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A numerical solution to the stability of a slope

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A NUMERICAL SOLUTION TO THE STABILITY OF A SLOPE

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ABSTRACT

A numerical solution to Lowe's graphical method of slope stability analysis which is based on limit equilibrium is presented. Both a dimensional and dimensionless solution are discussed and formulated in detail. A computer solution to the trial and error procedure used to determine the factor of safety is outlined.

An explanation of the meaning and significance of the factor of safety in slope stability analyses is included.

Water forces and their influence on the method of analysis are subject to a general discussion. The "contour method", which involves determination of the water pressure distribution along the contour of an arbitrarily chosen soil volume, is suggested to be suitable for the numerical solution developed.

I. INTRODUCTION

1.1 Slope Stability Analyses

Slopes exist in many different forms. They may be the result of early geological activity (natural slopes) or may be artificially constructed (dams, highway and railroad embankments, etc.).

In general, slope stability analyses are concerned with the failure state of a slope. Loss of stability (failure) occurs when an outer portion of a slope starts to move in an outward and downward direction. If it is assumed that the separation zone along which failure occurs is a continuous line, it is possible to investigate the stability of a slope in a relatively simple manner.

The existing methods of analysis most commonly used today are based on the well known limit equilibrium concept. These methods include the ϕ-circle method, the sliding block method, the logarithmic spiral method and the slices method.

Lowe's method of analysis, which is subject to treatment in this thesis, is based on the slices method and thus furnishes one type of the many existing limit equilibrium solutions.

1.2 Objective

The framework of a complete slope stability analysis requires consideration of the following three aspects:

1. Methods of analyzing forces causing and resisting failure
2. Methods of expressing the shear strength of the materials of a slope or embankment and its foundation
3. Conditions of loading of the slope or the embankment

Only the first of these will be subject to detailed examination. The remaining two features involve thoughts for another independent work.

The method suggested by Lowe (1967) involves a graphical solution to the problem. However, it was found that the method is well suited to numerical solution by computer.

The major objective of this work, therefore, is to translate Lowe's interpretation into an analytical form which can be solved on the computer.

The minor objectives include the explanation of the meaning of the factor of safety applied to slope stability analyses and the discussion of the influence water forces in stability problems.

1.3 Scope

The scope of this project, while primarily intended to present an explicit form of a numerical solution to Lowe's method of analysis, includes both a qualitative and quantitative discussion about the factor of safety as well as a general discussion about the interaction problem between soil and water.

II. FACTOR OF SAFETY

2.1 Historical Review

Concern for soil behavior dates back to the dam and irrigation problems dealt with by ancient cultures such as China and Egypt. Up to the eighteenth century the state of art in soil mechanics was based exclusively on experience and intuition. Soil mechanics obtained some scientific basis when Coulomb (1773) published his famous essay on the classical earth pressure theory. Coulomb assumed the law of friction, applicable to solid bodies, was also valid for a granular material such as soil. In the course of time, as heavier structures were built, additional problems became subject to analytical treatment. Rankine (1857), Boussinesq (1855) and Culman (1866) contributed much to the knowledge of the mechanics of pouring bodies and the theory of elasticity as applied to soil mechanics problems.

It is worthwhile to mention that Coulomb's law is still used today when considering failure states of a soil mass. Particularly, the fundamentals of the 20th century theories of plasticity as applied to soil mechanics problems are based on the idealized Coulomb failure envelope.

Even though concepts of analyzing the stability of earth structures in the early 20th century were available, few engineers relied on theoretical stability analyses. Up until 1935 (Sherard, Woodward, Giziensky and Clevenger, 1963) dam designers still were of the opinion that new dams could most successfully be designed on the basis of past experience. There appeared to be two reasons to support this view point. First, there was almost no opportunity to confirm or verify the analysis of a structure at a real state of failure. Secondly, an understanding of the behavior of the soil mass, even in an idealized manner, was not available to most engineers at that time.

It was Fellenius (1927), Taylor (1937), and Terzaghi (1944), who made the first pioneering efforts in establishing stability analyses of slopes and provided explanations for the complex behavior of soils in an idealized manner.

2.2 Significance of Factor of Safety

An understanding of the significance of the factor of safety involves consideration of two important features:

One of them may be termed the required factor of safety. It is primarily dependent on two aspects:

1. Economy
2. Public Safety

The magnitude of the required factor of safety may be quite difficult to establish.

The second feature to be considered is the computed factor of safety. It involves the quantitative interpretation of the given material parameters with respect to the chosen method of analysis. The computed factor of safety is, in part, governed by the mathematical law of errors. This important fact is frequently overlooked in stability problems. Instead, the method of analysis, which is merely an arbitrary means to solve the problem, is overemphasized. For example, the presentation of safety factors showing three or more significant figures may indicate unfamiliarity of the engineer with the problem to be solved. Such a solution may be of academic interest when comparing different concepts based on the same physical assumptions.

The quality of a computed factor of safety in any stability analysis depends by and large on the following three points:

1. Method of analysis chosen
2. Soil strength
3. Loading conditions.

Since the three points each involve arithmetic errors, errors due to physical considerations and errors due to engineering judgments, it follows that the computed factor of safety must have a limited number of significant figures. The computed factor of safety does not lose significance if the three points mentioned above are properly included in the analysis.

2.3 Methods to Determine Factor of Safety

Soil is a complicated heterogeneous material, the properties of which deviate greatly from those of elastic solids which can be described by the well developed theories of elasticity.

All the theories for computing the factor of safety in problems involving soil are therefore based on simplified assumptions. The real behavior of a soil is far more complex than the assumed behavior of the physical model used for existing stability solutions. Thus, the results obtained are far from being exact and, hence, should always be understood as more or less rough estimations.

The present concept for determining the factor of safety is based on the law of Coulomb friction. The investigation is restricted to considering potential failure zones only. It should be noted that the limit equilibrium method does not consider a stress-strain relationship. The limiting state of equilibrium is reached when the critical stress state is obtained at each point along the continuous failure zone. In other words, each point in the stress space σ-τ lies on the Coulomb failure envelope.

It should be noted that there are other methods of computing the factor of safety. For example, a point to point analysis based on the theory of elasticity which characterizes lines of equal factors of safety against flow may be used. The theory of perfectly plastic soils may also be applied to the problem. However, these lie outside the scope of the present discussion.

2.4 Definition of Factor of Safety

Froehlich (1955) suggested that a reasonable definition of the factor of safety, F, is:

$$F = \frac{\text{available shear strength}}{\text{required shear strength}} \tag{1}$$

where the expression for F is valid at each point within the potential failure zone. The available shear strength, τ, may be computed from Coulomb's law:

$$\tau = c + \sigma \tan \phi \tag{2}$$

where c = cohesion of the soil

ϕ = internal friction angle of the soil

σ = natural stress on the failure plane

Two different ways for determining F are possible:

The indirect method, which establishes a cohesion-friction relationship on the basis of F equaling unity, was widely used until several years ago. By assuming F = 1 and a given value for the cohesion, a critical friction angle, ϕ_k, may be found which is necessary to arrive at a limiting state of equilibrium. Conversly the same concept may be applied by assuming a friction angle and subsequently determining the critical cohesion, c_k, in order to produce a limiting state of equilibrium.

Fellenius (1927) suggested an approximate rule for indirectly finding F. This rule was later adopted by Krey (1936), Taylor (1937), Terzaghi (1946), Chugeav (1964) and others. Fellenius (1927) postulated the following approximation:

$$\frac{\tan \phi}{\tan \phi_k} = \frac{c}{c_k} = \eta \simeq F \qquad (3)$$

where c_k = critical cohesion of the soil when F = 1

ϕ_k = critical friction angle of the soil when F = 1

η = approximate value for F

For a given geometry slope and an assumed soil parameter, c_k, it is possible to determine the magnitude of ϕ_k which is required to arrive at the limiting state of equilibrium of a specified soil wedge which is part of the slope. A minimization process yields a pair of (c_k, ϕ_k) min which is based on the criteria that F = 1. Several such pairs finally determine the so called friction angle curve

$$\phi_k = f\ (\sigma_k) \tag{4}$$

as illustrated in Fig. 1.

Eqs. (3) and (4) may be combined to determine η.

The validity of this rule is demonstrated below (see Fig. 2).

if

$$R_c = c\ ab$$
$$R_\phi = N \tan \phi \tag{5a}$$

and

$$R_{c_k} = c_k\ ab$$
$$R_{\phi_k} = N_k \tan \phi_k \tag{5b}$$

Eq. (1) can be rewritten in terms of moment equilibrium in the following form:

$$F = \frac{R_c\ \rho_c + R_\phi\ \rho_\phi}{R_{c_k}\ \rho_{c_k} + R_{\phi_k}\ \rho_{\phi_k}} \tag{6}$$

where N = existing normal force on slip surface

N_k = critical normal force on slip surface

R_c = resisting cohesive force

R_{c_k} = critical cohesive force with F equals unity

R_ϕ = critical friction force

R_{ϕ_k} = critical friction force with F equals unity

ab = chord

ρ_c = radius vector normal to R_c

ρ_{c_k} = radius vector normal to R_{c_k}

ρ_ϕ = radius vector normal to R_ϕ

ρ_{ϕ_k} = radius vector normal to R_{ϕ_k}

The parameters c_k and $\tan \phi_k$ may be expressed by Eq. (3) as c/η and $\tan \phi/\eta$, respectively. These values may then be substituted into Eq. (5b) for c_k and $\tan \phi_k$ and the resulting forces, as expressed by Eqs. (5a) and (5b) then, in turn, substituted into Eq. (6) resulting in:

$$F = \frac{ab\ c\ \rho_c + N \tan \phi\ \rho_c}{ab\ \frac{c}{\eta}\ \rho_{c_k} + N_k\ \frac{\tan \phi}{\eta}\ \rho_{\phi_k}} \tag{7}$$

If the difference between N and N_k is small and ρ_c, ρ_ϕ are approximately equal to ρ_{c_k} and ρ_{ϕ_k}, respectively, Eq. (7) reduces to

$$F \cong \eta \tag{8}$$

It has been shown by various researchers, including Taylor (1937), that this particular approximation is good if F lies in the neighborhood of unity.

The second approach is the direct method, and was suggested by Froelich (1955), Bishop and Morgenstern (1960) and Lowe (1967) among others.

The direct way of determining F is consistent with the definitions shown in Eq. (1). Explicitly it assumes the following form:

$$F = \frac{\int^{L} \tau \, dL}{S_D} \tag{9}$$

where L = arc length

S_D = developed resisting shear force

Both the direct and indirect methods discussed above are in use today.

2.5 Choice of Factor of Safety

Lowe's analysis (1967) arrives at the factor of safety on a direct basis in which the unknown force components are expressed in terms of the factors of safety F. Eq. (9) of Chapter 2.4 therefore was chosen by Lowe (1967). The implication of Eq. (9) is explained in detail in Chapter 3.

It should be noted that the indirect method of analysis could also have been applied. The direct method, however, involves only one computational process, and furthermore, does not raise the questions of accuracy which may arise if Eq. (8) is applied.

III. ANALYSIS

3.1 General Aspects

The stability of an earth mass is guaranteed if no continuous zone exists along which the shear strength of the material is exceeded at each point such that a portion close to the free boundary of the earth mass is able to move in an outward and downward direction.

To investigate whether instability or failure may or may not occur, it is necessary to analyze the state of stress at least within possible failure zones. Furthermore, the so-called "condition of flow failure" must be known. If, at a point within the soil mass, the condition of flow failure is attained, then the critical ratio of the two principle stresses is reached. If the maximum principle ratio increases further, movement of the surrounding particles starts what is known as the beginning of plastic flow.

At the point in question, which may be located some distance away from the free boundary, stresses are mobilized to counteract the plastic flow. Ultimate failure finally takes place when a continuous zone is formed in which all points have the same condition as described above. All planes parallel to the considered plane in the soil mass may be assumed to have the same stress state. Thus, it is a plane stress problem in which the mean principle stresses normal to the considered cross section are neglected. This assumption may be justified in order to reduce the complexity of the problem. Several methods of approaching the problem have been proposed.

Brahtz (1936) introduces stress functions and uses the theory of elasticity to compute a stress field. From this field he determines lines of equal factors of safety which are defined as the ratio between available shearing strength and developed shear stress. Similar approaches were advocated by Bennett (1951).

Other authors, including Frontard (1922), Jaky (1936) and Drucker and Prager (1952) have suggested methods in which the shape and the location of the sliding surface may be rigorously determined.

The third approach, which is probably the most widely used, is to choose any continuous failure surface and place it within the soil mass in such a way that, relatively speaking, a minimum factor of safety against sliding exists.

Fellenius (1927), developed a procedure for investigating the stability of slopes which is known as the Swedish Method or Slip Circle Method. It is based on the concept of limit equilibrium and assumes a circular failure plane. Fellenius (1927) proposed the new approach in order to analyze the safety of railway embankments against instability. Circular failure surfaces were more or less in good agreement to what was observed in nature.

3.2 Geometrical and Physical Concepts

The geometrical concept of the method suggested by Lowe (1967) assumes circular failure planes composed of one or more radii (Fig. 3). The geometry of the slope, or free boundary, may be irregular but should agree with the general shape of existing slopes, i.e., no overhanging portions.

The physical concept is based on the well-known slices method (Fellenius, 1927) which essentially subdivides the assumed failure wedge (Fig. 3) into an arbitrarily chosen number of elements (Fig. 4). With certain postulates about the distribution of the side forces acting upon a slice, which in fact corresponds to assuming a particular type of stress distribution along the failure zone, it is possible to utilize two equilibrium equations. These are namely: the sum of the horizontal and sum of the vertical forces, respectively, must be equal to zero.

Lowe works primarily with force vectors and uses the factor of safety as a constant parameter. This parameter assumes the correct value at the moment when certain boundary conditions are satisfied within the given equilibrium system.

Fig. 5 shows an individual slice element as part of the assumed failure wedge. Each element is described by two vertical planes, the free boundary and the failure surface as shown in Fig. 4.

The forces acting upon each slice may be categorized into the following groups:

1. external forces
2. internal forces.

The force R acting upon a slice (Fig. 5) may be considered to be composed of externally applied loads, the weight of the soil wedge, "W", and the water forces, "U", which will be discussed in more detail in Chapter 4. The internal or reaction forces, on the other hand, are presented by the vectors E_L, E_R and P (Fig. 5). Both the magnitude and direction of the external forces are normally known. However, neither magnitude nor direction of the internal forces are available. All that is known is that the corresponding force vectors must exist because of equilibrium considerations.

It should not be too difficult to imagine that the force system as presented in Fig. 5 represents a statically highly indeterminate problem and simplifications are necessary in order to arrive at a solution.

A first step in the simplification is to reduce the force system from six to four unknowns by assuming the directions of E_L and E_R. Taylor (1937) postulated that a reasonable approximation is to assign E_L a direction which is the average of the boundary slopes at point 1 and 2 and to assign the direction of E_R the average of slope 1' and 2'. This postulate was adapted by Lowe. Furthermore, Lowe makes use of Eq. 9 in Chapter 2 by concluding that the necessary shearing force can always be expressed as a fraction of the available shearing force along the arc 2-2' (Fig. 5) in terms of the factor of safety F. Thus

$$S_D = \frac{cL}{F} + \frac{N \tan \phi}{F} \tag{10}$$

or

$$S_D = C_D + N \tan \phi_D \tag{11}$$

where

C_D = developed cohesive force

ϕ_D = developed friction angle of the soil

Hence, vector P can be written in terms of a component normal to the failure surface, i.e., normal force, and a component tangential to the failure surface, i.e., shear force as

$$P = S_D + N \tag{12}$$

In Eq. (12), the directions of N and S_D are known. N passes through the center of the failure circle and S_D is perpendicular to N. The direction of C_D in Eq. (11) coincides with the direction of S_D. It's magnitude is known when the length, L, of the arc is specified.

The locations of the forces are not known because of a lack on information concerning the stress distribution along the contour of a slice. Therefore the equilibrium condition with respect to moments cannot be satisfied. Hence, no use can be made of moment equilibrium to determine the unknowns.

The remaining four unknowns are N, E_R, E_L and F. At this stage Lowe assumes the magnitudes E_R and F are given. Therefore, from equilibrium considerations, the force polygon will close and the two unknowns, E_L and N, may be determined (Fig. 6b). However, the problem is not solved because the assumed values E_R and F are still involved in the equilibrium system of the slice under consideration.

On the basis of the model described above, the following steps are possible. From the boundary condition of the first slice (Fig. 6c), E_R must be equal to zero because 1'-2' vanishes. Thus, if F is arbitrarily assumed, E_L and N of the first slice can be determined. Subsequently, from Newton's law of "action = reaction", the interslice forces on two adjacent planes must have the same magnitude with opposite directions. Hence, E_R of the next slice corresponds to $-E_L$ of the previous one.

The implication of this is that, with a common value for F within the given equilibrium system, all slice forces may be found by proceeding stepwise from one slice to the next until the last slice is reached (Fig. 6a). At the last slice, a secondary boundary condition requires that E_L equals zero. If the computed E_L of the last slice is not zero, the arbitrary value for F has to be changed and the process of going from slice to slice must be repeated all over again until the second boundary condition is satisfied.

This approach, which has been solved graphically by Lowe, is suitable for convenient formulation on the computer. The use of the computer appears to be more efficient than the graphical approach if one recalls that, to find a minimum factor of safety F, several failure circles have to be investigated for a specified problem. In addition, no mention has been made about the variation of the external loading conditions. These may be of significance in practical cases. Multiple loading conditions may be readily examined by the computer once a program is established.

3.3 Formulation of Solution

The numerical formulation is based on the same principles which were followed by Lowe. The following section therefore, involves only an algebraic development of those principles. The geometrical concept on which the formulation is based on is detailed in Fig. 7.

Fig. 7 represents a slice which is considered to be part of the total failure wedge. The subscripts "i" refer to the number of the individual slice within the given numbering system as illustrated in Fig. 4. From the figure it can be deduced that there is a differentiation between slices and vertical boundaries. The slice numbers follow the sequence 2, 4 ... $2N_s$, whereas the vertical boundaries follow the sequence 1, 3, 5 .... $2N_s+1$, where N_s is the total number of slices.

The formulas which serve as the basic framework of the analysis are stated below:

$$\Sigma \text{ (horizontal forces)} = 0 \qquad (13)$$

$$\Sigma \text{ (vertical forces)} = 0 \qquad (14)$$

and from Eq. (10)

$$S_{D,i} = \frac{cL_i}{F} + N_i \frac{\tan \phi}{F} \qquad (15)$$

$S_{D,i}$ = developed resisting shear force of slice i

L_i = arc length of slice i

N_i = normal force at slip surface of slice i

From Taylor's postulate it may be seen that:

$$\alpha_{L,i+1} = \frac{\alpha_{T,i+1} + \alpha_{B,i+1}}{2} \tag{16}$$

and similarly

$$\alpha_{R,i-1} = \frac{\alpha_{T,i-1} + \alpha_{B,i-1}}{2} \tag{17}$$

where $\alpha_{L,i+1}$ = direction of side force $E_{L,i+1}$

$\alpha_{T,i+1}$ = slope of surface at the vertical boundary i+1 of slice i

$\alpha_{B,i+1}$ = slope of failure plane at the vertical boundary i+1 of slice i

$\alpha_{R,i-1}$ = direction of side force $E_{R,i-1}$

$\alpha_{T,i-1}$ = slope of surface at the vertical boundary i-1 of slice i

$\alpha_{B,i-1}$ = slope of failure plane at the vertical boundary i-1 of slice i

If the free boundary which intersects with a vertical plane of a slice has a corner, the α'_Ts are expressed as an average value of the slopes to the left and to the right side of the corner. The remaining angles may be determined directly from the geometry.

For reasons of simplicity the resultant force R is expressed in terms of its horizontal and vertical components.

Thus,

$$R_{V,i} \gtrless 0 \tag{18}$$

$$R_{H,i} \gtrless 0 \tag{19}$$

where $R_{V,i}$ = vertical projection of R of slice i

$R_{H,i}$ = horizontal projection of R of slice i

3.3.1 Implicit Form

For slice i, the following two equations may be obtained by combining Eq. (13), (14), (15):

$$E_{L,i+1} \cos \alpha_{L,i+1} + N_i \left(\cos \alpha_{N,i} + \frac{\tan \phi}{F} \cos \alpha_{S,i}\right)$$

$$= - \left(R_{H,i} + \frac{cL_i}{F} \cos \alpha_{S,i} + E_{R,i-1} \cos \alpha_{R,i-1}\right) \tag{20}$$

and

$$E_{L,i+1} \sin \alpha_{L,i+1} + N_i \left(\sin \alpha_{N,i} + \frac{\tan \phi}{F} \sin \alpha_{S,i}\right)$$

$$= - \left(R_{V,i} + \frac{cL_i}{F} \sin \alpha_{S,i} + E_{R,i-1} \sin \alpha_{R,i-1}\right) \tag{21}$$

where

$\alpha_{N,i}$ = direction of normal force N_i of slice i

$\alpha_{S,i}$ = direction of shear force $S_{D,i}$ of slice i

Equations (20) and (21) represent two simultaneous equations, linear in the two unknown quantities N_i and $E_{L,i+1}$.

This system of equations can be written as:

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} E_{L,i+1} \\ N_i \end{bmatrix} = \begin{Bmatrix} B \end{Bmatrix} \qquad (22)$$

with A = coefficient matrix

B = column matrix

and by making the substitution

$$\alpha_{S,i} = \alpha_{N,i} + 90° \qquad (23)$$

the direction cosines l_{ij}, associated with the two unknowns N_i, $E_{L,i+1}$, of the coefficient matrix A may be written:

$$l_{11} = \cos \quad {}_{L,i+1}$$

$$l_{12} = \cos \alpha_{N,i} + \frac{\tan \phi}{F} \sin \alpha_{N,i}$$

$$l_{21} = \sin \alpha_{N,i} \qquad (24a)$$

$$l_{22} = \sin \alpha_{N,i} + \frac{\tan \phi}{F} \cos \alpha_{N,i}$$

The coefficients of the column matrix B assume the form:

$$b_1 = -\left(R_{H,i} - \frac{cL_i}{F} \sin \alpha_{N,i} + E_{R,i-1} \cos \alpha_{R,i-1}\right)$$

$$b_2 = -\left(R_{V,i} + \frac{CL_i}{F} \cos \alpha_{N,i} + E_{R,i-1} \sin \alpha_{R,i-1}\right) \quad (24b)$$

The implicit form presented in Eq. (22) is valid for any slice "i". The interdependence of the individual equilibrium systems require that the solution vectors must always be found in the sequence i = 2, 4, ... $2N_S$. This rather simple formulation is thought to be suitable for the computer. If there is a matrix subroutine available for this, no further refinements of the basic formulation need be made.

3.3.2 Explicit Form

After some algebraic manipulations of Eqs. (20) and (21), the quantities N_i and $E_{L,\ i+1}$ may be expressed, separately, in terms of the given elements as follows:

$$E_{L,i+1} = \frac{E_{R,i-1}\, g_1 + R_{v,i}\, g_2 + R_{H,i}\, g_3 + cL_i}{g_4} \quad (25)$$

and

$$N_i = \frac{E_{R,i-1}\, g_5 + R_{V,i}\, g_6 + R_{H,i}\, g_7 + cL_i\, g_8}{g_4} \quad (26)$$

where:

$$g_1 = F \sin (\alpha_{R,i-1} - \alpha_{N,i}) - \tan \phi \cos (\alpha_{R,i-1} - \alpha_{N,i})$$

$$g_2 = F \cos \alpha_{N,i} - \tan \phi \sin \alpha_{N,i}$$

$$g_3 = -(F \sin \alpha_{N,i} + \tan \phi \cos \alpha_{N,i})$$

$$g_4 = \tan \quad \cos (\alpha_{L,i+1} - \alpha_{N,i}) - F \sin (\alpha_{L,i+1} - \alpha_{N,i}) \tag{27}$$

$$g_5 = F \sin (\alpha_{L,i+1} - \alpha_{N,i})$$

$$g_6 = F \cos (\alpha_{L,i+1} - \alpha_{N,i})$$

$$g_7 = F \cos \alpha_{L,i+1}$$

$$g_8 = -F \sin \alpha_{L,i+1}$$

It can be seen from Eqs. (25) and (26) that the two quantities $E_{L,i+1}$ and N_i are independent of each other. Therefore, it should be possible to obtain F by applying Eq. (25) alone. It was found, as will be seen later, that F is conveniently determined by using Eqs. (25) and (26) simultaneously. For this reason N_i is carried along throughout the analysis.

It may also be noted that N_i can be used to determine the approximate stress distribution along the potential failure zone, once F is determined. This may have importance in soil mechanics problems when different types of shear strength are to be considered. The normal stress along the failure zone of slice i can be written as follows:

$$\sigma_i = \frac{N_i}{L_i} \tag{28}$$

A rather attractive idea is to make the system, Eqs. (25) and (26), dimensionless. If a set of vectors is multiplied by a scaler λ, a new set of vectors is described all of which have the original directions, but with magnitudes proportionally reduced by λ. Hence, the new set of vectors will obey the same laws as the original one. Thus it is possible to write:

$$\varepsilon_{L,i+1} = \frac{E_{L,i+1}}{\lambda} \tag{29}$$

and

$$\mu_i = \frac{N_i}{\lambda} \tag{30}$$

where $\varepsilon_{L,i+1}$ = reduced side force at the vertical boundary i+1 of slice i

μ_i = reduced normal force at the failure surface of slice i

λ is valid for the total equilibrium system as well as for each individual one.

It is possible to obtain relationships for the external components in such a way, that they can be expressed in terms of the slope geometry, including the chosen failure mechanism, as:

$$R_{V,i} = \sum_{1}^{n} f_{V,n}\, \gamma_n \tag{31a}$$

$$R_{H,i} = \sum_{1}^{m} f_{H,m}\, \gamma_m \tag{31b}$$

$$\lambda = \xi\, \lambda \tag{32}$$

The numbers n and m are reference numbers associated with the corresponding components within the slope geometry, where $f_{V,n}$, $f_{H,m}$, and ξ, are geometrical functions and a dimensional parameter, respectively, and the γ's represent the related unit weights. Eqs. (29) and (30) can now be modified to read:

$$\varepsilon_{L,i+1} = \frac{\varepsilon_{R,i-1}\, g_1 + g_2 \sum_1^n \left(\frac{F_{V,n}}{\xi}\right)\left(\frac{\gamma_n}{\gamma}\right) + g_3 \sum_1^m \left(\frac{f_{H,m}}{\xi}\right)\left(\frac{\gamma_m}{\gamma}\right) + \frac{cLi}{\xi \cdot \gamma}}{g_4} \qquad (33)$$

$$\mu_i = \frac{\varepsilon_{R,i-1}\, g_5 + g_6 \sum_1^n \left(\frac{f_{V,n}}{\xi}\right)\left(\frac{\gamma_n}{\gamma}\right) + g_7 \sum_1^m \left(\frac{f_{H,m}}{\xi}\right)\left(\frac{\gamma_m}{\gamma}\right) + \frac{cLi}{\xi \cdot \gamma}\, g_8}{g_4} \qquad (34)$$

where $\varepsilon_{R,i-1}$ = reduced side force at the vertical boundary i-1 of slice i and the functions "g" are explicitly given by Eq. (27).

It should be noticed that the relationship given by Eq. (31) cannot always be established. For example, there is no way to express an external point load in any one of the terms $R_{H,i}$ or $R_{V,i}$ in terms of the geometry of the failure mechanism and the geometry of the slope.

The cohesion term in classical problems is usually handled as follows:

$$c = N_c\, \gamma\, H \qquad (35)$$

N_c = cohesion number

H = characteristic height of embankment

However, Eq. (35) requires the existence of a characteristic height.

The following is a demonstration of the formulation discussed above for a homogeneous slope (Fig. 8) which does not have a characteristic height.

For reasons of simplicity it is assumed that $R_{V,i} = W_i$ and $R_{H,i} = 0$. $R_{V,i}$ may then be expressed as follows:

$$R_{V,i} = \int_{x_{i-1}}^{x_{i+1}} (Y_2(x) - Y_1(x))\, dx\, \gamma \tag{36}$$

x, y = cohesion coordinates

x = independent variable

x_{i+1} = coordinate of slice boundary i+1

x_{i-1} = coordinate of slice boundary i-1

$Y_1(x)$ = function which describes the free boundary of the external slope

$Y_2(x)$ = function which describes the failure surface

In order to eliminate dimensions, $R_{V,i}$ is conveniently expressed in polar coordinates.

$$Y_2(x) - Y_1(x) = r\left(\sin\theta - \frac{Y_1(x)}{r}\right) \quad (37a)$$

where

θ = independent variable of the polar coordinate system

r = radius of failure circle

$\frac{Y_1(x)}{r}$ = function of θ

Hence we can write

$$Y_2(x) - Y_1(x) = r\,\psi(\theta) \quad (37b)$$

when $\psi(\theta)$ = function of θ and the slope geometry with

$$dx = -r\sin\theta\,d\theta \quad (38)$$

Eq. (36) assumes the form:

$$R_{y,i} = r^2\gamma\int_{\theta_{i+1}}^{\theta_{i-1}} \psi(\theta)\sin\theta\,d\theta \quad (39)$$

θ_{i-1} = unit angle of vertical boundary $i-1$

θ_{i+1} = unit angle of vertical boundary $i+1$

which is a special case of Eq. (31a). The function $Y_1(x)/r$ always can be expressed in terms of "θ" if the free boundary (Fig. 8) is given. The dimensions of Eq. (38)

can be eliminated by choosing an expression for ξ of the form

$$\xi = \delta\ r^2 \tag{40}$$

Thus, Eq. (32) reads:

$$\lambda = \delta\ r^2\ \gamma \tag{41}$$

where $1 \leq \delta < \infty$ for numerical considerations.

Furthermore, the arc length L_i of a slice can be expressed as

$$L_i = r\ (\theta_{i+1}^{*} - \theta_{i-1}) \tag{42}$$

The limit angles themselves depend upon the chosen number of slices and may be determined in the following way:

$$\Delta = \frac{d \cos \omega}{N_s} \tag{43}$$

where

Δ = slice width

$$d = 2\ r\ \sin \frac{(\theta_H - \theta_o)}{2} \tag{44a}$$

d = distance from points P_1 and P_2 (Fig. 8)

and

$$\cos \omega = \sin \frac{(\theta_H + \theta_o)}{2} \tag{44b}$$

ω = arbitrary angle

It follows that $\cos \theta_{i-1} = \cos \theta_o - \frac{(i-2)}{2} \frac{\Delta}{r}$ (45a)

or

$$\cos \theta_{i-1} = \cos \theta_o - \frac{(i-2)}{N_s} \sin \frac{(\theta_H + \theta_o)}{2} \sin \frac{(\theta_H - \theta_o)}{2} \tag{45b}$$

θ_H, θ_o = limit angles which bound the failure wedge (Fig. 8)

Noting that:

$$\cos \theta_i = \frac{\int_{\theta_{i-1}}^{\theta_{i+1}} \psi(\theta) \sin \theta \cos \theta \, d\theta}{\int_{\theta_{i-1}}^{\theta_{i+1}} \psi(\theta) \sin \theta \, d\theta} \tag{46}$$

$$\alpha_{N,i} = (\theta_i + \pi) \tag{47}$$

The last expression which has to be considered is "L_i c". Dividing this term in its explicit form by $\lambda = \delta r^2 \gamma$, it can be seen that there still is one "r" left in the denominator. In the case of an embankment with a characteristic height H, Eq. (35) may be introduced to yield an expression H/r which is always expressible in terms of θ. In general, to eliminate all r - terms in the equations, another parameter has to be found which is closely related to r. This may be accomplished by introducing "d" which is known through the given coordinates of P_1 and P_2 (See Fig. 8). Hence, if the numerator and denominator of the left hand side of Eq. (42) are multiplied by d

$$\frac{L_i\ c}{\lambda} = \frac{c\ (d/r)\ (\theta_{i+1} - \theta_{i-1})}{d\ \gamma\ \delta} \tag{48}$$

which, by applying Eq. (44a), may be written as

$$\frac{L_i\ c}{\lambda} = \left(\frac{c}{d\gamma}\right) \frac{2 \sin \frac{(\theta_H - \theta_o)}{2} (\theta_{i+1} - \theta_{i-1})}{\delta} \tag{49}$$

Thus the expressions are parameter bound dimensionless forms.

Considering at this point all the elements discussed above as being given, Eqs. (33) and (34) can be rewritten in the following form:

$$\varepsilon_{L,i+1} = [\varepsilon_{R,i-1}\ g_1 + \frac{g_2}{\delta} \int_{\theta_{i+1}}^{\theta_{i-1}} \psi(\theta) \sin\theta\ d\theta + \tag{50}$$

$$\left(\frac{c}{d\gamma}\right) \frac{2}{\delta} \sin \frac{(\theta_H - \theta_o)}{2} (\theta_{i+1} - \theta_{i-1})]/g_4$$

and

$$\mu_i = [\varepsilon_{R,i-1}\, g_5 + \frac{g_6}{\delta} \int_{\theta_{i+1}}^{\theta_{i-1}} \psi(\theta) \sin\theta\, d\theta + \tag{51}$$

$$(\frac{c}{d\gamma})\, g_8\, \frac{2}{\delta} \sin \frac{(\theta_H - \theta_o)}{2}\, (\theta_{i+1} - \theta_{i-1})]/g_4$$

This form is dimensionless with respect to the given points P_1 and P_2 described by their coordinates. The most critical circle passing through the two points may now be determined by varying the variables θ_H and θ_o independently. To obtain the most critical failure circle of the slope in the absolute sense, a subsequent variation in the two defined points P_1 and P_2 on the surface boundary has to be made.

An investigation has indicated that the critical failure circle of a slope may be found directly in terms of θ_H and θ_o. However, this is possible only if the free boundary is clearly defined, as in the case of a classical type of embankment. The problem is similar to the one treated above, requiring the introduction of more complex geometrical relations. The scope of the analysis is beyond the framework of this thesis.

IV. WATER FORCES

4.1 General Aspects

In soil mechanics problems, water forces frequently are important factors which influence the stability of a slope or dam. Examples include, the overlaying water mass of a slope which acts on an impervious facing and stabilizes the soil, and the internal pore pressure, which may contribute to a possible slide by pressing the soil outwards and thus reduces the internal friction. Many researchers including Casagrande (1934), Ivanov (1940) and Polubarinova Kochina (1962), have tried to determine solutions to practical problems involving water forces.

The graphical approach for finding the forces which act upon the porous media soil is probably more effective than any of the existing hand calculation methods. Even though it is rather time consuming, the graphical method is easy to perform. However, with the development of computer science, it is now possible to more efficiently account for seepage on a numerical basis. Recently, computer solutions to seepage problems have been published by Malcholuf (1966), and Finn (1967). However, no mention has been made regarding the correlation of given flow nets with failure mechanisms such as those which appear in slope stability problems.

The purpose of this chapter, therefore, is to suggest a general idea for taking into account water forces in conjunction with slope stability problems.

4.2 Uplift and Seepage

The important water component in a stability problem is the pore water pressure distribution in the soil mass. It can be estimated by using the theory of flow through porous media. This theory is applicable to anisotropic as well as to isotropic soils and is based on the following assumptions:

1. The soil skeleton which constitutes the seepage media is incompressible.
2. The hydraulic gradient which causes flow of water is due to gravity head loss only.
3. The degree of saturation remains constant, (flow out equals flow in).
4. The boundary conditions are known.

It is relatively easy to solve the differential equation of fluid flow:

$$k_x \frac{\partial h^2}{\partial x^2} + k_y \frac{\partial h^2}{\partial y^2} = 0 \qquad (52)$$

where k_x, k_y = permeabilities in the x and y directions

h = potential head

x, y = cartesian coordinates

by means of a computer using finite difference or finite element techniques.

Several methods may be applied to account for water forces in slope stability problems. The purpose of the following discussion is to illustrate the "contour method" proposed by Ivanov (1940). It is essentially the same method as Lowe (1967) uses in his analysis. It is, however, more detailed and in particular, contributes to the understanding of what is meant by the terms Uplift and Seepage.

The main advantage of this method is that it requires knowledge of the pore pressure distribution only along the contour of the wedge or slice to be considered at the limiting state of failure and not through the entire soil media. The problem is considered to be two dimensional and the soil particles are assumed to be ideal spheres. Fig. 9 shows an arbitrarily chosen soil volume abcd, (V_S), through which seepage flow occurs. Consider that the pressure distribution along abcd is known. It is possible to express the forces acting upon the water contained in the pores of the volume abcd in terms of the forces U_T, U_L, U_R, U_B, (Fig. 9).

If Φ is the force with which the soil skeleton acts upon the water in the enclosed volume, V_S, and Φ_V, Φ_H

are the corresponding components in the vertical and horizontal direction, the dynamic equilibrium equation for the water contained in the soil volume V_S, when inertia forces are neglected, is:

$$\Phi_V = - U_B + U_T + W_W \tag{53}$$

or

$$\Phi_V = - (U_B - U_T - V_S \gamma_W n) \tag{54}$$

U_B, U_T, U_R, U_L = water forces (Fig. 9)

γ_W = unit weight of water

n = porosity

Furthermore, the equation for Φ_H is:

$$\Phi_H = - (U_L - U_R) \tag{55}$$

The total hydraulic reaction on the soil skeleton may now be written:

$$\Omega = - \Phi \tag{56}$$

where the projections Ω_V and Ω_H are identical with Φ_V and Φ_H respectively in quantity but have opposite directions.

Thus, we can deduce from Eqs. (53) and (54) that

$$\Omega_V = U_B - U_T - V_S \gamma_W n \tag{57}$$

and from Eqs. (56) and (57)

$$\Omega_H = U_L - U_R \tag{58}$$

The components constituting Ω_V and Ω_H can now conveniently be described by means of the following graphical interpretation. Let us first consider the vertical projection Ω_V of Ω. Fig. 10 shows a plot of the piezometric heads of each point along the contour of the soil mass V_S.

It can be seen that

$$U_T = A_T \gamma_W \tag{59}$$

and

$$U_B = A_B \gamma_W \tag{60}$$

where A_T is the area abcc'b'a'a and A_B is the area adcc'd'a'a.

Hence it follows that

$$\Omega_V = (A_T - A_B) \gamma_W - V_S \gamma_W n \tag{61}$$

The area $(A_T - A_B)$, however, can now be written in another way, as:

$$(A_T - A_B) = A_{a'b'c'd'a'} + V_S \qquad (62)$$

it follows that:

$$\Omega_V = \gamma_w A_{a'b'c'd'a'} + V_S \gamma_w (1.0 - n) \qquad (63)$$

The first term of the right hand side of Eq. (63) is known as the seepage component and the second term as the hydrostatic uplift. $A_{a'b'c'd'a'} \gamma_w$ may be positive as well as negative in its direction whereas the uplift is always positive.

Applying the same concept to the side forces one can write:

$$\Omega_H = \gamma_w A_{a^\circ b^\circ c^\circ d^\circ a^\circ} \qquad (64)$$

It should be noticed that the hydrostatic side forces cancel each other since the pressure at all points along a horizontal plane are equal and opposite. Thus a pure seepage force can exist only between two points lying in the same horizontal plane.

4.3 Force Components Ω_V and Ω_H

The explanation above does not necessarily contribute directly to a method of accounting for water forces. It does, however, provide a physical understanding of the water forces which may act upon a soil mass when seepage flow occurs. For example, from Eq. (64) it can be concluded immediately that in a static water condition only uplift has to be considered.

Another way of interpreting the water forces acting on the soil is to plot the piezometric heads with reference to 0 - 0 for each point along the contour of the arbitrarily chosen soil volume (Fig. 10). All points of the contour abcd then lie within the projection of "ac" and 0 - 0, and the resulting pressure area ($A_{a'b'c'd'a'}$ + V_S (1.0 - n)) becomes $A_{a'b'c'd'a'}$ such that

$$\Omega_V = \bar{A}_{a'b'c'd'a'} \gamma_w \qquad (65)$$

Seepage and Uplift in Eq. (65) apparently are resolved to a single quantity. The same principle applies to Ω_H as can be seen by Eq. (58).

The concept above is suggested for the use in computer solutions to slope stability problems. The force components Ω_V and Ω_H may be determined by using a computed potential field in place of the flow net normally used in graphical approaches. The potential field may be computed on the basis of finite differences or finite elements, Malcholuf (1967), Finn (1966).

As shown in Fig. 11, points on the contour of the individual slices are selected, and the piezometric head is computed on the basis of the given potential field using one of the many existing interpolation methods. The number of points to be chosen between the corner points abcd essentially has to be found on the basis of experience. The number will not be large since it is known that the pressure distribution along a contour between any of the two points does not change markedly.

After having computed all the pressure heads at the selected points on the contour, numerical integration of the pressure areas remains. For this purpose either the trapezoidal rule or Gauss' quadrature formula are applicable. Many computers have such a subroutine available for problem solution by these methods.

V. TRIAL AND ERROR PROCEDURE

5.1 General Aspects

In Chapter 3 the two functions $E_{L,i+1}$ and N_i, as well as the corresponding functions $\varepsilon_{L,i+1}$ and μ_i, involve the factor of safety F which for convenience was arbitrarily assumed to be known. In reality F remains to be determined on the basis of a trial and error procedure. As was explained in Chapter 3, the value for the factor of safety has to satisfy:

$$E_{L,2N_S+1} = 0 \tag{66}$$

where N_S is the chosen number of slices.

$E_{L,i+1}$ and N_i are magnitudes of vectors all of which must be positive within the given equilibrium system. This furnishes another important criterion:

$$\left.\begin{aligned} E_{L,i+1} &\geq 0 \\ N_i &\geq 0 \end{aligned}\right\} \quad \text{for } i = 2,4,6,\ldots\ldots 2N_S \tag{67}$$

A numerical investigation has shown that if $E_{L,2N_S+1}$ is plotted against F, the function $E_{L,2N_S+1}$ oscillates along F from plus to minus so that Eq. (66) may be satisfied several times. Furthermore, the variation of N_{2N_S} as a function of F shows distinct positive and negative ranges for N_{2N_S}, to the left and the right side respectively of F_{real} (Fig. 12).

The findings above lead to a third criterion which may be expressed as

$$\left.\begin{array}{l} N_{2N_S} > 0 \\ E_{L,2N_S+1} > 0 \end{array}\right\} \quad F_p < F_{real} \qquad \text{(68a)}$$

and

$$\left.\begin{array}{l} N_{2N_S} \gtrless 0 \\ E_{L,2N_S+1} > 0 \end{array}\right\} \quad F_p > F_{real} \qquad \text{(68b)}$$

F_p = predicted factor of safety

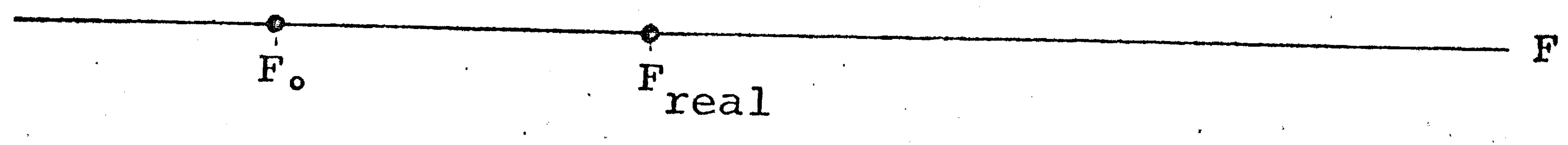
F_{real} = factor of safety at equilibrium state

hence, Eqs. (66), (67) reduces to: (68a) and (68b). Thus it is possible to predict whether $F_p < F_{real}$ or $F_p > F_{real}$. Note that the same criteria are valid for the dimensionless functions $\varepsilon_{L,i+1}$ and μ_i and therefore need no further discussion.

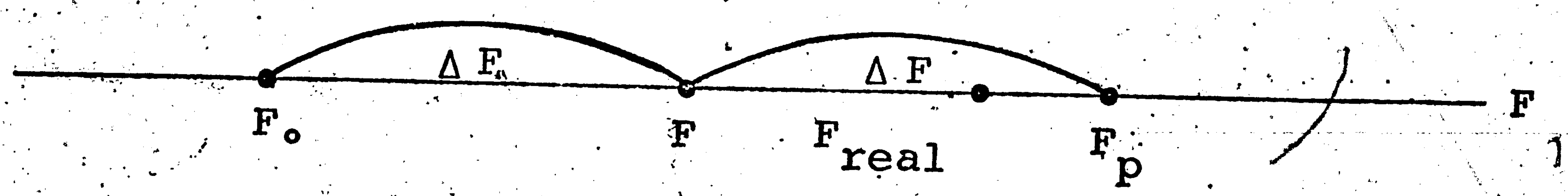
5.2 Procedure

The procedure for arriving at the desired factor of safety may be designed in many different ways. One such example is given diagrammatically below to contribute more to the understanding of the problem.

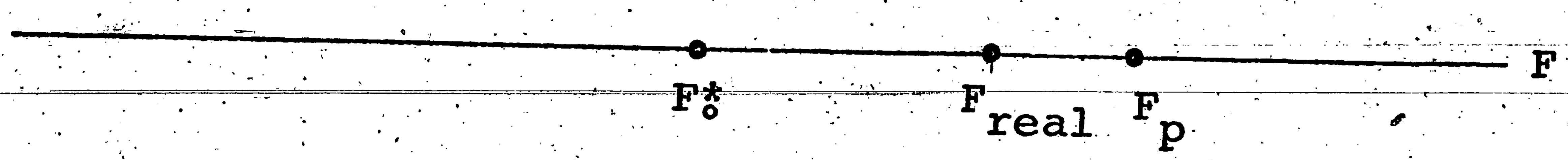
Step 1: choose initial value $F = F_o$



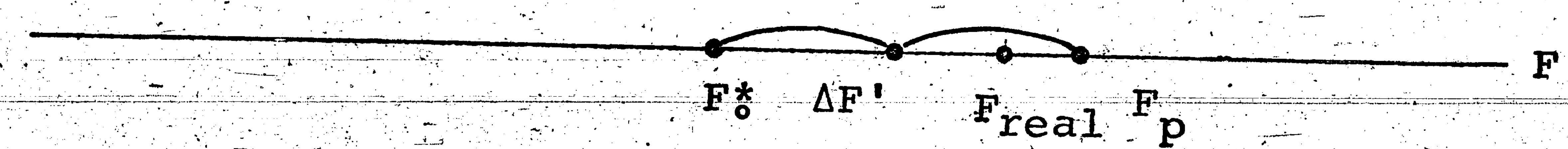
Step 2: choose increment ΔF, and increment F until $F_p > F_{real}$ (68b) (67)



Step 3: put $F_o = F_o{}^* = F$ which immediately follows F_p



Step 4: choose new increment $\Delta F' < \Delta F$ and increment on F until $F_p > F_{real}$ (68b) (67)



and repeat further steps analogically to the point where F lies at or within the required accuracy. F_{real} according to (68a), (68b) then must exist within the limits

$$F_p - \Delta F \leq F_{real} \leq F_p \qquad (69)$$

For reasons of simplicity there are only two increments indicated. The number of increments to be made with one interval depend on the position of the initial points relative to F_{real} as well as on the magnitude of the chosen intervals.

VI. CONCLUSION

It has been demonstrated that Lowe's method of analysis, which utilizes the two equilibrium equations with respect to vertical and horizontal forces respectively, can be suitably formulated on the computer. The theoretical implications in Lowe's approach are simple to understand. The solution, however, is not straight forward because of the trial and error process required to arrive at the desired factor of safety.

A strong point of the particular approach is, however, the fact that some aspects of "soil-water interaction" can be conveniently included in the analysis.

VII. FIGURES

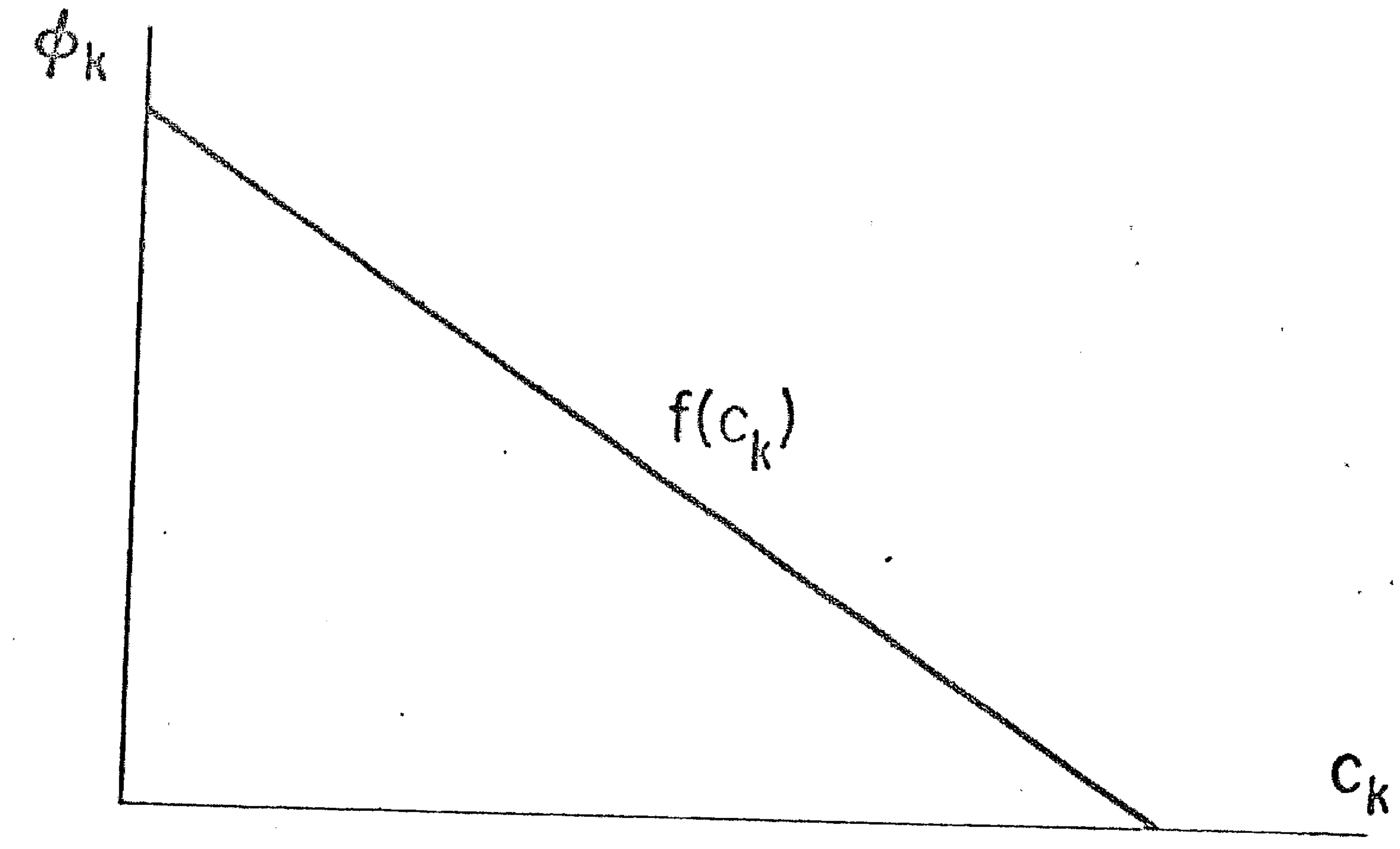


Fig. 1: Friction Angle Curve

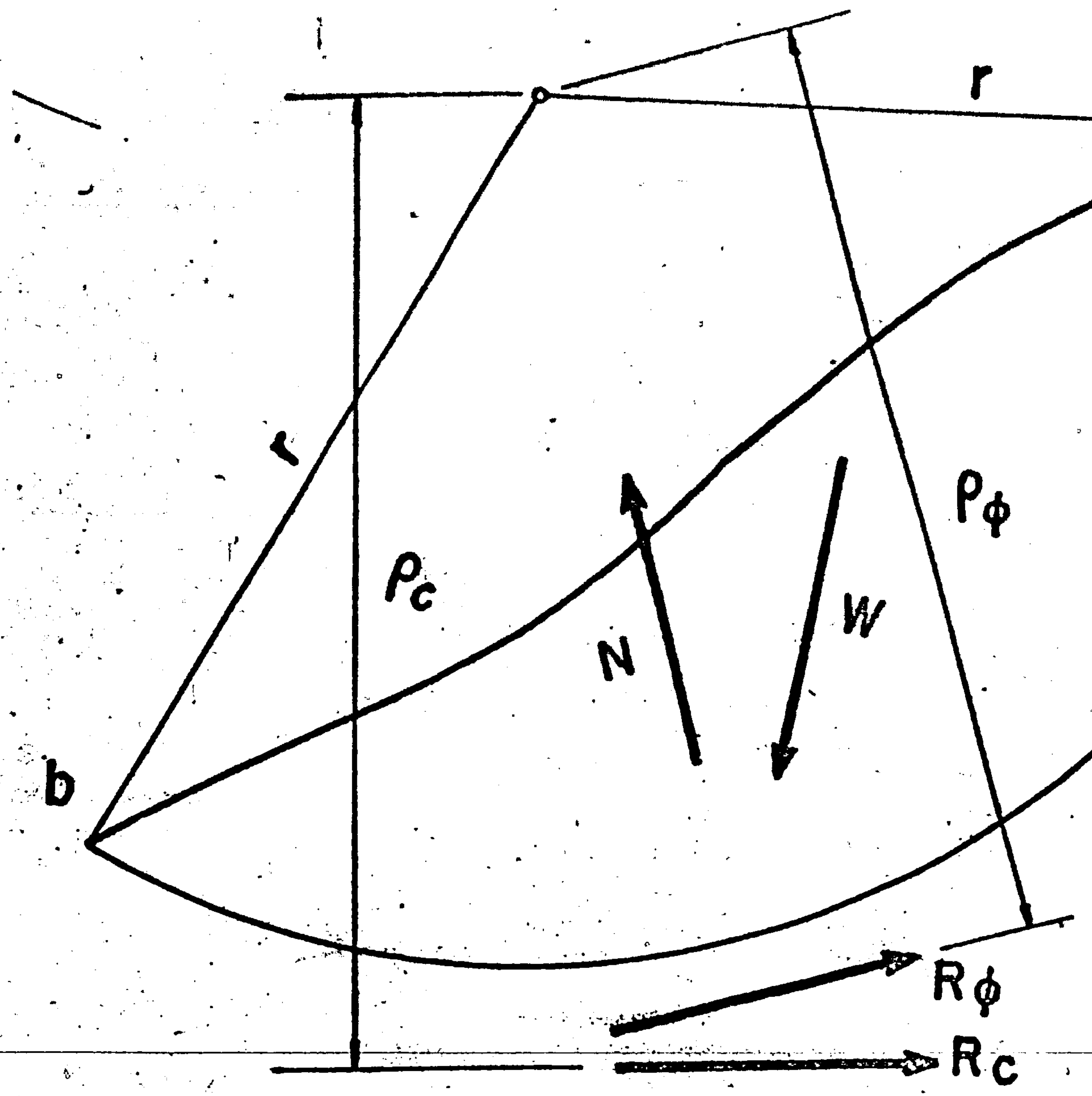


Fig. 2: Simple Failure Mechanism with Forces

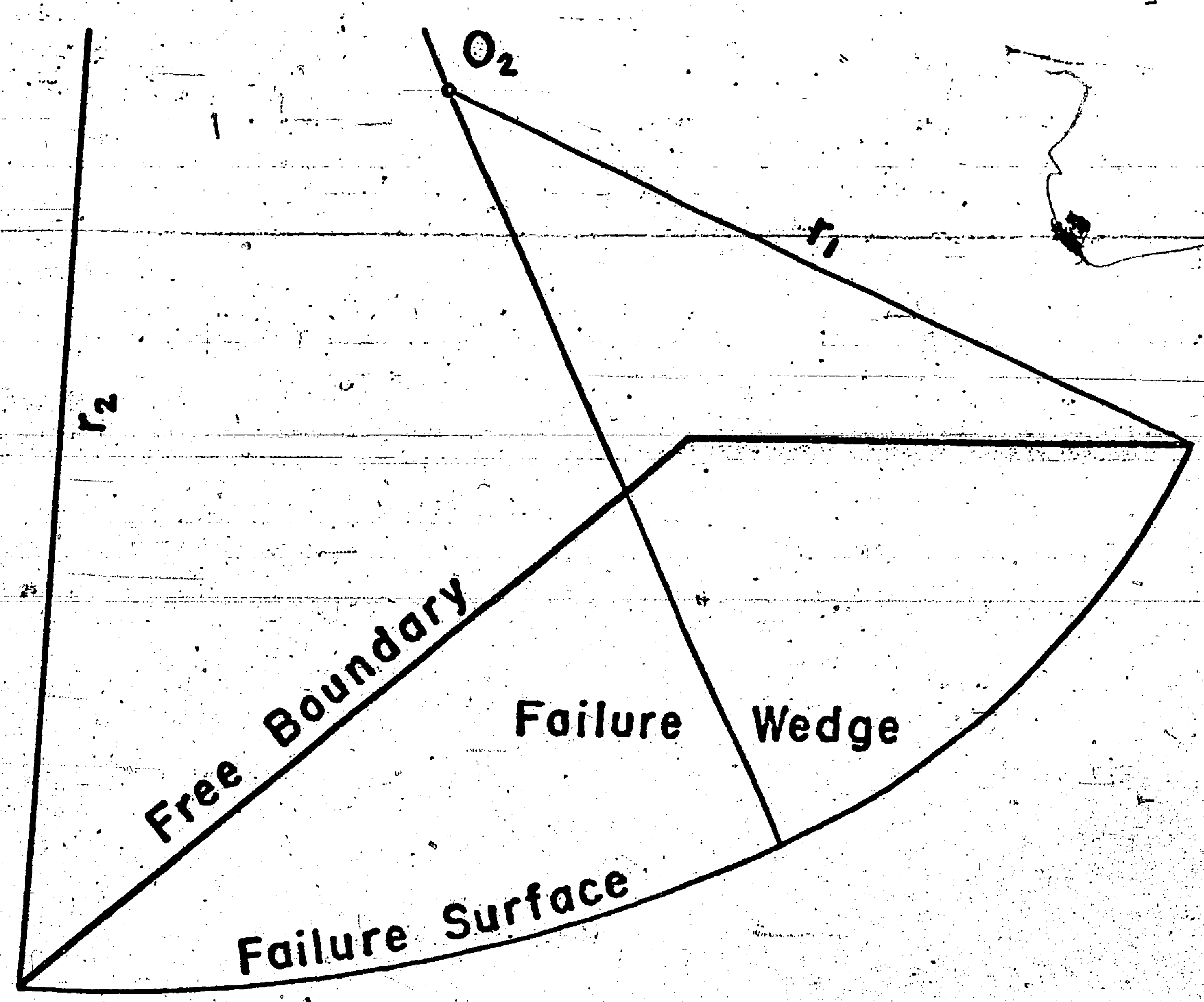


Fig. 3: Concept of Failure Mechanism for Method of Slices

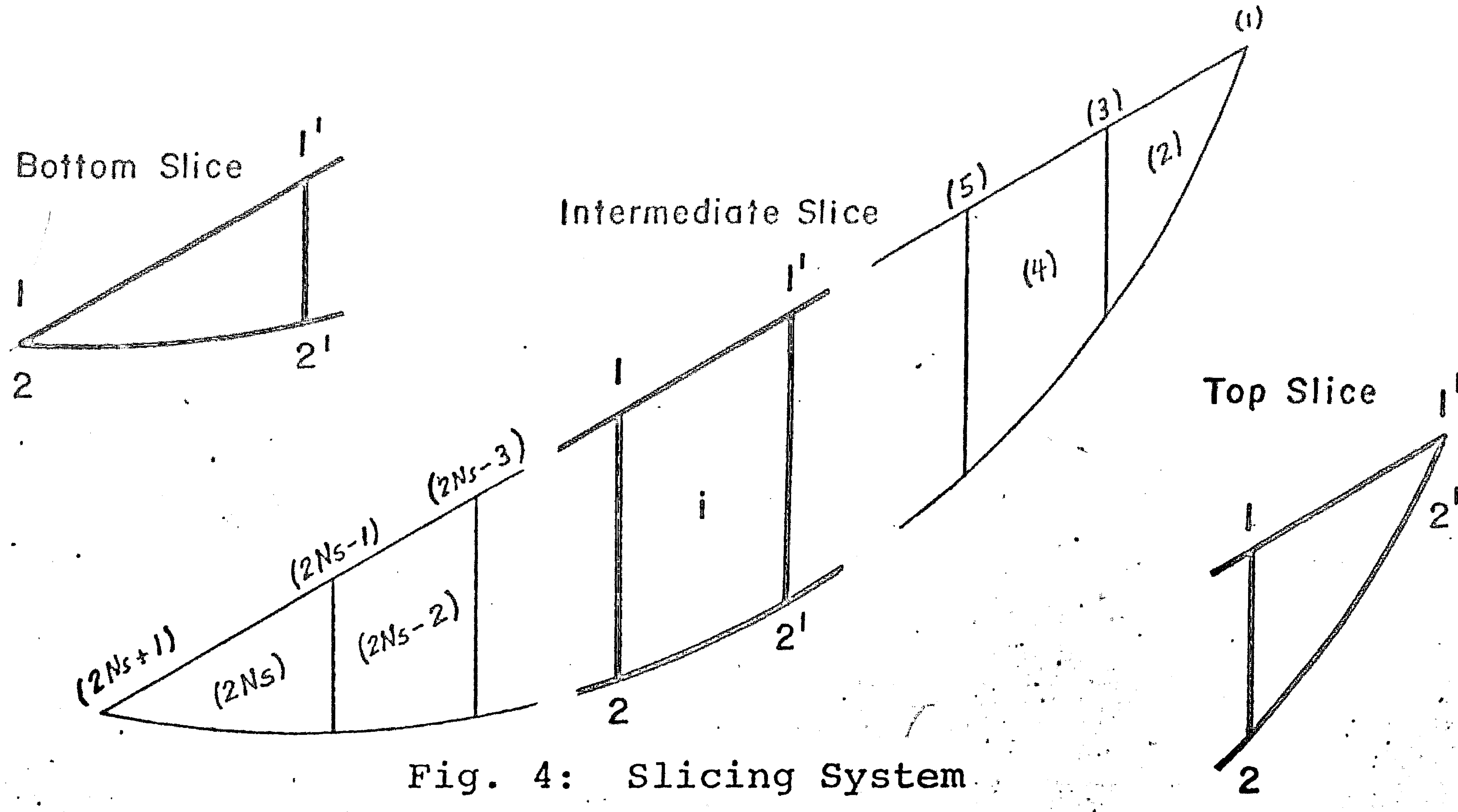


Fig. 4: Slicing System

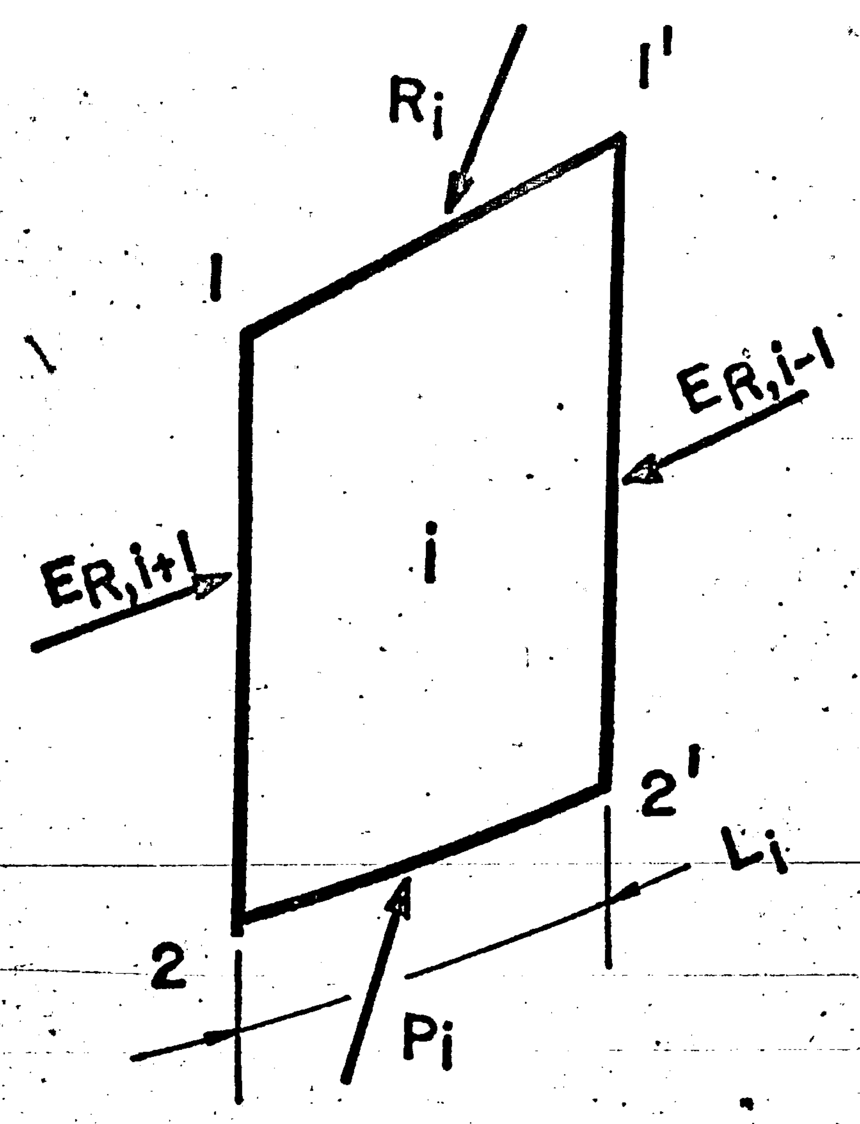


Fig. 5: Force System Slice i

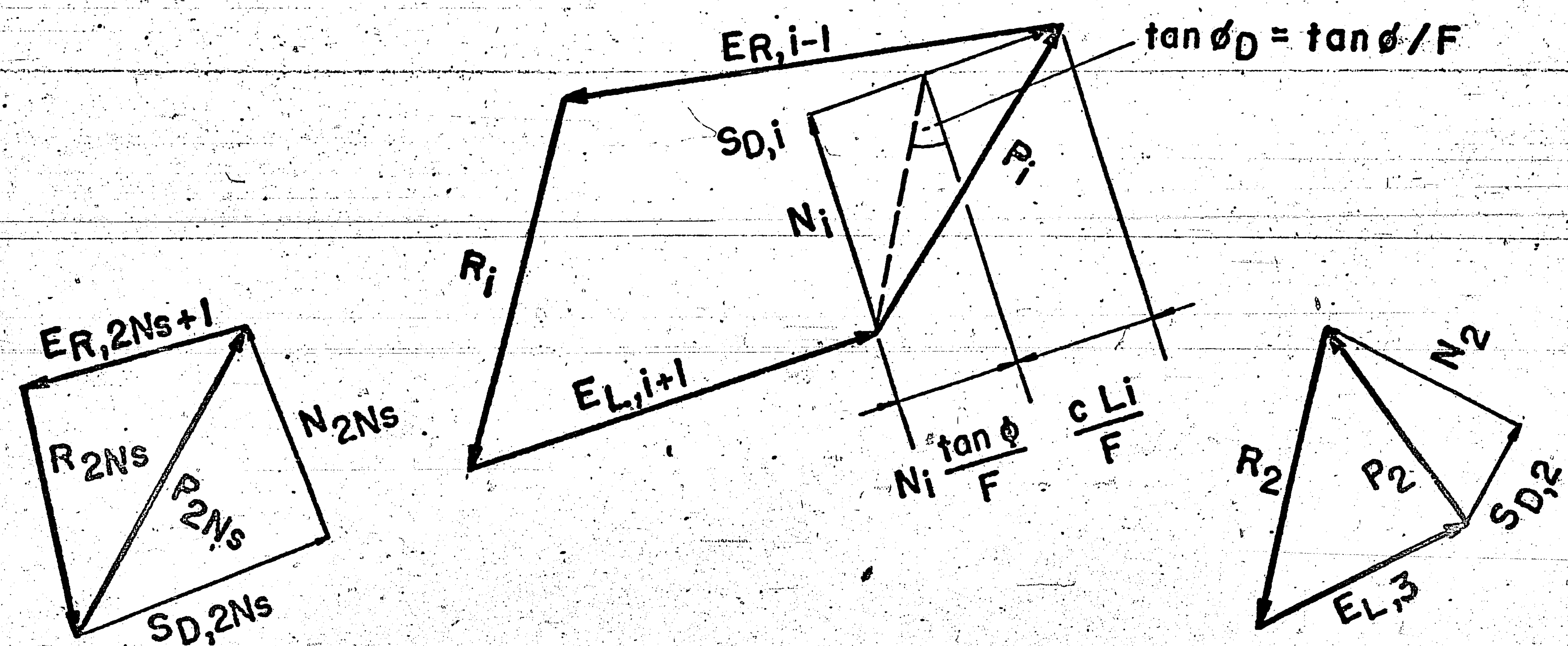


Fig. 6: Force Polygons

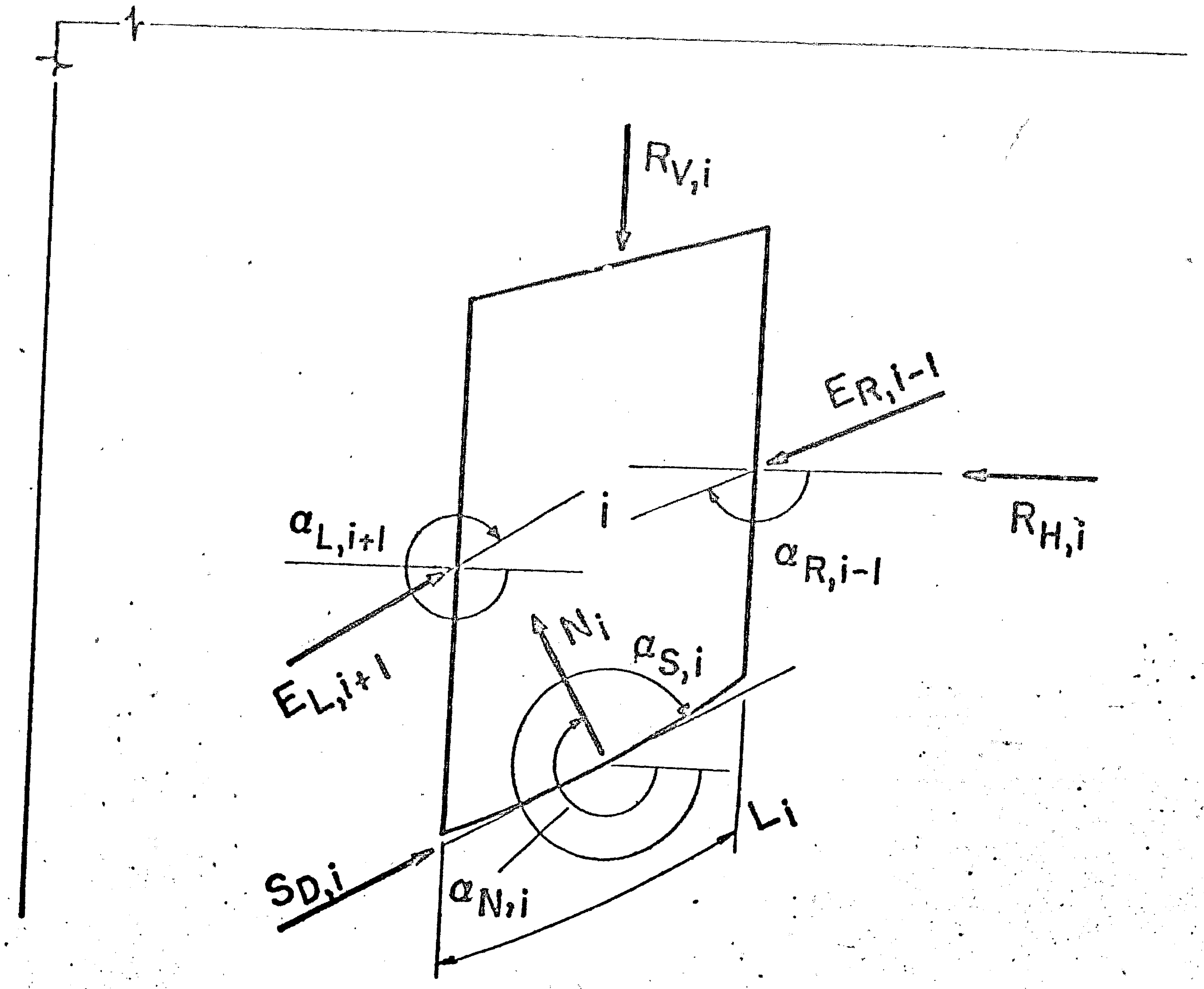


Fig. 7: Orientation of Slice Forces in x/y System as used in Analysis

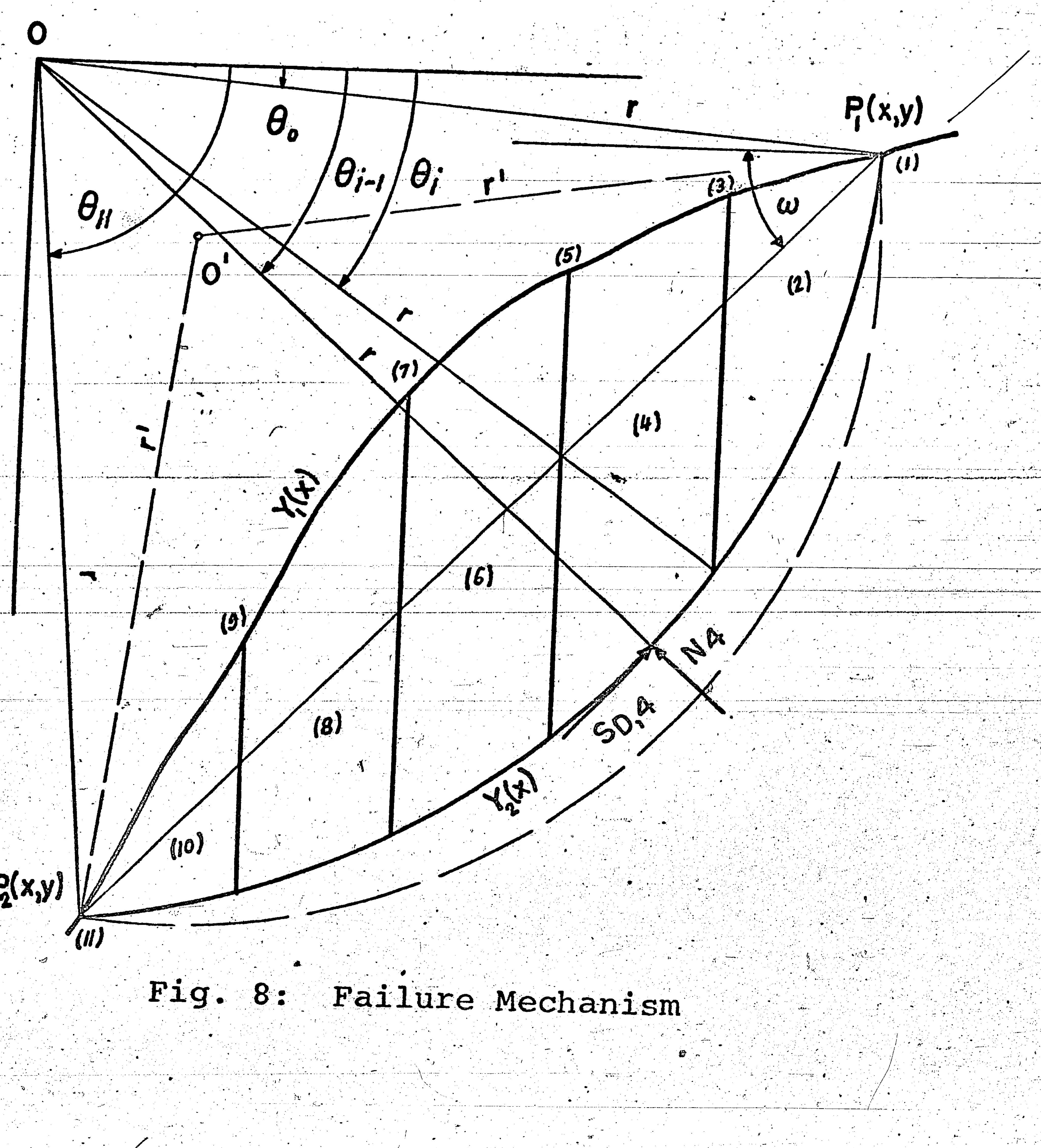


Fig. 8: Failure Mechanism

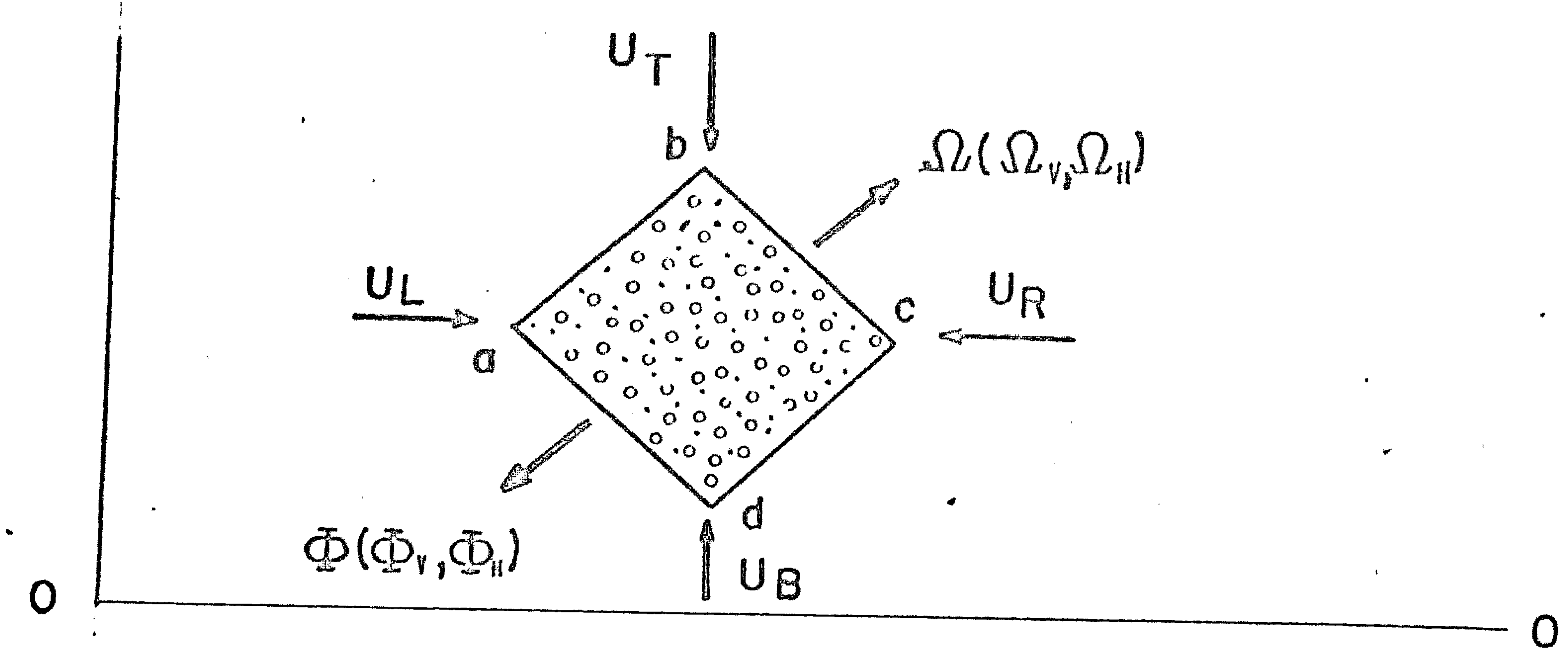


Fig. 9: Water Forces

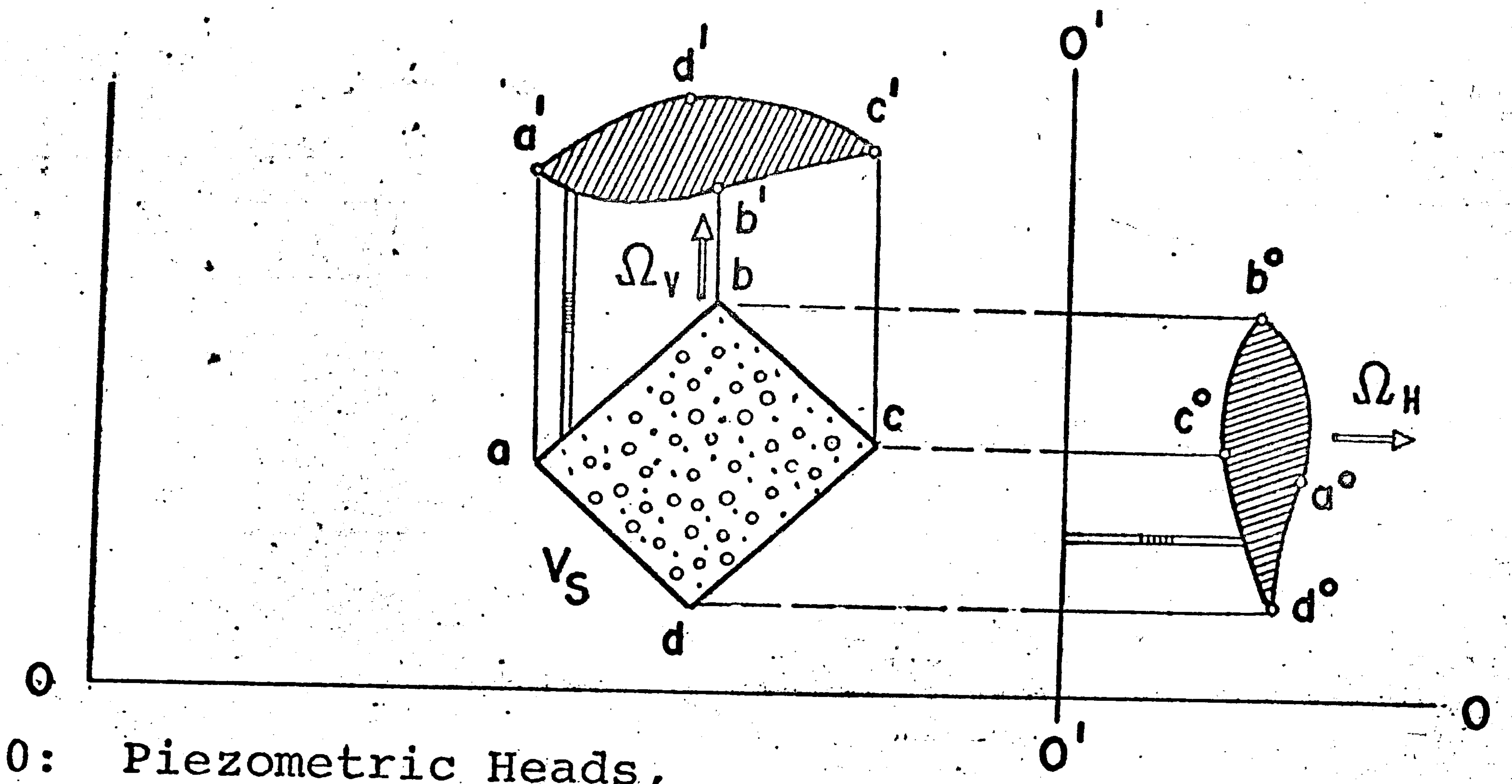


Fig. 10: Piezometric Heads,
h_V plotted along Contour of Soil Volume V_S
h_H plotted with Reference to o'-o'

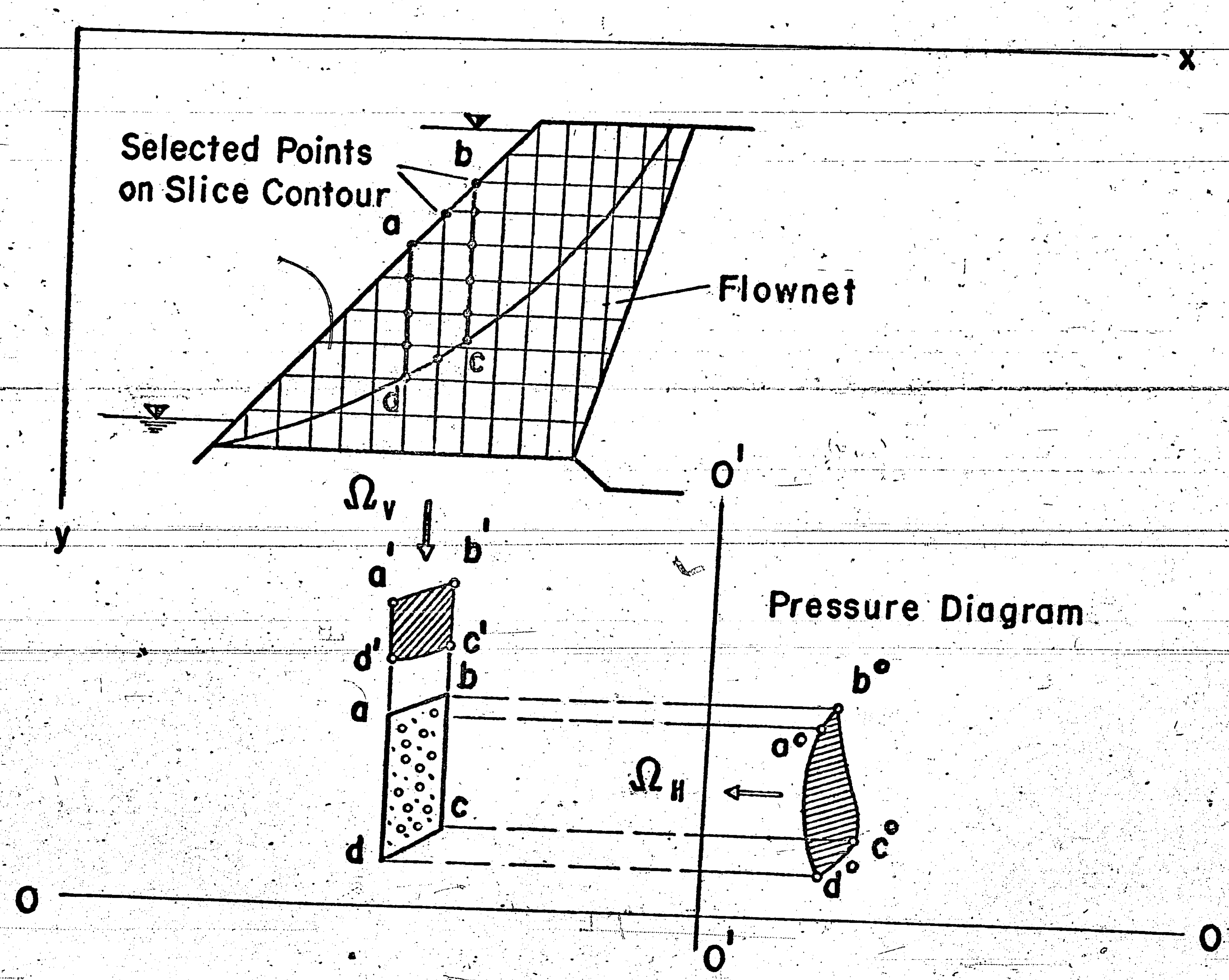


Fig. 11: Seepage through Dam on Upstream Side

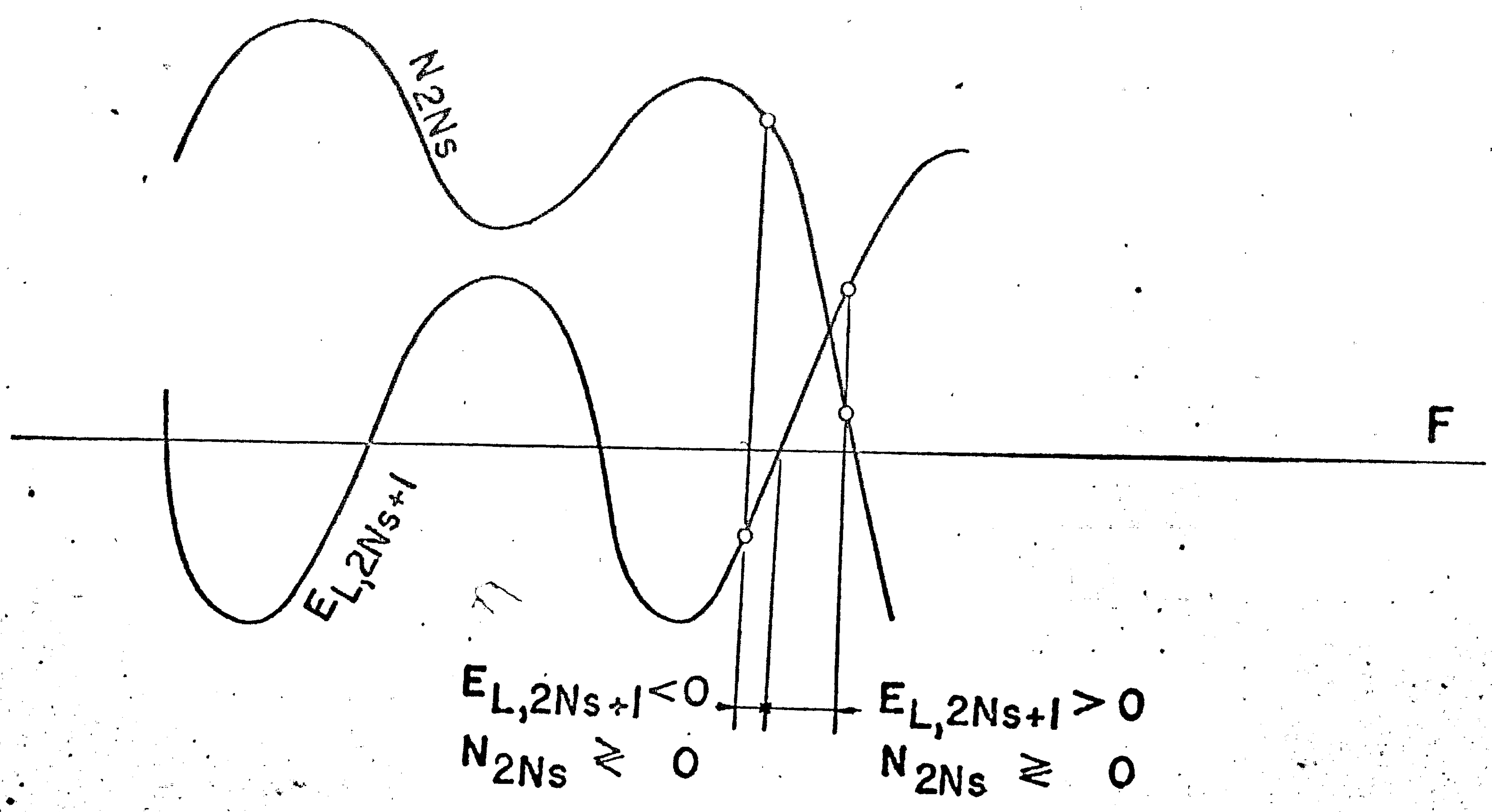


Fig. 12: Graph illustrating Criteria (68a) and (68b)

VIII. NOMENCLATURE

A	Coefficient matrix
A_T, A_B	Areas
$A_{a'b'c'd'a'}$	Area
$Aa°b°c°d°a°$	Area
B	Column matrix
C_D	Developed cohesion force
E_L, E_R	Side forces
$E_{L,\,i+1}$	Side forces at the boundary i+1 of slice i
$E_{R,i-1}$	Side force at the boundary i-1 of slice i
F	Factor of safety
F_p	Predicted factor of safety
F_o F_o^*	Improved factor of safety
F_{real}	Factor of safety at equilibrium state
ΔF	Increment on factor of safety
H	Characteristic height of embankment
L	Arc length
L_i	Arc length of slice i
N	Existing normal force on slip surface
N_k	Critical normal force on slip surface
N_i	Normal force at slip surface of slice i
N_c	Cohesion number
N_s	Total number of slices
P	Force resultant at slip surface
$P_{V,i}$	Vertical projection of P of slice i
$P_{H,i}$	Horizontal projection of P of slice i

R	Force resultant acting upon slice i
$R_{V,i}$	Vertical projection of R of slice i
$R_{H,i}$	Horizontal projection of R of slice i
R_{c_k}	Critical cohesive force with F equals unity
R_{ϕ_k}	Critical friction force with F equals unity
R_c	Resisting cohesive force
R_ϕ	Resisting friction force
S_D	Developed resisting shear force
$S_{D,i}$	Developed resisting shear force of slice i
U_T, U_B, U_R, U_L	Water forces
$U_{T,i}$	Vertical component of water force on top of slice 1
$U_{B,i}$	Vertical component of water force on bottom of slice i
$U_{L,i+1}$	Horizontal component of water force at the boundary i+1 of slice i
$U_{R,i+1}$	Horizontal component of water force at the boundary i-1 of slice i
V_s	Arbitrarily chosen soil volume
W	Weight of the soil mass
W_i	Weight of the soil mass of slice i
W_w	Weight of water within the arbitrarily chosen soil volume
Y_1	Function which describes the free boundary of the slope
Y_2	Function which describes the failure surface
abcd	Volume
ab	Chord
b_i's	Coefficient of B

c	Cohesion of the soil
c_k	Critical cohesion of the soil when $F = 1$
d	Distance from points P_1 to P_2
$f_{v,n}$	Geometrical function
$f_{H,i}$	Geometrical function
g_i's	Arbitrarily assigned functions
h	Potential head
i	Slice number
k_x, k_y	Permeabilities
l_{ij}'s	Direction cosines
n	Porosity
r, r_1, r_2	Radius vectors
x,y	Cartesian coordinates
x	Independent variable
x_{i+1}	Coordinate of slice boundary in x-direction i+1
x_{i-1}	Coordinate of slice boundary in x-direction i-1
$\alpha_{N,i}$	Direction of N_i of slice i
$\alpha_{S,i}$	Direction of $S_{D,i}$ of slice i
$\alpha_{L,i+1}$	Direction of side force $E_{L,i+1}$
$\alpha_{T,i+1}$	Slope of surface at the vertical boundary i+1 of slice i
$\alpha_{B,i+1}$	Slope of failure plane at the vertical boundary i+1 of slice i
$\alpha_{R,i-1}$	Direction of side force $E_{R,i-1}$
$\alpha_{T,i-1}$	Slope of surface at the vertical boundary i-1 of slice i

$\alpha_{B,i-1}$	Slope of failure plane at the vertical boundary i-1 of slice i
γ	Total unit weight of soil
γ_w	Unit weight of water
δ	Positive real number
$\varepsilon_{L,i+1}$	Reduced side force at the vertical boundary i+1 of slice i
$\varepsilon_{R,i-1}$	Reduced side force at the vertical boundary i-1 of slice i
μ	Approximate value for F
θ	Independent variable of the polar coordinate system
θ_{i-1}	Limit angle of vertical boundary i-1
θ_{i+1}	Limit angle of vertical boundary i+1
θ_i	Limit angle of vertical through center of gravity of slice i
θ_H, θ_O	Limit angles which bound the failure wedge
λ	Multiplier
μ_i	Reduced normal force at the failure surface of slice i
ξ	Dimensional parameter
$\S_c$	Radius vector normal to R_c
$\S_{ck}$	Radius vector normal to R_{ck}
$\S_\phi$	Radius vector normal R_ϕ
$\S_{\phi k}$	Radius vector normal $R_{\phi k}$
σ	Normal stress on the failure plane
σ_i	Normal stress on the failure plane of slice i
τ	Shear strength
ϕ	Internal friction angle of the soil

ϕ_o	Developed friction angle of the soil
ϕ_k	Critical friction angle when F = 1
w	Arbitrary angle
Δ	Slice width
Φ	Resulting force of soil on water
Φ_V	Vertical projection of
Φ_h	Horizontal projection of
$\psi(\theta)$	Function of θ
Ω	Waterforce
Ω_V	Vertical projection of
Ω_H	Horizontal projection of

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X. VITA

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He received his surveyor degree from the Swiss government in 1961. During 1960 and part of 1961 he served in the Swiss army. He enrolled at Technikum Winterthur, Zurich where he received the diploma in civil engineering in 1965.

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