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An introduction to two norm spaces

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AN INTRODUCTION TO TWO NORM SPACES

by
Lee W. Baric

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INTRODUCTION

In this paper I present some of the concepts that result when one considers a linear space with two norms on it. A two-norm convergence is defined, called γ-convergence, and from this, in a more or less natural manner, arise questions of γ-linear functionals, γ-conjugate spaces, γ-reflexivity, and γ-topological problems. In Chapter 2 it is shown that there exists a locally convex linear separated topology generating the γ-convergence. Finally, in Chapter 4 I include a brief generalization to the case of a space with a norm topology and a locally convex linear topology.

The system of reference is a standard one, e.g., u.v.w refers to statement w in section v of chapter u.

The following results, for which I give only references to the proofs, will be used later.

(a) The Mackey-Arens Theorem: Let L be a locally convex linear separated space and denote its conjugate space by L'. Let $\sigma(L,L')$ and $\beta(L,L')$ denote, respectively, the weak and strong topologies on L. Let t be any locally convex linear separated topology on L and denote the conjugate space of L under t by L'_t. Then $L'_t = L'$ if and only if $\sigma(L,L') \leq t \leq \beta(L,L')$.

This is Arens' refinement of a theorem by Mackey. The above is not exactly Arens' statement but may be deduced from the results of his paper, [6].

(b) **Wiweger's Lemma:** Let F be a family of functionals defined on an arbitrary set Q and such that $\sup \{|f(q)| : f \in F\} < \infty$ for every q Q, and let (q_n) be a sequence in Q. The following propositions are equivalent:

(1) for each sequence of positive numbers (a_n) tending to ∞ and for each sequence (f_n) in F there is an M such that

$$|f_n(q_m)| < a_n \text{ for } n = 1,2,\ldots \quad \text{and} \quad m > M$$

(2) for each $f \in F$ we have $\lim_n f(q_n) = 0$ and $\{|f(q_n)| : f \in F,\ n = 1,2,\ldots\}$ is bounded.

See [18] p. 126.

(c) A theorem of Alexiewicz: Let X be a metric space, and call any subset whose complement is of Category I residual. Let $(U_n(x))$ be a sequence of operations on X convergent in a residual set A. Then $(U_n(x))$ is equicontinuous in a residual set $B \subset X$.

See p. 5 of [1].

(d) A result of Dixmier: Let E be a normed space and E' its conjugate space. Let A and B be subspaces of E' which are dense in E' under $\sigma(E',E)$ and such that $\sigma(E,A)$ and $\sigma(E,B)$ are separated. Then $\sigma(E,A)$ and $\sigma(E,B)$ agree on S, the unit ball of E, if and only if $\overline{A} = \overline{B}$ where these closures are taken in E' under the norm topology.

See p. 1059 of [10].

Chapter 1

The Two Norm Convergence

1. The Convergence Defined

Let X be a linear space and $||.||$ an F-norm on X, i.e.,

(1) $||x|| \geqq 0$ for all x in X,

(2) $||x|| = 0$ if and only if $x = 0$,

(3) $||x+y|| \leqq ||x|| + ||y||$ for all x and y in X,

(4) if $a_n \to a$ and $||x_n - x|| \to 0$, then $||a_n x_n - ax|| \to 0$ where (a_n) is a scalar sequence and (x_n) a sequence in X.

If, in place of (4), $||.||$ satisfies the stronger condition that $||ax|| = |a|.||x||$ where a is any scalar and x is in X, we shall call $||.||$ a norm on X. Clearly any norm is an F-norm.

Suppose a second F-norm $||.||^*$ is defined on X weaker than $||.||$, i.e.,

<u>Postulate (i)</u>: $||x_n|| \to 0$ implies that $||x_n||^* \to 0$.

A sequence (x_n) in X will be called "γ-convergent" to x if it is bounded with respect to $||.||$ and if $||x_n - x||^* \to 0$. We shall write

$$\gamma\text{-lim } x_n = x \quad \text{or} \quad x_n \xrightarrow{\gamma} x.$$

This convergence will also be called the "two-norm" convergence.

The space X supplied with this convergence will be denoted by $(X, ||\cdot||, ||\cdot||^*)$ and called a "two-norm space". In this notation the weaker F-norm is always the last element of the triple.

2. γ-Convergence and Norm Convergence

In general, γ-convergence is equivalent to an F-norm convergence only in the trivial case where $||\cdot||$ and $||\cdot||^*$ are equivalent. (Two convergences are said to be "equivalent" when the classes of convergent sequences coincide and the limits are identical.)

1.2.1. Lemma: Let $||\cdot||$ be an F-norm on a linear space X and let (x_n) be a sequence in X with $||x_n|| \to 0$. Then there is a scalar sequence (a_n) with $|a_n| \to 0$ such that $||\frac{x_n}{a_n}|| \to 0$.

Proof: For any fixed positive integer k, $||kx_n|| \to 0$ by property (4) of an F-norm. Choose N_k such that $||kx_n|| < \frac{1}{k}$ whenever $n > N_k$. We may assume that $N_1 < N_2 < N_3 < \ldots$. Let $a_i = 1$ for $i = 1, 2, \ldots, N_2$, and in general let $a_i = \frac{1}{k}$ for $i = N_k + 1, N_k + 2, \ldots, N_{k+1}$. Then $||\frac{x_n}{a_n}|| \to 0$.

1.2.2. Lemma: Suppose there is a scalar sequence $a_n \to 0$ such that $\frac{x_n}{a_n} \xrightarrow{\gamma} x$. Then $||x_n|| \to 0$.

Proof: $(\frac{x_n}{a_n})$ is $||\cdot||$-bounded. Hence $||x_n|| = ||a_n \cdot \frac{x_n}{a_n}|| \to 0$.

<u>1.2.3. Corollary</u>: If γ-convergence is equivalent to an F-norm convergence, it is equivalent to $||.||$-convergence.

Proof: Let $x_n \xrightarrow{\gamma} 0$. Since it's an F-norm convergence there is a scalar sequence $a_n \to 0$ such that $\frac{x_n}{a_n} \xrightarrow{\gamma} 0$. Hence by 1.2.2., $||x_n|| \to 0$.

<u>1.2.4. Theorem</u>: Let X be a Banach space under $||.||$. Each of the following is necessary and sufficient that γ-convergence be equivalent to a norm convergence:

(a) $||x_n||^* \to 0$ implies boundedness of (x_n) under $||.||$,

(b) $||.||$ and $||.||^*$ are equivalent,

(c) X is complete under $||.||^*$.

Proof: (a) implies (b): Let $||x_n||^* \to 0$, then, as in the proof of the corollary, we can show that $||x_n|| \to 0$.

(b) implies (c): Obvious since $||.||^*$ is a linear metric on X.

(c) implies (a): $||.||$ and $||.||^*$ are complete comparable linear metrics on the same space and hence are equivalent.

It is clear that (b) is a sufficient condition. Furthermore, (a) is a necessary condition as we now show. Let $||x_n||^* \to 0$ and suppose $||x_n|| \to \infty$. Let $x_n' = \frac{x_n}{||x_n||}$. Then $x_n' \xrightarrow{\gamma} 0$ and by lemma 1.2.2. $||x_n'|| \to 0$. But this is impossible.

Theorem 1.2.4. is true also if X is a B_0-space, see Mazur and Orlicz [12].

We will say γ-convergence is "metrical" if it is possible to introduce a metric on X such that the metric convergence is equivalent to γ-convergence.

<u>1.2.5. Remark</u>: If γ-convergence is metrical, then it is equivalent to $||.||$ convergence.

Proof: Let $x_n \xrightarrow{\gamma} 0$ and let (a_n) be a scalar sequence with $a_n \rightarrow 0$. Clearly, then, $a_n x_n \xrightarrow{\gamma} 0$ so that the metric is an F-norm, and the remark follows by corollary 1.2.3..

3. Some Examples

In the following, $(X, ||.||)$ denotes X under the topology induced by $||.||$. Similar notation will be used throughout the remainder of this paper.

We now state two more postulates:

<u>Postulate (ii)</u>: The unit disc, $S = \{x: ||x|| \leq 1\}$, in $(x, ||.||)$ is $||.||^*$-complete.

<u>Postulate (iii)</u>: If x is in X and (x_n) is a sequence in X converging to x under $||.||^*$, then $||x|| \leq \liminf ||x_n||$, i.e., $||.||$ is a lower semi-continuous function on $(X, ||.||^*)$.

<u>1.3.1. Lemma</u>: $(X, ||.||, ||.||^*)$ satisfies (iii) if and only if S is closed in $(X, ||.||^*)$. Hence, in the non-trivial case, $(X, ||.||^*)$ is of Category I in itself.

Proof: A necessary and sufficient condition for $||.||$ to be lower semi-continuous on $(X, ||.||^*)$ is that $\{x: ||x|| \leq k\}$ be closed for all k.

The lemma shows that (iii) follows from (ii). However, there will be times in the sequel where (iii) alone will be assumed. Spaces satisfying (iii) will be termed "normal".

Now let us consider some examples of two-norm spaces satisfying (i) and (ii).

1.3.2. Remark: Let $p > q \geq 1$ be integers and let $||.||_p$ denote the usual norm on $L^p = L^p\,[0,1]$ and $||.||_q$ that on $L^q = L^q\,[0,1]$. Then $(L^p, ||.||_p, ||.||q)$ satisfies (i).

Proof: $L^p \subset L^q \subset S$, where S is the space of measurable function on $[0,1]$. For f in S, let $|f| = \int_0^1 \frac{|f(t)|}{1+|f(t)|}\,dt$. It is easily seen that S is Hausdorff and L^p and L^q are FH subspaces of S. Hence $||.||_p$ is stronger than $||.||_q$.

1.3.3. Example: $(L^2, ||.||_2, ||.||_1)$. We have already seen that (i) is satisfied. Let A be the set of simple functions on $[0,1]$. Then $A \subset L^2 \subset L^1$. Furthermore A is dense in $(L^1, ||.||_1)$, so that $L^{2*} = L^1$. Let (x_n) be a sequence in the unit disc of $(L^2, ||.||_2)$ which converges to x, x in L^1, under $||.||_1$. Since (x_n) converges to x in the mean of order one, there is a subsequence (x_{n_k}) which converges to x pointwise a.e.. Thus $x_{n_k}^2 \rightarrow x^2$ pointwise a.e. and by Fatou's lemma, $||x||_2^2 \leq \liminf ||x_{n_k}||^2 \leq 1$. Thus (ii), and hence also (iii), are satisfied. It is clear that γ-convergence may be characterized as: $x_n \xrightarrow{\gamma} x$ if and only if $\int_0^1 |x_n(t)|^2 dt < K$ for all n and $\int_0^1 |x_n(t)-x(t)|\,dt \rightarrow 0$.

1.3.4. Example: $(L^1, ||.||, ||.||^*)$, where $||x||^* = \int_0^1 \frac{|x(t)|}{1+|x(t)|}\, dt$. It can then be shown that $L^{1*} = S$ and that (i) and (ii) are satisfied. Moreover, γ-convergence may be characterized as: $x_n \xrightarrow{\gamma} x$ if and only if $\int_0^1 |x_n(t)|dt < K$ for all n and $x_n \rightarrow x$ in measure.

Chapter 2

The Linear Functionals

1. Preliminaries

Unless otherwise stated, it is supposed throughout this chapter that $(X, ||.||, ||.||^*)$ is a two-norm space satisfying postulates (i) and (iii). It is further assumed that $||.||$ is a norm and that $(X, ||.||^*)$ is a B_0^*-space, i.e., there is a sequence of seminorms $([.]_i)$ such that $\sum_{i=1}^{\infty} [x]_i = 0$ implies $x = 0$. If we set

$$||x||^* = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{[x]_k}{1+[x]_k}$$

then $||.||^*$ is an F-norm and $||x_n - x||^* \rightarrow 0$ if and only if $[x_n - x]_i \rightarrow 0$ for each i. It may be assumed that $[x]_1 \leq [x]_2 \leq \ldots$, and, in fact, we shall do this later.

The spaces conjugate to $(X, ||.||)$ and $(X, ||.||^*)$ will be denoted by C and C^* respectively, i.e., the set of functionals linear and continuous under the respective topologies. By a "γ-linear" functional we mean a functional f on X which is linear and satisfies:

$$x_n \xrightarrow{\gamma} x \text{ implies } f(x_n) \to f(x)$$

The set of γ-linear functionals will be written as C_γ. It is clear that $C^* \subset C_\gamma \subset C$.

2.1.1. Lemma: Let X_o be a dense subspace of a normed space $(X, ||.||)$ and let $||.||^*$ be an F-norm on X coarser than $||.||$ and satisfying (iii) in X_o. Then (iii) is also satisfied in X. If $||.||^*$ is a norm in X_o, it is a norm in X too.

The proof is elementary.

We are not assuming $(X, ||.||)$ to be complete. However, in many cases of interest, this may be assumed for the completion $\tilde{X}$ is again a two-norm space satisfying (i) and (iii). Let $\tilde{x} \epsilon \tilde{X}$ and let (x_n) be an element of the equivalence class given by $\tilde{x}$. Since (X_n) is $||.||$-Cauchy, it is also $||.||^*$-Cauchy, and so we define $||\tilde{x}||^* = \lim ||x_n||^*$. This value is clearly independent of the representative chosen for $\tilde{x}$.

2.1.2. Theorem: $(\tilde{X}, ||.||, ||.||^*)$ is a two-norm space satisfying (i) and (iii). Every continuous linear functional on $(X, ||.||)$ may be uniquely extended to a continuous linear functional on $(\tilde{X}, ||.||)$. Furthermore, the γ-linear functionals and the continuous linear functionals on $(X, ||.||^*)$ may also be uniquely extended to γ-linear functionals on

$(\tilde{X}, ||.||, ||.||^*)$ and continuous linear functionals on $(\tilde{X}, ||.||^*)$ respectively.

Proof: The first assertion follows from the previous lemma. The proof of the second is well-known, and the others follow readily.

2. The Null Sets

The "γ-closure" of a subset, Y, of $(X, ||.||, ||.||^*)$ is the set of all γ-limits of sequences in Y, and every set including its γ-closure will be termed "γ-closed". We shall denote the γ-closure of Y by γ(Y). Note that normality need not be assumed in this section.

2.2.1. Lemma: Let X_1 be a linear subspace of $(X, ||.||, ||.||^*)$ and suppose that x_o is not in the γ-closure of X_1. Let X_2 be the linear span of (X_1, x_o) and let f be defined on X_2 by $f(x) = \lambda$ where $x = z + \lambda x_o$. Then f is γ-linear on X_2.

Proof: Let (x_n) be a sequence in X_2 with $x_n \xrightarrow{\gamma} 0$. Suppose $f(x_n) \not\rightarrow 0$. There is a subsequence, say (x_n'), and a $\delta > 0$ such that $|f(x_n')| > \delta$. Now there is a subsequence (x_n''), of (x_n') such that $[f(x_n'')]^{-1}$ converges to a limit C. Write $x_n'' = z_n'' + f(x_n'')x_o$. Then $z_n'' + f(x_n'')x_o \xrightarrow{\gamma} 0$ and so $z_n'' [f(x_n'')]^{-1} + x_o \xrightarrow{\gamma} C.0 = 0$. But this asserts that x_o is in $\gamma(X_1)$ contrary to our hypothesis.

2.2.2. Theorem: A subset H of $(X, \|.\|, \|.\|^*)$ is the null set of a non-trivial γ-linear functional if and only if it is γ-closed, linear, and of deficiency 1.

Proof: Necessity is obvious, and sufficiency follows from the lemma.

For any subset A of X, write $A^I = \bigcup_{n=1}^{\infty} \overline{A \cap S_n}^*$ where $S_n = \{x : \|x\| \leq n\}$ and $\overline{B}^*$ is the closure of B in $(X, \|.\|^*)$. We then have:

2.2.3. Theorem: A is γ-closed if and only if $A = A^I$.

Proof: First consider the necessity. Fix n and let $x \in \overline{A \cap S_n}^*$. There is a sequence (x_n) in $A \cap S_n$ with $x_n \to x$ in $\|.\|^*$. Then $x_n \xrightarrow{\gamma} x$, so $x \in A$. Thus $\overline{A \cap S_n}^* \subset A$ for each n, and so $A^I \subset A$. But $A = \bigcup_{n=1}^{\infty} A \cap S_n \subset A^I$. Hence $A = A^I$.

Now the sufficiency. $A = A^I$ implies that $A \supset \overline{A \cap S_n}^*$ for any n. Let (x_p) be a sequence in A with $x_p \xrightarrow{\gamma} x$. Then for some M, $x_p \in S_M$ for all p, so that $\|x_p - x\|^* \to 0$ implies $x \in A$.

As we shall see in section five of this chapter, Mazurkiewicz has constructed an example for which γ(A) is not equal to γ(γ(A)). However, since the intersection of any family of γ-closed sets is γ-closed, there exists for any set A a smallest γ-closed set, $\overline{\gamma}(A)$, containing A. Now write $\gamma_0(A) = A$ and $\gamma_a(A) = \gamma\left(\bigcup_{b<a} \gamma_b(A)\right)$ for any ordinal $a \geq 1$. We then have:

2.2.4. Lemma: $\overline{\gamma}(A)$ is identical with $\gamma_{\omega_1}(A)$ where ω_1 is the smallest uncountable ordinal.

Proof: By the construction of $\gamma_{\omega_1}(A)$ it is clear that $\overline{\gamma}(A) \supset \gamma_{\omega_1}(A)$. Let (x_n) be a sequence in $\gamma_{\omega_1}(A)$ with $x_n \xrightarrow{\gamma} x$. Each x_n is a γ-limit of a sequence in $\gamma_b(A)$ for some countable ordinal b. Thus $x \in \gamma(\gamma_{\aleph_0}(A))$ and $x \in \gamma_{\omega_1}(A)$. Hence $\gamma_{\omega_1}(A)$ is γ-closed and so includes $\overline{\gamma}(A)$.

3. Representation of the γ-linear Functionals

2.3.1. Lemma: Let H be a closed linear subspace of $(X,||\cdot||)$ and x_o not in H. There is a constant A such that $h \in H$ and $||a'x_o + h|| \leqq 1$ for a scalar a' imply $||h|| \leqq A$.

Proof: Let Y be the linear span of (H,x_o). Every element of Y may be uniquely written as $x = h(x) + a(x)x_o$ where h(x) is in H and the functional a is continuous on $(Y,||\cdot||)$ since its null set is closed in $(Y,||\cdot||)$. Thus, letting $x = a'x_o + h = a(x)x_o + h(x)$, $||h|| = ||x-a(x)x_o||$ $\leqq ||x|| + ||a||\cdot||x||\cdot||x_o|| \leqq 1 + ||a||\cdot||x_o|| = A$.

2.3.2. Theorem: The general form of the γ-linear functionals in $(X,||\cdot||,||\cdot||^*)$ is

$$f(x) = \lim g_n(x)$$

where $g_n \in C^*$ and $||f-g_n|| \to 0$.

Proof: It is sufficient to consider only non-trivial functionals. Let $f \in C_\gamma$, $f \neq 0$. Denote the null set of f by H and let $x_o \in X$ with $f(x_o) = 1$. Now note that for any positive integer n, $\overline{H \cap S_n}^* \subset S_n$, since if x is a $||\cdot||^*$-limit point of

$H \cap S_n$ it is in S_n by the normality of X. It then follows that $H \cap S_n = S_n \cap H^I = S_n \cap (\overline{H \cap S_n}^*) = \overline{H \cap S_n}^*$, i.e., the set $Z_n = H \cap S_n$ is closed in $(X, ||.||^*)$. Furthermore Z_n is convex and $x_o \notin Z_n$. Hence there is a $g_n \epsilon C^*$ such that

$$g_n(x) \begin{cases} < 1 \text{ for } x \epsilon Z_n \\ = 1 \text{ for } x = x_o \end{cases} .$$

This implies that $|g_n(x)| < 1/n$ for $x \epsilon Z_1$, for assume $|g_n(x)| \geq \frac{1}{n}$ for some $x \epsilon Z_1$. Then $nx \epsilon Z_n$ and $|g_n(nx)| \geq 1$, a contradiction. Let $f_n(x) = f(x) - g_n(x)$, then

$$|f_n(x)| \begin{cases} \leqq \frac{1}{n} \text{ for } x \epsilon Z_1 \\ = 0 \text{ for } x = x_o \end{cases} .$$

Let $x \epsilon X$ with $||x|| \leqq 1$. Then $x = h + ax_o$ with $h \epsilon H$ and, by the lemma, there is an A such that $||h|| \leqq A$. Thus $h/A \in Z_1$ and it follows that

$$|f_n(x)| = |f_n(h)| = A\ |f_n(h/A)| < A/n.$$

Hence $||f_n|| \leqq A/n$ which yields $||f - g_n|| \to 0$ and $f(x) = \lim g_n(x)$ for all x.

We shall now show the sufficiency of the representation. Let $x_p \xrightarrow{\gamma} 0$. Then $\sup ||x_p|| = K < \infty$ and $||x_p||^* \to 0$, so that for each n, $|f(x_p)| \leqq |g_n(x_p)| + |(g_n - F)(x_p)| \leqq |g_n(x_p)| + K\ ||g_n - F||$ and so $\lim\sup_p |f(x_p)| \leqq K\ ||g_n - f||$ since $g_n \in C^*$. Letting n tend to ∞ we have $\lim_p |f(x_p)| = 0$ and the theorem follows.

The previous theorem shows that C_γ is equal to the closure of C^* in $(C, ||.||)$. This assertion is not true when $||.||^*$

is only an F-norm. Let X be the space L^{∞} of essentially bounded measurable functions on [0,1], and let

$$||x|| = \text{ess sup } |x(t)|, \quad ||x||^* = \int_0^1 \frac{|x(t)|}{1+|x(t)|}\, dt.$$

In this case $C^* = \{0\}$. However, these are non-trivial functionals in C_γ since $||x||^* = \int_0^1 |x(t)|dt$ leads to the same γ-convergence as $||x||^*$. Note that the lack of local convexity causes the difficulties here.

The theorem also shows that f is γ-linear if and only if for each $E > 0$ it may be represented in the form

$$f(x) = g(x) + h(x)$$

where $g \in C^*$, $h \in C$, and $||h|| < E$. The necessity is clear by the theorem. Now consider the sufficiency. For every positive integer n there are $g_n \in C^*$ and $h_n \in C$ with $||h_n|| < \frac{1}{n}$ such that $f = g_n + h_n$. Then $||f-g_n|| = ||h_n|| \to 0$ as $n \to \infty$. Hence $f \in C_\gamma$ since C_γ is the closure of C^* in $(C, ||.||)$.

<u>2.3.3. Lemma:</u> If Y_o is a linear subspace of a normed space $(Y, ||.||)$, and if $y_o \in \overline{Y}_o$, then there is (y_n) in Y_o such that

$$y_o = \sum_{n=1}^{\infty} y_n \quad \text{and} \quad \sum_{n=1}^{\infty} ||y_n|| < \infty .$$

Moreover, for any $\varepsilon > 0$, (y_n) may be chosen so that

$$\sum_{n=1}^{\infty} ||y_n|| \leqq ||y_o|| + \varepsilon$$

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Proof: Let $\varepsilon > 0$ and let (y_p) be a sequence in Y_o converging to y_o. For each positive integer i there is an M_i such that $m,n \geqq M_i$ imply $||y_m - y_n|| < \varepsilon/2^{i+1}$ with $M_j > M_p$ for $j > p$. Let $K > M_1$ and such that $||y_K - y_o|| < \varepsilon/4$. Then $||y_K|| < \varepsilon/4 + ||y_o||$. Now let

$$y_1' = y_K$$
$$y_2' = y_{M_1} - y_K$$
$$y_3' = y_{M_2} - y_{M_1}$$

and, in general, for $n \geqq 3$, let $y_n' = y_{M_{n-1}} - y_{M_{n-2}}$. Then

$$\sum_{k=1}^{n} y_k' = y_{M_{n-1}} \to y_o \quad \text{and}$$

$$\sum_{k=1}^{\infty} ||y_k'|| = ||y_K|| + ||y_{M_1} - y_K|| + \sum_{n=2}^{\infty} ||y_{M_n} - y_{M_{n-1}}||$$
$$\leqq \varepsilon/4 + ||y_o|| + \varepsilon/4 + \varepsilon/2 = ||y_o|| + \varepsilon$$

This lemma immediately yields the following alternative form of theorem 2.3.2.

2.3.4. Theorem: The general form of the γ-linear functionals in $(X, ||\cdot||, ||\cdot||^*)$ is

$$f(x) = \sum_{n=1}^{\infty} g_n(x)$$

where $g_n \in C^*$ and $\sum_{n=1}^{\infty} ||g_n|| < \infty$. For any $\varepsilon > 0$ this representation may be chosen so that $\sum_{n=1}^{\infty} ||g_n|| \leqq ||f|| + \varepsilon$

Now let us suppose that the seminorms $[.]_i$ form a non-decreasing sequence. Let $C^{(n)}$ be the set of linear functionals continuous when X is given the $[.]_n$-topology. I assert that $C^* = \bigcup_{n=1}^{\infty} C^{(n)}$. It is rather clear that $C^* \subset \bigcup_{n=1}^{\infty} C^{(n)}$. Let $f \epsilon \bigcup_{n=1}^{\infty} C^{(n)}$. There is an m such that $f \epsilon C^{(m)}$; so also $f \epsilon C^{(p)}$ for $p > m$ since $C^{(1)} \subset C^{(2)} \subset \ldots$. Hence f is continuous under $||x||^{**} = \sum_{k=m}^{\infty} \frac{1}{2^k} \cdot \frac{[x]_k}{1+[x]_k}$. But $||.||^{**} \leqq ||.||^{*}$, and so f is continuous under $||.||^{*}$, i.e., $f \epsilon C^*$.

2.3.5. Theorem: The functionals g_n in 2.3.4. may be chosen so that $g_n \epsilon C^{(n)}$.

Proof: $C^{(1)} \subset C^{(2)} \subset \ldots$ implies there are k_n such that $g_n \epsilon C^{(k_n)}$ and $k_1 < k_2 < \ldots$. Let

$$g'_n = \begin{cases} g_i & \text{for } n=k_i,\ i=1,2,\ldots \\ 0 & \text{elsewhere} \end{cases}$$

Then $f = \sum_{n=1}^{\infty} g'_n$ is the desired representation.

4. Wiweger's Theorem

In this section we shall show there is a locally convex linear separated topology μ on X such that sequential convergence under μ is equivalent to γ-convergence and such that μ-conjugate space is equal to C_γ. Such topologization is essentially due to A. Wiweger, see [18], but the one given here is not Wiweger's.

A subset A of C is called "norming" in $(X,||.||)$ if the norm

$$||x||_A = \sup\{f(x) : f\in A, ||f|| \leq 1\}$$

is equivalent to the given norm. A is termed "strictly norming" in $(X,||.||)$ if for every sequence (x_n) such that $\sup_n|f(x_n)| < \infty$ for every f in A, (x_n) is necessarily bounded. Every strictly norming set is norming, but the converse is, in general, not true.

2.4.1. Lemma: The set $Y = \{f : f\in C^*, ||f|| = 1\}$ is norming in $(X,||.||)$, in fact, $||x|| = \sup\{f(x) : f\in Y\}$ for $x\in X$.

Proof: Let $||x_o|| = 1$ and let $\mathcal{E} > 0$. The set $S = \{x : ||x|| \leq 1\}$ is closed in $(X,||.||^*)$ since X is normal. Now $(1+\mathcal{E})x_o$ is not in S and hence can be separated from S, i.e., there is an f in C^* such that

$$f(x) \begin{cases} < 1 & \text{for } x\in S \\ = 1 & \text{for } x = (1+\mathcal{E})x_o. \end{cases}$$

Then $||x|| \leq 1$ implies $|f(x)| < 1$ and so $f\in Y$. However, $f(x_o) = \frac{||x_o||}{1+\mathcal{E}}$, where $\mathcal{E}$ was arbitrary. Thus $\sup\{g(x_o) : g\in Y\} \geq ||x_o||$. The reverse inequality is obvious, and since both norms are homogeneous the lemma is established.

2.4.2. Lemma: C_γ is strictly norming for $(X,||.||)$.

Proof: Let (x_n) be a sequence in X such that $\sup_n|f(x_n)| < \infty$ for every $f\in C_\gamma$. Let B denote the space of all bounded scalar-valued functions g defined on (x_n). It is elementary that B is a Banach space under $||g|| = \sup_n |g(x_n)|$. Let $T: C_\gamma \to B$ be defined by $Tf(x_n) = f(x_n)$. T is clearly linear, and $||f-f_n|| \to 0$ implies that $f_n(x) \to f(x)$ for each $x \in X$. Thus if $Tf_n \to g$, then $g = Tf$. Hence T is closed. Since C_γ and B are complete, it follows that T is continuous. Thus for any n and any $f \in C_\gamma$, $|f(x_n)| \leq \sup_n |f(x_n)| = ||Tf|| \leq ||T||.||f||$. But $||x_n|| = \sup \{f(x_n): f \in C_\gamma, ||f|| \leq 1\}$, as one can readily deduce from the previous lemma. Hence $||x_n|| \leq ||T||$ for all n.

Let $(X, |||.|||, ||.||^*)$ be the two-norm space under consideration and let T denote the topology induced by $||.||^*$. Further, let $W = \sigma(X, C_\gamma)$ be the weak topology on X under C_γ. Now let the μ topology be TvW, i.e., μ has as subbase TvW. Recall that μ is the weakest topology stronger than T and W. It is clear that μ is a locally convex linear separated topology on X. Furthermore, we have:

<u>2.4.3. Theorem</u>: $x_n \xrightarrow{\gamma} x$ if and only if $x_n \xrightarrow{\mu} x$.

Proof: If $x_n \xrightarrow{\mu} 0$, then $x_n \to 0$ in T, and $\{|||x_n|||\}$ is bounded since C_γ is strictly norming on $(X, |||.|||)$. Hence $x_n \xrightarrow{\gamma} 0$.

If $x_n \xrightarrow{\gamma} 0$, then $x_n \to 0$ in W and in T. Let G be a μ-neighborhood of 0. There are G_1 and G_2, T and W neighborhoods of 0 respectively, such that $G \supset G_1 \cap G_2$. But (x_n) is eventually in $G_1 \cap G_2$. Hence $x_n \xrightarrow{\mu} 0$.

2.4.4. Theorem: A functional is γ-linear if and only if it is μ-continuous and linear.

Proof: Let f be linear and μ-continuous, and let $x_n \xrightarrow{\gamma} 0$. Then $x_n \xrightarrow{\mu} 0$ and so $f(x_n) \to 0$, i.e., f is γ-linear.

Let f be γ-linear and let N be a neighborhood of 0 in the scalars. There is a W-neighborhood of 0, G_2, such that $f(G_2) \subset N$ since f is W-continuous. Let G_1 be any T-neighborhood of 0. Then $G_1 \cap G_2$ is a μ-neighborhood of 0 and $f(G_1 \cap G_2) \subset N$, i.e., f is μ-linear.

As we shall see later, the requirements that the topology satisfy the conclusions of the above two theorems do not determine the topology uniquely.

2.4.5. Theorem: If $C^* = C_\gamma$, then $||.||$ and $||.||^*$ are equivalent.

Proof: By normality and theorem 2.1.2., we may assume that $(X,||.||)$ is complete. Let us first consider the case where $||.||^*$ is a norm. We may assume that $||x||^* \leqq ||x||$ for all x in X. By the Mackey-Arens theorem, of all the locally convex linear separated topologies on X having as conjugate space C^* there is a weakest and strongest and these are $\sigma(X,C^*)$ and $\tau(X,C^*)$ respectively. Recall that the basis of closed neighborhoods of 0 for $\tau(X,C^*)$ consists of all sets of the form $\bigcap_{f\in\phi} \{x: |f(x)| \leqq 1\}$ where ϕ is a subset of C^* compact in $\sigma(C^*,X)$. We shall now show that ϕ is bounded in $(C^*,||.||^*)$. Note that the topology $\sigma(C^*,X)$ is identical with the topology induced by $\sigma(C,X)$ on C^*. Hence ϕ is compact in $\sigma(C,X)$ and it follows that ϕ is bounded in

$\sigma(C,X)$. So for each x in X there is an M such that $|f(x)| < M$ for each $f \in \phi$, i.e., ϕ is pointwise bounded on $(X, ||.||)$, a tonelé space. Moreover, each f in ϕ is in C. Hence ϕ is bounded in $(C, ||.||)$ by the Uniform Boundedness Principle, and hence in $(C^*, ||.||)$. Since X is normal, $C^* = C_\gamma$ is closed in $(C, ||.||)$ so that C^* is complete with respect to both $||.||$ and $||.||^*$. But then $(C^*, ||.||^*)$ and $(C^*, ||.||)$ are topologically isomorphic since each is an FH subspace of $(C, ||.||)$. So the identity map, $i:(C^*, ||.||) \to (C^*, ||.||^*)$ is a linear homeomorphism, and hence preserves bounded sets. Thus ϕ is bounded in $(C^*, ||.||^*)$, i.e., ϕ is included in a sphere $\Sigma_r = \{f : ||f||^* \leqq r\}$. Thus

$$\bigcap_{f\in\phi} \{x : |f(x)| \leqq 1\} \supset \bigcap_{f\in\Sigma_r} \{x : |f(x)| \leqq 1\} = \{x : ||x||^* < \tfrac{1}{r}\}$$

Thus $\tau(X, C^*)$ is weaker than the $||.||^*$-topology on X. The converse follows by the Mackey-Arens theorem.

Now suppose that $(X, ||.||^*)$ is a B_o^*-space with the sequence of seminorms $([.]_n)$ defining its topology. Since each of the seminorms $[.]_n$ is weaker than $||.||$, there exist constants K_n such that $[x]_n \leqq K_n||x||$ for x in X. Let

$$||x||_o^* = \sum_{n=1}^{\infty} \frac{1}{2^n K_n} \cdot [x]_n .$$

Then $||.||_o^*$ is a norm. Let (x_p) be a null sequence in $(X, ||.||_o^*)$. It is clear that $[x_p]_n \to 0$ as $p \to \infty$ for each n. Hence (x_p) is null in $(X, ||.||^*)$, and therefore $||.||_o^*$ is stronger than $||.||^*$. It is obvious, though, that $||x||_o^* \leqq ||x||$ for all x in X.

We shall now show that $||.||_o^*$ and $||.||^*$ are equivalent on bounded subsets of $(X,||.||)$. Let B be a bounded subset of $(X,||.||)$. Let $x \in B$ and consider $B-x = \{y-x : y \in B\}$. Let (x_p) be a sequence in B-x such that $||x_p||^* \to 0$. Then $[x_p]_n \to 0$ as $p \to \infty$ for each n. Note that

$||x||_o^* = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{[x]_n}{K_n}$ is uniformly convergent on B-x by the Weierstrass M-test, since B-x is bounded in $(X,||.||)$. Hence

$$\lim_{p\to\infty} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{[x_p]_n}{K_n} = \sum_{n=1}^{\infty} \left(\lim_{p\to\infty} \frac{1}{2^n} \cdot \frac{[x_p]_n}{K_n}\right) = 0.$$

Hence $||.||^* \geq ||.||_o^*$ on B, but we have already seen the reverse inequality. Therefore $||.||^*$ and $||.||_o^*$ are equivalent on bounded subsets of $(X,||.||)$. Hence the γ-convergence in $(X,||.||,||.||^*)$ and $(X,||.||,||.||_o^*)$ are identical.

We shall now show that γ-convergence in $(X,||.||,||.||^*)$ is metrical. It will then follow by 1.2.5 and 1.2.4 that $||.||$ and $||.||^*$ are equivalent. Let $x_n \xrightarrow{\gamma} x$ in $(X,||.||,||.||^*)$. Then $x_n \to x$ in $(X,||.||_o^*)$. Now let C_o^* denote the conjugate space to $(X,||.||_o^*)$. Then $C^* \subset C_o^* \subset C_\gamma$ and so $C^* = C_\gamma$ implies that $C_o^* = C_\gamma$. By the first part of the proof, the $||.||_o^*$-topology is stronger than the μ-topology. Therefore $x_n \to x$ in $(X,||.||_o^*)$ implies $x_n \xrightarrow{\mu} x$, which in turn implies $X_n \xrightarrow{\gamma} x$. Hence γ-convergence is metrical, and so $||.||$ and $||.||^*$ are equivalent.

Thus $C_\gamma = C^*$ only in the trivial case. On the other hand, we may have $C_\gamma = C$ in a non-trivial case. The space

$(L^2, ||\cdot||_2, ||\cdot||_1)$ gives an example of this case, (see 1.3.3.).

The above theorem is not true if $(X, ||\cdot||^*)$ is only an F-space, i.e., a complete linear metric space. A counter-example is given by $(L^1, ||\cdot||_1, ||\cdot||^*)$, (see 1.3.4). In this case C^* and C_γ consist only of the identically zero functional. Obviously, however, the two norms are not equivalent, for otherwise C would also be just the zero functional. Again, it's the absence of local convexity that causes the trouble.

5. Some Pathological Remarks

In this section we shall use the space $(\ell^1, ||\cdot||, ||\cdot||^*)$ with $||x|| = \sum_{n=1}^{\infty} |x_n|$ and $||x||^* = \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n|$ to produce some rather disturbing facts. For a sequence (x^n) in ℓ^1 it is rather easily seen that $x^n \xrightarrow{\gamma} x^o$ if and only if $\sup_n ||x^n|| < \infty$ and $\lim_{n\to\infty} x_p^n = x_p^o$ for each positive integer p.

I shall first show that the space is normal. Let $||x^n - x^o||^* \to 0$ with $||x^n|| \leqq 1$ for all n. Assume $||x^o|| > 1$, i.e., $\sum_{p=1}^{\infty} |x_p^o| > 1$. There is an M such that $\sum_{p=1}^{M} |x_p^o| > 1$. Let $(\sum_{p=1}^{M} |x_p^o|) - 1 = \delta > 0$. Let n be fixed but arbitrary. Then $\sum_{p=1}^{M} |x_p^n| \leqq 1$, and so

$$0 < \delta \leqq \sum_{p=1}^{M} |x_p^o| - \sum_{p=1}^{M} |x_p^n| \leqq |\sum_{p=1}^{M} (|x_p^o| - |x_p^n|)|$$

$$\leqq \sum_{p=1}^{M} ||x_p^o| - |x_p^n|| \leqq \sum_{p=1}^{M} |x_p^o - x_p^n| .$$

So that

$$\delta/2^M \leq \frac{1}{2^M}\left(\sum_{p=1}^{M} |x_p^o - x_p^n|\right) < \sum_{p=1}^{M} \frac{1}{2^p} |x_p^o - x_p^n|, \text{ i.e.,}$$

for any n, $||x^n - x^o||^* > \delta/2^M$, a flagrant contradiction. Thus $||x^o|| \leq 1$ and the space is normal by 1.3.1.

Recall that $(\ell^1, || \ ||)$ is conjugate to c_o, the space of null sequences.

2.5.1. Lemma: For sequences, γ-convergence is equivalent to weak * convergence in ℓ^1, i.e., to convergence in $\sigma(\ell^1, c_o)$.

Proof: Let $x^n \xrightarrow{\gamma} x^o$ and let $a \epsilon c_o$. Then $a = \sum_{k=1}^{\infty} a_k \delta^k$ where $\delta^k = (\delta_i^k)$. It is easily seen that $\delta^k \epsilon C^*$ so that $a \epsilon C_\gamma$. Thus $a(x^n) \to a(x^o)$ for each $a \epsilon c_o$, and so $x^n \to x^o$ in $\sigma(\ell^1, c_o)$.

Let $x^n \to x^o$ in $\sigma(\ell^1, c_o)$. For each $a \epsilon c_o$, $\sum_{k=1}^{\infty} a_k x_k^n \xrightarrow{n} \sum_{k=1}^{\infty} a_k x_k^o$. Hence $\sum_{i=1}^{\infty} \delta_i^k x_i^n \xrightarrow{n} \sum_{i=1}^{\infty} \delta_i^k x_i^o$, i.e., $x_k^n \xrightarrow{n} x_k^o$ for each k. Thus $||x^n - x^o||^* \to 0$. Furthermore, (x^n) is $\sigma(\ell^1, c_o)$-bounded and hence is equicontinuous, i.e., there is an M such that $||x^n|| < M$ for all n.

Mazurkiewicz, in [13], has constructed a linear subspace of ℓ^1 which we may use to advantage here. Arrange all pairs of positive integers (i,k) in a sequence and let N(i,k) denote the position of the pair (i,k). Let

$$x^{ik} = \frac{\delta^1}{2^1} + \dots + \frac{\delta^{2i-1}}{2^i} + i\delta^{2N(i,k)}$$

The set M, of Mazurkiewicz, is the linear span of the set of all x^{ik}. Now write

$$x^{o} = \sum_{k=1}^{\infty} \frac{\delta^{2k-1}}{2^{k}} \quad \text{and} \quad x^{n} = \sum_{k=1}^{n} \frac{\delta^{2k-1}}{2^{k}}$$

<u>2.5.2. Fact</u>: $x^{n} \epsilon \gamma(M)$ and $x^{n} \xrightarrow{\gamma} x^{o}$. However $x^{o} \notin \gamma(M)$. Thus, in general, $\gamma(\gamma(A)) \neq \gamma(A)$.

Proof: I shall first show that for any n, $x^{n} \epsilon \gamma(M)$. Consider the subsequence of the sequence of integer pairs consisting of (n,k) with $N(n,k) > n$. It is clear that for this subsequence $||x^{nk}|| \leqq n+1$ for all k. For each positive integer p it is equally clear that $\lim_{k\to\infty} x_p^{nk} = x_p^n$. Hence this subsequence γ-converges to x^n.

It is trivial that $x^{n} \xrightarrow{\gamma} x^{o}$, and it remains only to show that $x^{o} \notin \gamma(M)$. Suppose there is a sequence (x^{p}) in M with $x^{p} \xrightarrow{\gamma} x^{o}$. For each p, x^{p} is a certain linear combination of elements of the basis of M. Let a_p be the maximum of the absolute values of the scalar coefficients in this linear combination and let i_p be the maximum of the first elements of the integer pairs (i,j) such that x^{ij} is in this linear combination (with non-zero scalar coefficient). It is clear that $i_p \to \infty$ as $p \to \infty$. We may also write $x^{p} = x^{p_1} + x^{p_2}$ where x^{p_1} is an element of ℓ^1 with 0 in each even position and x^{p_2} is an element of ℓ^1 with 0 in each odd position, i.e.,

$$x^{p_1} = \sum_{i=1}^{\infty} b_i \delta^{2i-1} \quad \text{and} \quad x^{p_2} = \sum_{i=1}^{\infty} b_i \delta^{2i}$$ where the bi's are scalars. Now $x^{p} \xrightarrow{\gamma} x^{o}$ implies $x_n^p \to x_n^o$ as $p \to \infty$ for each n.

Hence $x^{p_2} \overset{\gamma}{\to} 0$. This implies $a_p \to 0$. For suppose not; there is an $\varepsilon > 0$ such that a_p is frequently greater than ε, i.e., for any Q there is $r > Q$ such that $a_r > \varepsilon$. Then $||x^{p_2}|| > \varepsilon i_p$. But $i_p \to \infty$ as $p \to \infty$, which contradicts $x^{p_2} \overset{\gamma}{\to} 0$. Hence $a_p \to 0$, and then $x^{p_1} \overset{\gamma}{\to} 0$ and so $x^p \overset{\gamma}{\to} 0$. Contradiction to our assumption that $x^p \overset{\gamma}{\to} x^o$. Hence x^o is not in $\gamma(M)$.

2.5.3. Remark: In $(\ell^1, ||.||, ||.||^*)$ there is a linear subspace H and a γ-linear functional on $(H, ||.||, ||.||^*)$ which cannot be extended to ℓ^1 with the preservation of γ-linearity.

Proof: Let (x^n) and x^o be as above and let H be the linear span of (M, x^o). Then $(H, ||.||, ||.||^*)$ is normal. Since x^o is not in $\gamma(M)$, there is, by 2.2.1., a γ-linear functional f on H such that $f(x) = 0$ for x in M and $f(x^o) = 1$. However, there is no extension F of f onto ℓ^1 which is γ-linear. Suppose there is; then $F(x^n) = 0$ for all n. But $x^n \overset{\gamma}{\to} x^o$ and so $F(x^o) = 0 \neq f(x^o)$.

The remark shows that the extension theorem is, in general, not true for the γ-linear functionals.

Let T denote the $||.||^*$-topology on X for a two-norm space $(X, ||.||, ||.||^*)$ and let V be the topology generated by all sets of the form $\{x: |f_n(x)| < a_n \text{ for all } n\}$ where (f_n) is a sequence in C^* with $||f_n||^* \leqq 1$ and where $0 < a_n$ and $a_n \to \infty$, i.e., sets of this form compose a basis of neighborhoods of 0. Let $\mu^* = T \vee V$.

<u>2.5.4. Lemma</u>: μ^* is a locally convex linear separated topology and μ^* is weaker than μ.

Proof: I shall first show that each neighborhood of 0 is absorbing. Let G be a μ^*-neighborhood of 0. There are G_1 and G_2, T and V neighborhoods of 0 respectively, such that $G \supset G_1 \cap G_2$. Let $x \in X$. There is an $\varepsilon > 0$ such that $|b| < \varepsilon$ implies bx is in G_1 since the T-topology is linear. G_2 is of the form $\{x: |f_n(x)| < a_n \text{ for all } n\}$ where $||f_n||^* \leqq 1$ and $0 < a_n \to \infty$. Let $m = g.\ell.b.\{a_n: n=1,2,\dots\}$. For $|b| < \frac{m}{||x||}$, we have $|f_n(bx)| \leqq ||bx|| < m \leqq a_n$ for all n, i.e., bx is in G_2. Letting $\delta = \min\left(\frac{m}{||x||}, \varepsilon\right)$ we have the fact that $|b| < \delta$ implies bx is in $G_1 \cap G_2$.

Now G_1 contains a balanced neighborhood of 0, N, and G_2 is balanced. Thus $N \cap G_2$ is a balanced μ^*-neighborhood of 0 included in G.

It is clear that G_2 is convex and we may assume also that N is convex. Thus $N \cap G_2$ is convex, and it follows that $\frac{1}{2} N \cap G_2 + \frac{1}{2} N \cap G_2 = N \cap G_2 \subset G$.

Thus μ^* is a locally convex linear topology. It is also separated since μ^* is stronger than T.

It remains to show that μ^* is weaker than μ. Let (x_α) be a net converging to 0 in the μ-topology, and let G be a μ^*-neighborhood of 0. Then $G \supset G_1 \cap G_2$, G_1 and G_2 as above. Clearly (x_α) is eventually in G_1 since μ is stronger than T. Let m be as above. Since $x_\alpha \to 0$ in the μ-topology,

$f(x_\alpha) \to 0$ for every $f \epsilon C_\gamma$. Thus there is an α_0 such that $\alpha > \alpha_0$ implies $|f(x_\alpha)| < m$. Therefore (x_α) is eventually in G_2, and so (x_α) is eventually in G. Thus μ is stronger than μ^*.

2.5.5. Lemma: $x_n \overset{\gamma}{\to} x$ if and only if $x_n \overset{\mu^*}{\to} x$.

Proof: Necessity is clear since μ is stronger than μ^*. Let $x_n \overset{\mu^*}{\to} 0$. Then $x_n \to 0$ in T and in V. By Wiweger's lemma $\{|f(x_n)| : f\ C^*, ||f|| \leq 1,\ n = 1,2,\ldots\}$ is bounded. Since C^* is norming on $(X, ||\cdot||)$ this implies that (x_n) is $||\cdot||$-bounded. Hence $x_n \overset{\gamma}{\to} x$.

Note that μ^* depends only on C^*. Hence for every linear subspace A the μ^*-topology constructed for $(A, ||\cdot||, ||\cdot||^*)$ is identical with the induced μ^*-topology for $(X, ||\cdot||, ||\cdot||^*)$. We then have:

2.5.6. Remark: The μ^*-topology on $(\ell^1, ||\cdot||, ||\cdot||^*)$ is such that $x_n \overset{\gamma}{\to} x$ is equivalent to $x_n \overset{\mu^*}{\to} x$. The γ-linear functionals, however, are not identical with the μ^*-continuous linear functionals.

Proof: The first assertion is simply 2.5.5.. If every γ-linear functional were μ^*-continuous, then every γ-linear functional could be extended to all of ℓ^1. But we have seen this is not so.

A topology, τ, on $(X, ||\cdot||, ||\cdot||^*)$ will be termed "appropriate" if

(i) it is a locally convex linear separated topology for X,

(ii) for sequences, γ-convergence is equivalent to $\mathcal{T}$-convergence,

(iii) C_γ is equal to the set of linear functionals continuous in the topology $\mathcal{T}$.

2.5.7. Remark: There may exist different appropriate topologies for the space $(X, ||.||, ||.||^*)$.

Proof: I shall first show that $\sigma = \sigma(\ell^1, c_0)$ is appropriate for $(\ell^1, ||.||, ||.||^*)$. Property (i) is clear and property (ii) was shown in Lemma 2.5.1.. Let $f \in C_\gamma \subset C$. Then $f \in m$, the space of bounded sequences, i.e.,

$f(x) = \sum_{k=1}^{\infty} a_k x_k$ where (a_k) is in m. Consider $\delta^n \xrightarrow{\gamma} 0$ and

$f(\delta^n) = \sum \delta^n_k a_k = a_n \to 0$, so that actually, $(a_k) \in c_0$. Hence f is σ-continuous. Since the two convergences are equivalent, sequentially, it is clear that a σ-continuous linear functional is also γ-linear. Thus $\sigma(\ell^1, c_0)$ is appropriate, and $C_\gamma = c_0$.

Thus we have two appropriate topologies, σ and μ, on $(\ell^1, ||.||, ||.||^*)$. Since each $a \in c_0$ is γ-linear, and hence μ-continuous, it is clear that σ is coarser than μ. We shall show that μ is strictly finer than σ. Let $V = \{x: ||x||^* \leq 1\} \cap G$ where G is any $\sigma(\ell^1, c_0)$ neighborhood of 0. Then V is a μ-neighborhood of 0. Furthermore, there are no σ-neighborhoods of 0 included in V since every σ-neighborhood of 0 includes a linear subspace of finite deficiency. Hence μ is strictly finer than σ.

2.5.8. Remark: There is a linear space X with norm $||.||$ and two weaker norms, $||.||^*$ and $||.||_1^*$, such that the γ-convergences in $(X,||.||,||.||^*)$ and $(X,||.||,||.||_1^*)$ are different but $(X,||.||,||.||^*)$ and $(X,||.||,||.||_1^*)$ have the same set of γ-linear functionals.

Proof: Again consider ℓ^1 and let $||x||_1^* = \sup|x_n|$. Denote the γ-convergence in $(X,||.||,||.||_1^*)$ by γ_1. The γ_1-convergence implies γ-convergence, but not conversely, since (δ^n) is γ-convergent to zero but is not γ_1-convergent

We have previously seen, in 2.5.7., that $f\epsilon C_\gamma$ implies $f\epsilon c_o$ and conversely. Let f be γ_1-linear. Since $f\epsilon C$, $f\epsilon m$. Suppose $f\epsilon m \sim c_o$, say $f = (a_n)$. There is an $\varepsilon > 0$ and a subsequence (a_{n_i}) such that $a_{n_i} > \varepsilon$ for every i or $-a_{n_i} > \varepsilon$ for every i. If it is the latter case, just consider -f rather than f. I assume the former. Now consider the sequence (x^n) in ℓ^1 where

$$x_j^n = \begin{cases} 0 & j < n \text{ or } j \neq n_i \text{ for some } i \\ \frac{1}{2^j} & j \geq n \text{ and } j = n_i \text{ for some } i \end{cases}$$

Then $x^n \xrightarrow{\gamma_1} 0$, but

$$f(x^n) \geq \frac{\varepsilon}{2^{n_1}}$$

Thus $f(x^n) \not\to 0$, a contradiction. Hence $f\epsilon c_o$, i.e., $C_{\gamma_1} \subset c_o$. Since γ_1-convergence forces γ-convergence, it is easily seen that $C_\gamma \subset C_{\gamma_1}$, i.e., $c_o \subset C_{\gamma_1}$. Hence we have $C_\gamma = C_{\gamma_1} = c_o$.

In the next section we shall see sufficient conditions for the pointwise limit of γ-linear functionals to be γ-linear. In general, however, the set of functionals which are pointwise limits of a sequence of γ-linear functionals is larger than C_γ. Consider, for example, $(\ell^1, ||.||, ||.||^*)$. In this case, $C = m$ and $C_\gamma = c_0$, so that $C_\gamma \subsetneqq C$, but every element of C is the pointwise limit of a sequence in C_γ. Let $f = (a_n) \epsilon m$, i.e., $f(x) = \sum_{k=1}^{\infty} a_k x_k$. Let $f_n = \{a_1, a_2, \ldots, a_n, 0, 0, 0 \ldots\} \epsilon c_0$. Clearly $f(x) = \lim f_n(x)$.

6. γ-γ Continuous Operations

For this section drop the assumptions mentioned at the beginning of the chapter. The necessary assumptions will be mentioned as we proceed.

Suppose X and Y are Banach spaces and each is provided with a γ-convergence. A linear operation U from X into Y will be called "γ-γ continuous" if it satisfies the condition that $x_n \overset{\gamma}{\to} x$ implies $U(x_n) \overset{\gamma}{\to} U(x)$. If the γ-convergence in Y is metrical (and hence equivalent to the $||.||$-convergence), any linear γ-γ continuous operation will be called "γ-continuous".

Now consider a sequence (U_n) of linear γ-γ continuous operations with $(U_n(x))$ γ-convergent for each $x \epsilon X$. The question naturally arises as to whether or not the operation defined by the pointwise limit is γ-γ continuous. As we have seen by the example of the γ-linear functionals on $(\ell^1, ||\cdot||, ||\cdot||^*)$, the answer is "not necessarily". However, we can state sufficient conditions to provide this result. To this end we state:

Postulate (iv): Let S be the unit disc in $(X, ||\cdot||)$ and give S the metric $||\cdot||^*$. Then for any open set G in S, G-G is a neighborhood of 0 in S, where $G-G = \{x-y : x\in G, y\in G, x-y\in S\}$.

2.6.1. Theorem: Let $(X, ||\cdot||)$ and $(Y, ||\cdot||)$ be Banach spaces and X a two-norm space satisfying (i), (ii), and (iv). Let $(\mathbf{U}_n)$ be a sequence of linear γ-continuous operations from X into Y which is pointwise convergent. Then the operation defined by $\mathbf{U}(x) = \lim \mathbf{U}_n(x)$ is linear and γ-continuous.

Proof: I shall first show that $||\cdot||$ and $||\cdot||^*$ are equivalent on S. By (ii), $(S, ||\cdot||^*)$ is complete, ~~but also~~ $(S, ||\cdot||)$ is complete since $(X, ||\cdot||)$ is a Banach space. Thus $||\cdot||$ and $||\cdot||^*$ are equivalent on S since they are complete comparable linear metrics on the same space.

For $x\in S$ let $V_n(x) = \mathbf{U}_n(x)$ and $V(x) = \mathbf{U}(x)$. Note that $x_1, x_2, x_1 \pm x_2$ in S implies that $V_n(x_1 \pm x_2) = V_n(x_1) \pm V_n(x_2)$ and $V(x_1 \pm x_2) = V(x_1) \pm V(x_2)$. It is clear that the V_n are continuous in $(S, ||\cdot||^*)$ and converge to V. Thus they are equicontinuous at a point x_0 (see the introduction). Thus given $\delta > 0$ there is an $\varepsilon > 0$ such that

$||x-x_o||^* < \varepsilon$, $x \in S$ implies $||V_n(x)-V_n(x_o)|| < \delta$ for $n=1,2,\ldots$. Let G be an open set of $(S,||.||^*)$ with $G \subset \{x: ||x-x_o||^* < \varepsilon\}$. By (iv) $G-G$ is a neighborhood of 0 in $(S,||.||^*)$. Let $x \in G-G$. Then $x = x_1 - x_2$ where x_1 and x_2 are in G, and we have $||V_n(x)|| = ||V_n(x_1) - V_n(x_2)|| \leqq ||V_n(x_1) - V_n(x_o)|| + ||V_n(x_2) - V_n(x_o)|| < 2\,\delta$. So V is continuous at 0. Let $x \in S$ and (x_n) a sequence with $||x_n-x||^* \to 0$. Then also $||x_n-x|| \to 0$, so there is an M such that $n > M$ implies $x_n-x \in S$. Considering the sequence thus "truncated" we have $V(x_n-x) \to 0$ so that $V(x_n) \to V(x)$ and V is continuous on $(S,||.||^*)$. The γ-continuity of U now follows quite readily.

<u>2.6.2. Theorem</u>: Let $(X,||.||)$, $(Y,||.||)$ and $(Y,||.||^*)$ be Banach spaces, and let X and Y be two-norm spaces satisfying (i). Further, suppose that $(X,||.||,||.||^*)$ satisfies also (ii) and (iv). Let (U_n) be a sequence of linear γ-γ continuous operations from X into Y which is pointwise γ-convergent. Then the operation defined by $U(x) = \gamma\text{-}\lim U_n(x)$ is also linear and γ-γ continuous.

~~Proof: It is sufficient to show that $x_n \xrightarrow{\gamma} 0$ implies~~ $U_n(x_n) \xrightarrow{\gamma} 0$.

Let $x_n \xrightarrow{\gamma} 0$ and $a_n \to 0$ where (a_n) is a sequence of scalars We may write $a_n = b_n c_n^2$ with $b_n = \pm 1$. Let $V_n(x) = c_n U_n(x)$.

Fix p and let $||z_n|| \to 0$. Choose $t_n \to \infty$ such that $||t_n z_n|| \to 0$. Then $V_p(t_n z_n) \xrightarrow{\gamma} 0$ so that $||t_n^{-1} V_p(t_n z_n)|| = ||V_p(z_n)|| \to 0$ since $t_n^{-1} \to 0$ and $(V_p(t_n z_n))$ is $||.||$-bounded. Thus for each p, V_p is $||.||$-continuous.

For $x \in X$, $||V_n(x)|| \to 0$ since $c_n \to 0$ and $(U_n(x))$ is $||.||$-bounded. Hence by the Uniform Boundedness Principle there is a K such that $||V_n|| < K$ for all n. Thus $||a_n U_n(x_n|| = ||c_n V_n(b_n x_n)|| \to 0$, and we have the fact that $(U_n(x_n))$ is $||.||$-bounded in Y.

Now the operations U_n are γ-continuous as operations from X to $(Y, ||.||^*)$. Thus, as in the previous theorem, there is a neighborhood of 0, N, in $(S, ||.||^*)$ such that $||U_n(x)||^* < \varepsilon$ if x is in N, and the theorem follows.

7. Some Examples

For an abundance of example utilizing theorem 2.4.4., the reader is referred to section 6 of [4]. I shall give only two rather short examples.

2.6.1. Example: Let $1 \leqq b < a \leqq \infty$ and $X = L^a$. Let $||x|| = ||x||_a$ and $||x||^* = ||x||_b$ where in general, $||x||_c = \left(\int_0^1 |x(t)|^c\right)^{1/c}$. Marking the conjugate exponent by an apostrophe we have

$$C = L^{a'}$$

Recall that L^a is dense in L^b under $||.||_b$ so that

$$C^* = L^{b'}$$

Now $L^{b'} \subset L^{a'}$ and is dense in $L^{a'}$ under $||.||_{a'}$, so that

$$C_\gamma = L^{a'} = C$$

<u>2.6.2. Example</u>: Let $1 < a \leqq \infty$, $X = L^a$, $b_n \to a$ with $b_n > 1$ for every n and b_n a non-decreasing sequence. Let

$$||x|| = ||x||_a \quad \text{and} \quad [x]_n = ||x||_{b_n}$$

We then have

$$C^{(n)} = L^{b'_n}, \quad C = L^{a'}, \quad C^* = \bigcup_{n=1}^{\infty} L^{b'_n}$$

Evidently, C^* is dense in $L^{a'}$ under $||.||_{a'}$, and again $C_\gamma = L^{a'} = C$.

Chapter 3

The Conjugate Spaces

1. Preliminaries

Throughout this chapter we shall assume that $||.||$ and $||.||^*$ are both norms. In the previous chapter we had assumed only that $(X, ||.||^*)$ was a B_o^*-space. However, our restriction is not too great since (as shown in the proof of 2.4.5.) one may always introduce a norm $||.||_1^*$ in X, finer than $||.||^*$ such that γ-convergence in $(X, ||.||, ||.||_1^*)$ is equivalent to γ-convergence in $(X, ||.||, ||.||^*)$. In the present chapter normality, i.e., postulate (iii), is not assumed unless specifically stated.

We assume, naturally, that $||.||^*$ is weaker than $||.||$, and, in fact, we assume that for each $x \epsilon X$, $||x||^* \leqq ||x||$.

A sequence, (x_n), will be termed "γ-bounded" if $t_n x_n \xrightarrow{\gamma} 0$ for any null sequence of scalars, (t_n). A useful fact now arises:

<u>3.1.1. Lemma</u>: (x_n) is γ-bounded if and only if $\sup||x_n|| < \infty$.

Proof: Sufficiency is obvious. Let (x_n) be γ-bounded and suppose $\sup||x_n|| = \infty$. There is a subsequence (X_{n_k}) such that $||x_{n_k}|| \to \infty$. Consider the sequence of reals

$$t_m = \begin{cases} 0, & m \neq n_k \text{ for some } k \\ \dfrac{1}{||x_{n_k}||}, & m = n_k \text{ for some } k . \end{cases}$$

Then $t_m \to 0$, but $||t_{n_k} X_{n_k}|| \to 1$ so that $t_n X_n \not\xrightarrow{\gamma} 0$.

Since $||.||$ and $||.||^*$ are norms it follows that their conjugate spaces, $(C,||.||)$ and $(C^*,||.||^*)$ respectively, are Banach spaces. Recall that $C^* \subset C$, and it is clear that for $f \in C^*$, $||f|| \leqq ||f||^*$. We shall also use the following notation:

$$S = \{x : x \in X, \ ||x|| \leqq 1\}$$
$$S^* = \{x : x \in X, \ ||x||^* \leqq 1\}$$
$$\Sigma = \{f : f \in C, \ ||f|| \leqq 1\}$$
$$\Sigma^* = \{f : f \in C^*, \ ||f||^* \leqq 1\}$$

Obviously, $S \subset S^*$ and $\Sigma^* \subset \Sigma$.

We now state a lemma which will be of use later:

<u>3.1.2. Lemma</u>: Let U be a linear γ-γ continuous operation from $(X,||.||,||.||^*)$ to $(Y,||.||,||.||^*)$, then U is

continuous as an operation from $(X, ||.||)$ to $(Y, ||.||)$.

Proof: Consider $\mathcal{U}(S)$ where S is the unit disc of $(X, ||.||)$. Any countable subset of $\mathcal{U}(S)$ is bounded in $(Y, ||.||)$ since $\mathcal{U}$ preserves γ-boundedness and γ-boundedness is equivalent to $||.||$-boundedness for sequences. Thus $\mathcal{U}(S)$ is bounded in $(Y, ||.||)$ and so $||\mathcal{U}|| < \infty$.

2. The Conjugate Spaces

For a given two-norm space, $(X, ||.||, ||.||^*)$, the space $(C^*, ||.||^*, ||.||)$ is also a two-norm space with $||.||^*$ the stronger norm. $(C^*, ||.||^*, ||.||)$ will be called the "γ-conjugate space" to $(X, ||.||, ||.||^*)$.

3.2.1. Theorem: $(C^*, ||.||^*, ||.||)$ is a normal, γ-complete two-norm space.

Proof: Let $||f_n||^* \leqq K$ for all n and let $||f_n - f_m|| \to 0$ as $n, m \to \infty$. Then for any $x \in X$, $f_n(x) - f_m(x) \to 0$ and we define $f_o(x) = \lim f_n(x)$. So for any $x \in X$ we have $|f_o(x)|$ $= |\lim f_n(x)| \leqq \lim\inf |f_n(x)| < \lim\inf ||f_n||^* ||x||^*$. This shows that $||f_o||^* \leqq \lim\inf ||f_n||^* \leqq K$ so that the space is γ-complete and also normal.

Now let $(D, ||.||)$ and $(D^*, ||.||^*)$ denote the spaces conjugate to $(C, ||.||)$ and $(C^*, ||.||^*)$ respectively. Thus

$$||f|| = \sup \{f(g) : g \in C \cap \Sigma\} \text{ for } f \in D$$

and $$||f||^* = \sup \{f(g) : g \in C^* \cap \Sigma^*\} \text{ for } f \in D^*.$$

$(D, ||.||)$ and $(D^*, ||.||^*)$ are the second conjugate spaces to

$(X,||.||)$ and $(X,||.||^*)$ respectively. Let $(D^{(\gamma)},||.||)$ be the space conjugate t $(C^*,||.||)$; in this case

$$||f|| = \sup \{f(g): g\in C^* \cap \Sigma\} \quad \text{for} \quad f\in D^{(\gamma)} .$$

It is clear then that $(D^{(\gamma)},||.||,||.||^*)$ is γ-conjugate to $(C^*,||.||^*,||.||)$ and so it is the second γ-conjugate to $(X,||.||,||.||^*)$.

Finally, in accordance with our earlier notation, let D_γ denote the set of γ-linear functionals on $(C^*,||.||^*,||.||)$. Since this space is normal, D_γ is the closure of $D^{(\gamma)}$ in $(D^*,||.||^*)$. Also, we obviously have $D^{(\gamma)} \subset D_\gamma \subset D^*$.

<u>3.2.2. Theorem</u>: Let $(X,||.||,||.||^*)$ be normal. Then $(D^{(\gamma)},||.||)$ may be identified with the space conjugate to $(C_\gamma,||.||)$.

Proof: Since $(X,||.||,||.||^*)$ is normal, $(C_\gamma,||.||)$ is identical with the completion of $(C^*,||.||)$. Hence their conjugate spaces are, in a sense, identical, and

$$||f|| = \sup \{f(g): g\in C^* \cap \Sigma\} = \sup \{f(g): g\in C_\gamma \cap \Sigma\}$$

Note, however, that in general the topologies $\sigma(D^{(\gamma)},C^*)$ and $\sigma(D^{(\gamma)},C_\gamma)$ are not equivalent. As we shall see later $(X,||.||,||.||^*)$ is canonically embedded in $(D^{(\gamma)},||.||,||.||^*)$. Thus if these two topologies were equivalent they would induce equivalent topologies on X and it would follow that $C_\gamma = C^*$, a remark not true in general. Notice also that $\sigma(D^{(\gamma)},C^*)$ is weaker than $||.||^*$ on $D^{(\gamma)}$ but $\sigma(D^{(\gamma)},C_\gamma)$ is not.

The first γ-conjugate space depends on the norm $||.||^*$ essentially. Given a strong norm $||.||$ and two weaker norms $||.||_1^*$ and $||.||_2^*$ which give rise to the same γ-convergence, the spaces C_1^* and C_2^*, conjugate to $(X,||.||_1^*)$ and $(X,||.||_2^*)$ respectively, need not be equal. The second γ-conjugate space, $D^{(\gamma)}$, however, depends only on $(C_\gamma,|| \,||)$, i.e.,

<u>3.2.3. Theorem</u>: Let $||.||_1^*$ and $||.||_2^*$ be two coarser norms in a normed space $(X,||.||)$ satisfying (iii) and leading to the same class C_γ of γ-linear functionals. Then the spaces $D^{(\gamma)}$ are equal in both cases.

Proof: Theorem 3.2.2..

Recall 2.5.8. which showed that two coarser norms could lead to the same class C_γ but determine different γ-convergences. On the other hand, the norm $||.||^*$ in $D^{(\gamma)}$ determines $||.||^*$ in X uniquely.

It is well known that the canonical mapping which takes x into f_x where $f_x(g) = g(x)$ embeds $(X,||.||)$ isometrically and isomorphically into $(D,||.||)$ and likewise ~~$(X,||.||^*)$ into $(D^*,||.||^*)$. Since this mapping defines~~ linear functionals on $(C^*,||.||)$, it also embeds $(X,||.||,||.||^*)$ into $(D^{(\gamma)},||.||,||.||^*)$. As thus restricted, the mapping will be termed "γ-canonical".

<u>3.2.4. Theorem</u>: If $(X,||.||,||.||^*)$ is normal, then the γ-canonical mapping embeds $(X,||.||,||.||^*)$ into

$(D^{(\gamma)}, ||.||, ||.||^*)$ with the preservation of $||.||$ and $||.||^*$, i.e., $||f_x|| = ||x||$ and $||f_x||^* = ||x||^*$. Conversely, the preservation of the norms by the γ-canonical mapping implies $(X, ||.||, ||.||^*)$ is normal.

Proof: By Lemma 2.4.1. we have

$$||f_x|| = \sup \{g(x) : g \in C^*, ||g|| \leqq 1\} = ||x||.$$

To show the preservation of $||.||^*$, note that $f_x \in D^{(\gamma)} \subset D^*$. Thus $||f_x||^* = ||x||^*$ since $(X, ||.||^*)$ is isometrically embedded in $(D^*, ||.||^*)$.

The converse statement is apparent since $(D^{(\gamma)}, ||.||, ||.||^*)$ is normal by 3.2.1 and any subspace of a normal space is also normal.

Now let $(\#^*, ||.||^*, ||.||)$ denote the γ-conjugate space to $(D^{(\gamma)}, ||.||, ||.||^*)$. The canonical mapping of C^* into $\#^*$ is given by

$$g \rightarrow z_g(f) = f(g)$$

Let $\#_\gamma$ be the space of γ-linear functionals on $(D^{(\gamma)}, ||.||, ||.||^*)$. Since the γ-canonical mapping of $(C^*, ||.||^*, ||.||)$ embeds C^* in $(\#^*, ||.||^*, ||.||)$ with the preservation of both norms, and since the canonical mapping embeds $(C^*, ||.||^*)$ into $(\#^*, ||.||^*)$ conjugate to $(D^{(\gamma)}, ||.||^*)$ with the preservation of $||.||^*$, we have:

<u>3.2.5. Theorem</u>: Let $(X, ||.||, ||.||^*)$ be normal, then the canonical mapping of $(C^*, ||.||^*)$ into $(\#^*, ||.||^*)$

embeds C_γ into $\#_\gamma$.

Proof: The closure of $\#^*$ under $||.||$ is equal to $\#_\gamma$. Thus the closure of C^*, as a subset of $\#^*$, under $||.||$ is contained in $\#_\gamma$.

3. γ-Reflexive Spaces

$(X,||.||,||.||^*)$ will be called "γ-reflexive" if it is normal and if the γ-canonical mapping embeds $(X,||.||,||.||^*)$ onto $(D^{(\gamma)},||.||,||.||^*)$, or equivalently, if each linear functional on $(C_\gamma,||.||)$ is of the form $f(g) = g(x)$ for an $x \in X$. Every γ-reflexive space is obviously γ-complete.

Let $(X,||.||,||.||^*)$ be normal. When X is given the topology $\sigma(X,C_\gamma)$, (recall C_γ is the conjugate space for the topology μ) the conjugate space is C_γ. We shall now consider the strong topology $\beta(C_\gamma,X)$ on C_γ. Recall that in this topology the basis of neighborhoods of zero is composed of the polar sets of all $\sigma(X,C_\gamma)$-bounded subsets of X.

3.3.1. Fact: If $(X,||.||,||.||^*)$ is normal, then the strong topology $\beta(C_\gamma,X)$ on C_γ is equivalent to the $||.||$-topology on C_γ.

Proof: Let A be a $\sigma(X,C_\gamma)$-bounded subset of X. Then $\sup\{|g(x)| : x \in A\} < \infty$ for every $g \in C_\gamma$. Thus $A \subset n\,S$ for some n since C_γ is strictly norming. Denoting the polar, under $\sigma(X,C_\gamma)$, by $\overset{\circ}{A}$ we have $\overset{\circ}{A} \supset (n\overset{\circ}{S}) = (\frac{1}{n}\,\Sigma) \cap C_\gamma$. Hence $\beta(C_\gamma,X)$ is weaker than the $||.||$-topology.

Now consider λS for any non-zero scalar λ. Let V be a basic neighborhood of zero for $\sigma(X,C_\gamma)$. Then V may be written as

$$V = \bigcap_{i=1}^{n} \{x : |g_i(x)| \leqq 1\}$$

with $g_i \in C_\gamma$ for $i=1,\ldots n$. Now $C_\gamma \subset C$ implies that for all x $|g_i(x)| \leqq M_i ||x||$ for some M_i. It then follows that $\lambda S \subset \lambda(M_1 +\ldots+ M_n)V$, and so λS is $\sigma(X,C_\gamma)$-bounded. Thus $(\mathring{\lambda S})$ is a $\beta(C_\gamma,X)$ neighborhood of zero and so the $||.||$-topology is weaker than $\beta(C_\gamma,X)$

<u>3.3.2. Theorem</u>: $(X,||.||,||.||^*)$ is γ-reflexive if and only if the ball S is compact for the weak topology $\sigma(X,C_\gamma)$.

Proof: If $(X,||.||,||.||^*)$ is γ-reflexive, then $\sigma(X,C_\gamma) = \sigma(D^{(\gamma)},C_\gamma)$. Now S is the unit ball of $(D^{(\gamma)},||.||)$ which is conjugate to $(C_\gamma,||.||)$, and hence S is $\sigma(X,C_\gamma)$-compact by the Tychonoff-Alaoglu theorem.

Now suppose S is $\sigma(X,C_\gamma)$-compact. Then S is $\sigma(X,C^*)$-compact and so S is $||.||^*$-closed. Thus $(X,||.||,||.||^*)$ is normal. Let S' be the unit ball of $(D^{(\gamma)},||.||)$. Clearly ~~$S' \supset S$. Suppose there is $x_0 \in S' \setminus S$. S is closed in $\sigma(D^{(\gamma)},C_\gamma)$~~ since this topology agrees with $\sigma(X,C_\gamma)$ on S. Hence there is a linear functional g in the conjugate of $\sigma(D^{(\gamma)}, C_\gamma)$ such that $g(x_0) \neq 0$ while $g(S) = 0$. Then also $g(X) = 0$. But X is $\sigma(D^{(\gamma)},C_\gamma)$-dense in $D^{(\gamma)}$, so that g is identically zero. A contradiction, so that $S' = S$. It follows that $X = D^{(\gamma)}$.

<u>3.3.3. Remark</u>: In the previous theorem the topology $\sigma(X,C_\gamma)$ may be replaced by $\sigma(X,C^*)$.

Proof: Recall that C_γ is the closure of C^* in $(C,||.||)$, and that any total linear subspace of C is dense in C under $\sigma(C,X)$. The statement then follows from the theorem of Dixmier (see introduction).

<u>3.3.4. Theorem</u>: A γ-closed subspace of a γ-reflexive space is γ-reflexive.

Proof: Let X_o by a γ-closed subspace of $(X,||.||,||.||^*)$, a γ-reflexive space. Thus S is compact in $\sigma(X,C^*)$. The unit ball, S_o, of $(X_o,||.||)$ is $X_o \cap S$ which is convex and $||.||^*$-closed (recall S is $||.||^*$-closed by normality of X). Hence S_o is closed in $\sigma(X,C^*)$ and so, as a subset of S, it is compact in $\sigma(X,C^*)$. Let C_o^* be the conjugate to $(X_o,||.||^*)$. By the Hahn-Banach theorem, $\sigma(X_o,C_o^*)$ is identical with the topology induced by $\sigma(X,C^*)$ on X_o. Thus S_o is compact in $\sigma(X_o,C_o^*)$ and the theorem follows by 3.3.2. and 3.3.3..

<u>3.3.5. Theorem</u>: A space which is γ-conjugate to a γ-reflexive space is γ-reflexive.

~~Proof: Let $(X,||.||,||.||^*)$ be γ-reflexive. Thus~~ $\sigma(C^*,X)$ and $\sigma(C^*,D^{(\gamma)})$ are equivalent. Now Σ^* is compact in $\sigma(C^*,X)$ by the Tychonoff-Alaoglu theorem and, again, our statement follows by 3.3.2. and 3.3.3..

<u>3.3.6. Theorem</u>: Let $(X,||.||,||.||^*)$ be normal and γ-complete and let $(C^*,||.||^*,||.||)$ be γ-reflexive, then $(X,||.||,||.||^*)$ is γ-reflexive.

Proof: By the previous theorem $(D^{(\gamma)}, ||.||, ||.||^*)$ is γ-reflexive. Now $(X, ||.||, ||.||^*)$, since it is γ-complete, is a γ-closed subspace of this space and hence is γ-reflexive by 3.3.4..

3.3.7. Theorem: The following assertions are equivalent:

(1) $(X, ||.||, ||.||^*)$ is γ-reflexive and $C_\gamma = C$.

(2) $(X, ||.||)$ is reflexive.

[Note: In particular this says that the reflexivity of $(X, ||.||)$ implies $(X, ||.||, ||.||^*)$ is normal and γ-complete for any norm, $||.||^*$, weaker than $||.||$.]

Proof: (2) implies (1): The reflexivity of $(X, ||.||)$ implies S is $\sigma(X,C)$-compact which in turn implies that S is $\sigma(X,C_\gamma)$-compact. Thus, by 3.3.2., $(X, ||.||, ||.||^*)$ is γ-reflexive. Since $(X, ||.||, ||.||^*)$ is then normal, it follows that C_γ is closed in $(C, ||.||)$. C_γ is total with respect to X since C^* is total. Assume $C_\gamma \subsetneqq C$. Let f $C \sim C_\gamma$. By the Hahn-Banach theorem there is an $x \epsilon X$ such that x = 0 on C_γ and x(f) = 1. But x = 0 since C_γ is total. Hence $C_\gamma = C$.

(1) implies (2): Since $C_\gamma = C$, the spaces conjugate to $(C, ||.||)$ and $(C_\gamma, ||.||)$ are equal so that $(X, ||.||)$ is reflexive.

4. The γ-Completion

If $(X, ||\cdot||, ||\cdot||^*)$ is normal, we have already seen that it can be embedded in $(D^{(\gamma)}, ||\cdot||, ||\cdot||^*)$ which is γ-complete, with the preservation of both norms. However, we would also like to say that every γ-linear functional on $(X, ||\cdot||, ||\cdot||^*)$ is uniquely extendable to the γ-complete space. To do this, we take an appropriate subset of $D^{(\gamma)}$. Before stating the theorem, let us also note that the usual Cantor method of completion is not suitable for our case since $\gamma(\gamma(A)) \neq \gamma(A)$ in general.

3.4.1. Theorem: If $(X, ||\cdot||, ||\cdot||^*)$ is normal, there is a normal, γ-complete space $(X^c, ||\cdot||, ||\cdot||^*)$ containing $(X, ||\cdot||, ||\cdot||^*)$ as a subspace. Moreover, every γ-linear functional on X may be extended in one and only one way onto X^c preserving γ-linearity and the norm, $||\cdot||$.

Proof: Let D_o be the γ-canonical image of X in $D^{(\gamma)}$. Since $D^{(\gamma)}$ is γ-complete, then so is $\bar{\gamma}(D_o)$. Reference to 2.2.4. shows that $\bar{\gamma}(D_o)$ is a linear subspace and we have X "contained" in $\bar{\gamma}(D_o)$. Hence $(\bar{\gamma}(D_o), ||\cdot||, ||\cdot||^*)$ is a normal γ-complete space containing a subspace equivalent to $(X, ||\cdot||, ||\cdot||^*)$.

Let $g_o \epsilon C_\gamma$. Recall that $(D^{(\gamma)}, ||\cdot||)$ may be considered as the conjugate to $(C_\gamma, ||\cdot||)$. Thus we extend g_o as follows:

$$\bar{g}_o(f) = f(g_o) \quad \text{for} \quad f \epsilon \bar{\gamma}(D_o) .$$

We shall now show that $\bar{g}_0$ is γ-linear on $(\bar{\gamma}(D_0), ||\cdot||, ||\cdot||^*)$. Let $f_n \overset{\gamma}{\to} 0$. Then $\sup \{f_n(g): g\epsilon C_\gamma, ||g|| \leqq 1, n=1,2,....\} < \infty$, i.e., $\sup ||f_n|| < \infty$, and $\sup_n \{f_n(g): g\epsilon C^*, ||g||^* \leqq \}1 \overset{n}{\to} 0$, i.e., $||f_n||^* \to 0$. Thus for every $g\epsilon C^*$, $\bar{g}(f_n) = f_n(g) \to 0$. Hence (f_n) is convergent to zero in C^* which is dense in C_γ under $||.||$ and $\sup ||f_n|| < \infty$. Therefore $f_n(g) \to 0$ for all $g\epsilon C_\gamma$, and in particular $\bar{g}_0(f_n) = f_n(g_0) \to 0$.

Let us now consider the uniqueness of the extension. Clearly, the demand that γ-linearity be preserved implies that $g\epsilon C_\gamma$ can be extended in only one way to $\gamma(X)$. By induction it follows that the extension is unique to $\gamma_n(X)$ for n a positive integer. Now consider $\gamma_{\aleph_0}(X) = \gamma(\bigcup_n \gamma_n X)$. Suppose $\bar{g}$ and g' are two extensions of g. Let $x\epsilon\gamma_{\aleph_0}(x)$; so there is (x_n) in $\bigcup_n \gamma_n(X)$ with $X_n \overset{\gamma}{\to} X$. For each n there is m_n such that $x_n \epsilon \gamma_{m_n}(X)$. Hence $\bar{g}(x_n) = g'(x_n)$ for each n and again the γ-linearity implies that $\bar{g}(x) = g'(x)$. So the extension is unique to $\gamma_{\aleph_0}(X)$. We then have

$$\bar{\gamma}(X) = \gamma_{\omega_1}(X) = \gamma(\gamma_{\aleph_0}(X))$$

and the extension is unique to $\bar{\gamma}(X)$.

Finally consider that

$$||\bar{g}|| = \sup \{f(g): f\epsilon\bar{\gamma}(D_0), ||f|| \leqq 1\} \geqq \sup\{x(g): x\epsilon X, ||x|| \leqq 1\}$$
$$= ||g||$$

but also $||\bar{g}|| = \sup \{f(g): f\epsilon\bar{\gamma}(D_0), ||f|| \leqq 1\}$ $\leqq \sup \{f(g): f\epsilon D^{(\gamma)}, ||f|| \leqq 1\} = ||g||$ where the last equality follows by 3.2.2.. Hence $||\bar{g}|| = ||g||$.

X is not strictly contained in $\overline{\gamma}(D_o)$. However, we may define $X^c = X \cup [\overline{\gamma}(D_o) \sim D_o]$. Then X^c, with the norms induced by $\overline{\gamma}(D_o)$, will be denoted by $(X^c,||\cdot||,||\cdot||^*)$ and called the "γ-completion" of $(X,||\cdot||,||\cdot||^*)$. It is obvious that X^c, as a subset of $D^{(\gamma)}$, is the smallest γ-closed set containing X. However, as we shall now see, this condition does not determine X^c uniquely.

Consider ℓ^1 with $||x|| = \sum_{n=1}^{\infty}|x_n|$ and $||x||^* = \sum_{n=1}^{\infty}\frac{1}{2^n}|x_n|$. Let M be the set of Mazurkiewicz and x^o as in section 2.5, and let $X = \{y+tx^o : y \in M,\ t \text{ a scalar}\}$. Let λ be the functional on X defined by $\lambda(y+tx^o) = t$. Further let $(X_1,||\cdot||,||\cdot||^*)$ denote the γ-completion of $(X,||\cdot||,||\cdot||^*)$ and let $X_2 = \overline{\gamma}(X)$ in $(\ell^1,||\cdot||,||\cdot||^*)$. Now λ is γ-linear on $(X,||\cdot||,||\cdot||^*)$ but it cannot be extended to X_2 with the preservation of γ-linearity (see 2.5.3.). Thus X_1 and X_2 are essentially different. More precisely, we have shown that there are normal two-norm spaces $(X,||\cdot||,||\cdot||^*)$, $(X_1,||\cdot||,||\cdot||^*)$, and $(X_2,||\cdot||,||\cdot||^*)$ such that:

(1) $X \subset X_1$; $X \subset X_2$.

(2) $||\cdot||$ and $||\cdot||^*$ are identical on $X, X \cap X_1$, $X \cap X_2$

(3) $(X_1,||\cdot||,||\cdot||^*)$ and $(X_2,||\cdot||,||\cdot||^*)$ are γ-complete.

(4) $\overline{\gamma}(X) = X_1$ considering X as a subset of X_1.

(5) $\overline{\gamma}(X) = X_2$ considering X as a subset of X_2.

(6) there is no linear γ-γ continuous one-to-one mapping of $(X_1, ||.||, ||.||^*)$ onto $(X_2, ||.||, ||.||^*)$ equal on X to the identity mapping.

Now let us consider a more positive approach.

3.4.2. Lemma: There is a natural isomorphical embedding of the set X^c into the completion X^* of the space $(X, ||.||^*)$. Hence X^c may be identified with a part of X^* and every functional linear on $(X, ||.||^*)$ may be extended uniquely to X^c with the preservation of $||.||^*$.

Proof: Using the notation of 3.4.1., we have $D_o \subset \overline{\gamma}(D_o) \subset D^{(\gamma)} \subset D^*$.

The canonical map of X^* in $(D^*, ||.||^*)$ is the closure of D_o in $(D^*, ||.||^*)$ and obviously this closure contains $\overline{\gamma}(D_o)$.

3.4.3. Theorem: Let $(X^A, ||.||, ||.||^*)$ be a normal γ-complete space containing $(X, ||.||, ||.||^*)$ as a subspace and such that

(1) $\overline{\gamma}(X) = X^A$

(2) every γ-linear functional on $(X, ||.||, ||.||^*)$ may be uniquely extended to $(X^A, ||.||, ||.||^*)$ with preservation of $||.||$ and of γ-linearity.

Then there is an isomorphism from X^A onto the γ-completion, X^c, of X. Moreover, this map is isometric with respect to

$||\cdot||$ and $||\cdot||^*$.

Proof: By our hypothesis (2), $(C_\gamma, ||\cdot||)$ is congruent to $(C_\gamma^A, ||\cdot||)$, i.e., there is an isometric isomorphism from the first space onto the second. It then follows by 3.2.2. that $(D_A^{(\gamma)}, ||\cdot||) \cong (D^{(\gamma)}, ||\cdot||)$, where $\cong$ denotes congruence. It is clear that $C_A^* \subset C^*$. By the Hahn-Banach theorem, $C^* \subset C_A^*$ and, moreover, $||\cdot||^*$ is preserved. Thus $(C^*, ||\cdot||^*) \cong (C_A^*, ||\cdot||^*)$, and this in turn implies that $(D_A^{(\gamma)}, ||\cdot||, ||\cdot||^*) \cong (D^{(\gamma)}, ||\cdot||, ||\cdot||^*)$ where the congruence here means that $||\cdot||$ and $||\cdot||^*$ are both preserved. Hence we have

$$X^c = \overline{\gamma}(X) \text{ in } D^{(\gamma)} \cong \overline{\gamma}(X) \text{ in } D_A^{(\gamma)} = X^A.$$

Thus the γ-completion X^c is defined uniquely, within congruences, by requiring that it be normal, γ-complete and such that $\overline{\gamma}(X) = X^c$, and the γ-linear functionals be extensible in only one manner with the preservation of $||\cdot||$ and γ-linearity.

If $(X, ||\cdot||, ||\cdot||^*)$ is normal, as we observed in section 2.1. then $(\tilde{X}, ||\cdot||, ||\cdot||^*)$ is also a normal two-norm space, where $(\tilde{X}, ||\cdot||)$ is the completion of $(X, ||\cdot||)$. However, this completion is not necessarily equal to the γ-completion, since the completeness of $(X, ||\cdot||)$ does not imply γ-completeness. On the contrary we have:

<u>3.4.4. Lemma:</u> If $(X, ||\cdot||, ||\cdot||^*)$ is γ-complete and normal, then $(X, ||\cdot||)$ is complete.

Proof: Let (x_n) be a Cauchy sequence in $(X, ||.||)$. Then $x_n \leq M$ for all n; so that $\frac{x_n}{M}$ is in S. By the γ-completeness, $\frac{x_n}{M} \xrightarrow{\gamma} \frac{x}{M}$ which is in S, since S is $||.||^*$-closed by normality. Now, as in the proof of theorem 2.6.1., γ-convergence in S is equivalent to $||.||$-convergence in S, since γ-convergence is clearly equivalent, in S, to $||.||^*$-convergence. Thus $X_n \to X$ under $||.||$ with $x \epsilon X$.

It then follows that $\tilde{X} \subset X^c$. Moreover, it is not hard to see that $(\tilde{X}, ||.||, ||.||^*) \subset (X^c, ||.||, ||.||^*)$, i.e., the norms are identical on $\tilde{X}$.

<u>3.4.5. Theorem</u>: Let $(X, ||.||, ||.||^*)$ be normal and let $(\tilde{X}, ||.||)$ denote the completion of $(X, ||.||)$. Then the γ-completion of $(X, ||.||, ||.||^*)$ coincides with $(\tilde{X}, ||.||, ||.||^*)$ if and only if every continuous linear functional on $(X, ||.||)$ has a unique extension to a continuous linear functional on $(X^c, ||.||)$.

Proof: The necessity is trivial. Now note that $\tilde{X} \subset X^c$ and $\tilde{X}$ is a $||.||$-closed linear subspace. Suppose there is $x \epsilon X^c \sim \tilde{X}$. Then there is a linear functional g, continuous in $(X^c, ||.||)$, which is zero on $\tilde{X}$ but $g(x) = 1$. Now g is an extension of the zero functional on $(X, ||.||)$. But the zero functional on $(X^c, ||.||)$ is another such extension, which contradicts the uniqueness.

Chapter 4

On Generalizing the Two Norm Spaces

1. A Generalization

In this chapter we shall consider $(X, ||.||, \tau)$, a triple in which X is a linear space, $||.||$ a seminorn, and τ a locally convex linear topology on X weaker than the norm topology. In addition we shall assume that the ball $S = \{x: ||x|| \leq 1\}$ is closed in (X, τ). $(X, ||.||, \tau)$ will be termed a space with a "mixed topology". Such generalizations were first considered by A. Wiweger [18].

The pseudonorms continuous with respect to τ will be denoted by $[x]_\alpha$, and $\{[.]_\alpha\}_{\alpha \in A}$ will denote the totality of all these pseudonorms. (Recall that the topology τ can actually be given by a family of seminorms.)

A linear functional f on X will be called

γ_s-linear, if $\lim_{n\to\infty} [x_n]_\alpha = 0$ for all $\alpha \in A$ and $\sup_n ||x_n|| < \infty$ implies $f(x_n) \to 0$,

γ_t-linear, if it is continuous on S with respect to the topology τ.

Let C_{γ_s} and C_{γ_t} denote the sets of γ_s- and γ_t-linear functionals respectively, and let C and C_τ denote the class of functionals continuous with respect to the norm $||.||$ and τ respectively.

<u>4.1.1. Lemma</u>: $C_\tau \subset C_{\gamma_t} \subset C_{\gamma_s} \subset C$.

Proof: $C_\tau \subset C_{\gamma_t}$: Obvious.

$C_{\gamma_t} \subset C_{\gamma_s}$: Let $\{[.]_\beta\}_{\beta \in B}$ denote the family of seminorms which gives the topology τ, and let $f \in C_{\gamma_t}$. Let $\lim_n [x_n]_\alpha = 0$ for each $\alpha \in A$. Then (x_n) converges to 0 in τ since $B \subset A$. Thus $f(x_n) \to 0$.

$C_{\gamma_s} \subset C$: Let $f \in C_{\gamma_s}$ and let $||x_n|| \to 0$. For each $\alpha \in A$, $[.]_\alpha \leqq ||.||$ since $[.]_\alpha$ is continuous with respect to τ which is weaker than the norm topology. Then $[x_n]_\alpha \to 0$ for each $\alpha \in A$ and $\sup_n ||x_n|| < \infty$. Hence $f(x_n) \to 0$.

<u>4.1.2. Theorem:</u> C_{γ_t} is the closure of C_τ in $(C, ||.||)$, equivalently, each γ_t-linear functional is of the form

$$f(x) = \sum_{n=1}^{\infty} g_n(x)$$

where $g_n \in C_\tau$ and $\sum_{n=1}^{\infty} ||g_n|| < \infty$.

Proof: Similar to that of 2.3.2. and 2.3.4.

The following theorem allows us to restrict considerations of countable sets of γ_t-linear functionals to the case of normable topology τ.

<u>4.1.3. Theorem:</u> For each sequence (f_n) of γ_t-linear functionals there is a sequence of seminorms $([.]_n)$ and a sequence (b_n) of positive numbers such that

$$||x||^* = \sum_{k=1}^{\infty} b_k [x]_k < \infty \quad \text{for all } x \in X$$

and such that

$$\sup_n ||x_n|| < \infty, \ ||x_n||^* \to 0 \text{ imply } f_m(x_n) \xrightarrow{n} 0$$

for m=1,2,.... .

Proof: For each m, let $f_m = \sum_{k=1}^{\infty} g_{km}$ where $g_{km} \epsilon C_{\tau}$ and $\sum_{k=1}^{\infty} ||g_{km}|| < \infty$. Then $|g_{km}(x)| \leqq [x]_{\beta_{km}}$ with certain $\beta_{km} \epsilon A$, (moreover these may actually be chosen as seminorms rather than pseudonorms). By arranging the seminorms into a single sequence $([.]_k)$ we have $[x]_k \leqq M_k ||x||$ for each x and, setting $b_k = (2^k M_k)^{-1}$, we are led to the conclusion of the theorem.

4.1.4. Example: Let X be the conjugate to a Banach space Z. Let $||.||$ denote the usual norm on X and τ be the weak * topology, $\sigma(X,Z)$. Then C is the second conjugate of Z, and C_{τ} is the canonical image of Z since Z is equal to its second conjugate when X is given the weak * topology. It follows that C_{τ} is closed in $(C,||.||)$. Hence the general form of γ_t-linear functionals in $(X,||.||, \tau)$ is $f(x) = x(z)$ with $z \epsilon Z$ and independent of x, and $C_{\tau} = C_{\gamma_t}$.

We shall now show that, in general, $C_{\gamma_t} \neq C_{\gamma_s}$. Let X_o be the space of all bounded measurable functions $x = x(t)$ with $t \in [0,1]$, and let

$$||x|| = \sup_{0 \leq t \leq 1} |x(t)| , \; [x]_t = |x(t)| \text{ for } t \in [0,1].$$

Let τ be the locally convex linear topology given by the family of seminorms $\{[.]_t\}_{t \epsilon [0,1]}$. I now assert that if f is γ_t-linear on X_o it is of the form

$$f(x) = \sum_{k=1}^{\infty} a_k x(t_k)$$

with fixed $t_1, t_2 \dots$ and $\sum_{k=1}^{\infty} |a_k| < \infty$. The proof follows:

Let $f \in C_{\gamma_t}$. By Theorem 4.1.3. there is a sequence $([.]_i)$ of seminorms in the above family and a sequence (b_n) of positive numbers such that

$$||x||^* = \sum_{i=1}^{\infty} b_i [x]_i$$

is a norm on X_o and $f \in C_\gamma$ for $(X_o, ||.||, ||.||^*)$.

Now $(X_o, ||.||^*)$ has a Schauder basis, for let $[.]_i = [.]_{t_i}$ and define

$$x_i(u) = \begin{cases} 1, & u = t_i \\ 0, & u \neq t_i \end{cases} \qquad \text{for } u \in [0,1].$$

We then have

$$x = \sum_{i=1}^{\infty} \alpha_i x_i \qquad \text{where} \qquad \alpha_i = x(t_i)$$

since

$$||x - \sum_{i=1}^{n} \alpha_i x_i||^* = \sum_{k=1}^{\infty} b_k [x - \sum_{i=1}^{n} \alpha_i x_i]_k$$

$$= \sum_{k=1}^{\infty} b_k |x(t_k) - \sum_{i=1}^{n} \alpha_i x_i(t_k)|$$

and for $n > k$ the sum is zero, i.e., $\sum_{i=1}^{n} \alpha_i x_i \xrightarrow{||.||^*} x$. The uniqueness of the expansion is apparent. Thus (x_i) is a Schauder basis for $(X_o, ||.||^*)$.

Furthermore, for each n, $||\sum_{i=1}^{n} \alpha_i x_i|| \leq ||x||$ so that $(\sum_{i=1}^{n} \alpha_i x_i)$ γ-converges to x in $(X_o, ||.||, ||.||^*)$. Since

$f \epsilon C_{\gamma}$, we have

$$f(x) = f[\sum_{i=1}^{\infty} \alpha_i x_i] = \sum_{i=1}^{\infty} \alpha_i f(x_i) = \sum_{i=1}^{\infty} f(x_i) x(t_i) .$$

Letting $f(x_i) = a_i$ we have $f(x) = \sum_{i=1}^{\infty} a_i x(t_i)$ and it remains only to show that (a_i) is in ℓ^1. Consider the function $1(t) = 1$ for $t \epsilon [0,1]$. Then

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} f(x_i) = f(\sum_{i=1}^{\infty} x_i) = f(1) < \infty.$$

It is clear that every rearrangement of $\sum a_i$ converges so that $\sum |a_i|$ converges.

Thus every f C_{γ_t} is of the prescribed form. However, the functional $\int_0^1 x(t)dt$ is clearly γ_s-linear but is not in C_{γ_t}. Hence, in general, $C_{\gamma_t} \subsetneq C_{\gamma_s}$.

Under Ulam's hypothesis (that there are no inaccessible alephs between $\aleph_0$ and $2^{\aleph_0}$) it may be shown that each γ_s-linear functional on this space is of the form

$$f(x) = f_0(x) + \int_0^1 x(t)\phi(t)dt$$

where $f_0 \epsilon C_{\gamma_t}$ and $\int_0^1 |\phi(t)|dt < \infty$. However, if X is the space of all bounded functions on [0,1] with the same $||.||$ and $[.]_t$, then, also under Ulam's hypothesis, the spaces C_{γ_s} and C_{γ_t} coincide and consist of all the functionals of the form

$$f(x) = \sum_{i=1}^{\infty} a_i x(t_i)$$

with fixed $t_1, t_2, \ldots.$ and $\sum_{i=1}^{\infty} |a_i| < \infty$. Comparing this space with the preceeding one as a subspace, we may conclude that the γ_s-linear functionals do not have the extension property.

APPENDIX

In this section I shall allow myself the privilege of stating results only, without their supporting arguments. The proofs may be found in the references cited in the Bibliography, [5] being of particular usefulness.

I shall assume throughout this section that $(X,||\cdot||,||\cdot||^*)$ is a two-norm space for which $||\cdot||$ and $||\cdot||^*$ are both norms. A subset A of X will be called "γ-dense" if $\gamma(a) = X$, and $(X,||\cdot||,||\cdot||^*)$ will be termed "γ-separable" if it includes a countable γ-dense subset.

<u>Theorem</u>: $(X,||\cdot||,||\cdot||^*)$ is γ-separable if and only if $(X,||\cdot||^*)$ is separable.

$(X,||\cdot||,||\cdot||^*)$ will be called "γ-compact" if every γ-bounded sequence contains a γ-convergent subsequence, and $(X,||\cdot||,||\cdot||^*)$ will be called "γ-precompact" if every γ-bounded sequence contains a γ-Cauchy subsequence.

<u>Theorem</u>: If $(X,||\cdot||,||\cdot||^*)$ is normal and γ-compact, then it is γ-reflexive.

<u>Theorem</u>: The γ-separability of $(C^*,||\cdot||^*,||\cdot||)$ implies the γ-separability of $(X,||\cdot||,||\cdot||^*)$.

<u>Theorem</u>: $(X,||\cdot||,||\cdot||^*)$ is γ-precompact if and only if $(C^*,||\cdot||^*,||\cdot||)$ is γ-compact.

<u>Theorem</u>: If $(X,||\cdot||,||\cdot||^*)$ is normal, then it is γ-precompact if and only if its γ-completion $(X^c,||\cdot||,||\cdot||^*)$

is γ-compact.

A normed space $(X, ||.||)$ will be called "prereflexive" if its completion is reflexive.

Theorem: If $(X, ||.||^*)$ is prereflexive and separable, then all γ-conjugate spaces to $(X, ||.||, ||.||^*)$ are γ-separable.

A one-to-one linear γ-γ continuous operation from a γ-complete two-norm space onto another does not, in general, satisfy the Banach inversion property. However the inverse is linear and γ-γ continuous if the domain space is γ-compact, i.e.,

Theorem: Let $(X, ||.||, ||.||^*)$ be γ-compact, and let $(X, ||.||)$ and $(Y, ||.||)$ be complete. Let U be a linear γ-γ continuous one-to-one mapping of $(X, ||.||, ||.||^*)$ onto $(Y, ||.||, ||.||^*)$. Then U^{-1} is linear and γ-γ continuous and U establishes a linear homeomorphism between $(X, ||.||)$ and $(Y, ||.||)$.

Theorem: Let $(X, ||.||, ||.||^*)$ be a γ-compact, let U be a linear γ-γ continuous operation from $(X, ||.||, ||.||^*)$ onto $(Y, ||.||, ||.||^*)$, and let $(X, ||.||)$ and $(Y, ||.||)$ be complete. Then $(Y, ||.||, ||.||^*)$ is γ-compact.

Theorem: $(X, ||.||, ||.||^*)$ is γ-precompact if and only if it is γ-separable and γ-convergence in $(C^*, ||.||^*, ||.||)$ is equivalent to convergence with respect to $\sigma(C^*, X)$.

Theorem: Let $(X, ||.||)$ be a Banach space. Then the following conditions are equivalent:

(1) there is a coarser norm, $||.||^*$, such that $(X,||.||,||.||^*)$ is γ-precompact,

(2) there exists a total sequence of continuous linear functionals on $(X,||.||)$.

Furthermore, if $(X,||.||)$ is separable, then (1) and (2) are always satisfied.

$(X,||.||,||.||^*)$ is termed "saturated" if $C_\gamma = C$. Recall that, for normal spaces, $C_\gamma = C^*$ only in the trivial case. However, there are non-trivial saturated spaces.

<u>Theorem</u>: Any saturated two-norm space is normal.

<u>Theorem</u>: The following conditions are equivalent:

(1) $C_\gamma = C$,

(2) C^* is dense in $(C,||.||)$,

(3) any γ-convergent sequence is convergent in $\sigma(X,C)$,

(4) for every $f \in C$ and for every $\varepsilon > 0$ there is a K such that $f(x) \leq \varepsilon + K||x||^*$ for all $x \in S$,

(5) $C_\gamma \cap \Sigma$ is closed in $\sigma(C,X)$,

(6) every convex closed subset of $(X,||.||)$ is γ-closed.

<u>Theorem</u>: If $(X,||.||,||.||^*)$ is γ-compact and saturated then $(X,||.||)$ is reflexive and separable and γ-convergence in $(X,||.||,||.||^*)$ is equivalent to convergence in $\sigma(X,C)$.

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