1965

Some basic properties of isolators as circuit elements

Howard A. Seid
Lehigh University

SOME BASIC PROPERTIES OF ISOLATORS AS CIRCUIT ELEMENTS

by

Howard Alan Seid

A Thesis

Presented to the Graduate Faculty

of Lehigh University

in Candidacy for the Degree of

Master of Science

Lehigh University

1965

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

May 24, 1965
(date)

Nikolai Chedanov
Professor in charge

J. L. Karakash
Head of the Department

Acknowledgments

The author gratefully acknowledges Dr. Nikolai Eberhardt for giving the motivation for this thesis and for his tireless efforts in making it a success; Mr. Gunnar Bagge for his valuable discussions and critique.

Table of Contents

Abstract	1
Introduction	3
Section I Compensation of Reflections in an Isolator	5
Section II The Theory of the Nonreciprocal Transmission Line - Impedance Concept, Reflection Coefficient, and Voltage Standing Wave Ratio	20
Section III Matrix Representation of a Nonreciprocal Transmission Line	27
List of Figures	
Figure 1	33
Figure 2	33
Figure 3	33
Figure 4	34
Figure 5	34
Figure 6	34
Figure 7	34
Figure 8	35
Figure 9	35
Figure 10	35
Appendix A	36
Appendix B	39
References	40
Bibliography	41
Biography	42

Abstract

In many systems applications employing isolators, maximum power transfer of desired signals is required. Therefore, the elimination of reflections from the isolator becomes a primary goal. If the isolator were ideal (infinite loss in one direction), it would be possible to completely compensate for the reflections merely by a single correction from each side, if matching devices are placed before and after the isolator. However, where finite losses exist, it will generally require more than one adjustment of each tuning device.

In order to determine the maximum number of adjustments needed to obtain any given tolerance for reflections, one must make use of the fact that the reflection coefficient at a junction between an isolator and a reciprocal line is the same when measured from either side of the junction. This fact is true for any reciprocal device. In order to prove this for an isolator, a thermodynamical argument is offered.

With the aid of this theorem, sequences giving an upper bound on the reflection coefficients which remain at each junction after $\underline{n}$ adjustments of each matching device is developed. The convergence of these sequences are investigated for various values of the constants of the isolator.

An isolator is really a special case of a nonreciprocal transmission line. Therefore, it may be described by means of the theory of such a line. The impedance concept, reflection coefficient and voltage standing wave ratio are derived for this theoretical device. A major result is that the theory of the lossless and lossy reciprocal lines may be completely described as special cases of the nonreciprocal transmission line.

Finally, an alternate way of describing the nonreciprocal line in terms of matrix elements is presented. The ABCD, Z, and Y parameters are derived. The applications of these matrix representations are also described.

Introduction

Passive nonreciprocal devices are becoming very important because of their varied systems applications. For instance, where klystrons are used, an isolator prevents "pulling" which might otherwise occur due to the reflected power from other parts of the system. Also, delayed echoes which might cause "ghost" images are eliminated by proper isolation of components. Multiplexed signals are separated into proper channels with circuits incorporating isolators. Many measuring schemes include isolators in the circuits used.

In general, an isolator is a two port device which has a low loss between ports one and two and a high loss between ports two and one. One major requirement for this device is that it have low reflections from either port. When placed in a system, at least a spurious reflection of power will occur at both junctions. At first glance, it would seem possible to compensate the reflection at one junction with a matching device and then likewise compensate the reflection at the other junction and, with one adjustment of each matching device, achieve reflectionless power transmission. This is true for an ideal isolator with infinite loss in one direction. However, in general, due to the finite losses, one faces the more involved problem of matching a lossy, nonreciprocal two port from both ends simultanously. A practical matching procedure for this case is discussed in section one.

A specific nonreciprocal device, closely related to the isolator, is the nonreciprocal lossy transmission line. Although at present this is not a practical device in itself, from theoretical considerations, it is of interest because of the ease of handling of transmission line equations as compared with those concerning the theory of ferrite loaded waveguide. And, it seems to be of some merit to have a theory of the nonreciprocal lossy transmission line available. This theory will be developed in section two.

As an alternate way of describing a nonreciprocal two port, the matrix approach is presented in section three. The <u>ABCD</u>, <u>Z</u>, and <u>Y</u> parameters of pieces of nonreciprocal transmission line are derived. And, their usefulness in certain applications is discussed.

Section I

Compensation of Reflections in an Isolator

Reflections caused by the insertion of an isolator may be due to two reasons. First, the geometry of the device may be such that its cross section may differ from the cross section of the waveguide or transmission line into which it was inserted. For instance, if the isolator were placed in a coaxial line assembly, its inner and outer conductors might differ in radius or position from the inner and outer conductors of the rest of the coaxial system. In a rectangular waveguide application, the height or width of the waveguide assembly of the device might differ from that of the rest of the system. The result is that at the junction between the device and the system, a reactive storage of energy occurs and this is responsible for the reflection of power.

The second cause is due to the dielectric loading of the device. In many devices, there is a large amount of ferrite and alumina with dielectric and magnetic properties much distinct from the rest of the system into which the device is placed. This results in an abrupt transition from one medium to another with accompanying reflections.

The reflection at a junction is described by the junction reflection coefficient, Γ.

$$\Gamma \equiv \frac{V_{ref.}}{V_{inc.}}$$

where $V_{ref.}$ is the reflected voltage from the junction and $V_{inc.}$ is the incident voltage at the junction. Γ is only a function of the geometry and loading and is a fixed quantity associated with the junction between the device and system, at the given reference plane.

In the case of junctions between reciprocal lines it is known that Γ is the same when viewed from either side of the junction. In the case considered here, however, a separate proof is needed. This will be based on a thermodynamical argument derived from second law, which may be stated: "No process is possible whose sole result is the removal of heat from a reservoir at one temperature and the absorption of an equal quantity of heat by a reservoir at a higher temperature."[1]

Consider Figure 1. The figure represents a general closed thermodynamical system at temperature, T. It contains only passive elements having noise bandwidth, B. Then, only noise power is able to flow in the system. At equilibrium, the flux of power through any plane cutting the system, (e.g., A-A'), must be zero. If there were a net flux, then this would imply that some part of the system is generating more noise power than it is absorbing thereby losing thermal energy and hence temperature. Another part of the system is absorbing more power than it is generating thereby gaining thermal energy and hence temperature. If such a condition were possible, one could construct a perpetual motion machine out of the system in clear violation of the second law of thermodynamics. Hence, there must be zero flux through any plane cutting the system.

Figure 2 shows an infinitely long uniform isolator. The device is at temperature, T and has noise bandwidth, B. The device has losses L, in the forward direction, and L', in the reverse direction, in a section of length ℓ which will be under consideration. L is defined to be the ratio of the power absorbed by this section in the forward direction to the power incident at A-A'. $\underline{L}'$ is similarly defined for power flow in the reverse direction. By convention, $\underline{L} < \underline{L}' < 1$. By a theorem due to Nyquist and by the above discussion, the infinite device generates available noise power, kTB (where $\underline{k}$ is Boltzmann's constant), from right to left and vice versa. But, at any vertical plane (e.g., A-A' and B-B') the power flux is zero.

Consider just the power flow from left to right. At plane A-A' the incident power is kTB. Due to the uniformity of the device, there are no reflections. Therefore, the component of incident power which reaches plane B-B' is (1-L)kTB due to the loss, $\underline{L}$, in the forward direction. But at plane B-B', kTB power must be present. So the device must generate LkTB noise power in the region between A-A' and B-B' going from left to right. This is obviously a theorem by itself. It predicts the noise power generated by an attenuator and will be needed below. By similar considerations of power flow in the direction from right to left, one finds that the device must generate L'kTB in the region between A-A' and B-B' going from right to left.

With this knowledge, it is now possible to prove the above statement about the equality of the magnitude of the reflection coefficient at the junction between an isolator and the system.

Consider Figure 3. The device is just as in Figure 2, to the left of B-B'. At B-B', a reciprocal lossless transmission line, with characteristic impedance Z_o, is connected to the isolator. The line is terminated in Z_o, a termination with noise bandwidth, $\underline{B}$, and temperature, $\underline{T}$. At the junction, there are assumed reflection coefficients Γ_1, as seen from the device side, and Γ_2, as seen from the termination side.

As before, kTB noise power must flow from right to left and vice versa. So P_1 = kTB since it represents the power incident from the left on the part of the device under consideration. Then due to the forward loss, $P_2 = (1-L)P_1 = (1-L)kTB$. And, due to the reflection coefficient, $P_3 = |\Gamma_1|^2 P_2 = |\Gamma_1|^2 (1-L)kTB$. (When dealing with powers, one must use the square of the magnitude of Γ which was defined in terms of voltages.) $P_4 = (1-|\Gamma_1|^2)P_2 = (1-|\Gamma_1|^2)(1-L)kTB$. The power, P_5, is the component of P_3 which reaches the vertical plane A-A'. It must traverse the reverse direction of the device. So $P_5 = (1-L')P_3 = |\Gamma_1|^2 (1-L)(1-L')kTB$. The power, P_6, represents the power generated in the device traveling to the left, and, from the above discussion, has a value L'kTB. From similar considerations, there must be a power P_7 = LkTB traveling to the right. $P_8 = |\Gamma_1|^2 P_7 = |\Gamma_1|^2$ LkTB and $P_9 = (1-|\Gamma_1|^2) P_7 = (1-|\Gamma_1|^2)LkTB$. $P_{10} = (1-L') P_8 = |\Gamma_1|^2 L(1-L')kTB$. This concludes the consideration of powers from the device above.

Now, the powers arising from the termination are developed. P_{11} = kTB is the power generated by the termination. $P_{12} = |\Gamma_2|^2 P_{11} = |\Gamma_2|^2 kTB$ while $P_{13} = (1-|\Gamma_2|^2) P_{11} = (1-|\Gamma_2|^2)kTB$. Finally, $P_{14} = (1-L')P_{13} = (1-|\Gamma_2|^2)(1-L')kTB$.

At plane A-A', the net power flux is zero.

$$P_1 = P_5 + P_6 + P_{10} + P_{14}$$

$$kTB = |\Gamma_1|^2(1-L)(1-L')kTB + L'kTB + |\Gamma_1|^2 L(1-L')kTB + (1-|\Gamma_2|^2)(1-L')kTB$$

By dividing both sides by kTB and expanding, one obtains

$$1 = |\Gamma_1|^2 - L'|\Gamma_1|^2 - L|\Gamma_1|^2 + LL'|\Gamma_1|^2 + L' + L|\Gamma_1|^2 - LL'|\Gamma_1|^2 + 1$$
$$-L' - |\Gamma_2|^2 + L'|\Gamma_2|^2.$$

Or, on further simplification, the following result is obtained:

$$0 = (1-L')(|\Gamma_1|^2 - |\Gamma_2|^2)$$

But $L' < 1$. So $|\Gamma_1|^2 = |\Gamma_2|^2$ and $|\Gamma_1| = |\Gamma_2|$.

As a check on this result, one may write power balance equation at the vertical plane, C-C'.

$$P_{11} = P_4 + P_9 + P_{12}$$

$$kTB = (1-|\Gamma_1|^2)(1-L)kTB + (1-|\Gamma_1|^2)LkTB + |\Gamma_2|^2 kTB$$

On simplifying this equation, one obtains

$$0 = (|\Gamma_2|^2 - |\Gamma_1|^2). \text{ So as above, } |\Gamma_1| = |\Gamma_2|.$$

With the aid of this important theorem, it is possible to discuss the problem of matching an isolator into a lossless reciprocal transmission line or waveguide system.

In principle, if the $\underline{Z}$ matrix of the isolator is known, it is mathematically possible to determine the necessary compensating impedance to eliminate reflections and obtain maximum power transfer. Or, the method of Dechamps may be used to find experimentally the elements of the scattering matrix which also contain the information needed for compensation to eliminate reflections. But such procedures are not nearly as easily applied as the more practical approach of direct experimental matching procedures. It is the latter approach which will be considered here.

Before going into theoretical details this method shall be described briefly: The two port is inserted between a generator and a matched termination. With one of the standard techniques the two port is corrected at the input port so that no power will be reflected back to the generator. Now generator and load are interchanged and the same correction done at the other port. A new switch between generator and load however shows, that the initially established match now is destroyed. The procedure has to be repeated several times until satisfactory results are obtained.

Suppose one places a isolator in a circuit such as that shown in Figure 4. Here, the only reflections occur at the junctions between the isolator and the transmission lines. Reflection coefficients (hereafter called partial reflection coefficients), Γ_L and Γ_R, exist at the left and right junctions respectively. From the above theorem both Γ_L and Γ_R have the same values independent of whether one views the respective junctions from the device side or from the transmission line side. The device has a forward loss, L, and a reverse loss, L'.

If a component of the source power, P_s, is incident at the left junction, $|\Gamma_L|^2 P_s = P_1$ will be reflected and $(1-|\Gamma_L|^2)P_s$ is transmitted. At the right junction, $|\Gamma_R|^2(1-|\Gamma_L|^2)(1-L)P_s$ is reflected while $(1-|\Gamma_R|^2)(1-|\Gamma_L|^2)(1-L)P_s$ is absorbed by the load, Z_0. Then $(1-|\Gamma_L|^2)(|\Gamma_R|^2)(1-L)(1-L')P_s$ is the component of power which returns to the left junction at which point $(1-|\Gamma_L|^2)^2(|\Gamma_R|^2)(1-L)(1-L')P_s=P_2$ is transmitted back to the source. If L and L' are large enough so that any component of power which circulates through the device more than once may be neglected, the reflected component of power at the left junction, $|\Gamma_L|^2(1-|\Gamma_L|^2)(|\Gamma_R|^2)(1-L)(1-L')P_s$ will not be considered further.

The powers P_1 and P_2 are proportional to squares of the voltages V_1 and V_2 respectively. If the two voltage waves interfere constructively at the left junction, then the total returned power at the junction is proportional to $(V_1+V_2)^2$.*

$$(V_1+V_2)^2 = V_1^2+V_2^2+2V_1V_2 = V_1^2+V_2^2+2\sqrt{V_1^2V_2^2}$$

$|\Gamma_L|^2_{tot} \equiv \frac{\text{returned power}}{\text{incident power}}$ at the left junction. $|\Gamma_L|^2_{tot}$ as seen from the transmission line side of the left junction is $\frac{P_1 + P_2}{P_s} + 2\sqrt{\frac{P_1P_2}{P_s^2}}$.

Then since $|\Gamma_L|^2_{tot}\, P_s$ in the returned power, it is proportional to $(V_1 + V_2)^2$. Therefore,

$$|\Gamma_L|^2_{tot}\, P_s = P_1 + P_2 + 2\sqrt{P_1P_2}$$

$$= |\Gamma_L|^2 P_s + (1-|\Gamma_L|^2)^2(|\Gamma_R|^2)(1-L)(1-L')P_s$$

$$+ 2|\Gamma_L|(1-|\Gamma_L|^2)(|\Gamma_R|)\sqrt{(1-L)(1-L')}\, P_s$$

and

$$|\Gamma_L|^2_{tot} = \left[\,|\Gamma_L| + (1-|\Gamma_L|^2)(|\Gamma_R|)\sqrt{(1-L)(1-L')}\,\right]^2$$

or

$$|\Gamma_L|_{tot} = |\Gamma_L| + (1-|\Gamma_L|^2)(|\Gamma_R|)\sqrt{(1-L)(1-L')} \qquad (1)$$

Thus $|\Gamma_L|_{tot}$ represents the upper bound of the possible magnitudes of reflection coefficients at the left junction. Similarly, if the generator and load in Figure 4 were switched around, one can define $|\Gamma_R|_{tot}$ for the right junction as

---

*In this analysis, the interaction of voltage waves in the device due to the multiple reflections has been ignored. This is justified if the total losses in the device are large. However, if the Q of the device attains large values, this analysis no longer applies because of the resonance behavior of the cavity.

$$|\Gamma_R|_{tot} = |\Gamma_R| + (1-|\Gamma_R|^2)(|\Gamma_L|)\sqrt{(1-L)(1-L')} \qquad (2)$$

Therefore $|\Gamma_R|_{tot}$ is an upper bound for magnitudes of the reflection coefficients at the right junction.

It is true that in an actual situation, the values of the total reflection coefficients may be less than those given by equations (1) and (2). But unless one has a device which approaches the ideal ($L'=1$), the total reflection coefficients at both the left and right junctions can be greater than the partial reflection coefficients at those respective junctions.

If matching devices are placed before and after the nonreciprocal device, the resulting scheme will be as in Figure 5. Initially assume that matching devices are adjusted so that no compensation, for instance shunt susceptance, has been introduced. Then under the assumptions used to derive equations (1) and (2), the two partial reflection coefficients, Γ_L and Γ_R, and the losses, $\underline{L}$ and $\underline{L'}$, lead to a $|\Gamma_L|_{tot}$ which is greater or equal to the partial reflection coefficient at the left junction (which may be immediately seen from equation (1)). If the left matching device now is used for tuning, it will introduce a reflection coefficient, Γ_{L1} which is equal in magnitude to but 180° out of phase with Γ_{Ltot}. Thus the total reflection coefficient will be completely compensated, though not so for the partial reflection coefficient, Γ_L. In the worst case, there now remains a new partial reflection coefficient, of magnitude $|\Gamma_L|_{tot}-|\Gamma_L|$, at the left junction.

If the source and load of Figure 5 are now reversed so that power is incident on the right junction, a similar effect of overcompensating the partial coefficient at the right junction occurs due to the $|\Gamma_R|_{tot}$

found from equation (2) with $|\Gamma_L|$ replaced by $|\Gamma_{L1}|$. When compensation has been added by the right matching device (in a similar manner as was discussed with the left matching device) a resultant partial reflection coefficient of magnitude $|\Gamma_R|_{tot} - |\Gamma_R|$ (where $|\Gamma_{R1}|$ is the magnitude of the compensating reflection coefficient) now exists at the right junction.

This new reflection coefficient at the right junction disrupts the equilibrium at the left junction so that there are again reflections. These reflections necessitate the readjusting of the left matching device with the resultant need to retune the right matching device. Due to the fact that the total losses, L and L', are not sufficient to completely isolate the reflections at one junction from those at the other junction, there may be a need for many adjustments of each matching device before reflections are sufficiently compensated.

It is therefore valuable to establish an upper bound which will allow one to determine the maximum number of adjustments needed to satisfy the tolerance for reflections in a given situation.

It is convenient to define three quantities, $|\Gamma_{n-1}|_{tot}$, $|\Gamma_n|$, and $|\Gamma_n|_{max}$ for both the left and right junctions as follows:

$|\Gamma_{Ln-1}|_{tot} \equiv$ the total magnitude of the reflection coefficient from the left side after n-1 adjustments of both the left and right matching devices.

$|\Gamma_{Rn-1}|_{tot} \equiv$ the total magnitude of the reflection coefficient from the right side after n-1 adjustments of the right and n adjustments of the left matching devices.

$|\Gamma_{Ln}| \equiv$ the magnitude of the partial reflection coefficient at the left junction after n adjustments of the left and n-1 adjustments of the right matching devices.

$|\Gamma_{Rn}| \equiv$ the magnitude of the partial reflection coefficient at the right junction after n adjustments of both the left and right matching devices.

The definition of the third quantity will depend upon the consideration of the phase of the voltage wave reflected at the incident junction of the device with respect to the voltage wave at that junction due to reflections from the other junction. It should be obvious that when these voltage waves are in phase, the total reflection coefficient at the incident junction would be at least as large or larger than the total reflection coefficient under a more general phase relation. This condition is, in effect, a worst case approximation to the actual situation occurring in the device. After compensation, the partial reflection coefficient at the junction will also be a maximum under the condition that the waves are in phase. Then it becomes an obvious extention to define $|\Gamma_{Ln}|_{max} \equiv$ the magnitude of the partial reflection coefficient at the left junction after n adjustments of the left and n-1 adjustments of the right matching devices under this worst case condition.

$|\Gamma_{Rn}|_{max} \equiv$ the magnitude of the partial reflection coefficient at the right junction after n adjustments of both left and right matching devices under the worst case condition.

In general, then, $|\Gamma_{Ln}| \leq |\Gamma_{Ln}|_{max}$ and $|\Gamma_{Rn}| \leq |\Gamma_{Rn}|_{max}$. If the sequences, $\{|\Gamma_{Ln}|_{max}\}$ and $\{|\Gamma_{Rn}|_{max}\}$, converge to zero, then the sequences, $\{|\Gamma_{Ln}|\}$ and $\{|\Gamma_{Rn}|\}$ converge to zero at least as fast. This statement may be proved very simply:

Since by hypothesis, $\{|\Gamma_n|_{max}\}$ converges to zero, for any $\epsilon > 0$, one can find an $N(\epsilon)$ such that $||\Gamma_{Ln}|_{max} - 0| < \epsilon$ whenever $n > N(\epsilon)$. This is the definition of the convergence of a sequence. Therefore $|\Gamma_{Ln}|_{max} < \epsilon$ for $n > N(\epsilon)$. But $|\Gamma_{Ln}| \leq |\Gamma_{Ln}|_{max}$. So it follows immediately that $|\Gamma_{Ln}| < \epsilon$ for $n > N(\epsilon)$. And, $\{|\Gamma_{Ln}|\}$ converges to zero at least as fast as $\{|\Gamma_L|_{max}\}$. The proof for the convergence of $\{|\Gamma_{Rn}|\}$ is exactly similar. Q.E.D.

As a result of this proof, the sequences $\{|\Gamma_{Ln}|_{max}\}$ and $\{|\Gamma_{Rn}|_{max}\}$ can be used as upper bounds for sequences $\{|\Gamma_{Ln}|\}$ and $\{|\Gamma_{Rn}|\}$ respectively. The upper bound sequences will be developed under the following assumption: Any component of power that passes back and forth through the isolator more than once will be considered to be attenuated so much by the combined forward and reverse losses of the devices that it will be negligibly small when compared with any component of power which had made only one circuit in the device.

With no susceptance introduced by either matching devices, under the above assumption, the magnitude of the total reflection coefficient, $|\Gamma_{LO}|_{tot}$, is found from Figure 5 and equation (1) to be

$$|\Gamma_{LO}|_{tot} = |\Gamma_L| + (1-|\Gamma_L|^2)(|\Gamma_R|)\sqrt{(1-L)(1-L')}$$

since the worst case is being considered. If the left matching device is adjusted so as to be equal in magnitude to but 180° out of phase with $\Gamma_{LO\ tot}$, then complete compensation has been accomplished from the left side. However, due to the overcompensation of the partial reflection coefficient, Γ_L, there exists, in magnitude, a new partial reflection coefficient, $|\Gamma_{L1}|_{max} = |\Gamma_{L0}|_{tot} - |\Gamma_L|$.

$$|\Gamma_{L1}|_{max} = (1-|\Gamma_L|^2)(|\Gamma_R|)\sqrt{(1-L)(1-L')}$$

If the source and load in Figure 5 are reversed, then $|\Gamma_{R0}|_{tot}$ may be found from equation (2) with $|\Gamma_L|$ replaced by $|\Gamma_{L1}|_{max}$. So,

$$|\Gamma_{R0}|_{tot} = |\Gamma_R| + (1-|\Gamma_R|^2)(|\Gamma_{L1}|_{max})\sqrt{(1-L)(1-L')}$$

If the right matching device compensates completely for $|\Gamma_{R0}|_{tot}$ in a similar manner as was discussed when $|\Gamma_{L0}|_{tot}$ was considered, then the right matching device has overcompensated for the partial reflection coefficient at the right junction. If $|\Gamma_{R0}|_{tot}$ is the magnitude of the compensating reflection coefficient, then the new partial reflection coefficient at the right junction is equal in magnitude to $|\Gamma_{R0}|_{tot} - |\Gamma_R| = |\Gamma_{R1}|_{max}$.

$$|\Gamma_{R1}|_{max} = (1-|\Gamma_R|^2)(|\Gamma_{L1}|_{max})\sqrt{(1-L)(1-L')}$$

But by adjusting the right matching device, the change in the partial reflection coefficient from Γ_R to $\Gamma_{R1\ max}$ causes the initial equilibrium at the left junction to be disrupted, thereby introducing reflections at that junction. If the source and load are returned to the original position, a $\Gamma_{L1\ tot}$ exists at the left junction. Its magnitude may be found from equation (1) with $|\Gamma_L|$ replaced by $|\Gamma_{L1}|_{max}$ and $|\Gamma_R|$ replaced by $|\Gamma_{R1}|_{max}$.

$$|\Gamma_{L1}|_{tot} = |\Gamma_{L1}|_{max} + (1-|\Gamma_{L1}|^2_{max})(|\Gamma_{R1}|_{max})\sqrt{(1-L)(1-L')}$$

Since this quantity is a direct consequence of initial overcompensation from the left matching device, if the initial compensation were inductive, for instance, one would have to add capacitance to obtain zero total reflection. If the initial compensation were capacitive, then inductance would be needed. In either of the two possible cases, the magnitude of the compensation required must be equal to $|\Gamma_{L1}|_{tot}$. When this adjustment is made, the magnitude of the partial reflection coefficient remaining at the junction is $|\Gamma_{L2}|_{max}$ where

$$|\Gamma_{L2}|_{max} = |\Gamma_L| - (|\Gamma_{L0}|_{tot} - |\Gamma_{L1}|_{tot})$$

$$= |\Gamma_{L1}|_{tot} - |\Gamma_{L1}|_{max} .$$

So

$$|\Gamma_{L2}|_{max} = (1-|\Gamma_{L1}|^2_{max})\,(|\Gamma_{R1}|_{max})\,\sqrt{(1-L)(1-L')}$$

If the source and load are interchanged, there is a reflection at the right junction for reasons similar to those stated directly above in connection with the left junction. The magnitude of the total reflection coefficient, $|\Gamma_{R1}|_{tot}$, may be found from equation (2) with $|\Gamma_R|$ replaced by $|\Gamma_{R1}|_{max}$ and $|\Gamma_L|$ replaced by $|\Gamma_{L2}|_{max}$. Therefore

$$|\Gamma_{R1}|_{tot} = |\Gamma_{R1}|_{max} + (1-|\Gamma_{R1}|^2_{max})(|\Gamma_{L2}|_{max})\sqrt{(1-L)(1-L')}$$

This total reflection coefficient is due to the initial overcompensation by the right matching device. Therefore the comments stated before the derivation of $|\Gamma_{L2}|_{max}$ apply to this situation. Compensation of magnitude $|\Gamma_{R1}|_{tot}$ must be added by the right matching device. After adjustment, there is a new partial reflection coefficient, $\Gamma_{R2\ max}$, where

$$|\Gamma_{R2}|_{max} = |\Gamma_{R1}|_{tot} - |\Gamma_{R1}|_{max}$$

$$|\Gamma_{R2}|_{max} = (1-|\Gamma_{R1}|^2_{max})(|\Gamma_{L2}|_{max})\,\sqrt{(1-L)(1-L')}$$

However, at both the left and right junctions, the matching devices have now undercompensated for the original partial reflection coefficients, Γ_L and Γ_R. Due to the finite losses (L and L'), after the next adjustments, there will be overcompensation at each junction. After every odd numbered adjustment of either matching device, overcompensation of the original partial reflection coefficient at the junction occurs. And after every even numbered adjustment, undercompensation occurs. As a result, after n adjustments of the left and n-1 adjustments of the right matching devices, the partial reflection coefficient which remains at the left junction is

$$|\Gamma_{Ln}|_{max} = (1-|\Gamma_{Ln-1}|^2_{max})(|\Gamma_{Rn-1}|_{max})\sqrt{(1-L)(1-L')}$$

$n = 1, 2, 3, 4, \ldots$

After n adjustments of both right and left matching devices, the partial reflection coefficient at the right junction is

$$|\Gamma_{Rn}|_{max} = (1-|\Gamma_{Rn-1}|^2_{max})(|\Gamma_{Ln}|_{max})\sqrt{(1-L)(1-L')} \qquad (4)$$

$n = 1, 2, 3, 4, \ldots$

Equations (3) and (4) are subject to the condition that $|\Gamma_{LO}|_{max} = |\Gamma_L|$ and $|\Gamma_{RO}|_{max} = |\Gamma_R|$.

The convergence of the sequences $\{|\Gamma_{Ln}|_{max}\}$ and $\{|\Gamma_{Rn}|_{max}\}$ will be demonstrated in the Appendix A for values of $|\Gamma_L|$, $|\Gamma_R|$, L, and L' which obey the assumption used in deriving the sequences. The limits of these sequences (having n^{th} terms as in equations (3) and (4) respectively) are zero when the limits exist. Thus for and actual device, the sequences $\{|\Gamma_{Ln}|\}$ and $\{|\Gamma_{Rn}|\}$ converge at least as fast to the same limits as $\{|\Gamma_{Ln}|_{max}\}$ and $\{|\Gamma_{Rn}|_{max}\}$, namely to $|\Gamma_{R\infty}| = |\Gamma_{L\infty}| = 0$.

These limits are obviously the conditions for maximum power transfer.

In many applications, it is found that only one or two adjustments of each matching device are needed to obtain the almost total elimination of reflections. This result follows directly from the extremely fast convergence of $\{|\Gamma_{Ln}|_{max}\}$ and $\{|\Gamma_{Rn}|_{max}\}$ inside their regions of convergence. For example a typical isolator might have the following data associated with it:

forward loss: 0.5 db
reverse loss: .15db

$$|\Gamma_L| = 0.2$$

and $$|\Gamma_R| = 0.4$$

Then, the resulting terms of the sequences $\{|\Gamma_{Ln}|_{max}\}$ and $\{|\Gamma_{Rn}|_{max}\}$ are:

$$|\Gamma_{L1}|_{max} = 0.06458,\ |\Gamma_{L2}|_{max} = 0.00147,\ |\Gamma_{L3}|_{max} = 0.00004 \text{ and}$$
$$|\Gamma_{L4}|_{max},\ \ldots,\ |\Gamma_{Ln}|_{max},\ \ldots < 0^+$$

$$|\Gamma_{R1}|_{max} = 0.00877,\ |\Gamma_{R2}|_{max} = 0.00025,\ |\Gamma_{R3}|_{max} = 0.00001 \text{ and}$$
$$|\Gamma_{R4}|_{max},\ \ldots,\ |\Gamma_{Rn}|_{max},\ \ldots < 0^+$$

where the notation 0^+ means 0.00000^+.

Therefore, after four adjustments each of the left and right matching devices, the trend of convergence of the two sequences is established and the sequences are converging to zero.

Section II

The Theory of the Nonreciprocal Transmission Line - Impedance Concept, Reflection Coefficient, and Voltage Standing Wave Ratio

An isolator is a special case of the generalized nonreciprocal transmission line which is shown in Figure 6. The line is of length ℓ with load at $x = 0$ and generator at $x = -\ell$. Unlike a reciprocal line, this line may not be characterized by one propagation constant, γ, and one characteristic impedance, Z_0. Due to the difference in propagation properties and loss characteristics for x increasing and x decreasing, in the nonreciprocal line, two propagation constants and two characteristic impedances are needed. The propagation constant for x increasing is $\gamma_1 = \alpha_1 + j\beta_1$ where α_1 is the attenuation constant measured in nepers/meter and β_1 is the phase constant measured in radians/meter. The characteristic impedance is Z_{01} in this direction. For x decreasing, the propagation constant is $\gamma_2 = \alpha_2 + j\beta_2$ and the characteristic impedance is Z_{02}.

Because of the wave properties of voltage in a transmission line, the voltage in the forward direction (x increasing) is $V_f = V_1 e^{-\gamma_1 x} e^{j\omega t}$ where V_1 is a complex amplitude. And similarly, the voltage wave in the reverse direction is $V_r = V_2 e^{\gamma_2 x} e^{j\omega t}$ where V_2 is a complex amplitude. The total voltage is

$$V(x,t) = \left(V_1 e^{-\gamma_1 x} + V_2 e^{\gamma_2 x}\right) e^{j\omega t} \tag{5}$$

By definition, $\frac{V_f}{I_f} \equiv Z_{01}$ and $\frac{V_r}{I_r} \equiv -Z_{02}$. The reason for the negative sign in the reverse voltage to current ratio is due to the assumed forward direction of current (from left to right in Figure 6). The total current is $I = I_f + I_r$.

$$I(x,t) = \left(\frac{V_1}{Z_{01}} e^{-\gamma_1 x} - \frac{V_2}{Z_{02}} e^{\gamma_2 x}\right) e^{j\omega t} \qquad (6)$$

Due to the fact that only ratios of voltages and currents will be considered, there is no need to carry the $e^{j\omega t}$ terms in equations (5) and (6). Therefore only $V(x)$ and $I(x)$ will be used where $V(x) = V(x,t)$ and $I(x) = I(x,t)$ when the $e^{j\omega t}$ term is dropped.

The impedance is defined by

$$Z(x) \equiv \frac{V(x)}{I(x)} = \frac{V_1 e^{-\gamma_1 x} + V_2 e^{\gamma_2 x}}{\frac{V_1}{Z_{01}} e^{-\gamma_1 x} - \frac{V_2}{Z_{02}} e^{\gamma_2 x}}$$

from equations (5) and (6).

$$Z(x) = \frac{1 + \frac{V_2}{V_1} e^{(\gamma_1+\gamma_2)x}}{\frac{1}{Z_{01}} - \frac{V_2}{V_1} \frac{\exp(\gamma_1+\gamma_2)x}{Z_{02}}} \qquad (7)$$

It is convenient to define the load reflection coefficient, Γ_L.

$$\Gamma_L \equiv \frac{V_2}{V_1} e^{(\gamma_1+\gamma_2)x} \Big|_{x=0} = \frac{V_2}{V_1}$$

Then, by substituting into equation (7)

$$Z(x) = \frac{1 + \Gamma_L e^{(\gamma_1+\gamma_2)x}}{\frac{1}{Z_{01}} - \frac{\Gamma_L}{Z_{02}} e^{(\gamma_1+\gamma_2)x}} \qquad (8)$$

At $x = 0$, $Z(0) = Z_L$

$$Z_L = \frac{1 + \Gamma_L}{\frac{1}{Z_{01}} - \frac{\Gamma_L}{Z_{02}}}$$

Solve for Γ_L.

$$\Gamma_L = \frac{Z_{02}\,(Z_L - Z_{01})}{Z_{01}\,(Z_L + Z_{02})} \qquad (9)$$

Substituting equation (9) into (8), one obtains

$$Z(x) = \frac{Z_{01}(Z_L+Z_{02}) + Z_{02}(Z_L-Z_{01})\; e^{(\gamma_1+\gamma_2)x}}{(Z_L+Z_{02}) + (Z_{01}-Z_L)\; e^{(\gamma_1+\gamma_2)x}}$$

Now, for notational purposes, $\gamma' \equiv \frac{1}{2}(\gamma_1+\gamma_2)$

$$= \frac{1}{2}(\alpha_1+\alpha_2) + \frac{1}{2}\,j(\beta_1+\beta_2) = \alpha' + j\beta'$$

So

$$Z(x) = \frac{Z_{01}(Z_L+Z_{02}) + Z_{02}(Z_L-Z_{01})\; e^{2\gamma' x}}{(Z_L+Z_{02}) + (Z_{01}-Z_L)\; e^{2\gamma' x}} \qquad (10)$$

The quantity, λ_m is now defined such that $\lambda_m \equiv \frac{2\pi}{\beta'}$.

Then if α' were zero, $Z(x) = Z(x \pm \frac{1}{2}\lambda_m)$. This implies that $Z(x)$ would be a periodic function of x with period $\frac{1}{2}\lambda_m$ if it were not for losses. With α' not zero, $Z(x)$ is an aperiodic function. However, relative maximums or relative minimums of the impedance, $Z(x)$, are separated by $\frac{1}{2}\lambda_m$.

Equation (10) is for the impedance at a point, x, only if the generator, nonreciprocal line, and load are as shown in Figure 6. If the nonreciprocal line had been reversed in that figure, the power, flowing in the direction of increasing x, would "see" a transmission line having constants Z_{02}, γ_2. The power reflected by the load would "see" a transmission line with constants Z_{01}, γ_1. Therefore, in all of the formulas developed so far, the roles of Z_{01} and Z_{02} must be interchanged. The same applies to the propagation constants γ_1 and γ_2.

Then, the load reflection coefficient for the reversed line (denoted by Γ'_L) is

$$\Gamma'_L = \frac{Z_{01}(Z_L - Z_{02})}{Z_{02}(Z_L + Z_{01})} \,. \tag{9'}$$

The impedance function (denoted by $Z'(x)$) is

$$Z'(x) = \frac{Z_{02}(Z_L+Z_{01}) + Z_{01}(Z_L-Z_{02})\, e^{2\gamma' x}}{(Z_L+Z_{01}) + (Z_{02}-Z_L)\, e^{2\gamma' x}} \tag{10'}$$

One may wish to work with admittances rather than impedances. Therefore, it is convenient to define $Y_{01} \equiv 1/Z_{01}$, $Y_{02} \equiv 1/Z_{02}$, and $Y_L \equiv 1/Z_L$. Then for the line pictured in Figure 6, equation (9) expressed in terms of admittances is

$$\Gamma_L = \frac{Y_{01} - Y_L}{Y_{02} + Y_L} \,. \tag{11}$$

Substituting equation (11) into (8), one obtains $Y(x) \equiv 1/Z(x)$.

$$Y(x) = \frac{Y_{01}(Y_L+Y_{02}) + Y_{02}(Y_L-Y_{01})\, e^{2\gamma' x}}{(Y_L+Y_{02}) + (Y_{01}-Y_L)\, e^{2\gamma' x}} \tag{12}$$

Expressions may be written for Γ'_L and $Y'(x)$ for the reversed line merely by interchanging Y_{01} and Y_{02} in equations (11) and (12).

A very useful tool in plotting impedances or admittances in reciprocal lossy and lossless transmission lines is the Smith Chart. This graphical device was derived to condense the infinite impedance or admittance plane into the plane of the reflection coefficient which consists of points in a unit circle.[2,3] However, the transformation used to accomplish this conformal mapping is not the correct one to use for the case of a general nonreciprocal line. This fact can be immediately seen by noticing that the expression of the load reflection coefficient for the reciprocal line differs materially from that for the nonreciprocal line. Therefore, the Smith Chart is not applicable for the nonreciprocal line.

A second important concept is the reflection coefficient as a function of χ. The reflection coefficient measures the ratio of the reverse traveling voltage wave to the forward traveling voltage wave at any point, χ, on the line. Hence the reflection coefficient

$$\Gamma(x) \equiv \frac{V_r}{V_f} = \frac{V_2\ e^{\gamma_2 x}}{V_1\ e^{-\gamma_1 x}} = \Gamma_L\ e^{2\gamma' x} \qquad (13)$$

This derivation is for the line as shown in Figure 6. For the reversed line, the form of the reflection coefficient remains unchanged but, Γ_L must be replaced by Γ_L'.

By comparing equations (8) and (13), it becomes clear that $\Gamma(x)$ determines the impedance function. As with $Z(x)$, if α' were zero, $\Gamma(x)$ would be periodic with period $\frac{1}{2}\lambda_m$. But with α' not zero, $\Gamma(x)$ is aperiodic although its relative maximums or relative minimums occur at points which are separated by $\frac{1}{2}\lambda_m$. As one moves from the load in the direction of decreasing χ, each successive maximum is decreased by a factor of $e^{-\alpha'\lambda_m}$ and the magnitude of each successive minimum is similarly attenuated.

In a reciprocal lossless line, the voltage standing wave ratio (VSWR) is defined as,

$$\text{VSWR} \equiv \frac{\text{maximum voltage}}{\text{minimum voltage}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$

and is a constant function of χ along the line. But with any lossy line no such constant behavior is possible. So with lossy reciprocal or nonreciprocal lines, the VSWR is a function of χ and may be defined as follows for the nonreciprocal case:

$$\mathrm{VSWR}(x) = \frac{1 + |\Gamma(x)|}{1 - |\Gamma(x)|}$$

$$= \frac{1 + |\Gamma_L\, e^{2\gamma' x}|}{1 - |\Gamma_L\, e^{2\gamma' x}|} \quad \text{from equation (13)}$$

$$= \frac{1 + |\Gamma_L|\,|e^{2\alpha' + j2\beta'}|}{1 - |\Gamma_L|\,|e^{2\alpha' + j2\beta'}|}$$

$$\mathrm{VSWR}(x) = \frac{1 + |\Gamma_L|\, e^{2\alpha' x}}{1 - |\Gamma_L|\, e^{2\alpha' x}} \tag{14}$$

since $|e^{j2\beta' x}| = |\cos(2\beta' x) + j\sin(2\beta' x)| = 1$ and $e^{2\alpha' x} > 0$. As with the impedance function and the reflection coefficient, equation (14) is derived for the line as shown in Figure 6. For the reversed line, $\mathrm{VSWR}'(x)$ may be found from (14) with $|\Gamma_L|$ replaced by $|\Gamma_L'|$.

From its definition, it is clear that $0 \le |\Gamma_L| \le 1$. Therefore $1 \le \mathrm{VSWR}(x) < \infty$. If there is no reflected voltage at the load in Figure 6 ($\Rightarrow Z_L = Z_{01}$), $|\Gamma_L| = 0$ and the $\mathrm{VSWR}(x)$ is a constant function of χ of value 1. At the other extreme, if the load is a short circuit, $|\Gamma_L| = 1$ and at $x = 0$, $\mathrm{VSWR}(0) \to \infty$. (In the case of a lossy reciprocal line, $|\Gamma_L| = 1$ if the load is open circuited or short circuited. But in a nonreciprocal line, $|\Gamma_L| = \frac{Z_{02}}{Z_{01}}$ for an open circuited load.) However due to the loss factor, $e^{2\alpha' x}$, multplying the $|\Gamma_L|$ term in both numerator and denominator of (14), the VSWR(x) for $-\ell \leq x < 0$ is finite. At $x = -\ell$, the VSWR($-\ell$) $= \dfrac{1 + e^{-2\alpha'\ell}}{1 - e^{-2\alpha'\ell}}$ if $|\Gamma_L| = 1$. As $-\ell \to -\infty$, the VSWR($-\ell$) $\to 1$ implying perfect match at the generator ($x = -\ell$) even though the load is a short circuit.

It may be shown that if the VSWR($-\ell$) is known and if the phase of $\Gamma(-\ell)$ and the propagation constants of the nonreciprocal line are known, the impedance at any point on the line is completely determined. (The VSWR($-\ell$) and the phase of $\Gamma(-\ell)$ may be read from slotted line measurements while the propagation constants are determined from the dielectric and permeable properties of the medium.)

It should be noted that the reciprocal lossy and lossless lines appear now as special cases of the nonreciprocal line theory. If $\gamma_1 = \gamma_2$ and $Z_{01} = Z_{02}$, then the above equations reduce to those for the reciprocal lossy line. (The pairs of equations for impedance, admittance, reflection coefficient, and VSWR for the nonreciprocal line reduce to single equations for these quantities when $\gamma_1 = \gamma_2$ and $Z_{01} = Z_{02}$.) And further, if $\gamma_1 = \gamma_2 = j\beta$ and $Z_{01} = Z_{02}$ = a real number, then the theory reduces to that of the lossless reciprocal line. Just as lossy and lossless reciprocal line theory applies to waveguides, the nonreciprocal line theory also applies under the conditions of the proper definition of impedance. A few of these definitions for the reciprocal case with TE_{10} propagation are discussed by Montgomery[(4)] and they are easily extended to the non-reciprocal waveguide if the magnetic and dielectric properties of the medium are considered to be different in the different directions.

Section III

Matrix Representations of a Nonreciprocal Transmission Line

Instead of the theory introduced in Section II, one may wish to describe the nonreciprocal line in terms of the measurable quantities of the input voltage and current and load voltage and current.

Consider Figure 7 in which the convention for voltage and current used throughout this section is explained. The input current and voltage are i_1 and E_1 respectively while at the output, i_2 and E_2 are the current and voltage respectively. The matrix elements derived in this section are for the line as pictured in the figure. If the line is reversed, the roles of Z_{01} and Z_{02} are interchanged. The same is true for the propagation constants γ_1 and γ_2.

ABCD Parameters

It is of interest to have a direct relationship between the input quantities and the output quantities. This relationship is accomplished by the ABCD matrix as follows:

$$\begin{bmatrix} E_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_2 \\ i_2 \end{bmatrix} \tag{15}$$

The elements of the ABCD matrix are known as ABCD parameters. From equations (5) and (6) with x = 0 and dropping $e^{j\omega t}$,

$$V(0) = V_1 + V_2 = E_2$$

$$I(0) = \frac{V_1}{Z_{01}} - \frac{V_2}{Z_{02}} = i_2 \tag{16}$$

Solving the simultaneous equation (16) for V_1 and V_2 in terms of E_2 and i_2, one obtains

$$V_1 = \frac{Z_{01}E_2 + Z_{01}Z_{02}i_2}{Z_{01} + Z_{02}}\,; \quad V_2 = \frac{Z_{02}E_2 - Z_{01}Z_{02}i_2}{Z_{01} + Z_{02}}$$

Therefore on substituting for V_1 and V_2 in equations (5) and (6):

$$V(x) = \frac{Z_{01}E_2 + Z_{01}Z_{02}i_2}{Z_{01} + Z_{02}} e^{-\gamma_1 x} + \frac{Z_{02}E_2 - Z_{01}Z_{02}i_2}{Z_{01} + Z_{02}} e^{\gamma_2 x}$$

$$I(x) = \frac{E_2 + Z_{02}i_2}{Z_{01} + Z_{02}} e^{-\gamma_1 x} + \frac{Z_{01}i_2 - E_2}{Z_{01} + Z_{02}} e^{\gamma_2 x} \tag{17}$$

At $x = -\ell$, $V(-\ell) = E_1$ and $I(-\ell) = i_1$. Therefore at $x = -\ell$ (with terms on the right of (16) rearranged) one has:

$$E_1 = \frac{Z_{01}\, e^{\gamma_1 \ell} + Z_{02}\, e^{-\gamma_2 \ell}}{Z_{01} + Z_{02}} E_2 + \frac{Z_{01}Z_{02}(e^{\gamma_1 \ell} - e^{-\gamma_2 \ell})}{Z_{01} + Z_{02}} i_2$$

$$i_1 = \frac{e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}}{Z_{01} + Z_{02}} E_2 + \frac{Z_{02}\, e^{\gamma_1 \ell} + Z_{01}\, e^{-\gamma_2 \ell}}{Z_{01} + Z_{02}} i_2 \tag{18}$$

Comparing equations (18) with the matrix equation (15), one finds the ABCD parameters to be:

$$A = \frac{Z_{01}\, e^{\gamma_1 \ell} + Z_{02}\, e^{-\gamma_2 \ell}}{Z_{01} + Z_{02}}$$

$$B = \frac{Z_{01}Z_{02}(e^{\gamma_1 \ell} - e^{-\gamma_2 \ell})}{Z_{01} + Z_{02}}$$

$$C = \frac{e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}}{Z_{01} + Z_{02}} \tag{19}$$

$$D = \frac{Z_{02}\, e^{\gamma_1 \ell} + Z_{01}\, e^{-\gamma_2 \ell}}{Z_{01} + Z_{02}}$$

For reciprocal lines or nonreciprocal passive devices which may be characterized by one propagation constant, γ or one transfer constant,

θ, the relationship, AD - BC = 1, holds.[5] However, for a nonreciprocal line (or more generally, any device with two propagation constants, γ_1, γ_2 or two transfer constants, θ_1, θ_2,)

$$AD - BC = e^{(\gamma_1 - \gamma_2)\ell} \tag{20}$$

From equation (20), it is seen that if any three of the ABCD parameters are known, the fourth is determined.

The usefulness of the ABCD parameters is mainly found when one wishes to cascade two or more lines. Then, the input current and voltage are related to the load current and voltage by an ABCD matrix formed by the product of the ABCD matrices of each of the cascaded lines, line one with parameters, $A_1B_1C_1D_1$, and line two with parameters, $A_2B_2C_2D_2$. Then by consideration of Figure 8, one sees that the load voltage and current of the first line are the input voltage and current of the second line. So, if E_1 and i_1 are inputs to the first line, and E_2 and i_2 are the outputs of the first line, then with E_L and i_L the output voltage and current of the second line, the following equations hold:

$$\begin{bmatrix} E_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} E_2 \\ i_2 \end{bmatrix} ; \quad \begin{bmatrix} E_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} E_L \\ i_L \end{bmatrix}$$

or

$$\begin{bmatrix} E_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} E_L \\ i_L \end{bmatrix} . \tag{21}$$

Z Parameters

Very often, one may wish to relate the voltages, E_1, E_2, to the currents, i_1, i_2, in the line (where all quantities are defined as in Figure 7). These quantities are related by the Z parameter matrix as follows:

$$\begin{bmatrix} E_1 \\ -E_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{22}$$

The $-E_2$ element arises from consideration of Kirchhoff's laws. The form of the equation is discussed by Karakash.(6)

The ABCD matrix elements may be used to derive the Z parameters. If $i_2 = 0$, $\frac{E_1}{E_2} = -\frac{Z_{11}}{Z_{21}} = A$. With $E_2 = 0$, $\frac{i_1}{i_2} = -\frac{Z_{22}}{Z_{21}} = D$. If $i_2 = 0$, $\frac{i_1}{E_2} = -\frac{1}{Z_{21}} = C$. And if $E_2 = 0$, $E_1 = Z_{11}\left(\frac{-Z_{22}}{Z_{21}}\right)i_2 + Z_{12}i_2$ (since $\frac{i_1}{i_2} = \frac{-Z_{22}}{Z_{21}}$ when $E_2 = 0$).

$\frac{E_1}{i_1} = \frac{Z_{12}Z_{21} - Z_{11}Z_{22}}{Z_{21}} = B$. By systematically substituting from the known ABCD parameters, one finds:

$$\begin{aligned}
Z_{11} &= \frac{Z_{01}e^{\gamma_1 \ell} + Z_{02}\, e^{-\gamma_2 \ell}}{e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}} \\
Z_{12} &= -\frac{(Z_{01}+Z_{02})\; e^{(\gamma_1-\gamma_2)\ell}}{e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}} \\
Z_{21} &= \frac{-(Z_{01}+Z_{02})}{e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}} \\
Z_{22} &= \frac{Z_{02}\, e^{\gamma_1 \ell} + Z_{01}\, e^{-\gamma_2 \ell}}{e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}}
\end{aligned} \tag{23}$$

The main use of the Z parameter matrix is when two lines are connected, as shown in Figure 9, in series. The input voltages across each line add while the input current is the same for both lines. At the load end, the output voltages across each line add, while the output current is the same for both lines. Therefore, the total Z matrix for the series co ection of the two lines is the sum of the Z matrices for the two lines.[7]

Y Parameters

One may wish to express the input and output currents of Figure 7 as a function of the input and output voltages. Such a representation may be accomplished by means of the Y parameter matrix.

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} E_1 \\ -E_2 \end{bmatrix} \tag{24}$$

As might be expected, the Y matrix is the inverse of the matrix of Z parameters given by equation (23). However, the Y parameters may also be derived directly from the ABCD matrix elements (which will be left to the Appendix B).

By using the standard method of inverting the Z matrix,[8] one obtains:

$$\begin{aligned}
Y_{11} &= \frac{Z_{02}\, e^{\gamma_1 \ell} + Z_{01}\, e^{-\gamma_2 \ell}}{Z_{01} Z_{02}\left(e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}\right)} \\
Y_{12} &= \frac{(Z_{01} + Z_{02})\, e^{(\gamma_1 - \gamma_2)\ell}}{Z_{01} Z_{02}\left(e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}\right)} \\
Y_{21} &= \frac{(Z_{01} + Z_{02})}{Z_{01} Z_{02}\left(e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}\right)} \\
Y_{22} &= \frac{Z_{01}\, e^{\gamma_1 \ell} + Z_{02}\, e^{-\gamma_2 \ell}}{Z_{01} Z_{02}\left(e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}\right)}
\end{aligned} \tag{25}$$

The Y matrix is useful when two lines are connected in parallel as is shown in Figure 10. Here, the input voltage is the same for each line while the input current is the sum of the input currents to the two lines. At the output, the voltage is the same for each line while the current is the sum of the currents from the two lines. Therefore, the Y matrix of the parallel connected lines is the sum of the Y matrices of the two lines.[9]

The matrix elements reduce to those for the lossy or lossless reciprocal lines. If $Z_{01} = Z_{02}$ and $\gamma_1 = \gamma_2$, then the equations derived above apply to the lossy reciprocal line. If $Z_{01} = Z_{02}$ = a real number and $\gamma_1 = \gamma_2 = j\beta$, then the equations apply to the lossless reciprocal line. This is a direct result of the theory presented in Section II.

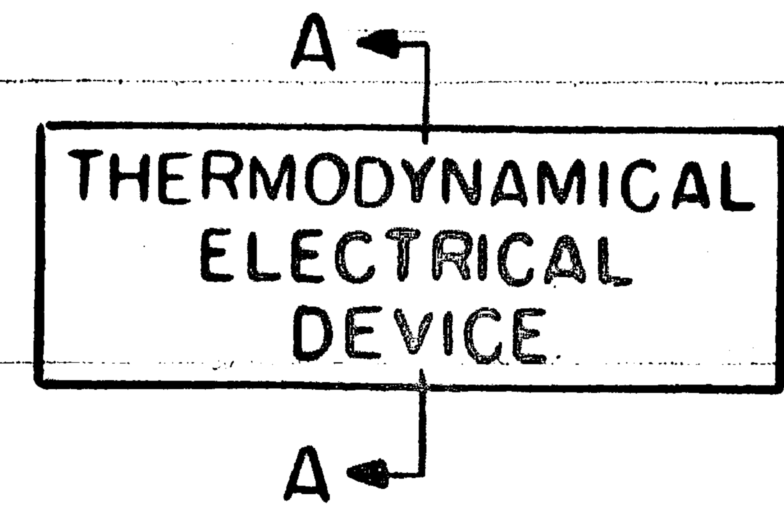


Fig. 1

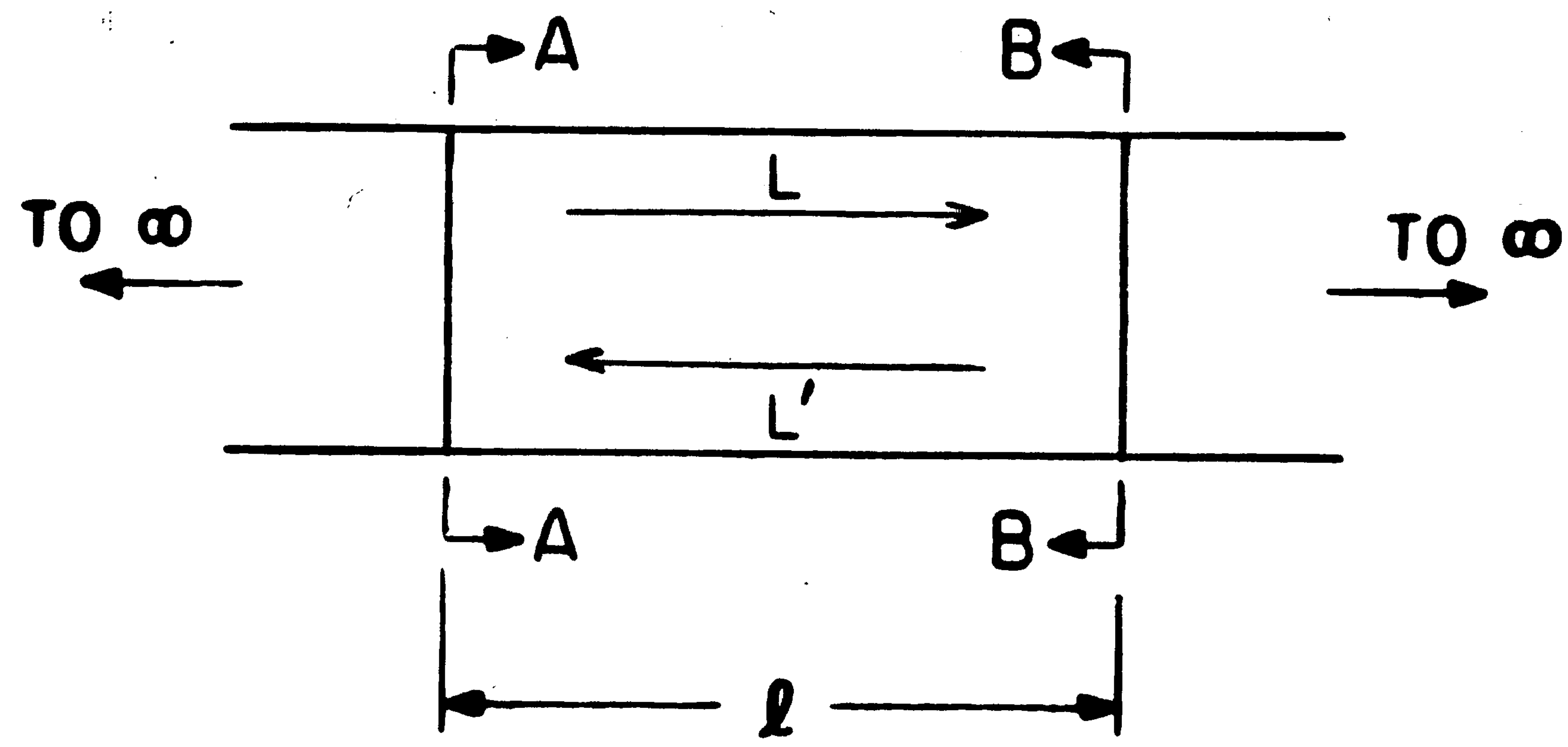
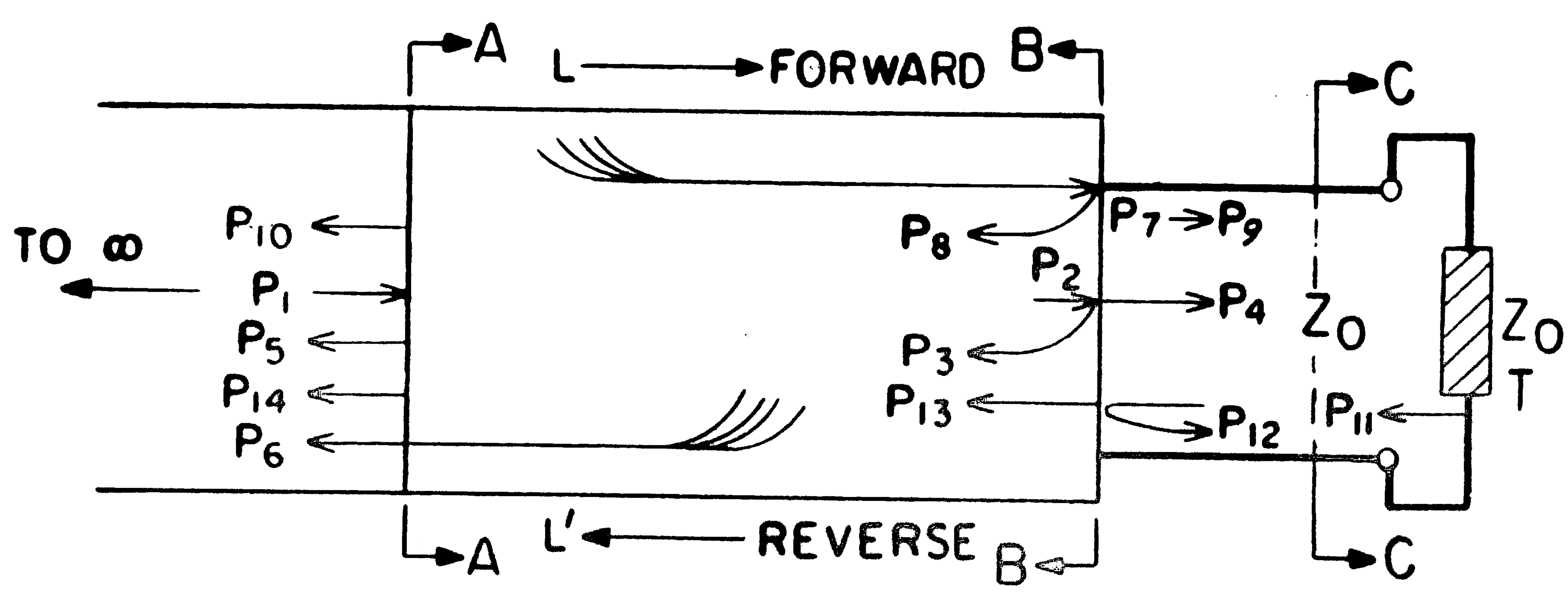


Fig. 2



$P_1 = kTB$, $P_2 = (1-L)kTB$, $P_3 = |\Gamma_1|^2(1-L)kTB$, $P_4 = (1-|\Gamma_1|^2)(1-L)kTB$,
$P_5 = |\Gamma_1|^2(1-L)(1-L')kTB$, $P_6 = L'kTB$, $P_7 = LkTB$, $P_8 = |\Gamma_1|^2 LkTB$,
$P_9 = (1-|\Gamma_1|^2)LkTB$, $P_{10} = |\Gamma_1|^2 L(1-L')kTB$, $P_{11} = kTB$, $P_{12} = |\Gamma_2|^2 kTB$,
$P_{13} = (1-|\Gamma_2|^2)kTB$, $P_{14} = (1-|\Gamma_L|^2)(1-L')kTB$

Fig. 3

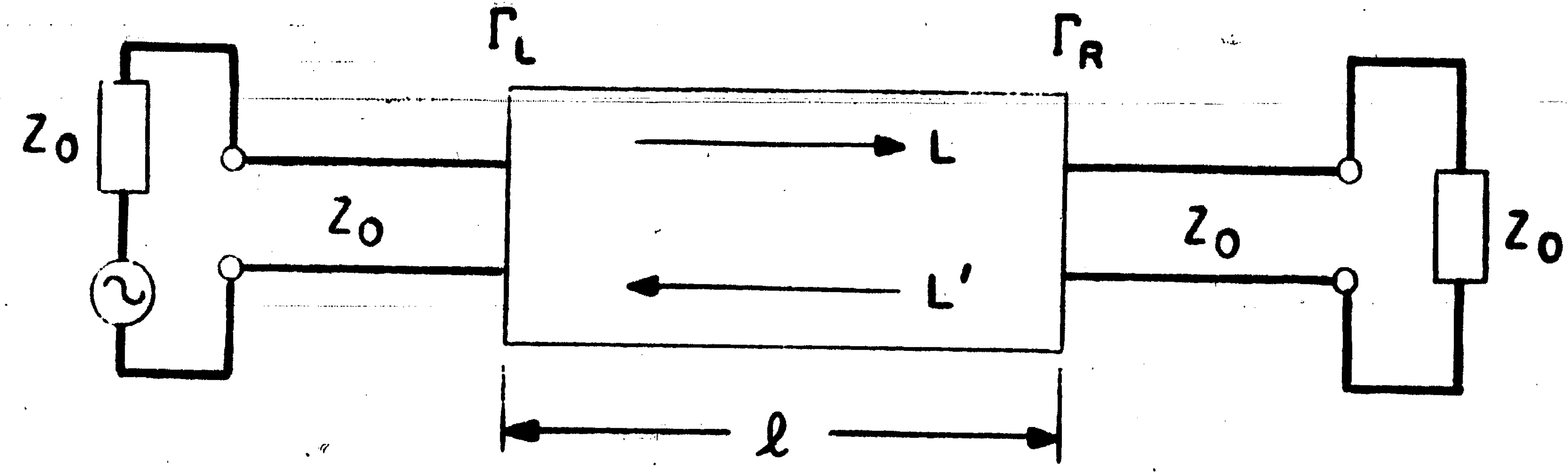


Fig. 4

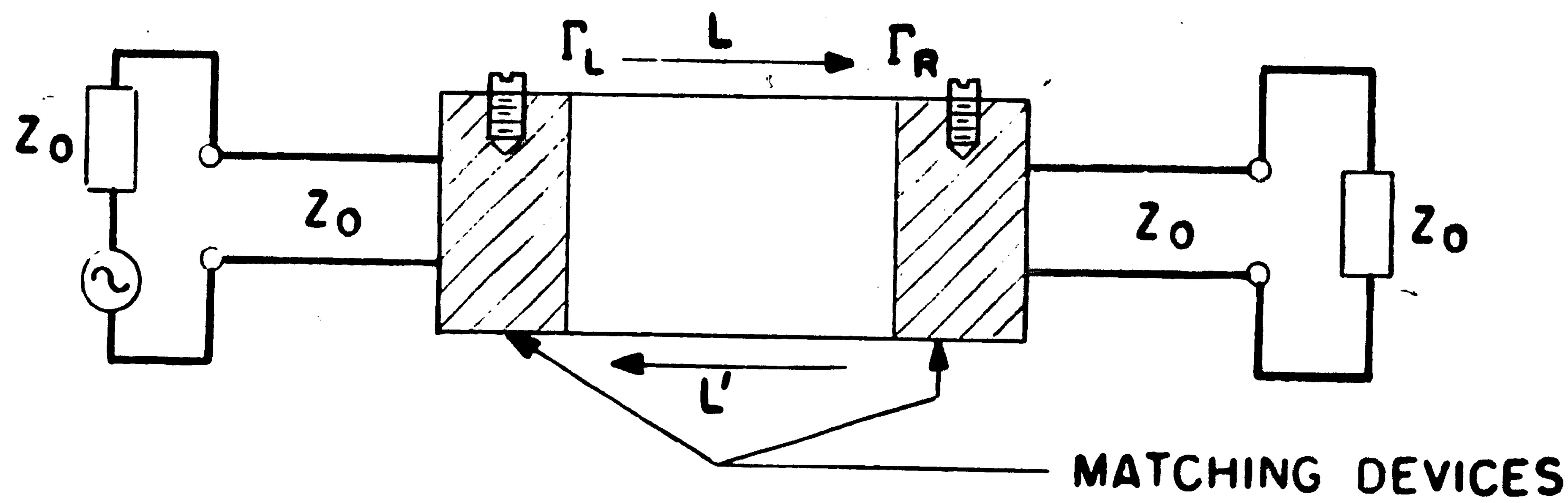


Fig. 5

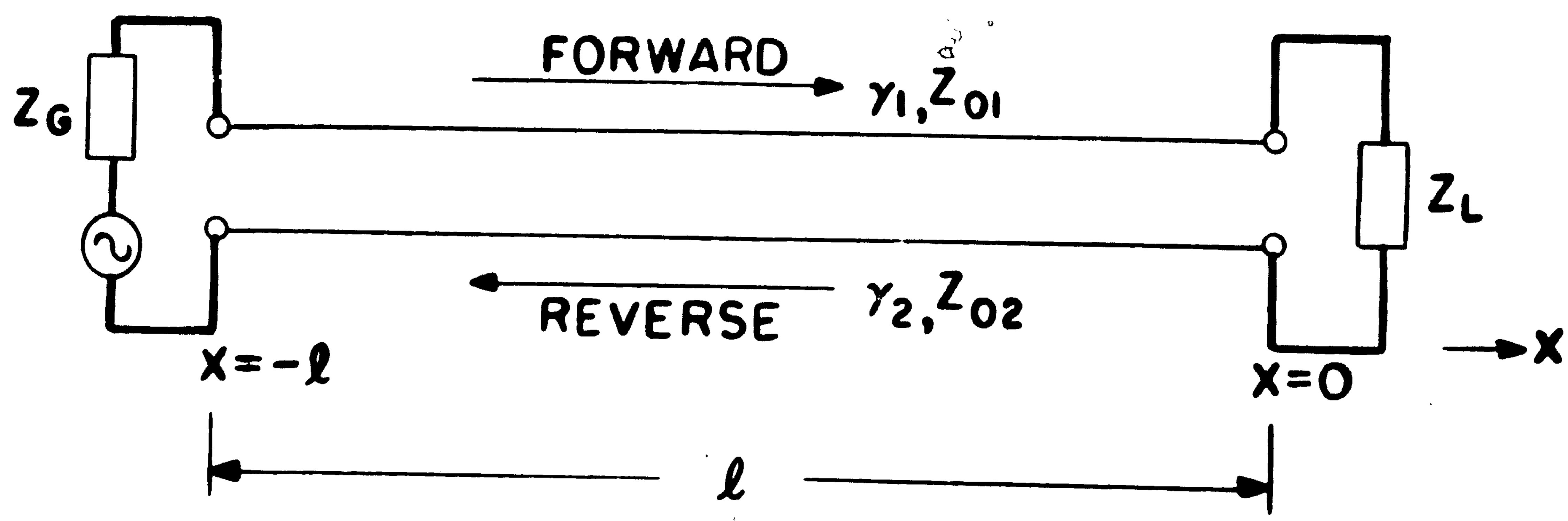


Fig. 6

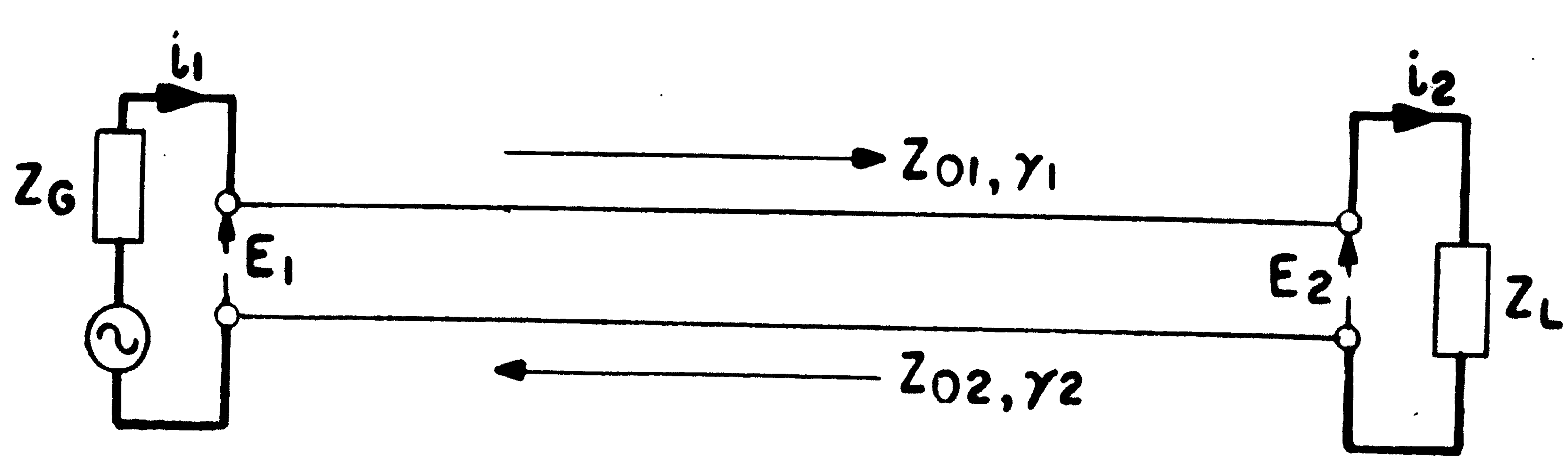


Fig. 7

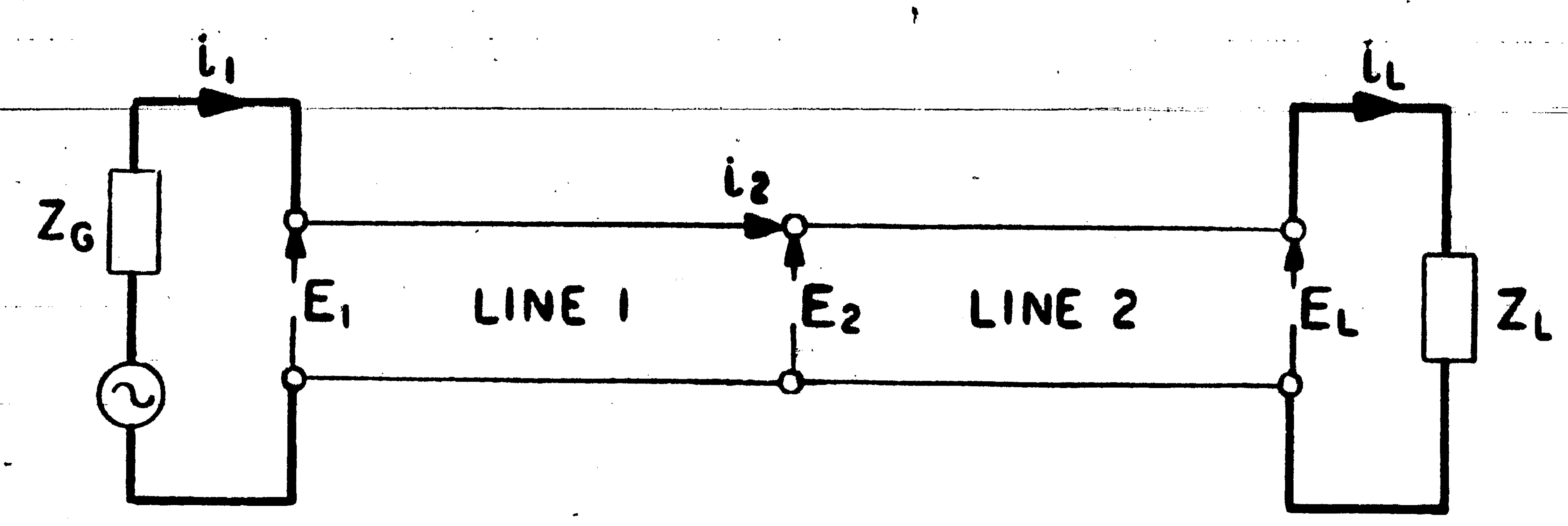


Fig. 8

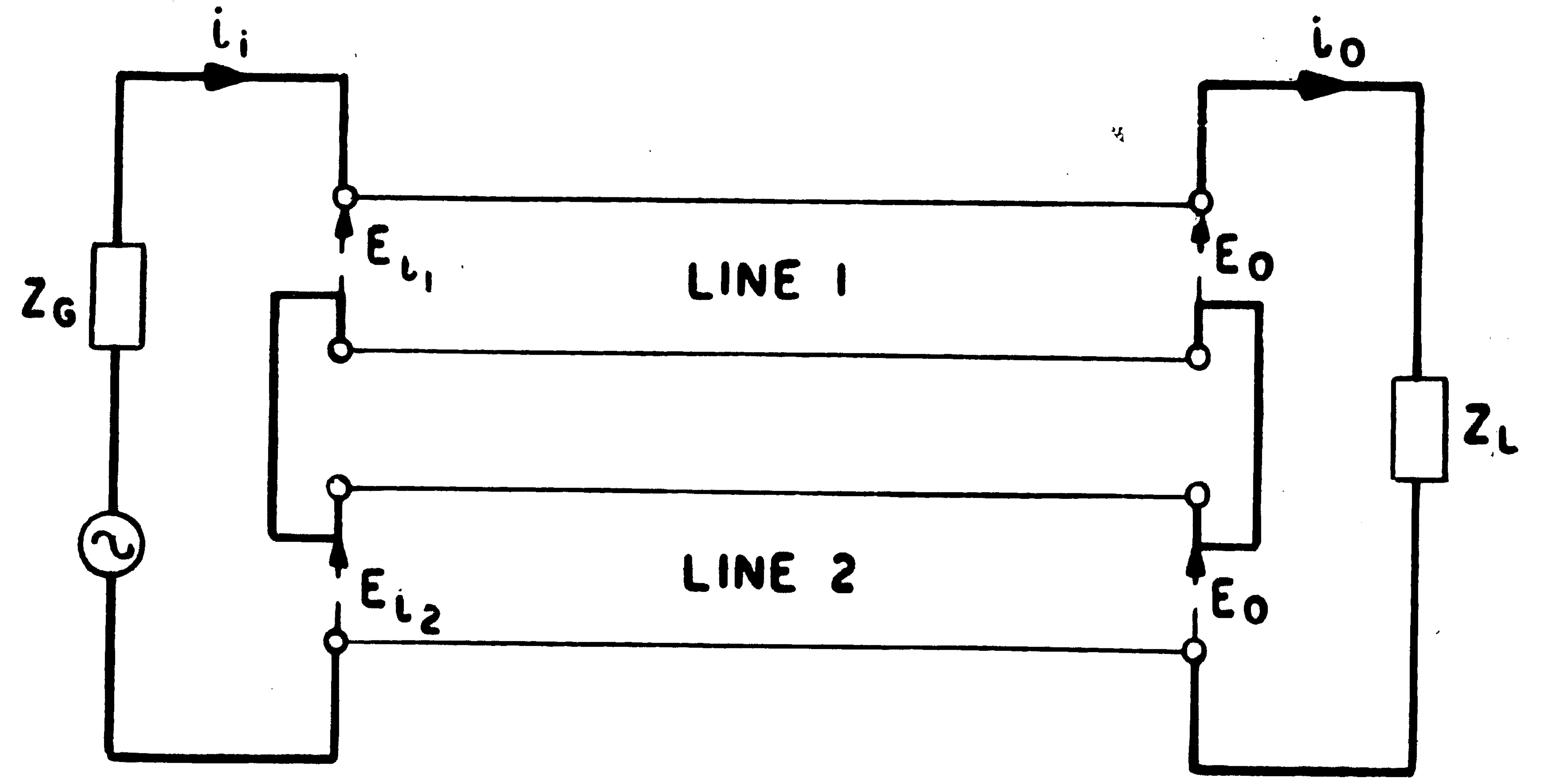


Fig. 9

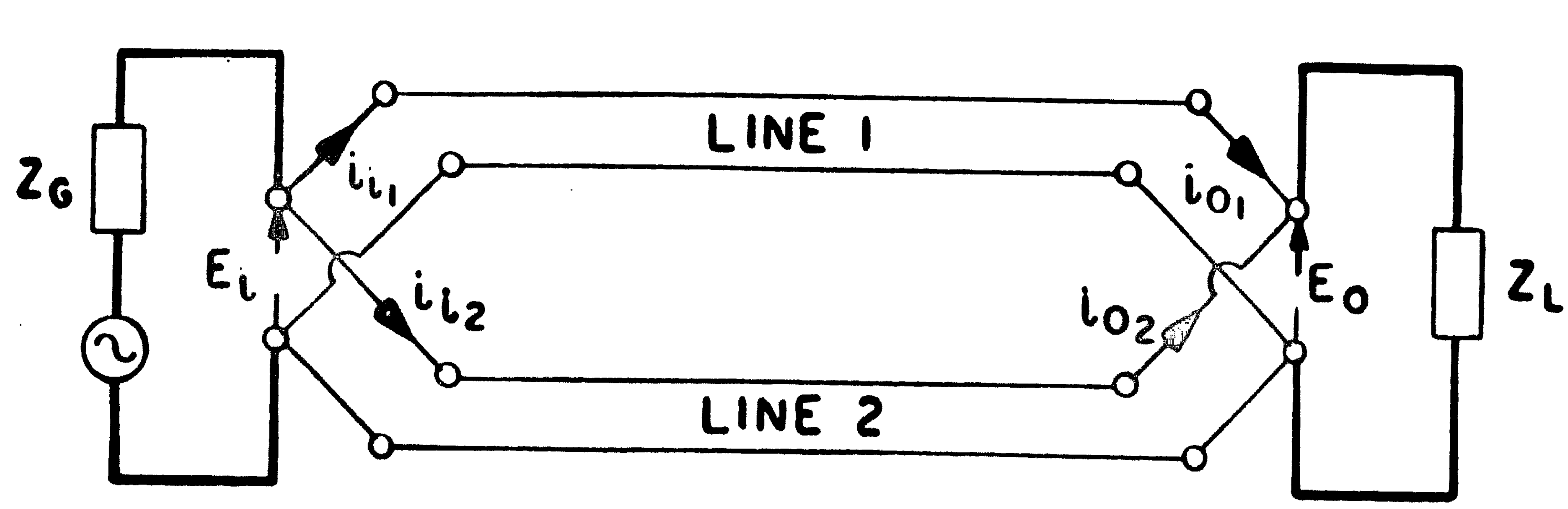


Fig. 10

Appendix A

Examples of Convergence of the Sequences $\{|\Gamma_{Ln}|_{max}\}$ and $\{|\Gamma_{Rn}|_{max}\}$

Consider power P_s incident at the left junction in Figure 5. Then if $(1-L)(1-L') = 0.05$ any component of P_s which passes back and forth through the isolator two or more times will at least be attenuated by a factor $(1-L)(1-L')$ in excess of a component of power which has circulated the device just once. Therefore, the components undergoing multiple reflections more than an order of magnitude smaller than the others and hence they will be neglected. If the sequences $\{|\Gamma_{Ln}|_{max}\}$, and $\{|\Gamma_{Rn}|_{max}\}$ converge with L, L' chosen so that $(1-L)(1-L') = 0.05$, then they converge for L, L' such that $(1-L)(1-L') < 0.05$. Therefore, only values of $|\Gamma_L|$ and $|\Gamma_R|$ will be varied and $(1-L)(1-L')$ will be fixed equal to 0.05. The values of $|\Gamma_L|$ and $|\Gamma_R|$ will be chosen to demonstrate that the sequences $\{|\Gamma_{Ln}|_{max}\}$ and $\{|\Gamma_{Rn}|_{max}\}$ converge under extreme conditions.

Case 1: $|\Gamma_L| = |\Gamma_R| = 0.9$

$$|\Gamma_{L1}|_{max} = (1-|\Gamma_L|^2)(|\Gamma_R|)\sqrt{(1-L)(1-L')}$$

$$= (0.19)(0.9)(0.22361)$$

$$|\Gamma_{L1}|_{max} = 0.03824$$

$$|\Gamma_{R1}|_{max} = (1-|\Gamma_R|^2)(|\Gamma_{L1}|_{max})\sqrt{(1-L)(1-L')}$$

$$= (0.19)(0.03824)(0.22361)$$

$$|\Gamma_{R1}|_{max} = 0.00162$$

$$|\Gamma_{L2}|_{max} = (1-|\Gamma_{L1}|^2_{max})(|\Gamma_{R1}|_{max})\sqrt{(1-L)(1-L')}$$

$$= (0.998538)(0.00162)(0.22361)$$

$$|\Gamma_{L2}|_{max} = 0.00036$$

$$|\Gamma_{R2}|_{max} = (1-|\Gamma_{R1}|^2_{max})(|\Gamma_{L2}|_{max})\sqrt{(1-L)(1-L')}$$

$$= (0.999998)(0.00036)(0.22361)$$

$$|\Gamma_{R2}|_{max} = 0.00008$$

$$|\Gamma_{L3}|_{max} = (1-|\Gamma_{L2}|^2_{max})(|\Gamma_{R2}|_{max})\sqrt{(1-L)(1-L')}$$

$$= (1.000000)(0.00008)(0.22361)$$

$$|\Gamma_{R3}|_{max} < 0^+$$

and from the nature of the $\underline{n}^{th}$ terms of the sequences $\{|\Gamma_{Ln}|_{max}\}$ and $\{|\Gamma_{Rn}|_{max}\}$, all the remaining elements of both sequences are $< 0^+$.

So, $|\Gamma_{L1}|_{max} = 0.03824$, $|\Gamma_{L2}|_{max} = 0.00036$,

$|\Gamma_{L3}|_{max} = 0.00002$, and $|\Gamma_{L4}|_{max}, \ldots, |\Gamma_{Ln}|_{max}, \ldots < 0^+$

$|\Gamma_{R1}|_{max} = 0.00162$, $|\Gamma_{R2}|_{max} = 0.00036$

$|\Gamma_{R3}|_{max} = 0.00008$, and $|\Gamma_{R4}|_{max}, \ldots, |\Gamma_{Rn}|_{max}, \ldots < 0^+$

The method of solution of the following cases is just the same as that used in Case 1. So only the results will be given.

Case 2: $|\Gamma_L| = 0.1$, $|\Gamma_R| = 0.9$

Results:

$|\Gamma_{L1}|_{max} = 0.19924$, $|\Gamma_{L2}|_{max} = 0.00182$, $|\Gamma_{L3}|_{max} = 0.00009$

and $|\Gamma_{L4}|_{max}, \ldots, |\Gamma_{Ln}|_{max}, \ldots < 0^+$

$|\Gamma_{R1}|_{max} = 0.00847$, $|\Gamma_{R2}|_{max} = 0.00041$, $|\Gamma_{R3}|_{max} = 0.00002$

and $|\Gamma_{R4}|_{max}, \ldots, |\Gamma_{Rn}|_{max}, \ldots < 0^+$

Case 3: $|\Gamma_L| = 0.9$, $|\Gamma_R| = 0.1$

Results:

$$|\Gamma_{L1}|_{max} = 0.00425,\ |\Gamma_{L2}|_{max} = 0.00021,\ |\Gamma_{L3}|_{max} = 0.00001,$$

and $|\Gamma_{L4}|_{max}, \ldots, |\Gamma_{Ln}|_{max}, \ldots < 0^+$

$$|\Gamma_{R1}|_{max} = 0.00094,\ |\Gamma_{R2}|_{max} = 0.00005$$

and $|\Gamma_{R3}|_{max}, \ldots, |\Gamma_{Rn}|_{max}, \ldots < 0^+$

Case 4: $|\Gamma_L| = |\Gamma_R| = 0.1$

Results:

$$|\Gamma_{L1}|_{max} = 0.02214,\ |\Gamma_{L2}|_{max} = 0.00101,\ |\Gamma_{L3}|_{max} = 0.00005$$

and $|\Gamma_{L4}|_{max}, \ldots, |\Gamma_{Ln}|_{max}, \ldots < 0^+$

$$|\Gamma_{R1}|_{max} = 0.00450,\ |\Gamma_{R2}|_{max} = 0.00023,\ |\Gamma_{L3}|_{max} = 0.00001$$

and $|\Gamma_{R4}|_{max}, \ldots, |\Gamma_{Rn}|_{max}, \ldots < 0^+$

In all of the above examples, no more than four adjustments of each matching device were needed to obtain partial reflection coefficients $\leq 0^+$ at each junction. Therefore the trend of convergence is established; and $\{|\Gamma_{Ln}|_{max}\}$ and $\{|\Gamma_{Rn}|_{max}\}$ are converging to zero.

Appendix B

Derivation of the Y Parameter Matrix from the ABCD Parameters

From equations (15) and (24), one obtains the following relation between the Y parameters and the ABCD parameters:

With $i_2 = 0$, $\dfrac{E_1}{E_2} = \dfrac{Y_{22}}{Y_{21}} = A$

and, $i_1 = Y_{11}\, E_1 - Y_{12}\, E_2 = Y_{11}\left(\dfrac{Y_{22}}{Y_{21}}\right) E_2 - Y_{12} E_2$

$$\frac{i_1}{E_2} = \frac{Y_{11}Y_{22} - Y_{21}Y_{22}}{Y_{21}} = B$$

and $\dfrac{i_1}{i_2} = \dfrac{Y_{11}}{Y_{21}} = D$

Solving for the Y parameters in terms of the ABCD parameters one obtains:

$$Y_{11} = \frac{Z_{02}\, e^{\gamma_1 \ell} + Z_{01}\, e^{-\gamma_2 \ell}}{Z_{01} Z_{02}\left(e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}\right)}$$

$$Y_{12} = \frac{(Z_{01} + Z_{02})\, e^{(\gamma_1 - \gamma_2)\ell}}{Z_{01} Z_{02}\left(e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}\right)}$$

$$Y_{21} = \frac{Z_{01} + Z_{02}}{Z_{01} Z_{02}\left(e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}\right)}$$

$$Y_{22} = \frac{Z_{01}\, e^{\gamma_1 \ell} + Z_{02}\, e^{-\gamma_2 \ell}}{Z_{01} Z_{02}\left(e^{\gamma_1 \ell} - e^{-\gamma_2 \ell}\right)}$$

These parameters are exactly the same as those in equation (25).

References

(1) F. W. Sears, p. 112.

(2) P. H. Smith in Electronics, vol. 12, pp. 29-31.

(3) P. H. Smith in Electronics, vol. 17, pp. 130-133, 318, 320, 322, 324-325.

(4) C. G. Montgomery in Montgomery et al. pp. 79-82.

(5) J. J. Karakash, pp. 66-68, 174-175.

(6) J. J. Karakash, pp. 150-152.

(7) J. J. Karakash, pp. 155-156.

(8) C. R. Wylie, Jr., pp. 19-21.

(9) J. J. Karakash, pp. 153-154.

Bibliography

Books

Ghose, R. N. Microwave Circuit Theory and Analysis. New York: McGraw-Hill Book Co., Inc., 1963.

Karakash, J. J. Transmission Line and Filter Networks. New York: Macmillan Co. 1950.

Montgomery, C. G., Dicke, R. H., and Purcell, E. M., editors. Principles of Microwave Circuits. 1st ed. New York: McGraw-Hill Book Co., Inc., 1948.

Ramo, S. and Whinnery, J. R. Fields and Waves in Modern Radio. 2nd ed. New York: John Wiley and Sons, Inc., 1953.

Sears, F. W. An Introduction to Thermodynamics, the Kinetic Theory of Gases, and Statistical Mechanics. 2nd ed. Reading, Massachusetts: Addison-Wesley Publishing Co., Inc., 1953.

Wylie, C. R., Jr. Advanced Engineering Mathematics. 2nd ed. New York: McGraw-Hill Book Co., Inc., 1960.

Articles

Smith, P. H. "Transmission Line Calculator," Electronics, vol. 12 (1939), pp. 29-31.

______. "An Improved Transmission Line Calculator," Electronics, vol. 17 (1944) pp. 130-133, 318, 320, 322, 324-325.

Biography

Howard Alan Seid, born October 11, 1942 in Newark, New Jersey, is the son of Esther P. and Irving L. Seid. He graduated with highest honors from Lehigh University in June 1964 receiving the degree of B.S. in E.E. Mr. Seid is a member of Tau Beta Pi, Eta Kappa Nu, and Pi Mu Epsilon. Since graduation, he has been a member of the technical staff of Bell Telephone Laboratories, Incorporated, Allentown, Pennsylvania.