1966

The smoothing and forecasting of discrete time series that occur at random review intervals of time

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THE SMOOTHING AND FORECASTING OF DISCRETE TIME SERIES THAT OCCUR AT RANDOM REVIEW INTERVALS OF TIME

by

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A Thesis

Presented to the Graduate Faculty

of Lehigh University

in Candidacy for the Degree of

Master of Science

Lehigh University

1966

CERTIFICATE OF APPROVAL

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

17 May 1966
Date

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ACKNOWLEDGEMENTS

Thanks are due to my thesis advisor, Dr. J. W. Adams of Lehigh University, for his guidance and suggestions during the preparation of this paper. The author wishes to thank Professors W. T. Richardson and W. A. Smith, thesis committee members, who participated in the initial phases of presenting the concept. Also, thanks are due to Professor A. F. Gould, Head of the Industrial Engineering Department, who first introduced me to the fascinating subject of smoothing and forecasting a time series. Finally, I would like to express my gratitude to Messrs. R. E. Rahikka and E. S. Bahary, of the Western Electric Co., for bringing the specific problem to my attention.

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ABSTRACT

Implicit in the classical techniques of smoothing and forecasting of discrete time series is the restriction that the data spans equal intervals of time. There exists a need to make forecasts at unequal time intervals for the transaction or event oriented business activity.

In the thesis a formal method of smoothing and forecasting is formulated for data that occurs informally with time; that is, the information is given at aperiodic review intervals. The model is of the recursive nature so that such useful characteristics as simplicity of computation, accuracy, and ease of data storage can be maintained. The model represents a generalization of classical exponential smoothing with the aforementioned departure from standard theory, and it logically reduces to the familiar exponential smoothing model for the special case of equal time intervals. The theory has been extended to include the smoothing and forecasting of data that exhibits either simple or higher order polynomial underlying processes.

Following the derivation, the statistical and dynamic properties are explored analytically so that the characteristics of the model can be appraised. The study suggests that the model as formulated in the thesis behaves, for the irregular review intervals, similar to that of classical exponential smoothing which in its original form is only applicable to periodic reviews. Finally, an example is included that will serve to demonstrate the behavior of the model under actual application and help lend support to the mathematical considerations.

I. INTRODUCTION

In this chapter of the thesis the purpose, the intended coverage, and the nature of the utility of the proposed model is introduced. Then, the mathematical formulation upon which the derivation will rest is summarized. Finally, a discussion covering the background upon which the argument to accomplish the generalization will be presented.

Purpose and Scope

The forecasting of a discrete time series is a smoothing process for which one considers known observations in a sequence of numbers and applies a method to extrapolate this time series into the future. There is a large class of problems that arises in industry for which it is economically appropriate to smooth and forecast discrete time series data on a short-term basis. For example, it may be financially advantageous to anticipate the demand for specific goods in the form of inventories. The forecasting methods are usually isolated from the actual control process and studied by themselves. Thus, the forecast is made by a smoothing technique based only on the time series data.

The traditional treatment is applied to data that arise in the form of a succession of observations that occur at equidistant points in time or covering equal intervals of time. However, there exists transaction oriented control systems that arise in industry that generate time series data that occur at irregular time intervals. The basic data for this subclass are inherently discrete but are only available at varying finite intervals of time. If a forecast of this particular non-periodic time series is considered valuable, it would be useful to generalize a representative forecasting method to include this type of problem. Necessarily the demands will accumulate during the irregular time interval. Then at some event clued point of time, a smoothing process will be applied and a forecast rendered.

Exponential smoothing is a forecasting technique that has gained wide acceptance since its exposition by Robert Goodell Brown (1). One of its advantages is its relative ease in carrying out the computations. Further, a recursive relationship alleviates the need for an exorbitant amount of storage capacity for historical data. It is intuitively satisfying in that the parameters of the fitting model, such as the extimate of the average level or the slope of a trend, are based on a geometric discounting of the past data with the greatest weight given to the most recent observation. Thus it is particularly applicable to the situation in which the parameters of the model are slowly varying in time. The fitting function reduces a time-series that is not strictly stationary to a pseudo-stationary series suitable for short-term forecasting. Its final characteristic is that a parameter in the smoothing function can be adjusted to regulate the rate of response of the system versus the stability of the system. However, the fundamental theorem of exponential smoothing in its present form asserts that the observations occur at equally spaced intervals. The advantages of exponential smoothing suggests that it would be valuable to generalize the technique to unequally spaced intervals, and the response-stability parameter characteristic suggests the method by which such a generalization may be incorporated.

The purpose of this thesis is to propose a formal model for smoothing and forecasting discrete time series data that occur informally with time. The data are triggered by an event or transaction oriented business activity; and therefore, the information from which the decisions and forecasts are made is considered at irregular time intervals. The formulation of the proposed model follows from a set of reasonable criteria, and an extensive analysis of the major characteristics of the method will be included in this paper. The nature of the formulation of the model represents a generalization of the widely applied exponential smoothing method to include the aforementioned departure from standard theory.

Mathematical Background - A Summary of the Fundamental Theorem of Exponential Smoothing

Since the formulation of the proposed model follows directly from an extension of exponential smoothing, a brief summary of the germane equations and philosophy will be presented.

An internal memorandum of the A. D. Little Co. was issued by Brown and Meyer in January, 1960 (3). The theorem proves, given an equally spaced time series $\{X_t\}$, that it is possible to estimate the n+1 coefficients in an n^{th} order polynomial model of the form:

$$X_t = a_o + a_1 t + \tfrac{1}{2} a_2 t^2 + \ldots + \frac{1}{n!} a_n t^n \qquad \text{(I-1)}$$

An estimate of the coefficients are expressed as a linear combination of n^{th} order recursive operators defined by the following equation:

$$S_t^{(n)}(X) = \alpha S_t^{(n-1)}(X) + (1-\alpha)\, S_{t-1}^{(n)}(X) \qquad \text{(I-2)}$$

where:

$S_t^0\ (X) = X_t$

$S_t^{(n)}(X)$ = The current n^{th} order operator

$S_{t-1}^{(n)}(X)$ = The previous n^{th} order operator

$S_t^{n-1}(X)$ = The current $(n-1)^{th}$ order operator

α = A smoothing parameter used to adjust the degree of the system stability - response; $0 \leq \alpha \leq 1$.

In the process of smoothing and forecasting $\{X_t\}$, one hypothesizes an underlying process. The observations are said to include this process plus random noise $\{\epsilon_t\}$ that has zero mean and variance σ_ϵ^2. The constant, linear, and quadratic processes with n = 0, 1, 2, respectively are summarized as follows:

<u>Hypothesized Model</u>	<u>Estimate of Coefficients</u>
$X_t = a_o + \epsilon_t$	$\hat{a}_o = S_t^{(1)}$
$X_t = a_o + a_1 t + \epsilon_t$	$\hat{a}_o = 2S_t^{(1)} - S_t^{(2)}$
	$\hat{a}_1 = \frac{\alpha}{1-\alpha}(S_t^{(1)} - S_t^{(2)})$
$X_t = a_o + a_1 t + \frac{1}{2}a_2 t^2 + \epsilon_t$	$\hat{a}_o = 3S_t^{(1)} - 3S_t^{(2)} + S_t^{(3)}$
	$\hat{a}_1 = \frac{\alpha}{2(1-\alpha)}(6-5\alpha)S_t^{(1)}$
	$-2(5-4\alpha)S_t^{(2)} + (4-3\alpha)S_t^{(3)}$
	$\hat{a}_2 = \frac{\alpha^2}{(1-\alpha)^2}\ S_t^{(1)} - 2S_t^{(2)} + S_t^{(3)}$

One merit of multiple smoothing is that one can recover the exact coefficients of the model when there is no noise in the data. Further, it is statistically an unbiased estimate. Finally, D'Esopo (8) has proved that for any sequence of observations, the polynomial of degree n obtained by appropriate multiple exponential smoothing is the solution that minimizes the discounted square error criterion. It is again reiterated that the above development strictly requires equally spaced data observations.

A Comment on the Stability-Response Parameter and The Sampling Interval

The basic unit of time is the sampling interval which is denoted by a subscript t. The next observation in a sequence will be denoted t+1. The problem of choosing some optimum sampling interval is not a trivial one. In practice the interval is usually selected by some arbitrarily convenient periodic reporting business activity and is not related to the interval being an optimum in some forecast error sense. The choice of the sampling interval will not be of concern to this thesis since we are considering an event triggered method of clueing the forecast; the intervals will be irregular. However, the notions surrounding the sampling interval are conceptually relevant. A long sampling interval tends to have a damping effect on the system. That is, a change in the data in the form of a high frequency contribution will not be reflected in the computations. Conversely, a short interval will be sensitive to random noise fluctuations.

Let us now observe the parameter α in the basic first order equation:

$$S_t^{(1)}(X) = \alpha X_t + (1-\alpha) S_{t-1}^{(1)}(X)$$

Clearly a higher α weights the latest observations more heavily and discounts the older data. Thus, the response to a changing pattern improves with a higher smoothing constant. However, it would also respond rapidly to any random noise fluctuations. The ability to smooth out random noise fluctuations is decreased by a higher smoothing constant; that is, the smoothing is less stable in the presence of random variations for a higher constant α. The converse is true for a smaller α. Brown has more rigorously demonstrated the property in reference (2), p. 57. He relates the variance of the output to the variance of the noise data for some α. For first order smoothing with random data, the output variance is given as:

$$\sigma_y^2 = \frac{\alpha}{2-\alpha} \sigma_\epsilon^2$$

If one interprets the variance of the output as a measure of stability, then the smaller the value of α, the more stable the estimate. Necessarily then, the less responsive to a true signal change.

One can observe that the same compromise between the degree of stability and the response rate exists for both the sampling interval length and for the choice of the smoothing constant α. It is obvious that a close relationship exists, and that the smoothing constant α could be a function of the interval length. We can relate these two ideas for the class of problems that requiring irregular sampling rates. For this relationship the smoothing constant α is better interpreted as a time dependent smoothing coefficient. It is to this formulation of the time dependent smoothing coefficient to which the next chapter will be directed.

II. THE FIRST-ORDER RANDOM REVIEW INTERVAL SMOOTHING MODEL

The formulation that will follow will build upon the mathematical summarizations and the concepts of the previous chapter. The mathematical approach will be formulated, and then its properties critically examined at various informative boundary conditions. The model will be related to the standard periodic form of exponential smoothing and shown to reduce to the standard form under the special condition of periodic reviews. In this chapter the discussion will be limited to a process in which a constant model is felt to be the underlying process. The observations are given by the general expressions:

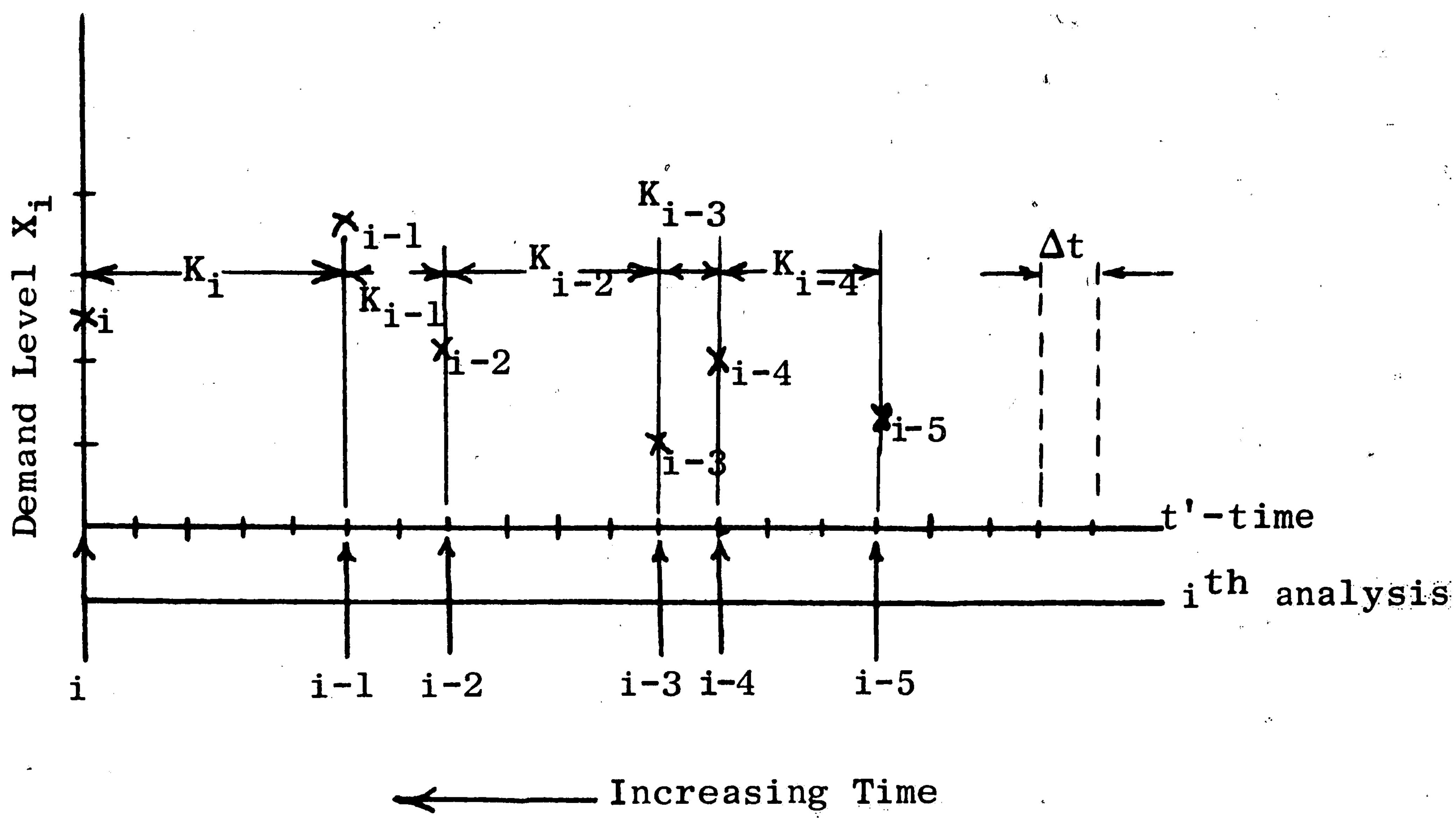
$$X_t = a_o + \epsilon_t \qquad \text{(II-1)}$$

with the estimate given by:

$$\hat{a}_o = S_t^{(1)}(X) = \alpha X_t + (1-\alpha) S_{t-1}^{(1)}(X) \qquad \text{(II-2)}$$

Derivation of a First Order Smoothing Function

The pictorial illustration to follow will serve as an example to depict both the nature of the problem and the applicable notation. Let X_i denote the accumulated demands during an irregular interval of time from i-1 to i. Also, count the number of arbitrary units of time, Δt, between the last review and the respective interval, and call this number K_i.



where:

i = i^{th} analysis

Δt = arbitrary unit of time on t' axis

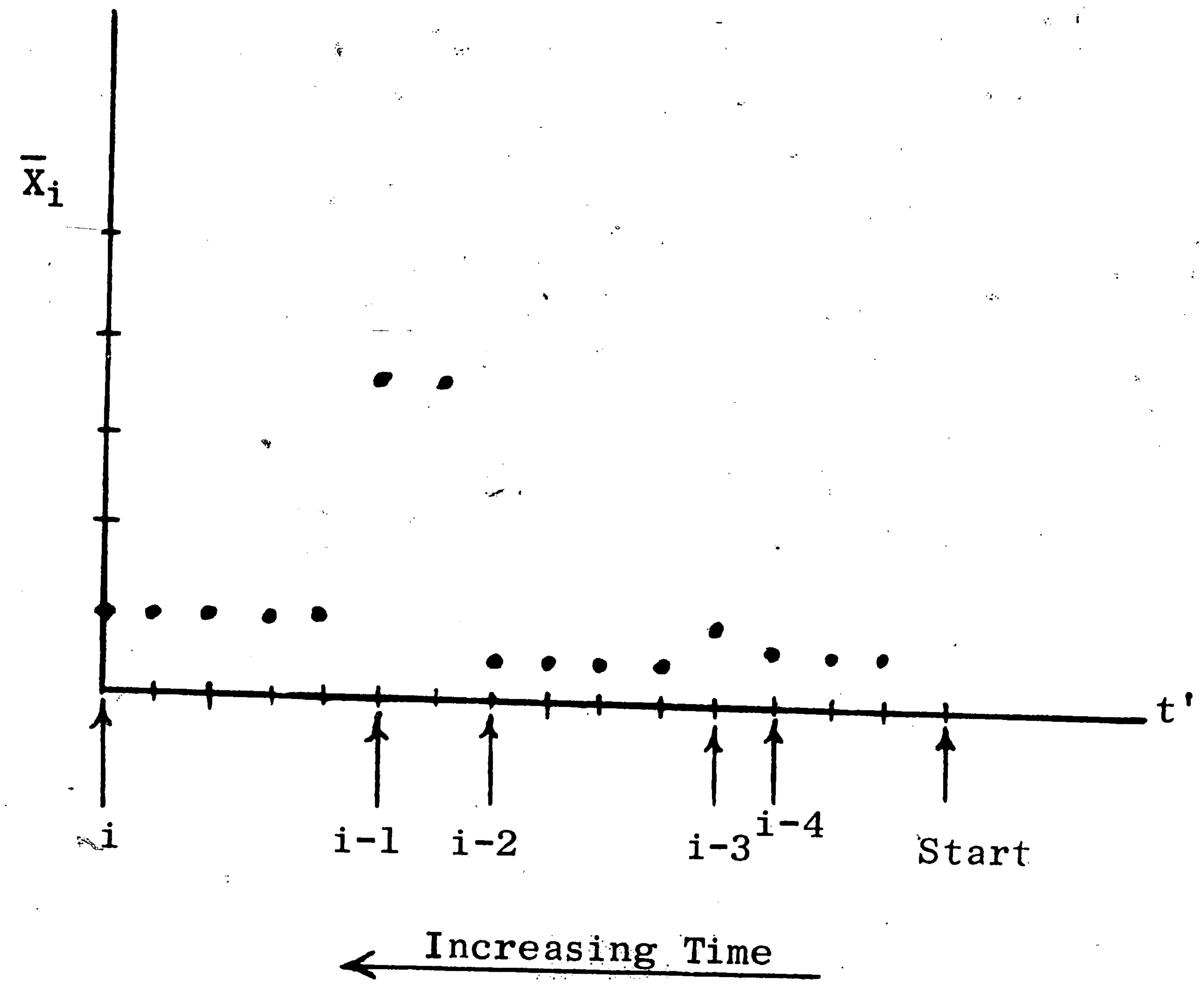
K_i = number of Δt units between i and i-1

X_i = accumulated demand during interval i and i-1.

Make the following definition:

$\overline{X}_i = \frac{X_i}{K_i}$ = the average demand for each Δt during the i^{th} analysis.

Under the above definition, the example can be replaced by the system to follow. Effectively the irregular review interval system is approximated by a pseudo periodic review interval system that is periodic on Δt time units.



Since the above system is now equally spaced, the fundamental theorems of exponential smoothing can be applied to it. However, certain adjustments will have to be made so that the pseudo system is representative of the situation under consideration. It is immediately apparent that in the example, 15 reviews are characterized in a length of time in which only 5 reviews were originally specified. Therefore, the "stability-response" interplay is altered. This suggests that the smoothing constant α should be appropriately adjusted. For now, let α_1 denote a smoothing constant different in value but the same in form from that characterized in Brown's model. We shall return to an interpretation of α_1 after the following discussion.

The pseudo system can be smoothed by:

$$S_{t'}^{(1)}(\bar{X}_i) = \alpha_1 \bar{X}_i + (1-\alpha_1) S_{t'-1}^{(1)}(\bar{X}_i) \qquad \text{(II-3)}$$

where: t' = the arbitrary time scale for pseudo period Δt.

However, it would be inconvenient to smooth the process every Δt. Instead, expand equation (II-3) back in time by repeated substitution from the present time to exactly $K_i - 1$ units of Δt to the last review at $i-1$.

$$S_{t'}^{(1)}(\bar{X}_i) = \alpha_1 \bar{X}_i + \alpha_1(1-\alpha_1)\bar{X}_i + \alpha_1(1-\alpha_1)^2 \bar{X}_i + \ldots + \alpha_1(1-\alpha_1)^{K_i - 1}\bar{X}_i + (1-\alpha_1)^{K_i} S_{t'-K_i}^{(1)}(\bar{X}_i). \qquad \text{(II-4)}$$

But by definition within the i^{th} interval, $\bar{X}_i$ can be factored.

$$S_{t'}^{(1)}(\bar{X}_i) = \left[\alpha_1 + \alpha_1(1-\alpha_1) + \alpha_1(1-\alpha_1)^2 + \ldots + \alpha_1(1-\alpha_1)^{K_i - 1}\right] X_i + \left[(1-\alpha_1)^{K_i}\right] S_{t'-K_i}^{(1)}(\bar{X}_i) \qquad \text{(II-5)}$$

One can recognize the coefficient of $\bar{X}_i$ as a finite geometric series of K_i terms. This can be shown to be identically expressed in the following closed form:

$$\boxed{S_{t'}^{(1)}(\bar{X}_i) = \left[1 - (1-\alpha_1)^{K_i}\right] \bar{X}_i + \left[(1-\alpha_1)^{K_i}\right] S_{t'-K_i}^{(1)}(\bar{X}_i)} \qquad \text{(II-6)}$$

The pseudo system can be smoothed by the above equation each time there is a review. Thus, there is one smoothing operation each review. Note that the coefficient of the current piece of data, $\bar{X}_i$, is time dependent on the number, K_i, of Δt units since the last review. The one remaining undefined parameter is α_1. One could merely suggest equation (II-6) as the method to smooth the irregularly reviewed time series data. The practitioner would have to select an α_1 by observing past data to give him a reasonable balance between stability and response. It is possible, however, to give an interpretation of α_1 in terms of the more familiar α. The advantages of doing this are obvious, and the discussion to follow in the next two sections will develop such an interpretation.

A Useful Property of the General Exponential Smoothing Form

The first step toward an interpretation of α_1 is to make an important observation about the general smoothing equation as given by (II-2). Expand equation (II-2) back in time to the initial value:

$$S_t^{(1)}(X) = \alpha X_t + \alpha(1-\alpha)X_{t-1} + \alpha(1-\alpha)^2 X_{t-2} + \ldots +$$

$$\alpha(1-\alpha)^n X_{t-n} + \ldots + (1-\alpha)^t X_o \qquad \text{(II-7)}$$

The notation t has been given to mean an interval period. Let us choose some arbitrary smaller unit of time, Δt, where $\mu \Delta t = t$. That is, there are μ units of Δt in the interval t.

Divide equation (II-7) by μ

$$\frac{S_t^{(1)}(X)}{\mu} = \alpha \frac{X_t}{\mu} + \alpha(1-\alpha)\frac{X_{t-1}}{\mu} + \alpha(1-\alpha)^2 \frac{X_{t-2}}{\mu} + \ldots +$$

$$\alpha(1-\alpha)^n \frac{X_{t-n}}{\mu} + \ldots + (1-\alpha)^t \frac{X_o}{\mu} \qquad \text{(II-8)}$$

Define: $\frac{X_t}{\mu} \equiv \bar{X}_t$ = The average demand per Δt units of time in interval t.

Substitute $\bar{X}_t$ in equation (II-8)

$$\frac{S_t^{(1)}(X)}{\mu} = \alpha \bar{X}_t + \alpha(1-\alpha)\bar{X}_{t-1} + \alpha(1-\alpha)^2 \bar{X}_{t-2} + \ldots$$

$$\alpha(1-\alpha)^n \bar{X}_{t-n} + \ldots + (1-\alpha)^t \bar{X}_o \qquad \text{(II-9)}$$

Recollect terms in a recursive manner.

$$\frac{S_t^{(1)}(X)}{\mu} = \alpha \bar{X}_t + (1-\alpha) S_{t-1}^{(1)}(\bar{X}) = S_t^{(1)}(\bar{X}) \qquad \text{(II-10)}$$

It is actually easier to expand (II-10) and to note that it is identical to (II-9).

Therefore, the useful property:

$$\frac{S_t^{(1)}(X)}{\mu} = S_t^{(1)}(\bar{X}) \qquad \text{(II-11)}$$

Because the smoothing function is a linear combination of all past observations and therefore equation (II-11) holds, one can smooth for the average demand per Δt units in the interval t. Although this is of no practical use for equally spaced data, this fact will prove useful in the following development. Brown's equation (II-2) can be modified by equation (II-11) to smooth the average demand per unit of Δt within the interval t; i.e.,

$$S_t^{(1)}(\bar{X}) = \alpha \bar{X}_t + (1-\alpha) S_{t-I}^{(1)}(\bar{X}) \qquad \text{(II-12)}$$

Formulation of α_1

It seems reasonable to suggest that the degree of response (stability) be equal in the two systems described by equations (II-6) and (II-12). The response is clearly related to the number of reviews made over a length of time. Under the irregular review condition, one may characterize the number of irregular reviews over a length of time in terms of an average number of reviews over that length of time; the average number of K_i units of Δt is μ over a long length of time.

Brown in reference (1) p. 107, has indicated a convenient measure of response; the average age of data. He uses this measure to compare the exponential smoothing α to the N of moving average theory. He discusses the relationship that concerns the rate of response to a changing pattern increasing with higher smoothing constant or with smaller values of N. Conversely, he mentions that the higher α or smaller N decreases the ability to smooth random fluctuations. Finally, he defines an exponential smoothing system that is equivalent to an N period moving average by equating the average age of the data.

The average age of the data is then a convenient method of measuring response (stability). The average age of the data will be compared in the two systems that are described by (II-12) and (II-6). The age of the current observation is 0; the age of the previous observation is presently 1; and the one before is 2. The average age is the age of each piece of data used in the average, weighted as the data of that age would be weighted. This formulation is like that of measuring the distance in time back to the centroid of the data.

The approach to formulate α_1 is then to let the data be equally spaced at μ units of Δt for both systems and to set the average age of both systems to be equal. This insures for an average number of reviews in a length of time that the response rate (stability) will be equal. The mathematics of the preceding discussion will now be formulated. It will be shown that α_1 can be expressed as a function of α and μ; $\alpha_1 = f(\alpha, \mu)$.

By using the weights assigned to the data, which can be seen from equation (II-9), and the age of the data concept, the average of the age data can be calculated for Brown's modified system of (II-12).

$$\bar{A}_\alpha = 0\alpha + 1\alpha(1-\alpha) + 2\alpha(1-\alpha)^2 + \ldots$$

$$\bar{A}_\alpha = \sum_{j=0}^{\infty} j\alpha(1-\alpha)^j$$

It can be shown that the summation for $\bar{A}_\alpha$ in a closed form is:

$$\bar{A}_\alpha = \frac{1-\alpha}{\alpha} \qquad \text{(II-13)}$$

Now consider the system represented by equation (II-6). Let $K_i = \mu$, for all i, then,

$$S_{t'}^{(1)}(\bar{X}) = \left[1 - (1-\alpha_1)^\mu\right] \bar{X}_i + \left[(1-\alpha_1)^\mu\right] S_{t'-\mu}^{(1)}(\bar{X}) \qquad \text{(II-14)}$$

Expand (II-14) back in time.

$$S_{t'}^{(1)}(\bar{X}) = \left[1 - (1-\alpha_1)^\mu\right] \bar{X}_i + \left[1 - (1-\alpha_1)^\mu\right]\left[1-\alpha_1\right]^\mu \bar{X}_{i-1} + \left[1 - (1-\alpha_1)^\mu\right]\left[1-\alpha_1\right]^{2\mu} \bar{X}_{i-2} + \ldots$$

Associating the above weights with the age of the data:

$$\bar{A}\,\alpha_1 = \sum_{j=0}^{\infty} j(1-\alpha_1)^{j\mu}\left[1-(1-\alpha_1)^{\mu}\right]$$

The summation can be expressed in closed form as:

$$\bar{A}\,\alpha_1 = \frac{(1-\alpha_1)^{\mu}}{1-(1-\alpha_1)^{\mu}} \qquad \text{(II-15)}$$

Equating (II-13) to (II-15):

$$\bar{A}_{\alpha} = \bar{A}\,\alpha_1$$

$$\frac{1-\alpha}{\alpha} = \frac{(1-\alpha_1)^{\mu}}{\left[1-(1-\alpha_1)^{\mu}\right]}$$

Solving for α_1:

$$\boxed{\alpha_1 = \left[1-(1-\alpha)^{1/\mu}\right]} \qquad \text{(II-16)}$$

<u>The Proposed Model</u>

Substitute equation (II-16)

$$\alpha_1 = 1-(1-\alpha)^{1/\mu} \qquad \text{(II-16)}$$

into equation (II-6)

$$S_{t'}^{(1)}(\bar{X}_i) = \left[1-(1-\alpha_1)^{K_i}\right]\bar{X}_i + \left[(1-\alpha_1)^{K_i}\right]S_{t'-K_i}^{(1)}(\bar{X}_i) \qquad \text{(II-6)}$$

If the above substitution is made, the following proposed model is expressed as follows:

$$S_{t'}^{(1)}\ (\bar{X}_i) = \left[1 - (1-\alpha)^{K_i/\mu}\right] \bar{X}_i + \left[(1-\alpha)^{K_i/\mu}\right] S_{t'-K_i}^{(1)}\ (\bar{X}_i)$$

(II-17)

where:

$S_{t'}^{(1)}\ (\bar{X}_i)$ = Current Smoothed demand in i^{th} irregular interval

$\bar{X}_i$ = Average demand for Δt in the i^{th} interval

$S_{t'-K_i}^{(1)}\ (\bar{X}_i)$ = Last Smoothed demand K_i units of Δt ago.

K_i = Number of units of Δt since the last review

μ = Average number of Δt units in an interval

α = Brown's Smoothing Constant, $0 \leq \alpha \leq 1$.

It is convenient to define:

$$\alpha_i = 1 - (1-\alpha)^{K_i/\mu} \qquad \text{(II-18)}$$

Then,

$$S_{t'}^{(1)}\ (\bar{X}_i) = \alpha_i\ \bar{X}_i + (1-\alpha_i)\ S_{t-K_i}^{(1)}\ (\bar{X}_i) \qquad \text{(II-19)}$$

This expression is similar in form to Brown's Model with a time dependent smoothing coefficient:

$$\alpha_i = f(K_i : \alpha, \mu)$$

The time dependent smoothing coefficient is expressed as a function of a number of irregular time lengths, K_i of Δt and the response (stability) is controlled by two parameters. One parameter is the familiar parameter as given by Brown. The other is the average number of Δt units in an interval. It is believed that equation (II-17), although a two parameter system, is more convenient than equation (II-6). The α parameter of (II-17) is given explicit meaning in terms of existing writings. All work that has been done to help interpret that parameter (1, 2, 3, 39) or the writing on adaptive exponential smoothing (4, 25, 33) for a non-stationary time series logically follow. The other parameter μ is rather easily ascertained from past history. The relative sensitivity of these two parameters with respect to each other will be studied later in the thesis.

There is one modification of the original form that may be computationally simpler. Rather than calculate $\overline{X}_i$ from the definition $\overline{X}_i = \frac{X_i}{K_i}$, substitute this back in the equations and express it in the following alternate form:

$$S_{t'}^{(1)}(\overline{X}_i) = \frac{1-(1-\alpha)^{K_i/\mu}}{K_i} X_i + (1-\alpha)^{K_i/\mu} S_{t'-K_i}^{(1)}(\overline{X}_i) \qquad \text{(II-17-A)}$$

or

$$S_{t'}^{(1)}(\overline{X}_i) = \frac{\alpha_i}{K_i} X_i + (1-\alpha_i) S_{t'-K_i}^{(1)}(\overline{X}_i) \qquad \text{(II-19-A)}$$

where as before:

$$\alpha_i = 1-(1-\alpha)^{K_i/\mu} \qquad \text{(II-18)}$$

Properties of the Model

These are some immediate observations that can be made about the proposed model.

1. If: $K_i \equiv \mu$ for all intervals; i.e., this case is the one where we have equally spaced data.

Then let $\frac{K_i}{\mu} = 1$ in equation (II-18).

Therefore: $\alpha_i = \alpha$

and $1-\alpha_i = 1-\alpha$

Then equation (II-17) becomes

$$S_{t'}^{(1)}(\bar{X}_i) = \alpha \bar{X}_i + (1-\alpha) S_{t'-\mu}^{(1)}(\bar{X}_i)$$

which is Brown's modified model given in equation (II-12).

2. If: $K_i > \mu$ in the interval from i-1 to i

then $\frac{K_i}{\mu} = 1 + \delta$

with $\delta > 0$

and $\alpha_i = 1 - (1-\alpha)^{1+\delta}$

Since $0 \leq \alpha \leq 1$

then $(1-\alpha) \leq 1$

$(1-\alpha)^{1+\delta} \leq (1-\alpha)^1$

then

$$\alpha_i = \left[1 - (1-\alpha)^{1+\delta}\right] \geq \left[1 - (1-\alpha)^1\right] = \alpha$$

i.e., $\alpha_i \geq \alpha$

For intervals greater than the average, the last observation is weighted more heavily than one occurring at the average interval length. This is intuitively the direction we would hope.

3. Conversely for $K_i < \mu$

$$\alpha_i \leq \alpha$$

For intervals less than the average, the current observation is weighted less than that occurring at the average interval length.

4. If: $\alpha = 0$

$$\alpha_i = 1 - (1)^{K_i/\mu} = 0$$

$$\alpha_i = 0 \quad \text{independent of } K_{i/\mu}$$

5. If: $\alpha = 1$

$$\alpha_i = 1 - (1-1)^{K_i/\mu} = 1$$

$$\alpha_i = 1 \quad \text{independent of } K_{i/\mu}$$

6. From 4 and 5 and since (II-18) is monotonic

$$0 \leq \alpha_i \leq 1$$

7. There is no systematic bias; i.e., the sum of the coefficients (II-17) or (II-19) equals 1.

$$\alpha_i + (1- \alpha_i) = 1$$

8. As $K_i \longrightarrow \infty$; the interval is getting very long

$\alpha_i \longrightarrow 1$; suggests using current data as can be seen from (II-19); i.e.,

$$S_{t'}^{(1)} (\bar{X}_i) \longrightarrow \bar{X}_i$$

9. As $K_i \longrightarrow 0$

$$\alpha_i \longrightarrow 0$$

We are going back to the last smoothing operation; i.e.,

$$S_{t'}^{(1)} (X_i) \longrightarrow S_{t'-K_i}^{(1)} (X_i)$$

10. From the formulation of the derivation, the geometric discounting of the data is implied.

The family of curves as illustrated in Figure (II-1) clearly shows the relationship of α_i as a function of K for parameters α and μ. In the figure, α_i is plotted against the ratio $\frac{K}{\mu}$ with Brown's α as a parameter. Obviously when K/μ is 1, the interval is equal to the average interval length and therefore $\alpha_i = \alpha$. If the ratio is greater than 1, the interval is larger than the average interval length and α_i is appropriately larger than α. The converse is true for an interval less than the average; that is, a ratio less than 1.

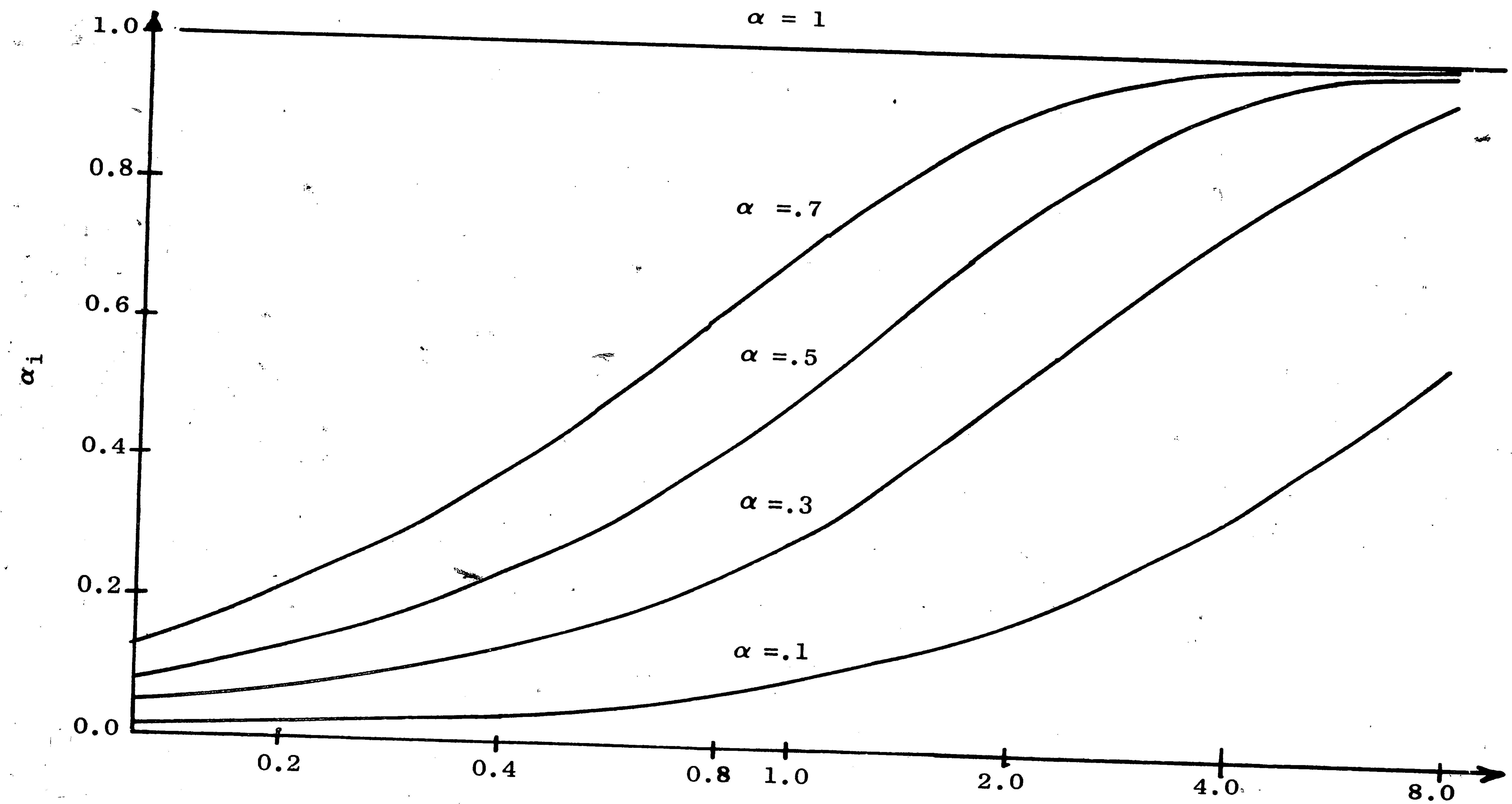
Interpretation of the Forecast

One point that has been tacitly implied throughout the development of the smoothing function is the forecast. Actually, the hypothesized constant model is:

$$\bar{X}_i = \bar{a}_o + \epsilon_i$$

with the estimate

$$\hat{a}_o = S_{t'}^{(1)} (\bar{X}_i)$$



RATIO OF INTERVAL LENGTH TO AVERAGE INTERVAL LENGTH - $\frac{K}{\mu}$

FIGURE (II-1) - TIME DEPENDENT SMOOTHING COEFFICIENT - $\alpha_i = f(K_i : \alpha, \mu)$

III. HIGHER-ORDER RANDOM REVIEW INTERVAL SMOOTHING MODELS

The formulation of the models in this chapter will be similar to that of the first-order model. Although the concepts will be the same, the mathematics will be more tedious. Unfortunately, the final model is not easily expressed in closed form. For the practitioner, approximations to the theory are suggested so that the equations remain simple for actual use.

Derivation of a Second-Order Smoothing Function

For the second order smoothing function, it is felt that the underlying process is linear. The observations are given by the general expression:

$$X_t = a_o + a_1 t + \epsilon_t \qquad \text{(III-1)}$$

with the estimates given by:

$$\hat{a}_o = 2S_t^{(1)}(X) - S_t^{(1)}(X) \qquad \text{(III-2)}$$

$$\hat{a}_1 = \frac{\alpha}{1-\alpha} \quad S_t^{(1)}(X) - S_t^{(2)}(X) \qquad \text{(III-3)}$$

where:

$$S_t^{(1)}(X) = \alpha X_t + (1-\alpha)\, S_{t-1}^{(1)}(X)$$

$$S_t^{(2)}(X) = \alpha S_t^{(1)}(X) + (1-\alpha)\, S_{t-1}^{(2)}(X)$$

The same philosophy is alluded to in this chapter as that of the case in the previous chapter. The irregular review interval system is approximated by a pseudo periodic review interval system; where as before: $\bar{X}_i = \frac{X_i}{K_i}$. The smoothing constant is denoted as α_2; and again, it must be stability-response altered. Thus, the equations for the second order system are:

$$S_{t'}^{(2)}(\bar{X}_i) = \alpha_2 S_{t'}^{(1)}(\bar{X}_i) + (1-\alpha_2)\, S_{t'-1}^{(2)}(\bar{X}_i) \qquad \text{(III-4)}$$

$$S_{t'}^{(1)}(\bar{X}_i) = \alpha_2 \bar{X}_i + (1-\alpha_2)\, S_{t'-1}^{(1)}(\bar{X}_i) \qquad \text{(III-5)}$$

Substituting back in time for one unit of Δt and using $\bar{X}_i$ within that interval.

$$S_{t'}^{(2)}(\bar{X}_i) = \alpha_2^2\, \bar{X}_i + 2\,\alpha_2^2\,(1-\alpha_2)\,\bar{X}_i + 2\,\alpha_2^2(1-\alpha_2)^2\, S_{t'-2}^{(1)}(\bar{X}_i) + (1-\alpha)^2\, S_{t'-2}^{(2)}(\bar{X}_i)$$

Substituting back in time for one more unit of Δt.

$$S_{t'}^{(2)}(\bar{X}_i) = \alpha_2^2\, \bar{X}_i + 2\,\alpha_2^2\,(1-\alpha_2)\,\bar{X}_i + 3\,\alpha_2^2\,(1-\alpha_2)^2\,\bar{X}_i + 3\,\alpha_2(1-\alpha_2)^3\, S_{t'-3}^{(1)}(\bar{X}_i) + (1-\ _2)^3\, S_{t'-3}^{(2)}(\bar{X}_i)$$

Now express this for K_i units of Δt. That is, expand back in time for exactly K_i-1 units of Δt. This is the number of Δt units back to the last review. By definition, $\bar{X}_i$ can be factored out:

$$S_{t'}^{(2)}(\bar{X}_i) = \left[\alpha_2^2 + 2\,\alpha_2^2\,(1-\alpha_2) + 3\,\alpha_2^2\,(1-\alpha_2)^2 + \ldots + K_i(1-\alpha_2)^{K-1}\right] X_i + K_i\,\alpha_2(1-\alpha_2)^{K_i} S_{t'-K_i}^{(1)}(\bar{X}_i) + (1-\alpha_2)^{K_i}\, S_{t'-K_i}^{(2)}(\bar{X}_i) \qquad \text{(III-6)}$$

It can be shown that the coefficient of $\overline{X}_i$ can be expressed in closed form.

Thus,

$$S_{t'}^{(2)}(\overline{X}_i) = \left[1-(1-\alpha_2)^{K_i}(1+\alpha_2 K_i) \right] \overline{X}_i + \left[K_i \, \alpha_2 (1-\alpha_2)^{K_i}\right] S_{t'-K_i}^{(1)}(X_i) + \left[(1-\alpha_2)^{K_i}\right] S_{t'-K_i}^{(2)}(\overline{X}_i) \qquad \text{(III-7)}$$

Thus the second order smoothing function is smoothed each irregular review interval. The coefficient of the current piece of data is time dependent. Equation (III-7) could be used to smooth the data. However, an interpretation of α_2 is presented in the next session.

Formulation of α_2

As in the case of the first-order smoothing function, the linear combinatorial property allows us to note the following:

$$\frac{S_t^{(2)}(X)}{\mu} = S_t^{(2)}(\overline{X}) \qquad \text{(III-8)}$$

From this, Brown's equations are modified as follows:

$$S_t^{(1)}(\overline{X}) = \alpha \overline{X}_t + (1-\alpha) \, S_{t-1}^{(1)}(\overline{X}) \qquad \text{(III-9)}$$

$$S_t^{(2)}(\overline{X}) = \alpha S_t^{(1)}(\overline{X}) + (1-\alpha) \, S_{t-1}^{(2)}(\overline{X}) \qquad \text{(III-10)}$$

Expressing (III-9) and (III-10) in a similar form to (III-7):

$$S_t^{(2)}(\overline{X}) = \alpha^2 \, \overline{X}_t + \alpha(1-\alpha) \, S_{t-1}^{(1)}(\overline{X}) + (1-\alpha)^2 \, S_{t-1}^{(2)}(\overline{X}) \qquad \text{(III-11)}$$

Once again, the response (stability) will be set equal in equations (III-7) and (III-11) by equating the average age data.

Taking equation (III-11) and expanding it back in time yields the following:

$$S_t^2(\bar{X}) = \alpha^2 \bar{X}_t + 2\alpha^2(1-\alpha)\bar{X}_{t-1} + 3\alpha^2(1-\alpha)^2\bar{X}_{t-2}$$

$$+ \dots + n\alpha^2(1-\alpha)^{n-1}\bar{X}_{t-(n-1)} + \dots$$

If we assign the weights to the data:

$$\bar{A}_\alpha = \sum_{j=0}^{\infty} j(j+1)\,\alpha^2(1-\alpha)^j$$

This can be shown to be equal to:

$$\bar{A} = \frac{2(1-\alpha)}{\alpha} \qquad \text{(III-12)}$$

Now expanding (III-7) back in time for $K_i = \mu$

$$S_{t'}^{(2)}(\bar{X}_i) = \left\{\left[1-(1-\alpha_2)^\mu\right] + \left[-(1-\alpha_2)^\mu\right]\left[\alpha_2\mu(1-\alpha_2)^0\right]\right\}\bar{X}_i$$

$$+ \left\{\left[1-(1-\alpha_2)^\mu\right]\left[1-\alpha_2\right]^{2\mu} + \left[1-2(1-\alpha_2)^\mu\right]\left[\alpha_2\mu(1-\alpha_2)^2\right]\right\}\bar{X}_{i-1}$$

$$+ \left\{\left[1-(1-\alpha_2)^\mu\right]\left[1-\alpha_2\right]^{2\mu} + \left[2-3(1-\alpha_2)^\mu\right]\left[\alpha_2\mu(1-\alpha_2)^{2\mu}\right]\right\}\bar{X}_{i-2}$$

Using the age of the data:

$$\bar{A}\alpha_2 = \sum_{j=0}^{\infty}\left\{ j(1-\alpha_2)^{j\mu}\left[1-(1-\alpha_2)^\mu\right]\right\} +$$

$$\sum_{j=0}^{\infty}\left\{ j^2\mu\,\alpha_2(1-\alpha_2)^{j\mu} - j(j+1)\mu\,\alpha_2(1-\alpha_2)^{\mu(j+1)}\right\}$$

(III-13)

$$\bar{A}_{\alpha_2} = \frac{(1-\alpha_2)[\mu\alpha_2 + 1 - (1-\alpha_2)^{\mu}]}{[1-(1-\alpha_2)^{\mu}]^2} \qquad \text{(III-14)}$$

Equating $\bar{A}_{\alpha} = \bar{A}_{\alpha_2}$

Unfortunately, α_2 is not easily expressed as a function of μ and α. However α can be expressed as a function of α_2 and μ as follows:

$$\alpha = \frac{2[1-(1-\alpha_2)^{\mu}]^2}{2[1-(1-\alpha_2)^{\mu}]^2 + (1-\alpha_2)^{\mu}[\mu\alpha_2 + 1 - (1-\alpha_2)^{\mu}]} \qquad \text{(III-15)}$$

With the aid of Figure (III-1), one can find α_2 for a particular α and μ. This value can be substituted back in equation (III-7). Since this is less convenient to the practitioner, the next two sections suggest possible approximations so that a final closed form can be used.

An Approximation

As a first step towards expressing a second-order model in closed form, one can make an approximation.

For small α_2,

$$(1-\alpha_2)^{K_i} \cong 1-K_i\alpha_2$$

Equation (III-7) becomes:

$$S_t^2(\bar{X}_i) \cong (\alpha_2 K_i)^2 \bar{X}_i + K_i\alpha_2(1-K_i\alpha_2) S_{t-K_i}^{(1)}(X_i) + (1-K_i\alpha_2) S_{t-K_i}^{(2)}(\bar{X}_i) \qquad \text{(III-16)}$$

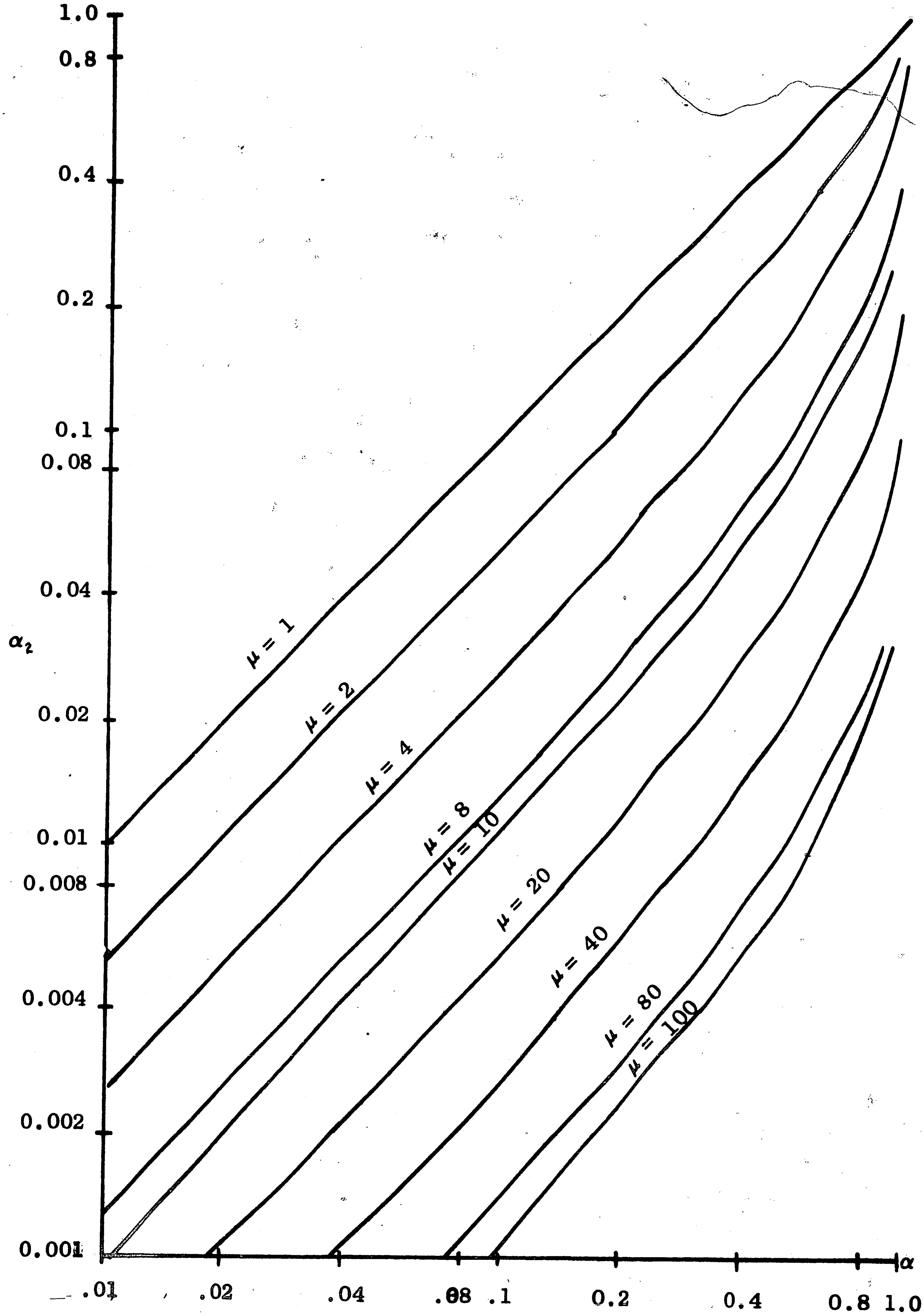


FIGURE (III-1) RESPONSE ADJUSTED SMOOTHING CONSTANT $\alpha_2 = f(\alpha, \mu)$

Equation (III-14) becomes:

$$\bar{A}_{\alpha_2} \cong \frac{(1-\mu\alpha_2)\left[\ 2\mu\alpha_2\ \right]}{(\mu\alpha_2)^2} \qquad \text{(III-17)}$$

Equating (III-17) to (III-12)

$$\frac{2\,\mu\alpha_2(1-\mu\alpha_2)}{(\mu\alpha_2)^2} \cong \frac{2\ (1-\alpha)}{\alpha}$$

$$\boxed{\alpha_2 \cong \frac{\alpha}{\mu}}$$

Substituting back in (III-7)

$$\boxed{\begin{aligned} S_t^2(X_i) \cong \left[\frac{\alpha^2}{\mu^2} K_i^2\right]\bar{X}_i + \left[\frac{K_i}{\mu}\alpha\left(1-\frac{K_i\alpha}{\mu}\right)\right] S^{(1)}_{t'-K_i}(\bar{X}_i) \\ + \left[\left(1 - \frac{K_i\alpha}{\mu}\right)\right] S^{(2)}_{t'-K_i}(\bar{X}_i) \end{aligned}} \qquad \text{(III-18)}$$

Equation (III-18) is thus a closed form approximation for smoothing the second order function.

<u>Another Approximation</u>

If the same approximation is made in another matter, a more convenient form can be used.

For small α_2,

$$(1-\alpha_2)^{K_i} \cong 1-K_i\alpha_2$$

then,

$$K_i\alpha_2 \cong \left[1 - (1-\alpha_2)^{K_i}\right]$$

If the above approximation is made for all $K_i\,\alpha_2$, then equation (III-7) becomes:

$$S_{t'}^{(2)}(\bar{X}_i) \cong \left\{ 1-(1-\alpha_2)^{K_i}\left[2-(1-\alpha_2)^{K_i}\right]\right\}\bar{X}_i$$

$$+\left[1-(1-\alpha_2)^{K_i}\right]\left[(1-\alpha_2)^{K_i}\right]S_{t'-K_i}^{(1)}(\bar{X}_i)$$

$$+(1-\alpha_2)^{K_i}\,S_{t'-K_i}^{(2)}(\bar{X}_i) \qquad \text{(III-19)}$$

Also equation (III-14) becomes:

$$\bar{A}_{\alpha_2} \cong \frac{(1-\alpha_2)^{\mu}\; 2\left[1-(1-\alpha_2)^{\mu}\right]}{\left[1-(1-\alpha_2)^{\mu}\right]^2}$$

Equating $\bar{A}_{\alpha} = \bar{A}_{\alpha_2}$

$$\frac{2(1-\alpha)}{\alpha} \cong \frac{2(1-\alpha_2)^{\mu}}{1-(1-\alpha_2)^{\mu}}$$

$$\boxed{\alpha_2 \cong 1-(1-\alpha)^{1/\mu}} \qquad \text{(III-20)}$$

Note that α_2 is the same as α_1. Substituting the approximation for α_2 into equation (III-7).

$$S_{t'}^{(2)}(\bar{X}_i) \cong \left[1-(1-\alpha)^{K_i/\mu}\right]^2 \bar{X}_i$$

$$+\left[1-(1-\alpha)^{K_i/\mu}\right]\left[1-\alpha\right]^{K_i/\mu} S_{t'-K_i}^{(1)}(\bar{X}_i)$$

$$+\left[1-\alpha\right]^{K_i/\mu} S_{t'-K_i}^{(2)}(\bar{X}_i)$$

Using the same definition for α_i; i.e.,

$$\alpha_i = 1-(1-\alpha)^{K_i/\mu}$$

$$S_{t'}^{(2)}(\bar{X}_i) \cong \alpha_i^2 \bar{X}_i + \alpha_i(1-\alpha_i) S_{t'-K_i}^{(1)}(\bar{X}_i)$$

$$+ (1-\alpha_i) S_{t'-K_i}^{(2)}(\bar{X}_i)$$

but this is the same as:

$$S_{t'}^{(2)}(\bar{X}_i) \cong \alpha_i S_t^{(1)}(\bar{X}_i) + (1-\alpha_i) S_{t'-K_i}^{(2)}(\bar{X}_i) \qquad \text{(III-21)}$$

Under this approximation, α_i is the same for both models. Further the equations are identical to Brown's with α_i replacing α. Obviously all characteristics of α_i that were listed in the last chapter follow.

There is a further satisfying reason for suggesting (III-21) as a good approximation. Divide Brown's basic equation for the second-order function by μ and use α_1 as the response (stability) parameter.

$$\frac{S_{t'}^{(2)}(X_i)}{\mu} = \alpha_1 \frac{S_{t'}^{(1)}(X_i)}{\mu} + (1-\alpha_1) \frac{S_{t'}^{(2)}(X_i)}{\mu}$$

then,

$$S_{t'}^{(2)}(\bar{X}_i) = \alpha_1 S_{t'}^{(1)}(\bar{X}_i) + (1-\alpha_1) S_{t'}^{(2)}(\bar{X}_i)$$

If we substitute $S_{t'}^{(1)}(\bar{X}_i)$ for each Δt back in time for exactly K_i units, we have an exactly analogous situation as in the first-order model.

$S^{(2)}_{t'}(\bar{X}_i)$ replaces $S^{(1)}_{t'}(\bar{X}_i)$ and $S^{(1)}_{t'}(\bar{X}_i)$ replaces $\bar{X}_i$. Under this argument,

$$S^{2}_{t'}(\bar{X}_i) = \alpha_i\, S^{(1)}_{t'}(\bar{X}_i) + (1-\alpha_i)\, S^{(2)}_{t'-K_i}(\bar{X}_i)$$

This is the same as the approximation (III-21) which followed from a slightly different philosophy. Further this second approach can be generalized for all higher order models. It is felt then that Brown's equations can be used in general for the irregular review interval with α_i replacing α.

$$\alpha_i = 1-(1-\alpha)^{K_i/\mu}$$

In general the recursive operator is given as follows:

$$S^{(n)}_{t'}(\bar{X}_i) = \alpha_i\, S^{(n-1)}_{t'}(\bar{X}_i) + (1-\alpha_i)\, S^{(n)}_{t'-K_i}(\bar{X}_i) \qquad \text{(III-22)}$$

Interpretation of the Forecasts

The second order hypothesized model is $\bar{X}_t = \bar{a}_o + \bar{a}_1 t' + \epsilon_t$ with the estimates

$$\hat{\bar{a}}_o = 2S^{(1)}_{t'}(\bar{X}_i) - S^{(2)}_{t'}(\bar{X}_i)$$

$$\hat{\bar{a}}_1 = \frac{\alpha}{1-\alpha}\left(S^{(1)}_{t'}(\bar{X}_i) - S^{(2)}_{t'}(\bar{X}_i)\right)$$

where:

$$S^{(1)}_{t'}(\bar{X}_i) = \alpha_i\, \bar{X}_i + (1-\alpha_i)\, S^{(1)}_{t'-K_i}(\bar{X}_i)$$

$$S^{(2)}_{t'}(\bar{X}_i) = \alpha_i S^{(1)}_{t'}(\bar{X}_i) + (1-\alpha_i) S^{(2)}_{t'-K_i}(\bar{X}_i)$$

$$\alpha_i = 1-(1-\alpha)^{K_i/\mu}$$

However, for the forecast of $\bar{X}_i$, one has a problem as to what t' to use since we are not forecasting regular intervals. This was no problem in the constant model case since X_i was not a function of time. It is felt that the best approximation is to use $t' = \mu$, the average interval length.

Then, the forecast is given by:

$$\bar{X}_i = \hat{a}_o + \hat{a}_1 \mu$$

In the chapter to follow some of the statistical and dynamic properties of the derived models will be explored.

IV. ANALYSIS OF THE CHARACTERISTICS OF THE MODELS

Having completed the derivation of the models, it seems appropriate to summarize prior to further study of these models. In the first chapter the need for a transaction triggered, formal smoothing and forecasting model was suggested. The second chapter was devoted to the formulation of a first order model, given by equation (II-17), that evidently performs the intended function for a constant underlying process. Particularly encouraging were the list of ten properties under all boundary conditions that further tended toward the intuitively proper magnitudes in the intervals between the boundary conditions. Finally the third chapter extended the basic arguments and notions to higher order models. The conclusion to this point is to suggest that the α in Brown's models be replaced by a time dependent smoothing coefficient for each interval given by:

$$\alpha_i = f(K_i : \alpha, \mu) = 1 - (1-\alpha)^{K_i/\mu}$$

It is the purpose of this chapter to further study the models so that one can gain a better understanding of their behavior. The characteristics explored in this chapter should further support the derivations and properties that have been discussed in the previous chapters. First it will be shown that the estimate, $\hat{\bar{a}}_0$ in the first order model is a statistically unbiased estimate. Second, a large portion of this chapter will be devoted to the response of the model to standard input signals. The response-stability interplay will be demonstrated for different magnitudes of α. Finally, the sensitivity of the coefficients α_i, or some mathematically convenient function of α_i, will be given with respect to the two parameters of the system α and μ .

$S^{1}_{t'}(\bar{X}_i)$ As An Estimate of the Average of the Data

If the expected value of the function $S^{(1)}_{t'}(\bar{X}_i)$ is equal to the expectation of the time series data $\bar{X}_i$, one is justified in calling it an average and $\hat{\bar{a}}_o = S^{(1)}_{t'}(\bar{X}_i)$. Recall that the model given by equation (II-17) is the closed form expression for equation (II-3) over a finite interval length with α_1 given by equation (II-16). The closed form expression was derived so that one need only perform the computation once for each review rather than K_i times in the interval i. We can then equivalently use equation (II-3) with equation (II-16) to prove the various properties.

Expanding equation (II-3) back in time, substitution (II-16), and taking the expected value:

$$E\left[S^{(1)}_{t'}(\bar{X}_i)\right] = \sum_{j=o}^{\infty}\left[1-(1-\alpha)^{1/\mu}\right](1-\alpha)^{j/\mu}\;E(\bar{X}_{t'-j})$$

$$E\left[S^{(1)}_{t'}(\bar{X}_i)\right] = \left[1-(1-\alpha)^{1/\mu}\right]E(\bar{X}_i)\;\sum_{j=o}^{\infty}(1-\alpha)^{j/\mu}$$

but the summation is an infinite geometric series

$$\sum_{j=o}^{\infty}(1-\alpha)^{j/\mu} = \frac{1}{1-(1-\alpha)^{1/\mu}}$$

$$E\left[S^{(1)}_{t'}(\bar{X}_i)\right] = E(\bar{X}_i)$$

Response to Standard Signals

In this section the way in which the proposed model responds to several typical standard time series generating functions will be studied. Further the effect on the model of the choice of the parameters, α and μ, will be explored. The basic approach will be through the technique of linear systems analysis. More specifically the methods of the z-transform will be employed. The output response of the system will be characterized for a specific input to the system.

The state of the mathematical art to treat this type of analysis, the z-transform, imposes implicit restrictions relevant to this thesis. The theory limits the analysis to linear, discrete, time invariant systems. The model as proposed in the form given by (II-17) violates these restrictions. We can again go back to (II-3) where the basic interval is periodic on the t' scale. The approach will be to apply the z-transform theory on equation (II-3) and to apply the same definitions to the input signal and the parameter α_1 as were employed in deriving (II-17).

Representing equation (II-3):

$$S_{t'}^{(1)}(\bar{X}_i) = \alpha_1 \bar{X}_i + (1-\alpha_1)\, S_{t'-1}^{(1)}(\bar{X}_i)$$

where: t' = the smallest convenient interval of time Δt.

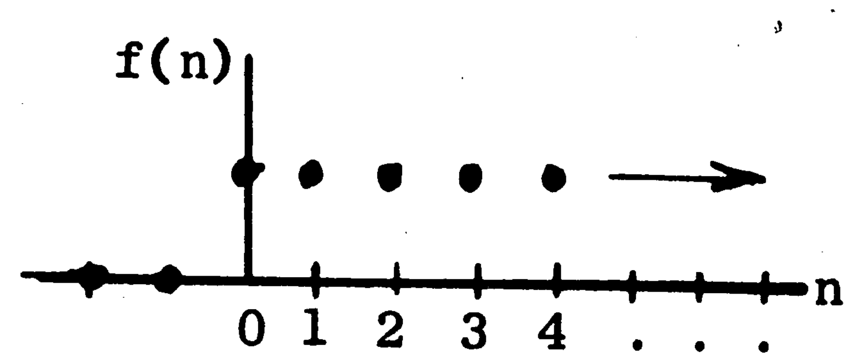
$\bar{X}_i = \frac{X_i}{K_i}$ = the average demand for each Δt during the i^{th} interval.

$\alpha_1 = 1 - (1-\alpha)^{1/\mu}$

The time series given by $\overline{X}_i$ will have to be interpreted as a modified standard signal which is a statistic of the standard continuous function X_t. The average demand $\overline{X}_i$ for each Δt is used for each Δt in the interval i. By the definition of $\overline{X}_i$ all inputs will be pulses or combinations of pulses. The pulses will vary in magnitude and length. For example if the input of the time series X_t is a ramp function, it is put into this system as $\frac{X_i}{K_i}$ which would be a staircase of equal "step heights" and varying "step lengths". A parabolic generating function will be a staircase of monotonically increasing "step heights" with varying "step lengths". A time series function that is a step input will also be a step in this system. "Noisy" data will tend to be somewhat averaged out and will be randomly varying up and down pulses. Clearly then, it is important to study the pulse in some depth.

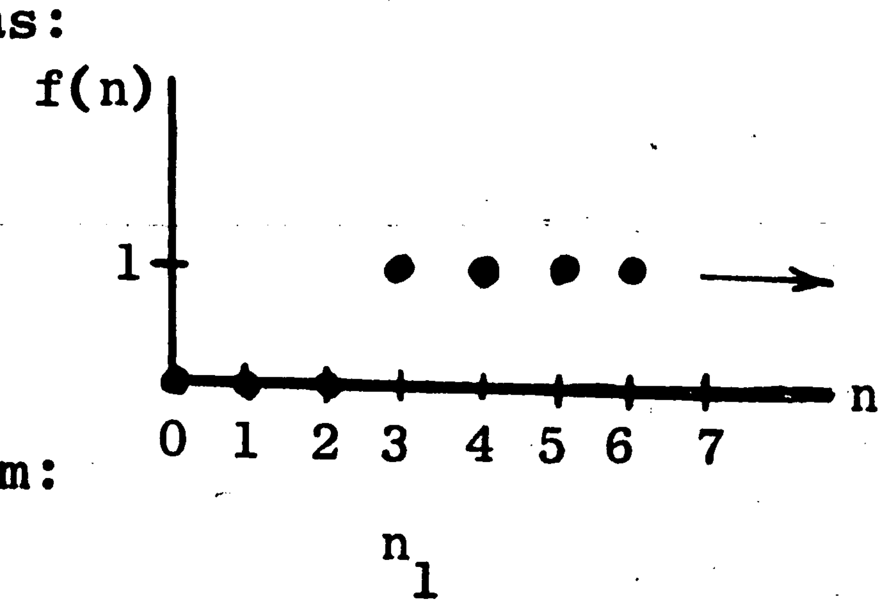
A pulse can be constructed from two unit step functions. The unit step is defined as:

$$f(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



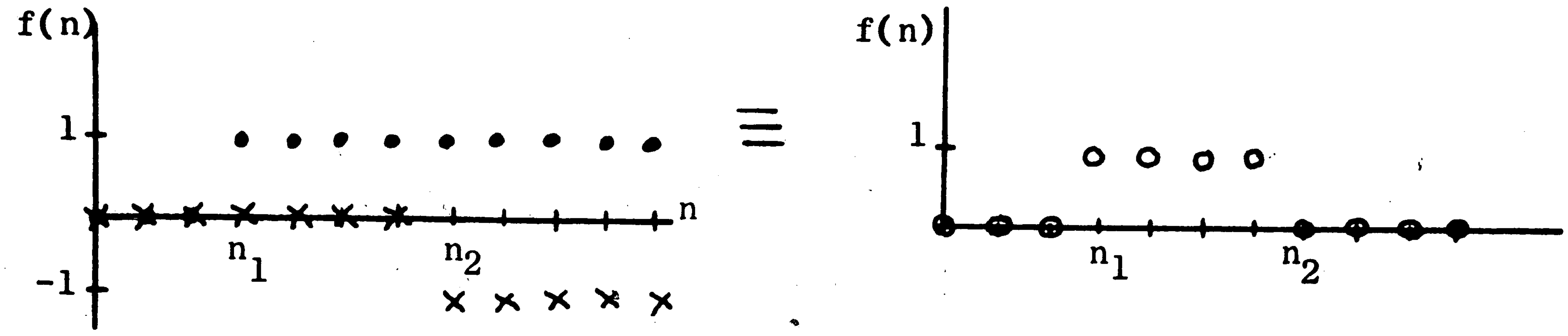
This expression is generalized as:

$$f(n-n_1) = \begin{cases} 1 & n \geq n_1 \\ 0 & n < n_1 \end{cases}$$



The pulse may be constructed from:

$$f(n-n_1) - f(n-n_2)$$



The use of the unit step functions with the definition as illustrated in the example is all that is needed to represent the typical nature of the signal entering this system. The unit step function can in this manner be used as the basic building blocks to represent the pulse or pulses. The magnitude of the pulse is merely set by a constant multiple given by $\frac{X_i}{K_i}$ over the i^{th} interval.

Having represented the nature of the incoming signal in the discrete time domain, let us build the necessary z-transform theory that is required. The z-transform of f(n) which takes on values at points n = 0, 1, 2... is defined as:

$$f^T(z) = F(z) = \sum_{n=o}^{\infty} f(n)\, z^n$$

Consider the unit step:

$$f(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$F(z) = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots$$

$$F(z) = \frac{1}{1 - z} \qquad \text{(IV-1)}$$

Clearly for $f(n) = \begin{cases} C & n \geq 0 \\ 0 & n < 0 \end{cases}$

$$F(z) = \frac{C}{1 - z} \qquad \text{(IV-2)}$$

The general transform of delay for any function f(n-K) is $z^K f^T(z)$. Thus for the delayed unit step:

$$f(n - K) = \begin{cases} C & n \geq K \\ 0 & n < K \end{cases}$$

$$F(z) = \frac{C z^K}{1 - z} \qquad \text{(IV-3)}$$

Another function that will be useful is the impulse function. This is defined by:

$$\begin{matrix} \delta(n) \\ 0 \end{matrix} \Bigg\} \begin{matrix} \text{for } n = 0 \\ n \neq 0 \end{matrix}$$

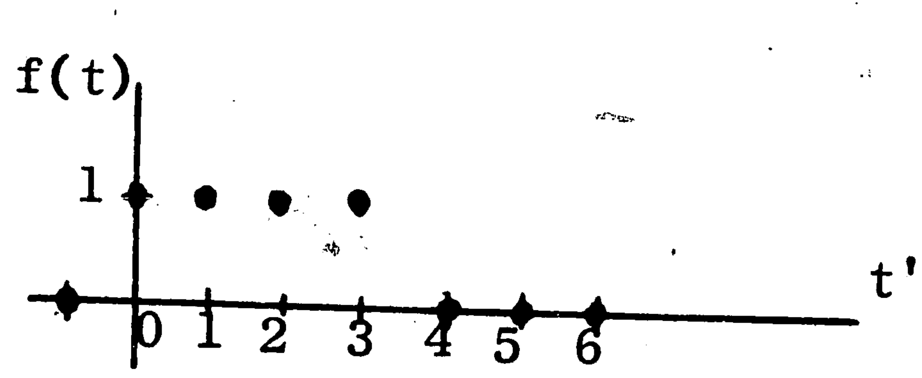
and the transform:

$$F(z) = 1$$

The transform of a delayed impulse for K units is clearly:

$$F(z) = z^K \qquad \text{(IV-4)}$$

Consider for example the following time pulse function:



$$u(t) = f(t') - f(t' - 4)$$

The transform from what has been outlined before is from (IV-3):

$$F(z) = \frac{1}{1 - z} - \frac{z^4}{1 - z}$$

$$F(z) = \frac{1 - z^4}{1 - z}$$

By long division:

$$F(z) = 1 + z + z^2 + z^3$$

The inverse of this is then 4 impulses which can be seen from (IV-4). Frequently, this long division technique is the quickest way to express the function.

A final necessary transform is that of the exponential decay. This will be used to interpret the output response after the input pass through the exponentially smoothed system.

$$f(n) = (1-\alpha)^n \qquad n \geq 0$$

$$F(z) = \sum_{n=0}^{\infty} (1-\alpha)^n z^n$$

This is a geometric series equivalent to:

$$F(z) = \frac{1}{1-(1-\alpha)z} \qquad \text{(IV-5)}$$

Brown in (1) has shown that the z-transform for simple exponential smoothing is:

$$H(z) = \frac{\alpha}{1-(1-\alpha)z} \qquad \text{(IV-6)}$$

Relating this to equation (II-3) with time interval t'.

$$H(z) = \frac{\alpha_1}{1-(1-\alpha_1)z}$$

where $\alpha_1 = 1 - (1-\alpha)^{1/\mu}$

Also recall that all time series $\overline{X}_i$ are a collection of pulses.

It is well known in z-transform theory that a discrete, linear, time invariant system can be completely described by its impulse function h_t. The impulse response h_t is a description of the output of the system t periods after an impulse is applied at the input. The convolution of the impulse response with any arbitrary signal X_t gives the output response y_t to that signal.

$$y_t = X_t * h_t$$

In transform theory, convolution can be interpreted as multiplication in the transform domain.

$$Y(z) = X(z) \cdot H(z)$$

Applying the unit step to the simple exponential smoothing function.

$$Y(z) = \frac{\alpha_1}{1-(1-\alpha_1)z} \cdot \frac{1}{1-z}$$

$$Y(z) = \frac{1}{(1-z)} - \frac{1-\alpha_1}{1-(1-\alpha_1)z}$$

then $y(t') = \left[1 - (1-\alpha_1)(1-\alpha_1)^{t'}\right]$

$$y(t') = 1 - (1-\alpha_1)^{t'+1}$$

substituting $\alpha_1 = 1 - (1-\alpha)^{1/\mu}$

$$y(t') = 1 - (1-\alpha)^{(t'+1)/\mu}$$

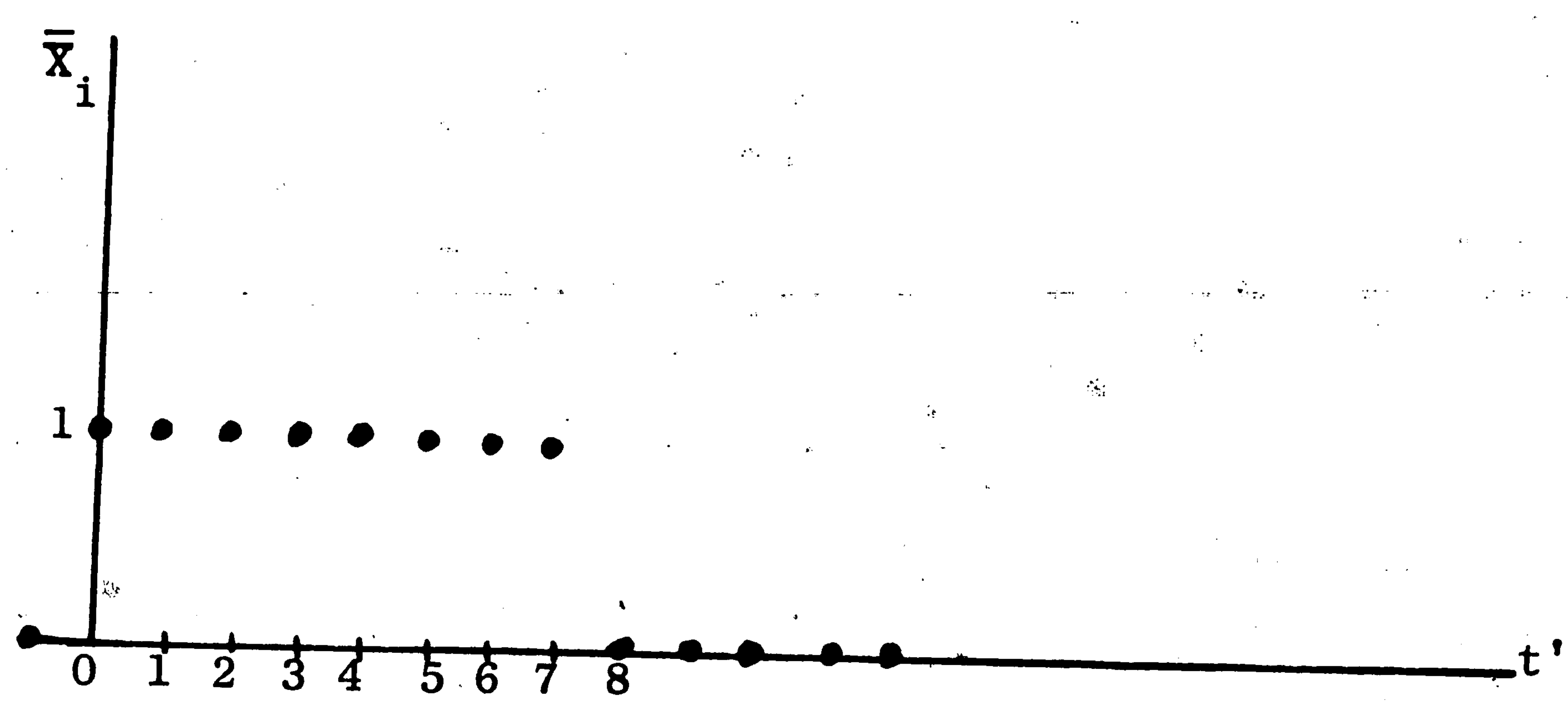
In general then the delayed step:

$$Y(z) = \frac{1}{1-(1-\alpha_1)z} \cdot \frac{z^K}{1-z}$$

finally reduces to

$$y(t') = 1 - (1-\alpha)^{(t'+1-K)/\mu} \tag{IV-7}$$

Let us now study in some depth the basic unit pulse of arbitrary length since it is the heart of the model. Consider the following example which is constructed from two unit steps and represents the nature of the signal during an interval. Notice that there are K_i time units in the interval i.



$\overline{X}_i(t') = f(t') - f(t'-8)$

$$F(z) = \frac{1}{1-z} - \frac{z^8}{1-z}$$

Passing this through the exponential smoothing system:

$Y(z) = F(z) \cdot H(z)$

From (IV-7)

$$y(t') = \left\{ 1 - (1-\alpha)^{(t'+1)/\mu} \right\} - \left\{ 1 - (1-\alpha)^{(t'+1-8)/\mu} \right\}$$

Therefore:

$$y(t') = 1 - (1-\alpha)^{(t'+1)/\mu} \qquad \text{for } 0 \leq t' \leq 7$$

$$y(t') = (1-\alpha)^{(t'-7)/\mu} - (1-\alpha)^{(t'+1)/\mu} \qquad \text{for } t' \geq 8$$

SUMMARIZING

t'	$y(t')$
0	$1 - (1-\alpha)^{1/\mu}$
1	$1 - (1-\alpha)^{2/\mu}$
2	$1 - (1-\alpha)^{3/\mu}$
.	.
.	.
.	.
7	$1 - (1-\alpha)^{8/\mu}$
8	$(1-\alpha)^{1/\mu} - (1-\alpha)^{9/\mu}$
.	
.	
.	
8 + K	$(1-\alpha)^{(1+K)/\mu} - (1-\alpha)^{(9+K)/\mu}$

For this example:

Let $\mu = 5$

$\alpha = .3$ and $.5$

$\mu = 3$

$\alpha = .3$ and $.5$

The results for this input are plotted for the 4 combinations of the parameters in Figure (IV-1). Clearly, the higher the α, the faster the response. The lower the μ, the faster the response. The interpretation of the parameters, however, should follow their definitions. The parameter μ is the average interval length. The closer this is estimated the closer α can be interpreted as Brown's α.

Let us consider an example of a string of pulses and observe the way in which this signal is tracked. The input and output is plotted in Figure (IV-2). The signal is given as:

$$\begin{aligned} f(t') = {} & C_1 f(t') - C_1 f(t'-3) + C_2 f(t'-3) \\ & - C_2(t'-8) + C_3(t'-8) - C_3(t'-10) \\ & + C_4(t'-10) - C_4(t'-15) \\ & + C_5(t'-15) + \ldots \end{aligned}$$

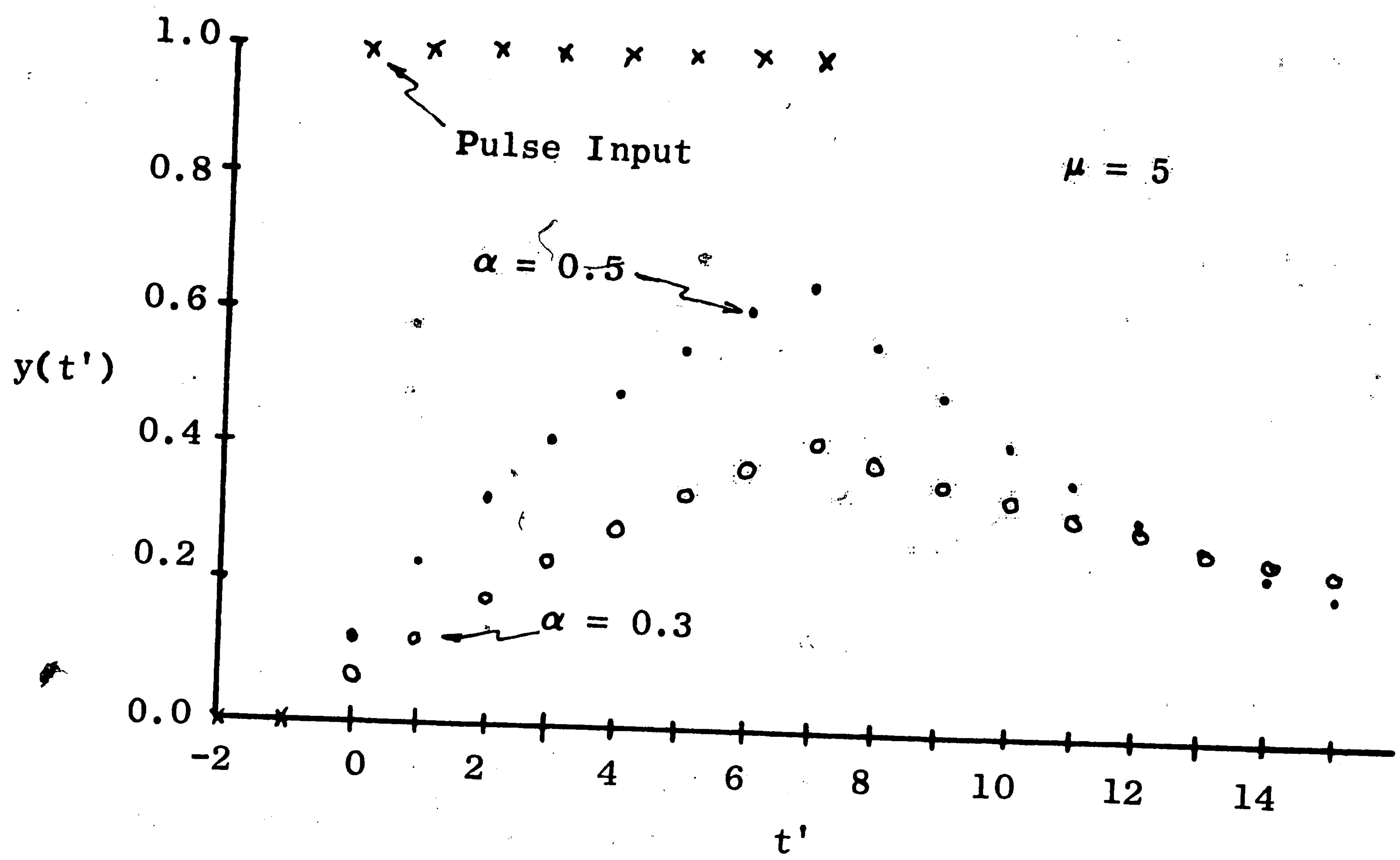
Let: $\mu = 4$

$\alpha = .3$

$C_1 = 1$

$C_2 = 2$

$C_3 = 3$



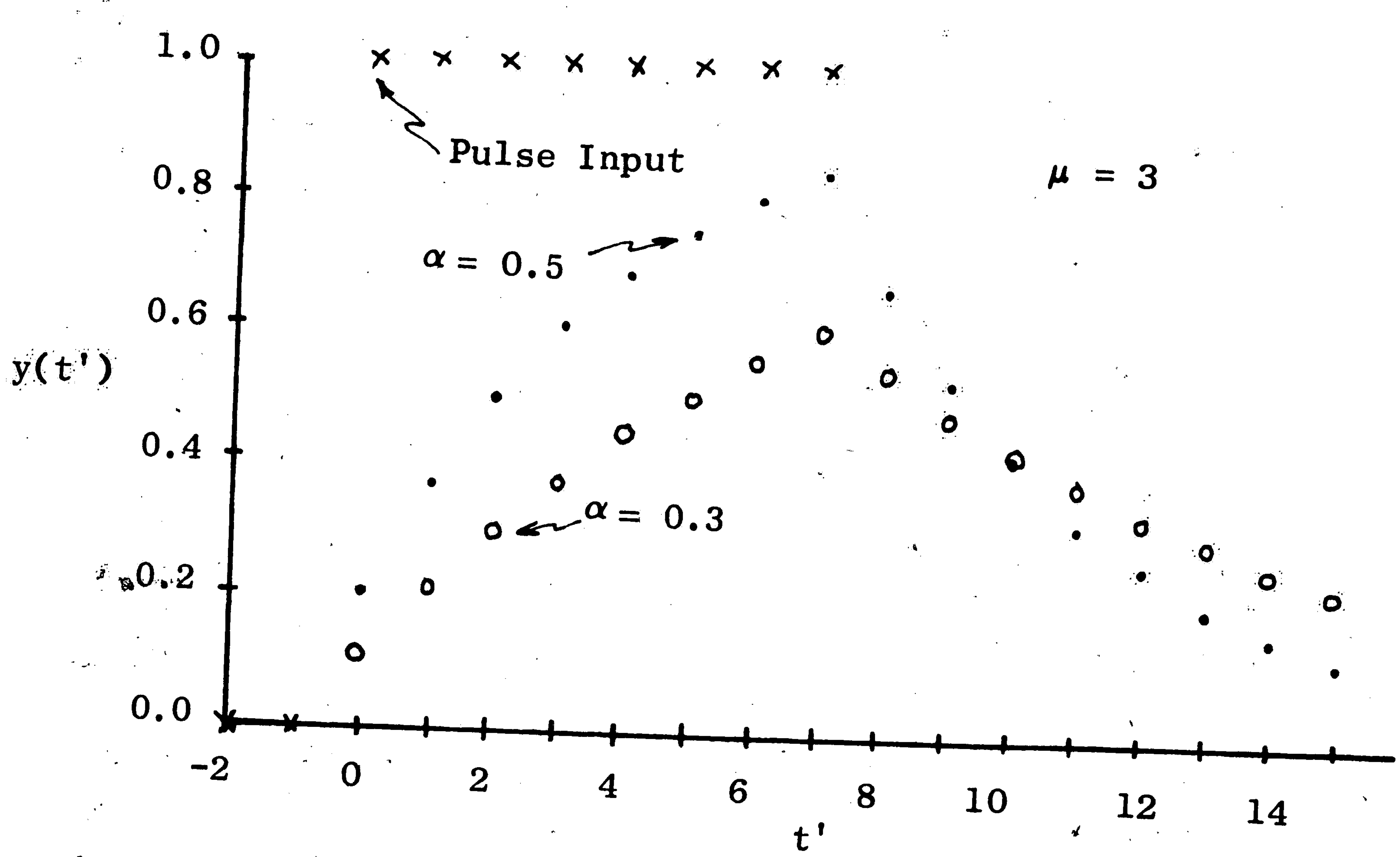


FIGURE (IV-1) NORMALIZED RESPONSE TO PULSE INPUT FOR DIFFERENT α and μ

$C_4 = 2.5$

$C_5 = .75$

The first three intervals represent a ramp generating function. Substituting:

$$f(t') = 1\ f(t') + 1 \cdot f(t'-3) + f(t'-8)$$
$$- .5\ f(t'-10) - 1.75\ f(t'-15) + \ldots$$

$$F(z) = \frac{1}{1-z} + \frac{z^3}{1-z} + \frac{z^8}{1-z}$$
$$- .5\ \frac{z^{10}}{1-z} - 1.75\ \frac{z^{15}}{1-z}$$

Passing the above signal through the exponential smoothing system and taking the inverse, as given by (IV-7)

$$y(t') = 1 - (1-\alpha)^{(t'+1)/\mu} \qquad 0 \le t' \le 2$$

$$y(t') = 2 - (1-\alpha)^{(t'+1)/\mu} - (1-\alpha)^{(t'-2)/\mu} \qquad 3 \le t' \le 7$$

$$y(t') = 3 - (1-\alpha)^{(t'+1)/\mu} - (1-\alpha)^{(t'-2)/\mu}$$
$$- (1-\alpha)^{(t'-7)/\mu} \qquad 8 \le t' \le 9$$

$$y(t') = 2.5 - (1-\alpha)^{(t'+1)/\mu} - (1-\alpha)^{(t'-2)/\mu}$$
$$- (1-\alpha)^{(t'-7)/\mu} + .5(1-\alpha)^{(t'-9)/\mu} \qquad 10 \le t' \le 14$$

$$y(t') = .75 - (1-\alpha)^{(t'+1)/\mu} - (1-\alpha)^{(t'-2)/\mu}$$
$$- (1-\alpha)^{(t'-7)/\mu} + .5(1-\alpha)^{(t'-9)/\mu} + 1.75(1-\alpha)^{(t'-14)/\mu}$$
$$t' \ge 15$$

Although these equations describe the response to the signal on the t' scale, let us recall that the formula suggested by equation (II-17) only generates a smoothed value at the irregular review interval. The only value calculated for each interval is the final one. Thus, the above calculations are given at: y(2), y(7), y(9), and y(14). The response to this input signal is plotted on Figure IV-2. The final values are shown as a circled dot.

A careful study of Figure IV-2 should demonstrate how the smoothing function follows the signal. The lag of the response to the signal is quite obvious for a deterministic signal. This is the price that must be paid for smoothing "noisy" data. It is a demonstration of the response - stability trade-off. That is, a system that responds faster to the deterministic system would also respond to random noise. Further, the "ramp" portion of the signal is building up a steady state bias. This is the reason a ramp signal suggests a second order model to follow the signal. The models under study evidently behave in a manner similar to the equally-spaced models suggested by Brown. Let us now take a look at the sensitivity of the parameters.

Sensitivity Analysis

The models that have been formulated express a time dependent smoothing coefficient with two parameters. Since the user must set these two parameters either by experience or past data, the optimal result may deviate from the nominal values chosen. Brown (1, pp. 106-107) has asserted without proof that the results are not very sensitive to the exact choice of α . The coefficient

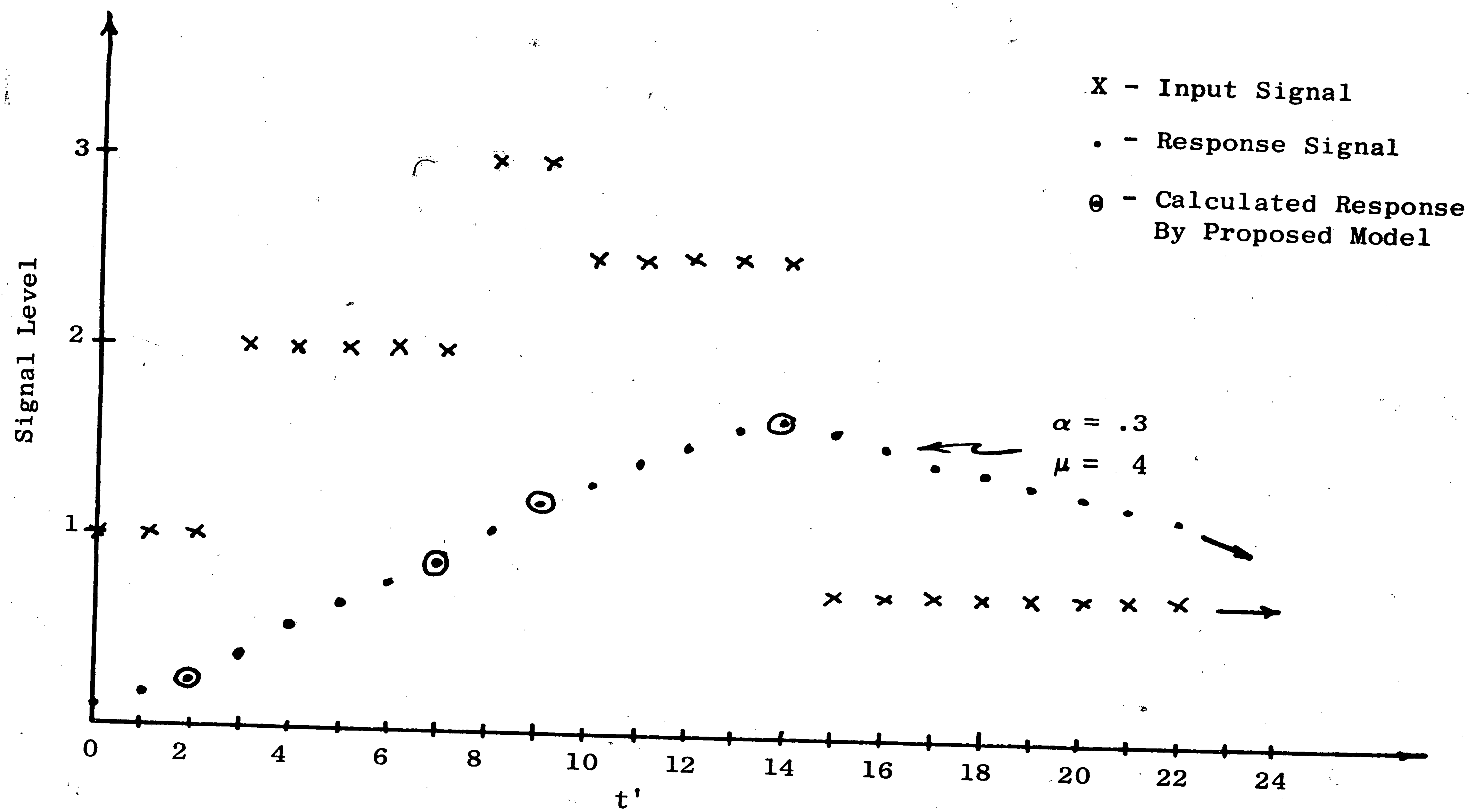


FIGURE (IV-2) RESPONSE TO A TYPICAL INPUT EXAMPLE

α_i holds the same relative position in the models of this thesis as α does in Brown's model. That is, the smoothing functions are not very sensitive to the choice of α_i.

It is instructive however to view the sensitivity of α_i as a function of α and μ. The sensitivity of a function is defined as a ratio of the percentage change in the function to the percentage change in the parameter. In this manner we can interpret the relative importance of the effects of errors or changes in the functions parameters on the function itself. By definition (see reference 9, p. 633) the percentage change in G with respect to a parameter b is given by:

$$S_b^G = \frac{\frac{\partial G}{G}}{\frac{\partial b}{b}} = \frac{\partial \ln G}{\partial \ln b}$$

Let us interpret a measure of the sensitivity of the function α_i with respect to the parameter α. It will become apparent, however, that the mathematical relationship is such that a related function of α_i is more convenient. Writing the relationship

$$\alpha_i = 1-(1-\alpha)^{K_i/\mu}$$

then $(1-\alpha_i) = (1-\alpha)^{K_i/\mu}$

For convenience let us interpret the sensitivity of $(1-\alpha_i)$ with respect to $(1-\alpha)$. Recall that $(1-\alpha_i)$ is the weight given to all past data.

Taking the natural log of both sides:

$$\ln(1-\alpha_i) = \frac{K_i}{\mu} \ln(1-\alpha)$$

then:

$$\partial \ln(1-\alpha_i) = \frac{K_i}{\mu} \partial \ln(1-\alpha)$$

$$\frac{\partial \ln(1-\alpha_i)}{\partial \ln(1-\alpha)} = \frac{K_i}{\mu}$$

therefore:

$$\frac{\dfrac{\partial(1-\alpha_i)}{(1-\alpha_i)}}{\dfrac{\partial(1-\alpha)}{(1-\alpha)}} = \frac{K_i}{\mu}$$

Thus the percent change in the function $(1-\alpha_i)$ with respect to an error in $(1-\alpha)$ is directly proportional to the ratio of the interval length to the average interval length. For an average interval length, a percent error in $(1-\alpha)$ gives the same percent error in $(1-\alpha_i)$.

Consider the parameter μ:

$$(1-\alpha_i) = (1-\alpha)^{K_i/\mu}$$

for small α

$$(1-\alpha_i) \cong 1 - \frac{K_i}{\mu}\alpha$$

$$\alpha_i \cong \frac{K_i}{\mu}\alpha$$

$$\partial\alpha_i \cong -\frac{K_i}{\mu^2}\alpha\,\partial\mu$$

$$\frac{\partial\alpha_i}{\alpha_i} \cong -\frac{1}{\mu}\partial\mu$$

Therefore:

$$\frac{\frac{\partial \alpha_i}{\alpha_i}}{\frac{\partial \mu}{\mu}} \cong -1$$

A percentage error in the parameter μ gives the same percentage error in α_i.

Up to this point in the thesis, the properties have been analytically explored. In the next chapter a demonstration of the model using real data is studied.

CHAPTER V AN APPLICATION OF THE PROPOSED MODEL

In the first four chapters of the thesis, a model was formulated and the characteristics studied in an analytical fashion. It is the purpose of this chapter to observe the behavior of the proposed model for a typical example time series for which the model was formulated; that is, data that has been accumulated over random interval lengths of time. It is felt that a demonstration of the model on some data would help bridge the gap between the theoretical investigation of the characteristics and the behavior of the model as it might respond to a real-life application.

The interaction between the model, the data, and the parameters are extremely complex. Although the mathematical analysis has given insight to the behavior, it is logical that an exhaustive simulation using real-life data would help make the application of the model more complete. Unfortunately, such a study is beyond the scope of this thesis. Further, it is suggested that such a study is needed for both the model as proposed in this thesis and the model for equally spaced data as given by Brown. It was the intent of this thesis to bring the formulation of the random review interval model to a similar position of study as that of Brown's model. The models appear to be ready for application but they are ripe for some generalized conclusions for a variety of classes of data that represent real-life application.

The model that has been formulated cannot be compared to any model performing the aforementioned function. No model exists for this type of forecast. However, throughout this paper, every attempt has been made to relate the proposed model to the traditional exponential smoothing model. The smoothing coefficient α_i is expressed as a function of the α given by Brown. Further, the proposed model identically reduces to Brown's model under the special case of equally spaced data. Every indication of the analytical treatment suggests similar type behavior. Thus, a relative comparison would be appropriate and informative. Example time series will be explained in the next section so that such a comparison can be demonstrated.

Time Series Data

Recall that the model is used on demand that accumulates over the interval length. In order that the comparison discussed in the previous section can be made, two sets of data will be generated. The sets will be generated from the same population. One set will sum the demand over equal interval lengths of time. The other set will sum the demand over unequal interval lengths of time that are generated from a Poisson distribution. The mean of the Poisson distribution is set equal to the interval length of the equally spaced data.

The Poisson distribution was arbitrarily selected since it does not allow time less than zero and because it is a one parameter distribution. The interval lengths were generated from the assumed theoretical population by a random number table. The common population selected was the first one hundred points of Warmdot Business Conditions given in Reference (1) page 434. The equal interval length and the mean of the Poisson distribution was arbitrarily selected to be four units of time. The two generated time series, each with 25 points, are given in Table V-1. The number of time units over which the demand was accumulated and their respective demands are illustrated.

Evaluation of Forecast Error

One way of measuring the behavior of the models is to evaluate the standard deviation of the errors. The errors are defined as the difference between the demand which was forecast and that which actually occurred. The standard deviation is a measure of the accuracy of the forecast. The average will be very close to zero. The standard deviation is a measure of how much the errors cluster around the mean. If the errors are small, the standard deviation will be small.

The approach suggested in the thesis is to forecast the average demand per smallest unit of time from the last piece of data until the next piece of data. For the evaluation of the forecast error, we will multiply the forecast, based on past data, per smallest unit of time by the number of units of time until the next piece of data. This number will be compared to the actual demand that has accumulated during that interval.

Evaluation of Results

There are actually two comparisons that can be made. One is to apply the equally spaced data to Brown's model and compare this to the unequally spaced data as applied to the model of this thesis. The analytical study intuitively suggests that the measure of forecast

TABLE V-1

Sample Time Series*

Equally Spaced: No. of Time Units	Equally Spaced: Demand	Unequally Spaced: No. of Time Units	Unequally Spaced: Demand
4	1081	4	1081
4	2028	5	2549
4	1927	6	2485
4	1479	6	2817
4	1968	2	1056
4	2005	7	3719
4	2219	2	993
4	1993	6	2883
4	1977	3	1570
4	1928	2	1013
4	1997	6	2804
4	1820	4	2561
4	2419	6	4344
4	2997	5	3184
4	2611	5	2506
4	2510	4	1858
4	2087	3	1438
4	1767	5	2605
4	1948	3	1246
4	2146	3	1487
4	1705	2	832
4	1916	2	839
4	1778	4	2140
4	2076	1	472
4	2280	4	2280

* Data Generated from Population given in Ref. (1), p. 434.

error should be about the same. Another approach is to apply the unequally spaced data to Brown's model; we recognize that this was not the intended function. If the models given in this thesis do not significally improve on this application, it would be a waste of computation time to use them.

The first comparison was made using first order exponential smoothing. The time series data as given in Table V-1 was applied to their respective models; the equally spaced data to Brown's, the unequally spaced data to equation (II-17-A). The variance of the forecast errors (square of the standard deviation) was calculated for different α's. The results are summarized on Table V-2. The ratio of the unbiased estimates of the population variances are also listed. This ratio has the F-distribution, and we can therefore test whether or not the two sets of errors come from the same normal population. At a significance level of $\epsilon = .01$ for 24 degrees of freedom for each sample, the F value is 2.66. Since the ratio for all α's are less than 2.66 (they range from 1.15 to 1.25) there is no reason to reject the hypothesis that the two samples came from the same normal population.

There are two more facts worth noting. The minimum forecast error occurred at about the same α for both types of application. Since the value of μ was set equal to four, the mean of the Poisson distribution that we assumed, one was led to believe from the mathematical analysis that the α of the recommended model should be the same as that of the α given by Brown's model. The fact that the

TABLE V-2

α	Brown's Model Equally Spaced Data Variance 1	Modified Model Unequally Spaced Data Variance 2	Ratio var 2/var 1
0.00	103347.67	121640.89	1.18
0.05	98046.79	114134.15	1.16
0.10	93450.53	107932.55	1.15
0.15	89558.92	102985.03	1.15
0.20	86371.89	99240.29	1.15
0.25	83889.49	96646.59	1.15
0.30	82111.70	95151.79	1.16
0.35	81038.53	94903.27 min.	1.17
0.40	min. 80669.97	95247.88	1.18
0.45	81006.02	96731.85	1.19
0.50	82046.69	99100.73	1.21
0.55	83791.97	102299.29	1.22
0.60	86241.86	106271.48	1.23
0.65	89396.38	110960.29	1.24
0.70	93255.50	116307.77	1.25
0.75	97819.23	122255.26	1.25
0.80	103087.57	128744.56	1.25
0.85	109060.55	135719.99	1.24
0.90	115738.11	143140.05	1.24
0.95	123120.32	151032.98	1.23
1.00	131207.17	161962.20	1.23

α's, using two separate sets of data (generated from the same parent population) on two smoothing equations respectively, were about the same is encouraging. Finally, the sensitivity of the α so far as forecast error is concerned is relatively minimal. A pictorial illustration of forecast error is given on Figure V-1. The standard deviation of the forecast errors are plotted as a function of α. The standard deviations are normalized by the minimum standard deviation of the unequally spaced data for convenience.

Let us now take a look at the second comparison that can be made. That is, the unequally spaced data will be used for both Brown's model and the model that was formulated in the thesis. The forecast error variances are summarized on Table V-3 for a range of values for $0 \leq \alpha \leq 1$. The ratio of the variances are taken so that the F-test can be made. The calculated value significantly exceeds the value F = 2.66 for $\epsilon = .01$ and 24 degrees of freedom for each sample. We can reject the hypothesis that the samples came from the same normal population. This comparison clearly shows the superiority of using the formulated model for this application. A pictorial illustration of this comparison is seen in Figure V-1. The standard deviations are again normalized by the minimum standard deviation of the unequally spaced data as applied to the model of this thesis,

The two aforementioned comparisons are very encouraging. The first comparison demonstrates that the model behaves similar to that of Brown's model as it is applied to equally spaced data. The second comparison demonstrates that the model of this thesis is significantly better than the case of arbitrarily applying unequally

TABLE V-3

α	Brown's Model Unequally Spaced Data Variance 1	Modified Model Unequally Spaced Data Variance 2	Ratio Var 1/Var 2
0.00	min. 966612.30	121640.89	7.95
0.05	968752.01	114134.15	8.49
0.10	975912.93	107932.55	9.04
0.15	988094.58	102985.03	9.59
0.20	1005297.40	99240.29	10.13
0.25	1027521.20	96646.59	10.63
0.30	1054765.70	95151.79	11.09
0.35	1087031.20	94703.27 min.	11.48
0.40	1124317.60	95247.88	11.80
0.45	1166625.10	96731.85	12.06
0.50	1213953.70	99100.73	12.25
0.55	1266302.60	102299.29	12.38
0.60	1323673.00	106271.48	12.46
0.65	1386064.20	110960.29	12.49
0.70	1453476.30	116307.77	12.50
0.75	1525909.50	122255.36	12.48
0.80	1603363.40	128744.56	12.45
0.85	1685838.30	135719.99	12.45
0.90	1773334.10	143140.05	12.39
0.95	1865851.00	151032.98	12.35
1.00	1963389.10	161962.20	12.12

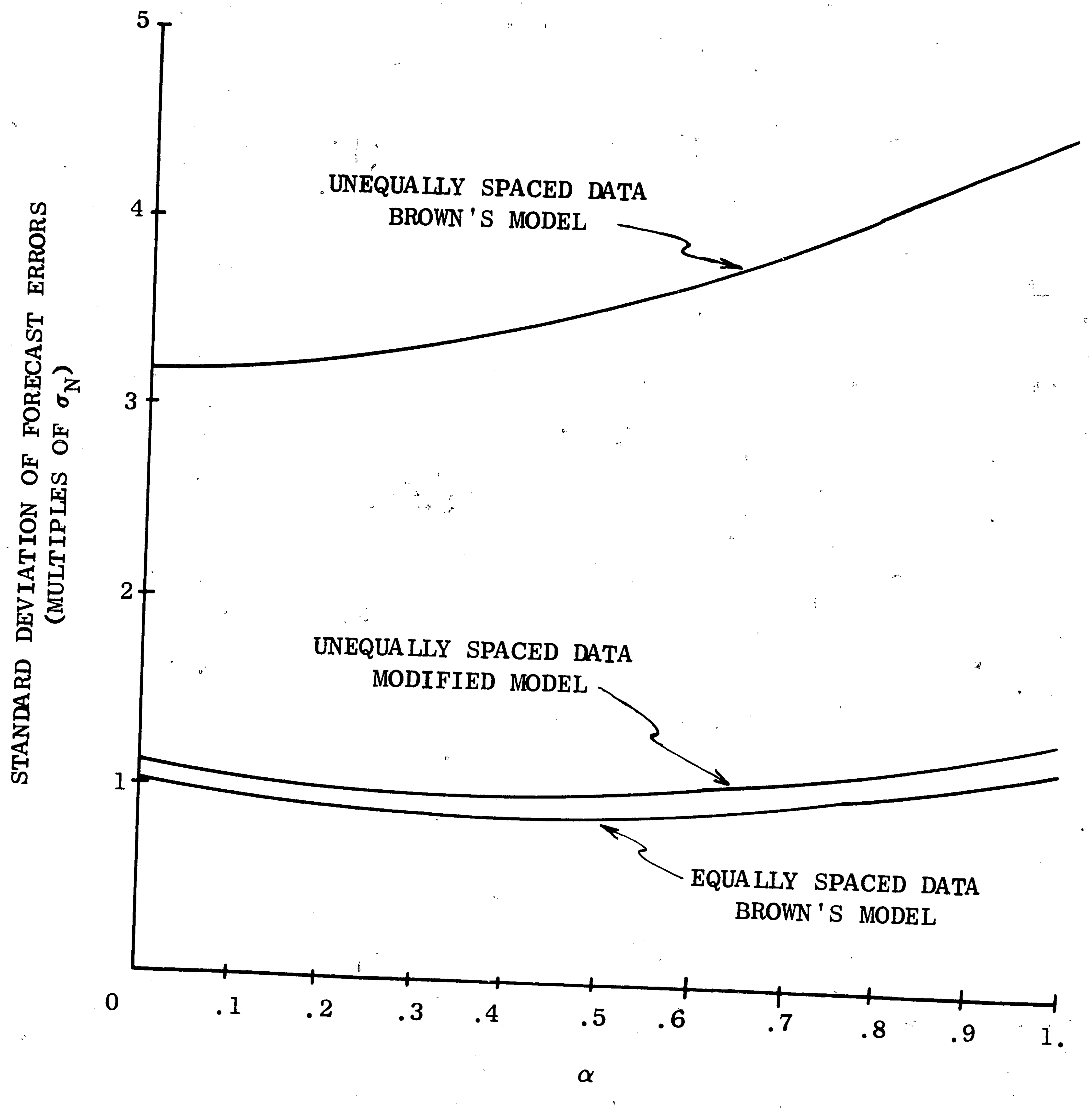


FIGURE (V-1) NORMALIZED FORECAST ERROR

spaced data to Brown's model.

Sensitivity of Parameters

Since the practitioner must select μ , it is important to investigate the sensitivity. Let us use the unequally spaced data and apply it to the model as given by equation (II-17-A).

The normalized forecast variances are calculated and tabulated in Table V-4 for a range of values of α and for μ equal to 3, 4, and 5. The variances are normalized with respect to the value given at $\mu = 4$ and $\alpha = .35$, the minimum. From the table, it is seen that the forecast variance is relatively insensitive to the parameters. It is seen from the table that if μ increases, then the value of that gives the minimum variance error is increased. Conversely if μ is estimated low, then the value of α that gives the minimum variance error is decreased.

Summary of Chapter V

It is to be emphasized that this chapter was only intended to serve as a demonstration of the model that has been recommended. It is included to help support the indications that were given by the analytical procedures.

Although only one set of data has been included for this demonstration, other sets have been run; and they all give comparable results. The behavior of the model both from the analytical study and from the examples tends to act in a manner like that of Brown's model.

TABLE V-4

Normalized Forecast Error Variances for Modified Model, Unequally Spaced Data

α	$\mu = 3$	$\mu = 3$	$\mu = 5$
0.00	1.285	1.285	1.285
0.05	1.182	1.205	1.220
0.10	1.106	1.140	1.164
0.15	1.051	1.088	1.115
0.20	1.017	1.048	1.076
0.25	1.002	1.020	1.044
0.30	min. 1.000	1.005	1.021
0.35	1.016	*1.000	1.007
0.40	1.043	1.006	min. 1.000
0.45	1.080	1.021	1.003
0.50	1.125	1.046	1.013
0.55	1.177	1.080	1.032
0.60	1.234	1.122	1.059
0.65	1.296	1.171	1.095
0.70	1.359	1.228	1.140
0.75	1.422	1.290	1.193
0.80	1.485	1.363	1.257
0.85	1.544	1.433	1.332
0.90	1.601	1.511	1.420
0.95	1.653	1.595	1.526
1.00	1.710	1.710	1.710

* Normalizing Value, Variance - 94703.27

CHAPTER VI SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS FOR FURTHER STUDY

It was the intent of the thesis to formulate a model that will smooth and forecast time series data that occur at aperiodic intervals of time from a set of logical arguments. The model suggested is a time dependent smoothing coefficient formulation that is directly analogus to the traditional exponential smoothing constant. That is, a smoothing coefficient, functional with the irregular time interval, replaces the exponential smoothing constant. The model recommended is a two parameter model. One parameter is equivalent to the α smoothing constant. The other parameter μ is equal to the average interval length. The advantages of exponential smoothing still hold. The computation is simple and accurate; and the file of historical data is small.

Following the derivation, the model was analytically explored. The smoothing function was shown to be an estimate of the average of the data. Then using z-transform theory, the response to standard signals was investigated. The signals were tracked by the model in the same manner as the equally spaced models. The response-stability trade-off is set by the parameter α in the model. Finally the smoothing coefficient was shown to be approximately linearly sensitive to the two parameters.

Following the derivation and analytical study, the model was tried on some data. Two sets of data were generated from the same parent population. One set accumulated demand over unequal intervals.

The other set accumulated demand over equal intervals of time. The forecast error was calculated for the unequally spaced data using the model of this thesis. This was compared to the forecast error of the equally spaced data as applied to Brown's model. There was no significant forecast error difference. This demonstrates that the model behaves for the irregular interval similar to that of Brown's model as applied to equally spaced data. A second comparison was made. The unequally spaced data was applied both to the model of this thesis and to Brown's model. The model of this thesis was shown to be significantly superior to Brown's model. This justifies its use for such an application.

It is recommended that the model be applied to an exhaustive test and application using either simulated or real-life data. Hopefully, such a study, which would be a formidable task, would give some generalized insight to the nature of the data where the model might best be applied.

There is one other area in which no method exists for smoothing data. Recall that in the model formulated in this thesis that demand was accumulated over the irregular review interval. However, there is another class of problems that involves irregular reviews. These are problems that sample a level at irregular intervals. For example, it may be desirable to sample the stock market level at irregular intervals of time. Clearly, the demands are not being accumulated over time for this case. For this class of problems another attack would have to be made.

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