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# An investigation of the statistical distribution of rises and falls in a stochastic process

Robert G. Wagner
*Lehigh University*

# AN INVESTIGATION OF THE STATISTICAL DISTRIBUTION OF RISES AND FALLS IN A STOCHASTIC PROCESS

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Robert Gene Wagner

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Frederick C. T. Allen
Professor in Charge

Frederick C. T. Allen
Head of Department

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# INTRODUCTION

This paper is a further investigation of a problem outlined in report number 6-389950-0942 to the Boeing Airplane Company by F. P. Beer, P. C. Paris and L. Y. Bahar. The purpose of the aforementioned report was to establish the necessary conditions under which two Gaussian processes of known power spectra will produce equivalent sequencings. Ultimately this would be used to find a way to predict crack growth rate under a random loading. One of the conditions given was that the statistical distribution of rises and falls for the two processes must be the same.

The purpose of this paper, then, is the study of the statistical distribution of the rises and falls for an arbitrary random process. A series solution for the distribution is developed and the results applied to the example of an ideal low pass filter. The computations were performed on an LGP-30 computer.

## 1. STATEMENT OF THE PROBLEM

The statistical distribution of rises and falls is given by the probability density function P(h) where P(h)dh is defined as the probability that a rise or fall in going from one extremum of a random curve to the following extremum will lie in the interval h, h+dh.

Consider the random curve y(t) of Figure 1 where h is now the distance from a given relative minimum to the following relative maximum of the curve. Introduce the following notation:

$Q_\alpha \, d\alpha$ = Expected number of minima per unit time in the interval
$$\alpha \leq y \leq \alpha + d\alpha \qquad (1\text{-}1)$$

$Q^{\alpha'} d\alpha'$ = Expected number of maxima per unit time in the interval
$$\alpha' \leq y \leq \alpha' + d\alpha' \qquad (1\text{-}2)$$

$Q$ = Expected number of minima (or maxima) per unit time (1-3)

$P_\alpha^{\alpha'} \, d\alpha \, d\alpha'$ = probability, given a minimum and the following maximum, that the minimum lies in the interval $\alpha \leq y \leq \alpha + d\alpha$ and the maximum in the interval $\alpha' \leq y \leq \alpha' + d\alpha'$ (1-4)

Then the probability density of having a rise h between a given minimum and the following maximum is obtained by setting $\alpha' = \alpha + h$ and forming

$$P(h) = \int_{-\infty}^{\infty} P_\alpha^{\alpha + h} \, d\alpha \qquad (1\text{-}5)$$

To evaluate $P_{\alpha}^{\alpha'}$ the "inclusion and exclusion principle" is applied. First consider the joint probability density of having a minimum at $y = \alpha$, $t = 0$ and a maximum at $y = \alpha'$, $t = \tau$. Denote this density by

$$\text{Prob (Min } t = 0,\ y = \alpha;\ \text{Max } t = \tau,\ y = \alpha'). \tag{1-6}$$

This is not, however, the probability density of having at $y = \alpha'$, $t = \tau$, the first maximum following the minimum at $y = \alpha$, $t = 0$, for it does not take into consideration the case where the first maximum occurs at $\sigma$, where $\sigma$ lies in the interval $0<\sigma<\tau$. Hence, this is an overestimate of the probability density of having the first maximum at $y = \alpha'$, $t = \tau$ and so the probability density of having one other maximum in the interval $0<t<\tau$ must be subtracted, excluding this case. This probability density is denoted by

$$\int_0^{\tau} \text{Prob.}(\text{Min } t=0, y=\alpha;\ \text{Max } t=\tau\ \ y=\alpha';\ \text{Max } t=\sigma)\,d\sigma \tag{1-7}$$

But now too much has been excluded since a curve having two maxima in the interval $0<t<\tau$ will be excluded 2! times. Hence (1/2!) times the probability of having two maxima in the interval $0<t<\tau$ must be added. Again too much has been included and (1/3!) times the probability of having three maxima in the interval $0<t<\tau$ must be excluded. Continuing in this way and then integrating the resulting series over the interval $0<\tau<T$, letting $T$ approach infinity and dividing the result by $\Omega$, the expected

number of minima per unit time, we have

$$P_\alpha^{\alpha'} = \frac{1}{Q} \lim_{T\to\infty} \left[ P_1(T) - P_2(T) + \frac{1}{2!} P_3(T) - \frac{1}{3!} P_4(T) + \cdots \right] \qquad (1\text{-}8)$$

where

$$P_1(T) = \int_0^T \text{Prob}\,(\text{Min } t=0,\ y=\alpha;\ \text{Max } t=\tau,\ y=\alpha')\,d\tau \qquad (1\text{-}9)$$

$$P_2(T) = \int_0^T d\tau \int_0^\tau \text{Prob}\,(\text{Min } t=0,\ y=\alpha;\ \text{Max } t=\tau,\ y=\alpha';\ \text{Max } t=\sigma)\,d\sigma \qquad (1\text{-}10)$$

$$P_3(T) = \int_0^T d\tau \int_0^\tau\!\!\int_0^\tau \text{Prob}\,(\text{Min } t=0,\ y=\alpha;\ \text{Max } t=\tau,\ y=\alpha';\ \text{max } t=\sigma;\ \text{Max } t=\sigma')\,d\sigma\,d\sigma' \qquad (1\text{-}11)$$

etc.

Letting $\alpha' = \alpha + h$ and substituting (1-8) into (1-5) we obtain the probability density function $P(h)$:

$$P(h) = \frac{1}{Q} \lim_{T\to\infty} \int_{-\infty}^{\infty} \left[ P_1(T) - P_2(T) + \frac{1}{2!} P_3(T) - - - - \right] d\alpha \qquad (1\text{-}12)$$

The resulting series, however, cannot be evaluated in this form since each of the terms $P_i$ (T) defined in expressions (1-8) and the following will approach infinity with T. In order to replace this series by a converging one, consider the results obtained by assuming that the probability density of having a maximum or minimum at any time is independent of the existence of any preceding or subsequent maxima or minima. For this uncorrelated case, then, all joint probability densities become the products of the individual probability densities. Using the expressions defined in (1-1), (1-2) and (1-3) and using $Q_1(T)$, $Q_2(T)$,... in place of $P_1(T)$, $P_2(T)$,... the following results are obtained:

$$Q_1(T) = \int_0^T \mathrm{Prob}[\min t=0, y=\alpha]\cdot \mathrm{Prob}[\max t=\tau, y=\alpha']\, d\tau$$
$$= Q_\alpha Q^{\alpha'} T \tag{1-13}$$

$$Q_2(T) = \int_0^T d\tau \int_0^\tau \mathrm{Prob}[\min t=0, y=\alpha]\,\mathrm{Prob}[\max t=\tau, y=\alpha']\,\mathrm{Prob}[\max t=\sigma]\, d\sigma$$
$$= Q_\alpha Q^{\alpha'} Q \frac{T^2}{2} \tag{1-14}$$

$$Q_3(T) = \int_0^T d\tau \int_0^\tau \int_0^\tau \mathrm{Prob}[\min t=0, y=\alpha]\,\mathrm{Prob}[t=\tau, y=\alpha']\cdot$$
$$\cdot\, \mathrm{Prob}[\max t=\sigma]\,\mathrm{Prob}[\max t=\sigma']\, d\sigma\, d\sigma' = Q_\alpha Q^{\alpha'} Q^2 \frac{T^3}{3} \tag{1-15}$$

and so on. Substituting the $Q_i$ for the $P_i$ in equation (1-8) we obtain the probability density of having a minimum at $y = \alpha$ followed by a maximum at $y = \alpha'$, under the assumption of uncorrelated maxima and minima, as

$$Q_\alpha^{\alpha'} = \frac{1}{Q} \lim_{T\to\infty} \frac{Q_\alpha Q^{\alpha'}}{Q}\left(QT - \frac{1}{2!}Q^2T^2 + \frac{1}{3!}Q^3T^3 - \cdots\right)$$
$$= \frac{Q_\alpha Q^{\alpha'}}{Q^2} \lim_{T\to\infty}\left(1 - e^{-QT}\right) = \frac{Q_\alpha Q^{\alpha'}}{Q^2} \tag{1-16}$$

Adding $\frac{Q_\alpha Q^{\alpha'}}{Q^2}$ to $P_\alpha^{\alpha'}$ and subtracting the series form of $Q_\alpha^{\alpha'}$ term by term gives

$$P_\alpha^{\alpha'} = \frac{Q_\alpha Q^{\alpha'}}{Q^2} + \frac{1}{Q}\lim_{T\to\infty}[P_1(T) - Q_1(T)] - \frac{1}{Q}\lim_{T\to\infty}[P_2(T) - Q_2(T)] + \cdots \qquad (1\text{-}17)$$

Substituting this into equation (1-5), the probability density function P(h) is expressed as

$$P(h) = \lim_{T\to\infty}\frac{1}{Q}\int_{-\infty}^{\infty}\left\{\frac{Q_\alpha Q^{\alpha+h}}{Q} + [P_1(T) - Q_1(T)] - [P_2(T) - Q_2(T)] + \cdots\right\} d\alpha \qquad (1\text{-}18)$$

$$= p_0(h) + p_1(h) - p_2(h) + \cdots \qquad (1\text{-}19)$$

where $p_0(h) = \frac{1}{Q^2}\int_{-\infty}^{\infty} Q_\alpha Q^{\alpha+h}\, d\alpha$ (1-20)

$$p_1(h) = \lim_{T\to\infty}\frac{1}{Q}\int_{-\infty}^{\infty}[P_1(T) - Q_1(T)]\, d\alpha \qquad (1\text{-}21)$$

etc.

The term $p_0(h)$ represents the distribution of rises or falls assuming that no correlation exists between successive maxima and minima. The following terms represent corrective terms taking into consideration the fact that the successive maxima and minima are actually correlated.

To obtain a better insight into the meaning of these corrective terms the form (1-18) is reverted to its form prior to any integrations in time. This gives us

$$P(h) = \frac{1}{Q}\int_{-\infty}^{\infty}\frac{Q_\alpha Q^{\alpha+h}}{Q}\, d\alpha + \frac{1}{Q}\lim_{T\to\infty}\int_{-\infty}^{\infty} d\alpha \,\cdot$$

$$\cdot \int_0^T \Big\{ [\mathrm{Prob}(\min y = \alpha, t = 0; \max y = \alpha + h, t = \tau) - Q_\alpha Q^{\alpha+h}]$$

$$- \int_0^\tau [\mathrm{Prob}(\min y = \alpha, t = 0; \mathrm{Max}\, y = \alpha + h, t = \tau; \mathrm{Max}\, t = \sigma) \qquad (1\text{-}22)$$

$$- Q_\alpha Q^{\alpha+h} Q]\, d\sigma + \cdots \Big\}\, d\tau$$

Three facts can be ascertained from equation (1-22). (1) As $\tau$ becomes larger the integrands corresponding to the Pi(h) terms ($i \neq 0$) approach 0 since the effects of correlation between maxima and minima become smaller. This is a necessary condition for convergence of the integrations over $\tau$. (2) Each of the terms $p_i(h)$ ($i \neq 0$) is more restrictive in its demands on the correlation between the maxima and minima than its preceding term. For example, $p_1(h)$ says nothing about whether or not any other maxima occur before $t = \tau$ whereas $p_2(h)$ considers the case where one maximum occurs before that at $t = \tau$. The correlation between a maximum and minimum when more and more maxima are in between will be smaller and smaller, hence each succeding corrective term will be smaller than its predecessor. (3) Since each $p_i(h)$ ($i \neq 0$) represents a correction to the oversimplified assumption of no correlation, each may be expected to be finite.

The result of these three facts is that the original series for P(h), equation (1-12), has been replaced by a series each term of which is finite and, after $p_1(h)$, smaller than the preceding term; thus, the series may be expected to converge rapidly.

Note that no assumption was made about the relative sizes of $p_1(h)$ and $p_0(h)$. Since $p_1(h)$ represents the first effects of assuming correlated maxima and minima $p_1(h)$ may be large or may be small as compared to $p_0(h)$, depending upon the random curve under consideration.

## 2. APPROXIMATING FORM FOR THE PROBABILITY DENSITY FUNCTION, P(h)

It has previously been noted that one can expect each of the corrective terms $p_i(h)$ $(i \geq 2)$ to be smaller than the preceding corrective term. It is also highly probable that in most cases the term $p_2(h)$ will be so much smaller than $p_1(h)$ that a reasonable approximation to the probability density function $P(h)$ will be given by considering the first two terms. Hence the remainder of this paper will deal with the approximation for the probability density function given by

$$P(h) \cong p_0(h) + p_1(h) = \tilde{P}(h) \qquad (2\text{-}1)$$

## 3. EVALUATION OF THE PROBABILITY DENSITY FUNCTION P(h)

### 3.1 Reduction to a Double Integral

In determining P(h) consider first $p_1(h)$ in the following form, obtained from equation (1-22):

$$p_1(h) = \frac{1}{Q} \lim_{T\to\infty} \int_{-\infty}^{\infty} \int_0^T [\mathrm{Prob}(\min y=\alpha, t=0;\ \max y=\alpha+h, t=\tau) - Q_\alpha Q^{\alpha+h}]\, d\tau\, d\alpha \qquad (3.1\text{-}1)$$

The integrations in $\tau$ and $\alpha$ are independent and hence may be interchanged, giving

$$p_1(h) = \frac{1}{Q} \lim_{T\to\infty} \int_0^T \int_{-\infty}^{\infty} [\mathrm{Prob}(\min y=\alpha, t=0;\ \max y=\alpha+h, t=\tau) - Q_\alpha Q^{\alpha+h}]\, d\alpha\, d\tau \qquad (3.1\text{-}2)$$

Introducing the following notation

$$F_1(\tau,h) = \int_{-\infty}^{\infty} \mathrm{Prob}(\min y=\alpha, t=0;\ \max y=\alpha+h, t=\tau)\, d\alpha \qquad (3.1\text{-}3)$$

$$F_0(\tau,h) = \int_{-\infty}^{\infty} Q_\alpha Q^{\alpha+h}\, d\alpha = \int_{-\infty}^{\infty} \mathrm{Prob}(\min y=\alpha)\,\mathrm{Prob}(\max y=\alpha+h)\, d\alpha \qquad (3.1\text{-}4)$$

$p_1(h)$ is given by

$$p_1(h) = \frac{1}{Q} \lim_{T\to\infty} \int_0^T [F_1(\tau,h) - F_0(h)]\, d\tau \qquad (3.1\text{-}5)$$

Note that $p_0(h)$ is expressed in terms of $F_0(h)$ by

$$p_0(h) = \frac{1}{Q^2} F_0(h) \qquad (3.1\text{-}6)$$

The evaluation of $p_1(h)$ and, in the process, of $p_0(h)$ now involves the calculation of a joint probability density function for $F_1$ and a product of two probability densities for $F_0$. It has already been noted that in the limit as $\tau$ approaches infinity the joint distribution approaches the product of the individual distributions. Hence, it will only be necessary to go through the evaluation of $F_1(\tau, h)$, using the limiting case of the final result to obtain an expression for $F_0(h)$.

The calculation of $F_1(\tau, h)$ involves the use of a joint probability density function in six random variables. The six random variables are the values of $y(t)$, $y'(t)$ and $y''(t)$ evaluated at $t = 0$ and $t = \tau$. Denote the density function by

$$f(y_1, y_1', y_1'' ; y_2, y_2', y_2'')$$

where the subscript 1 denotes the values at $t_1 = 0$ and the subscript 2 denotes the values at $t_2 = \tau$. Then

$$f(\alpha, 0, \gamma ; \alpha', 0, \gamma')$$

represents the joint probability density of having $y = \alpha$, $y' = 0$, $y'' = \gamma$ at $t = 0$ and of having $y = \alpha'$, $y' = 0$, $y'' = \gamma'$ at $t = \tau$. This density function is used to calculate the joint probability density, Prob(Min $y = \alpha$, $t = 0$; Max $y = \alpha'$, $t = \tau$), which is simply the joint probability that at $t = 0$ $y'$ will pass through 0 in the interval $\alpha$, $\alpha + d\alpha$ with a positive slope and that at $t = \tau$, $y'$ will pass through 0 in the interval $\alpha'$, $\alpha' + d\alpha'$ with a negative slope. That is,

$$\mathrm{Prob}(\min y=\alpha,\ t=0;\ \max y=\alpha',\ t=\tau)$$
$$= \int_0^{-\infty}\int_0^{\infty} \gamma\gamma' f(\alpha,0,\beta;\alpha',0,\beta')\, d\gamma\, d\gamma' \quad [1] \tag{3.1-7}$$

Setting $\alpha' = \alpha + h$ and substituting (3.1-7) into the expression (3.1-3) gives us

$$F_1(\tau, h) = \int_{-\infty}^{\infty} d\alpha \int_0^{-\infty} \gamma' d\gamma' \int_0^{\infty} \gamma f(\alpha,0,\gamma;\alpha+h,0,\gamma')\, d\alpha \tag{3.1-8}$$

In the case of a Gaussian process, the joint probability density function f may be expressed as follows:

$$f(\alpha,0,\gamma;\alpha',0,\gamma') = \frac{1}{(2\pi)^3\sqrt{|M|}} \exp\left[-\frac{1}{2|M|}\sum_{i,j=1}^{6} M_{ij}\alpha_i\alpha_j\right] \tag{3.1-9}$$

where

$\alpha_1 = \alpha$ $\quad\quad$ $\alpha_2 = 0$ $\quad\quad$ $\alpha_3 = \gamma$

$\alpha_4 = \alpha'$ $\quad\quad$ $\alpha_5 = 0$ $\quad\quad$ $\alpha_6 = \gamma'$

and where $|M|$ and $M_{ij}$ are respectively the determinant and the cofactors of the matrix

$$\|M\| = \left\| \begin{matrix} \varphi(0) & 0 & \varphi''(0) & \varphi(\tau) & \varphi'(\tau) & \varphi''(\tau) \\ 0 & -\varphi''(\tau) & 0 & -\varphi'(\tau) & -\varphi''(\tau) & -\varphi^{(3)}(\tau) \\ \varphi''(0) & 0 & \varphi^{(4)}(0) & \varphi''(\tau) & \varphi^{(3)}(\tau) & \varphi^{(4)}(\tau) \\ \varphi(\tau) & -\varphi'(\tau) & \varphi''(\tau) & \varphi(0) & 0 & \varphi''(0) \\ \varphi'(\tau) & -\varphi''(\tau) & \varphi^{(3)}(\tau) & 0 & -\varphi''(0) & 0 \\ \varphi''(\tau) & -\varphi^{(3)}(\tau) & \varphi^{(4)}(\tau) & \varphi''(0) & 0 & \varphi^{(4)}(0) \end{matrix} \right\| \tag{3.1-10}$$

where $\varphi$, $\varphi'$, $\varphi''$, $\varphi^{(3)}$, and $\varphi^{(4)}$ represent the auto-correlation function of the process under consideration and its successive

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1 Ref. 3, pp. 189 - 191 (See Bibliography)

derivatives.

The following identities arising from the symmetry of the matrix $\| M \|$ are useful in simplifying equation (3.1-8):

$$(1) \quad M_{ij} = M_{ji} \tag{3.1-11}$$

$$(2) \quad M_{11} = M_{44},\ M_{33} = M_{66},\ M_{34} = M_{16}, \text{ and } M_{46} = M_{13} \tag{3.1-12}$$

The latter equalities are readily proved by elementary manipulations of the related cofactor determinants.

Using the results (3.1-9) and (3.1-10) and expanding the series of (3.1-8), the following is obtained for the probability density function f:

$$f = \frac{1}{(2\pi)^3\sqrt{|M|}} \exp\Big[-\frac{1}{2|M|}\Big\{M_{33}(\gamma^2+\gamma'^2) + 2M_{36}\gamma\gamma' + 2M_{13}h\gamma' + 2M_{16}h\gamma + 2([M_{11}+M_{14}]h + [M_{13}+M_{16}][\gamma+\gamma'])\alpha + 2(M_{11}+M_{14})\alpha^2 + M_{11}h^2\Big\}\Big] \tag{3.1-13}$$

Substituting f in (3.1-13) into (3.1-8) we obtain

$$F_1(\tau, h) = \int_{-\infty}^{\infty} d\alpha \int_0^{-\infty} \gamma' d\gamma' \int_0^{\infty} \gamma \frac{1}{(2\pi)^3\sqrt{|M|}} \exp\Big[-\frac{1}{2|M|}\Big\{M_{33}(\gamma^2+\gamma'^2) + 2M_{36}\gamma\gamma' + 2M_{13}h\gamma' + 2M_{16}h\gamma + 2([M_{11}+M_{14}]h + [M_{13}+M_{16}][\gamma+\gamma'])\alpha + 2(M_{11}+M_{14})\alpha^2 + M_{11}h^2\Big\}\Big] d\gamma \tag{3.1-14}$$

All three integrations in (3.1-14) are independent and hence, their order may be interchanged. We may therefore perform the integration in $\alpha$ first. Rewriting $F_1(\tau, h)$, we have

$$F_1(\tau,h) = \frac{\exp\left[-\frac{M_{11}h^2}{2|M|}\right]}{8\pi^3\sqrt{|M|}} \int_0^{-\infty} \gamma' d\gamma' \int_0^{\infty} \gamma \exp\left[-\frac{1}{2|M|}\left\{M_{33}(\gamma^2+\gamma'^2)\right.\right.$$

$$\left.\left. + 2M_{36}\gamma\gamma' + 2M_{13}h\gamma' + 2M_{16}h\gamma\right\}\right] \cdot H \, d\gamma \qquad (3.1\text{-}15)$$

where

$$H = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2|M|}\left\{([M_{11}+M_{14}]h + [M_{13}+M_{16}][\gamma'+\gamma])\alpha + (M_{11}+M_{14})\alpha^2\right\}\right] d\alpha$$

$$= \int_{-\infty}^{\infty} e^{-a\alpha^2 - b\alpha} d\alpha = \frac{e^{b^2/4a}}{\sqrt{a}} \frac{\sqrt{\pi}}{2} \operatorname{erf} \sqrt{a}\left(\alpha + \frac{b}{2a}\right)\Big]_{-\infty}^{\infty}$$

$$= \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}$$

or

$$H = \frac{\sqrt{\pi}}{\sqrt{\frac{(M_{11}+M_{14})}{|M|}}} \exp\left[\frac{1}{4|M|(M_{11}+M_{14})}\left\{(M_{11}+M_{14})^2h^2 + (M_{13}+M_{16})^2(\gamma+\gamma')^2 + 2(M_{11}+M_{14})(M_{13}+M_{16})(\gamma+\gamma')\right\}\right]$$

Substituting H into (3.1-15), expanding the various terms and then gathering like terms gives

$$F_1(\tau,h) = \frac{\exp\left[-\frac{(M_{11}-M_{14})h^2}{4|M|}\right]}{8\pi^{\frac{5}{2}}\sqrt{M_{11}+M_{14}}} \int_0^{-\infty}\int_0^{\infty} \gamma\gamma' \exp\left[-\frac{1}{2|M|}\left\{\left(M_{33} - \frac{[M_{13}+M_{16}]^2}{2[M_{11}+M_{14}]}\right)\cdot\right.\right.$$

$$\left.\left. \cdot(\gamma^2+\gamma'^2) + 2\left(M_{36} - \frac{[M_{13}+M_{16}]^2}{2[M_{11}+M_{14}]}\right)\gamma\gamma' - (M_{13}-M_{16})h(\gamma-\gamma')\right\}\right] d\gamma \, d\gamma'$$

(3.1-16)

Apply now the following change of coordinates

Let

$$u = \sqrt{\frac{1}{2|M|}\left(M_{33} - \frac{[M_{13}+M_{16}]^2}{2(M_{11}+M_{14})}\right)}\ \gamma \qquad (3.1\text{-}17)$$

$$V = -\sqrt{\frac{1}{2|M|}\left(M_{33} - \frac{[M_{13}+M_{16}]^2}{2(M_{11}+M_{14})}\right)}\ \gamma' \qquad (3.1\text{-}18)$$

---

2 The error function, erf x, is defined as: $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$

Rewriting $F_1(\tau,h)$ in terms of integrals in the u v plane gives

$$F_1(\tau,h)=\frac{|M|^2\exp\left[-\frac{(M_{11}-M_{14})h^2}{4|M|}\right]}{2\pi^{\frac{5}{2}}\sqrt{(M_{11}+M_{14})}\left(M_{33}-\frac{[M_{13}+M_{16}]^2}{2[M_{11}+M_{14}]}\right)^2}\int_0^\infty\int_0^\infty uv\exp\left\{-\left[u^2-2\frac{\left\{M_{36}-\frac{(M_{13}+M_{16})^2}{2(M_{11}+M_{14})}\right\}}{\left\{M_{33}-\frac{(M_{13}+M_{16})^2}{2(M_{11}+M_{14})}\right\}}uv+v^2-\frac{(M_{13}-M_{16})h\,(u+v)}{\sqrt{2|M|\left(M_{33}-\frac{(M_{13}+M_{16})^2}{2(M_{11}+M_{14})}\right)}}\right]\right\}du\,dv \quad (3.1\text{-}19)$$

To reduce this to a more compact form, the following are introduced:

$$A_1(\tau)=\frac{|M|^2}{2\pi^{5/2}\sqrt{(M_{11}+M_{14})}\left(M_{33}-\frac{[M_{13}+M_{16}]^2}{2[M_{11}+M_{14}]}\right)^2} \quad (3.1\text{-}20)$$

$$B_1(\tau)=\frac{(M_{11}-M_{14})}{4|M|} \quad (3.1\text{-}21)$$

$$b_1(\tau)=-\frac{\left\{M_{36}-\frac{(M_{13}+M_{16})^2}{2(M_{11}+M_{14})}\right\}}{\left\{M_{33}-\frac{(M_{13}+M_{16})^2}{2(M_{11}+M_{14})}\right\}} \quad (3.1\text{-}22)$$

$$k_1(\tau)=\frac{M_{13}-M_{16}}{2\sqrt{2|M|\left(M_{33}-\frac{[M_{13}+M_{16}]^2}{2[M_{11}+M_{14}]}\right)}} \quad (3.1\text{-}23)$$

Substituting (3.1-20) to (3.1-23) into (3.1-19) the final form of $F_1(\tau,h)$ before integrating is determined as

$$F_1(\tau,h)=A_1e^{-B_1h^2}\int_0^\infty\int_0^\infty uv\,e^{-[u^2+2b_1uv+v^2-2k_1hu-2k_1hv]}\,du\,dv \quad (3.1\text{-}24)$$

## 3.2 Determination of a Certain Double Integral

Obtaining $F_1(\tau, h)$ depends on calculating a double integral which is a particular case of the following integral, defined as a function of r, s, b and m:

$$G_m(r, s, b) = \int_0^\infty\int_0^\infty x^m y^m e^{-[x^2+2bxy+y^2-rx-sy]}\,dx\,dy. \tag{3.2-1}$$

Integrals of the type found in (3.1-24) and (3.2-1) occur frequently in studies dealing with statistics. Usually, however, the linear terms (such as rx and sy) are missing, in which case a closed form solution is readily obtained.[3] If the cross term is missing and the linear terms either present or not, a closed form solution is again easily found. In this latter instance the double integral becomes simply the product of two single integrals, each of which gives rise to the error function, erf x. Considerable effort was made to obtain a closed form solution to the double integral of (3.1-22) but all attempts proved futile. A series solution in a surprisingly simple form was finally found which is generalized in the following analysis to give the solution to the double integral of (3.2-1).

The first step in obtaining the solution to (3.2-1) is to expand $e^{-2bxy}$ in a Maclaurin series, treating 2bxy as a single variable.

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3 See Appendix.

$$e^{-2bxy} = 1 - \frac{(2b)(xy)}{1!} + \frac{(2b)^2(xy)^2}{2!} - \cdots \frac{(-1)^n(2b)^n(xy)^n}{n!} + \cdots \tag{3.2-2}$$

This expansion is valid for any value of 2bxy. Substituting into (3.2-1):

$$G_m(r,s,b) = \int_0^\infty \int_0^\infty e^{-[x^2 - rx + y^2 - sy]} \left(x^m y^m - \frac{2b}{1!} x^{m+1} y^{m+1} + \cdots \right.$$

$$\left. \cdots + \frac{(-1)^n b^n}{n!} x^{m+n} y^{m+n}\right) dx\,dy \tag{3.2-3}$$

Consider the fundamental integral obtained by setting b = m = 0

$$G_0(r,s,0) = \int_0^\infty \int_0^\infty e^{-[x^2 - rx + y^2 - sy]}\, dx\,dy \tag{3.2-4}$$

and note that the general integral $G_m(r, s, b)$ as given in (3.2-3) can be written in terms of $G_0(r, s, 0)$ in the following manner, treating r, s as variable parameters of the integral $G_0$:

$$G_m(r,s,b) = \frac{\partial^{2m} G_0}{\partial r^m \partial s^m} - \frac{2b}{1!} \frac{\partial^{2(m+1)} G_0}{\partial r^{m+1} \partial s^{m+1}} + \cdots$$

$$+ \frac{(-1)^n (2b)^n}{n!} \frac{\partial^{2(n+m)} G_0}{\partial r^{m+n} \partial s^{m+n}} \tag{3.2-5}$$

where $\frac{\partial^{2p} G_0}{\partial r^p \partial s^p}$ is the partial derivative obtained by differentiating $G_0$, p times with respect to r and p times with respect to s. Since the limits are independent of r, s the partial derivatives of the integral may be taken under the integral sign.

$$G_0(r,s,0) = \int_0^\infty\!\!\int_0^\infty e^{-[x^2 - rx + y^2 - sy]}\,dx\,dy$$

$$= \int_0^\infty e^{-[y^2 - sy]}\,dy \int_0^\infty e^{-[x^2 - rx]}\,dx$$

$$= \frac{\sqrt{\pi}}{2}\, e^{\frac{s^2}{4}} \operatorname{erf}\left(y - \frac{s}{2}\right)\Big]_0^\infty \frac{\sqrt{\pi}}{2}\, e^{\frac{r^2}{4}} \operatorname{erf}\left(x - \frac{r}{2}\right)\Big]_0^\infty$$

or

$$G_0(r,s) = \left\{\frac{\sqrt{\pi}}{2} e^{\frac{s^2}{4}}\left(1 + \operatorname{erf}\frac{s}{2}\right)\right\}\left\{\frac{\sqrt{\pi}}{2} e^{\frac{r^2}{4}}\left(1 + \operatorname{erf}\frac{r}{2}\right)\right\} \quad (3.2\text{-}6)$$

where $\operatorname{erf} x = 2/\sqrt{\pi}\int_0^x e^{-t^2}dt$ and where use was made of the fact that $\operatorname{erf}\{-x\} = -\operatorname{erf} x$; i.e., that the error function is an odd function.

By letting

$$f(u) = \frac{\sqrt{\pi}}{2} e^{u^2}(1 + \operatorname{erf} u) \quad (3.2\text{-}7)$$

(3.2-6) can be put in the following simplified form:

$$G_0(r,s,0) = f\left(\frac{s}{2}\right) f\left(\frac{r}{2}\right) \quad (3.2\text{-}8)$$

Taking the various partials of $G_0$ with respect to r and s yields

$$\frac{\partial^2 G_0}{\partial r \partial s} = \frac{1}{2^2} f'\left(\frac{s}{2}\right) f'\left(\frac{r}{2}\right) \qquad \frac{\partial^4 G_0}{\partial r^2 \partial s^2} = \frac{1}{2^4} f''\left(\frac{s}{2}\right) f''\left(\frac{r}{2}\right)$$

$$\frac{\partial^{2n} G_0}{\partial r^n \partial s^n} = \frac{1}{2^{2n}} f^{(n)}\left(\frac{s}{2}\right) f^{(n)}\left(\frac{r}{2}\right) \quad (3.2\text{-}9)$$

Substituting (3.2-9) into the term of (3.2-5) gives the final series form for the integral $G_m$:

$$G_m(r,s,b) = \frac{1}{2^{2m}} f^{(m)}\left(\frac{s}{2}\right) f^{(m)}\left(\frac{r}{2}\right) - \frac{2b}{1!}\frac{1}{2^{2(m+1)}} f^{(m+1)}\left(\frac{s}{2}\right) f^{(m+1)}\left(\frac{r}{2}\right) + \cdots$$

$$+ \frac{(-1)^n (2b)^n}{n!\, 2^{2(n+m)}} f^{(n+m)}\left(\frac{s}{2}\right) f^{(n+m)}\left(\frac{r}{2}\right) + \cdots$$

$$G_m(r,s,b) = \frac{1}{2^{2m}} f^{(m)}\left(\frac{s}{2}\right) f^{(m)}\left(\frac{r}{2}\right) - \frac{b}{1!\,2^{(1+2m)}} f^{(m+1)}\left(\frac{s}{2}\right) f^{(m+1)}\left(\frac{r}{2}\right) + \cdots$$

$$+ \frac{(-1)^n b^n}{n!\,2^{n+2m}} f^{(n+m)}\left(\frac{s}{2}\right) f^{(n+m)}\left(\frac{r}{2}\right) + \cdots$$

$$G_m(r,s,b) = \sum_{n=0}^{\infty} \frac{(-1)^n b^n}{n!\,2^{(n+2m)}} f^{(n+m)}\left(\frac{s}{2}\right) f^{(n+m)}\left(\frac{r}{2}\right) \qquad (3.2\text{-}10)$$

with f(u) defined in (3.2-7)

## 3.3 Evaluation of $\tilde{P}(h)$

The double integral contained in $F_1(\tau, h)$ (equation 3.1-24) is a particular case of the general double integral just evaluated. Denoting $G_m(r, s, b)$ by $G_m(r, b)$ when r = s and setting m = 1, b = $b_1$ and r = s = $2k_1h$ in (3.2-10), substituting $G_m(r,b)$ for the double integral in (3.2-10) gives the final form for $F_1(\tau, h)$.

$$F_1(\tau,h) = A_1(\tau) e^{-B_1(\tau)h^2} \sum_{n=0}^{\infty} \frac{(-1)^n b_1^n(\tau)}{n!\,2^{n+2}} \left\{ f^{(n+1)}(k_1(\tau)h) \right\}^2 \qquad (3.3\text{-}1)$$

or simply $F_1(\tau,h) = A_1(\tau) e^{-B_1(\tau)h^2} G_1(2h k_1(\tau), b_1(\tau))$ (3.3-2)

As was previously mentioned, $F_0(h)$ may be obtained from $F_1(h)$ as $\tau$ approaches infinity. As $\tau$ approaches infinity $\varphi(\tau)$, $\varphi'(\tau)$, $\varphi^{(3)}(\tau)$ and $\varphi^{(4)}(\tau)$ all approach zero. Define the terms

$d_{11}$, $d_{22}$, $d_{33}$ by

$$d_{11} = \overline{y^2(t)} = \varphi(0) = \frac{1}{\pi}\int_0^\infty \Phi(\omega)d\omega$$
$$d_{22} = \overline{[y'(t)]^2} = -\varphi''(0) = \frac{1}{\pi}\int_0^\infty \omega^2 \Phi(\omega)\,d\omega$$
$$d_{33} = \overline{[y''(t)]^2} = \varphi^{(4)}(0) = \frac{1}{\pi}\int_0^\infty \omega^4 \Phi(\omega)\,d\omega$$

where $\Phi$ is the normalized power spectrum corresponding to the correlation function $\varphi$. Then in the limit the matrix $\|M\|$ will approach the matrix $\|M_L\|$ where

$$\|M_L\| = \begin{Vmatrix} d_{11} & 0 & -d_{22} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 & 0 & 0 \\ -d_{22} & 0 & d_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{11} & 0 & -d_{22} \\ 0 & 0 & 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & -d_{22} & 0 & d_{33} \end{Vmatrix} \qquad (3.3\text{-}3)$$

Denoting by $\|m\|$ the matrix for the probability density function $f(\alpha, 0, \gamma)$, $\|m\|$ is given as

$$\|m\| = \begin{Vmatrix} d_{11} & 0 & -d_{22} \\ 0 & d_{22} & 0 \\ -d_{22} & 0 & d_{33} \end{Vmatrix} \qquad (3.3\text{-}4)$$

Obtaining the cofactors of the determinant $|M_L|$ and writing these in terms of the cofactors of the determinant, $|m|$, the following results are readily found:

(1) $(M_L)_{14} \equiv (M_L)_{16} \equiv (M_L)_{36} \equiv 0$ (3.3-5)

(2) (a) $|M_L| = (|m|)^2$ (3.3-6)

(b) $(M_L)_{11} = m_{11}|m|$

(c) $(M_L)_{13} = m_{13}|m|$

(d) $(M_L)_{33} = m_{33}|m|$

Substituting these cofactors into (3.1-20) through (3.1-23) gives us the corresponding values for $F_0(h)$:

$$A_0 = \frac{|m|^{3/2}}{2\pi^{5/2}\sqrt{m_{11}}\left(m_{33} - \frac{m_{13}^2}{2m_{11}}\right)^2} \qquad (3.3\text{-}7)$$

$$B_0 = \frac{m_{11}}{4|m|} \qquad (3.3\text{-}8)$$

$$b_0 = \frac{m_{13}^2}{2m_{11}\left(m_{33} - \frac{m_{13}^2}{2m_{11}}\right)} \qquad (3.3\text{-}9)$$

$$k_0 = \frac{m_{13}}{2\sqrt{2|m|\left(m_{33} - \frac{m_{13}^2}{2m_{11}}\right)}} \qquad (3.3\text{-}10)$$

By expanding the determinant $|m|$ and its cofactors in terms of the elements $d_{11}$, $d_{22}$, $d_{33}$ we obtain the following forms for the constants of $F_0$:

$$A_0 = \frac{2\left[(d_{11}d_{33} - d_{22}^2)\,d_{33}\right]^{3/2}}{\pi^{5/2}\,d_{22}\,(2d_{11}d_{33} - d_{22}^2)^2} \qquad (3.3\text{-}11)$$

$$B_0 = \frac{d_{33}}{4(d_{11}d_{33} - d_{22}^2)} \qquad (3.3\text{-}12)$$

$$b_0 = \frac{d_{22}^2}{2d_{11}d_{33} - d_{22}^2} \qquad (3.3\text{-}13)$$

$$k_0 = \frac{d_{22}\sqrt{d_{33}}}{2\sqrt{(d_{11}d_{33} - d_{22}^2)(2d_{11}d_{33} - d_{22}^2)}} \qquad (3.3\text{-}14)$$

Hence, using either (3.3-6) through (3.3-9) or (3.3-10) through (3.3-13), $F_o(h)$ is given as

$$F_o(h) = A_o e^{-B_o h^2} \sum \frac{(-1)^n b_o^n}{n!\, 2^{n+2}} \left\{ f^{n+1}(k_o h) \right\}^2 \tag{3.3-15}$$

or $$F_o(h) = A_o e^{-B_o h^2} G_1(2k_o h, b_o) \tag{3.3-16}$$

Using the following expression for Q, the expected number of maxima or minima per unit time,[4]

$$Q = \frac{1}{2\pi} \sqrt{\frac{d_{33}}{d_{22}}} \tag{3.3-17}$$

and substituting (3.3-14) and (3.3-1) into (3.1-5) and (3.1-6) and then substituting the results into (2-1) gives the following approximation for P(h), the distribution of rises or falls:

$$\begin{aligned} \tilde{P}(h) = {} & \frac{A_o e^{-B_o h^2}}{Q^2} \sum \frac{(-1)^n b^n}{n!\, 2^{n+2}} \left\{ f^{(n+1)}(k_o h) \right\}^2 \\ & + \frac{1}{Q} \int_0^\infty \Big[ A_1 e^{-B_1 h^2} \sum_{n=0}^{\infty} \frac{(-1)^n b_1^n}{n!\, 2^{n+2}} \left\{ f^{(n+1)}(k_1 h) \right\}^2 \\ & - A_o e^{-B_o h^2} \sum_{n=0}^{\infty} \frac{(-1)^n b_1^n}{n!\, 2^{n+2}} \left\{ f^{(n+1)}(k_o h) \right\}^2 \Big] d\tau \end{aligned} \tag{3.3-18}$$

or, again, more simply in terms of the double integral $G_m(r, s, b)$ as

$$\begin{aligned} \tilde{P}(h) = {} & \frac{1}{Q^2} G_1(2h k_o, b_o) + \frac{1}{Q} \int_0^\infty \Big[ A_1(\tau) e^{-B_1(\tau) h^2} G_1(2h k_1(\tau), b_1(\tau)) \\ & - A_o e^{-B_o h^2} G_1(2h k_o, b_o) \Big] d\tau \end{aligned} \tag{3.3-19}$$

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4 Ref. 2, p. 133 (See Bibliography)

with

$$p_0 = \frac{1}{Q^2} G_1(2hk_0, b_0) \qquad (3.3\text{-}20)$$

$$p_1 = \frac{1}{Q} \int_0^\infty [A_1(\tau) e^{-B_1(\tau) h^2} G_1(2hk_1(\tau), b_1(\tau)) - A_0 e^{-B_0 h^2} G_1(2hk_0, b_0)] \, d\tau \qquad (3.3\text{-}21)$$

The final step, integrating over $\tau$, can only be done numerically due to the extreme complexity of the coefficients $A_1$, $B_1$, $b_1$, and $k_1$ as functions of $\tau$.

## 4. THE AVERAGE RISE OR FALL

Unlike the distribution of rises and falls, the average value of the rise or fall, h, can be computed quite easily and is, of course, an important characteristic of a random curve, particularly since it can be computed with no knowledge of the actual distribution of rises and falls.

The method of computation of $\bar{h}$ is to determine the sum of all the rises and falls per unit time. Division of this by 2Q, the number of maxima and minima per unit time will then give $\bar{h}$.

The sum of all the rises and falls is found by integrating the product of $N_\alpha$, the number of times the curve crosses the line $y = \alpha$, and of $d\alpha$. $N_\alpha$ is expressed by[5]

$$N_\alpha = \frac{1}{\pi}\sqrt{\frac{d_{22}}{d_{11}}}\, e^{-\frac{\alpha^2}{2d_{11}}}$$

Using Q from (3.3-15), then, $\bar{h}$ is formed as

$$\bar{h} = \frac{1}{2\frac{1}{2\pi}\sqrt{\frac{d_{33}}{d_{22}}}} \int_{-\infty}^{\infty} \frac{1}{\pi}\sqrt{\frac{d_{22}}{d_{11}}}\, e^{-\frac{\alpha^2}{2d_{11}}}\, d\alpha$$

$$\bar{h} = \frac{d_{22}}{2}\sqrt{\frac{2\pi}{d_{33}}}\, \operatorname{erf}\frac{\alpha}{\sqrt{2d_{11}}}\Bigg]_{-\infty}^{\infty} = d_{22}\sqrt{\frac{2\pi}{d_{33}}} \qquad (4 - 1)$$

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5 Ref. 2, p. 127 (See Bibliography)

Hence the average rise or fall can be simply found from the second and fourth moments of the power spectrum.

# 5. EXAMPLE OF THE IDEAL LOW PASS FILTER

## 5.1 The Power Spectrum and Autocorrelation Function

The remainder of this paper will deal with the problem of applying the results of the previous analysis to the particular case of an ideal low pass filter.

For a random curve the autocorrelation function is given in terms of the normalized power spectrum as

$$\varphi(\tau) = \mathrm{Re}\,\frac{1}{\pi}\int_0^\infty \Phi(\omega)\,e^{i\omega\tau}\,d\omega \qquad (5.1\text{-}1)$$

or $$\varphi(\tau) = \frac{1}{\pi}\int_0^\infty \Phi(\omega)\cos\omega\tau\,d\omega \qquad (5.1\text{-}2)$$

Letting $G(\omega)$ denote the nonnormalized power spectrum then

$$\Phi(\omega) = \frac{G(\omega)}{\int_0^\infty G(\omega)\,d\omega} \qquad (5.1\text{-}3).$$

For the ideal low pass filter $G(\omega)$ is equal to

$$\begin{array}{ll} K \text{ if} & 0 \le \omega \le \omega_c \\ 0 \text{ if} & \omega_c < \omega \end{array} \qquad (5.1\text{-}4)$$

where $\omega_c$ is the cutoff frequency. (See Figure 2). Hence

$$\Phi(\omega) = \frac{K}{\int_0^{\omega_c} K\,d\omega} = \frac{1}{\omega_c} \qquad (5.1\text{-}5)$$

Substituting into (5.1-2), the autocorrelation function is given by

$$\varphi(\tau) = \frac{1}{\pi\omega_c}\int_0^{\omega_c} \cos\omega\tau\,d\omega \qquad (5.1\text{-}6)$$

From this, then, we obtain the following

$$\varphi(\tau) = \frac{\sin \omega_c \tau}{\pi \omega_c \tau} \tag{5.1-7}$$

$$\varphi'(\tau) = \frac{\cos \omega_c \tau}{\pi \tau} - \frac{\sin \omega_c \tau}{\pi \omega_c \tau^2} \tag{5.1-8}$$

$$\varphi''(\tau) = -\frac{2\cos \omega_c \tau}{\pi \tau^2} - \omega_c \frac{\sin \omega_c \tau}{\pi \tau}\left(1 - \frac{2}{\omega_c^2 \tau^2}\right) \tag{5.1-9}$$

$$\varphi^{(3)}(\tau) = -\omega_c^2 \frac{\cos \omega_c \tau}{\pi \tau}\left(1 - \frac{6}{\omega_c^2 \tau^2}\right) + 3\omega_c \frac{\sin \omega_c \tau}{\pi \tau^2}\left(1 - \frac{2}{\omega_c^2 \tau^2}\right) \tag{5.1-10}$$

$$\varphi^{(4)}(\tau) = 4\omega_c^2 \frac{\cos \omega_c \tau}{\pi \tau^2}\left(1 - \frac{6}{\omega_c^2 \tau^2}\right) + \omega_c^3 \frac{\sin \omega_c \tau}{\pi \tau}\left(1 - \frac{12}{\omega_c^2 \tau^2} + \frac{24}{\omega_c^4 \tau^4}\right) \tag{5.1-11}$$

$$d_{11} = \varphi(0) = \frac{1}{\pi} \tag{5.1-12}$$

$$d_{22} = -\varphi''(0) = \frac{\omega_c^2}{3\pi} \tag{5.1-13}$$

$$d_{33} = \varphi^{(4)}(0) = \frac{\omega_c^4}{5\pi} \tag{5.1-14}$$

## 5.2 Reduction of the Role of the Cutoff Frequency in the Computation

Set $\tau = t/\omega_c$. Then the values $t = 2n\pi$, $n = 1, 2, \ldots$, represent n periods of the highest frequency, the cutoff frequency $\omega_c$. Substitute this value into (5.1-7) through (5.1-11), insert the results into the matrix $\|M\|$ and solve for the determinant and pertinent cofactors. Finally, solve for the values $A_i$, $B_i$, $b_i$, $k_i$, $i = 1, 2$ in equations (3.1-20) through (3.1-23) and (3.3-7) through (3.3-10). The results of this show that the coefficients $B_0$, $b_0$, $k_0$, $B_1$, $b_1$, $k_1$, are independent of the cutoff frequency $\omega_c$ while $A_1$ is proportional to the square of $\omega_c$. Substituting expressions (5.1-13) and (5.1-14) into (3.3-17) and determining Q, the expected number of maxima, shows that Q is proportional to $\omega_c$. The net result of this is that $p_0(h)$ is independent of the cutoff frequency while $p_1(h)$ varies directly as the cutoff frequency, $\omega_c$. It has already been noted that (1) the series for P(h) is alternating in sign and (2) each of the terms $p_i(h)$ for $i \geqslant 2$ are smaller than their predecessors, so that $p_0(h)$ being independent of $\omega_c$ and $p_1(h)$ varying directly as $\omega_c$ enables us to immediately draw the conclusion that for small values of $\omega_c$ $p_0(h)$ alone will represent a reasonable approximation to P(h).

## 5.3 Results for the Low Pass Filter

Attempts to carry out the numerical evaluation of $p_0(h)$ and $P_1(h)$ for the low pass filter were only partially successful. Use was made of the Lehigh University LGP-30 electronic computer in this phase of study. The principal result obtained was an accurate plot of $p_0(h)$, the probability density function derived by assuming no correlation between maxima and minima. (See Figure 3). This curve was checked at the value h = 0 by evaluating the double integral $G_0(0, b_0)$ in closed form as shown in the Appendix. The centroid of the area under the curve, giving $\bar{h}_0$, the average rise or fall assuming no correlation, was found to be 1.16 $\bar{h}$, $\bar{h}$ being the actual average rise or fall as found by equation (4-1). This result indicates that ignoring correlation between maxima and minima tends to exaggerate the number of large rises and falls. Thus it appears that by choosing at random a maximum and a minimum, chances are better to get a large rise than by choosing a minimum and the following maximum.

It should be noted that the evaluation of $P_0(h)$ without the use of an electronic computer would be an extremely long and tedious process. The evaluation of $G_0(2k_0h, b_0)$ for one value of h, as the series defined in general in equation (3.2-10) required in one instance as many as seventy-one terms.

Unfortunately, similar success was not encountered in the case of $p_1(h)$ which could not be obtained due either to the limited capabilities of the LGP-30 Computer or to the determinant routine. $p_1(h)$ is given by

$$p_1(h) = \lim_{T\to\infty} \frac{1}{Q} \int_0^\infty [F_1(\tau, h) - F_0(\tau, h)]\, d\tau \qquad (3.1\text{-}6)$$

where $$F_1(\tau, h) = A_1(\tau)\, e^{-B_1(\tau)h^2}\, G_1(2hk_1(\tau), b_1(\tau)) \qquad (3.3\text{-}2)$$

$$F_0(h) = A_0\, e^{-B_0h^2}\, G_0(2hk_0, b_0) \qquad (3.3\text{-}16)$$

Trouble was encountered in the evaluation of $A_1$, $B_1$, $b_1$, $k_1$ for values of $t < 4\pi$, i.e. for those values of time less than two times the period of the cutoff frequency. For these values loss of significant figures gave results which were completely unreliable. For larger values if $t$ accuracy was as good as up to five significant figures. A plot of $F_1(\tau, h)$ is shown in figure 4 as a function of $\tau$ (or t) for the value $h = 0$. Figure 5 is an estimate of the form of $p_1(h)$ and its effect on the approximate form of $P(h)$. It should be noted that a limit is placed on the maximum value of $\omega_c$ for which the approximating form $\tilde{P}(h) = p_0(h) + p_1(h)$ is valid. This limit arises from the fact that part of $p_1(h)$ is negative. Since $p_1(h)$ varies directly as $\omega_c$, this means that as $\omega_c$ becomes larger, there will exist some $\omega_c$ such that for some value of h, $\tilde{P}(h) = p_0(h) + p_1(h)$ is negative, an impossible occurrence.

Although checks on the calculations of $p_1(h)$ must necessarily be limited due to the extensive number of computations involved, a method is indicated in the appendix which can be used to provide a check on the value of $p_1(h)$ at $h = 0$.

## 6. SUMMARY

Using the first two terms of the convergent series expansion given by equation (1-18), the statistical distribution of rises and falls is given approximately by $\tilde{P}(h)$ where

$$\tilde{P}(h) = \frac{1}{Q_2} F_0(h) + \frac{1}{Q} \int_0^\infty [F_1(\tau, h) - F_0(h)]\, d\tau$$

with $\dfrac{F_0(h)}{Q_2}$ = probability density of having a rise $h$ between a given minimum and the following maximum assuming no correlation between maxima and minima.

$$= A_0 e^{-B_0 h^2} G_0(2hk_0, b_0) \qquad (3.3\text{-}16)$$

$F_1(\tau, h)$ = joint probability density of having a minimum at $t = 0$, and a maximum of $t = \tau$, a distance $h$ above the minimum

$$= A_1(\tau) e^{-B_1(\tau) h^2} G_1(2hk_1(\tau), b_1(\tau)) \qquad (3.3\text{-}2)$$

and where

$$G_m(r, b) = \sum_0^\infty \frac{(-1)^n b^n}{2^{n+2} n!} \left\{ f^{(n+1)}\left(\frac{r}{2}\right) \right\}^2$$

with

$$f(u) = \frac{\sqrt{\pi}}{2} e^{u^2} (1 + \operatorname{erf} u)$$

The values of $A_i$, $B_i$, $b_i$, $k_i$ for $i = 1, 2$ are given in equations (3.1-20) through (3.1-23) and (3.3-11) through (3.3-14), the former in terms of the cofactors of the 6 x 6 matrix corresponding to a Gaussian probability distribution function in six random variables, the latter in terms of the area, and second and fourth moments of the power spectrum of the random process under consideration.

Further study should be along the following lines:

(1) Add a function to the autocorrelation function of the low pass filter which is even and whose first six derivatives are zero at $t = 0$. The effect of this would be to provide two different random curves with the same $d_{11}$, $d_{22}$, $d_{33}$, (area, etc. of the power spectrum).

(2) Calculate $\tilde{P}(h)$.

(3) Compare results to determine whether or not the distribution of rises and falls are similar enough to disregard any differences. If this is so, it is an indication that only a similarity between the area, and second and fourth moments of the power spectrum are necessary in order that two random curves have the same distribution of rises and falls.

(4) Follow the same procedure for other random curves, either verifying the results of three or, if necessary, determining other factors, the variation of which will have a significant effect on the distribution of rises and falls.

## APPENDIX

The purpose of this appendix is twofold: (1) as a partial check on the validity of the series solution for $G_m(r, s, b)$ and (2) to outline a method by which a check can be made on the computer solution for $p_1(h)$.

Consider the double integral given by

$$G_0(0,b) = \int_0^\infty \int_0^\infty e^{-[x^2 + 2bxy + y^2]} \, dx \, dy$$

A closed form solution to this is found in the following manner.[6]

(1) Apply a transformation of coordinates which will eliminate the cross term.

Let $F = x^2 + 2bxy + y^2$

Let $\xi_1 = x + by$

Then $F - \xi_1^2 = (1-b^2)y^2$

Therefore let $\xi_2 = \sqrt{1 - b^2}\, y$

Then the transformation

$$\xi_1 = x + by$$
$$\xi_2 = \sqrt{1 - b^2}\, y$$

reduces F to

$$F = \xi_1^2 + \xi_2^2$$

The Jacobian of the transformation is given by

$$|J| = (1-b^2)^{-1/2}$$

so that the area dx dy in the xy plane transforms into

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6 Ref. 3, p. 205. (See Bibliography)

the area $(1 - b^2)^{-1/2} d\xi_1 \, d\xi_2$ in the $\xi_1, \xi_2$ plane. (See Figure 6).

The line $y = 0$ transforms into the line $\xi_2 = 0$ while the line $x = 0$ transforms into the line given by

$$\xi_1 = \frac{b\,\xi_2}{\sqrt{1-b^2}}$$

Hence the area of integration is that included between the lines

$$\xi_2 = 0 \qquad\qquad \xi_1 = \frac{b\,\xi_2}{\sqrt{1-b^2}}$$

The double integral now becomes

$$G_0(0, b) = \iint e^{-(\xi_1^2+\xi_2^2)} (1-b^2)^{-\frac{1}{2}} d\xi_1 \, d\xi_2$$

Changing to polar coordinates,

let $\xi_1 = r \cos\theta$

$\xi_2 = r \sin\theta$

then $d\xi_1 d\xi_2 = r \, dr \, d\theta$

so that $G_0(0, b) = \int_0^{\theta_0}\int_0^{\infty} e^{-r^2} \sqrt{1-b^2} \, r \, dr \, d\theta$

where $\tan\theta_0 = \frac{\sqrt{1 - b^2}}{b}$ $(|b|<1)$

Evaluating $G_0(0, b)$:

$$G_0(0,b) = (1-b^2)^{-\frac{1}{2}} \int_0^{\theta_0} \left[-\frac{1}{2} e^{-r^2}\right]_0^{\infty} d\theta$$

$$= \frac{(1-b^2)^{-\frac{1}{2}}}{2} \arctan \frac{b}{\sqrt{1-b^2}}$$

If $b<1$, $\arctan \frac{\sqrt{1 - b^2}}{b} = \arccos b$, hence

$$G_0(0, b) = \frac{\arccos b}{2\sqrt{1 - b^2}} \tag{1}$$

Expanding this in a power series

$$G_0(0, b) = \frac{\pi}{4} - \frac{b}{2} + \frac{b^2\pi}{1!2^2} - \frac{b^3 16}{2!2^3} + \dots$$

From the definition of $G_m(r, s, b)$ in equation (3.2-10) the following is obtained:

$$G_0(0, b) = \sum_0^\infty \frac{(-1)^m b^m}{m!\,2^m} \left(f^m(0)\right)^2$$

(Recall that $G_m(r, s, b) = G_m(r, b)$ when $r = s$).

The following are easily shown, using f as defined in equation (3.2-7):

$$f(0) = \frac{\sqrt{\pi}}{2}$$

$$f^1(0) = 1$$

$$f^{11}(0) = \sqrt{\pi}$$

$$f^{111}(0) = 4$$

$$f^{(n)}(0) = 2(n-1)\, f^{(n-2)}(0)$$

Substituting these for $G_0$:

$$G_0(0, b) = \frac{\pi}{4} - \frac{b}{2} + \frac{b^2\pi}{1!2^2} - \frac{b^3 16}{2!2^3} + \dots$$

A more general comparison of the series is possible by noting the formation in the first series of a recurrence formula for the derivatives at $b = 0$ similar to that of f in the second.

This method of evaluation of the double integral when the linear terms are missing can be carried a step further, providing the means for a check on $p_1(h)$ at the value $h = 0$.

Consider

$$G_1(0, b) = \int_0^\infty \int_0^\infty xy\, e^{-[x^2+2bxy+y^2]}\, dx\, dy$$

and note that

$$-\frac{1}{2}\frac{\partial G_0(0,b)}{\partial b} = \int_0^\infty \int_0^\infty xy\, e^{-[x^2+2bxy+y^2]}\, dx\, dy$$

$$= G_1(0, b).$$

Hence $G_1(0, b) = -\frac{1}{4}\frac{d}{db}\left(\frac{\arccos b}{\sqrt{1 - b^2}}\right)$

or $G_1(0, b) = \dfrac{1}{4(1 - b^2)}\left(1 - \dfrac{b \arccos b}{\sqrt{1 - b^2}}\right)$ (2)

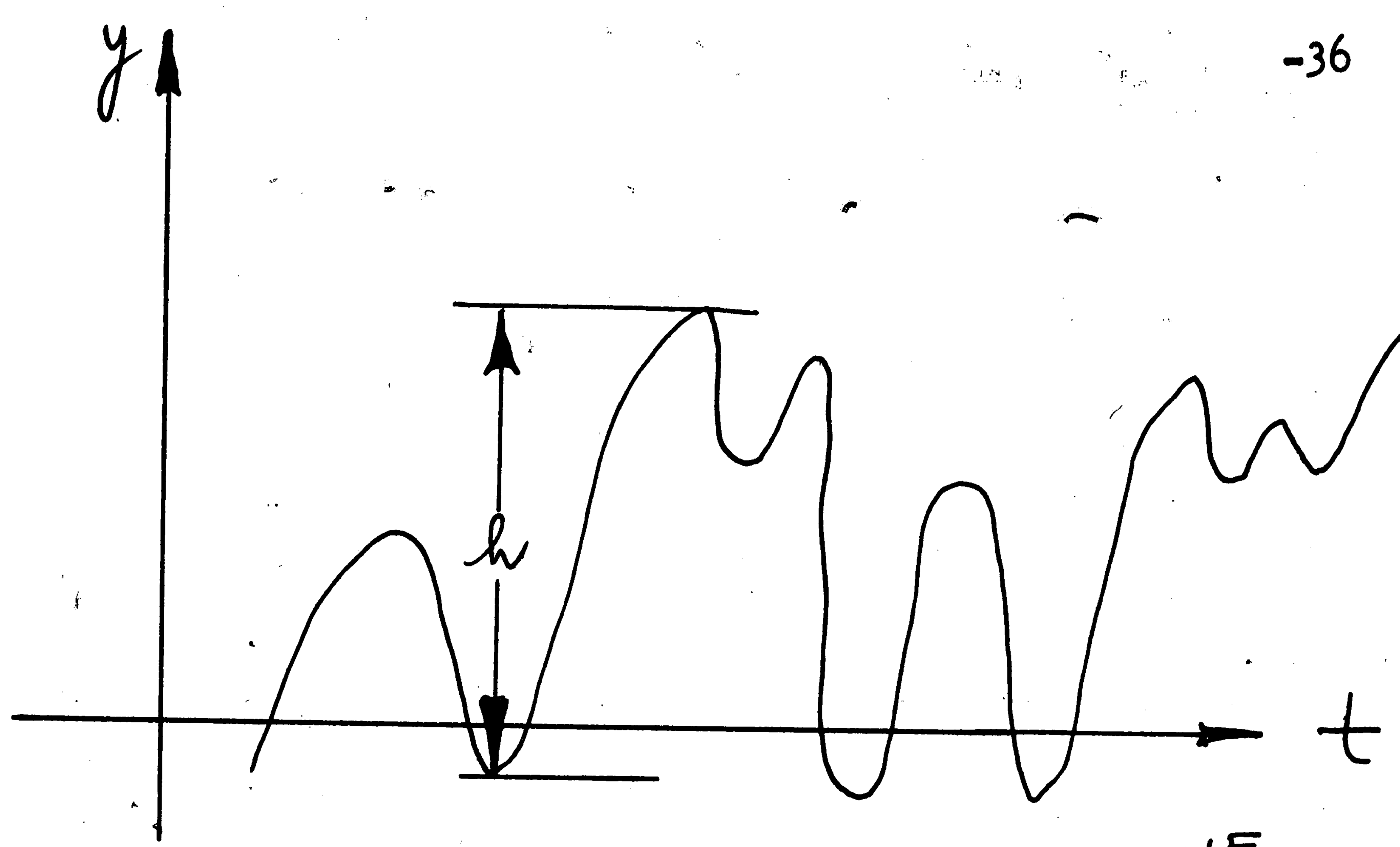


FIG. 1 RANDOM CURVE

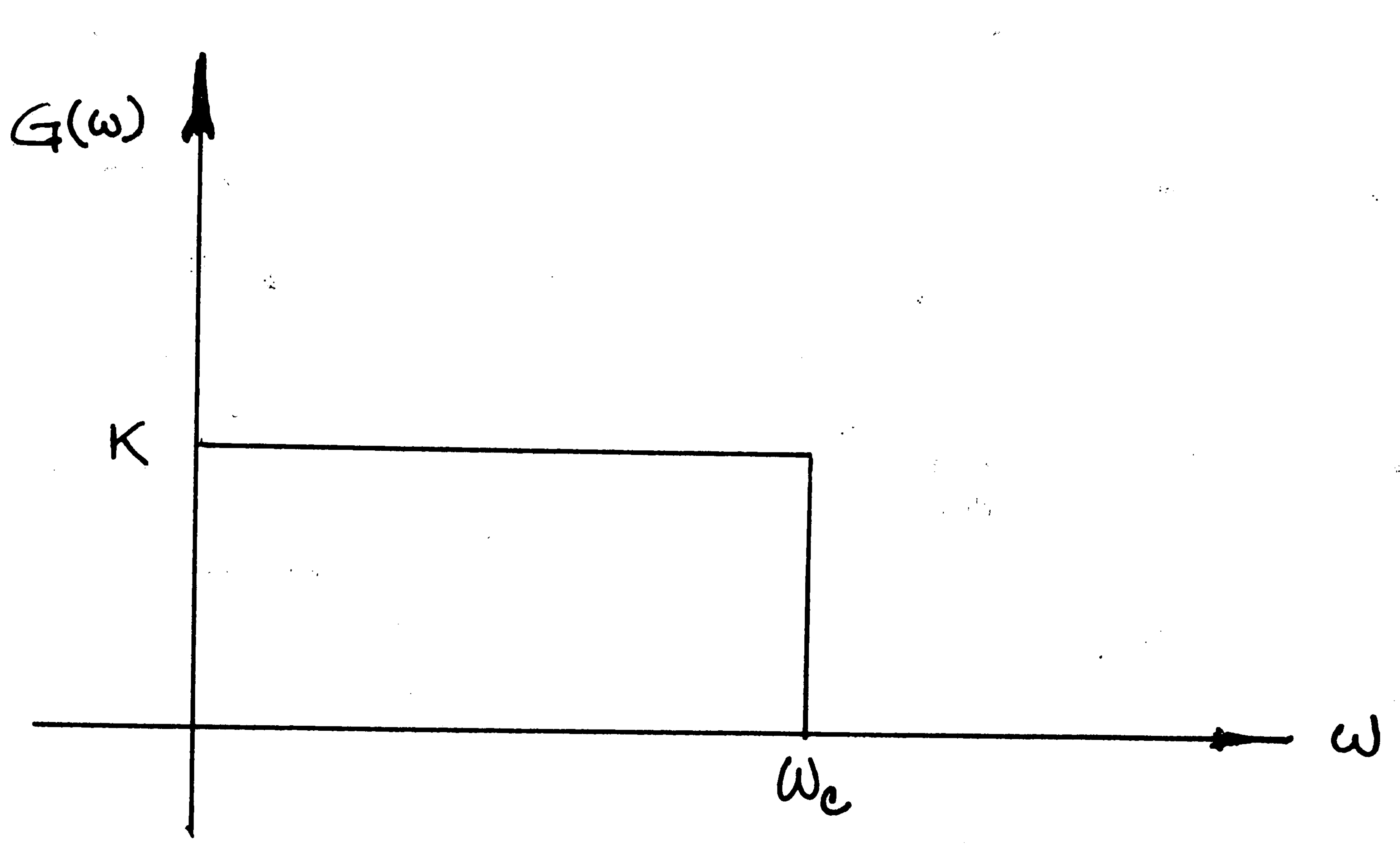


FIG. 2

POWER SPECTRUM FOR A LOW PASS FILTER

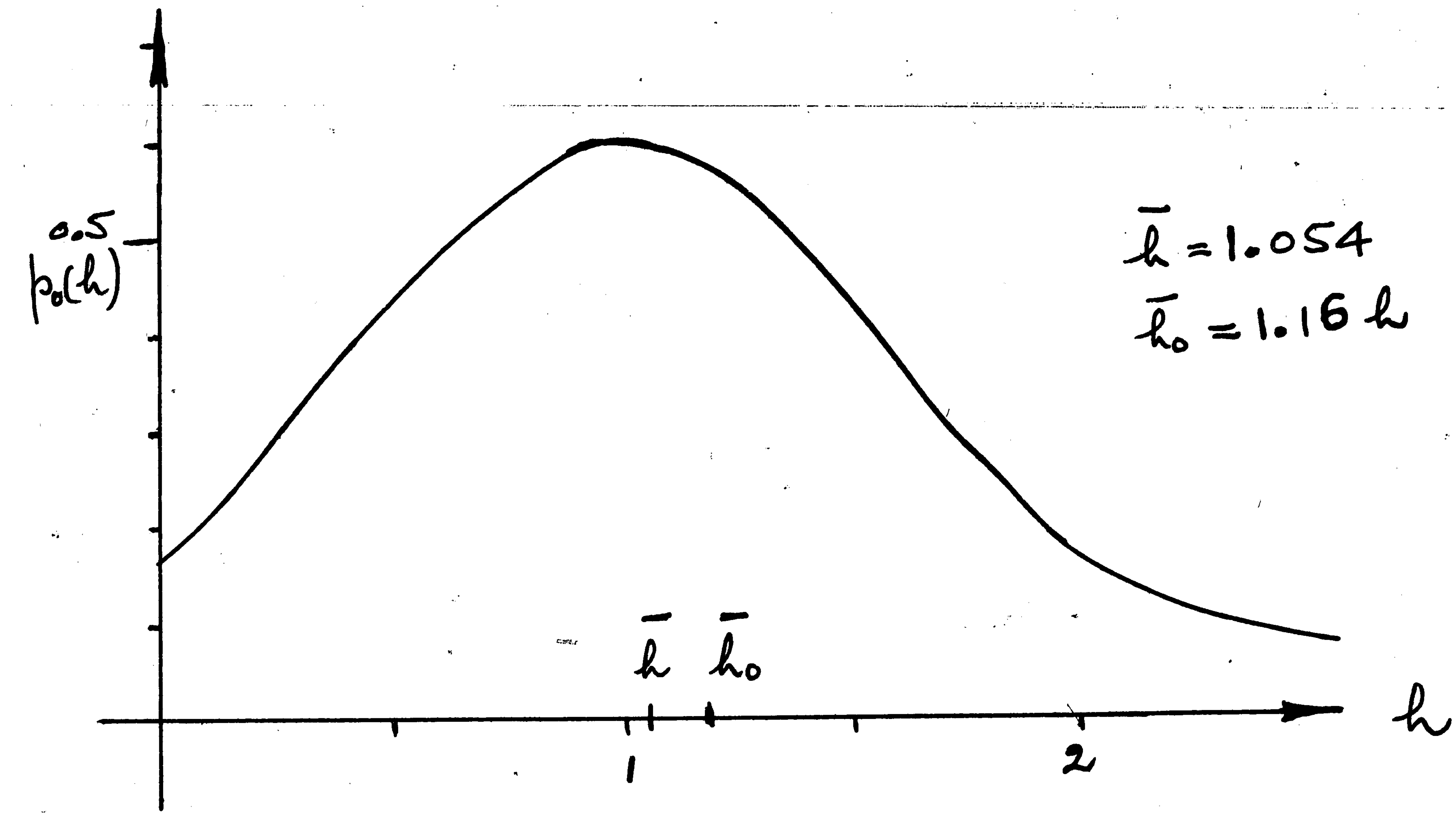


FIG. 3

PROBABILITY DENSITY $p_o(h)$
(No correlation)

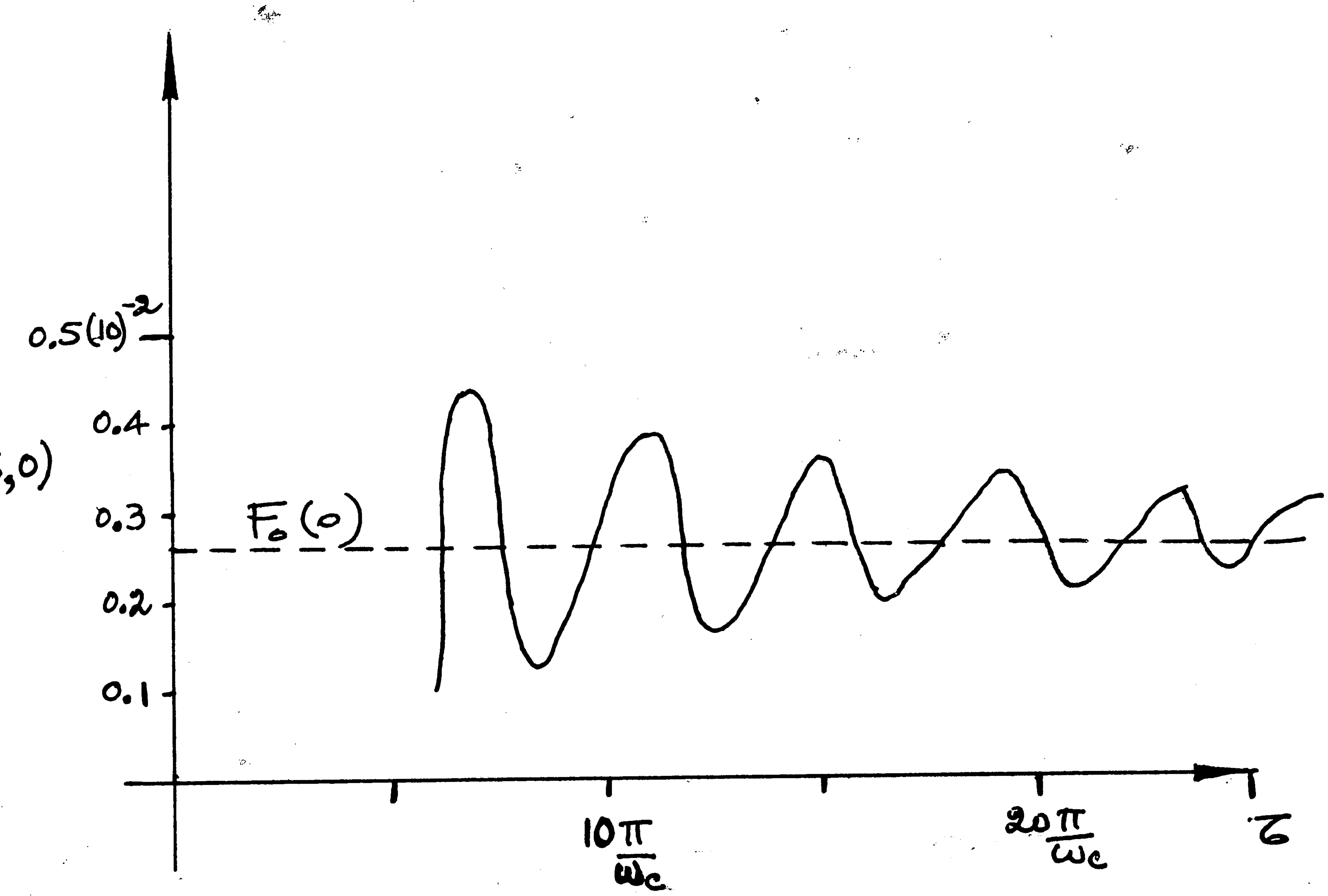


FIG. 4
JOINT PROBABILITY DENSITY
FOR
A MINIMUM AT O
A MAXIMUM AT $\tau$

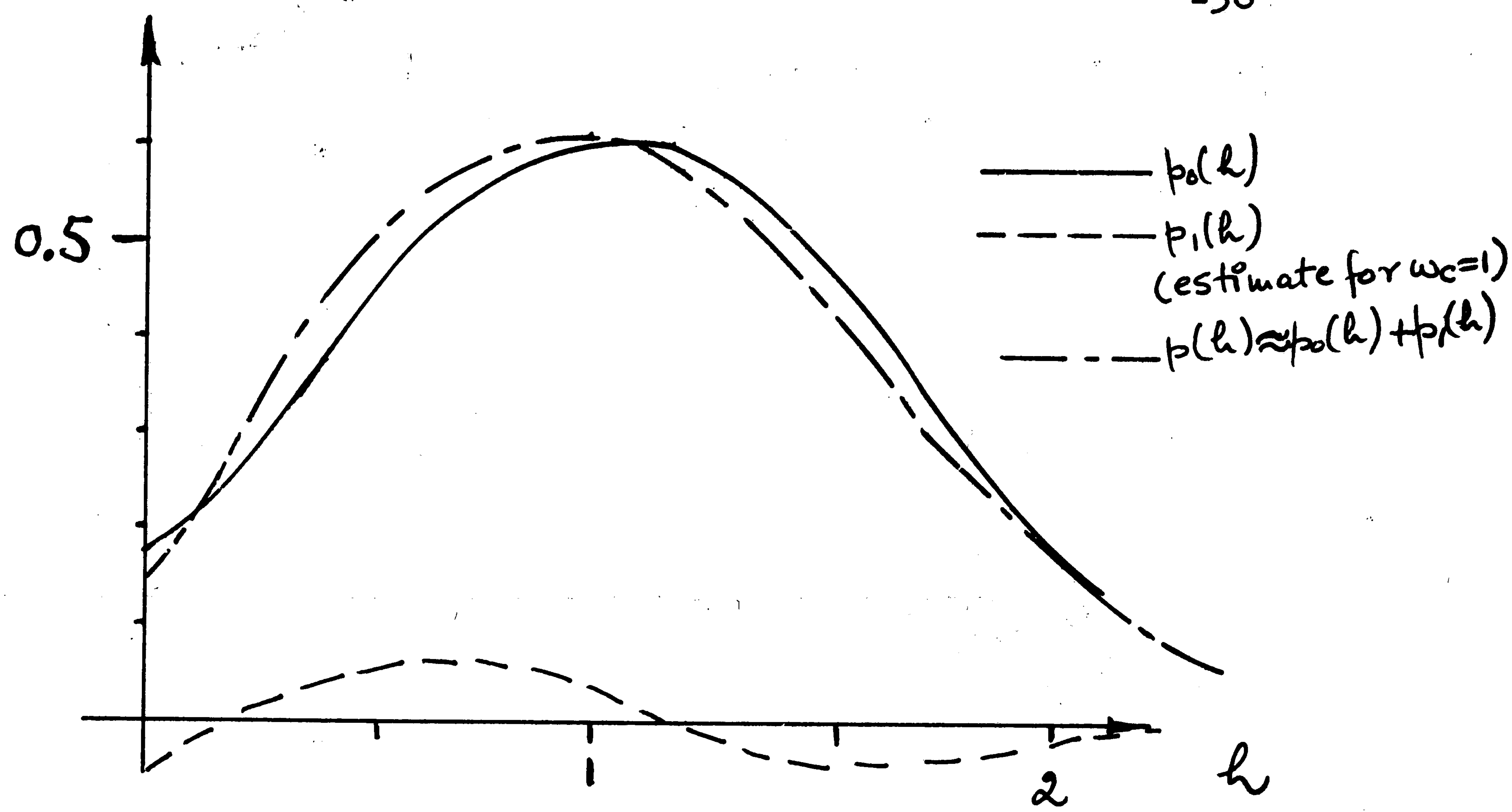


FIG. 5

DISTRIBUTION OF RISES AND FALLS

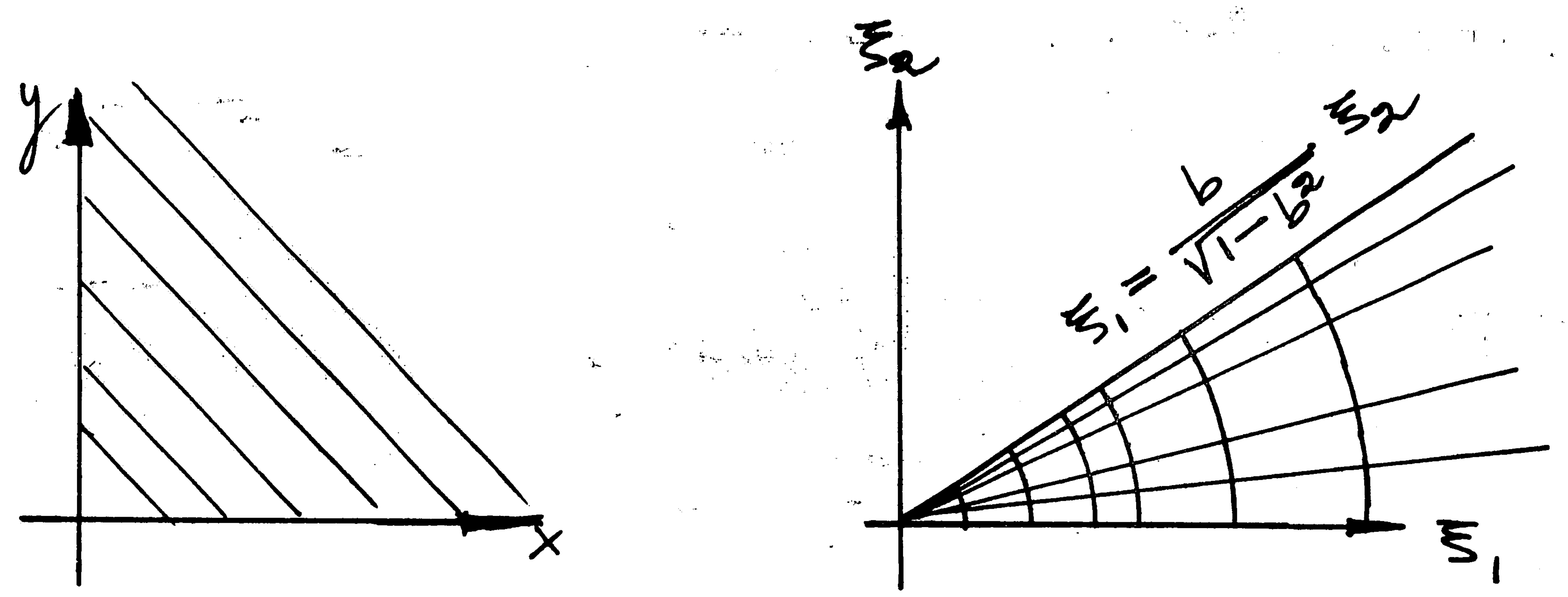


FIG. 6

COORDINATE TRANSFORMATION

AND

DOMAIN OF INTEGRATION

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## VITA

The author, Robert Gene Wagner, was born March 3, 1938 in Johnstown, Pennsylvania. He is the son of Donald L and Kathleen M. Wagner.

He graduated from Westmont-Upper Yoder High School in June, 1956 and then entered Lehigh University in September, 1956 where he had received a General Motors Scholarship.

In June, 1960 he graduated with high honors from Lehigh University, receiving the degree Bachelor of Arts.

In September, 1960 he reentered Lehigh University to study Applied Mechanics under a National Science Foundation Cooperative Fellowship, at the same time working as a quarter time graduate assistant in the Department of Mechanics.