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A polynomial approach to topological analysis

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A POLYNOMIAL APPROACH TO TOPOLOGICAL ANALYSIS

by

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CONTENTS

ABSTRACT OF A POLYNOMIAL APPROACH TO TOPOLOGICAL ANALYSIS

by

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The paper is a report of some work of Professor Kenneth Leland which may be found in his papers (1) A Polynomial Approach to Topological Analysis and (2) Topological Analysis of Analytic Functions. Section 2 of the thesis expands the development of (1) while Section 3 amplifies a portion of (2).

In Section 2 many basic results of topological analysis are obtained for twice continuously differentiable functions using only fundamental techniques. A form of the Maximum Principle and Schwarz's Lemma are proved leading to an extremely useful growth rate estimate for polynomials. Then, using the Stone-Weierstrass Theorem, we obtain a polynomial sequence approximating functions f which are twice continuously differentiable on the unit disc; and finally we convert this sequence into a power series expansion for f on the unit disc. We also verify the open mapping theorem and the Fundamental Theorem of Algebra.

In Section 3 we prove a useful Maximum Principle for the difference quotient and the Vitali-Stieltjes Theorem. Then we use an adaptation of a standard proof of the Riemann Mapping Theorem to obtain a polynomial sequence approximating functions f which are differentiable on the unit disc. Finally, we convert this sequence into a power series expansion for f on the unit disc.

1. Introduction

In the introduction to what has become the standard English reference work in the field of topological analysis, appropriately entitled <u>Topological Analysis</u> [15], Professor Whyburn has described the subject in the following manner

> "Topological analysis consists of those basic theorems of analysis, especially of the functions of a complex variable, which are essentially topological in character, developed and proved entirely by topological and pseudo-topological methods."

The phrase "by topological and pseudo-topological methods" was further clarified in the same introduction in a more or less negative fashion by Professor Whyburn; he remarked that in topological analysis, a minimum use is to be made of all such machinery and tools of analysis as derivatives, integrals, and power series. However, much of the more recent work done in the field has made great use of the derivative and the closely related function, the difference quotient. But in general no use of the line integral is made, and it has evolved today that the fundamental differentiating aspect of topological analysis as opposed to classical analysis is that it makes no use of any form of integration.

In classical complex analysis the usual development for deriving the fundamental properties of analytic functions revolves around the integral and the Cauchy Integral Formula. Having established the Cauchy Integral Formula, one can then obtain the infinite differentiability of analytic functions, Morera's Theorem, Liouville's Theorem, and the Fundamental Theorem of Algebra as veritable corollaries. Rouché's Theorem, the open mapping theorem, the Maximum Principle, the power series expansion for analytic functions and other basic results of complex analysis then follow quite naturally.

In topological analysis as it has thus far unfolded, the most fundamental theorem has become the open mapping theorem for non-constant differentiable functions. Professor Whyburn has proven this theorem in the above mentioned text using the topological analogue of the winding number, the topological or circulation index. In the journals over the past few years, mathematicians, active in research in the field of topological analysis, have used the open mapping theorem as a starting point for their developments. For example Porcelli and Connell ([3],[5]) have derived the infinite differentiability and anti-differentiability of analytic functions, the existence of a power series expansion for such functions in certain regions of the complex plane, Liouville's Theorem, and the removable singularity theorem using only the open mapping theorem and the difference quotient.

Most recently Connell [2] has demonstrated how one fundamental property of analytic functions can be proved quite succinctly using only the techniques of algebraic topology. Using homology theory and an analogue of the winding number, he has produced a very simple proof that the existence of one derivative for a function of a complex variable implies the existence of the second derivative.

In a polynomial approach to topological analysis the basic tools are, of course, the complex polynomials and the central result sought is the existence of a power series expansion for functions analytic in some bounded region R. In this paper two developments leading to such a power series expansion will be elucidated. Both of these developments have their origins in the work of K.O. Leland ([9],[10]). In the first development (Section 2) the family of complex valued functions $A = \{f : f \in C^2 \text{ on } U, f \in C^0 \text{ on } \overline{U}\}$ will be the focal point where U is the open unit disc in the complex plane and $\overline{U}$ is its closure. Using this attack many basic results of topological analysis will be obtained for twice continuously differentiable functions without the use of the integral, measure theory, topological indexes or algebraic topology. The entire development is self-contained and uses only elementary methods and the Stone-Weierstrass Theorem. The second development (Section 3) is more complicated and not self-contained. Certain basic results will be taken as independently proved by the methods of topological analysis. Polynomials will again be central to the development however. At the beginning of each of the sections a brief outline of the development will be given. The theorems used without proof in Section 3 are listed in the appendix with references. These theorems will be referred to by letter, e.g., Theorem B.

In the paper the following notation will be used. Let K denote the complex plane and ω the positive integers. For $r > 0$, let $U(r)$ denote the interior of the circle $C(r)$ with center at the origin and radius r. As noted above $U(1)$ is denoted simply by U; further $C(1)$ will be denoted by C. For real numbers a and b, $R(a+bi)=a$ and $I(a+bi)=b$. If M and N are subsets of K, we let $\delta(M,N)$ denote the real number inf $\{|x-y|:x\in M, y\in N\}$. If R is a subset of K then we shall denote the set $\bar{R}-R$ by $B(R)$, the boundary of R. If V is a simple closed curve, let $E(V)$ denote the exterior (unbounded component of the complement) of V and $I(V)$ the interior (bounded component of the complement) of V.

Let f and g be functions on subsets of K such that the range of g lies in the domain of f. Then $f\circ g$ shall denote the function h such that $h(z)=f[g(z)]$ for all z in the domain of g. For $z\in K$, let $I_0(z)$ be the identity mapping. Let f be a function defined on a set S in K, into K. If f is continuous, we shall call f a map of S into K; f is called an open map, if $f(V)$ is open in K for all open sets $V\subseteq S$.

2. The Stone-Weierstrass Approach

In Theorem 2.1, we prove the Maximum Modulus Theorem for the elements of the family A described in the introduction. In Theorem 2.2, applying Theorem 2.1, we adapt a theorem of Porcelli and Connell [5] to show that all functions which are uniform limits on $\bar{U}$ of sequences of polynomials lie in A and are infinitely differentiable on U. Theorem 2.3 is a particular case of Schwarz's Lemma and is used to prove Theorem 2.4, the key to the Section. It makes use of a simple auxiliary function to obtain growth rate extimates for polynomials which depend only on the magnitude of their real parts.

Employing the Stone-Weierstrass Theorem we show that every continuous function on the boundary C of U may be extended to a function on $\overline{U}$ which is the uniform limit on $\overline{U}$ of the real parts of a sequence of polynomials. Applying Theorem 2.4, we show that these sequences must converge on U to complex differentiable functions.

Given a twice continuously differentiable function f on U, we then readily obtain a polynomial sequence approximating f on U. Simple arguments of Leland ([9],[10]) and of Porcelli and Connell [4] are then used to convert this sequence into a power series expansion for f on U.

In the last part of the development we obtain as a by-product the theory of harmonic functions in the two-dimensional case, including the existence of conjugate harmonic functions and the resolution of the Dirichlet problem for the circle. The Fundamental Theorem of Algebra and the open mapping theorem are also verified.

Finally it should be noted that the restriction of our attention to twice continuously differentiable functions is not as narrow as might appear. For the results may be readily extended to complex differentiable functions in general by making use of Whyburn's Maximum Modulus Theorem [15].

<u>Theorem 2.1</u>: <u>Let</u> $f \in A$ <u>and set</u> $u = Rf$, $v = If$. <u>Then for</u> $z \in U$:

(1) $|u(z)| \leq \sup \{|u(t)| : t \in C\}$,

(2) $|v(z)| \leq \sup \{|v(t)| : t \in C\}$,

(3) $|f(z)| \leq \sup \{|f(t)| : t \in C\}$.

<u>Proof</u>: For the functions u and v, the Cauchy-Riemann equations hold, i.e., $u_x = v_y$ and $u_y = -v_x$.

On U, since $u, v \in C^2$ on U, we have

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = v_{xy} - v_{xy} = 0.$$

Similarly, $v_{xx} + v_{yy} = 0$ on U.

Given $\varepsilon > 0$, let $r(z) = (\varepsilon/4) \cdot |z|^2$ for $z \in \overline{U}$. Let $w = u + r$. The function w is a real valued one of two variables defined on $\{(x,y) : x^2+y^2 \leq 1\}$. Further,

$$w_{xx} + w_{yy} = u_{xx} + u_{yy} + r_{xx} + r_{yy} .$$

But $r = (\varepsilon/4) \cdot |z|^2 = (\varepsilon/4)(x^2+y^2)$, where $z = x+iy$.

Then $r_x = (\varepsilon/4)(2x) = (\varepsilon/2)x$ and $r_{xx} = \varepsilon/2$.

Similarly, $r_{yy} = \varepsilon/2$.

Finally, $w_{xx} + w_{yy} = 0 + \varepsilon/2 + \varepsilon/2 = \varepsilon$ for $z \in U$.

Assume that for some $z \in U$, $w(z) \geq \sup \{w(t) : t \in C\}$. Since w is a real valued continuous function defined on the closed set $\overline{U}$, it assumes its maximum at some point of $\overline{U}$. By the assumption w assumes its maximum at some point of U. Hence, there exists $z_0 \in U$ such that $w(z_0) = \sup \{w(t) : t \in \overline{U}\}$. Thus $w_x(z_0) = w_y(z_0) = 0$ which is a necessary condition for a real valued function of two variables to have a maximum at an interior point of its domain. If $w(z_0)$ is a maximum value of the function $w(z)$, then w restricted to the real axis and w restricted to the imaginary axis must also have a maximum at z_0. However, for all $z \in U$, $w_{xx} + w_{yy} = \varepsilon$. Thus either $w_{xx} > 0$ or $w_{yy} > 0$ and w restricted to the real axis or w restricted to the imaginary axis must have a minimum at z_0. Since w is not a constant function, we have a contradiction. Thus $w(z) < \sup \{w(t): t \in C\}$ for $z \in U$. Letting $\varepsilon \to 0$, we have $u(z) \leq \sup \{u(t): t \in C\}$ for $z \in U$.

In a similar manner by using the function $w_1 = -u + r$, we can show that $w_1(z) < \sup \{w_1(t): t \in C\}$ for $z \in U$ and that

$$-u(z) \leq \sup \{-u(t): t \in C\} \text{ for } z \in U.$$

But $u(z) \leq \sup \{u(t): t \in C\} \leq \sup \{|u(t)|: t \in C\}$ and

$$-u(z) \leq \sup \{-u(t): t \in C\} \leq \sup \{|u(t)|: t \in C\}.$$

Hence, $|u(z)| \leq \sup \{|u(t)|: t \in C\}$ for $z \in U$.

The argument for (2) is analogous since v is also a real valued harmonic function. Simply replace u by v wherever it appears and (2) follows immediately.

To prove (3), assume there exists $z_0 \in U$ such that $|f(z_0)| > M = \sup \{|f(t)|: t \in C\}$.

Consider the function $g(z) = f(z)/M$ for $z \in \overline{U}$.

$|g(z_0)| = |f(z_0)|/M > 1$, and for $t \in C$,

$|g(t)| = |f(t)|/M \leq 1$.

There exists $n \in \omega$ such that $|g(z_0)^n| > 2$. Clearly the function $[g(z)]^n \in A$. By the triangle inequality,

$$|g(z_0)^n| \leq |R[g(z_0)^n]| + |I[g(z_0)^n]|.$$

Then by parts (1) and (2),

$|g(z_0)^n| \leq \sup\{|R[g(t)^n]| : t\in C\} + \sup\{|I[g(t)^n]| : t\in C\}$,

and $|g(z_0)^n| \leq 2 \sup\{|g(t)^n| : t\in C\}$.

Finally, $|g(z_0)^n| \leq 2$ since $|g(t)| \leq 1$ for $t\in C$. This contradiction proves (3).

<u>Theorem 2.2</u>: <u>Let</u> $P_1, P_2, \ldots$ <u>be a sequence of polynomials, which converges uniformly on</u> $\overline{U}$ <u>to a limit function</u> f. <u>Then,</u> $f\in A$ <u>and all derivatives of</u> f <u>on</u> U <u>exist</u>.

<u>Proof</u>: Without loss of generality, we may take $P_i(0) = 0$ for $i\in\omega$. For $z\in\overline{U}$ and $i\in\omega$ set

$$Q_i(z) = \begin{cases} P_i(z)/z & \text{for } z \neq 0 \\ P_i'(0) & \text{for } z = 0 . \end{cases}$$

Let $z\in\overline{U}$, $z \neq 0$. Then $|z| \neq 0$ and there is $N\in\omega$ such that for $n \geq N$, $n\in\omega$,

$$|P_n(z) - f(z)| < \varepsilon\cdot|z| ,$$

where ε is an arbitrary positive number. Hence, $|P_n(z)/z - f(z)/z| < \varepsilon$ for $n \geq N$ and we have shown that $\{Q_i\}_{i=1}^{\infty}$ is a sequence of polynomials which converges pointwise to $f(z)/z$ for all $z\in\overline{U}$, $z \neq 0$.

Furthermore, if $z\in C$, then there is $M\in\omega$ such that for $n \geq M$, $n\in\omega$, and $\varepsilon>0$, $|P_n(z) - f(z)| < \varepsilon$.
Hence, $|P_n(z)/z - f(z)/z| < \varepsilon/|z|$ for $n \geq M$. But for $z\in C$, $|z| = 1$ and we have

$$|Q_n(z) - f(z)/z| < \varepsilon \text{ for } n \geq M \text{ and } z\in C.$$

Thus $\{Q_i\}_{i=1}^{\infty}$ is a sequence of polynomials which converges uniformly on C to $f(z)/z$.

Since Q_i, $i \in \omega$ is a polynomial, $Q_i \in A$. Also $(Q_m - Q_n)$ is an element of A for all $m,n \in \omega$. By Theorem 2.1,

$$|Q_m(z) - Q_n(z)| \leq |Q_m(t_\circ) - Q_n(t_\circ)|$$

for all $z \in U$ and some $t_\circ \in C$.

We note here that the complex plane K and all closed subsets of K are complete. Hence, a sequence is convergent (in K or in the closed subset of K) if and only if the sequence is a Cauchy sequence. Thus there exists $N \in \omega$ such that for $m,n \geq N$ and all $t \in C$,

$$|Q_m(t) - Q_n(t)| < \varepsilon.$$

Then, $|Q_m(z) - Q_n(z)| < \varepsilon$ for all $m,n \geq N$ and $z \in \overline{U}$.

By the remark above, there exists a limit function $Q_\circ$ such that the sequence $\{Q_i\}_{i=1}^{\infty}$ converges uniformly to $Q_\circ$ on $\overline{U}$. Thus $f'(0)$ exists and in fact is equal to $Q_\circ(0) = \lim_{n\to\infty} Q_n(0) = \lim_{n\to\infty} P_n'(0)$. Finally, f is differentiable on U.

Let $0 < \rho < 1$, $m,n \in \omega$, $z \in U(\rho)$. Then,

$$\begin{aligned} |P_m'(z) - P_n'(z)| &\leq \sup\left\{\left|\frac{P_n(t)-P_n(z)}{t-z} - \frac{P_m(t)-P_m(z)}{t-z}\right| : t \in C\right\} \\ &\leq (1-\rho)^{-1} \sup\{|P_n(t)-P_n(z)-P_m(t)+P_m(z)| : t \in C\} \\ &\leq (1-\rho)^{-1} \sup\{|P_n(t)-P_n(z)| + |P_m(t)-P_m(z)| : t \in C\} \\ &\leq 2(1-\rho)^{-1} \sup\{|P_n(t)-P_m(t)| : t \in C\}. \end{aligned}$$

Let $\varepsilon > 0$. There exists $N \in \omega$ such that for $m,n \geq N$, $m,n \in \omega$,

$$|P_n(t) - P_m(t)| < \varepsilon \cdot (1-\rho)/2 .$$

Finally,

$$|P'_m(z)-P'_n(z)| < [2/(1-\rho)][(1-\rho)/2]\cdot\varepsilon = \varepsilon,$$

and the sequence of polynomials $\{P'_i\}_{i=1}^{\infty}$ converges uniformly on compact subsets of U to f'.

An application of the above process to the polynomials $\{P'_i\}_{i=1}^{\infty}$ will show that f" exists and that the sequence $\{P''_i\}_{i=1}^{\infty}$ converges uniformly on compact subsets of U to f". Furthermore, continued application of the same technique leads to the conclusion that f possesses derivatives of all orders on U. Trivially, we also have that $f\in A$.

Theorem 2.3: Let $f\in A$ such that $f(0) = 0$ and such that there exists a sequence of polynomials $\{P_i\}_{i=1}^{\infty}$ which converges uniformly on $\overline{U}(\delta)$ to f for some $0<\delta<1$. Then for $z\in U$, $|f(z)| \leq M\cdot|z|$ where $M = \sup\{|f(t)| : t\in C\}$.

Proof: For $z\in\overline{U}$, set $g(z) = \begin{cases} f(z)/z & \text{for } z \neq 0 \\ f'(0) & \text{for } z = 0 \end{cases}$.

If $z \neq 0$, $g'(z) = [zf'(z)-f(z)]/z^2$ and

$$g''(z) = [z^2f''(z)-2zf'(z)+2f(z)]/z^3 .$$

Thus for $z\in U$, $z \neq 0$, $g'(z)$ and $g''(z)$ exist and are continuous since $f\in A$.

Let us now consider the case of $z = 0$. For $z\in\overline{U}(\delta)$, define $Q_i(z) = \begin{cases} P_i(z)/z & \text{for } z \neq 0 \\ P'_i(0) & \text{for } z = 0 \end{cases}$.

Following the proof of Theorem 2.2, we conclude that $\{Q_i\}_{i=1}^{\infty}$ converges uniformly on $\overline{U}(\delta)$ to g. Hence, $g\in A$ on $\overline{U}(\delta)$ and both $g'(0)$ and $g''(0)$ exist and are continuous in particular. Thus we have shown that $g \in A$.

Applying Theorem 2.1, for $z \in \overline{U}$, $z \neq 0$, we have

$$|f(z)/z| = |g(z)| \leq \sup \{|g(t)| : t \in C\}$$
$$\leq \sup \{|f(t)/t| : t \in C\} = M.$$

Finally, $|f(z)| \leq M \cdot |z|$.

If $z=0$, $|f(0)| = 0$ and $0 = M \cdot |z| = 0$.

Corollary (Fundamental Theorem of Algebra): Let P be a Polynomial and suppose that P has no roots. Then P is a constant.

Proof: Let P be a polynomial and assume P has no roots. Then 1/P is a bounded, twice differentiable function in K. From Theorem 2.3, for $z \in K$ and $|z| < r$,

$$|P(z) - P(0)| \leq \sup \{|P(t)| : t \in C(r)\} \cdot |z|/r$$
$$\leq \sup \{|P(t)| : t \in K\} \cdot |z|/r.$$

If $\sup \{|P(t)| : t \in K\} = M$, we have

$$|P(z) - P(0)| \leq M \cdot |z|/r.$$

Letting $r \to \infty$, we see that $|P(z) - P(0)| \to 0$. Hence, $|P(z) - P(0)| = 0$ and $P(z) = P(0)$ as was to be shown.

Lemma 2.1: Let $f \in A$ and set $M = \sup \{|Rf(t)| : t \in C\}$. Then,

$$|f(z)|^2 \leq |2M + f(z)|^2 \text{ for } z \in \overline{U}.$$

Proof: Let $u = Rf$. From Theorem 2.1, $-u \leq |u| \leq M$ for $z \in \overline{U}$.

Hence, $0 \leq M + u$

$$\leq 4M^2 + 4Mu$$

and $|f(z)|^2 \leq 4M^2 + 4Mu + |f(z)|^2$.

Consider the product $T = [2M + f(z)][2M + \overline{f(z)}]$.

$$T = 4M^2 + 2Mf(z) + 2M\overline{f(z)} + |f(z)|^2$$
$$= 4M^2 + 4M[f(z)+\overline{f(z)}]/2 + |f(z)|^2$$
$$= 4M^2 + 4Mu + |f(z)|^2 .$$

Hence, $|f(z)|^2 \leq [2M+f(z)][2M+\overline{f(z)}] = [2M+f(z)][\overline{2M+f(z)}]$
$$\leq |2M + f(z)|^2 .$$

<u>Theorem 2.4</u>: <u>Let</u> $0 < r < 1$, <u>and let</u> P <u>be a polynomial such that</u> $P(0) = 0$. <u>Then for</u> $z \in U(r)$, $|P(z)| \leq 2M \cdot |z|/(1-r)$ <u>where</u> $M = \sup \{|RP(t)| : t \in C\}$.

<u>Proof</u>: For $z \in \overline{U}$, set $g(z) = P(z)/[2M + P(z)]$. Unless $P \equiv 0$, $g \in A$. By Lemma 2.1, $|P(z)|^2 \leq |2M + P(z)|^2$. Thus,

$$|P(z)/[2M + P(z)]|^2 \leq 1 \quad \text{and}$$
$$|g(z)| \leq 1.$$

For $z \in K$, $n \in \omega$, set $Q_n(z) = -\sum_{i=1}^{n} [-P(z)/2M]^i$. $P \in A$ and the sequence $P_i \equiv P$ for $i \in \omega$ converges uniformly to P on $\overline{U}$. Also $P(0) = 0$. Thus we may apply Theorem 2.3 to P and we have

$$|P(z)| \leq N \cdot |z| \text{ for } z \in U \text{ where } N = \sup \{|P(t)| : t \in C\} .$$

Hence, $|-P(z)/2M| = |P(z)|/2M \leq (N/2M)|z|$ for $z \in U$. Since $N/2M$ is a fixed constant, we can find $\delta > 0$ such that $(N/2M) \cdot \delta \leq 1/2$. Consequently, for $z \in \overline{U}(\delta)$,

$$|-P(z)/2M| \leq (N/2M) \cdot |z| \leq (N/2M) \cdot \delta \leq 1/2 .$$

The sequence of polynomials $\{Q_i\}_{i=1}^{\infty}$ converges uniformly on $\overline{U}(\delta)$ to $-[-P(z)/2M][1 + P(z)/2M]^{-1}$ since $|-P(z)/2M| < 1$ for $z \in \overline{U}(\delta)$. Thus, $\{Q_i\}_{i=1}^{\infty}$ converges uniformly on $\overline{U}(\delta)$ to

$$[P(z)/2M][2M + P(z)/2M]^{-1} = P(z)/[2M + P(z)] = g(z).$$

Further, since $g(0) = 0$, we may apply Theorem 2.3 to $g(z)$.

Let $z \in U(r)$. We already know that $|g(z)| \leq 1$ for $z \in \overline{U}$. Hence, $\sup \{|g(t)| : t \in C\} \leq 1$ and $|g(z)| \leq |z|$ for $z \in U(r)$ by Theorem 2.3. We now have

$$|g(z)| = |P(z)/[2M + P(z)]| \leq |z| .$$

Therefore,

$$|P(z)| \leq |2M + P(z)| \cdot |z| \leq 2M \cdot |z| + |P(z)| \cdot |z| , \text{ and}$$

$$\begin{aligned} |P(z)| \cdot (1-|z|) &= |P(z)| - |P(z)| \cdot |z| \\ &\leq 2M \cdot |z| + |P(z)| \cdot |z| - |P(z)| \cdot |z| \\ &\leq 2M \cdot |z| . \end{aligned}$$

Since $|z| \leq r < 1$, $1 - |z| > 0$. Thus,

$$|P(z)| \leq (2M \cdot |z|)/(1-|z|) \leq (2M \cdot |z|)/(1-r).$$

Let X be a compact Hausdorff space. Let $C^*(X)$ be the set of all continuous complex valued functions on X. The set $C^*(X)$ is a linear space since any constant multiple of a continuous function is continuous and the sum of two continuous functions is continuous. The space $C^*(X)$ becomes a normed linear space if we define $\|f\| = \sup \{|f(x)| : x \in X\}$, and a metric space if we set $\rho(f,g) = \|f - g\|$. As a metric space $C^*(X)$ is complete.

Definition 2.1: A linear space A of functions in $C^*(X)$ is called an algebra if the product of any two elements in A is again in A. A family T of functions in $C^*(X)$ is called a subalgebra if for any two functions f and g in T and any complex numbers a and b, we have $af + bg \in T$ and $fg \in T$.

Definition 2.2: An algebra A in $C^*(X)$ is called a Banach algebra if A has a norm such that A is a complete normed linear space and multiplication satisfies the condition: $\|fg\| \leq \|f\| \cdot \|g\|$.

The boundary of the unit disc, C, is a compact Hausdorff space. $C^*(C)$ becomes a Banach algebra if we define $\|f\| = \sup \{|f(z)| : z \in C\}$.

Lemma 2.2: Let $z \in C$. Then

(1) $z^n \bar{z}^m = z^{n-m}$ if $n > m$,

(2) $z^n \bar{z}^m = \bar{z}^{m-n}$ if $n < m$,

(3) $z^n \bar{z}^m = |z|^{2n}$ which is a real number.

Proof: If $z \in C$, then $z = e^{i\theta}$, $\bar{z} = e^{-i\theta}$ where $0 \leq \theta < 2\pi$.

Thus, $z^n = e^{in\theta}$ and $\bar{z}^m = e^{-im\theta}$.

(1) If $n > m$, then,

$$z^n \bar{z}^m = (e^{in\theta})(e^{-im\theta}) = e^{i(n-m)\theta} = z^{n-m} .$$

(2) If $n < m$, then,

$$z^n \bar{z}^m = (e^{in\theta})(e^{-im\theta}) = e^{-i(m-n)\theta} = \bar{z}^{m-n} .$$

(3) If $n=m$, then,

$$z^n \bar{z}^m = z^n \bar{z}^n = (z\bar{z})^n = (|z|^2)^n = |z|^{2n} .$$

Lemma 2.3: Let $C^*(C)$ be the Banach algebra of continuous complex valued functions on C. Then the family T generated by functions of the form $P(z)$ and $\overline{P(z)}$ where $z \in C$ and P is a complex polynomial is a subalgebra of $C^*(C)$.

Proof: Let S(C) be the set of complex polynomials over C. Let $a, b \in K$, and $P, Q \in S(C)$. Then we have

(1) $aP + bQ = R$ where $R \in S(C)$,

(2) $a\bar{P} + b\bar{Q} = \bar{R}$ where $R \in S(C)$,

(3) $aP + b\overline{Q} = R + \overline{N}$ where $R, N \in S(C)$,

(4) $a\overline{P} + bQ = \overline{R} + N$ where $R, N \in S(C)$.

Thus for $a,b \in K$, $f,g \in T$, we can conclude that $(af+bg) \in T$. Furthermore, $P \cdot Q = R$ where $R \in S(C)$ and $\overline{P} \cdot \overline{Q} = \overline{R}$ where $R \in S(C)$. If we now consider the remaining product possibility $P \cdot \overline{Q}$, by Lemma 2.2 terms involving $z^n \bar{z}^m$ reduce to terms of the form z^{n-m}, $\bar{z}^{m-n}$, or to real numbers. Hence, $P \cdot \overline{Q} = R + \overline{Q}$ where $R, Q \in S(C)$. We now have that if $f,g \in T$, then $(fg) \in T$. Thus T is a subalgebra of $C^*(C)$.

Lemma 2.4: Let $C^*(C)$ be the Banach algebra of continuous complex valued functions on C. If T is the subalgebra of $C^*(C)$ generated by functions of the form $P(z)$ and $\overline{P(z)}$ where $z \in C$ and P is a complex polynomial, then $\overline{T} = C^*(C)$.

Proof: By Lemma 2.3, T is a subalgebra of $C^*(C)$. If $f \in T$, then Rf and If are elements of T since $Rf = (f+\overline{f})/2$ and $If = (f-\overline{f})/2i = (f-\overline{f})i/2$.

Let $a,b \in C$, $a \neq b$. The identity function $I_0(z)$ is an element of T. Hence, Rz and Iz are real valued elements of T. Since $a \neq b$, either $Ra \neq Rb$ or $Ia \neq Ib$.

Define the function $g(z)$ as follows for $z \in C$:

$$g(z) = \begin{cases} Rz & \text{if } Ra \neq Rb \\ Iz & \text{if } Ra = Rb \end{cases} .$$

We observe that $g \in T$ and $g(a) \neq g(b)$. Thus we have found an element of T which separates points. Since the complex valued and real valued constant functions on C are elements of T, we may apply the Stone-Weierstrass Theorem to T and conclude that $\overline{T}$ must contain all continuous real valued functions on C. Finally, if $f \in C^*(C)$, $f = u+iv$, where u and v are real valued continuous functions on C. Thus, $\overline{T} = C^*(C)$.

Theorem 2.5 (Leland [9]): *Let ϕ be a continuous real valued function on C. Then:*

(1) *There exists a continuous real valued function h on $\overline{U}$ such that $h(z) = \phi(z)$ for all $z \in C$.*

(2) *There exists a complex valued function w on U such that all derivatives of w on U exist and such that $h(z) = Rw(z)$ for all $z \in U$.*

(3) *If $f \in A$, there exists a sequence of polynomials $\{P_i\}_{i=1}^{\infty}$ which converge uniformly on $\overline{U}$ to f.*

Proof: (1) Let $\phi(z)$ be a continuous real valued function on C. By Lemmas 2.3 and 2.4, there exist sequences of polynomials $\{P_i\}_{i=1}^{\infty}$ and $\{Q_i\}_{i=1}^{\infty}$ such that the sequence $\{P_i+\overline{Q}_i\}_{i=1}^{\infty}$ converges uniformly on C to ϕ. Without loss of generality, we may take $P_i(0) = Q_i(0) = 0$ for $i \in \omega$.

$$
\begin{aligned}
\text{For } i \in \omega,\ R(P_i+\overline{Q}_i) &= [(P_i+\overline{Q}_i) + \overline{(P_i+\overline{Q}_i)}]/2 \\
&= [(P_i+\overline{Q}_i) + (\overline{P}_i+Q_i)]/2 \\
&= [(P_i+Q_i) + (\overline{P}_i+\overline{Q}_i)]/2 \\
&= [(P_i+Q_i) + \overline{(P_i+Q_i)}]/2 \\
&= R(P_i+Q_i)\ .
\end{aligned}
$$

Let $h_i = R(P_i+Q_i)$ for $i \in \omega$. Since $\{P_i+\overline{Q}_i\}_{i=1}^{\infty}$ converges uniformly on C to the real valued function ϕ, the sequence $\{R(P_i+\overline{Q}_i)\}_{i=1}^{\infty}$ converges uniformly to ϕ on C.

But $h_i = R(P_i+Q_i) = R(P_i+\bar{Q}_i)$ for $i \in \omega$. Hence, $\{h_i\}_{i=1}^{\infty}$ converges uniformly to ϕ on C.

Let $\varepsilon > 0$. There exists $N \in \omega$ such that for $m,n \geq N$, $t \in C$,

$$|R[P_m(t)+Q_m(t)] - R[P_n(t)+Q_n(t)]| < \varepsilon .$$

But the function $P_m + Q_m - P_n - Q_n \in A$ for all $m,n \in \omega$. Thus from Theorem 2.1, for $m,n \geq N$, $z \in \bar{U}$, we have

$$|h_m(z)-h_n(z)| = |R[P_m(z)+Q_m(z)]-R[P_n(z)+Q_n(z)]|$$

$$\leq \sup\{|R[P_m(t)+Q_m(t)]-R[P_n(t)]-Q_n(t)]| : t \in C\}.$$

Hence, $|h_m(z) - h_n(z)| < \varepsilon$ for $m,n \geq N$, $z \in \bar{U}$, and the sequence $\{h_i\}_{i=1}^{\infty}$ converges uniformly on $\bar{U}$ to a limit function h such that $h(z) = \phi(z)$ for $z \in C$.

(2) Let us now consider the sequence $\{P_i+Q_i\}_{i=1}^{\infty}$ formed in part (1) of this theorem. Since $P_i(0) = Q_i(0) = 0$, $P_i(0)+Q_i(0) = 0$ for $i \in \omega$. Thus we may apply Theorem 2.4 to the polynomials $P_i(z)+Q_i(z)$, $i \in \omega$.

Let $0 < r < 1$ and $\varepsilon > 0$. There exists $N \in \omega$ such that for $t \in C$, $m,n \geq N$, $m,n \in \omega$,

$$|h_m(t)-h_n(z)| = |R[P_m(t)+Q_m(t)] - R[P_n(t)+Q_n(t)]|$$

$$< (\varepsilon/2)[(1-r)/r]$$

Let $M = \sup\{|R[P_m(t)+Q_m(t)-P_n(t)-Q_n(t)| : t \in C\}$. By Theorem 2.4 we have for $m,n \in \omega$ and $z \in U(r)$,

$$|[P_m(z)+Q_m(z)] - [P_n(z)+Q_n(z)]| \leq (2M \cdot |z|)/(1-r).$$

We note that $M = \sup\{|R[P_m(t)+Q_m(t)]-R[P_n(t)+Q_n(t)| : t \in C\}$. Thus, $M < (\varepsilon/2)(1-r)/r$ since the supremum is assumed for some $t_0 \in C$.

Hence, for $z \in U(r)$, $m,n \geq N$,

$$|[P_m(z)+Q_m(z)]-[P_n(z)+Q_n(z)]| < [2r\varepsilon(1-r)]/[2r(1-r)]$$
$$< \varepsilon.$$

Thus, we see that the sequence $\{P_i+Q_i\}_{i=1}^{\infty}$ converges uniformly on compact subsets of U to a limit function w. By Theorem 2.2, all derivatives of w on U exist. Further, $h(z) = Rw(z)$ for $z \in U$ as desired.

(3) Let $f \in A$. Set $\phi(z) = Rf(z)$ for $z \in C$.
From Theorem 2.1 we have

$$|Rf(z)-h(z)| \leq \sup_{0<r<1} \{|R[f(t)-h(t)]| : t \in C(r)\} = 0.$$

Thus $Rf(z) = h(z)$ for $z \in \overline{U}$.

Set $u = R(f-w)$ and $v = I(f-w)$.

$R(f-w) = Rf - Rw = Rf - h = 0$ for $z \in \overline{U}$. Thus $u \equiv 0$ on $\overline{U}$. Consequently, $0 = u_x = v_y$ and $0 = u_y = v_x$ on U.
Let $z \in U$, $z = x+iy$. Since $v_x = v_y = 0$ on U, $v \equiv c$ on $\overline{U}$, where c is a real constant. Thus, $v = I(f-w) = If - Iw = If(0) - If(0) = c$. Since $Iw(0) = 0$ from part (2), $v \equiv c$ on $\overline{U}$ where $If(0) = c$. Finally, $f-w = u+iv = ic$ and $f = w+d$ where d is a complex constant.

If $\{P_i+Q_i\}_{i=1}^{\infty}$ is the sequence considered in part (2) which converges uniformly on compacta of U to w, set $R_i(z) = P_i(z) + Q_i(z) + d$ for $i \in \omega$, $z \in \overline{U}$. Clearly the sequence $\{R_i\}_{i=1}^{\infty}$ converges uniformly to $f = w+d$ on compacta of U.

Let $\varepsilon > 0$. Then there exists $0 < \delta < 1$ such that for all $z,t \in \overline{U}$ with $|z-t| \leq 1-\delta$, $|f(z)-f(t)| < \varepsilon/2$. This follows from the fact that, since f is continuous on the compact set $\overline{U}$, it is uniformly continuous there.

Further, there exists $N \varepsilon \omega$ such that $|f(z)-R_n(z)| < \varepsilon/2$ for $n \geq N$, $n \varepsilon \omega$ and for all $z \varepsilon \overline{U}(\delta)$.

Let $A_i(z) = R_i(\delta z)$ for all $z \varepsilon K$, $i \varepsilon \omega$. If $z \varepsilon \overline{U}$, then $\delta z \varepsilon \overline{U}(\delta)$. Also $|z-\delta z| = |(1-\delta)\cdot|z| \leq 1-\delta$ for $z \varepsilon \overline{U}$.

Thus for $n \geq N$, $z \varepsilon \overline{U}$,

$$\begin{aligned} |f(z)-A_n(z)| &= |f(z)-f(\delta z)+f(\delta z)-R_n(\delta z)| \\ &\leq |f(z)-f(\delta z)| + |f(\delta z)-R_n(\delta z)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon . \end{aligned}$$

Hence, the sequence $\{A_i\}_{i=1}^{\infty}$ converges uniformly to f on $\overline{U}$ as desired.

Remark: Since $h_{xx} + h_{yy} = 0$ on U, $h(z) = \phi(z)$ for $z \varepsilon C$, we have solved the Dirichlet problem for the circle [6]. Iw is called the conjugate harmonic function of h.

Theorem 2.6 (Porcelli and Connell [4, p.232]): If $P(z) = \sum_{p=0}^{n} a_p z^p$ for $z \varepsilon K$, and $|P(z)| \leq 1$ for $z \varepsilon \overline{U}$, then $|a_i| \leq 1$ for $i = 0,1,2,...,n$.

Proof: Trivially, the theorem holds for polynomials of degree zero. Suppose for $n \varepsilon \omega$, it holds for polynomials of degree n or less. Let $P(z) = \sum_{p=0}^{n+1} a_p z^p$ be a polynomial of degree n+1 with $|P(z)| \leq 1$ for $z \varepsilon \overline{U}$.

Let $\theta \varepsilon C$, i.e., $|\theta| = 1$.

Set $Q(z) = 1/2[P(z)-P(\theta z)]$. Then $Q(0) = 0$.

$$\text{Set } Q_0(z) = \begin{cases} Q(z)/z & \text{for } z \neq 0 \\ Q'(0) & \text{for } z = 0 \end{cases}.$$

Q_0 is a polynomial of degree n, and from Theorem 2.1,

$$|Q_0(z)| \leq \sup\{|Q_0(t)| : t\in C\} = \sup\{|Q(t)| : t\in C\} .$$
$$\leq \sup\{1/2\,|P(t)-P(\theta t)| : t\in C\}$$
$$\leq 1/2 \sup\{|P(t)| : t\in C\} + 1/2 \sup\{|P(\theta t)| : t\in C\}$$
$$\leq 1/2\,(1+1) = 1 .$$

By the induction hypothesis, for $p = 0,1,2,\dots,n$,

$$|(a_{p+1}/2)(1-\theta^{p+1})| \leq 1.$$

Choose $\theta \in C$ such that $\theta^{p+1} = -1$.

Then we have for $p = 0,1,2,\dots,n$,

$$|(a_{p+1}/2)(1-\theta^{p+1})| = |(a_{p+1}/2)(1+1)| = |a_{p+1}| \leq 1 .$$

Finally, $|a_0| = |P(0)| \leq 1$ by hypothesis. Hence, $|a_p| \leq 1$ for $p = 0,1,2,\dots,n+1$ and the proof is complete.

Corollary 1: *If* $P(z) = \sum_{p=0}^{m} a_p z^p$ *for* $z\in K$, *and* $|P(z)| \leq 1$ *for* $z\in \bar{U}(1-1/n)$ *where* $n\in\omega$, *then* $|a_p| \leq (1-1/n)^{-p}$ *for* $p = 0,1,2,\dots,m$.

Proof: Let $P(z) = \sum_{p=0}^{m} a_p z^p$ for $z\in K$, and $|P(z)| \leq 1$ for $z\in\bar{U}(1-1/n)$.

For $z\in K$ set $Q(z) = \sum_{p=0}^{m} a_p[(n-1)z/n]^p = P[(n-1)z/n]$.

Let $z\in\bar{U}$. Since $|z| \leq 1$, $[(n-1)/n]\cdot|z| \leq (n-1)/n$. Hence, $|[(n-1)/n]z| \leq (n-1)/n$ and $[(n-1)/n]z\in\bar{U}(1-1/n)$. Thus, $|Q(z)| \leq 1$ for $z\in\bar{U}$.

Further, $Q(z) = \sum_{p=0}^{m} a_p[(n-1)/n]^p\, z^p =$

From Theorem 2.6 we have, for $p=0,1,2,\ldots,m$,

$$|a_p[(n-1)/n]^p| \le 1 .$$

Finally,

$$|a_p| \le [(n-1/n]^{-p} = (1-1/n)^{-p} \text{ for } p=0,1,2,\ldots,m.$$

Corollary 2. *If* $P(z) = \sum_{p=0}^{n} a_p z^p$ *for* $z \in K$ *and* $|P(z)| \le \theta$ *for some* $\theta > 0$ *and all* $z \in \bar{U}$, *then* $|a_p| \le \theta$ *for* $p=0,1,\ldots n$.

Proof: Let $P(z) = \sum_{p=0}^{n} a_p z^p$ for $z \in K$, and suppose that $|P(z)| \le \theta$ for some $\theta > 0$ and all $z \in \bar{U}$. Let $Q(z) = P(z)/\theta$. $Q(z)$ is a polynomial of degree n and $|Q(z)| \le 1$ for $z \in \bar{U}$. Hence, from Theorem 2.6, we have

$$|a_p/\theta| \le 1 \text{ and } |a_p| \le \theta \text{ for } p=0,1,2,\ldots,n.$$

Theorem 2.7 (Leland [10,p.171]): *Let* $f \in A$. *Then there exists a power series* $T(z) = \sum_{p=0}^{\infty} a_p z^p$, *which converges uniformly on compact subsets of* U *to* f.

Proof: Let $n \in \omega$. By Theorem 2.5 there exists a sequence of polynomials $\{P_i\}_{i=1}^{\infty}$ which converges uniformly on $\bar{U}(1-1/n)$ to f. Hence, there is $p \in \omega$ such that

$$|P_p(z)-f(z)| < (1/2)^{n+1} \text{ for } z \in \bar{U}(1-1/n).$$

Further, $|P_i(z)-P_j(z)| \le (1/2)^n$ for $i,j \ge p$, $i,j \in \omega$, and $z \in \bar{U}(1-1/n)$. Let $Q_n = P_p$, $Q_{n+1} = P_{p+1}, \ldots$. Then for $n \in \omega$, $i,j \ge n$, and $z \in \bar{U}(1-1/n)$,

$$|Q_i(z)-Q_j(z)| \le (1/2)^n .$$

By the Corollaries to Theorem 2.6,

$$|a_{ik}-a_{jk}| \le (1/2)^n(1-1/n)^{-k},$$

for all $k \in \omega$ where $(a_{ik}-a_{jk})$ denotes the kth coefficient of the polynomial $(Q_i - Q_j)$ when that coefficient is defined, and $(a_{ik} - a_{jk}) = 0$ when k is strictly greater than the degree of the polynomial. $\{a_{ij}\}_{j=1}^{\infty}$ is a sequence in K such that $Q_j(z) = \sum_{k=0}^{\infty} a_{jk} z^k$ for $j \epsilon \omega$. But for a fixed $k \epsilon \omega$, this sequence is a Cauchy sequence in K and hence a convergent sequence. Thus there exists $a_k \epsilon K$ such that for $n \epsilon \omega$, $i \epsilon \omega$, $i > n$,

$$|a_k - a_{ik}| \leq 2(1/2)^n (1- 1/n)^{-k} = (1/2)^{n-1}(1/ 1/n)^{-k}$$

Let $n \epsilon \omega$ and let n_0 denote the degree of Q_n. For $k > n_0$, $k \epsilon \omega$, $a_{nk} = 0$ and hence,

$$|a_k| = |a_k - a_{nk}| < (1/2)^{n-1}(1- 1/n)^{-k}.$$

Thus $\limsup_{k \to \infty} |a_k|^{1/k} \leq (1- 1/n)^{-1}$. And since n is arbitrary, we have $\limsup_{k \to \infty} |a_k|^{1/k} \leq 1$.

If we consider the power series $T(z) = \sum_{k=0}^{\infty} a_k z^k$, we see that its radius of convergence is 1. Hence, T(z) converges on compact subsets of U in a uniform manner.

Let $z \epsilon U$. There exists $n \epsilon \omega$ such that $|z| < 1- 2/n$. Thus,

$$\begin{aligned} |z| \cdot (1- 1/n)^{-1} &< [(1- 2/n)][(n-1)/n]^{-1} \\ &< [(n-2)/n][n/(n-1)] \\ &< (n-2)/(n-1) < 1 \quad . \end{aligned}$$

Consider now $|T(z) - Q_n(z)| = |\sum_{k=0}^{\infty} (a_k - a_{nk}) z^k|$.

$$\begin{aligned} |T(z) - Q_n(z)| &\leq \sum_{k=0}^{\infty} |a_k - a_{nk}| \cdot |z|^k \\ &\leq \sum_{k=0}^{\infty} (1/2)^{n-1}(1- 1/n)^{-k} |z|^k \quad . \end{aligned}$$

Since $|z|\cdot(1-1/n)^{-1} < 1$, we have

$$|T(z)-Q_n(z)| \leq (1/2)^{n-1}\{1-[n/(n-1)]\cdot|z|\}^{-1}.$$

But $|z| < (n-2)/n$. Hence,

$$\begin{aligned} |T(z)-Q_n(z)| &\leq (1/2)^{n-1}\{1-[n/(n-1)]\cdot[(n-2)/n]\}^{-1} \\ &\leq (1/2)^{n-1}\{1-[(n-2)/(n-1)]\}^{-1} \\ &\leq (1/2)^{n-1}[(n-1-n+2)/(n-1)]^{-1} \\ &\leq (1/2)^{n-1}[1/(n-1)]^{-1} = (n-1)(1/2)^{n-1}. \end{aligned}$$

Thus $\lim_{n\to\infty} Q_n(z) = T(z)$. Since $\lim_{n\to\infty} Q_n(z) = f(z)$ for all $z\in U$, we have $T(z) = f(z)$ for all $z\in U$. Also, $T(z)$ converges uniformly on compact subsets of U to f.

Theorem 2.8 (Open Mapping Theorem): Let f be a non-constant element of A. Then $F(U)$ is an open set.

Proof: Given $z_0\in U$, we wish to show that $f(z_0)$ is an element of the interior of $f(U)$. Without loss of generality, we may take z_0 to be 0. Let $f\in A$. From Theorem 2.7, f can be expanded in a power series $\sum_{p=0}^{\infty} a_p z^p$. Since f is non-constant, there exists $n\in\omega$ such that $a_n \neq 0$ and $f(z) = a_0 + \sum_{p=n}^{\infty} a_p z^p$ for $z\in U$.

Set $g(z) = \sum_{p=0}^{\infty} a_{n+p} z^p$ for $z\in U$.

$$\text{Thus, } a_0+z^n g(z) = a_0+z^n \sum_{p=0}^{\infty} a_{n+p} z^p = a_0 + \sum_{p=0}^{\infty} a_{n+p} z^{n+p}$$
$$= a_0 + \sum_{p=n}^{\infty} a_p z^p = f(z) \text{ for } z\in U.$$

We note that $g(0) = a_n \neq 0$. The function g is continuous. Hence, there exists $0<\delta<1$ such that $g(z) \neq 0$ for all $z\in\overline{U}(\delta)$. To see this, suppose the contrary. Since

$a_n \neq 0$, then $|a_n| > 0$. From the continuity of g, there is $0 < \delta_1 < 1$ such that $|g(0)-g(z)| < |a_n|$ for $|z| < \delta_1$. Let $\delta = \delta_1/2$. Then $|g(z)-g(0)| < |a_n|$ for $|z| \leq \delta$. If $g(t) = 0$ for some $t \in \bar{U}(\delta)$, then $|g(0)-g(t)| = |g(0)| < |a_n|$. This is a contradiction and we have thus veriried that there exists $0 < \delta < 1$ such that $g(z) \neq 0$ for all $z \in \bar{U}(\delta)$.

Thus, $f(z)-a_0 = z^n g(z) \neq 0$ for all $z \in \bar{U}(\delta)$, $z \neq 0$. Also, $f(z) \neq a_0$ for all $z \in \bar{U}(\delta)$, $z \neq 0$. Since $f(0) = a_0$, $f(0) \notin f[C(\delta)]$.

Let $M = f[\bar{U}(\delta)]$. Then B(M) denotes the set of boundary points of the set M. Suppose now, by way of contradiction, that $f(0) \in B(M)$.

The set M is compact since $\bar{U}(\delta)$ is compact and f is a continuous function. Therefore, M is closed and bounded in K. For the same reason $f[C(\delta)]$ is a compact, closed and bounded subset of K.

Let $r_1 = \delta(\{f(0)\}, f[C(\delta)])$. Since $f(0) \notin f[C(\delta)]$ and since $f[C(\delta)]$ is a closed set, $r_1 > 0$.

Let $w_1 \in K - M$ such that $|w_1-f(0)| < r_1/2$. We are assured of the existence of such a point since the set $\{w: |f(0)-w| < r_1/2\}$ is a neighborhood of f(0) in the w-plane and, since $f(0) \in B(M)$, it contains at least one point w_1 which does not lie in M.

Let $r_2 = \delta(\{w_1\}, M)$. Since $f(0) \in M$, $r_2 < r_1/2$; since M is closed, $0 < r_2$. Furthermore, since M is closed, there exists $w_2 \in M$ such that $|w_1-w_2| = r_2$.

$|f(z)-w_1| \geq r_2$ for all $z \in \bar{U}(\delta)$. Let $t_o \in C(\delta)$ and suppose that $|f(t_o)-w_1| = r_2 < r_1/2$. Then,

$$\begin{aligned} |f(t_o)-f(0)| &= |f(t_o)-w_1+w_1-f(0)| \\ &\leq |f(t_o)-w_1| + |f(0)-w_1| \\ &\leq r_2 + r_1/2 < r_1 . \end{aligned}$$

But this is a contradiction since $|f(t_o)-f(0)| \geq r_1$. Thus $|f(t)-w_1| > r_2 > 0$, for $t \in C(\delta)$.

Let $h(z) = r_2 \cdot [f(z)-w_1]^{-1}$ for $z \in \bar{U}(\delta)$. Since $w_1 \notin M$, $f(z)-w_1 \neq 0$ for all $z \in \bar{U}(\delta)$. Hence, $h \in C^2$ on $U(\delta)$ and $h \in C^\circ$ on $C(\delta)$. We note that

$$\sup \{|h(t)| : t \in C(\delta)\} = \sup \{|r_2 \cdot [f(t)-w_1]^{-1}| : t \in C(\delta)\}$$
$$< 1$$

since $|f(t)-w_1| > r_2$ for all $t \in C(\delta)$.

But there exists $z_2 \in U(\delta)$ such that $f(z_2) = w_2$. Thus

$$|h(z_2)| = |r_2 \cdot [f(t)-w_1]^{-1}| = |r_2 \cdot [w_2-w_1]^{-1}| = r_2/r_2 = 1 .$$

This contradicts Theorem 2.1; hence $f(0) \notin B(M)$. Since $f(0) \in M$, this implies that $f(0)$ is an element of the interior of M and consequently an element of the interior of $f(U)$.

3. A Riemann Mapping Theorem Approach

In this section a somewhat more complicated attack is followed to derive a power series expansion for functions differentiable on the unit disc. The open mapping theorem and the difference quotient are basic tools in the development; the key intermediate theorem is an adaptation of a standard proof of the Riemann Mapping Theorem [13] in Theorem 3.3. Most of the work done previous to the proving of this theorem is simply auxiliary in function although some of it has independent import. For example, a useful Maximum Modulus Principle for the difference quotient is obtained in Theorem 3.1 and a basic theorem concerning families of uniformly bounded differentiable functions is verified in Theorem 3.2.

Following Theorem 3.3, the inverse function of the function obtained in that theorem, which maps U in a one-to-one manner onto a bounded region S, is shown to be unique. Consequently, it is proved that given an arbitrary differentiable function f on U, there exists a sequence of polynomials $\{P_i\}_{i=1}^{\infty}$ which converges uniformly on compact subsets of U to f. Finally, a power series expansion is obtained in Theorem 3.6 in exactly the same fashion as in Theorem 2.7 completing the development of the section.

Lemma 3.1: *Let T be a bounded open set and f a function which is continuous on $\overline{T}$ and open on T. Then if W is a complementary domain (a component of the complement) of f[B(T)],*

$$f(T) \cap W \neq \phi \Rightarrow W \subseteq f(T).$$

Proof: Suppose $f(T) \cap W \neq \phi$. Since T is an open set, f(T) is open in K and $f(T) \cap W$ is open in W. Since $\overline{T}$ is closed and bounded in K, $\overline{T}$ is compact. Thus $f(\overline{T})$ is compact and hence closed and bounded in K. But $f(T) \cap W = f(\overline{T}) \cap W$. Hence, $f(T) \cap W$ is closed in W. Finally, since W is connected, $f(T) \cap W = W$ and $W \subseteq f(T)$.

Lemma 3.2: Let V be an open set and $p \in V$. If f is continuous on V and open on $V-\{p\}$, then f is open on V.

Proof: Let D be an open set of V containing p. We wish to show that f(p) is an element of the interior of f(D). Let S be a circle with center p such that $S \cup I(S) \subseteq D$.

Suppose $f(p) \in f(S)$. $S \subseteq D-\{p\}$. Hence, $f(S) \subseteq F[D-\{p\}]$. Since $D-\{p\}$ is an open set of $V-\{p\}$, $f[D-\{p\}]$ is an open set in K and an open subset of f(D). Thus f(p) is in the interior of f(D).

Now suppose $f(p) \notin f(S)$. Let $T = I(S)-\{p\}$; this is an open set of V and $V-\{p\}$. Let I(C) be the interior of a circle containing f(p) with $I(C) \cap f(S) = \phi$. The existence of such a circle is always assured since $f(p) \in I[f(S)]$. We note that $I(C)-\{f(p)\} \cap f(T) \neq \phi$. Let W be the complementary domain of f[B(T)] containing $I(C)-\{f(p)\}$. By Lemma 3.1, $W \subseteq f(T)$. Hence, $I(C)-\{f(p)\} \subseteq f(T)$. Thus, $I(C) \subseteq f(\overline{T})$ and f(p) is an element of the interior of $f(\overline{T})$. Finally, f(p) is in the interior of f(D).

Lemma 3.3 (Connell[3]): Let R be a bounded region and $p \in R$. If h is a function which is continuous on $\overline{R}$ and differentiable on $R-\{p\}$, then for all $z \in R$,

$$|h(z)| \leq \sup\{|h(t)| : t \in B(R)\} .$$

Proof: Suppose h is constant on $R-\{p\}$. Then h is constant on $\overline{R}$ and equality holds above.

Suppose that h is non-constant on $R-\{p\}$. By Theorem B, h is open on $R-\{p\}$. By Lemma 3.2, h is open on R. The function $|h(z)|$ is continuous on $\overline{R}$ and hence attains its maximum at some point $z_0 \in \overline{R}$. Suppose that $z_0 \in R$. Then $|h(z_0)|$ would be a maximum of $|h(z)|$ in a neighborhood $|z-z_0| < \delta \leq R$. But this is impossible unless h(z) is constant in this neighborhood. Hence, h is constant in R, contradicting the assumption, and $|h(z)| \leq \sup\{|h(t)| : t \in B(R)\}$, for all $z \in R$.

Theorem 3.1: Let T be a simple closed curve, R = I(T), and S a set containing $\overline{R}$. Let f be a continuous function on S such that f is differentiable on R. For $p \in R$, $z \in S$ let

$$Q_{f,p}(z) = \begin{cases} [f(z)-f(p)]/(z-p) & \text{if } z \neq p \\ f'(p) & \text{if } z = p \,. \end{cases}$$

Finally, if $Q(z) = Q_{f,p}(z)$, then for all $z \in R$,

$$|Q(z)| \leq \sup\{|Q(t)| : t \in T\} \,.$$

Proof: From the definition of the derivative, Q is continuous on $\overline{R}$. Furthermore, Q is differentiable on $R-\{p\}$. Hence, by Lemma 3.3, for all $z \in R$,

$$|Q(z)| \leq \sup\{|Q(t)| : t \in T\} \,.$$

Corollary: If $Q_{f,p}$ is a non-constant function, then for all $z \in R$,

$$|Q_{f,p}(z)| < \sup\{|Q_{f,p}(t)| : t \in T\} \,.$$

Lemma 3.4: Let F be a uniformly bounded collection of differentiable functions on an open set S. Then F is an equicontinuous family of functions.

Proof: Let $p \in S$ and T be a circle with radius r and center p such that $\overline{R} = T \cup I(T) \subseteq S$. There exists $M > 0$ such that $|f(z)| < M$ for all $z \in S$ and all $f \in F$. If $f \in F$ and $z \in \overline{R}$, then from Theorem 3.1,

$$|Q_{f,p}(z)| \leq \sup \{|Q_{f,p}(t)| : t \in T\}$$

$$\leq \sup \{|f(t)-f(p)|/r : t \in T\} \leq 2M/r \ .$$

Thus, $|f(z)-f(p)| \leq |z-p| \cdot (2M/r)$. Hence, if $|z-p| < (r/3M)\varepsilon$ for $z \in \overline{R}$,

$$|f(z)-f(p)| \leq (r/3M)(2M/r)\varepsilon < \varepsilon \ .$$

Thus F is an equicontinuous family.

Theorem 3.2: Let F be a uniformly bounded collection of differentiable functions on a region R. Then if $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions from F, there exists a subsequence $\{f_{p_k}\}$, $p_1 < p_2 < \cdots$ which converges uniformly on compact subsets of R.

Proof: By Lemma 3.4, F is an equicontinuous family of functions. A constructive proof of the theorem now follows using the equicontinuity of F and the "rational points" of R. The prototype of such a proof may be found in Nehari [11], pp.141 ff.

Lemma 3.5: Let f and g be polynomials such that for each $z \in U$ $g(z) \neq 0$. Let S be an open set in U and $h(z) = f(z)/g(z)$ for $z \in \overline{U}$. Then if $\{P_i\}_{i=1}^{\infty}$ is a sequence of polynomials converging uniformly on compact subsets of h(S) to a limit function F, there exists a sequence of polynomials $\{Q_i\}_{i=1}^{\infty}$ converging uniformly on compact subsets of S to $F \circ h$.

<u>Proof</u>: Since $P_n \circ h = P_n \circ (f/g)$ where f and g are polynomials with $g(z) \neq 0$ for each $z \in \bar{U}$, $n \in \omega$, clearly there exist polynomials f_n and g_n such that $g_n(z) \neq 0$ for each $z \in \bar{U}$, and $P_n \circ h(z) = f_n(z)/g_n(z)$ for $z \in \bar{U}$.

By Theorem A, there exists a finite collection of numbers $a_o, z_1, \ldots, z_p$ such that $g_n(z) = a_o(z_1 - z)(z_2 - z) \cdots (z_p - z)$ for all $z \in K$. Further, since $g_n(z) \neq 0$ for each $z \in \bar{U}$, we have $|z_k| > 1$ for $k = 1, 2, \ldots, p$. Thus if $z \in \bar{U}$,

$$|z/z_k| = |z|/|z_k| < 1.$$

Hence, for $k = 1, 2, \ldots, p$ the sequence of polynomials $\{T_{km}\}_{m=1}^{\infty}$ converges uniformly on $\bar{U}$ to

$$(1/z_k)(1 - z/z_k)^{-1} = (z_k - z)^{-1}$$

where $T_{km}(z) = \sum_{j=0}^{m} (z/z_k)^j$ for $k = 1, 2, \ldots, p$, $m \in \omega$, $z \in K$.

For $m \in \omega$, $z \in K$, let $Q_{nm} = (1/a_o) f_n(z) T_{1m}(z) \cdots T_{pm}(z)$.

$\{Q_{nm}\}_{m=1}^{\infty}$ is a sequence of polynomials converging uniformly on $\bar{U}$ to f_n/g_n .

Suppose that $p_1 < p_2 < \ldots$ is an increasing sequence in ω such that for $n \in \omega$, $z \in \bar{U}$,

$$|Q_{np_n}(z) - P_n \circ h(z)| < 1/n \quad .$$

Let M be a compact subset of S and let $\varepsilon > 0$ be given. Since h is continuous, h(M) is compact. Also, there exists $N > 0$ such that $n > N$, $n \in \omega$ implies that $1/n < \varepsilon/2$ and $|F(z) - P_n(z)| < \varepsilon/2$ for all $z \in h(M)$. Thus for $z \in M$ and $n > N$,

$n \in \omega$, we have

$$|F\circ h(z)-Q_{np_n}(z)| \le |F\circ h(z)-P_n\circ h(z)| + |P_n\circ h(z)-Q_{np_n}(z)|$$

$$\le 1/n + \varepsilon/2 < \varepsilon .$$

Hence, the sequence $\{Q_{np_n}\}_{n=1}^{\infty}$ converges uniformly to $F\circ h$ on compact subsets of S.

Lemma 3.6 (Connell and Porcelli, [3] and [5]): Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of differentiable functions defined on an open set S, converging uniformly on compact subsets of S to a limit function F. Then F is differentiable and $\{f_n'(z)\}_{n=1}^{\infty}$ converges to $F'(z)$ for all $z \in S$.

Proof: Let $p \in S$ and let T be a circle with center p and radius r such that $D = T \cup I(T) \subseteq S$.

Set $Q(z) = [F(z)-F(p)]/(z-p)$ for $z \in K - \{p\}$. For $n \in \omega$, let $Q_n = Q_{f_n,n}$. Since the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on compact subsets of S to F, $\{Q_n\}_{n=1}^{\infty}$ converges uniformly on the compact subset $D - \{p\}$ to Q.

By Theorem 3.1 for $z \in D$, $m,n \in \omega$,

$$|Q_n(z)-Q_m(z)| \le \sup \{|Q_n(t)-Q_m(t)| : t \in T\}$$

$$\le \sup \{|f_n(t)-f_m(t)|/r : t \in T\} .$$

But since $\{f_n\}_{n=1}^{\infty}$ converges uniformly on T, there exists $N \in \omega$ such that $|f_n(t)-f_m(t)| < r\cdot\varepsilon$ for $m,n \ge N$, $m,n \in \omega$ and all $t \in T$.

Hence, for $m,n \geq N$, $m,n \in \omega$ and $z \in D$,

$|Q_n(z) - Q_m(z)| < r \cdot \varepsilon / r = \varepsilon$, and we see that $\{Q_n\}_{n=1}^{\infty}$ converges uniformly on D to a limit function Q_0. Clearly $Q_0(z) = Q(z)$ for all $z \in D - \{p\}$. Hence, F is differentiable at p and $F'(p) = Q_0(p) = \lim_{n\to\infty} Q_n(p) = \lim_{n\to\infty} f_n'(p)$.

Lemma 3.7: Let S be a simply connected open set in U, such that $0 \in S$ and $S \neq U$. Then there exist polynomials f and g, such that for each $z \in \bar{U}$ $g(z) \neq 0$, and a one-to-one differentiable function h on S into U such that $h'(0) > 1$, $h(0) = 0$, and $[f/g] \circ h(z) = z$ for all $z \in S$.

Proof: Let $t \in U - S$. If $z \in \bar{U}$,

$$|1 - \bar{t}z| \geq 1 - |\bar{t}z| \geq 1 - |t| > 0.$$

We can thus define the function

$A(z) = (t-z)(1-\bar{t}z)^{-1}$ for $z \in \bar{U}$. We have that

$$A \circ A(z) = \frac{t - \frac{t-z}{1-\bar{t}z}}{1 - \bar{t}\left(\frac{t-z}{1-\bar{t}z}\right)} = \frac{\frac{t - |t|^2 z - t + z}{1-\bar{t}z}}{\frac{1 - \bar{t}z - |t|^2 + \bar{t}z}{1-\bar{t}z}}$$

$$= \frac{z(1-|t|^2)}{1-|t|^2} = z .$$

Thus, A is one-to-one and $A(\bar{U}) = \bar{U}$. Trivially, $A(0) = t$ and $A(t) = 0$, and hence $0 \notin A(S)$. A is differentiable on S. By Theorem B, A is an open map on S. Hence, A(S) is a simply connected, open set in U not containing 0.

From Theorem D, there exists a one-to-one differentiable function H on A(S) into K such that

$[H(z)]^2 = z$ for all $z \in A(S)$. Then $|[H(t)]^2| = |t| < 1$ and $|H(t)| < 1$. Further, for $z \in \overline{U}$, $[1-\overline{H(t)}]z \neq 0$. Hence, for $z \in \overline{U}$ we can define $B(z) = [H(t)-z][1-\overline{H(t)}z]^{-1}$. Performing the same computation as we did for $A(z)$, we find that $B \circ B(z) = z$ and hence B is also one-to-one. We note too that $B \circ H(t) = 0$.

For $z \in S$, set $P(z) = B \circ H \circ A(z)$.

Since B, H, and A are one-to-one differentiable functions in their respective domains, we have that $P(z)$ is one-to-one and differentiable. Further, $P(S) \subseteq U$ and $P(0) = B \circ H \circ A(0) = B \circ H(t) = 0$.

For $z \in \overline{U}$, let $K(z) = z^2$. Then for $z \in \overline{U}$, let $Q(z) = A \circ K \circ B(z)$. Since A, K, and B are differentiable functions in their respective domains, Q is differentiable. Also, $Q(\overline{U}) \subseteq \overline{U}$ since $|z|^2 \leq 1$ if and only if $|z| \leq 1$. Noting that $B \circ B(z) = z$ for each $z \in \overline{U}$ and that $K \circ H(z) = [H(z)]^2 = z$ for each $z \in A(S)$, we have for $z \in \overline{U}$,

$$Q \circ P(Z) = [A \circ K \circ B] \circ [B \circ H \circ A](z) = A \circ K \circ [B \circ B] \circ H \circ A(z)$$
$$= A \circ [K \circ H] \circ A(z) = A \circ A(z) = z \ .$$

Thus, $[Q \circ P(z)]' = z' = 1$ for $z \in \overline{U}$.

But $[Q \circ P(0)]' = Q'[P(0)] \cdot P'(0) = Q'(0) \cdot P'(0)$. Hence, $Q'(0) \cdot P'(0) = 1$.

For $z \in \overline{U}$, set

$$Q_o(z) = \begin{cases} Q(z)/z & \text{for } z \neq 0 \\ Q'(z) & \text{for } z = 0 \ . \end{cases}$$

Q_o is continuous on $\overline{U}$ and differentiable on $\overline{U} - \{0\}$.

$K(z) = z^2$ is not one-to-one and hence Q is not a one-to-one function on U. Thus, Q_0 cannot be constant on U. From the Corollary to Theorem 3.1 for all $z \in U$,

$$|Q_0(z)| < \sup \{|Q_0(t)| : t \in C\}.$$

But, $\sup \{|Q_0(t)| : t \in C\} = \sup \{|Q(t)/t| : t \in C\}$

$$= \sup \{|Q(t)| : t \in C\} \leq 1 .$$

Hence, $|Q_0(z)| < 1$ for $z \in \bar{U}$ and in particular $|Q_0(0)| = |Q'(0)| < 1$. Since $Q'(0) \cdot P'(0) = 1$, we have that $|P'(0)| > 1$.

Now let $s = \overline{P'(0)}/|P'(0)|$ and set $h(z) = sP(z)$ for $z \in S$. Since P is one-to-one and differentiable, $h = sP$ is also one-to-one and differentiable. We note that

$$|s| = |\overline{P'(0)}/|P'(0)|| = |P'(0)|/|P'(0)| = 1.$$

Hence, $|h(z)| = |sP(z)| = |s||P(z)| = |P(z)|$; since $P(S) \subseteq U$, $h(S) \subseteq U$.

Furthermore, $h'(0) = [sP(0)]' = sP'(0) =$

$$= \overline{P'(0)}P'(0)/|P'(0)|$$

$$= |P'(0)|^2/|P'(0)| = |P'(0)| > 1.$$

Also, $h(0) = sP(0) = s \cdot 0 = 0$. Finally, consider the function $Q \circ F$, where F is defined to be $F(z) = z/s$ for $z \in K$. $Q \circ F = f/g$ where f and g are polynomials and $g(z) \neq 0$ for each $z \in \bar{U}$.

But $Q \circ F \circ h(z) = [A \circ K \circ B] \circ F \circ [s\{B \circ H \circ A\}](z)$

$$= [A \circ K \circ B \circ B \circ H \circ A](z) = z$$

by an earlier computation.

Thus we have verified that $h(z)$ is the desired function.

The proof of the following theorem is adapted from a proof of the Riemann Mapping Theorem given by Saks and Zygmund [11], pp.225-230.

Theorem 3.3(Leland [7, p.169]): Let S be a bounded simply connected open set and $z_0 \in S$. Then there exists a one-to-one differentiable map F of S onto U such that $F(z_0) = 0$, $F'(z_0) > 0$, and a sequence of polynomials $\{P_i\}_{i=1}^{\infty}$ converging uniformly on compact subsets of U to F^{-1}.

Proof: Let E be the set of all one-to-one differentiable maps f of S into U such that $f(z_0) = 0$, $f'(z_0) > 0$ and such that there exists a sequence of polynomials $\{Q_i\}_{i=1}^{\infty}$ converging uniformly on compact subsets of f(S) to f^{-1}.

Since S is bounded, there exists $M > 0$ such that $|z| \leq M$ for all $z \in S$. Let $A(z) = z - z_0$ for $z \in S$. $A(z_0) = 0$ and $|A(z)| = |z-z_0| \leq |z| + |z_0| \leq 2M < 3M$ for $z \in S$.

Let $B(z) = kA(z)$ where $0 < k < 1/3M$.

$B(0) = kA(0) = 0$ and $B'(z_0) = [k(z-z_0]' = k > 0$.

$|B(z)| = |kA(z)| = k \cdot |A(z)| < k3M < (1/3M)(3M) < 1$.

Hence, B maps S into U. Clearly both A and B are one-to-one differentiable maps and trivially there exists a sequence of polynomials converging uniformly on compact subsets of B(S) to B^{-1} since $B^{-1}(z) = (1/k)z$ for $z \in B(S)$. Thus $B \in E$ and E is not empty.

If $s = \sup\{f'(z_0): f \in E\}$, then s is finite since the functions in E are uniformly bounded by 1. Also, $s > 0$

since $B'(z_0) > 0$ for the function B constructed above. Further, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in E such that $\lim_{n\to\infty} f_n'(z_0) = s$. The family E is uniformly bounded and hence by Theorem 3.2, there exist $p_1 < p_2 < \cdots$ in ω such that $\{f_{p_n}\}_{n=1}^{\infty}$ converges uniformly on compact subsets of S to a limit function F. From Lemma 3.6, F is differentiable; $F'(z_0) = s > 0$ and hence F is non-constant. From Theorem C, F is one-to-one and from Theorem F, $F(S) \subseteq U$.

For $n \in \omega$ let $C_n = \{z \in F(S) : \delta[\{z\}, K - F(S)] \geq 1/n\}$. C_n is closed and bounded and hence compact for $n \in \omega$. Further, $F(S) = \bigcup_{n=1}^{\infty} C_n$. Given the sequence $p_1 < p_2 < \cdots$ in ω determined above, then by Theorem F, there exists $n \in \omega$ such that if $m \geq p_n$, $m \in \omega$, then $C_n \subseteq f_m(S)$. Also, for $n \in \omega$, there exists a sequence of polynomials $\{P_{ni}\}_{i=1}^{\infty}$ converging on compact subsets of $f_{p_n}(S)$ to $f_{p_n}^{-1}$ since $f_{p_n} \in E$ for each n. Finally, there exists $q_1 < q_2 < \cdots$ in ω such that for $n \in \omega$,

$$\left| P_{nq_n}(z) - f_{p_n}^{-1}(z) \right| < 1/n \text{ for } z \in C_n.$$

If D is a compact subset of F(S), then there exists an integer n_0, such that $D \subseteq C_{n_0}$. Let $\varepsilon > 0$ be given. Then by Theorem F, there exists $M > n_0$ such that $m > M$, $m \in \omega$ implies that $\left| f_{p_m}^{-1}(z) - F^{-1}(z) \right| < \varepsilon$ for all $z \in C_{n_0}$. Hence, for $z \in D \subseteq C_{n_0}$ and $m > M$, $m \in \omega$, we have

$$\left| F^{-1}(z) - P_{mq_m}(z) \right| \leq \left| F^{-1}(z) - f_{p_m}^{-1}(z) \right| + \left| f_{p_m}^{-1}(z) - P_{mq_m}(z) \right|$$

$$< \varepsilon/2 + 1/m .$$

Thus, $|F^{-1}(z) - P_{mq_m}(z)| < \varepsilon$ for m sufficiently large, and we have found a sequence of polynomials converging uniformly on compact subsets of U to F^{-1}. Finally, $F(z_0) = 0$ and we have shown that $F \varepsilon E$.

Suppose now that $F(S) \neq U$. By the continuity and differentiability of F, F(S) is a simply connected open set of U containing 0. From Lemma 3.7, there exist polynomials f and g such that for each $z \varepsilon \bar{U}$ $g(z) \neq 0$ and a one-to-one differentiable function h on F(S) into U such that $[f/g] \circ h(z) = z$ for all $z \varepsilon F(S)$ and such that $h'(0) > 1$. We note that $h^{-1}(z) = [f(z)]/[g(z)]$ for $z \varepsilon h[F(S)] = W$.

Consider $[h \circ F(z_0)]'$; we have

$$[h \circ F(z_0)]' = h'[F(z_0)] \cdot F'(z_0) = h'(0) \cdot s > s .$$

Now $W = h[F(S)]$ is an open set in U since h is a differentiable function on F(S) and F(S) is an open set. Also $F(S) = h^{-1}(W)$ since h is one-to-one. Since $F \varepsilon E$, there exists a sequence of polynomials $\{P_i\}_{i=1}^{\infty}$ converging uniformly on compact subsets of $h^{-1}(W)$ to F^{-1}. From Lemma 3.5, there exists a sequence of polynomials $\{Q_i\}_{i=1}^{\infty}$ converging uniformly on compact subsets of W to $F^{-1} \circ h^{-1}$. But $(F^{-1} \circ h^{-1}) = (h \circ F)^{-1}$. Thus $h \circ F \varepsilon E$, which is a contradiction. Hence $F(S) = U$.

Theorem 3.4: Let S be a bounded simply connected open set in K, $z_0 \in S$, and $x_0 \in U - \{0\}$. Then there exists a unique differentiable one-to-one function f on U onto S such that (1) f^{-1} is differentiable, $f(0) = z_0$, $f'(0) > 0$, and there exists a sequence of polynomials $\{P_i\}_{i=1}^{\infty}$ converging uniformly on compact subsets of U to f, and such that (2) if g is a one-to-one map of U onto S, such that g is differentiable on $U - \{x_0\}$, $g(0) = z_0$, and $g'(0) > 0$, then $g = f$.

Proof: The existence of at least one function f satisfying (1) is assured by Theorem 3.3.

Let g be a function satisfying (2) and set $Q(z) = f^{-1} \circ g(z)$ for all $z \in U$. Then Q is a one-to-one map of U onto U such that Q is differentiable on $U - \{x_0\}$. Also,

$$Q(0) = f^{-1} \circ g(0) = f^{-1}(z_0) = 0, \text{ and}$$

$$Q'(0) = [f^{-1} \circ g(0)]' = f^{-1'}[g(0)] \cdot g'(0) = f^{-1'}(z_0) \cdot g'(0).$$

But $f'(0) > 0$ and $f^{-1}(z_0) = 0$. Hence, $f^{-1'}(z_0) > 0$. Also, $g'(0) > 0$ and thus $Q'(0) > 0$.

For $z \in U$ let $T = Q_{Q,0}$. Thus,

$$T(z) = \begin{cases} \dfrac{f^{-1} \circ g(z) - f^{-1} \circ g(0)}{z} = \dfrac{f^{-1} \circ g(z)}{z} & \text{if } z \neq 0 \\ Q'(0) \text{ if } z = 0 . \end{cases}$$

From Theorem 3.1, we have

$$\sup \{|T(z)| : z \in U\} \leq \sup_{0<r<1} \sup \{|T(t)| : t \in C(r)\} \leq \sup_{0<r<1} 1/r = 1.$$

Thus $|T(z)| \leq 1$ for all $z \in U$.

Since Q is one-to-one and $Q(0) = 0$, $Q(z) \neq 0$ for $z \in U - \{0\}$. But $Q'(0) > 0$. Hence, $T(z) \neq 0$ for all $z \in U$.

Applying Theorem 3.1 to $1/T$, we have that $|T(z)| \geq 1$ for all $z \in U$. Consequently, $|T(z)| = 1$ for all $z \in U$.

If $z \neq 0$, $T(z) = [f^{-1} \circ g(z)]/z$. Since $|T(z)| = 1$, we have that $[f^{-1} \circ g(z)]/z = e^{i\alpha}$, where α is fixed. From the continuity of $T(z)$ at 0, $T(0) = e^{i\alpha}$.

But $|T(0)| = |Q'(0)| = 1$. Also, $Q'(0) > 0$. Hence, $T(0) = Q'(0) = e^{i\alpha} = 1$. Thus for $z \in U - \{0\}$, $[f^{-1} \circ g(z)]/z = 1$ or $f^{-1} \circ g(z) = z$. Also, $f^{-1} \circ g(0) = 0$. Therefore, for all $z \in U$, $f^{-1} \circ g(z) = z$ and $f = g$.

<u>Theorem 3.5</u>: <u>Let $x_0 \in U$, $x_0 \neq 0$, and f be a continuous function on U such that f is differentiable on $U - \{x_0\}$. Then f is differentiable on U, and there exists a sequence of polynomials $\{P_i\}_{i=1}^{\infty}$ which converges uniformly on compact subsets of U to f.</u>

<u>Proof</u>: Let $0 < \varepsilon < 1$ and $r = 1 - \varepsilon$. Then from Theorem E, there exists $p > 0$ such that the function $g(z) = f(z) + pz$ for $z \in U$, is one-to-one on $\overline{U}(r)$. Choose r such that $x_0 \in U(r)$. Let $S = g[U(r)]$. Since f is differentiable on $U - \{x_0\}$, it is differentiable on $U(r) - \{x_0\}$. Hence, g is differentiable on $U(r) - \{x_0\}$, and by Theorem B, g is open on $U(r) - \{x_0\}$. From Lemma 3.2, g is open on $U(r)$. Hence, S is an open set in K. From the continuity of g, we see that S is also simply connected. Furthermore, $U(r)$ is a bounded region; thus by Lemma 3.3, for all $z \in U(r)$,

$|g(z)| \leq \sup \{|g(t)| : t \in C(r)\}$. However, a continuous function assumes its maximum M on a compact set such as $C(r)$.

Hence, $|g(z)| \leq M$ for all $z \in U(r)$ and the set $S = g[U(r)]$ is bounded in K.

Since g is one-to-one on $U(r)$, $g'(z) \neq 0$ for $z \in U(r)-\{x_o\}$. In particular $g'(0) \neq 0$. Hence, $|g'(0)| > 0$ and, without loss of generality, we may take $g'(0) > 0$. Simply consider the function $h(z) = sg(z)$ where $s = \overline{g'(0)}/|g'(0)|$ and note that $h'(0) = sg'(0) = |g'(0)|^2/|g'(0)| = |g'(0)| > 0$.

Thus, from Theorem 3.4, g is differentiable and there exists a sequence of polynomials $\{P_i\}_{i=1}^{\infty}$ which converges uniformly on compact subsets of $U(r)$ to g. Hence, $\{P_i - pI_o\}_{i=1}^{\infty}$ converges uniformly on compact subsets of $U(r)$, and in particular, $\overline{U}_{1-2\varepsilon}$, to f. By a diagonal process we obtain a sequence of polynomials $\{Q_i\}_{i=1}^{\infty}$, such that $|Q_n(z)-f(z)| < (1/2)^n$ for $z \in U_{1-1/n}$. This is the desired sequence.

<u>Theorem 3.6</u>: <u>If</u> f <u>is</u> <u>a</u> <u>differentiable</u> <u>function</u> <u>on</u> U, <u>then</u> <u>there exists</u> <u>a</u> <u>power series</u> $\sum_{n=0}^{\infty} a_n z^n$ <u>which converges uniformly</u> <u>on compact subsets</u> <u>of</u> U <u>to</u> f.

<u>Proof</u>: Let $n \in \omega$. From Theorem 3.5, there exists a sequence of polynomials $\{P_i\}_{i=1}$ which converges uniformly on $U(1-1/n)$ to f. The constructive proof is now identical to that of Theorem 2.7.

APPENDIX

We list below the theorems used without proof in Section 3 with references.

Theorem A. (Fundamental Theorem of Algebra) (Whyburn[15,p.77]): Every non-constant polynomial has at least one zero in the complex plane.

Theorem B. (Open Mapping Theorem) (Whyburn[15,p.76]): If f is a non-constant differentiable function defined on a region R, then f is an open map.

Theorem C. (Hurwitz Theorem) (Whyburn[15,p.110]): Let the sequence of functions f_n, each continuous and differentiable in a region R, converge uniformly in R to a function f not identically zero. Then if $z_0 \in R$ is an m-fold zero of f, every sufficiently small neighborhood D of z_0 contains exactly m zeros of f_n for $n > N(D)$.

Corollary: Let the sequence of functions f_n, each differentiable and univalent in a region R, converge uniformly in R to a non-constant function f. Then f is also univalent in R.

Theorem D. (Leland[10,p.162]): Let $z_0 \in K$ and let S be a simply connected open set excluding z_0. Then there exists a map k of S into K, such that $[k(z)]^2 = z-z_0$ for all $z \in S$.

Theorem E. (Leland[10,p.165]): Let A be a finite subset of U and $0 < r < 1$. If f is a map of U such that f is differentiable on U - A, then there exists $p > 0$ such that the function

$g(z) = pf(z) + z$ for $z \in U$ is one-to-one on $\overline{U}(r)$.

Corollary: Let A be a finite subset of U and $0 < r < 1$. If f is a map of U such that f is differentiable on U - A, then there exists $p > 0$ such that the function $g(z) = f(z) + pz$ for $z \in U$ is one-to-one on $\overline{U}(r)$.

Theorem F. (Leland [10,pp.167-168]): Suppose that $\{f_i\}_{i=1}^{\infty}$ is a sequence of one-to-one differentiable functions on a simply connected bounded open set S into U, converging uniformly on compact subsets of S to a limit function F non-constant on each component of S. Then F is a one-to-one differentiable function such that $F(S) \subseteq U$. Moreover if M is a compact subset of F(S), there exists $N > 0$ such that $n \geq N$, $n \in \omega$, implies $M \subseteq f_n(S)$. Furthermore, $\{f_n^{-1}\}_{n=N}^{\infty}$ converges uniformly to F^{-1} on M.

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VITA

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