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On the bending of rectilinearly anisotropic plates with cracks

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I. Abstract

Plate-bending problems of anisotropic plates with cracks are considered. The material of the medium is assumed to be homogeneous and rectilinearly anisotropic. The beharmonic equation for this problem is solved using the complex variable method developed by Lekhnitzkii.

The bending stresses near the crack tip possess singulalities of the order of $r^{-\frac{1}{2}}$, where r is the radial distance from the crack tip. The fracture angles at which the circumferential stress ahead of the crack tip becomes a maximum are calculated to exhibit the phenomenon of possible forking of the crack.

Two examples of fundamental interest are worked out.

II. Introduction

Many materials such as wood, reinforced concrete, rolled materials with grain orientation etc. are anisotropic in nature. While a number of previous publications [1, 2, 5] have considered the stress distribution around hole of various shapes in anisotropic plates, the plate bending problem of plates with crack-like imperfections has yet to be investigated. This dissertation is concerned with the determination of crack tip stress field in a rectilinearly anisotropic plate subjected to out-of-plane bending. The result are useful in the development of fracture theories.

It is well known that the stress field near a crack tip governs the onset of rapid crack propagation. This concept has been explored by Sih et al [6] for plane extension and bending problems of cracks in homogeneous and isotropic bodies.

In the case of cracked bodies possessing directional properties, Sih et al [7] have proposed a fracture criterion in consistent with the concept of stress-intensity factors for cracks in isotropic bodies. The results in [7], however, are valid only if the body is subjected to in-plane stretching and longitudinal shear loads. The order of the crack-tip stress singularities is

$r^{-\frac{1}{2}}$, r being the distance measured from the crack front, while the angular distribution of the stress depends upon the elastic constants of the anisotropic material. The inverse square-root stress singularity appears to be typical of all crack problem in which surface traction are prescribed. This behavior has ever-been observed in situations where the crack is along the bond line between two dissimilar materials (3).

In what follows, the problem of a through crack in an anisotropic plate of infinite extent will be formulated and solved. Special attention will be given to the bending stresses in the neighborhood of the crack point. The maximum circumferential stresses that cause branching of the crack are determined for certain values of the elastic constants.

III. Statement of the problem

Consider the problem of an infinite homogeneous and anisotropic plate, containing a through crack of finite length. The bending and twisting couples at infinity are to be specified. The crack configuration will be taken as the degenerate case of an elliptical opening. The problem will be solved in a rectangular cartesian coordinate system as shown in Fig. 1.

Using the Poisson-Kirchhoff theory of plate bending, the usual assumptions will be made.

a) Linear elements which are perpendicular to the mid-plane before deformation remain linear and perpendicular to that plane after deformation. See Fog. 2.

b) The elements of the mid-plane of the plate remain unstrained at all times.

Basic Equations in Anisotropic Elasticity.

The generalized Hooke law in terms of rectangular components of stress and strain is (5).

$$
\begin{aligned}
\epsilon_x &= C_{11}\sigma_x + C_{12}\sigma_y + C_{13}\sigma_z + C_{16}\tau_{xy} \\
\epsilon_y &= C_{12}\sigma_x + C_{22}\sigma_y + C_{23}\sigma_z + C_{26}\tau_{xy} \\
\epsilon_z &= C_{13}\sigma_x + C_{23}\sigma_y + C_{33}\sigma_z + C_{36}\tau_{xy} \\
\gamma_{yz} &= C_{44}\tau_{yz} + C_{45}\tau_{zx} \\
\gamma_{zx} &= C_{45}\tau_{yz} + C_{55}\tau_{zx} \\
\gamma_{xy} &= C_{16}\sigma_x + C_{26}\sigma_y + C_{36}\sigma_z + C_{66}\tau_{xy}
\end{aligned} \qquad (1)
$$

where,

$$C_{14} = C_{15} = C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = C_{56} = 0$$

as the problem possesses elastic symmetry with respect to the mid-plane of the plate. Solving for the stresses, eqs. (1) yield

$$
\begin{aligned}
\sigma_x &= A_{11}\epsilon_x + A_{12}\epsilon_y + A_{13}\epsilon_z + A_{16}\gamma_{xy} \\
\sigma_y &= A_{12}\epsilon_x + A_{22}\epsilon_y + A_{23}\epsilon_z + A_{26}\gamma_{xy} \\
\sigma_z &= A_{13}\epsilon_x + A_{23}\epsilon_y + A_{33}\epsilon_z + A_{36}\gamma_{xy} \\
\tau_{yz} &= A_{44}\gamma_{yz} + A_{45}\gamma_{zx} \\
\tau_{zx} &= A_{45}\gamma_{yz} + A_{55}\gamma_{zx} \\
\tau_{xy} &= A_{16}\epsilon_x + A_{26}\epsilon_y + A_{36}\epsilon_z + A_{66}\gamma_{xy}
\end{aligned} \qquad (2)
$$

For thin plates, the additional assumption of $\sigma_z = 0$ throughout the plate is introduced. This implies that

$$\epsilon_z = -\frac{1}{A_{33}}(A_{13}\epsilon_x + A_{23}\epsilon_y + A_{36}\gamma_{xy}) \quad (3)$$

The strains can be expressed in terms of the plate deflection w (x, y) as (Fig. 22).

$$\epsilon_x = -\delta\frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_y = -\delta\frac{\partial^2 w}{\partial y^2} \quad (4)$$

$$\gamma_{xy} = -2\delta\frac{\partial^2 w}{\partial x \partial y}$$

where δ is the thickness coordinate.

The stress components must satisfy the equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \quad (5)$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

On multiplying the first and second of eqs. (5) by z and then integrating all three equations from $-\frac{h}{2}$ to $\frac{h}{2}$ through the plate thickness, it is found that

$$\frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y} - N_x = 0$$

$$\frac{\partial H_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - N_y = 0 \qquad (6)$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

in which Mx, My are the bending moments, Hxy the twisting moment and Nx, Ny the shear stresses per unit length. By means of eqs. (2) and (3), these quantities can be expressed in terms of w (x,y):

$$M_x = -\frac{h^3}{12}\left(a_{11}\frac{\partial^2 w}{\partial x^2} + a_{12}\frac{\partial^2 w}{\partial y^2} + 2a_{16}\frac{\partial^2 w}{\partial x \partial y}\right)$$

$$M_y = -\frac{h^3}{12}\left(a_{12}\frac{\partial^2 w}{\partial x^2} + a_{22}\frac{\partial^2 w}{\partial y^2} + 2a_{26}\frac{\partial^2 w}{\partial x \partial y}\right)$$

$$H_{xy} = -\frac{h^3}{12}\left(a_{16}\frac{\partial^2 w}{\partial x^2} + a_{26}\frac{\partial^2 w}{\partial y^2} + 2a_{66}\frac{\partial^2 w}{\partial x \partial y}\right) \qquad (7)$$

$$N_x = -\frac{h^3}{12}\left[a_{11}\frac{\partial^3 w}{\partial x^3} + 3a_{16}\frac{\partial^3 w}{\partial x^2 \partial y} + (a_{12} + 2a_{66})\frac{\partial^3 w}{\partial x \partial y^2} + a_{26}\frac{\partial^3 w}{\partial y^3}\right]$$

$$N_y = -\frac{h^3}{12}\left[a_{16}\frac{\partial^3 w}{\partial x^3} + (a_{12} + 2a_{66})\frac{\partial^3 w}{\partial x^2 \partial y} + 3a_{26}\frac{\partial^3 w}{\partial x \partial y^2} + a_{22}\frac{\partial^3 w}{\partial y^3}\right]$$

where, $a_{ij} = A_{ij} - \frac{A_{i3}A_{3j}}{A_{33}}$, $a_{ij} = a_{ji}$, $A_{ij} = A_{ji}$

Inserting the last two of equations (7) into the third of equations (6), the differential equation governing the deflection of thin anisotropic plates is

$$a_{11}\frac{\partial^4 w}{\partial x^4} + 4a_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2\,(a_{12} + 2a_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2}$$

$$+ 4a_{26}\frac{\partial^4 w}{\partial x \partial y^3} + a_{22}\frac{\partial^4 w}{\partial y^4} = 0, \qquad (8)$$

whose roots can be computed from the characteristic equation

$$a_{22}\mu^4 + 4a_{26}\mu^3 + 2\,(a_{12} + 2a_{66})\mu^2 + 4a_{16}\,\mu + a_{11} = 0 \qquad (9)$$

In the case of general anisotropy these roots can be determined with the stipulation that they are not equal. The four distinct roots may be written as

$$\mu_1 = \alpha_1 + i\beta_1, \qquad \mu_2 = \alpha_2 + i\beta_2$$
$$\mu_3 = \bar{\mu}_1 \qquad\quad \mu_4 = \bar{\mu}_2 \qquad (10)$$

$$(\beta_1 > 0,\ \beta_2 > 0)$$

IV. Method of Solution

1. Complex representation

The general solution of w may be expressed in terms of two complex functions of z_1 and z_2 :

$$w(x,y) = 2\mathrm{Re}\,[F_1(z_1) + F_2(z_2)] \qquad (11)$$

where the variables z_1 and z_2 are related to x, y by the expressions

$$\begin{aligned} z_1 &= x + \mu_1 y = x + (\alpha_1 + i\beta_1)y \\ z_2 &= x + \mu_2 y = x + (\alpha_2 + i\beta_2)y \end{aligned} \qquad (12)$$

putting eq, (11) into eqs. (7) renders

$$\begin{aligned} M_x &= -\frac{h^3}{6}\mathrm{Re}\,[p_1\varphi'(z_1) + q_1\phi'(z_2)] \\ M_y &= -\frac{h^3}{6}\mathrm{Re}\,[p_2\phi'(z_1) + q_2\phi'(z_2)] \\ H_{xy} &= -\frac{h^3}{6}\mathrm{Re}\,[p_3\varphi'(z_1) + q_3\phi'(z_2)] \\ N_x &= -\frac{h^3}{6}\mathrm{Re}\,[\mu_1 p_4\varphi'(z_1) + \mu_2 q_4\phi'(z_2)] \\ N_y &= -\frac{h^3}{6}\mathrm{Re}\,[p_1\varphi'(z_1) + q_4\phi'(z_2)] \end{aligned} \qquad (13)$$

where,

$$p_1 = a_{11} + a_{12}\mu^{2} + 2\mu_1 a_{16}$$

$$p_2 = a_{12} + a_{22}\mu_1^2 + 2\mu_1 a_{26}$$

$$p_3 = a_{16} + a_{26}\mu_1^2 + 2\mu_1 a_{66}$$

$$p_4 = \frac{a_{11}}{\mu_1} + 3a_{16} + \mu_1(a_{12} + 2a_{66}) + a_{26}\mu_1^2$$

$$q_1 = a_{11} + a_{12}\mu_2^2 + 2\mu_2 a_{16} \qquad (14)$$

$$q_2 = a_{12} + a_{22}\mu_2^2 + 2\mu_2 a_{26}$$

$$p_3 = a_{16} + a_{26}\mu_2^2 + 2\mu_2 a_{66}$$

$$q_4 = \frac{a_{11}}{\mu_2} + 3a_{16} + \mu_2(a_{12} + 2a_{66}) + a_{26}\mu_2^2$$

$$\varphi(z_1) = \frac{dF_1(z_1)}{dz_1}$$

$$\phi(z_2) = \frac{dF_2(z_2)}{dz_2}$$

2. The first fundamental problem

The classical theory of thin plates requires that the bending moment m(s) and the equivalent shear force p(s) per unit length of the plate contour are known over a portion or the entire contour of the plate.

From (1), the boundary conditions for the first fundamental problem are

$$\begin{aligned} &\mathrm{Re}\left[\frac{p_1}{\mu_1}\varphi(z_1) + \frac{q_1}{\mu_2}\psi(z_2)\right] = f_1 \\ &\mathrm{Re}\left[p_2\varphi(z_1) + q_2\psi(z_2)\right] = f_2 \end{aligned} \tag{15}$$

where,

$$\begin{aligned} f_1 &= -\frac{6}{h^3}\int_0^s \left[m(s)dy + f(s)dx\right] - cx + c_1 \\ f_2 &= -\frac{6}{h^3}\int_0^s \left[m(s)dx - f(s)dy\right] + cy + c_2 \end{aligned} \tag{16}$$

and,

$$\begin{aligned} f(s) &= \int_0^s p(s)ds \\ p(s) &= Nn + \frac{\partial Hnt}{\partial s} \quad \text{Kirchhoff's condition} \\ m(s) &= Mn \end{aligned} \tag{17}$$

The subscripts n and t refer to the normal and transverse directions, respectively. See Fig. 3. If rectangular coordinates are used, it is obvious that

$$p(s) = Ny + \frac{\partial Hxy}{\partial x}, \quad m(s) = My \tag{18}$$

3. The properties of $\varphi(z_1)$ and $\psi(z_2)$

The functions $\varphi(z_1)$ and $\psi(z_2)$ for a simply-connected region such as an infinite plate weakened by a crack or hole of some kind have been derived in [2]. If the moments Mx, My and Hxy at the infinity are uniform, the functions $\varphi(z_1)$ and $\psi(z_2)$ take the forms

$$\begin{aligned} \varphi(z_1) &= A \log z_1 + B^* z_1 + \varphi_o(z_1) \\ \psi(z_2) &= B \log z_2 + (B'^* + iC'^*) z_2 + \psi_o(z_2) \end{aligned} \tag{19}$$

where

$$\begin{aligned} \varphi_o(z_1) &= \sum_{n=o}^{\infty} \frac{a_n}{z_1^n} = a_o + \frac{a_1}{z_1} + \frac{a_2}{z_1^2} + \ldots \\ \psi_o(z_2) &= \sum_{n=o}^{\infty} \frac{b_n}{z_2^n} = b_o + \frac{b_1}{z_2} + \frac{b_2}{z_2} + \ldots \end{aligned} \tag{20}$$

are holomorphic everywhere in the complex plane. The constants A and B are complex, whereas B^*, B'^* and C'^* are real constants. The coefficients a_o and b_o can be set to zero for the first fundamental problems.

If the tractions on the edge of the hole zero or self-equilibrating, the constants A and B vanish identically.

Hence, $\varphi(z_1)$ and $\varphi(z_2)$ can be written as

$$\varphi(z_1) = B^* z_1 + \varphi_o(z_1)$$
$$\varphi(z_2) = (B'^* + iC'^*) z_2 + \varphi_o(z_2) \qquad (21)$$

The constants B', B'* and C'* can be related to the prescribed moments Mx, My and Hxy at the infinity. Combining eqs. (13) and (21) for large values of $|z_1|$ and $|z_2|$, it is found that

$$-\frac{6}{h^3} Mx^\infty = Re[p_1 B^* + q_1 (B'^* + iC'^*)]$$
$$-\frac{6}{h^3} My^\infty = Re[p_2 B^* + q_2 (B'^* + iC'^*)] \qquad (22)$$
$$-\frac{6}{h^3} Hxy^\infty = Re[p_3 B^* + q_3 (B'^* + iC'^*)]$$

Upon defining

$$p_j = p_j^{(1)} + ip_j^{(2)} \qquad (j = 1, 2, 3) \qquad (23)$$
$$q_j = q_j^{(1)} + iq_j^{(2)}$$

equations (22) become

$$-\frac{6}{h^3} Mx^\infty = p_1^{(1)} B^* + q_1^{(1)} B'^* - q_1^{(2)} C'^*$$
$$-\frac{6}{h^3} My^\infty = p_2^{(1)} B^* + q_2^{(1)} B'^* - q_2^{(2)} C'^* \qquad (24)$$

$$-\frac{6}{h^3}\,Hxy^{\infty} = p_3^{(1)}B^* + q_3^{(1)}B'^* - q_3^{(2)}C'^* \tag{24}$$

The three non-homogeneous algebraic equations may be solved to render the three unknowns B*, B'* and C'*. The result is

$$B^* = \frac{6}{\Delta h^3}\begin{vmatrix} Mx^{\infty} & q_1^{(1)} & q_1^{(2)} \\ My^{\infty} & q_2^{(1)} & q_2^{(2)} \\ Hxy^{\infty} & q_3^{(1)} & q_3^{(2)} \end{vmatrix}$$

$$B'^* = \frac{6}{\Delta h^3}\begin{vmatrix} p_1^{(1)} & Mx^{\infty} & q_1^{(2)} \\ p_2^{(1)} & My^{\infty} & q_2^{(2)} \\ p_3^{(1)} & Hxy^{\infty} & q_3^{(2)} \end{vmatrix}$$

$$C'^* = \frac{6}{\Delta h^3}\begin{vmatrix} p_1^{(1)} & q_1^{(1)} & Mx \\ p_2^{(1)} & q_2^{(1)} & My \\ p_3^{(1)} & q_3^{(1)} & Mxy \end{vmatrix}$$

where Δ stands for the determinant

$$\Delta = -\begin{vmatrix} p_1^{(1)} & q_1^{(1)} & q_1^{(2)} \\ p_2^{(1)} & q_2^{(1)} & q_2^{(2)} \\ p_3^{(1)} & q_3^{(1)} & q_3^{(2)} \end{vmatrix} \qquad \Delta \neq 0$$

4. Schwartz formula

The elliptical hole with semi-axes a and b on ox and oy axes each as shown in Fig. 4 will be examined.

The regions outside the ellipse, $S^{(j)}$ (j = 1, 2) in the z_j-planes correspond to the inside of the unit circles γj by (2, 4)

$$\begin{aligned} z1 = \omega_1(\zeta) &= \frac{a + i\mu_1 b}{2}\zeta + \frac{a - i\mu_1 b}{2}\cdot\frac{1}{\zeta} \\ z_2 = \omega_2(\zeta) &= \frac{a + i\mu_2 b}{2}\zeta + \frac{a - i\mu_2 b}{2}\cdot\frac{1}{\zeta} \end{aligned} \qquad (26)$$

For this problem, the boundary conditions given by eq. (15) becomes

$$\begin{aligned} &\mathrm{Re}\left[\frac{p_1}{\mu_1}\varphi_o(z1) + \frac{q_1}{\mu_2}\phi_o(z_2)\right] = f_1^o \\ &\mathrm{Re}\,(p_2\,\varphi_o(z1) + q_2\,\phi_o(z_2)) = f_2^o \end{aligned} \qquad \text{at the boundary} \quad (27)$$

where,

$$\begin{aligned} f_1^o &= f_1 - \mathrm{Re}\left\{\frac{p_1}{\mu_1}A\log z1 + \frac{q_1}{\mu_2}B\log z_2 + \frac{p_1}{\mu_1}B^* z_1 + \frac{q_1}{\mu_2}(B'^* + iC'^*)z_2\right\} \\ f_2^o &= f_2 - \mathrm{Re}\left\{p_2 A\log z_1 + q_2 B\log z_2 + p_2 B^* z_1 + q_2(B'^* + iC'^*)z_2\right\} \end{aligned} \qquad (28)$$

In eqs. (28), f_1 and f_2 are given by eqs. (16). If no external forces are applied to the contour of the hole, then $A = B = o$, $f_1 = f_2 = o$.

By changing the variables z_j ($j = 1, 2$) in eqs. (27) to the variable σ which is the value of ζ on the unit circle defined in eqs. (27), the boundary conditions may be arranged in the forms

$$\begin{aligned} &\mathrm{Re}\left[\frac{p_1}{\mu_1}\Phi_o(\sigma) + \frac{q_1}{\mu_2}\Psi_o(\sigma)\right] = f_1^o(\theta) \\ &\mathrm{Re}\left[p_2\Phi_o(\sigma) + q_2\Psi_o(\sigma)\right] = f_2^o(\theta) \end{aligned} \tag{29}$$

where,

$$\Phi_o(\zeta) = \varphi_o[\omega_1(\zeta)], \Psi_o(\zeta) = \phi_o[\omega_2(\zeta)] \tag{30}$$

The functions $\Phi(\zeta)$ and $\Psi(\zeta)$, which are holomorphic inside the unit circle γ, may be determined from eqs. (29) by means of the following Schwartz formula [4]:

$$F(\zeta) = \frac{1}{2\pi i}\int_\gamma f(\theta)\frac{\sigma+\zeta}{\sigma-\zeta}\frac{d\sigma}{\sigma} + i\alpha_o \tag{31}$$

where $f(\theta)$ is the value of the real part of the function $F(\zeta)$ on γ, and α_o is a real constant.

Applying eq. (31) to eqs. (29) gives

$$\frac{p_1}{\mu_1}\Phi_0(\zeta) + \frac{q_1}{\mu_2}\Psi_0(\zeta) = \frac{1}{2\pi i}\int_\gamma f_1^0(\theta)\frac{\sigma+\zeta}{\sigma-\zeta}\frac{d\sigma}{\sigma} + i\alpha_0$$

$$p_2\Phi_0(\zeta) + q_2\Psi_0(\zeta) = \frac{1}{2\pi i}\int_\gamma f_2^0(\theta)\frac{\sigma+\zeta}{\sigma-\zeta}\frac{d\sigma}{\sigma} + i\beta_0 \qquad (32)$$

Noting the following relations of integration,

$$\int_\gamma \sigma\frac{\sigma+\zeta}{\sigma-\zeta}\frac{d\sigma}{\sigma} = 4\pi i\zeta$$

$$\int_\gamma \bar{\sigma}\frac{\sigma+\zeta}{\sigma-\zeta}\frac{d\sigma}{\sigma} = 0$$

$\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ can be solved using the simultaneous equations (32).

Hence,

$$\Phi_0(\zeta) = \frac{\mu_1\zeta}{2(p_1q_2\mu_2 - p_2q_1\mu_1)}\Big\{B^*[(a+i\mu_1 b)(p_2q_1 - \frac{\mu_2 p_1 q_2}{\mu_1}) + (a+i\bar{\mu}_1 b)(q_1\bar{p}_2 - \frac{\mu_2\bar{p}_1q_2}{\bar{\mu}_1})] + (B'^* - iC'^*) \times (a+i\bar{\mu}_2 b)(q_1\bar{q}_2 - \frac{\bar{q}_1 q_2\mu_2}{\bar{\mu}_2})\Big\} + \lambda_1$$

$$\psi_0(\zeta) = -\frac{\mu_2\zeta}{2(p_1q_2\mu_2 - p_2q_1\mu_1)}\Big\{B^*(a+i\bar{\mu}_1 b)(p_1\bar{p}_2 - \frac{\mu_1\bar{p}_1p_2}{\bar{\mu}_1}) + (B'^* + iC'^*)(a+i\mu_2 b)(p_1q_2 - \frac{p_2q_1\mu_1}{\mu_2}) + (B'^* - iC'^*)(a+i\bar{\mu}_2 b)(p_1\bar{q}_2 - \frac{p_2\bar{q}_1\mu_1}{\bar{\mu}_2})\Big\} + \lambda_2 \qquad (33)$$

where

$$\lambda_1 = \frac{i\mu_1(\alpha_0 q_2\mu_2 - \beta_0 q_1)}{p_1 q_2\mu_2 - p_2 q_1\mu_1}, \qquad \lambda_2 = \frac{i\mu_2(\beta_0 p_1 - \alpha_0 p_2\mu_1)}{p_1 q_2\mu_2 - p_2 q_1\mu_1}$$

In the limit as $b \to o$, the complex functions for a line crack of length $2a$ are obtained:

$$\begin{aligned} \phi_o(\zeta) &= N_1\zeta + \text{constant} \\ \psi_o(\zeta) &= N_2\zeta + \text{constant} \end{aligned} \tag{34}$$

where

$$\frac{N_1}{a} = \frac{\mu_1}{2(p_1 q_2\mu_2 - p_2 q_1\mu_1)}\left\{B^*\left[\left(p_1 q_2 - \frac{p_1 q_2\mu_2}{\mu_1}\right) + \left(\bar{p}_2 q_1 - \frac{\bar{p}_1 q_2\mu_2}{\bar{\mu}_1}\right)\right]\right.$$
$$\left. + (B'^* - iC'^*)\left(q_1\bar{q}_2 - \frac{\bar{q}_1 q_2\mu_2}{\bar{\mu}_2}\right)\right\} \tag{35}$$

$$\frac{N_2}{a} = -\frac{\mu_2}{2(p_1 q_2\mu_2 - p_2 q_1\mu_1)}\left\{B^*\left(p_1\bar{p}_2 - \frac{\mu_1\bar{p}_1 p_2}{\bar{\mu}_1}\right) + (B'^* + iC'^*)\right.$$
$$\left. \times\left(p_1 q_2 - \frac{p_2 q_1\mu_1}{\mu_2}\right) + (B'^* - iC'^*)\left(p_1\bar{q}_2 - \frac{p_2\bar{q}_1\mu_1}{\bar{\mu}_2}\right)\right\}$$

Using the inverse of the mapping function

$$\zeta = \frac{a}{z_j + \sqrt{z_j^2 - a^2}} \tag{36}$$

the functions

$$\begin{aligned} \varphi_o(z_1) &= \frac{N_1 a}{z_1 + \sqrt{z_1^2 - a^2}} + \text{constant} \\ \varphi_o(z_2) &= -\frac{N_2 a}{z_2 + \sqrt{z_2^2 - a^2}} + \text{constant} \end{aligned} \tag{37}$$

are obtained. Note that $\varphi_o(z_1)$ and $\psi_o(z_2)$ become infinitely large as $|z_j| \to a$, i.e., at the crack tips.

5. The forms of $\varphi(z_1)$ and $\phi(z_2)$ and moments

Substituting eqs. (37) into eqs. (21), the final forms of $\varphi(z_1)$ and $\phi(z_2)$ are found:

$$\varphi(z_1) = B^* z_1 + \frac{N_1 a}{z_1 + \sqrt{z_1^2 - a^2}} + \text{constant}$$
$$\phi(z_2) = (B'^* + iC'^*) z_2 - \frac{N_2 a}{z_2 + \sqrt{z_2^2 - a^2}} + \text{constant} \qquad (38)$$

Once the complex functions are known, the bending and twisting moments follow immediately from the relations

$$Mx = Mx^{\infty} - \frac{h^3}{6} \operatorname{Re}\left[p_1 \varphi_0'(z_1) + q_1 \phi_0'(z_2)\right]$$
$$My = My^{\infty} - \frac{h^3}{6} \operatorname{Re}\left[p_2 \varphi_0'(z_1) + q_2 \phi_0'(z_2)\right] \qquad (39)$$
$$Hxy = Hxy^{\infty} - \frac{h^3}{6} \operatorname{Re}\left[p_3 \varphi_0'(z_1) + q_3 \phi_0'(z_2)\right]$$

in which

$$\varphi_0'(z_1) = \frac{N_1}{a}\left[1 - \frac{z_1}{\sqrt{z_1^2 - a^2}}\right]$$
$$\phi_0'(z_2) = -\frac{N_2}{a}\left[1 - \frac{z_2}{\sqrt{z_2^2 - a^2}}\right] \qquad (40)$$

6. Crack-tip stress field

A knowledge of the crack-tip stress field is pertinent to the formulation of fracture theories. To facilitate the analysis polar coordinate (r, θ) measured from the crack tip, as shown in Fig. 5, will be introduced, i.e.,

$$
\begin{aligned}
z_1 &= a + r\,(\cos\theta + \mu_1 \sin\theta) \\
z_2 &= a + r\,(\cos\theta + \mu_2 \sin\theta)
\end{aligned}
\tag{41}
$$

Substituting eqs. (41) into eqs. (40) for values of $\frac{r}{a} \ll 1$, $\varphi_0(z_1)$ and $\phi_0'(z_2)$ may be approximated by

$$
\begin{aligned}
\varphi_0'(z_1) &\cong -\frac{N_1}{a}\sqrt{\frac{a}{2r}}\,\frac{1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} \\
\phi_0'(z_2) &\cong \frac{N_2}{a}\sqrt{\frac{a}{2r}}\,\frac{1}{\sqrt{\cos\theta + \mu_1 \sin\theta}}
\end{aligned}
\tag{42}
$$

It follows that the moments close to the crack tip become

$$
\begin{aligned}
Mx &= \sqrt{\frac{a}{2r}}\,\frac{h^3}{6}\,\mathrm{Re}\left[\frac{p_1\frac{N_1}{a}}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{q_1\frac{N_2}{a}}{\sqrt{\cos\theta + \mu_2 \sin\theta}}\right] + Mx^{\infty} \\
My &= \sqrt{\frac{a}{2r}}\,\frac{h^3}{6}\,\mathrm{Re}\left[\frac{p_2\frac{N_1}{a}}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{q_2\frac{N_2}{a}}{\sqrt{\cos + \mu_2 \sin\theta}}\right] + My^{\infty} \\
Hxy &= \sqrt{\frac{a}{2r}}\,\frac{h^3}{6}\,\mathrm{Re}\left[\frac{p_3\frac{N_1}{a}}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{q_3\frac{N_2}{a}}{\sqrt{\cos\theta + \mu_2 \sin\theta}}\right] + Hxy^{\infty}
\end{aligned}
\tag{43}
$$

The bending stresses are distributed linearly through the thickness of the plate and, are related to Mx, My and Hxy as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \frac{12\delta}{h^3} \begin{Bmatrix} Mx \\ My \\ Mz \end{Bmatrix}$$

V. Special Cases

1. Orthotropic material

When the planes of elastic symmetry coincide with the coordinate axes x, y and z, the material is said to be orthotropic. In such a case, $A_{16} = A_{26} = A_{36} = 0$ in eq. (2) and hence $a_{16} = a_{26} = 0$ in eqs. (7) and (8). The characteristic equation (9) simplifies to

$$a_{22}\mu^4 + 2(a_{12} + 2a_{66})\mu^2 + a_{11} = 0 \tag{45}$$

The roots of this equation are

$$\mu_1 = \alpha + i\beta, \qquad \mu_2 = -\alpha + i\beta, \qquad \mu_3 = \bar{\mu}_1, \qquad \mu_4 = \bar{\mu}_2 \tag{46}$$

in which

$$\begin{aligned} \alpha &= \frac{1}{2}\left[-\frac{a_{12}+2a_{66}}{a_{22}} + \sqrt{\frac{a_{11}}{a_{22}}}\right]^{\frac{1}{2}} \\ \beta &= \frac{1}{2}\left[\frac{a_{12}+2a_{66}}{a_{22}} + \sqrt{\frac{a_{11}}{a_{22}}}\right]^{\frac{1}{2}} \end{aligned} \tag{47}$$

Under these considerations, p_j and q_j ($j = 1, 2, 3$) may be expressed in terms of the elastic constants. Putting eqs. (47) into eqs. (14) gives

$$
\begin{aligned}
p_1 &= p_1^{(1)} + i p_1^{(2)} \\
p_1^{(1)} &= \frac{a_{11}a_{22} - a_{12}^2 - 2a_{12}a_{66}}{a_{22}} \\
p_1^{(2)} &= \frac{a_{12}}{a_{22}}\sqrt{a_{11}a_{22} - (a_{12}+2a_{66})^2} \\
q_1 &= q_1^{(1)} + i q_1^{(2)} \\
q_1^{(1)} &= p_1^{(1)}, \quad q_1^{(2)} = -p_1^{(2)} \\
p_2 &= p_2^{(1)} + i p_2^{(2)} \\
p_2^{(1)} &= -2a_{66} \\
p_2^{(2)} &= \sqrt{a_{11}a_{22} - (a_{12}+2a_{66})^2} \\
q_2 &= q_2^{(1)} + i q_2^{(2)} \\
q_2^{(1)} &= p_2^{(1)}, \quad q_2^{(2)} = -p_2^{(2)} \\
p_3 &= p_3^{(1)} + i p_3^{(2)} \\
p_3^{(1)} &= \sqrt{2}\, a_{66}\left(-\frac{a_{12}+2a_{66}}{a_{22}} + \sqrt{\frac{a_{11}}{a_{22}}}\right)^{\frac{1}{2}} \\
p_3^{(2)} &= \sqrt{2}\, a_{66}\left(\frac{a_{12}+2a_{66}}{a_{22}} + \sqrt{\frac{a_{11}}{a_{22}}}\right)^{\frac{1}{2}} \\
q_3 &= q_3^{(1)} + i q_3^{(2)} \\
q_3^{(1)} &= -p_3^{(1)}, \quad q_3^{(2)} = p_3^{(2)}
\end{aligned}
\tag{48}
$$

For later use, the determinant as defined in eq. (25) will be evaluated for the orthotropic case:

$$= 2\sqrt{2}\,\frac{a_{66}}{a_{22}}(a_{11}a_{22} - a_{12}^{2})(-a_{12} - 2a_{66} + \sqrt{a_{11}a_{22}})\left(\frac{a_{12}+2a_{66}}{a_{22}} + \sqrt{\frac{a_{11}}{a_{22}}}\right)^{\frac{1}{2}}$$

2. $M_y^{\infty} \neq 0$, $M_x^{\infty} = H_{xy}^{\infty} = 0$

Let the boundary conditions at infinity be

$$M_y^{\infty} \neq 0, \quad M_x^{\infty} = H_{xy}^{\infty} = 0 \tag{49}$$

The corresponding constants B^*, B'^* and C'^* depend upon M_y^{∞} and the material constants given by

$$
\begin{aligned}
B^* &= \frac{3M_y^{\infty}}{h^3} \cdot \frac{a_{11}a_{22} - a_{12}\sqrt{a_{11}a_{22}}}{(a_{11}a_{22}-a_{12}^2)(a_{12}+2a_{66}-\sqrt{a_{11}a_{22}})} \\
B'^* &= -\frac{3M_y^{\infty}}{h^3} \cdot \frac{a_{11}a_{22}+a_{12}[-2(a_{12}+2a_{66})+\sqrt{a_{11}a_{22}}]}{(a_{11}a_{22}-a_{12}^2)(a_{12}+2a_{66}-\sqrt{a_{11}a_{22}})} \qquad (50) \\
C'^* &= -\frac{3M_y^{\infty}}{h^3} \cdot \frac{a_{11}a_{22}-a_{12}(a_{12}+2a_{66})}{(a_{11}a_{22}-a_{12}^2)\sqrt{a_{11}a_{22}-(a_{12}+2a_{66})^2}}
\end{aligned}
$$

Hence, the coefficients N_1 and N_2 are determined

$$\frac{N_1}{a} = \frac{3M_y}{h^3}^{\infty} \cdot \frac{1}{(\sqrt{a_{11}a_{22}} - a_{12} - 2a_{66})^2 (a_{11}a_{22} - a_{12}^2)(a_{11}a_{22} - a_{12}^2 + 4a_{66}\sqrt{a_{11}a_{22}})}$$

$$\times \Big\{ (\sqrt{a_{11}a_{22}} - a_{12} - 2a_{66}) [-(a_{11}a_{22} - a_{12}^2 - 4a_{66}\sqrt{a_{11}a_{22}})$$

$$\times (a_{11}a_{22} - a_{12}^2 - 2a_{12}a_{66}) + 2a_{66}\sqrt{a_{11}a_{22}}(a_{11}a_{22} - a_{12}^2)]$$

$$+ i \sqrt{a_{11}a_{22} - (a_{12} + 2a_{66})^2} [(a_{11}a_{22} - a_{12}^2 - 4a_{66}\sqrt{a_{11}a_{22}})$$

$$\times (a_{11}a_{22} - a_{12}^2 - 2a_{12}a_{66}) + 2a_{66}\sqrt{a_{11}a_{22}}(a_{11}a_{22} - a_{12}^2)] \Big\}$$

$$\frac{N_2}{a} = -\frac{3M_y}{h^3}^{\infty} \cdot \frac{1}{\sqrt{a_{11}a_{22}}(\sqrt{a_{11}a_{22}} - a_{12} - 2a_{66})(a_{11}a_{22} - a_{12}^2 + 4a_{66}\sqrt{a_{11}a_{22}})(a_{11}a_{22} - a_{12}^2)}$$

$$\times \Big\langle \Big\{ -(a_{11}a_{22} - a_{12}\sqrt{a_{11}a_{22}})(a_{11}a_{22} - a_{12}^2 - 4a_{66}\sqrt{a_{11}a_{22}})(a_{12} + 2a_{66} + \sqrt{a_{11}a_{22}})$$

$$-(a_{11}a_{22} - 2a_{12}^2 - 4a_{12}a_{66} + a_{12}\sqrt{a_{11}a_{22}}) \Big[(a_{11}a_{22} - a_{12}^2 + 4a_{66}\sqrt{a_{11}a_{22}})$$

$$\times (\sqrt{a_{11}a_{22}} - a_{12} - 2a_{66}) + 2(a_{11}a_{22} - a_{12}^2 - 4a_{12}a_{66})(a_{12} + 2a_{66}) \Big]$$

$$-2 (a_{11}a_{22} - a_{12}^2 - 2a_{12}a_{66})(\sqrt{a_{11}a_{22}} - a_{12} - 2a_{66})$$

$$\times (3a_{11}a_{22} - 3a_{12}^2 - 8a_{12}a_{66} + 4a_{66}\sqrt{a_{11}a_{22}})$$

$$+ (\sqrt{a_{11}a_{22}} + a_{12} + 2a_{66}) [(a_{11}a_{22} - a_{12}\sqrt{a_{11}a_{22}})(a_{11}a_{22} - a_{12}^2 - 4a_{66}\sqrt{a_{11}a_{22}})$$

$$+ (a_{11}a_{22} - 2a_{12}^2 - 4a_{12}a_{66} + a_{12}\sqrt{a_{11}a_{22}})(a_{11}a_{22} - a_{12}^2 - 4a_{12}a_{66} - 4a_{66}\sqrt{a_{11}a_{22}})]$$

$$-(a_{11}a_{22} - a_{12}^2 - 2a_{12}a_{66}) [(a_{11}a_{22} - a_{12}^2 + 4a_{66}\sqrt{a_{11}a_{22}})(a_{12} + 2a_{66} - \sqrt{a_{11}a_{22}})$$

$$+ 4(a_{11}a_{22} - a_{12}^2 - 4a_{12}a_{66})(a_{12} + 2a_{66})] \Big\}$$

(Continued)

$$-i\left\{\frac{\sqrt{a_{11}a_{22}-(a_{12}+2a_{66})^2}}{\sqrt{a_{11}a_{22}}-a_{12}-2a_{66}}\Big[-(a_{11}a_{22}-a_{12}\sqrt{a_{11}a_{22}})(a_{11}a_{22}-a_{12}^2-4a_{66}\sqrt{a_{11}a_{22}})\right.$$

$$\times(a_{12}+2a_{66}+\sqrt{a_{11}a_{22}})$$

$$-(a_{11}a_{22}-2a_{12}^2-4a_{12}a_{66}+a_{12}\sqrt{a_{11}a_{22}})[(a_{11}a_{22}-a_{12}^2+4a_{66}\sqrt{a_{11}a_{22}})$$

$$\times(\sqrt{a_{11}a_{22}}-a_{12}-2a_{66})\ +2(a_{11}a_{22}-a_{12}^2-4a_{12}a_{66})(a_{12}+2a_{66})]$$

$$-2(a_{11}a_{22}-a_{12}^2-2a_{12}a_{66})(\sqrt{a_{11}a_{22}}-a_{12}-2a_{66})$$

$$\times(3a_{11}a_{22}-3a_{12}^2-8a_{12}a_{66}+4a_{66}\sqrt{a_{11}a_{22}})\Big]$$

$$-\sqrt{a_{11}a_{22}-(a_{12}+2a_{66})^2}\,[(a_{11}a_{22}-a_{12}\sqrt{a_{11}a_{22}})\,(a_{11}a_{22}-a_{12}^2-4a_{66}\sqrt{a_{11}a_{22}})$$

$$+(a_{11}a_{22}-2a_{12}^2-4a_{12}a_{66}+a_{12}\sqrt{a_{11}a_{22}})$$

$$\times(a_{11}a_{22}-a_{12}^2-4a_{12}a_{66}-4a_{66}\sqrt{a_{11}a_{22}})]$$

$$+\frac{(a_{11}a_{22}-a_{12}^2-2a_{12}a_{66})(\sqrt{a_{11}a_{22}}-a_{12}-2a_{66})}{\sqrt{a_{11}a_{22}-(a_{12}+2a_{66})^2}}$$

$$\times[(a_{11}a_{22}-a_{12}^2+4a_{66}\sqrt{a_{11}a_{22}})(a_{12}+2a_{66}-\sqrt{a_{11}a_{22}})$$

$$\left.+4(a_{11}a_{22}-a_{12}^2-4a_{12}a_{66})(a_{12}+2a_{66})]\right\}\Big\rangle, \quad (51)$$

3. $H_{xy}^{\infty} \neq 0,\ M_x^{\infty} = M_y^{\infty} = 0$

Alternatively, it is possible to specify the conditions

$$H_{xy}^{\infty} \neq 0, \quad M_x^{\infty} = M_y^{\infty} = 0 \tag{52}$$

In a similar manner, the constants B^*, B'^*, C'^*, N_1 and N_2 are obtained :

$$B^* = -\frac{3H_{xy}^{\infty}}{\sqrt{2}\,h^3}\cdot\frac{\sqrt{a_{22}}}{a_{66}\sqrt{\sqrt{a_{11}a_{22}}-a_{12}-2a_{66}}}$$

$$B'^* = -B^* \tag{53}$$

$$C'^* = 0$$

$$\frac{N_1}{a} = \frac{3H_{xy}^{\infty}}{2h^3}\cdot\frac{\sqrt{a_{22}}\left(a_{11}a_{22}+a_{12}\sqrt{a_{11}a_{22}}\right)\left(2a_{66}+i\sqrt{a_{11}a_{22}-(a_{12}+2a_{66})^2}\right)}{\sqrt{a_{66}}\;a_{11}a_{22}-a_{12}-2a_{66}\left(4a_{11}a_{22}a_{66}+(a_{11}a_{22}-a_{12}^2)\sqrt{a_{11}a_{22}}\right)}$$

$$\frac{N_2}{a} = -\frac{3H_{xy}^{\infty}}{2h^3}\cdot\frac{\sqrt{a_{22}}\left(a_{11}a_{22}+a_{12}\sqrt{a_{11}a_{22}}\right)\left(2a_{66}-i\sqrt{a_{11}a_{22}-(a_{12}+2a_{66})^2}\right)}{a_{66}\sqrt{\sqrt{a_{11}a_{22}}-a_{12}-2a_{66}}\left(4a_{11}a_{22}a_{66}+(a_{11}a_{22}-a_{12}^2)\sqrt{a_{11}a_{22}}\right)} \tag{54}$$

4. Stationary value of M

If the preferred direction of the orthotropic material is not directly along the x - axis, i.e., ahead of the crack tip, there is a tendency for the crack to extend side ways. Such a phenomenon may be referred to as "branching" of the crack. The precise directions of the branches may be predicted by the angles at which M_θ is a maximum.

The quantities M_ρ, M_θ, $H_{\rho\theta}$, N_θ and N_ρ referred to polar coordinates r, θ are related to Mx, My, Hxy, Nx and Ny in the coordinate system x, y by the formulas

$$\begin{aligned} M_\rho + M_\theta &= M_x + M_y \\ M_\theta - M_\rho + 2iH_{\rho\theta} &= (M_y - M_x + 2iH_{xy})\, e^{2i\alpha} \\ N_\rho - iN_\theta &= (N_x - iN_y)\, e^{i\alpha} \end{aligned} \qquad (55)$$

where α is the angle between the x - axis and the radial direction, Fig. 6.

Equations (55) may be solved to give

$$M_\theta = M_x \sin^2\alpha + M_y \cos^2\alpha - 2H_{xy} \sin\alpha \cos\alpha \qquad (56)$$

Upon substitution of Mx, My and Hxy in eqs. (43) into eq. (56) yields

$$M_\theta = \sqrt{\frac{a}{2r}}\ \frac{h^3}{6}\ \mathrm{Re}\left\{ \frac{p_1 \frac{N_1}{a}}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{q_1 \frac{N_2}{a}}{\sqrt{\cos\theta + \mu_2 \sin\theta}} \right\} \sin^2\theta$$

$$+ \left\{ \frac{p_2 \frac{N_1}{a}}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{p_2 \frac{N_2}{a}}{\sqrt{\cos\theta + \mu_2 \sin\theta}} \right\} \cos^2\theta$$

$$-2 \left\{ \frac{p_3 \frac{N_1}{a}}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{p_3 \frac{N_2}{a}}{\cos\theta + \mu_2 \sin\theta} \right\} \sin\theta \cos\theta$$

$$+ Mx^\infty \sin\theta + My^\infty \cos^2\theta - 2Hxy^\infty \sin\theta\cos\theta \qquad (57)$$

The real and imaginary parts of the quantities in the brackets of eq. (57) may be best separated by introducing the following relations

$$z = x + iy = a + re^{i\theta}$$

$$\bar{z} = x + \mu_1 y = a + r_1 e^{i\theta_1} \qquad (58)$$

$$z = x + \mu_2 y = a + r_2 e^{i\theta_2}$$

where, r_j and θ_j can be expressed in terms of r and θ as follows:

$$\theta_1 = \arctan\left(\frac{\beta}{\alpha + \cot\theta}\right), \quad \theta_2 = \arctan\left(\frac{\beta}{-\alpha + \cot\theta}\right)$$

$$r_1 = r\beta \frac{\sin\theta}{\sin\theta_1} \qquad r_2 = r\beta \frac{\sin\theta}{\sin\theta_2} \qquad (59)$$

With the knowledge that

$$(\cos\theta+\mu_1\sin\theta)^{-\frac{1}{2}} = e^{-\frac{1}{2}\log(\cos\theta+\mu_1\sin\theta)}$$

$$= e^{-\frac{i}{2}\arctan\left(\frac{\beta}{\alpha+\cot\theta}\right)}$$

$$\times e^{-\frac{1}{4}\log\left[(\cos\theta+\alpha\sin\theta)^2+\beta^2\sin^2\theta\right]}$$

$$= e^{-\frac{1}{2}\left(i\theta_1+\log\frac{r_1}{r}\right)}$$

$$= \left(\frac{r}{r_1}\right)^{\frac{1}{2}}\left(\cos\tfrac{1}{2}\theta_1 - i\sin\tfrac{1}{2}\theta_1\right) \qquad (60)$$

$$(\cos\theta+\mu_2\sin\theta)^{-\frac{1}{2}} = \left(\frac{r}{r_2}\right)^{\frac{1}{2}}\left(\cos\tfrac{1}{2}\theta_2 - i\sin\tfrac{1}{2}\theta_2\right)$$

$M\theta$ is further reduced to the form

$$M\theta = \sqrt{\frac{a}{2r}}\cdot\frac{h^3}{6}\cdot r^{\frac{1}{2}}\Big\{r_1^{\frac{1}{2}}\big[(D_1\sin^2\theta + D_2\cos^2\theta - D_3\sin 2\theta)\cos\tfrac{1}{2}\theta_1$$

$$+ (E_1\sin^2\theta + E_2\cos^2\theta - E_3\sin 2\theta)\sin\tfrac{1}{2}\theta_1\big]$$

$$- r_2^{-\frac{1}{2}}\big[(d_1\sin^2\theta + d_2\cos^2\theta - d_3\sin 2\theta)\cos\tfrac{1}{2}\theta_2$$

$$+ (e_1\sin^2\theta + e_2\cos^2\theta - e_3\sin 2\theta)\sin\tfrac{1}{2}\theta_2\big]\Big\}$$

$$+ Mx^{\infty}\sin^2\theta + My^{\infty}\cos^2\theta - Hxy^{\infty}\sin 2\theta \qquad (61)$$

where,

$$p_1 \frac{N_1}{a} = D_1 + iE_1 \qquad q_1 \frac{N_1}{a} = d_1 + ie_1$$

$$p_2 \frac{N_1}{a} = D_2 + iE_2 \qquad q_2 \frac{N_2}{a} = d_2 + ie_2 \tag{62}$$

$$p_3 \frac{N_1}{a} = D_3 + iE_3 \qquad q_3 \frac{N_2}{a} = d_3 + ie_3$$

D_j, E_j, d_j, e_j $(j = 1, 2, 3)$ are all real.

Now, differentiating eq. (61) with respect to θ renders

$$\frac{\partial M_\theta}{\partial \theta} = \sqrt{\frac{a}{2r}} \cdot \frac{h^3}{6} \cdot r^{\frac{1}{2}}$$

$$\times \Big\langle (r_1^{-\frac{1}{2}})_\theta \big[(D_1 \sin^2\theta + D_2 \cos^2\theta - D_3 \sin 2\theta) \cos \tfrac{1}{2}\theta_1$$

$$+ (E_1 \sin^2\theta + E_2 \cos^2\theta - E_3 \sin 2\theta) \sin \tfrac{1}{2}\theta_1 \big]$$

$$-(r_2^{-\frac{1}{2}})_\theta \big[(d_1 \sin^2\theta + d_2 \cos^2\theta - d_3 \sin 2\theta) \cos \tfrac{1}{2}\theta_2$$

$$+ (e_1 \sin^2\theta + e_2 \cos^2\theta - e_3 \sin 2\theta) \sin \tfrac{1}{2}\theta_2 \big]$$

$$+ r_1^{-\frac{1}{2}} \Big\{ [(D_1 - D\ ) \sin 2\theta - 2D_3 \cos 2\theta] \cos \tfrac{1}{2}\theta_1$$

$$+ [(E_1 - E_2) \sin 2\theta - 2E_3 \cos 2\theta] \sin \tfrac{1}{2}\theta_1$$

$$- r_2^{-\frac{1}{2}} \Big\{ [(d_1 - d_2) \sin 2\theta - 2d_3 \cos 2\theta] \cos \tfrac{1}{2}\theta_2$$

$$+ [(e_1 - e_2) \sin 2\theta - 2e_3 \cos 2\theta] \sin \tfrac{1}{2}\theta_2 \Big\}$$

(Continued)

$$-\tfrac{1}{2}(\theta_1)_\theta r_1^{-\frac{1}{2}}\left[(D_1\sin^2\theta + D_2\cos^2\theta - D_3\sin 2\theta)\sin\tfrac{1}{2}\theta_1 - (E_1\sin^2\theta + E_2\cos^2\theta - E_3\sin 2\theta)\cos\tfrac{1}{2}\theta_1\right]$$

$$+\tfrac{1}{2}(\theta_2)_\theta r_2^{-\frac{1}{2}}\left[(d_1\sin^2\theta + d_2\cos^2\theta - d_3\sin 2\theta)\sin\tfrac{1}{2}\theta_2 - (e_1\sin^2\theta + e_2\cos^2\theta - e_3\sin 2\theta)\cos\tfrac{1}{2}\theta_2\right]\Big\rangle$$

$$+(Mx^\infty - My^\infty)\sin 2\theta - 2\,Hxy^\infty\cos 2\theta \qquad (63)$$

where

$$(r_1^{-\frac{1}{2}})_\theta = \frac{\partial}{\partial\theta}(r_1^{-\frac{1}{2}})$$

$$= -\tfrac{1}{2}r^{-\frac{1}{2}}\left(\beta\frac{\sin\theta}{\sin\theta_1}\right)^{-\frac{3}{2}}\left(\beta\frac{\cos\theta}{\sin\theta_1} - \frac{\cos\theta_1}{\sin\theta}\right)$$

$$(r_2^{-\frac{1}{2}})_\theta = \frac{\partial}{\partial\theta}(r_2^{-\frac{1}{2}})$$

$$= -\tfrac{1}{2}r^{-\frac{1}{2}}\left(\beta\frac{\sin\theta}{\sin\theta_2}\right)^{-\frac{3}{2}}\left(\beta\frac{\cos\theta}{\sin\theta_2} - \frac{\cos\theta_2}{\sin\theta}\right)$$

$$(\theta_1)_\theta = \frac{\partial\theta_1}{\partial\theta}$$

$$= \frac{1}{\beta}\cdot\frac{\sin^2\theta_1}{\sin^2\theta}$$

$$(\theta_2)_\theta = \frac{\partial\theta_2}{\partial\theta}$$

$$= \frac{1}{\beta}\cdot\frac{\sin^2\theta_2}{\sin^2\theta}$$

The stationary values of M_θ can be obtained by having

$$\frac{\partial M_\theta}{\partial \theta} = 0$$

The angle θ_0 at which M_θ is a maximum will be solved numerically in the next section.

VI. Numerical Examples

Numerical values of the elastic constants for anisotropic materials will be assigned in accordance with the work in [2]. They are

$$Ex = 582 \text{ kg/mm}^2$$

$$Ey = 219 \text{ kg/mm}^2$$

$$Gxy = 132 \text{ kg/mm}^2$$

$$\nu_{yx} = 0.313$$

$$\nu_{xy} \quad \nu_{yx}\frac{Ey}{Ex} = 0.122$$

Where Ex, Ey are the Young's modulii along the respective coordinate axes; Gxy is the shear modulus in the xoy and parallel planes and ν_{xy} is Poisson's ratio accounting for the contraction along the ox-axis due to the expansion along the oy-oxis, or vice-versa.

Using these values, the anisotropic coefficients a_{ij} are computed:

$$a_{11} = \frac{1}{\frac{1}{Ex} - \frac{\nu_{xy}^2}{Ey}} = 606 \text{ kg/mm}^2$$

$$a_{12} = \frac{\nu_{xy}}{\frac{1}{Ex} - \frac{\nu_{xy}^2}{Ey}} = 74 \text{ kg/mm}^2$$

$$a_{22} = \frac{1}{\frac{1}{E_y} - \frac{\nu_{xy}^2}{E_x}} = 227 \text{ kg/mm}^2$$

$$a_{66} = G_{xy} = 132 \text{ kg/mm}^2$$

The remaining constants required for the calculation of M_θ are

$\alpha = 0.269, \quad \beta = 1.25$

$p_1^{(1)} = 496 \text{ kg/mm}^2 \qquad p_1^{(2)} = 49.9 \text{ kg/mm}^2$

$q_1^{(1)} = 496 \text{ kg/mm}^2 \qquad q_1^{(2)} = 49.9 \text{ kg/mm}^2$

$p_2^{(1)} = 264 \text{ kg/mm}^2 \qquad p_2^{(2)} = 153 \text{ kg/mm}^2$

$q_2^{(1)} = 264 \text{ kg/mm}^2 \qquad q_2^{(2)} = 153 \text{ kg/mm}^2$

$p_3^{(1)} = 71 \text{ kg/mm}^2 \qquad p_3^{(2)} = 318 \text{ kg/mm}^2$

$q_3^{(1)} = -71 \text{ kg/mm}^2 \qquad q_3^{(2)} = 318 \text{ kg/mm}^2$

$\Delta = 1.264 \times 10^7 \text{ (kg/mm}^2)^3$

Case 1 ; $M_y^{\infty} \neq 0, \; M_x = H_{xy} = 0$

The numerical values of the elastic constants are

$$B^* = -0.0758 \times \frac{M_y^{\infty}}{h^3}$$

$$B'^* = 0.0792 \times \frac{M_y^{\infty}}{h^3}$$

$$C'^* = -0.0336 \times \frac{M_y^{\infty}}{h^3}$$

$$\frac{N_1}{a} = (0.0424 - 0.143\, i)\, \frac{M_y^{\infty}}{h^3}$$

$$\frac{N_2}{a} = (-0.0504 + 0.0396\, i)\, \frac{M_y^{\infty}}{h^3}$$

$$D_1 = 21.8 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2, \qquad E_1 = -4.96 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2$$

$$D_2 = -9.02 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2, \qquad E_2 = 10.3 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2$$

$$D_3 = 7.55 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2, \qquad E_3 = 12.5 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2$$

$$d_1 = -23.0 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2, \qquad e_1 = 22.1 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2$$

$$d_2 = 19.4 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2, \qquad e_2 = -2.73 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2$$

$$d_3 = -8.98 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2, \qquad e_3 = -18.8 \frac{M_y^{\infty}}{h^3} \text{ kg/mm}^2$$

The angles θ_o corresponding to $(M_\theta)_{max.}$ are found to be approximately $\pm 18.5^\circ$ from the x -axis. See Fig. 7. Based on the hypothesis that crack propagates in a direction perpendicular to maximum tension, the above result suggests the possibility that the crack branches symmetrically with respect to a line coinciding with the crack itself.

Case 2: $H_{xy}^{\infty} \neq 0, \ M_x^{\infty} = M_y^{\infty} = 0$

The requisite constants for this case are

$$B^* = -0.434 \frac{H_{xy}^{\infty}}{h^3}$$

$$B'^* = 0.434 \frac{H_{xy}^{\infty}}{h^3}$$

$$C'^* = 0$$

$$\frac{N_1}{a} = (0.155 + 0.0906\, i) \frac{H_{xy}^{\infty}}{h^3}$$

$$\frac{N_2}{a} = (-0.155 + 0.0906\, i) \frac{H_{xy}^{\infty}}{h^3}$$

$$D_1 = 72.7 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2, \quad E_1 = 52.8 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2$$

$$D_2 = -54.9 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2, \quad E_2 = -0.20 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2$$

$$D_3 = -17.8 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2, \quad E_3 = 55.7 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2$$

$$d_1 = -72.7 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2, \quad e_1 = 52.8 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2$$

$$d_2 = 54.9 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2, \quad e_2 = -0.20 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2$$

$$d_3 = -17.8 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2, \quad e_3 = -55.7 \frac{H_{xy}^{\infty}}{h^3} \text{kg/mm}^2$$

Since the problem is skew-symmetric with respect to the line crack, it is expected that crack extends only in one direction as it would be in isotropic materials. This direction makes an angle of approximately 0.4° to the x-axis, Fig. 8.

VII. Discussion and Conclusions

The results obtained in this work is similar to those found in [7]. The bending stresses near the crack tip were found to be proportional to the inverse square-root of the radial distance measured from the singular crack point. The functional relationship of the stresses depends upon the elastic constants of the anisotropic material. It may be concluded that the details of the local stresses are intimately connected with the nature of anisotropy. For instance, the crack-tip stress field for a polarly anisotropic body would be quite different than that for a rectilinearly anisotropic medium discussed in the present analysis.

It should be pointed out that the present solution will not accurate in the region close to the crack boundary, since the Poisson-Kirchhoff theory of plate bending utilizes approximate boundary conditions. Nevertheless, the theory does predict the qualitative features of the physical problem sufficiently well. The phenomenon of branch cracks and the $1/\sqrt{r}$ stress sigularity will be found even had the problem be solved by more refined theory of the bending of plates.

For future work, a Reinssner type of plate theory should be formulated for anisotoropic plates where all the three natural boundary conditions can be satisfied. In this way, the stress distribution around the crack tip may be calculated for a better understanding of the fracture strength of cracks in anisotropic plates under bending.

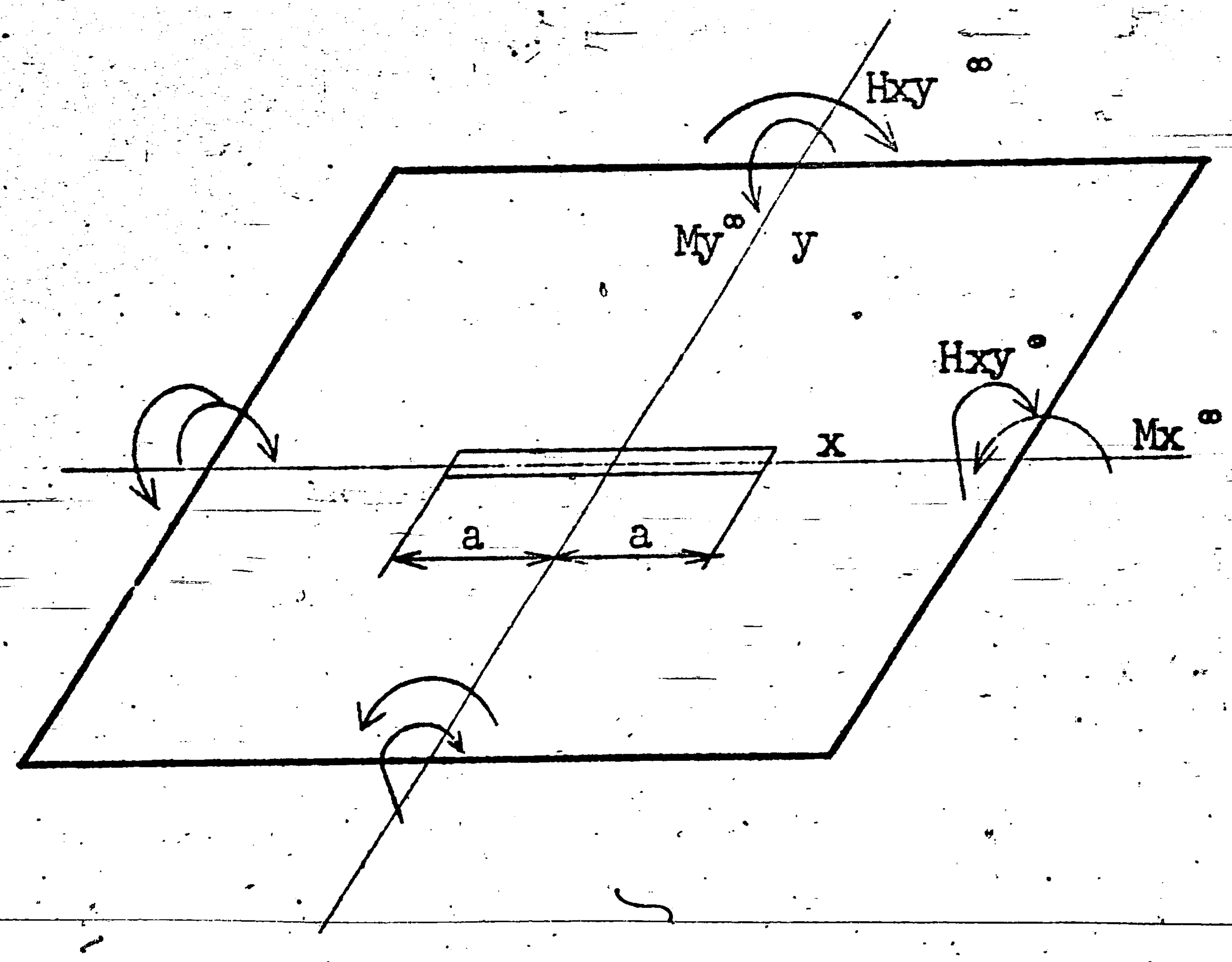


Fig. 1 Infinite plate with a crack subjected to moments at infinity.

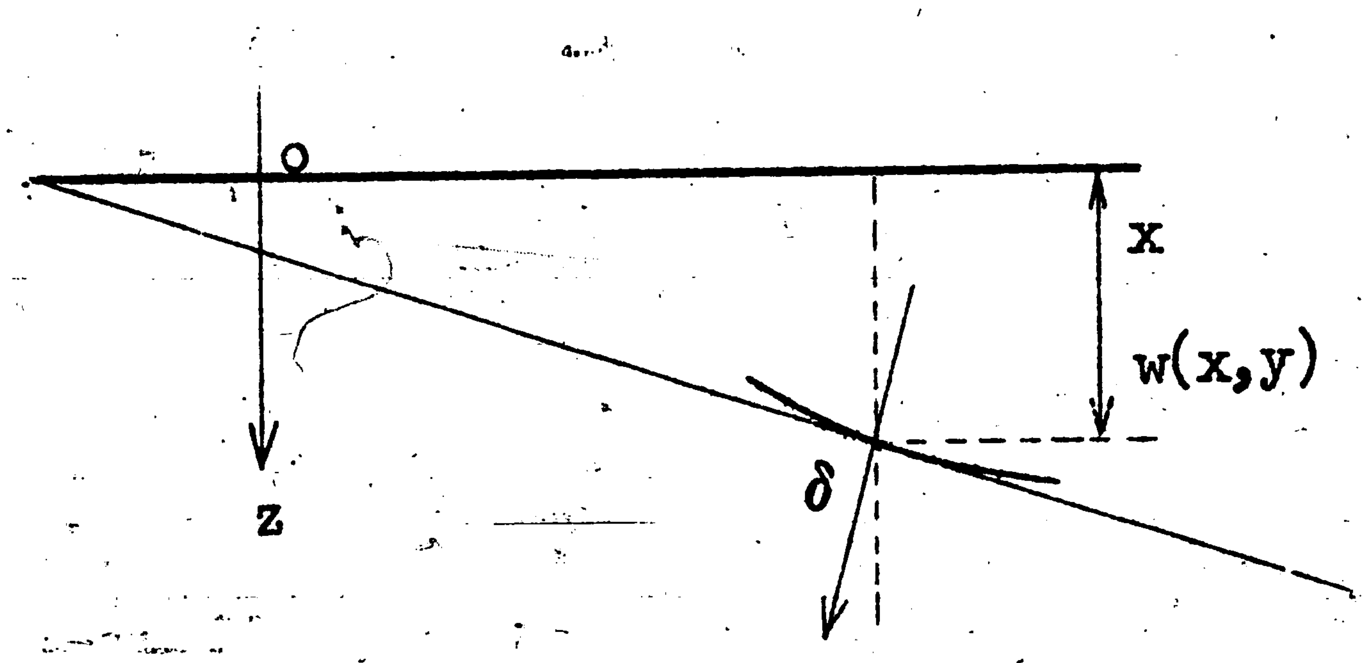
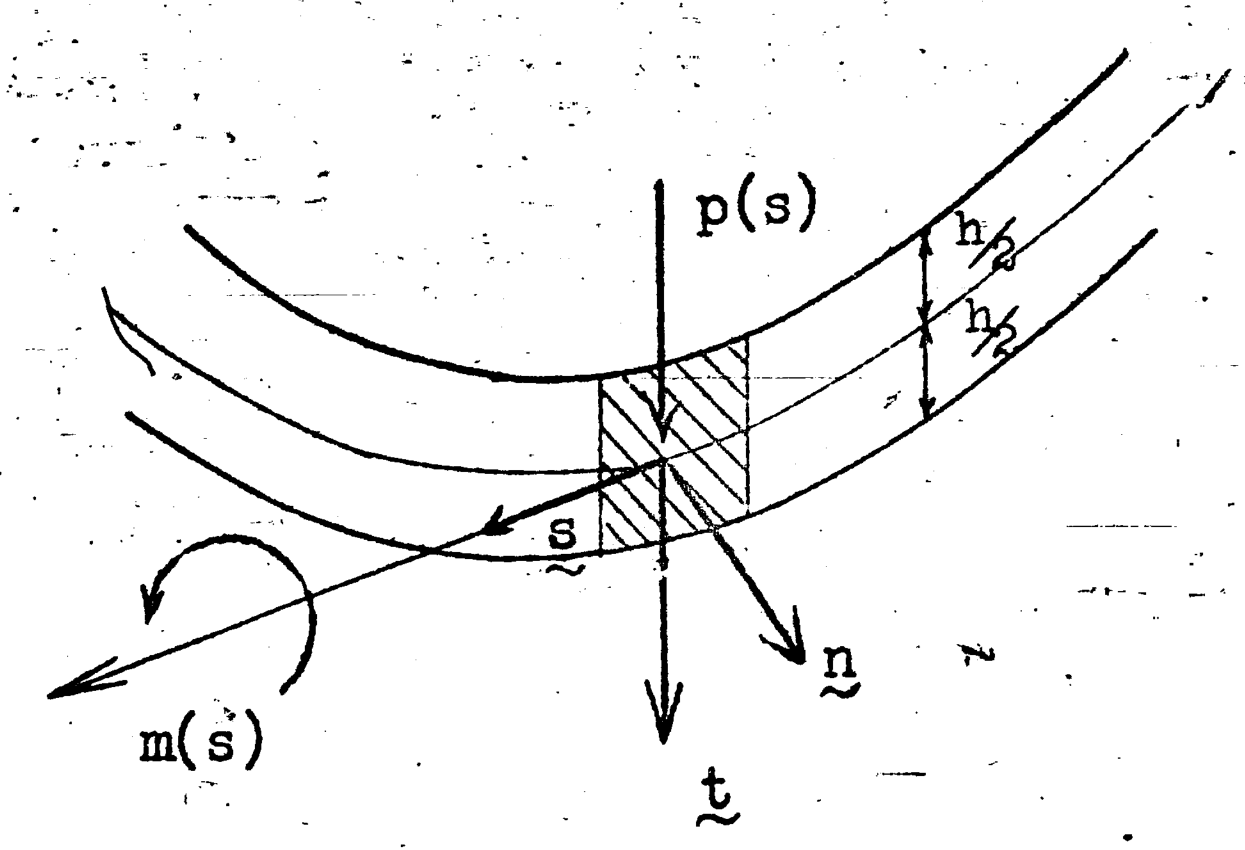


Fig. 2 Notation for displacement w.



m(s) : Bending moment

p(s) : Shear force

Fig. 3 Force and moment at contour

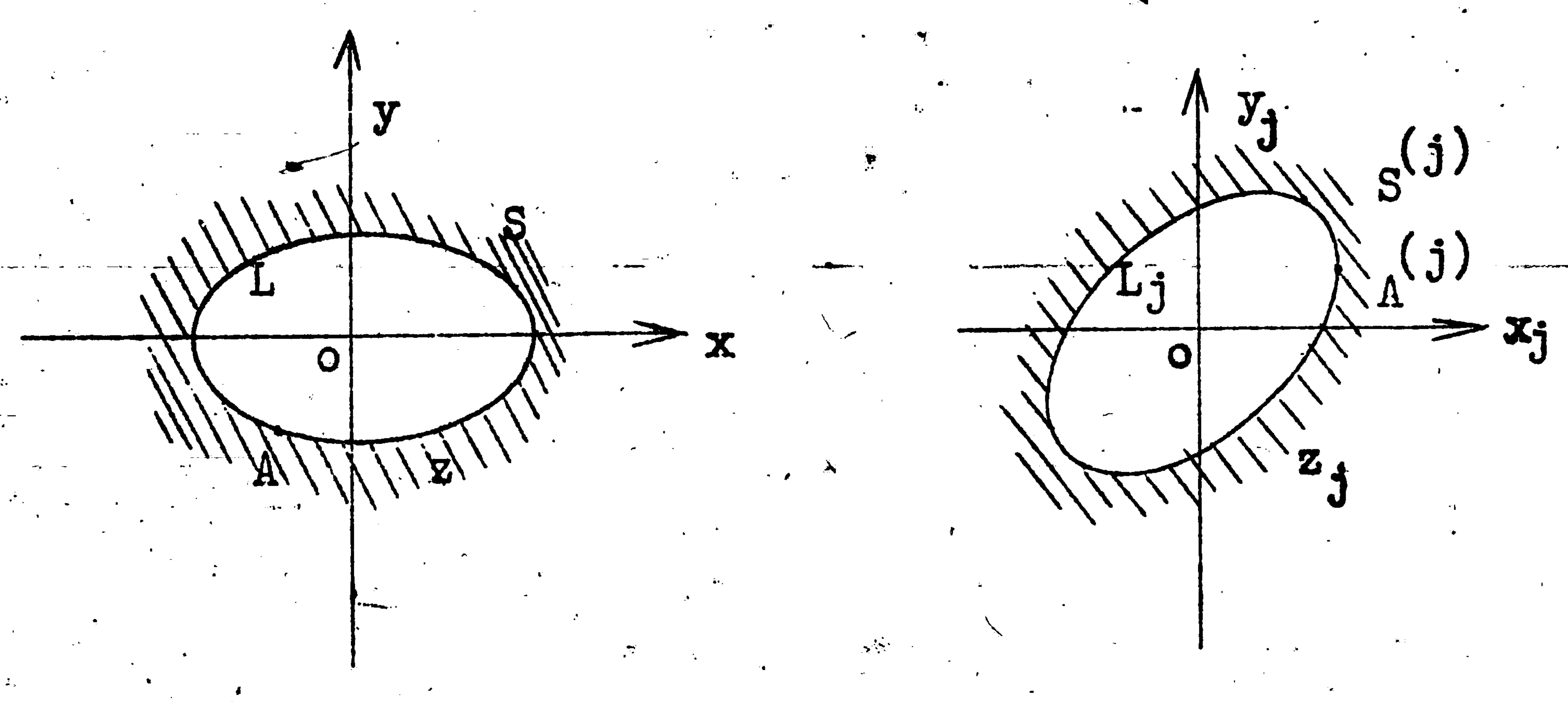


Fig. 4 Conformal transformation of ellipse

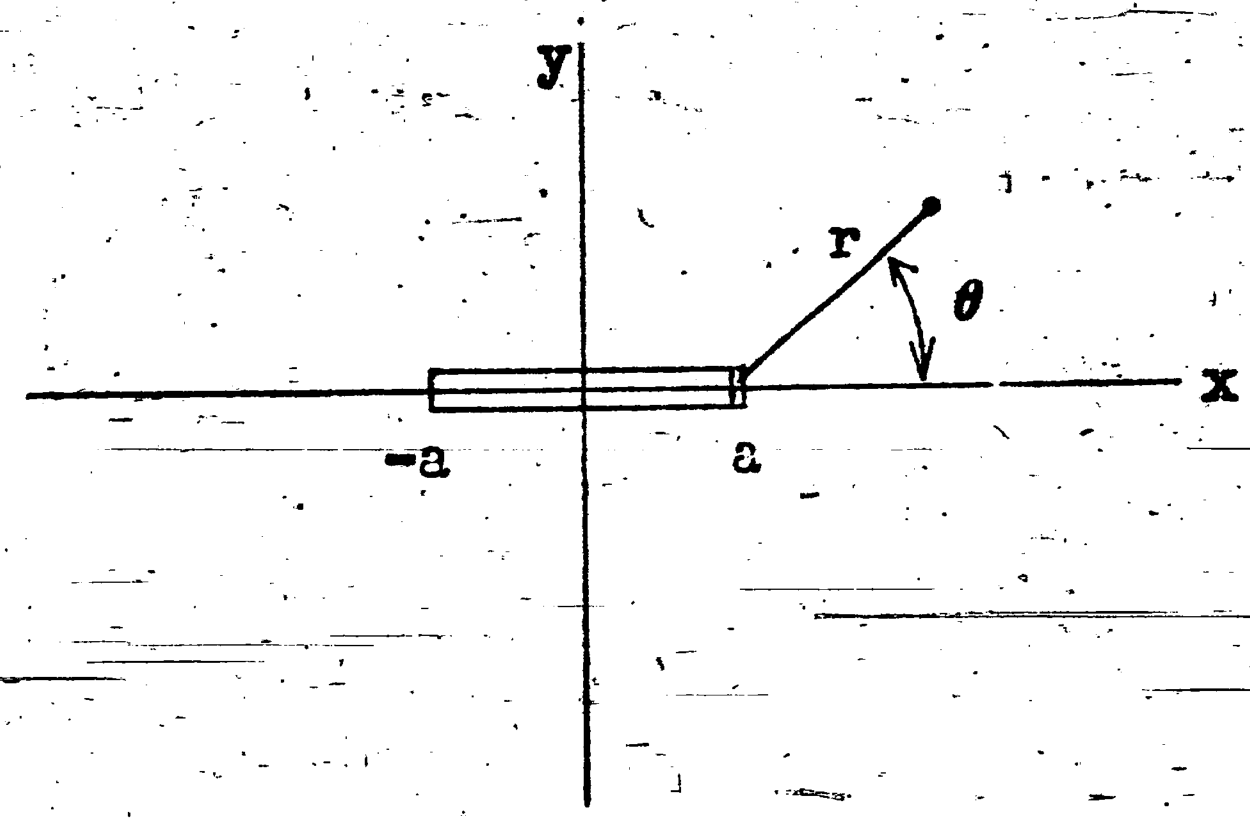


Fig. 5 Notation of polar coordinates

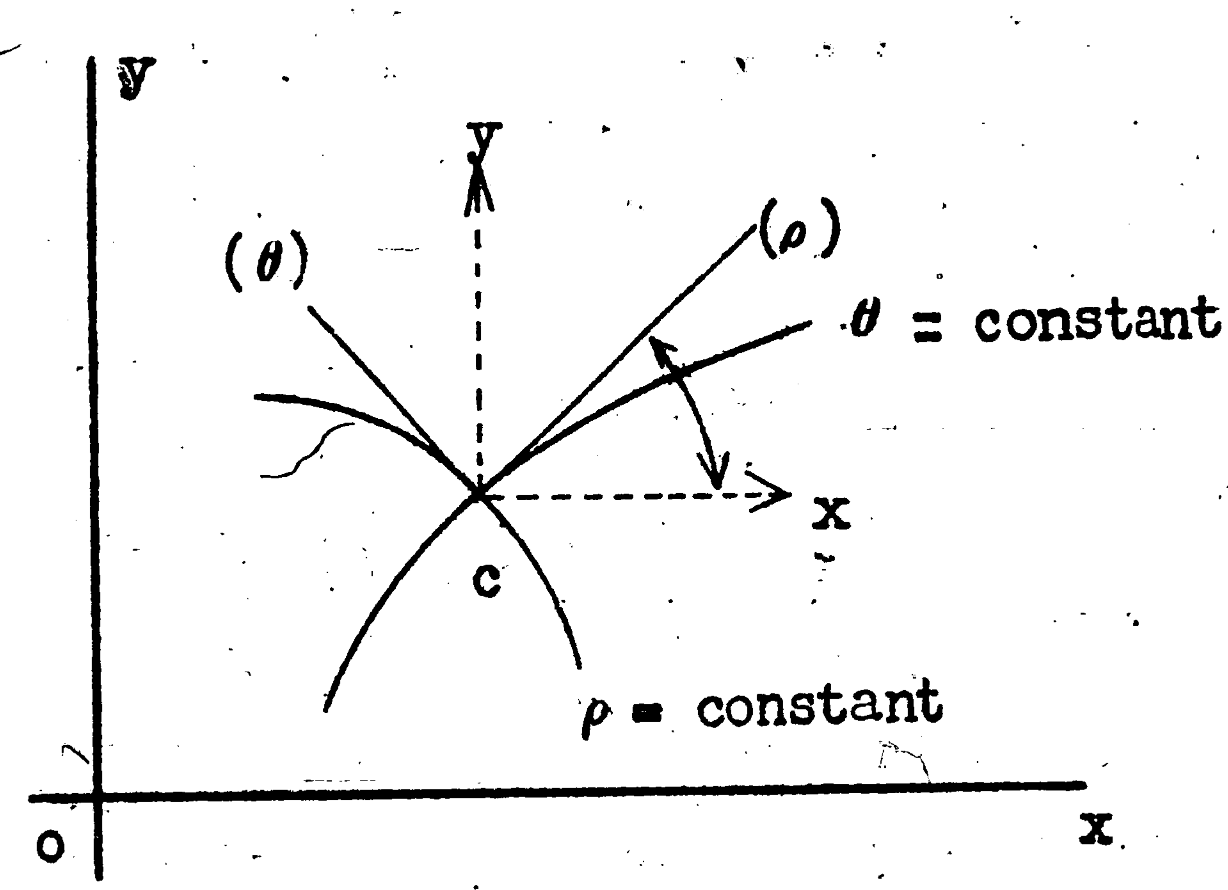
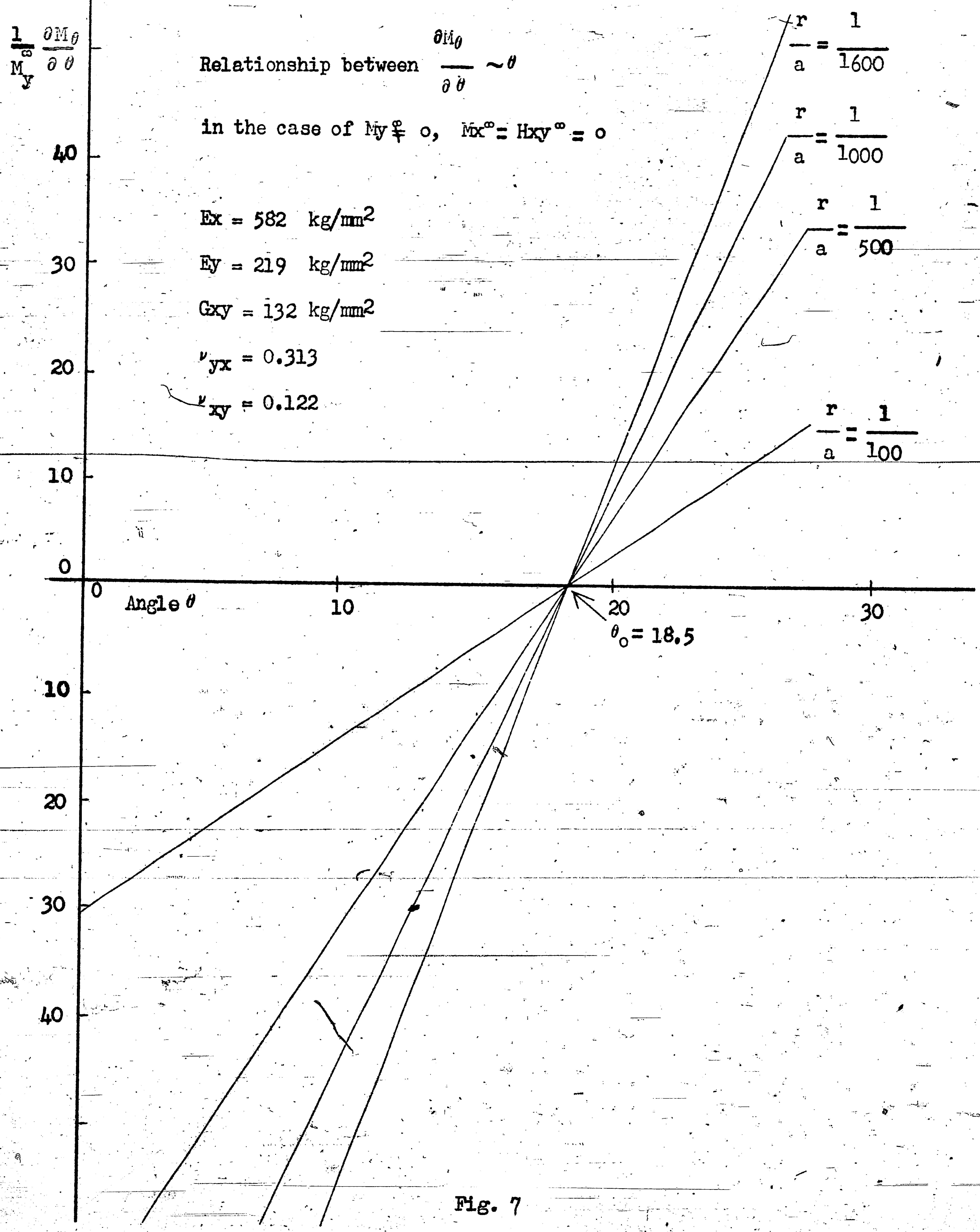
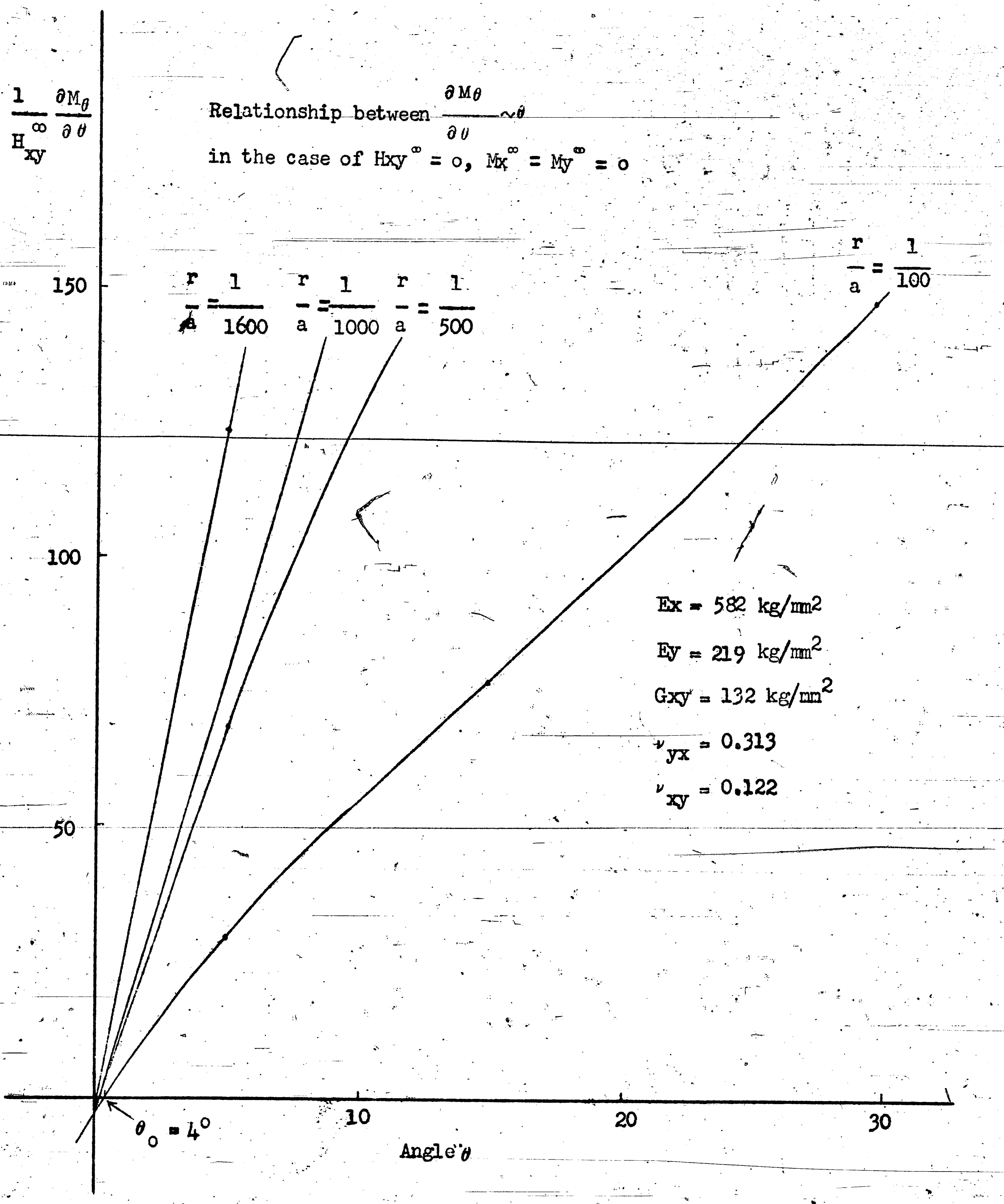


Fig. 6 Curvilinear co-ordinate



$\frac{1}{M_y^\infty}\frac{\partial M_\theta}{\partial \theta}$
Relationship between $\frac{\partial M_\theta}{\partial \theta} \sim \theta$
in the case of $My^\infty \neq 0$, $Mx^\infty = Hxy^\infty = 0$
$Ex = 582$ kg/mm²
$Ey = 219$ kg/mm²
$Gxy = 132$ kg/mm²
$\nu_{yx} = 0.313$
$\nu_{xy} = 0.122$
$\frac{r}{a} = \frac{1}{1600}$
$\frac{r}{a} = \frac{1}{1000}$
$\frac{r}{a} = \frac{1}{500}$
$\frac{r}{a} = \frac{1}{100}$
40
30
20
10
0
10
20
30
40
0
Angle θ
10
20
30
$\theta_0 = 18.5$

Fig. 7



Relationship between ∂Mθ/∂θ ~ θ
in the case of Hxy∞ = o, Mx∞ = My∞ = o
1/Hxy∞ ∂Mθ/∂θ
150
100
50
r/a = 1/1600
r/a = 1/1000
r/a = 1/500
r/a = 1/100
Ex = 582 kg/mm2
Ey = 219 kg/mm2
Gxy = 132 kg/mm2
νyx = 0.313
νxy = 0.122
θo = 4°
10
20
30
Angle θ

Fig. 8

References

1. S.G. Lekhnitzkii
"Theory of Elasticity of an Anisotropic Body"
Holden-Day, Inc. San Francisco, 1963.

2. G.N. Savin
"Stress Concentration around Holes"
Pergamon Press, 1961

3. G.C. Sih and J.R. Rice
"The Bending of Plates of Dissimilar Materials with Cracks".
Journal of Applied Mechanics No. 64-APM-5, Vol. 86, Series D-E,
Sept. 1964, pp 477-482.

4. N.I. Muskhelishvili
"Some Basic Problems in Mathematical Theory of Elasticity".
P. Noordhooff Ltd, 1963.

5. A.E.H. Love
"A Treatise on the Mathematical Theory of Elasticity"
Dover Publications, New York, 1944.

6. G.C. Sih et al

"Crack Tip Stress-Intensity Factors for Plane Extension and Plate Bending Problems"

Journal of Applied Mechanics, Vol. 29, Trans. ASME 84, Series E, 1962, pp. 306-312.

7. G.C. Sih, P.C. Paris and G.R. Irwin

"On Cracks in Rectilinear Anisotropic Bodies£

International Journal of Fracture Mechanics Vol. 1, No. 3, pp. 189-203.

Vita

The author was born on October 18, 1935 to yujiro and Gin Ishikawa, in Shizuoka, Japan. He attended primary and secondary school in Shizuoka and was graduated from Shizuoka High School. He entered Tokyo Institute of Technology in Tokyo in 1955, where he specialized in mechanical engineering. He passed the bachelor's examination and was awarded the Bachelor of Engineering degree in Mechanical Engineering in 1959. His thesis was on the Fundamental study of Fluid Vibration in Piping System.

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