eta) \equiv 0$ would imply that all the second derivatives of u were continuous at all points of the curve. Hence the discontinuity of the second derivative must exist at all points of the curve or at none.

2.4 As an example, suppose we are given the equation of a characteristic $x = g(t)$ where x(distance) and t(time) are the independent variables. Let a solution be given which is identically zero when $x < g(t)$.



If we suppose second order discontinuity of z on $g(t) = x$, then we can assume $u(x,t) \not\equiv 0$ for $x > g(t)$. Since a second order discontinuity at a single point on the curve implies second order discontinuities along the entire curve, we can think of the region $x > g(t)$ as being generated by a disturbence or wave $(u(x,t_o) \neq 0)$ propagated along $g(t) = x$ by the wave front $g(t_o) = x$.

2.5 The preceeding work on second order discontinuities can be extended in an analogous manner to higher order discontinuities.

2.6 Let us consider as an example the case of two dimensional fluid flow, where the velocity depends on x and y but not on the time t. Let u and v be the components of velocity along the x and y axes respectively and ρ the density of the fluid at point (x,y). Then we can express the pressure P in terms of density, the velocity of sound c, and a constant ρ_o by $P = c^2\rho + \rho_o$. Computing the acceleration at point (x,y) yields as x and y components:

$$\dot{u} = u_x\dot{x} + u_y\dot{y} = u_x u + u_y v$$
$$\dot{v} = v_x\dot{x} + v_y\dot{y} = v_x u + v_y v.$$

That is, the acceleration at a point (x,y) is given by

$$a = (u_x u + u_y v, v_x u + v_y v).$$

Differentiating the expression for pressure, we obtain

$$P_x = c^2\rho_x \quad \text{and} \quad P_y = c^2\rho_y.$$

We can express Newton's second law $F = ma$ in terms of its x and y components. The x-component of acceleration is $\dot{u}$ and of force per unit mass is $c^2\rho_x$. Hence, $\rho\frac{du}{dt} + c^2\rho_x = 0$. In the same manner we can consider the equation

$$\frac{d\rho}{dt} + \rho(u_x + v_y) = 0$$

Taking the x-component we have $\frac{\partial}{\partial x}(\rho u) = \rho u_x + \rho_x u = 0$ and the y-component $\frac{\partial}{\partial y}(\rho v) = \rho v_y + \rho_y v = 0$. Combining yields

$$\rho(u_x + v_y) + \rho_x u + \rho_y v = 0.$$

Thus we have the three equations:

$$\begin{aligned} \rho uu_x + \rho vu_y + c^2\rho_x &= 0 \\ \rho uv_x + \rho vv_y + c^2\rho_y &= 0 \\ \rho(u_x + v_y) + u\rho_x + v\rho_y &= 0, \end{aligned} \qquad (39)$$

from which we wish to determine u, v, ρ.

On changing independent variables from x, y to ξ, η these equations become

$$\begin{aligned} u_\xi(\rho u\xi_x + \rho v\xi_y) + \qquad\qquad\qquad\qquad + \rho_\xi c^2\xi_x &= \ldots \\ v_\xi(\rho u\xi_x + \rho v\xi_y) + \rho_\xi c^2\xi_y &= \ldots \\ u_\xi\rho\xi_x \qquad + v_\xi\rho\xi_y \qquad + \rho_\xi(u\xi_x + v\xi_y) &= \ldots \end{aligned}$$

The characteristic curves are those along which these equations do not determine u_ξ, v_ξ and ρ_ξ uniquely. This means that the determinent of the coefficients of u_ξ, v_ξ, ρ_ξ must equal zero. Thus

$$\begin{vmatrix} \rho(u\xi_x + v\xi_y) & 0 & c^2\xi_x \\ 0 & \rho(u\xi_x + v\xi_y) & c^2\xi_y \\ \rho\xi_x & \rho\xi_y & u\xi_x + v\xi_y \end{vmatrix} =$$

$$\rho^2(u\xi_x + v\xi_y)^3 - \rho^2c^2\xi_x^2(u\xi_x + v\xi_y) - \rho^2c^2\xi_y^2(u\xi_x + v\xi_y) =$$

$$\rho^2(u\xi_x + v\xi_y)\left[(u\xi_x + v\xi_y)^2 - c^2(\xi_x^2 + \xi_y^2)\right] = 0. \qquad (40)$$

Equation (40) will hold when either of the two factors is zero. If the first factor is zero, then $u\xi_x + v\xi_y = 0$ or $\frac{v}{u} = -\frac{\xi_x}{\xi_y}$.

This shows that the direction of the velocity specified by $\frac{v}{u}$ coincides with the tangent to $\xi(x,y) = \xi_0$ which shows that the family of characteristics has the same direction as the flow of the liquid.

If the second factor is zero, $(u\xi_x + v\xi_y)^2 - c^2(\xi_x^2 + \xi_y^2) = 0$, then

$$\xi_x^2(u^2 - c^2) + 2uv\xi_x\xi_y + \xi_y^2(v^2 - c^2) = 0.$$

If the discriminent of this equation is less than zero, we have what is called the elliptic case and there will be no real families of characteristics. On the other hand, if $4u^2v^2 - 4(u^2 - c^2)(v^2 - c^2) \geqq 0$, then $4c^2v^2 + 4u^2c^2 - 4c^4 = 4c^2(v^2 + u^2 - c^2) \geqq 0$, and $v^2 + u^2 \geqq c^2$. Hence, in the hyperbolic case the velocity of motion will exceed the speed of sound.

SECTION III

Problem of Cauchy for Hyperbolic Equations

3.1 We now wish to show under what conditions the general hyperbolic equation

$$u_{xy} = f(x,y,u,p,q) \tag{41}$$

will possess a unique solution. We shall assume that in a particular domain of the xy-plane and a domain of values of u,p,q, the function f is continuous with respect to x,y,u,p,q and has continuous first partial derivatives with respect to u,p,q, even though in our subsequent calculations it will be sufficient to assume f_u, f_p, f_q satisfy the Lipshitz condition. It will thus be sufficient for us to assume the difference quotient is bounded.

With this canonical form we know a = c = 0. Hence the defining equation for a characteristic curve,

$$a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 = 0, \text{ becomes}$$

$$\xi_x\xi_y = 0. \tag{42}$$

But this means that the characteristics are lines parallel to the x and y axes. Hence if we consider any plane curve C which meets no characteristic in more than one point, we can write the equation of C as $y = y(x)$, where $y(x)$ is monotonic. Let us assume that $y'(x)$ exists and is continuous for all x in question.

3.2 The appropriate initial condition consists of prescribed values of u, p and q at points of C. If x is used as parameter along C, these initial conditions are of the form,

$$u(x,y) = u(x,y(x)) = U_o(x),$$
$$p(x,y) = p(x,y(x)) = p_o(x), \qquad (43)$$
$$q(x,y) = q(x,y(x)) = q_o(x).$$

The condition that these initial values define a strip is

$$U_o' = p_o + q_o y'. \qquad (44)$$

If we assume that $p_o(x)$ and $q_o(x)$ are continuous with continuous derivatives, then we know by equation (44) that $U_o(x)$ is continuous and possesses continuous derivatives.

Further, it is possible to assign the initial values

$$U_o(x) \equiv p_o(x) \equiv q_o(x) \equiv 0 \qquad (45)$$

on the curve C without loss of generality. To justify this restriction, we shall show that by a change of dependent variable we are led to an equation of the same type as our original canonical hyperbolic equation, but with zero initial conditions. The change of variable is $u = v + \Phi(x,y)$, where

$$\Phi(x,y) = U_o(x) + (y - y(x))q_o(x). \tag{46}$$

Differentiating first with respect to x, we have

$$\begin{aligned}\Phi_x &= U_o'(x) - y'(x)q_o(x) + (y - y(x))\, q_o'(x)\\ &= p_o(x) + q_o(x)y'(x) - y'(x)q_o(x) + (y - y(x))q_o'(x)\\ &= p_o(x) + (y - y(x))q_o'(x).\end{aligned}$$

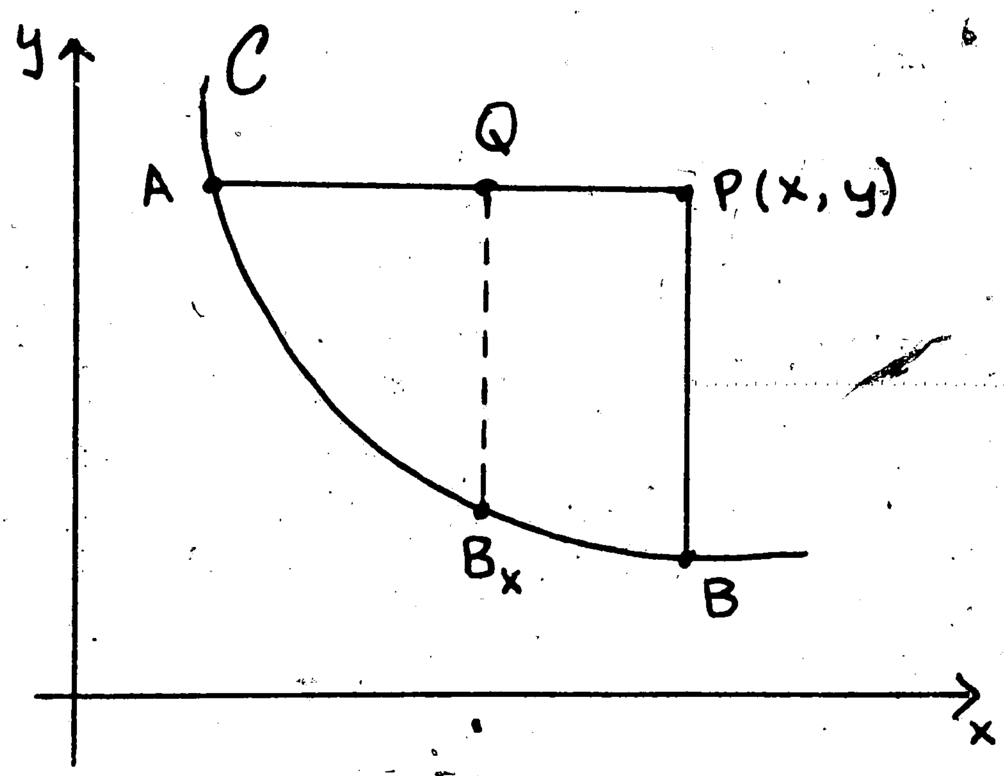
Differentiation with respect to y yields $\Phi_y = q_o(x)$. Hence, on the curve C, where $y = y(x)$,

$$\Phi(x,y) = U_o(x), \quad \Phi_x(x,y) = p_o(x), \quad \Phi_y(x,y) = q_o(x).$$

Hence, on the curve C, $v = 0$, $v_x = 0$, $v_y = 0$. Since $v_{xy} = u_{xy} - \Phi_{xy}$ we have $v_{xy} = f(x,y,u,p,q) - \Phi_{xy} = g(x,y,v,v_x,v_y)$. We know that the function g is continuous, so that the new equation is completely similar to equation (41). Thus we may without loss of generality make use of the above initial conditions in the following discussion.

3.3 We now construct an integral equation which we shall show is equivalent (41), but having the additional desirable feature of lending itself more readily to an existence discussion. In the case of the general hyperbolic equation we shall be able to find a unique solution valid for a finite region about the curve under consideration. Also, it will be shown that certain boundedness conditions on the function f, necessary in the general case, may be dispensed with for the linear hyperbolic equation and a unique solution will exist throughout a whole domain containing the curve.

Consider a curve C in the xy-plane and an arbitrary point $P(x,y)$ connected to C by lines parallel to the x and y axes. Let us integrate equation (41) with respect to y along the line BP and with respect to x along AP. Replace x and y by the variables of integration ξ and η respectively, and recall the initial conditions: $u = u_x = u_y = 0$, for all points on C. Then,



$$\int_A^P f(\xi,y,u,p,q)\,d\xi = u_y(P) - u_y(A) = u_y(P)$$

$$\int_B^P f(x,\eta,u,p,q)\,d\eta = u_x(P) - u_x(B) = u_x(P). \qquad (47)$$

To obtain an integral equation with unknown u, we first replace the upper limit P in the second integral above by a variable point $Q(\xi,y)$ on the line AP, and replace B by $B_x[\xi, f(\xi)]$. The resulting integral is then integrated with respect to ξ from A to P, that is from $\xi = a$ to $\xi = \xi_A$. This gives

$$\int_{\square} f(\xi,\eta,u,p,q)\,d\eta\,d\xi = \int_A^P\int_{B_x}^Q f(\xi,\eta,u,p,q)\,d\eta\,d\xi = \int_A^P \left(u_\xi\Big|_{B_x}^{Q}\right)d\xi =$$

$$\int_A^P \left[u_\xi(Q) - u_\xi(B_x)\right]d\xi = \int_A^P u_\xi\,d\xi = u(P) - u(A) = u(P).$$

Or,

$$\int_{\triangle} f(\xi,\eta,u,p,Q)\,d\eta\,d\xi = u(P) \tag{48}$$

We have shown that a necessary condition for u to be a solution of equation (41) satisfying $u_o(x) \equiv p_o(x) \equiv q_o(x) \equiv 0$ for all points on C, is that u be a solution of (48). To prove the sufficiency of this condition suppose $u(x,y)$ satisfies (48). We note first that $u(P) \equiv 0$ for any point P lying on the curve C, since the region of integration of the double integral degenerates to a point. Equations (47) then show that u_x and u_y vanish at points of C. Finally if we differentiate (48) with respect to x we obtain,

$$u_x(P) = \int_B^P f(x,\eta,u,p,q)\,d\eta = u_x(P) - u_x(B).$$

On differentiating this equation with respect to y we see that u must satisfy the original differential equation,

$$u_{xy} = f(x,y,u,p,q), \tag{41}$$

and hence sufficiency is proven.

3.4 We shall now proceed to construct a sequence of integrals which will converge uniformly to a limit which we shall prove to be the desired solution $u(x,y)$ of the initial value problem. This result will in general be applicable only in a restricted region of the xy-plane containing the curve C, since we will need to impose certain restrictions on the values of the function f.

We shall use Picard's method of successive approximations, an iterative method giving successive functions which in favorable cases tend (over a certain interval) toward the exact solution.

Let $u = 0$, $p = 0$, $q = 0$ be taken as the first approximation, and put

$$u_2(x,y) = \int_{\triangle} f(\xi,\eta,0,0,0)\,d\eta\,d\xi$$

$$p_2(x,y) = \frac{\partial}{\partial x}u_1(x,y) = \int_B^P f(x,\eta,0,0,0)\,d\eta \qquad (49)$$

$$q_2(x,y) = \frac{\partial}{\partial y}u_1(x,y) = \int_A^P f(\xi,y,0,0,0)\,d\xi\ .$$

Further, for 3,4,... define

$$u_n(x,y) = \int_{\triangle} f(\xi,\eta,u_{n-1},p_{n-1},q_{n-1})\,d\xi\,d\eta,$$

$$p_n(x,y) = \frac{\partial}{\partial x}u_n(x,y) = \int_B^P f(x,\eta,u_{n-1},p_{n-1},q_{n-1})\,d\eta, \qquad (50)$$

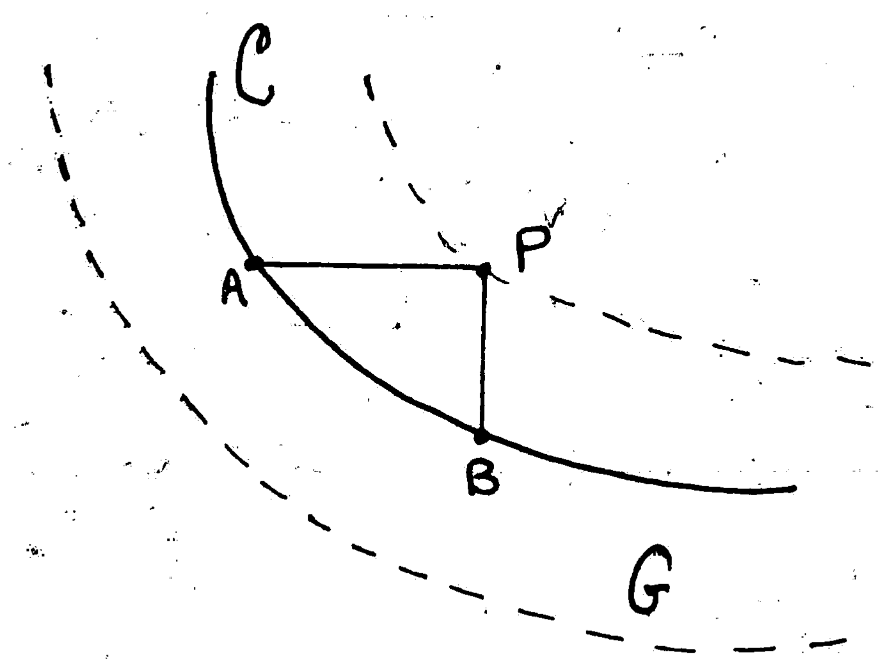
$$q_n(x,y) = \frac{\partial}{\partial y}u_n(x,y) = \int_A^P f(\xi,y,u_{n-1},p_{n-1},q_{n-1})\,d\xi.$$

First we must determine under what conditions these integrals will exist. We know that f is continuous with respect to x,y,u,p,q and further that f has continuous first partial derivatives with respect to u,p,q for a certain domain of these values. This implies that if we choose u,p,q within the region under consideration, that is described by $\max(|u|, |p|, |q|) < K$, then there will exist a positive number M such that f_u, f_p, f_q exist and $\max(|f|, |f_u|, |f_p|, |f_q|) < M$. This is true because a continuous function on a closed bounded set is itself bounded.

Let G be a region in the xy-plane containing the curve C and limited by the requirement that every length such as AP or BP from the curve to a point P of the region shall be less than a.

Let B be a region in $xyupq$-space in which (x,y) lies in G

and $|u|$, $|p|$, and $|q|$ are each less than K. Then if $u_{n-1}, p_{n-1}, q_{n-1}$ lie in B, equations (50) show that since $|f| < M$, $|u_n| < Ma^2$, $|p_n| < Ma$, $|q_n| < Ma$. Thus u_n, p_n, q_n will also lie in B provided we choose a such that $Ma^2 < K$ and $Ma < K$.



If we apply the law of the mean to the difference $u_{n+1}(P) - u_n(P)$ we have

$$u_{n+1}(P) - u_n(P) = \iint_{\triangle} [f(\xi,\eta,u_n,p_n,q_n) - f(\xi,\eta,u_{n-1},p_{n-1},q_{n-1})]\, d\xi d\eta$$

$$\leqq \iint_{\triangle} \{[u_n - u_{n-1}]\tilde{f}_u + [p_n - p_{n-1}]\tilde{f}_p + [q_n - q_{n-1}]\tilde{f}_q\}\, d\xi d\eta,$$

where $\tilde{f}_u$, $\tilde{f}_p$, $\tilde{f}_q$ denote values of the functions f_u, f_p, f_q from the intervals $[u_{n-1}, u_n]$, $[p_{n-1}, p_n]$, $[q_{n-1}, q_n]$ respectively. But from the preceding discussion we know that for all $P(x,y)$ in the region G, $\max(|u|, |p|, |q|) < K$, and hence $\max(|\tilde{f}_u|, |\tilde{f}_p|, |\tilde{f}_q|) < M$. Thus,

$$|u_{n+1}(P) - u_n(P)| < M \iint_{\triangle} (|u_n - u_{n-1}| + |p_n - p_{n-1}| + |q_n - q_{n-1}|)\, d\xi d\eta. \qquad (51.)$$

Next, investigating the difference $p_{n+1}(P) - p_n(P)$,

$$p_{n+1}(P) - p_n(P) = \int_B^P [f(x,\eta,u_n,p_n,q_n) - f(x,\eta,u_{n-1},p_{n-1},q_{n-1})]\, d\eta$$

$$\leqq \int_B^P \{[u_n - u_{n-1}]\tilde{f}_u + [p_n - p_{n-1}]\tilde{f}_p + [q_n - q_{n-1}]\tilde{f}_q\}\, d\eta$$

where $\tilde{f}_u$, $\tilde{f}_p$, $\tilde{f}_q$ are as above. Thus again

$$|p_{n+1}(P) - p_n(P)| < M\int_B^P (|u_n - u_{n-1}| + |p_n - p_{n-1}| + |q_n - q_{n-1}|)d\eta.$$

Similarly,

$$|q_{n+1}(P) - q_n(P)| < M\int_A^P (|u_n - u_{n-1}| + |p_n - p_{n-1}| + |q_n - q_{n-1}|)d\xi.$$

Let

$$S_n = \sup(|u_n - u_{n-1}| + |p_n - p_{n-1}| + |q_n - q_{n-1}|) \text{ for } n=1,2,\ldots \tag{52}$$

Then

$$|u_{n+1}(P) - u_n(P)| < M\int_{\square}(|u_n - u_{n-1}| + |p_n - p_{n-1}| + |q_n - q_{n-1}|)d\xi d\eta$$
$$< Ma^2 S_n.$$

Similarly,

$$|p_{n+1}(P) - p_n(P)| < MaS_n, \text{ and}$$

$$|q_{n+1}(P) - q_n(P)| < MaS_n.$$

Hence,

$$|u_{n+1}(P) - u_n(P)| + |p_{n+1}(P) - p_n(P)| + |q_{n+1}(P) - q_n(P)|$$
$$< Ma^2 S_n + 2MaS_n = MaS_n(a + 2), \tag{53}$$

an expression which is true for all $P(x,y)$ in the region G around C. But $S_{n+1} = \sup_n(|u_{n+1} - u_n| + |p_{n+1} - p_n| + |q_{n+1} - q_n|)$, hence $S_{n+1} \leqq MaS_n(a + 2)$. Let us choose a so small that $0 < Ma(a+2) = \alpha < 1$. Then $S_{n+1} \leqq S_n \cdot \alpha < S_n$.

Consider the series,

$$u_o + (u_1 - u_o) + (u_2 - u_1) + \ldots + (u_n - u_{n-1}) + \ldots$$

whose sum to n terms is u_{n-1}. For all x,y in the domain considered this is dominated by the convergent geometric series

$$S_1(1 + \alpha + \alpha^2 + \ldots) > S_1 + S_2 + S_3 + \ldots$$

Thus we can set

$$u(x,y) = u_o + (u_1 - u_o) + \ldots + (u_n - u_{n-1}) + \ldots = \lim_{n\to\infty} u_n(x,y) \tag{54}$$

where u(x,y) is the limit of a uniformly convergent sequence of continuous functions and hence is continuous. Analogously, if we consider $p_o + (p_1 - p_o) + \ldots$ and $q_o + (q_1 - q_o) + \ldots$ we know

$$p_o + (p_1 - p_o) + \ldots \leqq S_1 + S_2 + \ldots < S_1(1 + \alpha + \alpha^2 + \ldots)$$

yielding $p(x,y) = \lim_{n\to\infty} p_n(x,y)$, a continuous function; and similarly, $q(x,y) = \lim_{n\to\infty} q_n(x,y)$ also a continuous function. Hence,

$$\begin{aligned} u(x,y) &= \lim_{n\to\infty} u_n(x,y) = \iint f(\xi,\eta,u,p,q)\,d\xi\,d\eta \\ p(x,y) &= \lim_{n\to\infty} p_n(x,y) = \int_B^P f(x,\eta,u,p,q)\,d\eta \\ q(x,y) &= \lim_{n\to\infty} q_n(x,y) = \int_A^P f(\xi,y,u,p,q)\,d\xi \end{aligned} \tag{55}$$

Differentiation yields

$$u_x(x,y) = \int_B^P f(x,\eta,u,p,q)\,d\eta = p(x,y)$$

$$u_y(x,y) = \int_A^P f(\xi,y,u,p,q)\,d\xi = q(x,y).$$

Thus $u(x,y)$ is a solution to the Cauchy Problem in the region G around the curve C.

To complete the argument, it remains to show that $u(x,y)$ is a unique solution. Suppose there exist solutions $u^{(1)}$, $u^{(2)}$ of equation (48), such that

$$u^{(1)}(x,y) = \iint f(\xi,\eta,u^{(1)},p^{(1)},q^{(1)})\,d\xi d\eta \quad \text{and}$$

$$u^{(2)}(x,y) = \iint f(\xi,\eta,u^{(2)},p^{(2)},q^{(2)})\,d\xi d\eta.$$

We set $w(x,y) = u^{(1)} - u^{(2)}$ and proceed to show that w must be identically zero.

By definition we know,

$$w(x,y) = \iint \left[f(\xi,\eta,u^{(1)},p^{(1)},q^{(1)}) - f(\xi,\eta,u^{(2)},p^{(2)},q^{(2)})\right] d\xi d\eta$$

$$w_x(x,y) = \int_B^P \left[f(x,\eta,u^{(1)},p^{(1)},q^{(1)}) - f(x,\eta,u^{(2)},p^{(2)},q^{(2)})\right] d\eta$$

$$w_y(x,y) = \int_A^P \left[f(\xi,y,u^{(1)},p^{(1)},q^{(1)}) - f(\xi,y,u^{(2)},p^{(2)},q^{(2)})\right] d\xi$$

Applying the law of the mean and setting,

$$W = \sup(|w| + |w_x| + |w_y|)$$

we obtain

$$|w(x,y)| < \iint \left[(|u^{(1)} - u^{(2)}|\cdot\tilde{f}_u + |p^{(1)} - p^{(2)}|\cdot\tilde{f}_p + |q^{(1)} - q^{(2)}|\cdot\tilde{f}_q\right] d\xi d\eta$$
$$< WMa^2.$$

Analogously, $|w_x| < WMa$ and $|w_y| < WMa$.

It is then true that $W = \sup(|w| + |w_x| + |w_y|) \leqq WMa(a + 2) = W\alpha$ and since $\alpha < 1$ this implies $W = 0$ and $u^{(1)} \equiv u^{(2)}$.

3.5 As was previously mentioned, the linear hyperbolic equation yields more general results since the restrictions limiting the size of the domain integrated over may be omitted. This is true since if we consider

$$u_{xy} = \alpha u_x + \beta u_y + \gamma u + \delta \tag{56}$$

where $\alpha, \beta, \gamma, \delta$ are continuous functions of x and y, we may integrate the right member over an entire domain containing C and need not make the restrictions $Ma < K$, $Ma^2 < K$ which limit the discussion to a small strip containing C. Following the preceding development, we set

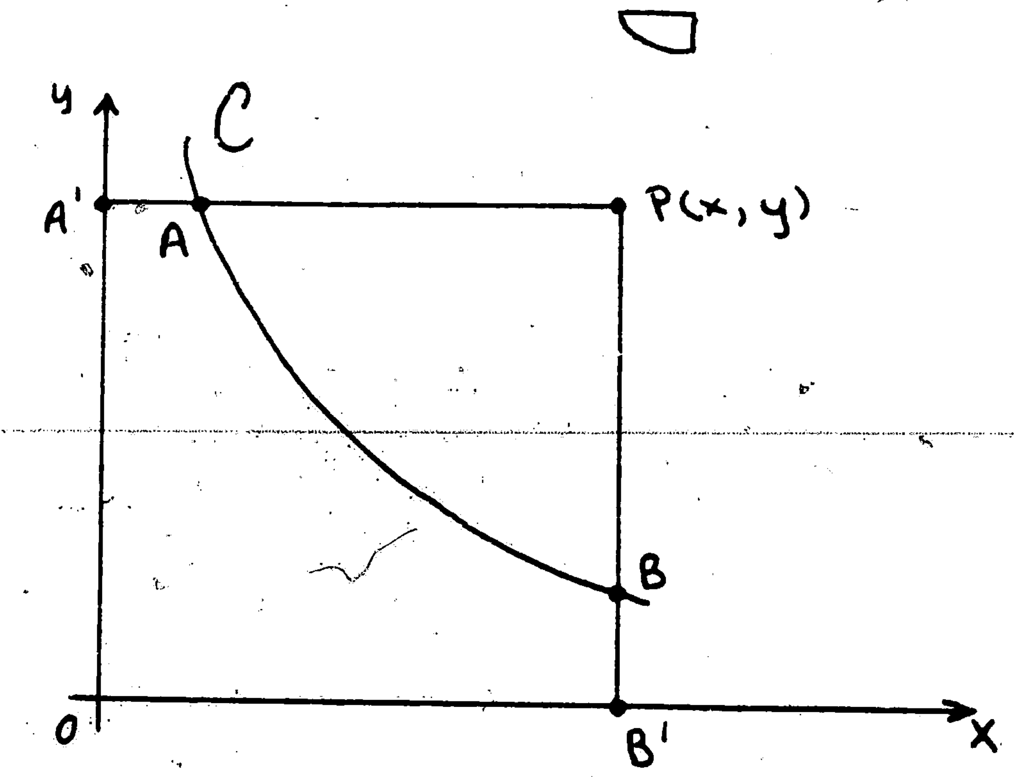
$$\begin{aligned}
u_n(P) &= \iint_{\triangledown} [\alpha p_{n-1} + \beta q_{n-1} + \gamma u_{n-1}]\, d\xi d\eta, \\
p_n(P) &= \int_B^P [\alpha p_{n-1} + \beta q_{n-1} + \gamma u_{n-1}]\, d\eta, \\
q_n(P) &= \int_A^B [\alpha p_{n-1} + \beta q_{n-1} + \gamma u_{n-1}]\, d\xi, \quad n = 1, 2, \ldots
\end{aligned} \tag{57}$$

Assuming we choose u_0 in such a fashion that

$$\sup \max(|u_0(P)|,\ |p_0(P)|,\ |q_0(P)|) < L,$$

then when $n=1$, and $\max[|\alpha|, |\beta|, |\gamma|] < M$ we find

$$u_1(P) = \int_{\triangledown} (\alpha p_0 + \beta q_0 + \gamma u_0)\, d\xi d\eta \text{ and } |u_1(P)| \leqq 3ML \int_{\triangledown} d\xi d\eta.$$



If we assume the domain under consideration about C lies in the first quadrant, as in the diagram, we shall not be introducing a new restriction since by translation of axes we can always reduce the problem to this case. Then $|u_1(P)| \leqq 3ML \int\int d\xi d\eta \leqq 3ML \int\int d\xi d\eta = 3MLxy < 3ML\frac{(x+y)^2}{2}$. Similarly, $p_1(P) = \int_B^P (\alpha p_o + \beta q_o + \gamma u_o) d\eta$ and

$|p_1(P)| \leqq 3ML \int_B^P d\eta \leqq 3ML \int_{B'}^P d\eta = 3MLy \leqq 3ML(x + y)$. Also,

$q_1(P) = \int_A^P (\alpha p_o + \beta q_o + \gamma u_o) d\xi$ and $|q_1(P)| \leqq 3ML(x + y)$.

That is,

$$|u_1(P)| < 3ML\frac{(x+y)^2}{2}, \quad |p_1(P)| < 3ML(x+y), \quad |q_1(P)| < 3ML(x+y). \tag{58}$$

Let $H = \sup(2 + \frac{x+y}{n+1})$, $K = MH$. Then we wish to prove by induction the inequalities,

$$|u_n(P)| \leqq 3MLK^{n-1}\frac{(x+y)^{n+1}}{(n+1)!}, \quad \left.\begin{matrix}|p_n(P)| \\ |q_n(P)|\end{matrix}\right\} = 3MLK^{n-1}\frac{(x+y)^n}{n!}. \tag{59}$$

We note that the above estimates for $|u_1(P)|$, $|q_1(P)|$, $|p_1(P)|$ satisfy these inequalities for $n = 1$. Assume that

$|u_{n-1}(P)| \leqq 3MLK^{n-2}\frac{(x+y)^n}{n!}$, $|p_{n-1}(P)| \leqq 3MLK^{n-2}\frac{(x+y)^{n-1}}{(n-1)!}$,

and $|q_{n-1}(P)| \leqq 3MLK^{n-2}\frac{(x+y)^{n-1}}{(n-1)!}$ hold. Then

$$\begin{aligned}|u_n(P)| &\leqq \int\int \left[\alpha|p_{n-1}(P)| + \beta|q_{n-1}(P)| + \gamma|u_{n-1}(P)|\right] d\xi d\eta \\ &\leqq M\cdot 3MLK^{n-2}\int\int\left[\frac{(\xi+\eta)^n}{n!} + 2\frac{(\xi+\eta)^{n-1}}{(n-1)!}\right]d\xi d\eta \\ &\leqq M\cdot 3MLK^{n-2}\int\int\left[\frac{(\xi+\eta)^n}{n!} + 2\frac{(\xi+\eta)^{n-1}}{(n-1)!}\right]d\xi d\eta \\ &\leqq MH\cdot 3MLK^{n-2}\frac{(x+y)^{n+1}}{(n+1)!}\end{aligned}$$

which was to be shown.

Similarly,

$$|p_n(P)| \leqq \int_B^P (\alpha|p_{n-1}(P)| + \beta|q_{n-1}(P)| + \gamma|u_{n-1}(P)|)d\eta$$

$$\leqq M\cdot 3MLK^{n-2}\int_B^P \left[\frac{(x+\eta)^n}{n!} + 2\frac{(x+\eta)^{n-1}}{(n-1)!}\right]d\eta$$

$$\leqq MH\cdot 3MLK^{n-2}\ \frac{(x+y)^n}{n!}.$$

Hence,

$$|p_n(P)| \leqq 3MLK^{n-1}\ \frac{(x+y)^n}{n!}.$$

In exactly the same manner,

$$|q_n(P)| \leqq 3MLK^{n-1}\ \frac{(x+y)^n}{n!}.$$

We have therefore verified the inequalities given in equation (59). Then

$$\sum_{n=0}^{\infty} u_n(x,y) \leqq u_o(x,y) + 3MLK^{-2}\sum_{n=0}^{\infty}\frac{[K(x+y)]^{n+1}}{(n+1)!}$$

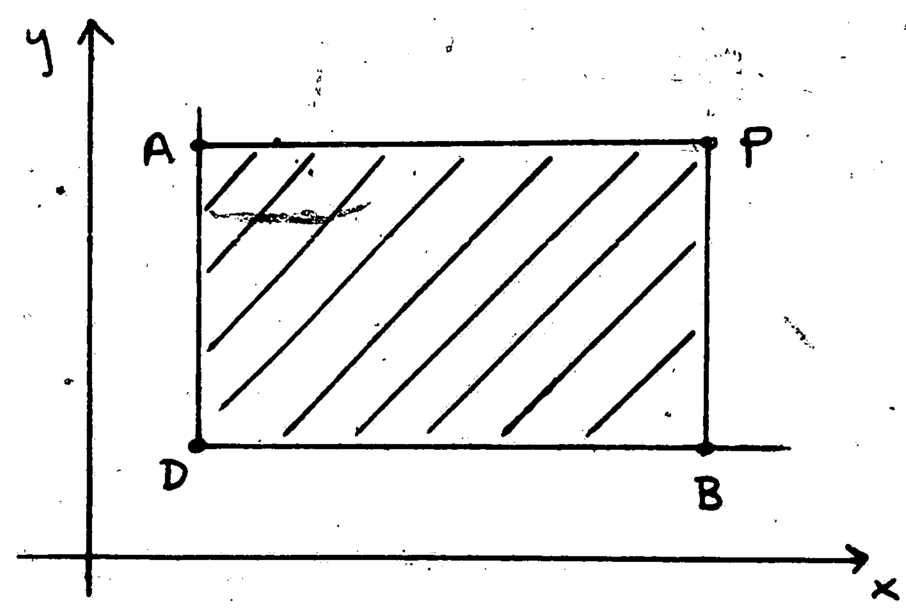
$$= u_o(x,y) + 3MLK^{-2}\left[e^{K(x+y)} - 1 - K(x+y)\right]$$

where the right member is an exponential series. Hence $\sum_{n=0}^{\infty} u_n(x,y)$ converges to a limit, $u(x,y)$, which is continuous along with its first partial derivatives in the entire domain containing C. Similarly,

$$\left.\begin{matrix}\sum_{n=0}^{\infty} p_n(P) \\ \sum_{n=0}^{\infty} q_n(P)\end{matrix}\right\} \leqq L + 3MLK^{-1}\sum_{n=1}^{\infty}\frac{[K(x+y)]^n}{n!} = L + 3MLK^{-1}\left[e^{K(x+y)} - 1\right],$$

and thus $p(x,y)$ and $q(x,y)$ exist and are continuous in the domain under consideration. Uniqueness of the solution can be shown in the same manner as in the preceding development.

3.6 Instead of giving boundary values for u, u_x, and u_y along the curve C, we can arrive at a unique solution to the linear hyperbolic equation by specifying the values of $u(x,y)$ along any two characteristics which intersect. All the previous development will still follow if we replace the region APB by the rectangle bounded by the two characteristics and by two lines through P perpendicular to these. This is the region PADB in the figure. By changing variables from x,y to $\hat{x},\hat{y}$, where $\hat{x} = x - x_D$, $\hat{y} = y - y_D$, we obtain a problem in which the positive x and y axes constitute the initial curve.



Finally, if we define

$$\Phi(x,y) = u(x,0) + u(0,y) - u(0,0), \quad v(x,y) = u(x,y) - \Phi(x,y), \tag{60}$$

then as before we can simply consider the case when $u(x,y) \equiv 0$ on the two positive axes.

3.7 At this point we would like to be able to express the unique solution to the Cauchy problem in terms of its coefficients. This will be possible for the linear hyperbolic equation if we introduce the method of Riemann operators. Consider

$$u_{xy} + \alpha u_x + \beta u_y + \gamma u + \delta = 0 . \tag{61}$$

Let $v(x,y)$ be an arbitrary function of x and y and set

$$L(u) = u_{xy} + \alpha u_x + \beta u_y + \gamma u = f_1(x,y) . \tag{62}$$

Define the adjoint operator M by

$$M(v) = v_{xy} - (v\alpha)_x - (v\beta)_y + \gamma v = v_{xy} - v\alpha_x - v_x\alpha - v\beta_y$$
$$-v_y\beta + \gamma v = v_{xy} - \alpha v_x - \beta v_y + v(\gamma - \alpha_x - \beta_y) . \tag{63}$$

Then by using the identities

$$vu_{xy} = v_y u_x + vu_{xy} - v_y u_x - v_{xy} u + uv_{xy} = (vu_x)_y - (v_y u)_x + uv_{xy}$$

$$v\alpha u_x = v_x \alpha u + v\alpha_x u + v\alpha u_x - uv_x \alpha - uv\alpha_x = (v\alpha u)_x - u(v\alpha)_x \qquad (64)$$

$$v\beta u_y = v_y \beta u + v\beta_y u + v\beta u_y - uv_y \beta + uv\beta_y = (v\beta u)_y - u(v\beta)_y$$

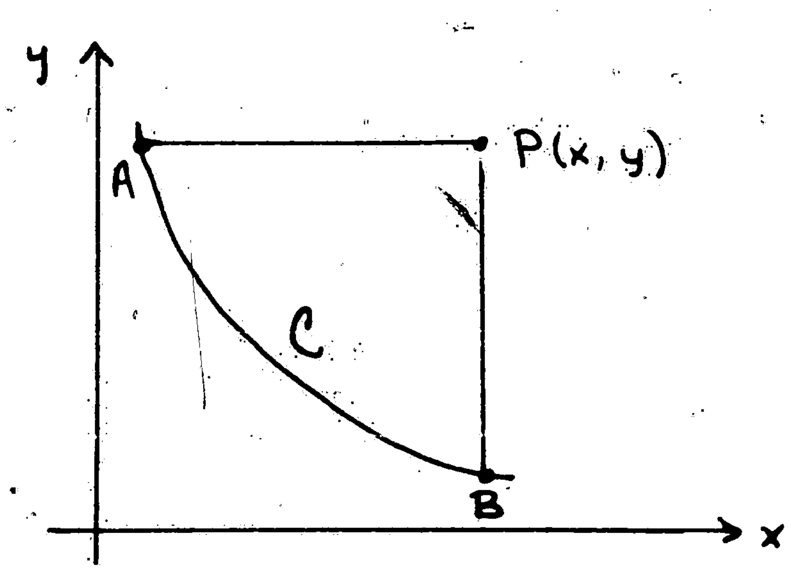
we obtain

$$vL(u) - uM(v) = (-uv_y + uv\alpha)_x + (vu_x + uv\beta)_y \equiv U_x + V_y, \qquad (65)$$

where

$$U = -uv_y + uv\alpha, \quad V = vu_x + uv\beta. \qquad (66)$$

Suppose we are given values of u, u_x, u_y along C, where P is arbitrary, and the straight lines PA and PB are characteristics. Then by Green's Theorem



$$\iint [vL(u) - uM(v)]\, d\xi d\eta = \int U d\eta - V d\xi \qquad (67)$$

and from equation (66),

$$\iint [vL(u) - uM(v)]\, d\xi d\eta = \int_A^B [-uv_\eta + uv\alpha]\, d\eta - [vu_\xi + uv\beta]\, d\xi +$$

$$\int_B^P [-uv_\eta + uv\alpha]\, d\eta - [vu_\xi + uv\beta]\, d\xi + \int_P^A [-uv_\eta + uv\alpha]\, d\eta - [vu_\xi + uv\beta]\, d\xi$$

$$= \int_A^B [u(-v_\eta + v\alpha) d\eta - v(u_\xi + u\beta) d\xi] + \int_B^P u(-v_\eta + v\alpha) d\eta - \int_P^A v(u_\xi + u\beta) d\xi.$$

However,

$$\int_P^A vu_\xi d\xi = uv \Big|_P^A - \int_P^A uv_\xi d\xi = u(A)v(A) - u(P)v(P) - \int_P^A uv_\xi d\xi .$$

Then

$$\iint_{\triangle} \left[vL(u) - uM(v)\right] d\xi d\eta = \int_A^B \left[u(-v_\eta + v\alpha)d\eta - v(u_\xi + u\beta)d\xi\right] \tag{68}$$

$$+ \int_B^P u(-v_\eta + v\alpha)d\eta - \int_P^A u(-v_\xi + v\beta)d\xi + u(P)v(P) - u(A)v(A).$$

Let us choose v as a function of ξ and η depending on the parameters x and y, expressed as $v = v(x,y;\xi,\eta)$, so that v satisfies the following three relations:

$$M(v) = 0, \quad \frac{\partial v(x,y;x,\eta)}{\partial \eta} = \alpha(x,\eta)\, v(x,y;x,\eta),$$

$$\frac{\partial v(x,y;\xi,y)}{\partial \xi} = \beta(\xi,y)\, v(x,y;\xi,y). \tag{69}$$

If we integrate the second equation in (69) we obtain,

$$\int_{\eta_o}^{\eta} \frac{\frac{\partial v(x,y;x,\eta)}{\partial \eta}}{v(x,y;x,\eta)}\, d\eta = \int_{\eta_o}^{\eta} \alpha(x,\eta)d\eta.$$

Hence $v(x,y;x,\eta) = v(x,y;x,\eta_o)\, e^{\int_{\eta_o}^{\eta} \alpha(x,\eta)d\eta}$.

Similarly the third equation yields:

$$v(x,y;\xi,y) = v(x,y;\xi_o,y)\, e^{\int_{\xi_o}^{\xi} (\xi,y)d\xi}.$$

Assume further that $v(x,y;x,y) = 1$, and substitute $x = \xi_o$ and $y = \eta_o$. Then we obtain the following set of conditions necessarily satisfied by v:

$$M(v) = 0, \quad v(x,y;x,\eta) = e^{\int_y^{\eta} \alpha(x,\eta)d\eta}, \quad v(x,y;\xi,y) = e^{\int_x^{\xi} \beta(\xi,y)d\xi}. \tag{70}$$

Notice that (70) consists of a second order differential equation in v with boundary conditions which v must satisfy along two characteristics AP and BP meeting at a point P. We have seen that such a problem has a unique solution valid for points in a domain containing C. Hence there will exist a function satisfying equation (70) say

$$R_L(x,y;\xi,\eta) = v(x,y;\xi,\eta). \tag{71}$$

We shall call $R_L(x,y;\xi,\eta)$ the Riemann Function of the operator L. Equation (68) then becomes

$$\iint_{\triangle} [vL(u) - uM(v)]\,d\xi d\eta = \int_A^B \left[u(-v_\eta + \alpha v)d\eta - (vu_\xi + uv\beta)d\xi\right] + u(P)v(P) - u(A)v(A).$$

Hence,

$$u(P) = u(P)v(P) = u(A)v(A) + \int_A^B \left[u(v_\eta - \alpha v)d\eta + v(u_\xi + u\beta)d\xi + \iint_{\triangle} f(\xi,\eta)v\,d\xi d\eta. \tag{72}$$

Note that $u(B)v(B) - u(A)v(A) = \int_A^B d(uv) = \int_A^B \left[\frac{\partial(uv)}{\partial\xi}\,d\xi + \frac{\partial(uv)}{\partial\eta}\,d\eta\right]$

$$= \int_A^B \left[uv_\xi + u_\xi v\right]d\xi + \left[uv_\eta + u_\eta v\right]d\eta.$$

Hence equation (72) becomes

$$u(P) = u(B)v(B) - \int_A^B \left[(uv_\xi - uv\beta)d\xi + (vu_\eta + uv\alpha)d\eta\right] + \iint_{\triangle} vf(\xi,\eta)\,d\xi d\eta. \tag{73}$$

Combining equations (72) and (73) yields

$$2u(P) = u(B)v(B) + u(A)v(A) + \int_A^B \left[(vu_\xi + vu\beta - uv_\xi + uv\beta)d\xi + (uv_\eta - uv\alpha - vu_\eta - uv\alpha)d\eta\right] + 2\iint_{\triangle} vf(\xi,\eta)\,d\xi d\eta.$$ Then

$$u(P) = \frac{u(A)R_L(A) + u(B)R_L(B)}{2} + \int_A^B \Big[-uR_L(\alpha d\eta - \beta d\xi) + ½R_L(u_\xi d\xi - u_\eta d\eta)$$

$$-½u(R_{L\xi}d\xi - R_{L\eta}d\eta) + \iint_{\triangle} R_L(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta. \qquad (74)$$

$u(P) = u(x,y)$ can then be computed since the right member depends on R_L and known functions. Integrating over the area ADBP, we have

$$\iint_{\square}\left[vL(u) - uM(v)\right]d\xi d\eta = \int_{AD+BP} u(-v_\eta + \alpha v)d\eta - \int_{DB+PA} v(u_\xi + \beta u)d\xi. \quad (75)$$

Taking into account the identities

$$\int_P^A vu_\xi d\xi = u(P)v(P) - u(A)v(A) - \int_P^A uv_\xi d\xi,$$

$$\int_A^D uv_\eta d\eta = u(D)v(D) - u(A)v(A) - \int_A^D vu_\eta d\eta$$ and recalling $v(D) = 1$,

we have

$$u(P) = u(D)v(D) + \int_D^A v(u_\eta + \alpha u)d\eta + \int_D^B v(u_\xi + \beta u)d\xi +$$

$$\iint_{\square} vf(\xi,\eta)d\xi d\eta. \qquad (76)$$

Choose $u(\xi,\eta) = R_M(x_o,y_o;\xi,\eta)$ where, analogously, R_M is the Riemann function of the operator M. Hence, $L(u) = 0$, $u_\eta - \alpha u = 0$, $u_\xi - \beta u = 0$, $u(D) = 1$. Applying these conditions to equation (76) we obtain

$$u(P) = R_M(x_o,y_o;x,y) = R_L(x,y;x_o,y_o) \qquad (77)$$

showing the reciprocity between R_L and R_M.

3.8 Now let us turn our attention to the case when the linear hyperbolic equation is expressed in terms of a parameter τ, that is,

$$u_{xy} = f(x,y,u,p,q,\tau). \tag{78}$$

Let us assume f is continuous with respect to all variables and has continuous first partial derivatives with respect to u, p, q in a certain domain of the xy-plane, for a particular domain of values of u, p, q and for all τ within a certain interval. Suppose further that f, f_u, f_p, f_q are continuous with respect to τ over the interval in question and that τ_0 is a point in that interval. It is possible to show that under these conditions a solution of equation (78) exists and is continuous with respect to τ for all τ in the interval. In addition we will show that there exists a function w defined in terms of a solution to equation (78) and satisfying the same conditions as a solution $u(x,y)$ to equation (41), that is, w, w_x, w_y will be identically zero on the curve C and w will satisfy a linear second order partial differential equation written in terms of w, w_x, w_y.

To justify the statement that continuity of f implies that there exists a solution to equation (78) continuous in τ, we note that for a sufficiently small interval containing τ_0, the integrals defining u_n, p_n, q_n in equations (50) will be again well-defined and hence the series in equation (54) will converge uniformly to a limit which we shall call $u(x,y,\tau)$ the solution in question. Similarly $p(x,y,\tau)$ and $q(x,y,\tau)$ exist. Hence if f is a continuous function of τ, so is $u(x,y,\tau)$. It is also true that the existence and continuity of f_τ implies the existence and continuity of u_τ.

To show this, set

$$u(x,y,\tau_o) = u_o, \quad p(x,y,\tau_o) = p_o, \quad q(x,y,\tau_o) = q_o. \tag{79}$$

If $u(x,y,\tau)$ is a solution to equation (78), then

$\dfrac{\partial^2 u(x,y,\tau)}{\partial x \partial y} = f(x,y,p(x,y,\tau),q(x,y,\tau),u(x,y,\tau),\tau)$ and when $\tau = \tau_o$,

$$\frac{\partial^2 u(x,y,\tau_o)}{\partial x \partial y} = f(x,y,p(x,y,\tau_o),q(x,y,\tau_o),u(x,y,\tau_o),\tau_o)$$

$$= f(x,y,p_o,q_o,u_o,\tau_o). \tag{80}$$

We have proved previously that arbitrary boundary conditions may be imposed on the values of u, p, q on the curve C. Thus let $p \equiv q \equiv u \equiv 0$ on C.

$$\text{Define } w(x,y,\tau) = \frac{u(x,y,\tau) - u(x,y,\tau_o)}{\tau - \tau_o}. \tag{81}$$

Differentiating,

$$\begin{aligned} w_x(x,y,\tau) &= \frac{u_x(x,y,\tau) - u_x(x,y,\tau_o)}{\tau - \tau_o} = \frac{p - p_o}{\tau - \tau_o} \\ w_y(x,y,\tau) &= \frac{u_y(x,y,\tau) - u_y(x,y,\tau_o)}{\tau - \tau_o} = \frac{q - q_o}{\tau - \tau_o} \end{aligned} \tag{82}$$

and differentiating once more, we obtain,

$$w_{xy}(x,y,\tau) = \frac{u_{xy}(x,y,\tau) - u_{xy}(x,y,\tau_o)}{\tau - \tau_o}$$

$$= \frac{f(x,y,p,q,u,\tau) - f(x,y,p_o,q_o,u_o,\tau_o)}{\tau - \tau_o}.$$

Adding and subtracting suitable terms and substituting the relations given in equation (82) yields

$$w_{xy}(x,y,\tau) = w_x \frac{f(x,y,p,q,u,\tau) - f(x,y,p_o,q,u,\tau)}{p - p_o} + w_y \frac{f(x,y,p_o,q,u,\tau) - f(x,y,p_o,q_o,u,\tau)}{q - q_o} + w \frac{f(x,y,p_o,q_o,u,\tau) - f(x,y,p_o,q_o,u_o,\tau)}{u - u_o} + \frac{f(x,y,p_o,q_o,u_o,\tau) - f(x,y,p_o,q_o,u_o,\tau_o)}{\tau - \tau_o}. \quad (83)$$

Naturally, if $p = p_o$ we define the coefficient of w_x to be $f(x,y,p_o,q,u,\tau)$ and similarly for the cases when $q = q_o$ and $u = u_o$. Hence, when $\tau \neq \tau_o$ equation (83) is a linear equation in w with continuous coefficients. Notice that on C, $u = p = q = 0$ for all τ and thus by definition $w(x,y,\tau) = 0$. It follows that on C, $w_x = w_y = 0$. We know $w(x,y,\tau_o)$ exists since in this case we allow $\tau \to \tau_o$ and the coefficients in equation (83) will take on their respective limits. That is,

$$w_{xy}(x,y,\tau) = w_x(x,y,\tau_o) f_p(x,y,p_o,q_o,u_o,\tau_o) + w_y(x,y,\tau_o) f_q(x,y,p_o,q_o,u_o,\tau_o) + w(x,y,\tau_o) f_u(x,y,p_o,q_o,u_o,\tau_o) + f_\tau(x,y,p_o,q_o,u_o,\tau_o).$$

Note also that

$$w(x,y,\tau_o) = \lim_{\tau \to \tau_o} \frac{u(x,y,\tau) - u(x,y,\tau_o)}{\tau - \tau_o} = (u_\tau)_{\tau = \tau_o}.$$

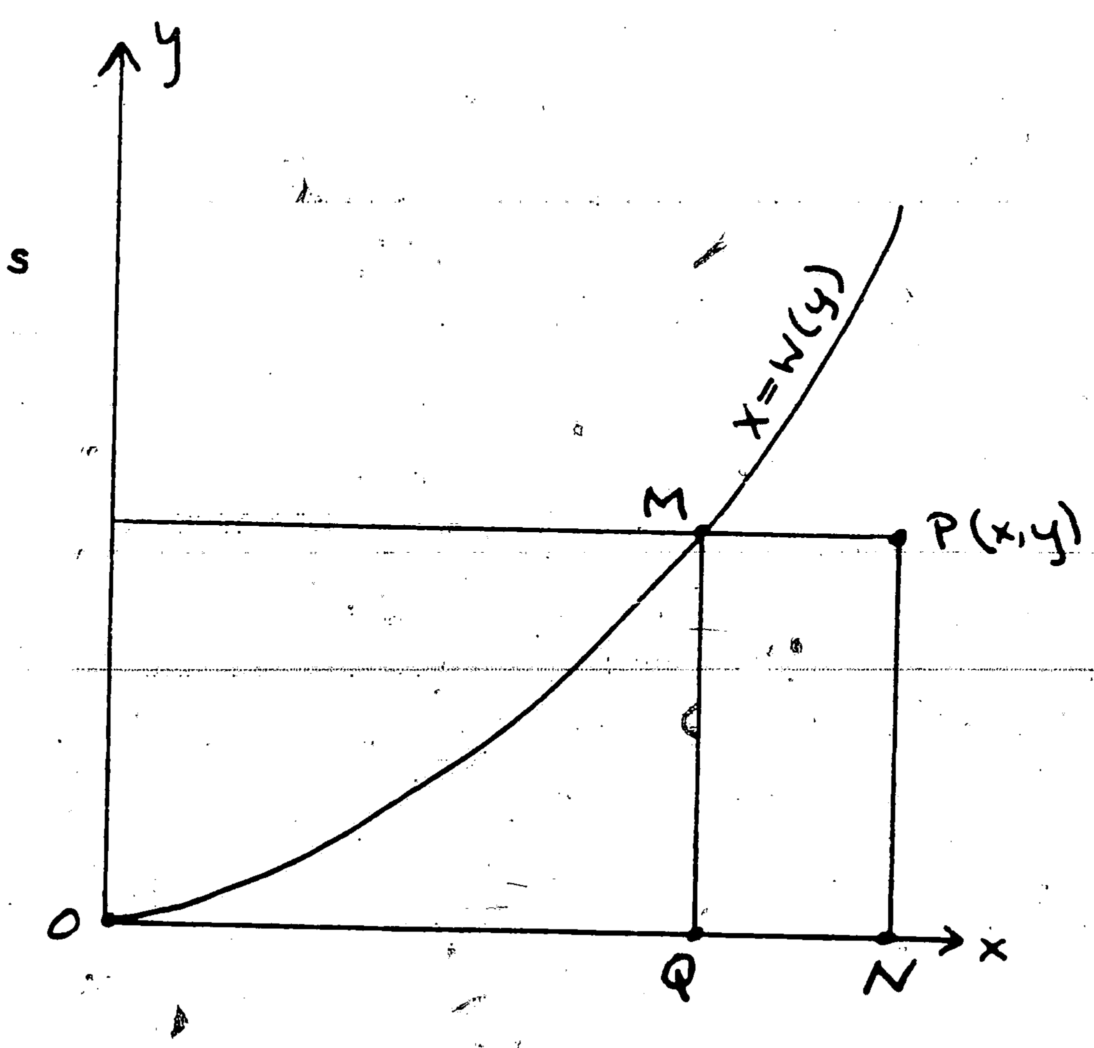
3.9 Earlier we determined that in our discussion of general hyperbolic equations we would allow the curve C to meet any characteristic curve in at most one point. To show that this is a necessary condition, let us consider a curve C meeting a characteristic $y = a$ in two points $P(x_1, a)$ and $Q(x_2, a)$. Let the hyperbolic equation in question be $u_{xy} = 0$. Then

$$\int_P^Q u_{\xi y} d\xi = u_y \Big|_P^Q = u_y(Q) - u_y(P) = 0 ,$$

which implies $u_y(Q) = u_y(P)$ and thus we can not arbitrarily specify the value of u_y along the curve C, a procedure we have relied heavily upon in the preceding discussions. Hence the requirement that C meet any characteristic in no more than one point must follow.

3.10 As we mentioned previously, the requirement that values of u, u_x, u_y be specified along C may be replaced by the condition that values of u are given along segments of two characteristics having a common endpoint, where u is continuous and possesses continuous derivatives along each characteristic. We can show that the latter condition may be generalized so that when values of u are given along a characteristic and a particular curve which crosses no characteristic twice, then we may determine u.

An example would be to specify values of u along the positive x-axis and the curve $x = w(y)$ in the first quadrant passing through the origin, as in the diagram. That is, set $u(x,0) = \varphi(x)$, $u(w(y),y) = \psi(y)$ and note that $\varphi(0) = \psi(0)$. Integrating



the hyperbolic equation $u_{xy} = 0$, we have

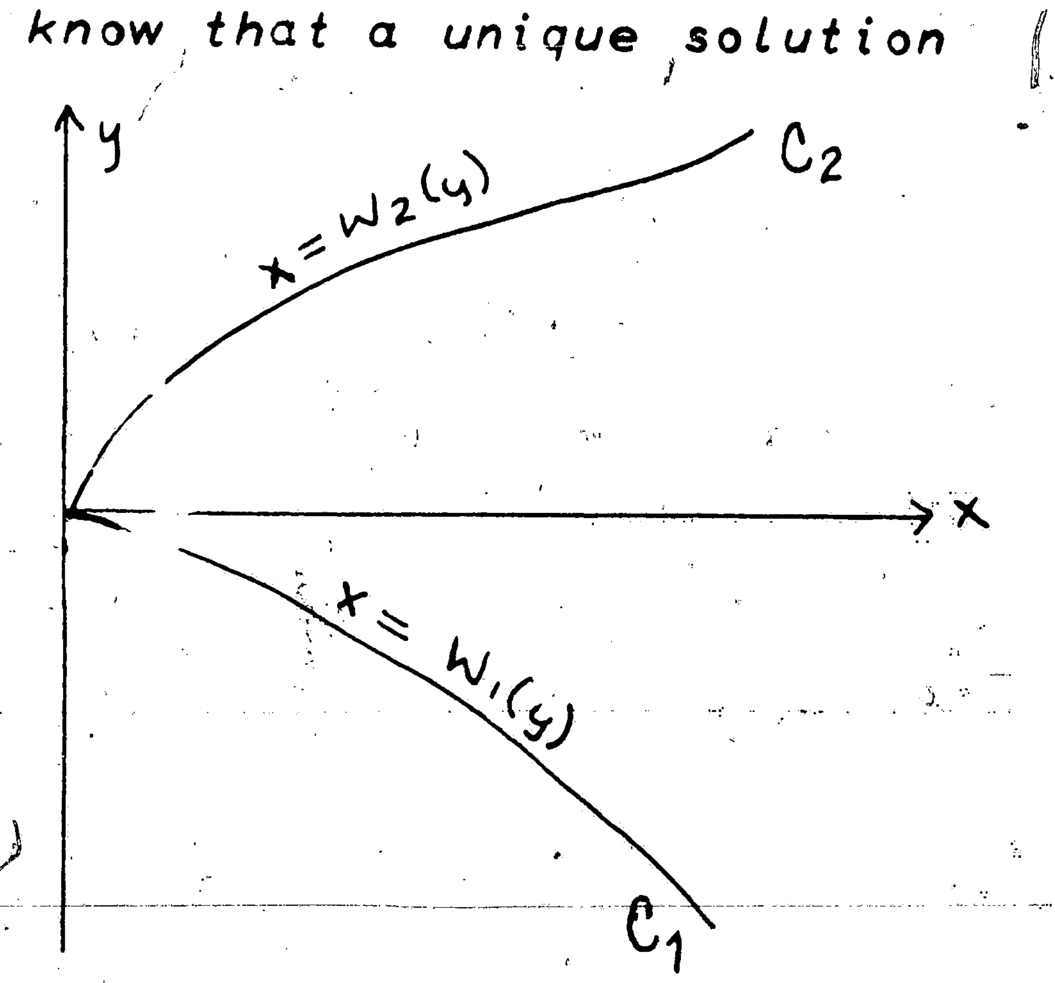
$$0 = \int_{QNPM} u_{\xi\eta}\, d\xi d\eta = \int_{w(y)}^{x} \int_{0}^{y} u_{\xi\eta}\, d\xi d\eta = \int_{w(y)}^{x} \left[u_\xi(\xi,y) - u_\xi(\xi,0)\right] d\xi$$

$$= u(x,y) - u(w(y),y) - u(x,0) + u(w(y),0).$$

Taking the above initial conditions into account this can be written

$$u(x,y) = \psi(y) + \varphi(x) - \varphi(w(y)) = u(M) + u(N) - u(Q) \qquad (84)$$

Suppose we attempt to extend this argument to two curves, neither crossing any characteristic more than once, and lying in adjacent quadrants. Suppose further we specify the values of u, u_x, u_y along one curve, say $x = w_1(y)$ and along the second $x = w_2(y)$ we give values for u. If we consider the equation $u_{xy} = 0$, we see that the coefficients of all terms are continuous with continuous derivatives. By our previous work we know that a unique solution $u_1(x,y)$ will exist everywhere in the region between the x-axis and C_1. Then using the value u_1 along the x-axis (a characteristic curve) and the specified value u along C_2 we obtain a unique solution $u_2(x,y)$ in the region between the x-axis and C_2.



Of course, we want the two values u_1 and u_2 to give a solution $u(x,y)$ with continuous first partial derivatives. In our previous discussion of discontinuities of the first partial derivative, we noted that the discontinuity of the first derivative must exist at all points of the curve or at none. Hence, it is sufficient to have $u_{1_y}(0,0) = u_{2_y}(0,0)$.

If we then let

$$\Psi(y) = u(w_2(y), y)$$

we find by equation (84)

$$u_{1_y}(0,0) = \Psi'(0) - u_{1_x}(w_2(0),0)\, w_2'(0), \tag{85}$$

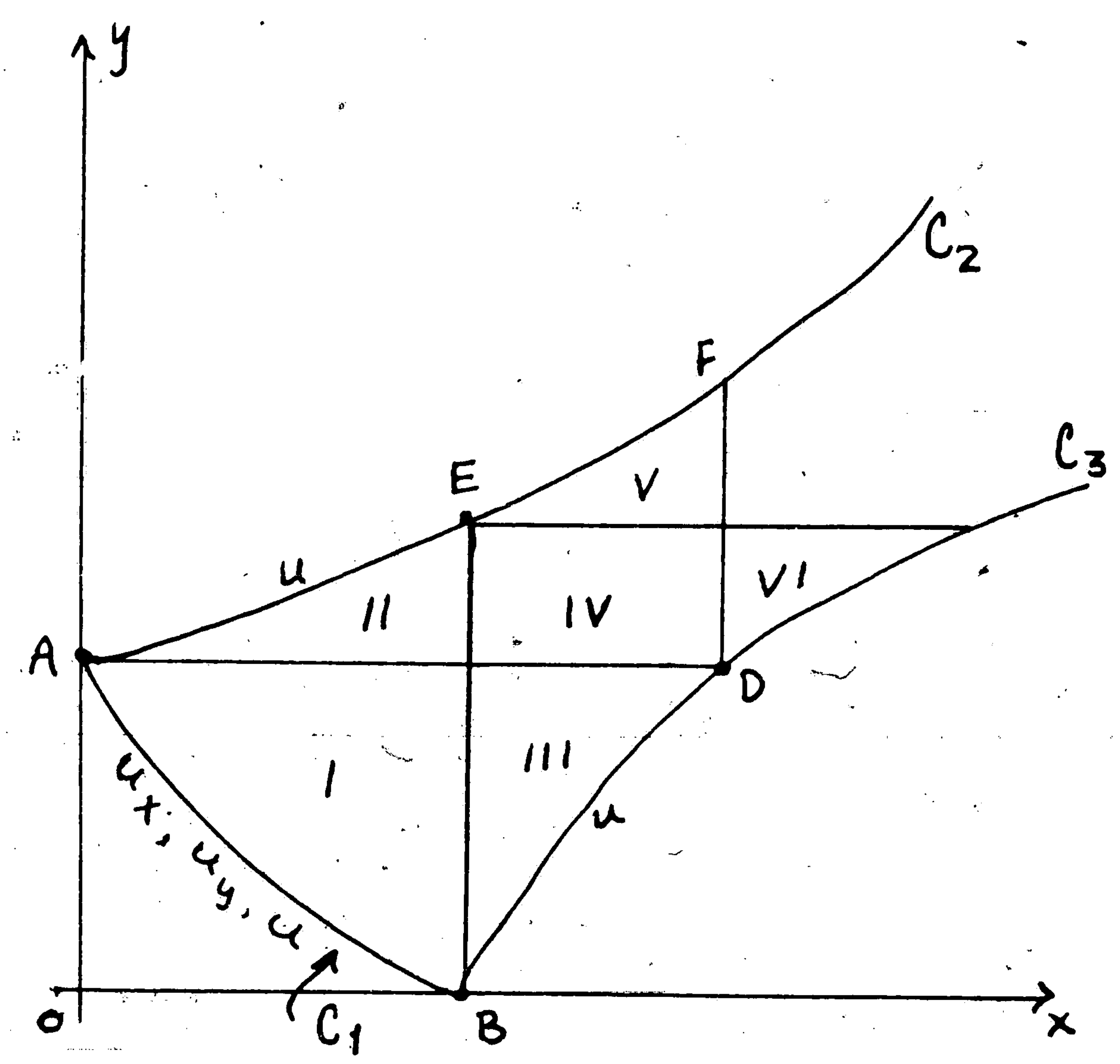
and then since $w_2(0) = 0$ and $w_2'(0) = 1$ at $(0,0)$ we have

$$u_{1_y}(0,0) = \Psi'(0) - u_{1_x}(0,0)$$

or (86)

$$p_1(0,0) + q_1(0,0) = \Psi'(0)$$

In the accompanying figure let us specify the Cauchy conditions u, u_x, u_y along C_1 and values of u along C_2 and C_3, and draw in the characteristics AD, BE, DF, etc. On section I there exists a unique solution $u(x,y)$ by our previous work. By the immediately preceding case, u will be determined on sections II and III; it will be determined on IV by the stipulation that a solution exists if values of u are given along two intersecting characteristics (AD, BE).



All that remains to do is to repeat these arguments for all other sections V, VI, ...

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VITA

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