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Some extremal problems in the theory of bounded analytic functions

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SOME EXTREMAL PROBLEMS IN THE THEORY OF BOUNDED ANALYTIC FUNCTIONS

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ABSTRACT

The purpose of this paper is to exhibit some extremal problems that abound in the literature of bounded analytic functions as well as to illustrate the methods of solving these problems. Much of the work in this field has been contributed by P. R. Garabedian [5] and by L.V. Ahlfors [1], and their results, in turn, have been expanded by Z. Nehari, see [6,7,8,9,10].

The first chapter of the thesis introduces basic concepts such as the Green's and Neumann's functions, harmonic functions and harmonic measure. Above all, a fundamental result in the theory of extremal problems concerning the existence of solutions to such problems is given at the conclusion of the chapter.

In Chapter II, within the confines of an extremal problem, we illustrate how, by means of a conformal map, we are able to transform our original problem into a more workable one. In the next chapter we set out to prove the existence of two specific functions $K_{\lambda}(z,\delta)$ and $L_{\lambda}(z,\delta)$ which are then used to solve subsequent extremal problems. Careful observation shows that, by limiting the positive function $\lambda(z)$, the extremal problem is nothing more than a generalization of Schwarz's lemma.

Chapter IV presents an extension of the

results in the previous two chapters to derivatives of bounded functions. These results are generalizations of results stated by Caratheodory [2], where we make explicit use of the functions $K_\lambda(z,\delta)$ and $L_\lambda(z,\delta)$ developed in the previous chapter.

In conclusion, we present two elementary theorems which illustrate the behavior of bounded functions in starlike and convex domains.

Chapter I

INTRODUCTION

SECTION 1. Definitions and Notation

Before embarking on the main topics of this thesis, we will first state some of the pertinent definitions and theorems from analytic function theory for ready reference.

Definition 1.1. A domain D is an open arc-wise connected set.

Definition 1.2. A domain D is simply-connected if its complement relative to the extended plane (Riemann sphere) is connected.

Definition 1.3. A domain D is of connectivity n, $n = 1,2,3,\ldots,$ if its complement relative to the Riemann sphere consists of n components. In other words, a domain of connectivity n is obtained by punching n holes in the Riemann sphere.

Definition 1.4. The boundary c of a domain D is the intersection of the closure of D with the closure of its complement relative to the Riemann sphere.

For our purposes the boundary c of a domain of connectivity n will always consist of n simple closed curves $c_1,\ldots,c_n$. The boundary curve c_k is called smooth if it has a continuously turning tangent.

Definition 1.5. A function $u(x,y)$ is said to be harmonic in a domain D if

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ exist and are continuous and if

$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at all points of D.

We shall have recourse to the maximum principle for harmonic functions: If $u(x,y)$ is a harmonic function in a domain D, it cannot attain its absolute maximum or minimum at an interior point of D unless $u(x,y) = \text{Const.}$

We now state Green's formula: Let u and v be functions with continuous first and second partial derivatives in the closure of a domain D, then

$$\iint_D (u\Delta v - v\Delta u)\,dA = \int_c \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS \qquad (1.1)$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the direction of the outward normal and S is the arc length parameter. The line integral in (1.1) is taken in the positive sense over each component of the boundary of D. We note, in particular, that when both $u(x,y)$ and $v(x,y)$ are harmonic,

$$\int_c \left(u\frac{\partial v}{\partial n}\right)dS = \int_c \left(v\frac{\partial u}{\partial n}\right)dS \qquad (1.2)$$

<u>Definition 1.6</u>. The Green's function $g(z,\delta)$ of a domain D with respect to a point $\delta \in D$ is of the form

$g(z,\delta) = -\log|z-\delta| + G(z,\delta)$ and has the following properties:

i) $g(z,\delta)$ is harmonic in $D + c$ except at $z = \delta$;

ii) $G(z,\delta)$ is regular harmonic in $D + c$;

iii) $g(z,\delta) = 0$ for $z \in c$, $\delta \in D$.

Now let D be a bounded domain with boundary c and let f be a given function defined and continuous on c. The Dirichlet problem is to find a function harmonic in D and continuous in D + c such that

$$u(\delta) = f(\delta) \quad , \quad \delta \varepsilon c .$$

We now state without proof the following important result, for details, see Nehari [10].

Theorem 1.1. If $g(z,\delta)$ is the Green's function of D relative to z ε D, then the solution of the Dirichlet problem is given by

$$u(z) = -\frac{1}{2\pi} \int_c f(\delta) \frac{\partial g(z,\delta)}{\partial n_\delta} ds_\delta \tag{1.3}$$

At this point we make the important observation that the Dirichlet problem is actually equivalent to a minimal problem in the calculus of variations. The solution of the Dirichlet problem is thus the extremal function for the corresponding variational problem. Riemann called this variational principle the Dirichlet principle, see Courant [3].

We now consider the related Neumann problem of finding a harmonic function with the property that the values of its normal derivative are prescribed on c.

Definition 1.7. The Neumann function $N(z,\delta)$ of a domain D with boundary curve c with respect to δ ε D is of the form

$N(z,\delta) = -\log|z-\delta| + N_1(z,\delta)$ and has the following properties:

i) $N(z,\delta)$ is harmonic in D + c except at $z = \delta$;

ii) $N_1(z,\delta)$ is regular harmonic in D + c;

iii) For all $z \in c$, $\frac{\partial N(z,\delta)}{\partial n}$ = constant.

To define the Neumann function uniquely, we will require

$$\int_c N(z,\delta)dS = 0 \quad .$$

Using this normalization, the constant of (iii) above is seen to be

$$\frac{\partial N(z,\delta)}{\partial n} = -\frac{2\pi}{L} \, ,$$

where $L = \int_c dS$.

SECTION 2. Harmonic Measure

Let D be a bounded domain with smooth boundary c. Let α denote an open arc which forms a part of c and let β denote the remainder of c excluding the end points of α. Then we define the harmonic measure $w(z,\alpha)$ of α with respect to D to be a harmonic function in D which takes on the boundary values 1 or 0 as z approaches α or β, respectively. Since we are concerned mainly with domains which are bounded by simple smooth closed curves $c_1,\ldots,c_n$, we shall define harmonic measure as follows:

Definition 1.8. Given a domain D bounded by n smooth curves c_k, $k = 1,\ldots,n$, the harmonic measure $w_j(z)$ of the component c_j relative to D at the point z is the function which is harmonic in D and has the boundary values 1 and 0 on c_j and c_i $(i \neq j)$ respectively.

Simple consequences of this definition are the following:

Lemma 1.1. Let $w_j(z)$ be the harmonic measure of a domain

D with boundary $c = c_1 + \dots + c_n$. Then

$$\sum_{j=1}^{n} w_j(z) \equiv 1 \quad \text{for } z \in D + c \quad .$$

Proof: $w_j(z) = 1$ for $z \in c_j$

$w_j(z) = 0$ for $z \in c_\ell$, $\ell \neq j$

Hence for all $z \in c$

$$w_1(z) + \dots + w_n(z) = 1$$

Applying the maximum principle we have that both the maximum and the minimum of

$w_1(z) + \dots + w_n(z)$ in D are equal to 1.

Hence

$$\sum_{j=1}^{n} w_j(z) \equiv 1 \quad , \quad z \in D + c \quad .$$

Q.E.D.

Lemma 1.2. The period of the harmonic conjugate of the Green's function $g(z,\delta)$ of a multiply connected domain D with respect to a circuit about the boundary component c_j is $-2\pi w_j(\delta)$, where $w_j(\delta)$ is the harmonic measure of c_j.

Proof: From eq. (1.3) and from the fact that the harmonic measure is nothing more than the solution of a Dirichlet problem, we may write

$$w_j(\delta) = -\frac{1}{2\pi} \int_c \frac{\partial g(z,\delta)}{\partial n} \, dS \qquad (1.4)$$

If D is of connectivity m, we are able to express the m-1 independent periods p_j of the harmonic conjugate by

$$p_j = \int_{c_j} \frac{\partial u}{\partial n} \, dS \quad , j = 1, \dots, m-1 \qquad (1.5)$$

of the inner boundaries of D.

For the outer boundary, namely the mth boundary,

$$p_m = \int_{c_m} \frac{\partial u}{\partial n} \, dS \qquad (1.6)$$

Hence we have that

$$\int_{c_j} \frac{\partial g(z,\delta)}{\partial n} \, dS = -2\pi \, w_j(\delta) \quad .$$

Q.E.D.

Although the harmonic measure functions have many more properties, the above-mentioned facts will suffice for our purposes.

SECTION 3. Extremal Problems

With the facts introduced above, we are now in a position to discuss briefly the theory of extremal problems. Such a problem requires the finding of the precise upper or lower bound of a functional relative to a given class of functions.

The existence of such a solution is asserted by the following basic result:

Theorem 1.2. Let $M(f)$ be a continuous functional defined in a normal and compact family $F = \{f(z)\}$, then the problem $|M(f)| = \max.$ has a solution within F.

Proof: We wish to show that there exists at least one $f_o(z) \varepsilon F$ such that for all $f(z) \varepsilon F$ the inequality

$|M(f)| \leq |M(f_o)|$ is satisfied.

The numbers $|M(f)|$ have a least upper bound, say A. By definition of least upper bound, there exists a sequence of functions $f_n(z) \varepsilon F$ such that $\lim_{n \to \infty} |M(f_n)| = A$.

Since F is normal, we can choose a subsequence $f_{n_k}(z)$ of $f_n(z)$ which converges to a function $f_o(z)$. Since F is compact $f_o(z) \varepsilon F$.

Since M(f) is continuous we have that $f_{n_k}(z) \to f_o(z)$ implies that $M(f_{n_k}) \to M(f_o)$. Hence we have that

$$|M(f_o)| = \lim_{k\to\infty} |M(f_{n_k})| = A .$$

Hence there exists a function $f_o(z) \varepsilon F$ such that $|M(f)| = A$. All we need to show is that A is finite and this follows easily from the fact that M(f) takes on only finite values by definition of a functional. Hence the theorem.

Q.E.D.

In Chapter II we shall investigate the method of solving a particular type of extremal problem under certain boundedness conditions. Chapter III will be concerned with the closely associated class of domain functions and Chapter IV will include an extension of the methods of Chapters II and III to the first derivatives of bounded functions as well as to derivatives of the kth order. We shall then conclude the paper with a brief discussion of the effect that certain domain properties can have on the solution of extremal problems.

Chapter II

A GENERALIZATION OF SCHWARZ'S LEMMA

We begin this chapter by stating without proof the classical Schwarz lemma:

Lemma 2.1. Let the analytic function $f(z)$ be regular in $|z| < R$ and let $f(0) = 0$. If, in $|z| < R$, $|f(z)| \leq M$, then

$$|f(z)| \leq \frac{|z| M}{R}, \qquad |z| < R,$$

where equality can hold only if

$$f(z) = \frac{M}{R} e^{i\theta} z$$

where θ is a real constant.

In particular, we see that the Schwarz lemma holds for bounded functions in the unit disk which vanish at the origin.

This result is easily generalized to bounded functions in $|z| < 1$ which do not vanish at the origin:

Lemma 2.2. Let $f(z)$ be regular in $|z| < 1$ and $|f(z)| \leq 1$, then

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2} \leq \frac{1}{1 - |z|^2}$$

for $|z| < 1$, where equality holds only if

$$f(z) = e^{i\theta} \frac{z - \delta}{1 - \delta^* z}, \qquad \theta \text{ real.}^{1}$$

This latter lemma states that among the class

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[1]The conjugate of δ is denoted by δ^*.

of functions which are bounded and regular in $|z| < 1$, the problem $|f'(z)| = \max.$ is solved by the function which maps the unit disk onto itself and maps $z = \delta$ into the origin. We are now interested in generalizing these results as follows:

Let D be a domain of connectivity n, and let B denote the family of single-valued analytic functions f(z) regular in D for which $|f(z)| \leq 1$, $z \in D$. Let each c_j be a closed analytic Jordan curve. We again designate the boundary of D as

$$c = c_1 + \dots + c_n \quad . \qquad \text{(See Fig. 1)}$$

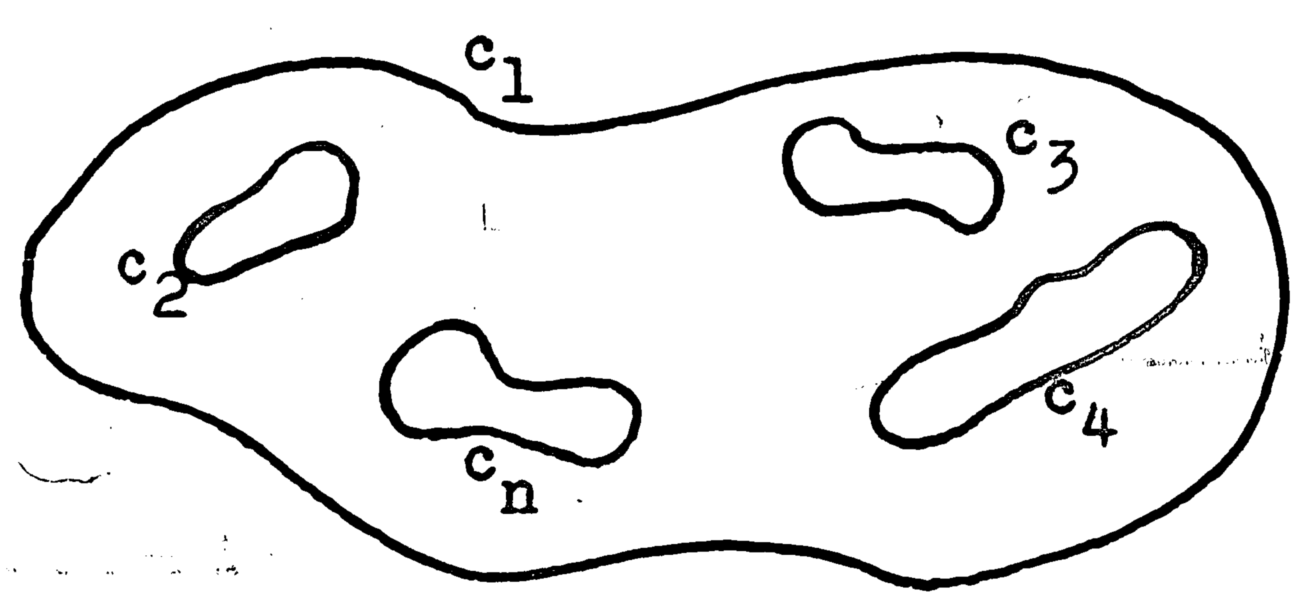

Figure 1

The results stated and methods used here are due principally to Nehari [6], [10]. For different approaches to the problem, see Ahlfors [1] and Garabedian [5].

In the proof of the theorem that follows we state an extremal problem for functions belonging to the class B and by means of a conformal map, we then arrive at an equivalent but simpler extremal problem. We now state and prove the following important generalization of Schwarz's lemma.

Theorem 2.1. The extremal problem

$$\max_B |f'(\delta)| \quad , \qquad \delta \varepsilon D$$

is solved by a function $F(z) \varepsilon B$ which yields a $(1,n)$ conformal map of D onto the unit disk. Moreover, there exists a function $Q(z)$ which is regular in $D + c$, except for a double pole at $z = \delta$ such that

$$- i F(z) Q(z) dz > 0 \quad , \qquad z \varepsilon c .$$

Proof:

For $F(z)$ to be a maximum we must have $F(\delta) = 0$. For if not, then we set

$$F_1(z) = \frac{F(z) - F(\delta)}{1 - F^*(\delta) \cdot F(z)}$$

which is clearly in B and

$$F_1'(z) = F'(z) \frac{1 - |F(\delta)|^2}{[1 - F^*(\delta)F(z)]} .$$

Thus, we find,

$$|F_1'(\delta)| = \frac{|F'(\delta)|}{1 - |F(\delta)|^2} > |F'(\delta)| .$$

This shows that we may confine ourselves to those functions of B for which $F(\delta) = 0$.

Let B_R be those functions $f(z)$ in B for which $f(\delta) = 0$ and

$$\left| \text{Re} \left\{ f(z) \right\} \right| \leq 1 . \qquad (2.1)$$

If $f(z) \varepsilon B$, then the function

$$\phi(z) = \frac{4}{\pi} \tan^{-1} f(z)$$

will clearly satisfy the inequality

$$\left| \mathrm{Re} \left\{ \phi(z) \right\} \right| \leq 1 \quad .$$

Moreover, $\phi(\delta) = 0$ whenever $f(\delta) = 0$ so that $\phi(z) \,\varepsilon\, B_R$.

Finally, if $F(z)$ is an extremal function, $e^{i\theta}F(z)$, θ real, will be one also. We may therefore suppose without loss of generality that $F'(\delta) > 0$. Since

$$\phi'(z) = \frac{4}{\pi} \frac{f'(z)}{1 + f^2(z)} \quad ,$$

we see that

$$\phi'(\delta) = \frac{4}{\pi} f'(\delta) > 0 \quad .$$

Our problem is therefore reduced to

$$\max_{B_R} \mathrm{Re} \left\{ \phi(z) \right\} \quad .$$

Without loss of generality we may restrict the functions $\phi(z)$ by requiring that they be continuous in D + c. Clearly,

$$\mathrm{Re} \left\{ \phi'(z) \right\} = \frac{\partial u(z)}{\partial x} \quad ,$$

where $\phi(z) = u(z) + i\, v(z)$ and $z = x + iy$. Let $\delta = \xi + i\eta$, then the problem

$$\phi'(\delta) = \frac{\partial u(\delta)}{\partial \xi} = \max.$$

is equivalent to our original problem.

Using eq. (1.3) we have

$$\emptyset'(\delta) = -\frac{1}{2\pi}\int_c u(z)\,\frac{\partial^2 g(z,\delta)}{\partial n\,\partial \xi}\,dS \qquad (2.2)$$

Since u(z) is a harmonic function which satisfies $|u(z)| \leq 1$ in D and which possesses a single-valued harmonic conjugate there, its periods must necessarily vanish. In fact,

$$\int_c u(z)\,\frac{\partial w_j(z)}{\partial n}\,dS = 0 \qquad j = 1,\ldots, n-1 \qquad (2.3)$$

To see this we note that, using eq. (1.2) we have

$$\int_c u(z)\,\frac{\partial w_j(z)}{\partial n}\,dS = \int_c w_j(z)\,\frac{\partial u(z)}{\partial n}\,dS$$

$$= \int_{c_j}\frac{\partial u(z)}{\partial n}\,dS = \int_{c_j}\frac{\partial v(z)}{\partial S}\,dS = \int_{c_j} dv(z)$$

since $w_j(z) = 1$ for $z \in c_j$ and the Cauchy-Riemann equation,

$$\frac{\partial u(z)}{\partial n} = \frac{\partial v(z)}{\partial S}$$ holds.

But $\int_{c_j} dv(z) = 0$ because $\emptyset(z)$ is single-valued in D.

It follows from (2.2) and (2.3) that

$$\emptyset'(\delta) = -\frac{1}{2\pi}\int_c u(z)\left[\frac{\partial^2 g(z,\delta)}{\partial n\,\partial \xi} + \sum_{j=1}^{n-1}\varepsilon_j\,\frac{\partial w_j(z)}{\partial n}\right]dS, \qquad (2.4)$$

where $\varepsilon_1,\ldots,\varepsilon_{n-1}$ are arbitrary real constants.

Since $|u(z)| \leq 1$ for $z \in D$,

$$|\emptyset'(\delta)| \leq \frac{1}{2\pi}\int_c \left|\frac{\partial^2 g(z,\delta)}{\partial n\,\partial \xi} + \sum_{j=1}^{n-1}\varepsilon_j\,\frac{\partial w_j(z)}{\partial n}\right| dS \qquad (2.5)$$

Hence for any choice of ε_j, the right hand side of (2.5) yields an upper bound for $|\emptyset'(\delta)|$.

We now wish to minimize the right hand side of (2.5) with respect to $\varepsilon_1,\ldots,\varepsilon_{n-1}$ in the following way:

Clearly $|\emptyset'(\delta)|$ is a lower bound in (2.5). We also know that a converging subsequence of a minimizing sequence must converge to a finite limit. For if not, then one or more of the ε_j would tend to $+\infty$ or $-\infty$. Hence there would exist an ε_k such that

$$\limsup \left|\frac{\varepsilon_j}{\varepsilon_k}\right| \leq 1 \quad , \qquad (j \neq k).$$

Since we are dealing with a minimizing sequence, the right hand side of (2.5) will be bounded. If we divide the integrand in (2.5) by $|\varepsilon_k|$ we have that

$$\frac{1}{2\pi}\int_C \varepsilon_k^{-1}\left|\frac{\partial^2 g(z,\delta)}{\partial n\,\partial\xi} + \sum_{j=1}^{n-1}\varepsilon_j\,\frac{\partial w_j(z)}{\partial n}\right| dS \longrightarrow 0 \; .$$

Since we can find a subsequence for which the integrand tends to a well-defined positive function, we have a contradiction. Hence the minimizing sequence converges to a finite limit.

We notice that the integrand cannot vanish identically since

$$\frac{\partial w_k}{\partial n} = 1 \quad , \qquad 1 \leq k \leq n-1 \; .$$

Now let

$$P = \frac{\partial^2 g(z,\delta)}{\partial n\, \partial \xi} + \sum_{j=1}^{n-1} \varepsilon_j \frac{\partial w_j(z)}{\partial n} \tag{2.6}$$

and $\varepsilon_1, \ldots, \varepsilon_{n-1}$ be the particular values of the parameters which minimize (2.5). Then $|P(\varepsilon_1,\ldots,\varepsilon_{n-1})| \leq |P(\varepsilon_1,\ldots,\varepsilon_j \pm \varepsilon,\ldots,\varepsilon_{n-1})|$ so that

$$\int_c |P|\, ds \leq \int_c \left| P \pm \varepsilon \frac{\partial w_j}{\partial n} \right| ds$$

$$= \int_c |P| \left| 1 \pm \frac{\varepsilon}{P} \frac{\partial w_j}{\partial n} \right| ds$$

$$= \int_c |P| \, [1 \pm \frac{\varepsilon}{P} \frac{\partial w_j}{\partial n} + O(\varepsilon)] ds$$

$$= \int_c |P|\, ds \pm \varepsilon \int_c \frac{\partial w_j}{\partial n} \frac{|P|}{P} ds + O(\varepsilon).$$

That is,

$$\pm \varepsilon \int_c \frac{|P|}{P} \frac{\partial w_j}{\partial n} ds + O(\varepsilon) \geq 0 .$$

Since ε is arbitrary, we conclude from this that

$$\int_c \frac{|P|}{P} \frac{\partial w_j}{\partial n} ds = 0 \quad , \quad j = 1,\ldots,n-1 \tag{2.7}$$

The function $\frac{|P|}{P}$ is evidently not defined for $P = 0$ but

$$-\frac{|P|}{P} = \begin{cases} -1 & , \quad P > 0 \\ +1 & , \quad P < 0 \end{cases} .$$

Using this as the boundary values in (1.3), we obtain a harmonic function

$$U(\delta) = \frac{1}{2\pi} \int_c \frac{|P|}{P} \frac{\partial g(z,\delta)}{\partial n} dS . \tag{2.8}$$

In view of (2.7)

$$\int_c U(z) \frac{\partial w_j}{\partial n} dS = 0 , \qquad j = 1,\ldots,n-1 .$$

Comparing this with (2.3) ~~we see~~ that the harmonic conjugate $V(z)$ of $U(z)$ is necessarily free of periods so that the analytic function

$$\Phi(z) = U(z) + i\,V(z)$$

is single-valued in D. Moreover, we have from the maximum principle that $|U(z)| \leq 1$ for $z \in D$. On c, we may write

$$|P| = -P\,U(z) \tag{2.9}$$

so that a comparison with (2.5) shows

$$|\phi'(\delta)| \leq -\frac{1}{2\pi} \int_c U(z)\left[\frac{\partial^2 g(z,\delta)}{\partial n\,\partial \xi} + \sum_{j=1}^{n-1} \varepsilon_j \frac{\partial w_j}{\partial n}\right] dS .$$

However, $\Phi(z)$ is single-valued so that

$$|u_\xi(\delta)| \leq U_\xi(\delta)$$

which shows $U(z)$ to be the solution of our extremal problem.

We note from (2.9) that for $z \in c$

$$U(z)\,P(z) \leq 0 \tag{2.10}$$

We now show that $W = \Phi(z)$ yields a (1,n) conformal map of D onto the strip

$$-1 \leq \operatorname{Re}\{W\} \leq 1 \qquad :$$

As pointed out before, $\left|\operatorname{Re}\{\Phi(z)\}\right| = |U(z)| \leq 1$ for $z \in D$. Moreover, $|U(z)| = 1$ on c except where $P(z) = 0$. As we pass through each zero of P, the value of U jumps from +1 to -1, or conversely. The image of D under the mapping $\Phi(z)$ is therefore a multiple covering of the infinite strip $\left|\operatorname{Re}\{W\}\right| \leq 1$. The number of sheets is evidently half the number of zeros of P on c. We will show that P has precisely 2n zeros on c.

Let p(z) and $W_j(z)$ be analytic functions such that $\operatorname{Re}\{p(z)\} = g(z,\delta)$ and $\operatorname{Re}\{W_j(z)\} = w_j(z)$, $j = 1,\ldots,n-1$.

Writing

$$p(z) = g(z,\delta) + i\, h(z,\delta)$$

and noting that $g = 0$ on c, we find, with the help of the Cauchy-Riemann equation $\frac{\partial g}{\partial n} = \frac{\partial h}{\partial s}$ that,

$$-i\, p'(z)\, dz = -i\left(\frac{\partial g}{\partial s}\, ds + i\, \frac{\partial h}{\partial s}\, ds\right)$$

$$= \frac{\partial g}{\partial n}\, ds \qquad .$$

Using definition (1.8) on c_k, a similar argument shows that

$$-i\, W_j'(z)\, dz = \frac{\partial w_j}{\partial n}\, ds \qquad , \qquad z \in c \qquad .$$

Combining these results with (2.6) we find that

$$P\,dS = -i\left[\frac{\partial}{\partial \xi} p'(z) + \sum_{j=1}^{n-1} \varepsilon_j W_j'(z)\right] dz$$

$$= i\, q(z)\, dz \qquad (2.11)$$

Since $g(z,\delta) = -\log|z-\delta| + G(z,\delta)$, where G is regular harmonic, the function $p'(z)$ evidently has a simple pole at $z = \delta$ with residue -1. The quantity $\frac{\partial}{\partial \xi} p'(z)$ is necessarily of the form

$$\frac{\partial p'(z)}{\partial \xi} = -\frac{1}{(z-\delta)^2} + t(z),$$

where $t(z)$ is regular in D. Combining with (2.11), we can therefore write

$$P(z)\,dS = i\,q(z)\,dz = i\left[\frac{1}{(z-\delta)^2} + t_1(z)\right]dz, \quad z \in c \qquad (2.12)$$

where $t_1(z)$ is regular in D. The zeros of P will thus coincide with those of $q(z)$ on c.

From (2.12) we have

$$[q(z)\,dz]^2 \leq 0$$

so that $\Delta_c \arg [q(z)\,dz]^2 = 0$ from which we conclude that

$$\Delta_c \arg [q(z)] + \Delta_c \arg [dz] = 0 .$$

Now, for a finite domain of connectivity n, it is well known that, [10, p. 135], $\Delta_c \arg (dz) = 2\pi(n-2)$. Since $q(z)$ has a double pole in D and the zeros of q on c are counted by half their multiplicity, it follows from the Argument Principle that $q(z)$, and hence P, has exactly 2n zeros on c. We have thus shown the image of D to be an n-fold covering of the infinite strip $-1 \leq \operatorname{Re}\{W\} \leq 1$.

We now wish to relate the inequality (2.10) with functions belonging to B in an obvious manner. We relate $F(z) \in B$ and $\Phi(z) \in B_R$ as follows:

$$F(z) = \tan \frac{\pi}{4} \Phi(z)$$

where $\Phi(z)$ is defined as before. Clearly, $F(z)$ maps D onto the multiply covered unit disk.

For $z \in c$, $\Phi(z) = \pm 1 + i\,t$ where t is real. Consider the expression

$$\frac{F(z)}{1 + F^2(z)} .$$

A computation shows that it is

$$\pm \frac{1}{2} \operatorname{Cosh} \frac{\pi}{2} t \quad , \quad z \in c .$$

Since $\operatorname{Cosh} \frac{\pi}{2} t$ is positive, this expression is positive or negative according as $U(z) = 1$ or -1. Thus,

$$\frac{F(z)}{U(z)\,[1 + F^2(z)]} \geq 0 \qquad \text{on } c .$$

Combining this with (2.10) and (2.12) we finally arrive at

$$-i\,\frac{F(z)\,q(z)}{1+F^2(z)} \geq 0 \,, \qquad z \in c \,,$$

or more simply

$$-i\,F(z)\,Q(z) \geq 0 \,, \qquad z \in c \,, \tag{2.13}$$

where $Q(z) = \dfrac{q(z)}{1+F^2(z)}$. A simple argument shows that the zeros of $q(z)$ and $1 + F^2(z)$ coincide so that (2.13) becomes

$$-i\,F(z)\,Q(z) > 0 \,, \qquad z \in c \,. \tag{2.14}$$

Since $Q(z)$ has a double pole at $z = \delta$, it follows from the Cauchy integral formula that

$$f'(\delta) = \frac{1}{2\pi i}\int_c f(z)\,Q(z)\,dz \,, \qquad f(z) \in B \,.$$

Hence,

$$\left|f'(\delta)\right| \leq \frac{1}{2\pi}\int_c \left|Q(z)\,dz\right| \,.$$

Since $\left|F(z)\right| = 1$ on c, we see from (2.14) that

$$\left|f'(\delta)\right| \leq \frac{1}{2\pi}\int_c \left|\frac{1}{i}\,F(z)\,Q(z)\,dz\right|$$

$$= \frac{1}{2\pi i}\int_c F(z)\,Q(z)\,dz \tag{2.15}$$

$$= F'(\delta)$$

Equality holds evidently for $f(z) = e^{i\theta}\,F(z)$, θ real.

This shows that

$$\max_{B} |f'(\delta)| = F'(\delta) , \qquad \delta \in D .$$

Q.E.D.

We have the following easy corollary to Theorem 2.1:

Corollary 2.1. The problem

$$\int_c |h(z)| \, dS = \min. ,$$

has a solution where h(z) is given by

$$h(z) = \frac{1}{(z - \delta)^2} + h_1(z) ,$$

$h_1(z)$ regular in D.

Proof:

$$\text{Let} \int_c |h(z)| \, dS < \infty .$$

Let F(z) be the extremal function of Theorem 2.1. Applying the residue theorem, we obtain

$$2\pi F'(\delta) = \left| \frac{1}{i} \int_c F(z) \, h(z) \, dz \right|$$

$$\leq \int_c |h(z)| dS$$

Since $F(z) \equiv 1$ for $z \in c$, (2.15) shows that

$$2\pi F'(\delta) = \left| \frac{1}{i} \int_c F(z) \, Q(z) \, dz \right|$$

$$= \int_c |Q(z)| dS .$$

Hence $\int_c |Q(z)| dS \leq \int_c |h(z)| dS$, and the function

Q(z) solves the extremal problem. It can easily be shown that the equality holds only if

$$h(z) \equiv Q(z) \quad .$$

Q.E.D.

Chapter III

THE CLASS B_λ

We are now interested in solving a slight variation of the extremal problem found in Chapter II. The difference between the two is found in the restrictions placed on the function $f(z)$ which we now give:

Again D denotes a multiply connected domain which is bounded by n closed analytic curves $c_j, j=1,\ldots,n$. We define the following class B_λ of functions:

$$B_\lambda = \left\{ f(z) \middle| f(\delta) = 0 \ , \ \limsup_{z\to z_0} |f(z)| \leq \lambda(z_0), \ z_0 \in c \right\},$$

where $\lambda(z)$ is a positive function continuous on each c_j and f is a single-valued analytic function regular in D.

<u>Theorem 3.1</u>. Given a domain D and a positive continuous function λ on the boundary c of D, there exist two analytic functions $K(z,\delta) = K_\lambda(z,\delta)$ and $L(z,\delta) = L_\lambda(z,\delta)$ uniquely determined by the following properties:

i) $K(z,\delta)$ and $L(z,\delta) - [2\pi(z-\delta)]^{-1}$ are regular in D,

ii) $|K(z,\delta)|$ is continuous in D + c and $|L(z,\delta)|$ is continuous in $D + c - c_\varepsilon$, where c_ε denotes an open disk about the point $\delta \in D$,

iii) $K(z,\delta)$ and $L(z,\delta)$ are connected by the identity

$$\lambda(z) \ [K(z,\delta)]^* dS = - i \ L(z,\delta) dz \ ,$$

for $z \in c$ and $dS = |dz|$.

<u>Proof</u>: We begin by investigating the existence of $K(z,\delta)$

and $L(z,\delta)$. Consider the harmonic function $u(z)$ defined as follows:

$$u(z) = -\frac{1}{4\pi}\int_c \log\lambda(t)\,\frac{\partial g(z,t)}{\partial n_t}\,ds_t \quad .$$

Since by hypothesis $\lambda(t)$ is positive and continuous on each c_j, $\log\ \lambda(t)$ is also continuous there. The boundary values of u are evidently $\frac{1}{2}\log\lambda(z)$, $z \in c$. The harmonic ~~conjugate $v(z)$ of u (z) is not~~ necessarily single-valued in D. We shall denote the periods of v (z) about c_j by $2\pi p_j$, $j=1,\ldots,n$. Consider the function $U(z)$ defined by

$$U(z) = u(z) + \sum_{k=1}^{n-1} \varepsilon_k\, g(z,z_k)$$

where $z_k \in D$, $k=1,\ldots,n-1$, $\varepsilon_k = \pm 1$.
Since $g(z,z_k) = 0$ for $z \in c$, $U(z)$ has the same boundary values as $u(z)$. The period of $g(z,z_k)$ about c_j with respect to D is $-(2\pi)^{-1} w_j(z_k)$, where $w_j(z)$ is the harmonic measure of c_j. The period of $2\pi P_j$ of the harmonic conjugate $V(z)$ of $U(z)$ about c_j is thus

$$P_j = p_j - \sum_{k=1}^{n-1} \varepsilon_k\, w_j(z_k) \qquad j=1,\ldots,n$$

The analytic function

$$q(z) = e^{U(z) + i\,V(z)}$$

will be single-valued in D if the periods P_j are integer-valued.

Since the sum of the n periods must be equal to the sum of the periods due to logarithmic poles, then $P_1 + \dots + P_n = n - 1$. Hence we must find the points $z_1, \dots, z_{n-1}$ in D which satisfy

$$\sum_{k=1}^{n-1} \varepsilon_k w_j(z_k) = p_j + m_j , \quad j=1,\dots,n-1 \qquad (3.1)$$

for suitable choices of ε_k and integers m_j.

We prove the existence of the n-1 points z_k which satisfy eq. (3.1):

First of all, we know that a univalent conformal map transforms the harmonic measures of D into the harmonic measures of its conformal image. Instead of handling the whole conformal equivalence class of domains, we shall consider only its representative domain. We choose the representative domain by considering the conformal mapping which takes the domain D with boundary c_n into the upper half plane D_p; c_n will correspond to the real axis.

Now, since $w_j(z) = 1,\dots,n-1$ are 0 on the real axis, they may be analytically continued beyond the real axis by means of the identity $w_j(z^*) = - w_j(z)$. The functions $w_j(z)$ are harmonic in D' where $D' = D_p + D_p^*$. (See Fig. 2)

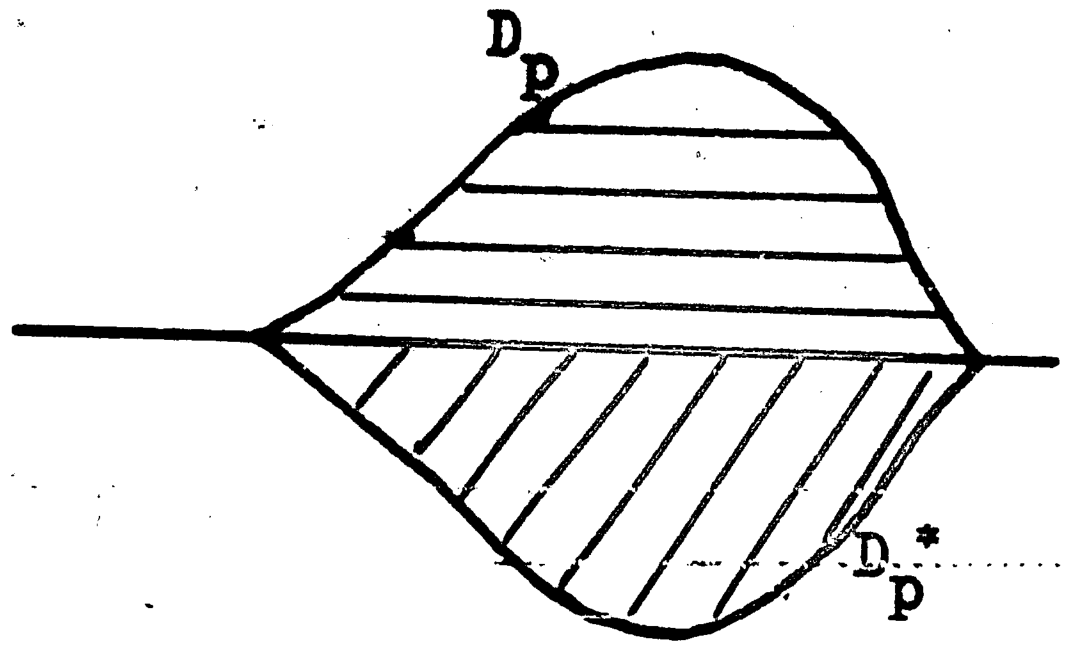

Figure 2

(Denote c' as the boundary of D'.)

By virtue of the above identity, eq. (3.1) is equivalent to

$$\sum_{k=1}^{n-1} w_j(z_k) = p_j + m_j, \ j=1,\ldots,n-1, \ z_k \in D' . \quad (3.2)$$

Now let

$$R = R(z_1,\ldots,z_{n-1})$$

$$= \sum_{j=1}^{n-1} \Big[\sum_{k=1}^{n-1} w_j(z_k) - p_j - m_j \Big]^2 .$$

We see that R is continuous in D' + c'. For some definite choice of integers m_j, R assumes a non-negative minimum for n-1 points $z_k \in D' + c'$.

Since $-1 \leq w_j(z) \leq 1$, there are a finite number of sets $\{m_j\}$; hence we are able to choose integers m_j such that the minimum of R is as small as possible.

Suppose $m_1,\ldots,m_{n-1}$ are the desired integers which minimize R. Then we may write

$$R = \sum_{j=1}^{n-1} \left[\sum_{k=1}^{n-1} w_j(z_k) - A_j \right]^2 = \min. \qquad (3.3)$$

where $A_j = p_j + m_j$ and $z_k \varepsilon D' + c'$.

It is easy to verify that there exists at least one minimizing set $Z = \{z_k\}$ entirely in D', and furthermore, eq. (3.3) shows that we may assume that the minimizing set does not contain both z_k and its conjugate z_k^*. Since $D' \supset Z$

$$\frac{\partial R}{\partial z_k} = 0 \qquad k = 1,\ldots,n-1 \ ,$$

where $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$, for all $z_k \varepsilon Z$.

The necessary condition for a minimum is thus

$$\sum_{j=1}^{n-1} a_j W_j'(z_k) = 0 \qquad k = 1,\ldots,n-1 \ , \qquad (3.4)$$

where $a_j = \sum_{k=1}^{n-1} w_j(z_k) - A_j$ and

$$\mathrm{Re}\left\{W_j(z)\right\} = w_j(z) \ .$$

We have two possibilities; either all the a_j vanish, or $W'(z_k) = 0$ for all $z_k \varepsilon Z$ where

$$W(z) = \sum_{j=1}^{n-1} a_j W_j(z) \ . \qquad (3.5)$$

In the first case,

$$0 = \sum_{k=1}^{n-1} w_j(z_k) - A_j$$

and hence

$$A_j = p_j + m_j = \sum_{k=1}^{n-1} w_j(z_k)$$

and the problem is solved.

For the second case, all we need to consider is the existence of at least one $a_j \neq 0$. For if so, then all $z_k \in Z$ are critical points of $W(z)$. If some of the z_k coincide, the critical points will then be counted according to their multiplicity.

Since the a_j are real, by the Schwarz reflection principle we have that $W_j'(z^*) = -[W_j'(z)]^*$ and Z^*, the set of conjugates of points in Z, is also a set of critical points of $W(z)$. If no $z_k \in Z$ lies on the real axis (See Fig. 2), then clearly Z and Z^* are disjoint and there are then $2(n-1)$ critical points of $W(z)$ of which only $n-1$ of these belong to D_p. But by the Argument Principle, $W(z)$ cannot have more than $n-2$ critical points in D_p which is impossible. Thus, we need only consider the case where at least one point of Z is on the real line and $a_k \neq 0$ for some k. In this case R is constant on the real line and all points of the real line will be critical points of $W(z)$ so that

$$W'(z) = \sum_{j=1}^{n-1} a_j W_j'(z) \equiv 0 .$$

This also holds for all $z \in D'$ if we proceed by analytic continuation. But since the $W_j'(z)$ are linearly independent, this implies that all the $a_j = 0$ which leads also to a contradiction.

Hence the assertion that eq. (3.1) has n-1 solutions holds.

Now let V(z) be the harmonic conjugate of U(z) above. The periods of U(z) + i V(z) are integral multiples of $2\pi i$, so that q(z) has no periods about c_j. We see that the singularities of z_k of U(z) are either simple zeros or simple poles of q(z), depending upon whether $\varepsilon_k = 1$ or -1. The function q(z) is thus single-valued in D_p, and the number of zeros and poles counted according to their multiplicities is less than n-1.

Let

$$|q(z)|^2 = \lambda(z) \qquad z \in c \quad . \tag{3.6}$$

We define the following functions:

$$K_1(z,\delta) = K'(z,\delta) + \sum_{j=1}^{m} \alpha_j K(z,z_j) + \sum_{j=m+1}^{n-1} \alpha_j L(z,z_j) \tag{3.7}$$

$$m \leq n-1 \quad ,$$

and

$$L_1(z,\delta) = L'(z,\delta) + \sum_{j=1}^{m} \alpha_j^* L(z,z_j) + \sum_{j=m+1}^{n-1} \alpha_j^* K(z,z_j) \tag{3.8}$$

where $z_1,\ldots,z_m$ are the zeros of q(z) and $z_{m+1},\ldots,z_{n-1}$ its poles, and K(z,t) and L(z,t) satisfy the equation

$$K^*(z,t)dS = -i\,L(z,t)dz \qquad z \in c \quad . \tag{3.9}$$

We further impose on eqs. (3.7) and (3.8) the requirement that the constants α_j shall be determined by the conditions

$$K_1(z_k,\delta) = 0 \quad , \quad k = 1,\ldots,m$$

and

$$L_1(z_k,\delta) = 0 \quad , \quad k = m+1,\ldots,n-1 \quad .$$

By virtue of eq. (3.9)

$$K_1^*(z,\delta)dS = -i\, L_1(z,\delta)dz \quad z \in c \quad . \tag{3.10}$$

We can define $K_\lambda(z,\delta)$ and $L_\lambda(z,\delta)$ as follows:

$$K(z,\delta) = K_\lambda(z,\delta) = \frac{K_1(z,\delta)}{q(z)\ [q(\delta)]^*} \tag{3.11}$$

and

$$L(z,\delta) = L_\lambda(z,\delta) = \frac{L_1(z,\delta)\ q(z)}{q(\delta)} \tag{3.12}$$

We assert that $K(z,\delta)$ is regular in D_p because the zeros of $q(z)$ are cancelled by the zeros of $K_1(z,\delta)$, and the poles of $q(z)$ are cancelled by the poles of $K_1(z,\delta)$. $L(z,\delta)$ is regular in D_p except for a simple pole at $z = \delta$. Since the residue of $L(z,\delta)$ is $(2\pi)^{-1}$, the same holds for $L_\lambda(z,\delta)$.

For $z \in c$ we have that

$$\left\{K(z,\delta)\ q(z)\ [q(\delta)]^*\right\}^* dS = \frac{1}{i}\,\frac{L(z,\delta)\ q(\delta)}{q(z)}\, dz \quad .$$

Hence,

$$[q(z)]^* K^*(z,\delta)\ q(\delta)dS = \frac{1}{i}\,\frac{L(z,\delta)\ q(\delta)}{q(z)}\, dz \quad ,$$

or

$$[q(z)]^* K^*(z,\delta)dS \searrow \frac{1}{i}\frac{L(z,\delta)}{q(z)}\,dz \quad .$$

Thus, $|q(z)|^2 K^*(z,\delta)dS \;=\; -\,i\,L(z,\delta)dz \qquad z \,\varepsilon\, c \quad .$

By (3.6)

$$\lambda(z) \;=\; -\,\frac{i\,L(z,\delta)dz}{K^*(z,\delta)dS}$$

This proves property (iii).

Now since $\lambda(z) > 0$,

$$-\,\frac{i\,L(z,\delta)\,\overline{K(z,\delta)}}{|K(z,\delta)|^2}\,dz \geq 0 \ ,$$

or

$$-\,i\,L(z,\delta)\,K(z,\delta) \geq 0, \qquad z \,\varepsilon\, c\ . \tag{3.13}$$

We now consider the function h(z) defined as follows:

$$h(z) \;=\; \int_z^{z_0} K_\lambda(t,\delta)\,L_\lambda(t,\delta)dt\ , \qquad z_0 \,\varepsilon\, c_j\ .$$

For $z \,\varepsilon\, c_j$ $\operatorname{Re}\{h(z)\}$ = constant.

By the Schwarz inversion principle h(z) therefore is regular and consequently, $K(z,\delta)\,L(z,\delta)$ is regular on c.

Both $|K(z,\delta)|$ and $|L(z,\delta)|$ are continuous separately, since

$$\lambda(z) \;=\; |\lambda(z)| \;=\; \left|\frac{L(z,\delta)}{K(z,\delta)}\right|, \qquad z \,\varepsilon\, c$$

and both the product and ratio of $|K(z,\delta)|$ and $|L(z,\delta)|$ are continuous on c. Since $L(z,\delta)$ has a pole at $\delta \,\varepsilon\, D$, $|L(z,\delta)|$ is continuous in D except at the point δ. This proves property (ii). To show uniqueness, let there be another pair of functions $K''(z,\delta)$ and $L''(z,\delta)$. Then for $z \in c$ we have that

$$\lambda(z)\,[K''(z,\delta)]^* dS = -i\, L''(z,\delta)dz$$

and hence,

$$\lambda(z)[K(z,\delta) - K''(z,\delta)]^* dS = -i[L(z,\delta) - L''(z,\delta)]dz, \quad z \in c \qquad (3.14)$$

where $L(z,\delta) - L''(z,\delta)$ is regular on D_p. From eq. (3.14) it follows that

$$\lambda(z)\left|K(z,\delta)-K''(z,\delta)\right|^2 dS = -i[L(z,\delta)-L''(z,\delta)][K(z,\delta)-K''(z,\delta)]dz.$$

Integrating both sides of the last equation over c we obtain, by Cauchy's theorem,

$$\int_c \left|K(z,\delta) - K''(z,\delta)\right|^2 dS = 0$$

Hence, $K(z,\delta) = K''(z,\delta)$,

and

$$0 = -i\,[L(z,\delta) - L''(z,\delta)]dz,$$

or

$$L(z,\delta) = L''(z,\delta)\ , \quad z \in c\ .$$

This completes the proof of the theorem.

Q.E.D.

We state without proof the following corollaries which follow immediately from the preceding theorem:

<u>Corollary 3.1.</u> Let $f(z)$ be regular and single-valued in D and let $\int_c \lambda(z)\left|f(z)\right|^2 dS$ exist, where the integral is

taken in the Lebesgue sense. Then

$$\int_C \lambda(z)\, f(z)\, [K(z,\delta)]^* dS = f(\delta) .$$

Corollary 3.2. $K(z,\delta)$ is hermitian; i.e.,

$$K(z,\delta) = [K(z,\delta)]^*$$

Corollary 3.3. $L(z,\delta) = -L_u(z,\delta)$,

where

$$\lambda(z)\, u(z) = 1 .$$

The following theorem illustrates how the function $K_\lambda(z,\delta)$ solves the extremal problem which is given in its hypothesis:

Theorem 3.2. Let $g(z)$ be regular and single-valued in D; let $\int_C \lambda(z) \left|g(z)\right|^2 dS < \infty$ and $g(\delta) = 1$, $\delta \varepsilon D$; then

$$\int_C \lambda(z) \left|M(z)\right|^2 dS \leq \int_C \lambda(z) \left|g(z)\right|^2 dS,$$

where $M(z) = \dfrac{K_\lambda(z,\delta)}{K_\lambda(\delta,\delta)}$. Equality holds only if $g(z) \equiv M(z)$.

Proof: By Corollary 3.1,

$$\int_C \lambda(z) \left|K(z,\delta)\right|^2 dS = K_\lambda(\delta,\delta) , \qquad (3.15)$$

where we have replaced $f(z)$ by $K_\lambda(z,\delta) = K(z,\delta)$. Also, by Corollary 3.1, we have that

$$\int_C \lambda(z)\, g(z)\, [K(z,\delta)]^* dS = g(\delta) .$$

Hence,

$$1 = |g(\delta)|^2 = \left|\int_C \lambda(z)\, g(z)\, [K(z,\delta)]^* ds\right|^2$$

$$\leq \int_C \lambda(z)|g(z)|^2 ds \int_C \lambda(z)|K(z,\delta)|^2 ds$$

by Schwarz's inequality.

But the latter expression is equal to

$$K_\lambda(\delta,\delta) \int_C \lambda(z)\,|g(z)|^2 ds \quad .$$

Hence,

$$1 \leq K_\lambda(\delta,\delta) \int_C \lambda(z)\,|g(z)|^2 ds ,$$

or

$$\int_C \lambda(z)\,|g(z)|^2 ds \geq [K_\lambda(\delta,\delta)]^{-1} . \qquad (3.16)$$

Since $M(z)\, K_\lambda(\delta,\delta) = K(z,\delta)$

$$|K(z,\delta)|^2 = |M(z)|^2\, |K_\lambda(\delta,\delta)|^2$$

From eq. (3.15) it follows that

$$\int_C \lambda(z)|M(z)|^2|K_\lambda(\delta,\delta)|^2 ds = |K_\lambda(\delta,\delta)|^2 \int_C \lambda(z)|M(z)|^2 ds$$

$$= K_\lambda(\delta,\delta) \; .$$

Therefore

$$\int_C \lambda(z)|M(z)|^2 ds = [K_\lambda(\delta,\delta)]^{-1} \qquad (3.17)$$

Hence, combining eqs. (3.16) and (3.17) we have that

$$\int_C \lambda(z)|M(z)|^2 ds \leq \int_C \lambda(z)|g(z)|^2 ds \; .$$

If the equality is to hold, we must have that

$$\int_C \lambda(z)\, g(z)\, [K(z,\delta)]^* dS \geq 0$$

which implies that $g(z)\,[K(z,\delta)]^* \geq 0$ and that $|g(z)| = |M(z)|$ for $z \varepsilon c$.

Since

$$M(\delta) = \frac{K_\lambda(\delta,\delta)}{K_\lambda(\delta,\delta)} = 1 ,$$

$$g(\delta) = M(\delta) , \quad \delta \varepsilon D .$$

Hence $\quad g(z) \equiv M(z)$ for equality to hold.

Q.E.D.

<u>Theorem 3.3</u>. Let the function $h(z)$ be regular in D except for a simple pole of residue $(2\pi)^{-1}$ at $z = \delta$; then

$$\int_C u(z)|h(z)|^2 dS \geq \int_C u(z)|L(z,\delta)|^2 dS,$$

where $u(z)\lambda(z) = 1, \quad z \varepsilon c$.

Equality holds only for $h(z) = L_\lambda(z,\delta) = L(z,\delta)$.

<u>Proof</u>:

$$h(z) = \frac{1}{2\pi\,(z-\delta)} + \text{regular terms.}$$

Then

$$\frac{1}{i}\int_C h(z)\, K(z,\delta) dz = \frac{1}{2\pi i}\int_C \frac{K(z,\delta)}{(z-\delta)} dz = K_\lambda(\delta,\delta)$$

by Cauchy's integral formula.

Using the Schwarz inequality and the hypothesis that $u(z)\lambda(z) = 1$ we obtain

$$K_\lambda^2(\delta,\delta) \leq \int_C u(z)|h(z)|^2 dS \int_C \lambda(z)|K(z,\delta)|^2 dS$$

$$= K_\lambda(\delta,\delta) \int_C u(z)|h(z)|^2 dS ,$$

the latter equality due to Corollary 3.1.

Hence,

$$K_\lambda(\delta,\delta) \leq \int_c u(z)\left|h(z)\right|^2 dS .$$

Using property (iii) in Theorem 3.1 and the inequality (3.13) we have

$$\int_c u(z)\left|L(z,\delta)\right|^2 dS = -i \int_c K(z,\delta)\, L(z,\delta) dz = K_\lambda(\delta,\delta) .$$

Hence

$$\int_c u(z)\left|L(z,\delta)\right|^2 dS \leq \int_c u(z)\left|h(z)\right|^2 dS .$$

Clearly, equality holds only if

$$-\, i\, h(z)\, K(z,\delta) dS \geq 0 \quad \text{and} \quad \left|h(z)\right| = \left|L(z,\delta)\right|$$

on c.

Q.E.D.

Chapter IV

BOUNDS FOR DERIVATIVES OF ANALYTIC FUNCTIONS

SECTION 1. Functions of the Class B_R

We now consider the following problem:

Let $\phi(z) \in B_R$, $\delta \in D$; $\mathrm{Re}\left\{e^{i\theta}\phi^{(k)}(\delta)\right\} = \max.$

where θ is an arbitrary fixed angle.

Theorem 4.1. The solution to the above stated problem is a function $W = \Phi(z)$ which yields a $(1,m)$ conformal map of D with boundary c onto the strip $-1 < \mathrm{Re}\{W\} < 1$ where $m \leq n+k-1$.

Proof:

Let $N(z,\delta)$ be the Neumann's function of D and M be its conjugate. Set $q(z,\delta) = N(z,\delta) + i\, M(z,\delta)$. From Chapter I, Section 1, for $z \in c$,

$$\frac{\partial N}{\partial n} = -\frac{2\pi}{L} .$$

Let $v(z)$ be harmonic in D. If we delete a circular neighborhood c_ε of δ and apply eq. (1.2) we arrive at the following computations:

$$v(\delta) = \frac{1}{2\pi}\int_c N(z,\delta)\frac{\partial v(z)}{\partial n}dS + \frac{1}{2\pi}\int_c v(z)\frac{\partial N(z,\delta)}{\partial n}dS$$

$$= \frac{1}{2\pi}\int_c N(z,\delta)\frac{\partial v(z)}{\partial n}dS - \frac{1}{L}\int_c v(z)dS .$$

Letting $\phi(z) = u(z) + i\, v(z)$ be an analytic function, applying the Cauchy-Riemann equations and integrating by parts we have

$$\int_C N(z,\delta)\frac{\partial v}{\partial n} = -\int_C N(z,\delta)\frac{\partial u}{\partial S}\,dS = -\int_C u\,\frac{\partial M(z,\delta)}{\partial n}\,dS\ .$$

Hence $$v(\delta) = -\frac{1}{2\pi}\int_C u(z)\,\frac{\partial M(z,\delta)}{\partial n}\,dS - \frac{1}{L}\int_C v(z)\,dS \qquad (4.1)$$

so that

$$\text{Re}\left\{e^{i\theta}\overline{\phi(\delta)}\right\} = \text{Re}\left\{(\text{Cos }\theta + i\,\text{Sin }\theta)[u(\delta) + i\,\overline{v}(\delta)]\right\}$$

$$= u(\delta)\,\text{Cos }\theta - v(\delta)\,\text{Sin }\theta$$

$$= -\frac{1}{2\pi}\int_C u(z)\,\text{Cos }\theta\,\frac{\partial g(z,\delta)}{\partial n}\,dS$$

$$+\frac{1}{2\pi}\int_C u(z)\,\text{Sin }\theta\,\frac{\partial M(z,\delta)}{\partial n}\,dS$$

$$+\frac{1}{L}\,\text{Sin }\theta\int_C v(z)\,dS$$

$$\text{Re}\left\{e^{i\theta}\phi(\delta)\right\} = -\frac{1}{2\pi}\int_C u(z)\left[\text{Cos }\theta\,\frac{\partial g(z,\delta)}{\partial n} - \text{Sin }\theta\,\frac{\partial M(z,\delta)}{\partial n}\right]dS$$

$$+\frac{1}{L}\,\text{Sin }\theta\int_C v(z)\,dS\ .$$

For $\delta = \xi + i\eta$, differentiating k times

$$\text{Re}\left\{e^{i\theta}\phi^{(k)}(\delta)\right\} = -\frac{1}{2\pi}\int_C u(z)\left[\text{Cos }\theta\,\frac{\partial^{k+1}g(z,\delta)}{\partial\xi^k\,\partial n} - \text{Sin }\theta\,\frac{\partial^{k+1}M(z,\delta)}{\partial\xi^k\,\partial n}\right.$$

$$\left. + \sum_{j=1}^{n-1}\lambda_j\,\frac{\partial w_j(z)}{\partial n}\right]dS \qquad (4.2)$$

where $\int_C u(z)\,\frac{\partial w_j(z)}{\partial n}\,dS = 0, \quad j = 1,\ldots,n-1$

and λ_j are arbitrary real parameters.

Since $\emptyset(z) \varepsilon B_R$, $|u(z)| \leq 1$.

Therefore

$$\text{Re}\left\{e^{i\theta}\emptyset^{(k)}(\delta)\right\} \leq \frac{1}{2\pi}\int_c \left| \cos\theta \frac{\partial^{k+1} g(z,\delta)}{\partial \xi^k \partial n} - \sin\theta \frac{\partial^{k+1} M(z,\delta)}{\partial \xi^k \partial n} \right.$$

$$\left. + \sum_{j=1}^{n-1} \lambda_j \frac{\partial w_j}{\partial n} \right| dS$$

$$= \frac{1}{2\pi}\int_c |P| \, dS .$$

As before (see eq.[2.7])

$$\int_c \frac{|P|}{P} \frac{\partial w_j}{\partial n} dS = 0$$

where $\int_c |P| \, dS$ is minimized and U(z) is defined in Chapter II.

Hence the harmonic conjugate V(z) of U(z) is single-valued in D. If we let $\Phi(z) = U(z) + i\,V(z)$ be regular and single-valued in D and if $|P| = -P\,U$, then

$$\text{Re}\left\{e^{i\theta}\emptyset^{(k)}(\delta)\right\} \leq \frac{1}{2\pi}\int_c |P| \, dS = -\frac{1}{2\pi}\int_c U\,P\, dS$$

$$= \text{Re}\left\{e^{i\theta}\,\Phi^{(k)}(\delta)\right\} .$$

Hence

$$\text{Re}\left\{e^{i\theta}\emptyset^{(k)}(\delta)\right\} \leq \text{Re}\left\{e^{i\theta}\,\Phi^{(k)}(\delta)\right\} .$$

Since $\Phi(z) \varepsilon B_R$, $W = \Phi(z)$ solves the extremal problem.

To show that $W = \Phi(z)$ is a conformal map we note that U(z) = 1 or -1 for all but a finite number of

$z \in c$ and $|U(z)| \leq 1$. We now determine the number of coverings as follows:

Since $|P| = -PU$,

$$U\,P\,dS \leq 0 \qquad z \in c\,.$$

We also have that

$$q'(z,\delta)dz = d\,q(z,\delta) = d\,N(z,\delta) + i\,d\,M(z,\delta)$$

$$= \frac{\partial N(z,\delta)}{\partial S}\,dS + i\,\frac{\partial M(z,\delta)}{\partial S}\,dS$$

$$= -\frac{\partial M(z,\delta)}{\partial n}\,dS + i\,\frac{\partial N(z,\delta)}{\partial n}\,dS$$

$$= -\frac{\partial M(z,\delta)}{\partial n}\,dS - \frac{2\pi}{L}\,i\,dS, \quad z \in c\,.$$

Hence

$$\frac{\partial M(z,\delta)}{\partial n}\,dS = -\,q'(z,\delta)dz - \frac{2\pi}{L}\,i\,dS\,, \quad z \in c\,; \qquad (4.3)$$

therefore for $p(z)$ defined in Chapter II,

$$-\,U[\cos\theta\,\frac{\partial^k}{\partial\xi^k}(-i\,p'\,dz) - \sin\theta\,\frac{\partial^k}{\partial\xi^k}(-\,q'\,dz - \frac{2\pi}{L}\,i)$$

$$+ \sum_{j=1}^{n-1}\lambda_j(-\,i\,w_j')]\,dz \geq 0\,.$$

It then follows that

$$-i\,U(z)\,\frac{\partial^k}{\partial\xi^k}[\cos\theta\,p'(z) + i\sin\theta\,q'(z,\delta) + \sum_{j=1}^{n-1}\lambda_j w_j'(z)]dz \leq 0 \quad (4.4)$$

If we make the obvious substitution of $R(z)$ in (4.4) we obtain

$$-\,i\,U(z)\,R(z)\,dz \leq 0 \qquad z \in c\,. \qquad (4.5)$$

Clearly, $R(z)$ is single-valued in D.

Since $p'(z) = i \frac{\partial g(z,\delta)}{\partial n}$, $R(z)$ has a pole at $z = \delta$. Expanding $R(z)$ in a neighborhood of $z = \delta$:

$$R(z) = -\frac{k!\,e^{i\theta}}{(z-\delta)^{k+1}} + \sum_{j=1}^{n-1} a_j (z-\delta)^j .$$

We describe the boundary of D in the positive sense with respect to the domain enclosed by it. However, the n-1 inner boundaries will be described in the negative sense. Hence arg(dz) on c is given by

$$2\pi - 2\pi(n-1) = -2\pi(n-2) = \Delta_c \arg(dz) .$$

Thus, since $\Delta_c \arg R(z) = -\Delta_c \arg (dz)$,

$$\Delta_c \arg R(z) = 2\pi(n-2) .$$

If we let n_1 denote the number of zeros in D and n_2 the number of zeros on c (recall that we must count the number of zeros on c with half of their multiplicities) we have that

$$n_1 = n-2$$

$$\frac{n_2}{2} = k+1$$

Hence $n_1 + \frac{n_2}{2} = n-2 + k+1$

or $2n_1 + n_2 = 2(n + k - 1)$.

Consequently,

$$n_2 \leq 2(n + k - 1) .$$

From (4.5) we have that each change in sign of $U(z)$ on c coincides with a zero of $R(z)$. Every two changes in sign corresponds to a single sheet of the multiply covered strip $-1 < \mathrm{Re}\{W\} < 1$.

The maximum number of sheets is therefore $n + k - 1$.

Hence the number of sheets $m \leq n + k - 1$.

Q.E.D.

These examples illustrate the procedures involved in solving extremal problems of the class B_R and equivalently of those of class B. Due to the repetitiveness and lengthy nature of the proofs involved, from now on we shall omit similar arguments.

SECTION 2. Univalent Functions

We shall say that a function $f(z)$ is univalent in a region D if it is analytic, single-valued and one-to-one.

Consider the following problem, solved by Landau: Let $f(z)$ be regular and bounded in the unit disk such that $|f'(0)| = a$, $0 < a < 1$. Then $f(z)$ yields a univalent mapping of the disk $|z| < \rho = a[1 + \sqrt{1-a}]^{-1}$, and in fact, the function $f(z)$ with the smallest radius of univalence yields a (1,2) mapping of the unit disk onto itself.

In order to see how this has been generalized we state without proof the following:

<u>Theorem 4.2</u>. Let $\phi(z) \in B_R$, $\phi'(\delta) = A$ where $A = a + ib$,

$\delta \in D$; $z_1, z_2 \in D$. Then

$$\left|\psi(z_1) - \psi(z_2)\right| \le \left|\phi(z_1) - \phi(z_2)\right|$$

where $\psi(z) \in B_R$ and $\psi'(\delta) = A$.
Also, $W = \psi(z)$ yields a $(1,m)$ conformal map of D onto the strip $-1 < \mathrm{Re}\{W\} < 1$ where $n \le m \le n+2$.

With this theorem in mind, we are now able to state and prove the following:

Theorem 4.3. Let $f(z) \in B$ and $f'(\delta) = A$ where $\delta \in D$. Let ρ be the largest number such that all functions $\phi(z)$ with the properties satisfying Theorem 4.2 are univalent in $|z-\delta| < \rho$. Then there exists a function $F(z) \in B$, with $F'(\delta) = A$ which yields a $(1,m)$ map of D onto the unit disk such that $\psi(z)$ is not univalent in any circle $|z-\delta| < \rho'$ with $\rho < \rho'$.

Proof: Let $\phi(z) = \frac{4}{\pi} \tan^{-1} f(z)$.
If $\phi(z_1) = \phi(z_2)$ then

$$\tan^{-1} f(z_1) = \tan^{-1} f(z_2) .$$

Hence,

$$f(z_1) = f(z_2) .$$

Since $f(z) \in B$, $f(z)$ is single-valued in D so that $z_1 = z_2$.

$\phi(z)$ is therefore a univalent function.

Let $\phi'(\delta) = A$ and define B_R^A as follows:

$$B_R^A = \left\{ \phi(z) \,\middle|\, \phi(z) \in B_R \text{ and } \phi'(\delta) = A \right\} \tag{4.6}$$

Clearly $B_R \supseteq B_R^A$.

Since B_R^A is closed and bounded, it is compact. Hence there exist real numbers $\rho_i > 0$, $(\rho_i < \rho_j\ ,\ i < j)$ such that for all $\phi(z) \in B_R^A$, $\phi(z)$ are univalent for $|z-\delta| < \rho_i$.

Let $\rho = \max \{ \rho_i \}$. Then ρ is the radius of univalence of B_R^A. Hence there exists a function $\phi_o(z) \in B_R^A$ which is univalent in $|z-\delta| < \rho$ but not in $|z-\delta| \leq \varepsilon$ where $\varepsilon > \rho$.

We now consider two cases for boundary points:

Case I: Let z_1 and z_2 be points such that $|z_1 - \delta| = \rho$ and $|z_2 - \delta| = \rho$ and for which $\phi_o(z_1) = \phi_o(z_2)$, $z_1 \neq z_2$. Then by Theorem 4.2 there exists a function $\psi(z) \in B_R^A$ such that

$$|\psi(z_1) - \psi(z_2)| \leq |\phi(z_1) - \phi(z_2)| = 0$$

Hence

$$\psi(z_1) = \psi(z_2) .$$

$\psi(z)$ therefore cannot be univalent in a disk about δ with a radius $\varepsilon > \rho$. For if it were so, $\psi(z_1) = \psi(z_2)$ would imply that $z_1 = z_2$ which contradicts the hypothesis.

Case II: Let z_3 be another point on the circumference $|z-\delta| = \rho$ at which $\phi'(z_3) = 0$. From Theorem 4.2 we have

$$\frac{|\psi(z_1) - \psi(z_2)|}{|z_1 - z_2|} \leq \frac{|\phi(z_1) - \phi(z_2)|}{|z_1 - z_2|}$$

Now let $z_1 \longrightarrow z_3$ and $z_2 \longrightarrow z_3$.

Then

$$\left|\psi_o'(z_3)\right| < \left|\phi'(z_3)\right|$$

where $\psi_o(z) = \lim_{k \to \infty} \psi_k(z)$ and where for each integer k, $\psi_k(z)$ maps D onto strips whose maximum number is n+2.

Since the set of all $\psi_k(z)$ is compact, the limit function $\psi_o(z)$ yields a mapping of the same type.

Now let $\phi_o(z) \in B_R^A$ such that $\phi_o'(z_3) = 0$. Then since $\left|\psi_o'(z_3)\right| \leq \left|\phi'(z_3)\right|$,

$$\psi'(z_3) = 0 .$$

Hence $\psi(z)$ is the required function; therefore, since $\phi(z)$ is univalent, the statement of the theorem is equivalent to the corresponding result for functions belonging to B_R^A. Hence the theorem.

Q.E.D.

SECTION 3. Functions of the Class B_λ

We now wish to extend the results of Chapter III to the first derivative of functions belonging to the class B_λ. To do this we state and prove

Theorem 4.4. Let $f(z) \in B_\lambda$, then

$$\left|f'(\delta)\right| \leq F'(\delta) = 2\pi K_u(\delta,\delta) ,$$

where $u(z_o)\lambda(z_o) = 1$, $z_o \in c$ and $F(z) = \dfrac{K_u(z,\delta)}{L_u(z,\delta)}$.

If $L_u(z,\delta) \neq 0$ for $z \in D$, $F(z) \in B_\lambda$ and the inequality $\left|f'(\delta)\right| \leq F'(\delta)$ is sharp; if $L_u(z,\delta) = 0$ at the points

$z_1, \ldots, z_k$, then the inequality holds for the wider class of functions f(z) which may have a simple or double pole at $z_1, \ldots, z_k$. For this wider class, the inequality is sharp and again, equality holds only if

$$F(z) \equiv \frac{K_u(z,\delta)}{L_u(z,\delta)} .$$

Proof: Since $f(z) \varepsilon B_\lambda$,

$$\limsup_{z \to z_o} |f(z)| \leq \lambda(z_o), \quad z_o \ \varepsilon \ c .$$

f(z) therefore must be bounded in D and hence $\lim_{z \to z_o} f(z)$ exists almost everywhere on c.

By property (i) of Theorem 3.1

$$L_u(z,\delta) - [2\pi(z-\delta)]^{-1} \text{ is regular in D.}$$

Hence $(2\pi)^{-1}$ is the residue of $L_u(z,\delta)$.

By Cauchy's theorem,

$$f'(\delta) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-\delta)^2} dz .$$

But $L_u(z,\delta) = \dfrac{(2\pi)^{-1}}{z-\delta}$

or $$L_u^2(z,\delta) = \frac{(4\pi^2)^{-1}}{(z-\delta)^2} = \frac{1}{4\pi^2(z-\delta)^2} .$$

Hence $f'(\delta) = \dfrac{4\pi^2}{2\pi i} \int_c f(z) L_u^2(z,\delta) dz$

or equivalently,

$$f'(\delta) = \frac{2\pi}{i} \int_c f(z) L_u^2(z,\delta) dz \qquad (4.7)$$

Therefore

$$\left|f'(\delta)\right| \leq 2\pi \int_c \lambda(z) \left|L_u^2(z,\delta)dz\right| \tag{4.8}$$

since lim sup $\left|f(z)\right| \leq \lambda(z)$.

By property (iii) of Theorem 3.1

$$\lambda(z)[K_u(z,\delta)]^*[L_u(z,\delta)]^*dS = -i\, L_u(z,\delta)[L_u(z,\delta)]^*dz$$

$$\lambda(z)[K_u(z,\delta)L_u(z,\delta)]^*dS = -i\, L_u^2(z,\delta)dz$$

$$\lambda(z)\left|K_u(z,\delta)\, L_u(z,\delta)dz\right| = \left|L_u^2(z,\delta)dz\right| .$$

Substituting into (4.8) we have

$$\left|f'(\delta)\right| \leq 2\pi \int_c \lambda^2(z) \left|K_u(z,\delta)\, L_u(z,\delta)dz\right|$$

or

$$\left|f'(\delta)\right| \leq 2\pi \int_c \left|K_u(z,\delta)\, L_u(z,\delta)dz\right| .$$

In view of (3.13) it then follows that

$$\left|f'(\delta)\right| \leq 2\pi \int_c K_u(z,\delta)\, L_u(z,\delta)dz \tag{4.9}$$

By Cauchy's theorem, the expression on the right of (4.9) is equal to $2\pi\, K_u(\delta,\delta)$. But $F'(\delta) = 2\pi\, K_u(\delta,\delta)$ and hence

$$\left|f'(\delta)\right| \leq F'(\delta) = 2\pi\, K_u(\delta,\delta) .$$

$$\left|F(z)\right| = \left|\frac{K_u(z,\delta)}{L_u(z,\delta)}\right| = \left|\frac{i\,[K_u(z,\delta)]^*dS}{L_u(z,\delta)dz}\right| = \frac{1}{\lambda(z)}$$

$$= u(z) , \quad z \in c .$$

If $L_u(z,\delta)$ has no zeros in D, we are done. If $L_u(z,\delta)$

has zeros $z_1,\ldots,z_k$, then everything still follows if $f(z)$ has simple or double poles at these points. Therefore the equality will now hold at points which will have simple poles.

Q.E.D.

Theorem 4.4 may be extended by further restricting the positive boundary function $\lambda(z)$. We do this by stating ~~the~~ following two corollaries to Theorem 4.4.

Corollary 4.1. Let $f(z)$ be regular and $|f(z)| \leq 1$ in D, and let $f(\delta) = f(a_1) = f(a_2) = \ldots = f(z_m) = 0$, $\delta, a_1, a_2, \ldots, a_m \in D$. Then

$$|f'(\delta)| \leq F'(\delta) = 2\pi K_\lambda(\delta,\delta) ,$$

where $\lambda(z) = \prod_{j=1}^{m} |z - a_j| \, |\delta - a_j|^{-1} \qquad z \in c, \ \delta \in D$

and $F(z) = \dfrac{K_\lambda(z,\delta)}{L_\lambda(z,\delta)} \prod_{j=1}^{m} \dfrac{(z - a_j)}{(\delta - a_j)}$.

Corollary 4.2. Let $g(z)$ be regular in D apart from m simple poles located at the points $a_1,\ldots,a_m$; let $g(\delta) = 0$, $\delta \in D$, and let $\limsup_{z\to c} |g(z)| \leq 1$. Then

$$|g'(\delta)| \leq G'(\delta) = 2\pi K_u(\delta,\delta) ,$$

where $u = u(z) = \prod_{j=1}^{m} |\delta - a_j| \, |z - a_j|^{-1} \qquad z \in c, \ \delta \in D$, and $$G(z) = \frac{K_u(z,\delta)}{L_u(z,\delta)} \prod_{j=1}^{m} \frac{(\delta - a_j)}{(z - a_j)} .$$

If $L_u(z,\delta) \neq 0$ in D, then G(z) satisfies the hypothesis of the theorem and the inequality is sharp. If $L_\lambda(z,\delta)$ has the zeros z_k in D, then the inequality is sharp for the wider class of functions g(z) which may have simple or double poles at the points z_k. G(z) solves the extremal problem in this case.

Chapter V

BOUNDED FUNCTIONS IN STARLIKE AND CONVEX DOMAINS

The theory presented earlier has an analogous development for functions belonging to the class B_R which is defined for starlike and convex domains. A domain D is starlike relative to an interior point z_o if every straight line segment joining z_o with a boundary point z_1 lies entirely in D. An example of a domain which is not starlike is illustrated in Fig. 3. Alternatively we may say that a domain D is starlike with respect to a point $z_o \in D$ if for any $z_1 \in D$ all $z = az_o + (1-a)z_1$, $0 < a < 1$ also belong to D.

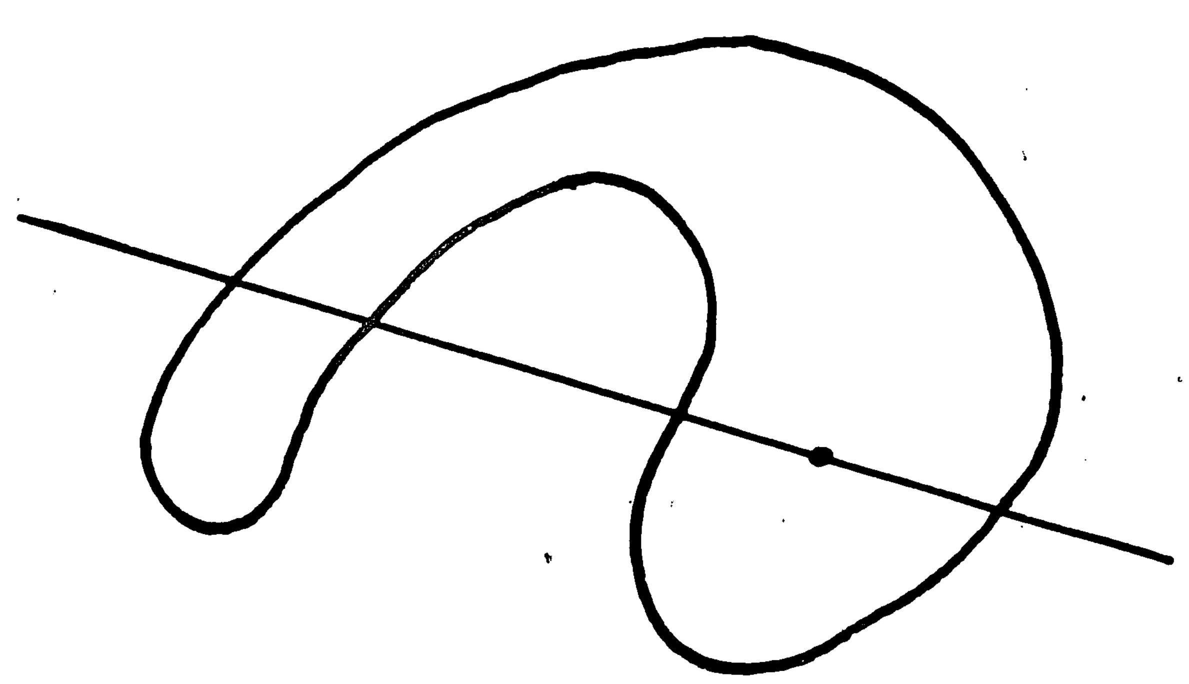

Figure 3

We may now state the following:

Definition 5.1. The radius of starlikeness $r_s(\delta)$ of an analytic function $f(z)$ with respect to the point δ at which $f(\delta) = 0$ is the radius of the largest circle about δ which is mapped by $f(z)$ onto a univalent domain that is starlike with respect to the origin.

Similarly we may say that a domain D is convex if for any two points $z_1, z_2 \in D$ the points of the set M also belong to D where

$$M = \left\{ z \,\middle|\, z = z_1 + a(z_2 - z_1), \quad 0 \le a \le 1 \right\}$$

We now define the radius of convexity as follows:

Definition 5.2. The radius of convexity $r_c(\delta)$ is defined as the radius of the largest circle about δ which is mapped by $f(z)$ onto a convex univalent domain.

With these definitions in mind, we may state the following:

Theorem 5.1. Let

$$B_R^{A'} = \left\{ \phi(z) \,\middle|\, \phi(z) \in B_R \,,\ \phi(\delta) = 0 \text{ and } \phi'(\delta) = A \right\}$$

and let $r_s(\delta)$ be the radius of starlikeness of $B_R^{A'}$ with respect to δ. Then $r_s(\delta)$ is the exact radius of starlikeness of a function $W = \psi(z) \in B_R^{A'}$ which yields a $(1,m)$ mapping of D onto the strip $-1 < \mathrm{Re}\left\{W\right\} < 1$.

Proof: Consider the quantity

$$\mathrm{Re}\left\{ e^{i\theta} [\phi(z_1) - a\, \phi(z_2)] \right\}, \quad a > 0 \ .$$

We then obtain that for

$$\phi(z) = u(z) + i\, v(z)$$

$$\mathrm{Re}\left\{ e^{i\theta} [\phi(z_1) - a\, \phi(z_2)] \right\} = [u(z_1) - a\, u(z_2)] \cos\theta$$

$$- [v(z_1) - a\, v(z_2)] \sin\theta$$

$$= -\frac{1}{2\pi}\int_C u(z)\left\{\cos\theta\left[\frac{\partial g(z,z_1)}{\partial n} - a\frac{\partial g(z,z_2)}{\partial n}\right]\right.$$

$$\left. - \sin\theta\left[\frac{\partial M(z,z_1)}{\partial n} - a\frac{\partial M(z,z_2)}{\partial n}\right] + \sum_{j=1}^{n-1}\lambda_j\frac{\partial w_j}{\partial n}\right\} dS$$

where λ_j are arbitrary real parameters.

Since $A = \phi'(\delta) = \frac{\partial u(\delta)}{\partial \xi} - i\frac{\partial u(\delta)}{\partial n}$

$$= -\frac{1}{2\pi}\int_C u(z)\left[\frac{\partial^2 g(z,\delta)}{\partial\xi\,\partial n} - i\frac{\partial^2 g(z,\delta)}{\partial\eta\,\partial n}\right] dS ,$$

we may write

$$\operatorname{Re}\left\{e^{i\theta}\left[\phi(z_1) - a\,\phi(z_2)\right]\right\} =$$

$$-\frac{1}{2\pi}\int_C u(z)\left[\cos\theta\frac{\partial g(z;z_1,z_2)}{\partial n} - \sin\theta\frac{\partial M(z;z_1,z_2)}{\partial n}\right.$$

$$\left. + \sum_{j=1}^{n-1}\lambda_j\frac{\partial w_j}{\partial n} + \alpha\frac{\partial^2 g(z,\delta)}{\partial\xi\,\partial n} + \beta\frac{\partial^2 g(z,\delta)}{\partial\xi\,\partial n}\right] dS$$

$$- B\alpha + C\beta \quad \text{where}$$

α, β are arbitrary real parameters, $A = B + i\,C$, and

$$\frac{\partial g(z;z_1,z_2)}{\partial n} = \frac{\partial g(z,z_1)}{\partial n} - a\frac{\partial g(z,z_2)}{\partial n} \quad \text{and}$$

$$\frac{\partial M(z;z_1,z_2)}{\partial n} = \frac{\partial M(z,z_1)}{\partial n} - a\frac{\partial M(z,z_2)}{\partial n} .$$

Since $|u(z)| \leq 1$ we have that

$$-\left|\phi(z_1) - a\,\phi(z_2)\right| < \mathrm{Re}\left\{e^{i\theta}\,[\phi(z_1) - a\,\phi(z_2)]\right\}$$

$$\leq \frac{1}{2\pi}\int_c \left|\cos\theta\,\frac{\partial g(z;z_1,z_2)}{\partial n} - \sin\theta\,\frac{\partial M(z;z_1,z_2)}{\partial n}\right.$$

$$\left. + \sum_{j=1}^{n-1}\lambda_j\,\frac{\partial w_j}{\partial n} + \alpha\,\frac{\partial^2 g(z,\delta)}{\partial\xi\,\partial n} + \beta\,\frac{\partial^2 g(z,\delta)}{\partial\eta\,\partial n}\right| dS$$

$$- B\alpha + C\beta .$$

Abbreviating this expression in the obvious manner we now have that

$$-\left|\phi(z_1) - a\,\phi(z_2)\right| \leq \frac{1}{2\pi}\int_c |P|\,dS - B\alpha + C\beta. \quad (5.1)$$

As before, we minimize the right hand side of (5.1) assuming that α and β have been so chosen as to yield the minimum of the expression. Again defining $U(z) = -\frac{|P|}{P}$ $z \in c$, $P \neq 0$ as before we obtain the necessary conditions for the existence of the minimum to be the following:

$$\int_c U(z)\,\frac{\partial w_j(z)}{\partial n}\,dS = 0 \quad (5.2)$$

$$-\frac{1}{2\pi}\int_c U(z)\,\frac{\partial^2 g(z,\delta)}{\partial\xi\,\partial n}\,dS = B \quad (5.3)$$

$$-\frac{1}{2\pi}\int_c U(z)\,\frac{\partial^2 g(z,\delta)}{\partial\eta\,\partial n}\,dS = -C \quad (5.4)$$

$$-\frac{1}{2\pi}\int_c U(z)[\sin\theta\,\frac{\partial g(z;z_1,z_2)}{\partial n} + \cos\theta\,\frac{\partial M(z;z_1,z_2)}{\partial n}]dS = 0 \quad (5.5)$$

Eq. (5.2) indicates that the harmonic conjugate V(z) of

U(z) is single-valued; therefore $\psi(z) = U(z) + i\,V(z)$ is single-valued in D.

Eqs. (5.3) and (5.4) indicate that $\frac{\partial U}{\partial x} = B$ and $\frac{\partial U}{\partial y} = -C$ at $z = \delta$. Hence by the Cauchy-Riemann equations,

$$\psi'(\delta) = B + iC = A .$$

Eq. (5.5) indicates that

$$\operatorname{Im}\left\{e^{i\theta}\left[\psi(z_1) - a\,\psi(z_2)\right]\right\} = 0 \qquad (5.6)$$

Hence

$$e^{i\theta}\left[\phi(z_1) - a\,\phi(z_2)\right] \text{ is real.}$$

We now show that

$$e^{i\theta}\left[\phi(z_1) - a\,\phi(z_2)\right] \leq 0$$

by considering an increment of θ. In the expression for P, if we replace θ by θ + ε and call this new expression P* we have that the following holds:

$$\int_c |P^*|\,dS = \int_c |P|\,dS - \varepsilon \int_c \frac{|P|}{P}\left[\operatorname{Sin}\theta \frac{\partial g(z;z_1,z_2)}{\partial n} + \operatorname{Cos}\theta \frac{\partial M(z;z_1,z_2)}{\partial n}\right] dS$$

$$- \frac{1}{2}\varepsilon^2 \int_c \frac{|P|}{P}\left[\operatorname{Cos}\theta \frac{\partial g(z;z_1,z_2)}{\partial n} - \operatorname{Sin}\theta \frac{\partial M(z;z_1,z_2)}{\partial n}\right] dS$$

$$+ \theta(\varepsilon^2)$$

By (5.5) the first variation vanishes.

Since $\int_c |P^*|\, dS \geq \int_c |P|\, dS$, the second variation is non-negative.

Therefore

$$\int_c U(z)\, [\cos\theta \frac{\partial g(z;z_1,z_2)}{\partial n} - \sin\theta \frac{\partial M(z;z_1,z_2)}{\partial n}]\, dS \geq 0$$

Hence $\operatorname{Re}\left\{e^{i\theta}\, [\psi(z_1) - a\,\psi(z_2)]\right\} \leq 0$

Since eq. (5.6) holds,

$$e^{i\theta}\, [\psi(z_1) - a\,\psi(z_2)] = -\left|\psi(z_1) - a\,\psi(z_2)\right|.$$

We also have that

$$-\left|\phi(z_1) - a\,\phi(z_2)\right| \leq \operatorname{Re}\left\{e^{i\theta}\, [\psi(z_1) - a\,\psi(z_2)]\right\}$$

Hence

$$-\left|\phi(z_1) - a\,\phi(z_2)\right| \leq \left|\psi(z_1) - a\,\psi(z_2)\right|$$

or equivalently,

$$\left|\phi(z_1) - a\,\phi(z_2)\right| \geq \left|\psi(z_1) - a\,\psi(z_2)\right| \qquad (5.7)$$

Now $W = \psi(z)$ maps D onto the strip $-1 < \operatorname{Re}\{W\} < 1$ m times where $m \leq n+2$ by Theorem 4.2.

Let $r_s(\delta)$ be the radius of starlikeness of $B_R^{A'}$ with respect to $z = \delta$.

Then for $\varepsilon > 0$ there exists a function $\psi_o(z) \,\varepsilon\, B_R^{A'}$ and points z_1, z_2 on $|z - \delta| = r_s(\delta) + \varepsilon$ such that $\phi_o(z_1) - a\,\phi_o(z_2) = 0$ for some $a > 0$.

Hence by (5.7), there exists a function $\psi(z)$ such that

$\psi(z_1) - \psi(z_2) = 0$ and which maps $|z - \delta| < r_s(\delta) + \varepsilon$ onto a non-starlike domain. Now let $\varepsilon \longrightarrow 0$. Since $B_R^{A'}$ is closed and bounded, it is compact. Hence $r_s(\delta)$ is the exact radius of starlikeness.

Q.E.D.

The analogous theorem concerning the radius of convexity will be stated without proof.

Theorem 5.2. Let $B_R^{A'}$ be defined as above and let $r_c(\delta)$ be the radius of convexity of $B_R^{A'}$ with respect to $z = \delta$. Then $r_c(\delta)$ is the exact radius of convexity of a function $W = \psi(z)$ of $B_R^{A'}$ which yields a $(1,m)$ mapping $(m \leq n+3)$ of D onto the strip $-1 < \mathrm{Re}\,\{W\} < 1$.

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