1968

Cracks in anisotropic bodies in a state of generalized plane deformation

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CRACKS IN ANISOTROPIC BODIES IN A STATE OF GENERALIZED PLANE DEFORMATION

by

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A Thesis

Presented to the Graduate Committee

of Lehigh University

in Candidacy for the Degree of

Master of Science in Mechanics

Lehigh University

1968

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

16 May, 1968
(date)

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Acknowledgement

The author wishes to express his appreciation for the invaluable guidance of Professor George Sih in the preparation of this thesis.

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ABSTRACT

Based on the theory of generalized plane deformation for a homogeneous anisotropic body, the general expressions for the six components of the stress tensor in the immediate vicinity of the tip of a sharpened crack are derived. The stress intensity factors are defined in a manner consistent with the current theories of brittle fracture.

The problem of a complex concentrated load applied to the crack surface is solved under the condition of plane stress, while the results for the case of plane strain may be obtained simply by changing the elastic coefficients.

Studied in detail is the influence of the rotation of the axes of elastic symmetry on the elastic strain energy stored in the cracked body undergoing plane deformation. It is found that when one of the preferred directions coincides with the crack line, the strain energy becomes a maximum or minimum depending upon the orthotropic nature of the material. Such information can assist in the understanding of the possible direction of crack propagation in materials possessing anisotropy.

INTRODUCTION

With the advent of modern technology, the materials sciences have become increasingly concerned with the importance of anisotropic materials and with the failures of materials due to fracture.

Lekhnitskii[1]* points out that our sophisticated technology no longer allows us to deal only with the simplified calculations resulting from assumptions of homogeneity and isotropy. Thus, differences in elastic properties of materials in different directions must be taken into account in many cases. The increased demand for advances in the study of anisotropic elasticity, in addition to better mathematical tools and experimental methods, has led to a profusion of papers dealing with the subject, a particularly good example of which is the book by Lekhnitskii[1].

Sudden failure of materials under severe conditions of loading and environment, such as the particulary catastrophic failures of the Comet airliners, has, in addition, created demand for progress in the area of fracture mechanics. Griffith[2] made the first important contribution to this area when he proposed a theory of brittle fracture based on an energy approach. Essentially, this approach postulated that the increase of crack surface is accompanied

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*Brackets indicate references on page 58.

by a corresponding change in the elastic energy of the system.

Sneddon[3] analyzed the stress field around a crack in two individual examples, however, it was Irwin[4] and Williams[5] who extended the local stress field equations to the general case of an isotropic elastic body. In addition, Irwin imparted much greater meaning to these field equations and reduced considerably the complexity of analysis of the crack propagation phenomenon when he showed that the elastic stress distribution about the crack tip can be completely described by three stress intensity factors, K_I, K_{II}, K_{III}, for any particular geometric and load configuration.

A general survey of the fracture field and, in particular, of the results of elastic stress analyses of cracked bodies was published by Sih and Paris in 1965[6].

Thus it is logical that the efforts made in the field of fracture mechanics would be extended to include anisotropic materials. The work of Savin[7], who has performed analyses in plane anisotropic elasticity of infinite plates containing an elliptical cavity by the method of conformal mapping, lends itself particularly well to analysis of cracked anisotropic bodies. A doctoral thesis, authored by Joseph Perna[8] at Lehigh University, uses Savin's work to extend the Hilbert problem to include anisotropic materials.

A paper by Sih, Paris, and Irwin[9] extended the local crack tip stress field for the general plane problem to the rectilinear anisotropic case and showed that the stress singularity associated with the crack was of the order of $r^{-1/2}$ just as in the isotropic case. The case of the normal non-equilibrated force on a crack in a rectilinear anisotropic body was also considered.

It will therefore be a purpose of this paper to extend the analysis of the crack tip stress field to the case of generalized plane deformation for a homogeneous anisotropic body. The case of a non-equilibrated complex concentrated load on a crack is also considered, and the stress concentration factors, K_I and K_{II}, are calculated for this configuration. The elastic constants in the anisotropic case are seen to play an important role in determination of the magnitude of the stress field in this case. The solution to this problem can be used as a Green's Function to facilitate the solutions of other problems involving tractions placed on the surface of the crack.

Since the susceptibility to fracture of a given material is a prime consideration, it is important to know what the effect of rotation of the axes of elastic symmetry with respect to a crack in an orthotropic material has on the amount of elastic energy available for fracture. The energy may be calculated for various angles of the axes of elastic symmetry by consideration of the general plane rectilinear anisotropic problem and use of the equations relating the rectilinear anisotropic elastic constants to the orthotropic constants. The only case considered is that of normal stress distribution on the crack, however, the cases of anti-plane and in-plane shear could be analyzed in a similar manner. The material presented here may be extended to provide a useful tool for analysis and prediction of fracture in anisotropic structures and crystalline structures. There is also possible application in the field of dislocations.

Part I

Generalized Plane Deformation

Consider a homogeneous body possessing anisotropy of a general form. Let the body be bounded by a cylindrical surface and be loaded with surface tractions and body forces. Assume the body is of infinite length and of arbitrary cross-section (infinite or finite; simply connected or multiply connected). A set of coordinate axes may be attached to this body with the origin at any point of any cross-section and with the z-axis directed parallel to the generator of the cylindrical surface. It may be asserted, in this case, that all components of stresses and displacement will not depend on z. Then, by generalized plane deformation we mean the deformation of the above described body (see Figure 1).

With the above assertions concerning stresses and displacement, Lekhnitskii[1] arrives at the following expressions for displacements for generalized plane deformation:

$$u = U(x, y) - \omega_3 y + u_o \tag{1a}$$

$$v = V(x, y) + \omega_3 x + v_o \tag{1b}$$

$$w = W(x, y) + \omega_o \tag{1c}$$

Where the second terms in the first two expressions correspond to a rigid body rotation about the z-axis and the last terms in each correspond to a rigid body displacement.

The generalized Hooke's Law for an anisotropic body is:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{zy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} \qquad (2)$$

From equations (1) we note that

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0$$

and therefore,

$$\sigma_z = -\frac{1}{a_{33}}(a_{13}\sigma_x + a_{23}\sigma_y + a_{34}\sigma_{yz} + a_{35}\sigma_{xz} + a_{36}\sigma_{xy}) \qquad (3)$$

Substituting equation (3) into (2) the number of independent elastic constants is reduced to fifteen and the constitutive relation becomes,

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{14} & \beta_{15} & \beta_{16} \\ \beta_{21} & \beta_{22} & \beta_{24} & \beta_{25} & \beta_{26} \\ \beta_{41} & \beta_{42} & \beta_{44} & \beta_{45} & \beta_{46} \\ \beta_{51} & \beta_{52} & \beta_{54} & \beta_{55} & \beta_{56} \\ \beta_{61} & \beta_{62} & \beta_{64} & \beta_{65} & \beta_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} \qquad (4)$$

where,

$$\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}} \qquad (i,j = 1,2,4,5,6) \tag{5}$$

Problems of this type can be formulated in terms of complex analytic functions $\Phi_k(z_k)$ $(k = 1,2,3)$ of the complex variable $z_k = x + \mu_k y$ where μ_k is a root of the equation:

$$\ell_4(\mu)\,\ell_2(\mu) - \ell_3^2(\mu) = 0 \tag{6}$$

where

$$\ell_2(\mu) = \beta_{55}\mu^2 - 2\beta_{45}\mu + \beta_{44} \tag{7a}$$

$$\ell_3(\mu) = \beta_{15}\mu^3 - (\beta_{14} + \beta_{56}) \tag{7b}$$

$$\ell_4(\mu) = \beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22} \tag{7c}$$

and the six complex roots of (6) are written,

$$\mu_k = \alpha_k + i\beta_k \qquad (k = 1,2,3) \tag{8a}$$

$$\overline{\mu}_k = \alpha_k - i\beta_k \qquad (k = 1,2,3) \tag{8b}$$

Thus the complex variable z_k can also be written $z_k = x_k + iy_k$ where,

$$x_k = x + \alpha_k y \tag{9a}$$

$$y_k = \beta_k y \tag{9b}$$

The general expressions for the components of stresses and displacements are:

$$\sigma_x = 2\text{Re}[\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2) + \mu_3^2\lambda_3\Phi_3'(z_3)] \tag{10a}$$

$$\sigma_y = 2\text{Re}[\Phi_1'(z_1) + \Phi_2'(z_2) + \lambda_3\Phi_3'(z_3)] \tag{10b}$$

$$\tau_{xy} = -2\text{Re}[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2) + \mu_3\lambda_3\Phi_3'(z_3)] \tag{10c}$$

$$\tau_{xz} = 2\text{Re}[\mu_1\lambda_1\Phi_1'(z_1) + \mu_2\lambda_2\Phi_2'(z_2) + \mu_3\Phi_3'(z_3)] \tag{10d}$$

$$\tau_{yz} = -2\text{Re}[\lambda_1\Phi_1'(z_1) + \lambda_2\Phi_2'(z_2) + \Phi_3'(z_3)] \tag{10e}$$

$$\sigma_z = -a_{33}'[a_{13}\sigma_x + a_{23}\sigma_y + a_{34}\tau_{yz} + a_{35}\tau_{xz} + a_{36}\tau_{xy}] \tag{10f}$$

and,

$$u = 2\text{Re}\sum_{k=1}^{3} p_k\Phi_k(z_k) \tag{11a}$$

$$v = 2\text{Re}\sum_{k=1}^{3} q_k\Phi_k(z_k) \tag{11b}$$

$$w = 2\text{Re}\sum_{k=1}^{3} s_k\Phi_k(z_k) \tag{11c}$$

where,

$$\lambda_1 = -\frac{\ell_3(\mu_1)}{\ell_2(\mu_1)} \tag{12a}$$

$$\lambda_2 = -\frac{\ell_3(\mu_2)}{\ell_2(\mu_2)} \tag{12b}$$

$$\lambda_3 = -\frac{\ell_3(\mu_3)}{\ell_4(\mu_3)} \tag{12c}$$

and,

$$p_k = \beta_{11}\mu_k^2 + \beta_{12} - \beta_{16}\mu_k + \lambda_k(\beta_{15}\mu_k - \beta_{14}) \quad (k = 1,2) \qquad (13a)$$

$$q_k = \beta_{12}\mu_k + \frac{\beta_{22}}{\mu_k} - \beta_{26} + \lambda_k(\beta_{25} - \frac{\beta_{24}}{\mu_k}) \quad (k = 1,2) \qquad (13b)$$

$$s_k = \beta_{14}\mu_k + \frac{\beta_{24}}{\mu_k} - \beta_{46} + \lambda_k(\beta_{45} - \frac{\beta_{44}}{\mu_k}) \quad (k = 1,2) \qquad (13c)$$

$$p_3 = \lambda_3(\beta_{11}\mu_3 + \beta_{12} - \beta_{16}\mu_3) + \beta_{15}\mu_3 - \beta_{14} \qquad (13d)$$

$$q_3 = \lambda_3(\beta_{12}\mu_3 + \frac{\beta_{22}}{\mu_3} - \beta_{26}) + \beta_{25} - \frac{\beta_{24}}{\mu_3} \qquad (13e)$$

$$s_3 = \lambda_3(\beta_{14}\mu_3 + \frac{\beta_{24}}{\mu_3} - \beta_{46}) + \beta_{45} - \frac{\beta_{44}}{\mu_3} \qquad (13f)$$

Thus the solution of a problem involving generalized plane deformation requires the determination of three functions, $\Phi_k(z_k)$, which satisfy the boundary conditions,

$$2\mathrm{Re}[\Phi_1 + \Phi_2 + \lambda_3\Phi_3] = \int_0^s Y_n ds \qquad (14a)$$

$$2\mathrm{Re}[\mu_1\Phi_1 + \mu_2\Phi_2 + \mu_3\lambda_3\Phi_3] = -\int_0^s X_n ds \qquad (14b)$$

$$2\mathrm{Re}[\lambda_1\Phi_1 + \lambda_2\Phi_2 + \Phi_3] = 0 \qquad (14c)$$

when the stresses X_n and Y_n are given on the inner boundary.

For the case of plane strain, the plane perpendicular to the generator of the cylindrical surface (the xy plane) is a plane of elastic symmetry. Therefore,

$$\beta_{14} = \beta_{24} = \beta_{64} = \beta_{15} = \beta_{25} = \beta_{65} = 0$$

and, since the constant ℓ_3 is equal to zero, the problem is reduced to one of two analytic functions $\Phi_k(z_k)$ $(k = 1,2)$ of the complex variable $z_k = x + \mu_k y$ where μ_k is a root of,

$$\beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{16})\mu^2 - 2\beta_{26}\mu + \beta_{22} = 0 \quad (15)$$

and the four complex roots of (16) are written,

$$\mu_k = \alpha_k + i\beta_k \qquad (k = 1,2) \quad (16a)$$

$$\overline{\mu}_k = \alpha_k - i\beta_k \qquad (k = 1,2) \quad (16b)$$

Thus the stresses and displacements for the case of plane strain are given by,

$$\sigma_x = 2\mathrm{Re}[\mu_1^2\Phi_1'(z_1) + \mu_2\Phi_2'(z_2)] \quad (17a)$$

$$\sigma_y = 2\mathrm{Re}[\Phi_1'(z_1) + \Phi_2(z_2)] \quad (17b)$$

$$\tau_{xy} = 2\mathrm{Re}[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2)] \quad (17c)$$

and

$$u = 2\mathrm{Re}\sum_{k=1}^{2} p_k\Phi_k(z_k) \quad (18a)$$

$$v = 2\mathrm{Re} \sum_{k=1}^{2} q_k \Phi_k(z_k) \tag{18b}$$

where

$$p_k = \beta_{11}\mu_k + \beta_{12} - \beta_{16}\mu_k \qquad (k = 1,2) \tag{19a}$$

$$q_k = \beta_{12}\mu_k + \frac{\beta_{22}}{\mu_k} - \beta_{26} \qquad (k = 1,2) \tag{19b}$$

For the case of plane stress the expressions for stresses and displacements are the same as above with the constants β_{ij} replaced by a_{ij}. The boundary conditions for the cases of plane strain and plane stress are given by equations(14) with λ_3 and μ_3 set equal to zero. It is obvious that, if a solution to a problem for the case of generalized plane deformation is known, the solution for the case of plane strain may be obtained simply by setting λ_3 and μ_3 equal to zero.

Part II

Stress and Displacement Fields Near Crack Tip

If the region occupied by the plane of $z = x + iy$ is denoted by S, then the stresses will depend on the three functions, $\Phi_j'(z_j)$ $(j = 1,2,3)$, which are determined, not in the region S, but in the three regions defined by the affine transformations $x_j = x + \alpha_j y$ and $y_j = \beta_j y$. As previously noted by Sih, Paris, and Irwin[9], for problems involving line discontinuities, $\Phi_j'(z_j)$ are sectionally holomorphic functions as defined by Muskhelishvili where, close to the crack tip at z_o, they may be written,

$$\Phi_j'(z_j) = \frac{\Psi_j^{(1)}(z_j)}{\sqrt{z_j - z_o}} + \Psi_j^{(2)}(z_j) \qquad (j = 1,2,3) \tag{20}$$

where $\Psi_j^{(\ell)}(z_j)$ are holomorphic functions at the crack tip z_o. That is,

$$\Psi_j^{(\ell)}(z_j) = \sum_{n=1}^{\infty} \lambda_{jn}^{(\ell)}(z_j - z_o)^n \qquad (\ell = 1,2, \text{ and } j = 1,2,3) \tag{21}$$

The stress and displacement fields near the crack tip are determined in terms of a polar coordinate system in the $z = x + iy$ plane with the origin at the crack tip (see figure 2). That is,

$$z - z_o = re^{i\theta} \tag{22}$$

Therefore,

$$z_j - z_o = r(\cos\theta + \mu_j \sin\theta) \tag{23}$$

And near the tip of the crack, equation(20) may be written,

$$\Phi_j'(z_j) = \frac{\lambda_{jo}^{(1)}}{\sqrt{r(\cos\theta + \mu_j \sin\theta)}} + O(r^{1/2}) \tag{24}$$

As r becomes small in comparison with other planar dimensions, the higher order terms in r may be neglected. The constant $\lambda_{jo}^{(1)}$ represents the intensity of the stress field near the crack tip and is dependent on the load and geometric configuration only. Thus the distribution of the crack tip stress field can be obtained in general without knowledge of the specific values of $\lambda_{jo}^{(1)}$. It is convenient, however, to redefine $\lambda_{jo}^{(1)}$ in terms of k_1 and k_2, the isotropic stress intensity factors, as is done in other papers dealing with the subject.[9] Therefore,

$$\lambda_{10}^{(1)} = \frac{1}{2\sqrt{2\pi}\,\Delta}\{(\mu_2 - \mu_3\lambda_1\lambda_2)k_1 + (1 - \lambda_2\lambda_3)k_2\} \tag{25a}$$

$$\lambda_{20}^{(2)} = \frac{1}{2\sqrt{2\pi}\,\Delta}\{(\mu_3\lambda_1\lambda_3 - \mu_1)k_1 + (\lambda_1\lambda_3 - 1)k_2\} \tag{25b}$$

$$\lambda_{30}^{(1)} = \frac{1}{2\sqrt{2\pi}\,\Delta}\{(\mu_1\lambda_2 - \mu_2\lambda_1)k_1 + (\lambda_2 - \lambda_3)k_2\} \tag{25c}$$

Note that the antiplane problem has no physical relevance to the problem considered and, therefore, k_3 is not included here.

The stress and displacements near the crack tip may now be evaluated and are as follows:

(a) Symmetric Loading:

$$\sigma_x = \frac{k_1}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\Delta} [\frac{\mu_1^2(\mu_2 - \mu_3\lambda_3\lambda_2)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\mu_2^2(\mu_3\lambda_1\lambda_2 - \mu_1)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{\mu_3^2\lambda_3(\mu_1\lambda_2 - \mu_2\lambda_1)}{\sqrt{\cos\theta + \mu_3 \sin\theta}}]\} \tag{26a}$$

$$\sigma_y = \frac{k_1}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\Delta} [\frac{\mu_2 - \mu_3\lambda_3\lambda_2}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\mu_3\lambda_1\lambda_2 - \mu_1}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{\lambda_3(\mu_1\lambda_2 - \mu_2\lambda_1)}{\sqrt{\cos\theta + \mu_3 \sin\theta}}]\} \tag{26b}$$

$$\tau_{xy} = -\frac{k_1}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\Delta} [\frac{\mu_1(\mu_2 - \mu_3\lambda_3\lambda_2)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\mu_2(\mu_3\lambda_1\lambda_3 - \mu_1)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{\mu_3\lambda_3(\mu_1\lambda_2 - \mu_2\lambda_1)}{\sqrt{\cos\theta + {}_3 \sin\theta}}]\} \tag{26c}$$

$$\tau_{xz} = \frac{k_1}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\Delta} [\frac{\mu_1\lambda_1(\mu_2 - \mu_3\lambda_3\lambda_2)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\mu_2\lambda_2(\mu_3\lambda_1\lambda_3 - \mu_1)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{\mu_3(\mu_1\lambda_2 - \mu_2\lambda_1)}{\sqrt{\cos\theta + \mu_3 \sin\theta}}]\} \qquad (26d)$$

$$\tau_{zy} = - \frac{k_1}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\Delta} [\frac{\lambda_1(\mu_2 - \mu_3\lambda_2\lambda_3)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\lambda_2(\mu_3\lambda_1\lambda_2 - \mu_1)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{\mu_1\lambda_2 - \mu_2\lambda_1}{\sqrt{\cos\theta + \mu_3 \sin\theta}}]\} \qquad (26e)$$

$$\sigma_z = \frac{k_1}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\Delta} [\frac{\gamma_1(\mu_2 - \mu_3\lambda_2\lambda_3)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\gamma_2(\mu_3\lambda_1\lambda_2 - \mu_1)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{\gamma_3(\mu_1\lambda_2 - \mu_2\lambda_1)}{\sqrt{\cos\theta + \mu_3 \sin\theta}}]\} \qquad (26f)$$

and,

$$u = k_1 \sqrt{\frac{2r}{\pi}} \mathrm{Re}\{\frac{1}{\Delta} [p_1(\mu_2 - \mu_3\lambda_2\lambda_3)\sqrt{\cos\theta + \mu_1 \sin\theta}$$

$$+ p_2(\mu_3\lambda_1\lambda_3 - \mu_1)\sqrt{\cos\theta + \mu_2 \sin\theta}$$

$$+ p_3(\mu_1\lambda_2 - \mu_2\lambda_1)\sqrt{\cos\theta + \mu_3 \sin\theta}]\}$$

$$(27a)$$

$$v = k_1 \sqrt{\frac{2r}{\pi}} \, \text{Re}\{\frac{1}{\Delta} [q_1(\mu_2 - \mu_3\lambda_2\lambda_3)\sqrt{\cos\theta + \mu_1 \sin\theta}$$

$$+ q_2(\mu_3\lambda_1\lambda_2 - \mu_1)\sqrt{\cos\theta + \mu_2 \sin\theta}$$

$$+ q_3(\mu_1\lambda_2 - \mu_2\lambda_1)\sqrt{\cos\theta + \mu_3 \sin\theta}\,]\} \quad (27b)$$

$$w = k_1 \sqrt{\frac{2r}{\pi}} \, \text{Re}\{\frac{1}{\Delta} [s_1(\mu_2 - \mu_3\lambda_2\lambda_3)\sqrt{\cos\theta + \mu_1 \sin\theta}$$

$$+ s_2(\mu_3\lambda_1\lambda_2 - \mu_1)\sqrt{\cos\theta + \mu_2 \sin\theta}$$

$$+ s_3(\mu_1\lambda_2 - \mu_2\lambda_1)\sqrt{\cos\theta + \mu_3 \sin\theta}\,]\} \quad (27c)$$

(b) Skew Symmetric Loading:

$$\sigma_x = -\frac{k_2}{\sqrt{2\pi r}} \, \text{Re}\{\frac{1}{\Delta} [\, \frac{\mu_1^2(\lambda_2\lambda_3 - 1)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\mu_2^2(1 - \lambda_1\lambda_3)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{\mu_3^2\lambda_3(\lambda_1 - \lambda_2)}{\sqrt{\cos\theta + \mu_3 \sin\theta}}\,]\} \quad (28a)$$

$$\sigma_y = -\frac{k_2}{\sqrt{2\pi r}} \, \text{Re}\{\frac{1}{\Delta} [\, \frac{(\lambda_2\lambda_3 - 1)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{(1 - \lambda_1\lambda_3)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{\lambda_3(\lambda_1 - \lambda_2)}{\sqrt{\cos\theta + \mu_3 \sin\theta}}\,]\} \quad (28b)$$

$$\sigma_{xy} = \frac{k_2}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\Delta} [\frac{\mu_1(\lambda_2\lambda_3 - 1)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\mu_2(1 - \lambda_1\lambda_3)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{\mu_3\lambda_3(\lambda_1 - \lambda_2)}{\sqrt{\cos\theta + \mu_3 \sin\theta}}]\} \tag{28c}$$

$$\sigma_{xz} = - \frac{k_2}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\Delta} [\frac{\mu_1\lambda_1(\lambda_2\lambda_3 - 1)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\mu_2\lambda_2(1 - \lambda_1\lambda_3)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{\mu_3(\lambda_1 - \lambda_2)}{\sqrt{\cos\theta + \mu_3 \sin\theta}}]\} \tag{28d}$$

$$\sigma_{yz} = \frac{k_2}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\Delta} [\frac{\lambda_1(\lambda_2\lambda_3 - 1)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\lambda_2(1 - \lambda_1\lambda_3)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{(\lambda_1 - \lambda_2)}{\sqrt{\cos\theta + \mu_3 \sin\theta}}]\} \tag{28e}$$

$$\sigma_z = - \frac{k_2}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\Delta} [\frac{\gamma_1(\lambda_2\lambda_1 - 1)}{\sqrt{\cos\theta + \mu_1 \sin\theta}} + \frac{\gamma_2(1 - \lambda_1\lambda_3)}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$+ \frac{(\lambda_1 - \lambda_2)\gamma_3}{\sqrt{\cos\theta + \mu_3 \sin\theta}}]\} \tag{28f}$$

and,

$$u = -k_2 \sqrt{\frac{2r}{\pi}}\, \mathrm{Re}\{\frac{1}{\Delta}[p_1(\lambda_2\lambda_3 - 1)\sqrt{\cos\theta + \mu_1 \sin\theta}$$

$$+ p_2(1 - \lambda_1\lambda_3)\sqrt{\cos\theta + \mu_2 \sin\theta}$$

$$+ p_3(\lambda_1 - \lambda_2)\sqrt{\cos\theta + \mu_3 \sin\theta}\,]\} \quad (29a)$$

$$v = -k_2 \sqrt{\frac{2r}{\pi}}\, \mathrm{Re}\{\frac{1}{\Delta}[q_1(\lambda_2\lambda_3 - 1)\sqrt{\cos\theta + \mu_1 \sin\theta}$$

$$+ q_2(1 - \lambda_1\lambda_3)\sqrt{\cos\theta + \mu_2 \sin\theta}$$

$$+ q_3(\lambda_1 - \lambda_2)\sqrt{\cos\theta + \mu_3 \sin\theta}\,]\} \quad (29b)$$

$$w = -k_2 \sqrt{\frac{2r}{\pi}}\, \mathrm{Re}\{\frac{1}{\Delta}[s_1(\lambda_2\lambda_3 - 1)\sqrt{\cos\theta + \mu_1 \sin\theta}$$

$$+ s_2(1 - \lambda_1\lambda_3)\sqrt{\cos\theta + \mu_2 \sin\theta}$$

$$+ s_3(\lambda_1 - \lambda_2)\sqrt{\cos\theta + \mu_3 \sin\theta}\,]\} \quad (29c)$$

The constants, γ_k, are given by the following expressions:

$$\gamma_k = a_{13}\mu_k^2 + a_{23} - a_{34}\lambda_k + a_{35}\mu_k\lambda_k - a_{36}\mu_k \quad (k=1,2) \quad (30a)$$

$$\gamma_3 = \lambda_3(a_{13}\mu_3^2 + a_{23} - a_{36}\mu_3) - a_{34} + a_{35}\mu_3 \tag{30b}$$

If the case of plane strain is considered, then,

$$\lambda_k = \delta_k = \mu_3 = \beta_{14} = \beta_{24} = \beta_{64} = \beta_{15} = \beta_{25} = \beta_{65} = 0$$

and equations (26) through (29) reduce to:

(a') Symmetric Loading:

$$\sigma_x = \frac{k_1}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{\mu_1\mu_2}{\mu_1 - \mu_2} [\frac{\mu_2}{\sqrt{\cos\theta + \mu_2 \sin\theta}} - \frac{\mu_1}{\sqrt{\cos\theta + \mu_1 \sin\theta}}]\} \tag{31a}$$

$$\sigma_y = \frac{k_1}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\mu_1 - \mu_2} [\frac{\mu_1}{\sqrt{\cos\theta + \mu_2 \sin\theta}} - \frac{\mu_2}{\sqrt{\cos\theta + \mu_1 \sin\theta}}]\} \tag{31b}$$

$$\tau_{xy} = \frac{k_1}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{\mu_1\mu_2}{\mu_1 - \mu_2} [\frac{1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{1}{\sqrt{\cos\theta + \mu_2 \sin\theta}}]\} \tag{31c}$$

and,

$$u = k_1 \frac{\sqrt{2\pi r}}{\pi} \mathrm{Re}\{\frac{1}{\mu_1 - \mu_2} [\mu_1 p_2 \sqrt{\cos\theta + \mu_2 \sin\theta}$$

$$- \mu_2 p_1 \sqrt{\cos\theta + \mu_1 \sin\theta}\,]\} \qquad (32a)$$

$$v = k_1 \frac{\sqrt{2\pi r}}{\pi} \mathrm{Re}\{\frac{1}{\mu_1 - \mu_2} [\mu_1 q_2 \sqrt{\cos\theta + \mu_2 \sin\theta}$$

$$- \mu_2 p_2 \sqrt{\cos\theta + \mu_1 \sin\theta}\,]\} \qquad (32b)$$

(b') Skew Symmetric Loading:

$$\sigma_x = \frac{k_2}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\mu_1 - \mu_2} [\frac{\mu_2^2}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$- \frac{\mu_1^2}{\sqrt{\cos\theta + \mu_1 \sin\theta}}]\} \qquad (33a)$$

$$\sigma_y = \frac{k_2}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\mu_1 - \mu_2} [\frac{1}{\sqrt{\cos\theta + \mu_2 \sin\theta}}$$

$$- \frac{1}{\sqrt{\cos\theta + \mu_1 \sin\theta}}]\} \qquad (33b)$$

$$\tau_{xy} = \frac{k_2}{\sqrt{2\pi r}} \mathrm{Re}\{\frac{1}{\mu_1 - \mu_2} [\frac{\mu_1}{\sqrt{\cos\theta + \mu_1 \sin\theta}}$$

$$- \frac{\mu_2}{\sqrt{\cos\theta + \mu_1 \sin\theta}}] \quad (33c)$$

and,

$$u = k_2 \frac{\sqrt{2\pi r}}{\pi} \mathrm{Re}\{\frac{1}{\mu_1 - \mu_2} [p_2 \sqrt{\cos\theta + \mu_2 \sin\theta}$$

$$- p_1 \sqrt{\cos\theta + \mu_1 \sin\theta}]\} \quad (34a)$$

$$v = k_2 \frac{\sqrt{2\pi r}}{\pi} \mathrm{Re}\{\frac{1}{\mu_1 - \mu_2} [q_2 \sqrt{\cos\theta + \mu_2 \sin\theta}$$

$$- q_1 \sqrt{\cos\theta + \mu_1 \sin\theta}]\} \quad (34b)$$

These equations are identical to the crack tip stress field equations previously derived for the case of plane strain of a rectilinear anisotropic body.[9] The only difference here is the definition of k_i which differs by a factor of $\frac{1}{\sqrt{2\pi}}$ with the k_i defined in the referenced paper by Sih, Paris, and Irwin.

Thus the stress singularity in the case of generalized plane deformation is of the order $r^{-1/2}$ just as if true for the case of plane strain. Other than the difference due to the elastic constants, the only major addition to the expressions for stresses and displacements for the case of generalized plane deformation is an additional term in θ, which is due to the contribution of a third function, $\Phi_3(z_3)$ of the complex variable $z_3 = x + \mu_3 y$.

Part III

Concentrated Load Solution

The solution to the problem involving a concentrated load normal to the surface of a crack in an anisotropic plate is given by Sih, Paris, and Irwin[9] for plane stress. Since the concentrated load solution may be used as a Green's function for other solutions involving crack line loading, it is useful to determine the stress function and stress intensity factors for the problem of a concentrated load of arbitrary orientation. The problem is, therefore, a concentrated load, $P_x + iP_y$, acting on a crack in an infinite body possessing rectilinear anisotropy as shown in Figure (5).

A. Stress Function

To determine the stress function $\Phi_1(z_1)$ for this loading* let,

$$\Phi_1(z_1) = \overset{1}{\Phi}_1(z_1) + \overset{2}{\Phi}_1(z_1) + \overset{3}{\Phi}_1(z_1) \tag{35}$$

Where $\overset{1}{\Phi}_1(z_1)$ represents the solution as z_1 approaches b, $\overset{2}{\Phi}_1(z_1)$ represents the solution for z_1>> and insures zero stresses at infinity, and $\overset{3}{\Phi}_1(z_1)$ is a holomorphic function.

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*The method of solution is similar to method used by Sih for solutions in isotropic case. See Reference [11].

As z_1 approaches b, the solution should approach that of a concentrated load on the half-plane and, therefore,

$$\Phi_1^1(z_1) = -\frac{P_x + \mu_2 P_y}{2\pi i(\mu_1 - \mu_2)} \log (z_1 - b) \tag{36}$$

The general structure of the stress function is the same for all problems of this type. And, therefore, sufficiently far from $z_1 = b$,

$$\Phi_1(z_1) = A_1 \log z_1 + B_1 z_1 + \overset{\circ}{\Phi}_1(z_1) \tag{37}$$

where A_1 is uniquely determined by consideration of the cyclic value of $\Phi_1(z_1)$ and single valuedness of the displacement field (see equations (45)).

For zero stresses at infinity, $B_1 = 0$. Equating equations (35) and (37) for $|z_1 - b| >> 0$,

$$-\frac{P_x + \mu_2 P_y}{2\pi i(\mu_1 - \mu_2)} \log z_1 + \Phi_1^2(z_1) = A_1 \log z_1 \tag{38}$$

yields the following result:

$$\Phi_1^2(z_1) = \left(\frac{P_x + \mu_2 P_y}{2\pi i(\mu_1 - \mu_2)}\right) + A_1 \log z_1 \tag{39}$$

Therefore, the value of the stress function $\Phi_1(z_1)$ for the entire plane is:

$$\Phi_1(z_1) = A_1 \log z_1 - \frac{P_x + \mu_2 P_y}{2\pi i(\mu_1 - \mu_2)} \log (z_1 - b)$$

$$+ \frac{P_x + \mu_2 P_y}{2\pi i(\mu_1 - \mu_2)} \log z_1 + \overset{3}{\Phi}_1(z_1) \qquad (40)$$

The stress function, $\Phi_2(z_2)$, may be evaluated in a similar manner, however, $\Phi_1(z_1)$ yields sufficient information to evaluate the stress intensity factors for this problem.

To map the crack problem onto the unit circle the following function is used:

$$z_j = \omega(\zeta_j) = \frac{a}{2}(\zeta_j + \frac{1}{\zeta_j}) \qquad (41)$$

The function $\Phi_1(\zeta_1)$, which corresponds to $\Phi_1(z_1)$ in the mapped plane, is:

$$\Phi_1(\zeta_1) = A_1 \log \zeta_1 - \frac{P_x + \mu_2 P_y}{2\pi i(\mu_1 - \mu_2)} \log (\zeta_1 - b)$$

$$+ \frac{P_x + \mu_2 P_y}{2\pi i(\mu_1 - \mu_2)} \log \zeta_1 + \overset{0}{\Phi}_1(\zeta_1) \qquad (42)$$

where $\overset{0}{\Phi}_1(\zeta_1)$ is evaluated by consideration of the boundary conditions on the crack and is found to be zero to insure zero tractions there. Therefore, $\Phi_1(\zeta_1)$ becomes:

$$\Phi_1(\zeta_1) = A_1 \log \zeta_1 - \frac{P_x + \mu_2 P_y}{2\pi i(\mu_1 - \mu_2)} \log (\zeta_1 - b)$$

$$+ \frac{P_x + \mu_2 P_y}{2\pi i(\mu_1 - \mu_2)} \log \zeta_1 \qquad (43)$$

As mentioned previously, A_1 may be evaluated by use of simultaneous equations based on cyclic value of stresses and single valuedness of displacements. These equations are:

$$A_1 + A_2 - \bar{A}_1 - \bar{A}_2 = \frac{P_y}{2\pi i} \qquad (44a)$$

$$\mu_1 A_1 + \mu_2 A_2 - \bar{\mu}_1 \bar{A}_1 - \bar{\mu}_2 \bar{A}_2 = - \frac{P_x}{2\pi i} \qquad (44b)$$

$$p_1 A_1 + p_2 A_2 - \bar{p}_1 \bar{A}_1 - \bar{p}_2 \bar{A}_2 = 0 \qquad (44c)$$

$$q_1 A_1 + q_2 A_2 - \bar{q}_1 \bar{A}_1 - \bar{q}_2 \bar{A}_2 = 0 \qquad (44d)$$

The solution to the above set of equations is:

$$A_1 = \frac{\Delta_1 + i\Delta_2}{\Delta} \qquad (45)$$

where,

$$\Delta = - \frac{4\beta_1\beta_2 a_{11} a_{22}}{(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)} [(\alpha_2 - \alpha_1)^4$$

$$+ 2(\alpha_2 - \alpha_1)^2(\beta_2^2 + \beta_1^2) + (\beta_2^2 - \beta_1^2)^2] \qquad (46)$$

and,

$$\Delta_1 = \frac{P_y}{\pi} \{p_2^{(2)} [\alpha_2 q_1^{(1)} - \alpha_1 q_2^{(1)}] + q_2^{(2)}[\alpha_1 p_2^{(1)} - \alpha_2 p_1^{(1)}]$$

$$+ \beta_2 [p_1^{(1)} q_2^{(1)} - p_2^{(1)} q_1^{(1)}]\}$$

$$+ \frac{P_x}{\pi} \{p_2^{(2)} [q_1^{(1)} - q_2^{(1)}] + q_2^{(2)} [p_2^{(1)} - p_1^{(1)}]\} \tag{47}$$

$$\Delta_2 = \frac{P_y}{\pi} \{q_2^{(1)} [\beta_1 p_2^{(2)} - \beta_2 p_1^{(2)}] + p_2^{(1)} [\beta_1 q_1^{(2)} - \beta_1 q_2^{(2)}]$$

$$+ \alpha_2 [p_1^{(2)} q_2^{(2)} - p_2^{(2)} q_1^{(2)}]\}$$

$$+ \frac{P_x}{\pi} \{p_1^{(2)} q_2^{(2)} - p_2^{(2)} q_1^{(2)}\} \tag{48}$$

Δ_1 and Δ_2 are expressed in terms of the real and imaginary parts of p_j and q_j $(j = 1,2)$ previously defined. That is,

$$p_j = p_j^{(1)} + i p_j^{(2)} \tag{49a}$$

$$q_j = q_j^{(1)} + i q_j^{(2)} \tag{49b}$$

B. Stress Intensity Factors

In order to evaluate the stress intensity factors we use the following expression.

$$k_1 + \frac{k_2}{\mu_2} = - 2\sqrt{2\pi}\ \frac{\mu_1 - \mu_2}{\mu_2} \lim_{\zeta \to \zeta_o} \ (\omega_1(\zeta_1) - \omega_1(\zeta_o))^{1/2} \frac{\Phi_1'(\zeta_1)}{\omega_1'(\zeta_1)} \tag{50}$$

Therefore,

$$k_1 + \frac{k_2}{\mu_2} = - 4\sqrt{\pi}\ \frac{\mu_1 - \mu_2}{\mu_2} \lim_{\zeta_1 \to a} \ \{[a(\zeta_1 + \frac{1}{\zeta_1} - 2)]^{1/2} \frac{1}{a(1 - \frac{1}{\zeta^2})}$$

$$(\frac{A_1}{\zeta_1} - \frac{P_x + \mu_2 P_y}{2\pi i(\mu_2 - \mu_1)} \frac{1}{(\zeta_1 - b)} + \frac{P_x + \mu_2 P_y}{2\pi i(\mu_1 - \mu_2)} \frac{1}{\zeta_1})\} \tag{51}$$

This can be reduced to,

$$k_1 + \frac{k_2}{2} = - \frac{4\sqrt{\pi}}{2\sqrt{a}} \ \{\frac{\mu_1 - \mu_2}{\mu_2} A_1 - \frac{P_x + \mu_2 P_y}{2\pi i \mu_2} (\frac{1}{2} + \frac{i}{2} (\frac{a + b}{a - b})^{1/2})\}$$

$$+ \frac{P_x + \mu_2 P_y}{2\pi i \mu_2} \tag{52}$$

Equation (52) may be greatly simplified in the orthotropic case where μ_1 and μ_2 are given either by $\mu_1 = \alpha_o + i\beta_o$ and $\mu_2 = - \alpha_o + i\beta_o$ or, $\mu_1 = i\beta_1$ and $\mu_2 = i\beta_2$.

In the first case, for $\mu_1 = -\bar{\mu}_2 = \alpha_o + i\beta_o$,

$$A_1 = \frac{P_y}{16\pi}\{\frac{1}{\alpha_o\beta_o}[\frac{a_{12}}{a_{11}} + (\alpha_o^2 - \beta_o^2)] - 2i\}$$

$$+ \frac{P_x}{16\pi}\{\frac{1}{\beta_o}(\frac{a_{12}}{a_{11}}(\alpha_o^2 + \beta_o^2) + 1) - \frac{i}{\alpha_o}(\frac{a_{12}}{a_{22}}(\alpha_o^2 + \beta_o^2) - 1)\} \quad (53)$$

and, by substitution into equation (52), k_1 and k_2 are found to be,

$$k_1 = \frac{P_y}{2(\pi a)^{1/2}}\{\frac{a+b}{a-b}\}^{1/2} + \frac{P_x}{2(\pi a)^{1/2}}\{\frac{1}{2\beta_o}[1 + \frac{a_{12}}{a_{22}}(\alpha_o^2 + \beta_o^2)]\} \quad (54a)$$

$$k_2 = -\frac{P_y}{2(\pi a)^{1/2}}\{\frac{1}{2\beta_o}[\frac{a_{12}}{a_{11}} + (\alpha_o^2 + \beta_o^2)]\} + \frac{P_x}{2(\pi a)^{1/2}}\{\frac{a+b}{a-b}\}^{1/2} \quad (54b)$$

In the second case, for $\mu_1 = i\beta_1$ and $\mu_2 = i\beta_2$,

$$A_1 = \frac{P_x}{4\pi}\frac{\beta_1(a_{12}\beta_2^2 - a_{22})}{(\beta_2^2 - \beta_1^2)a_{22}} - i\frac{P_y}{4\pi}\frac{\beta_2^2}{(\beta_2^2 - \beta_1^2)} \quad (55)$$

and,

$$k_1 = \frac{P_y}{2(\pi a)^{1/2}} \left(\frac{a + b}{a - b}\right)^{1/2} + \frac{P_x}{2(\pi a)^{1/2}} \left\{\frac{1}{\beta_2 + \beta_1} \left(\frac{a_{12}}{a_{22}} \beta_1\beta_2 + 1\right)\right\} \tag{56a}$$

$$k_2 = - \frac{P_y}{2(\pi a)^{1/2}} \left\{\frac{\beta_1\beta_2}{\beta_2 + \beta_1}\right\} + \frac{P_x}{2(\pi a)^{1/2}} \left(\frac{a + b}{a - b}\right)^{1/2} \tag{56b}$$

Part IV

Energy Consideration

For the case of generalized plane deformation, the strain energy stored in a body with normal pressure p applied to the crack surface, as shown in Figure 6, may be calculated.

A. Strain Energy Density Function

In this case the strain energy, W, is given by,

$$W = \frac{1}{2}\int_{-a}^{a} p[v^{+} - v^{-}]dx \qquad (57)$$

where v^+ and v^- are the displacements of the upper crack surface and lower crack surface respectively, and,

$$v = 2\mathrm{Re}\sum_{k=1}^{3} q_k\Phi_k \qquad (58)$$

For the load configuration and geometry under consideration, $\Phi_j(z_j)$ is found to be (see appendix A),

$$\Phi_j(z_j) = [\sqrt{z_j^2 - a^2} - z_j]\,\frac{p}{2\Delta}\,r_j \qquad (59)$$

where,

$$r_1 = \mu_2 - \mu_3\lambda_2\lambda_3 \tag{60a}$$

$$r_2 = \mu_3\lambda_2\lambda_3 - \mu_1 \tag{60b}$$

$$r_3 = \mu_1\lambda_2 - \mu_2\lambda_1 \tag{60c}$$

The values of $\Phi_j(z_j)$ on the upper and lower crack surfaces are, respectively,

$$\Phi_j^+(z_j) = [\sqrt{x^2 - a^2} - x]\,\frac{p}{2\Delta}\,r_j \tag{61a}$$

$$\Phi_j^-(z_j) = [-\sqrt{x^2 - a^2} - x]\,\frac{p}{2\Delta}\,r_j \tag{61b}$$

Therefore,

$$\sum_{k=1}^{3}(q_k\Phi_k^+ - q_k\Phi_k^-) = \frac{ip}{\Delta}\sqrt{a^2 - x^2}\sum_{k=1}^{3} q_k r_k \tag{62}$$

and,

$$v^+ - v^- = -2\,\mathrm{Im}\,\frac{p\sqrt{a^2 - x^2}}{\Delta}\sum_{k=1}^{3} q_k r_k \tag{63}$$

So, from equations (57) and (63), the strain energy is,

$$W = -p^2\,\mathrm{Im}\,\{\frac{\Sigma q_k r_k}{\Delta}\int_{-a}^{a}\sqrt{a^2 - x^2}\,dx\} \tag{64}$$

Integrating equation (64),

$$W = - \frac{\pi a^2 p^2}{2} \operatorname{Im}\{\frac{1}{\Delta} \sum_{k=1}^{3} q_k r_k\} \tag{65}$$

If we consider the case of generalized plane stress, the expression for w is reduced to,

$$W = - \frac{\pi a^2 p^2}{2} \operatorname{Im}\{\frac{q_1 \mu_2 - q_2 \mu_1}{\mu_2 - \mu_1}\}$$

or, simplifying further,

$$W = - \frac{\pi a^2 p^2}{2} \operatorname{Im}\{\frac{\mu_1 + \mu_2}{\mu_1 \mu_2}\} a_{11} \tag{66}$$

If the body is orthotropic, but the crack does not lie along an axis of symmetry (see Figure 7), the energy expression is (see appendix D):

$$W = \frac{\pi a^2 p^2}{2} [2\sqrt{a_{22} a_{11}} + 2a_{12} + a_{66}]^{1/2} [\sqrt{a_{22}} \cos^2 \phi + \sqrt{a_{11}} \sin^2 \phi] \tag{67}$$

where ϕ is the angle formed by the crack and the x-axis of elastic symmetry and the a_{ij}'s are the elastic constants for the orthotropic case.

We define a new variable W*, which is proportional to W, as:

$$W^* = 2W/\pi a^2 p^2 \left[2\sqrt{a_{22}a_{11}} + 2a_{12} + a_{66}\right]^{1/2} \sqrt{a_{11}} \tag{68}$$

Substituting the expression for W given by (67), W* becomes,

$$W^* = \frac{a_{22}}{a_{11}} \cos^2 \phi + \sin^2 \phi \tag{69}$$

Then, we see that the maximum strain energy occurs either at $\phi = 0$ or $\phi = 90°$; that is, it occurs along a principle elastic axis (see Figure 8).

B. Discussion

Equation (67) gives the total energy available for fracture in the body considered. This gives a fair estimate of the tendency toward fracture for various orientations of the crack with respect to the elastic axes.

For example, in the case of a sheet of three layer birch plywood glued to a Bakelite layer, the elastic constants are as follows*,

$$E_1 = 1.2 \cdot 10^5 \ \mathrm{kg/cm^2}$$
$$E_2 = 0.6 \cdot 10^5 \ \mathrm{kg/cm^2}$$
$$G_{12} = 0.07 \cdot 10^5 \ \mathrm{kg/cm^2}$$
$$\nu_{12} = 0.071$$

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*This data is taken from Reference [4], page 165.

and, therefore,

$$a_{11} = 8.33 \cdot 10^{-6} \text{ cm}^2/\text{kg}$$

$$a_{22} = 1.67 \cdot 10^{-5} \text{ cm}^2/\text{kg}$$

$$a_{12} = -5.91 \cdot 10^{-7} \text{ cm}^2/\text{kg}$$

$$a_{66} = 1.43 \cdot 10^{-4} \text{ cm}^2/\text{kg}$$

The strain energy is, therefore,

$$W = \frac{\pi a^2 p^2}{2} (3.60 \cdot 10^{-5} \text{ cm}^2/\text{kg}) (1.41 \cos^2 \phi + \sin^2 \phi) \quad (70)$$

Then, if we plot W* vs. ϕ, as in Figure 9, we see that the maximum energy is available in the body when the crack is along the x' axis of elastic symmetry which, in this example, is the grain direction. Thus, we would predict that the plate under consideration would have the greatest tendency to fracture if the crack were on the x'-axis.

If we consider the extreme case where the body is completely rigid in the x' direction then $E_1 \to \infty$ and,

$$W = \frac{\pi a^2 p^2}{2} \sqrt{a_{66}} \sqrt{a_{22}} \cos^2 \phi \quad (71)$$

and the crack would not run at all in the y' direction. That is the crack would have a tendency to run in the direction of greatest rigidity.

If we assume that the crack runs in the same direction as the alignment of the crack we can find an exact expression for the energy release rate, given by $G_I = \frac{1}{2}\frac{\partial W}{\partial a}$, and thus make a quantitative examination of the tendency toward fracture for a given material. Otherwise, using the energy expression, our examination is more qualitative in manner.

Appendix A

Stress and Displacement Fields Near Crack Tip For Case of Plane Extension

(1) Symmetric Loading

From Leknitskii[1] the stress functions for an ellipse loaded as shown in Figure (3) are given as,

$$\Phi_1'(z_1) = \frac{1}{\sqrt{z_1^2 - a^2 + \mu_1^2 b^2}} \{- \frac{(\mu_2 - \mu_3\lambda_2\lambda_3)\bar{a}_1 + (\lambda_2\lambda_3 - 1)\bar{b}_1}{\Delta_{\zeta_1}}\} \tag{A.1a}$$

$$\Phi_2'(z_2) = \frac{1}{\sqrt{z_2^2 - a^2 + \mu_2^2 b^2}} \{- \frac{(\mu_3\lambda_1\lambda_3 - \mu_1)\bar{a}_1 + (1 - \lambda_1\lambda_3)\bar{b}_1}{\Delta_{\zeta_1}}\} \tag{A.1b}$$

$$\Phi_3'(z_3) = \frac{1}{\sqrt{z_3^2 - a^2 + \mu_3 b^2}} \{- \frac{(\mu_1\lambda_2 - \mu_2\lambda_1)\bar{a}_1 + (\lambda_1 - \lambda_2)\bar{b}_1}{\Delta_{\zeta_1}}\} \tag{A.1c}$$

where,

$$\bar{a}_1 = - \frac{pa}{2} \qquad \bar{b}_1 = - \frac{pbi}{2} \tag{A.2a}$$

$$\Delta = \mu_2 - \mu_1 + \lambda_2\lambda_3(\mu_1 - \mu_3) + \lambda_1\lambda_3(\mu_3 - \mu_2) \tag{A.2b}$$

$$\zeta_j = \frac{z_j + \sqrt{z_j^2 - a^2 + \mu_j^2 b^2}}{a - i\mu_j b} \qquad (j = 1, 2, 3) \tag{A.2c}$$

For a crack, b = 0 and $\bar{b}_1 = 0$, and equations (A.1) reduce to,

$$\Phi_1'(z_1) = \frac{pa^2}{2\Delta} \frac{(\mu_2 - \mu_3\lambda_2\lambda_3)}{\sqrt{z_1^2 - a^2}\,[z_1 + \sqrt{z_1^2 - a^2}]} \qquad \text{(A.3a)}$$

$$\Phi_2'(z_2) = \frac{pa^2}{2\Delta} \frac{(\mu_3\lambda_1\lambda_3 - \mu_1)}{\sqrt{z_2^2 - a^2}\,[z_2 + \sqrt{z_2^2 - a^2}]} \qquad \text{(A.3b)}$$

$$\Phi'(z_3) = \frac{pa}{2\Delta} \frac{(\mu_1\lambda_2 - \mu_2\lambda_1)}{\sqrt{z_3^2 - a^2}\,[z_3 + \sqrt{z_3^2 - a^2}]} \qquad \text{(A.3c)}$$

If we consider the polar coordinate system shown in Figure (2),

$$z = a + re^{i\theta} \qquad \text{(A.4a)}$$

$$z_j = a + r\cos\theta + \mu_j r\sin\theta \qquad \text{(A.4b)}$$

And, near the crack tip,

$$\sqrt{z_j^2 - a^2} \cong \sqrt{2ar(\cos\theta + \mu_j\sin\theta)} \qquad \text{(A.5)}$$

Thus, near the crack tip, the stress functions reduce to,

$$\Phi_1'(z_1) = \frac{p\sqrt{a}}{2\sqrt{2r}} \frac{(\mu_2 - \mu_3\lambda_2\lambda_3)}{\Delta} \frac{1}{\sqrt{\cos\theta + \mu_1\sin\theta}} \qquad \text{(A.6a)}$$

$$\Phi_2'(z_2) = \frac{p\sqrt{a}}{2\sqrt{2r}} \frac{(\mu_3\lambda_1\lambda_3 - \mu_1)}{\Delta} \frac{1}{\sqrt{\cos\theta + \mu_2 \sin\theta}} \tag{A.6b}$$

$$\Phi_3'(z_3) = \frac{p\sqrt{a}}{2\sqrt{2r}} \frac{(\mu_1\lambda_2 - \mu_2\lambda_1)}{\Delta} \frac{1}{\sqrt{\cos\theta + \mu_3 \sin\theta}} \tag{A.6c}$$

where $p\sqrt{\pi a}$ is equal to k_1, the isotropic stress intensity factor.

The stresses may now be evaluated by substitution of Equations (A.6) into Equations (10) and are given by Equations (26).

The integrals of the $\Phi_j(z_j)$'s are ,

$$\Phi_1(z_1) = \frac{p}{2\Delta} (\mu_2 - \mu_3\lambda_2\lambda_3) \left[\sqrt{z_1^2 - a^2} - z_1\right] \tag{A.7a}$$

$$\Phi_2(z_2) = \frac{p}{2\Delta} (\mu_3\lambda_1\lambda_3 - \mu_1) \left[\sqrt{z_2^2 - a^2} - z_2\right] \tag{A.7b}$$

$$\Phi_3(z_3) = \frac{p}{2\Delta} (\mu_1\lambda_2 - \mu_2\lambda_1) \left[\sqrt{z_3^2 - a^2} - z_3\right] \tag{A.7c}$$

Near the crack tip the above expressions reduce to,

$$\Phi_1(z_1) = \frac{k_1\sqrt{2r}}{2\Delta\sqrt{\pi}} (\mu_2 - \mu_3\lambda_2\lambda_3)\sqrt{\cos\theta + \mu_1 \sin\theta} \tag{A.8a}$$

$$\Phi_2(z_2) = \frac{k_1\sqrt{2r}}{2\Delta\sqrt{\pi}} (\mu_3\lambda_1\lambda_3 - \mu_1) \sqrt{\cos\theta + \mu_2 \sin\theta} \qquad \text{(A.8b)}$$

$$\Phi_3(z_3) = \frac{k_1\sqrt{2r}}{2\Delta\sqrt{\pi}} (\mu_1\lambda_1 - \mu_2\lambda_2) \sqrt{\cos\theta + \mu_3 \sin\theta} \qquad \text{(A.8c)}$$

By Equations (11) and (A.8) the displacements may be calculated and are given by Equations (27).

(2) Skew Symmetric Loading

For a body loaded as shown in Figure (4), the stress functions are again given by Equations (A.1) where the constants $\bar{a}_1$ and $\bar{b}_1$ are now,

$$\bar{a}_1 = \frac{qbi}{2} \qquad \bar{b}_1 = -\frac{qa}{2} \qquad \text{(A.9)}$$

In this case, when b = 0, Equations (A.1) reduce to,

$$\Phi_1'(z_1) = \frac{qa^2}{2\Delta} \frac{(\lambda_2\lambda_3 - 1)}{\sqrt{z_1^2 - a^2}\,[z_1 + \sqrt{z_1^2 - a^2}\,]} \qquad \text{(A.10a)}$$

$$\Phi_2'(z_2) = \frac{qa^2}{2\Delta} \frac{(1 - \lambda_1\lambda_3)}{\sqrt{z_2^2 - a^2}\,[z_2 + \sqrt{z_2^2 - a^2}\,]} \qquad \text{(A.10b)}$$

$$\Phi_3'(z_3) = \frac{qa^2}{2\Delta} \frac{(\lambda_1 - \lambda_2)}{\sqrt{z_3^2 - a^2}\,[z_3 + \sqrt{z_3^2 - a^2}\,]} \qquad \text{(A.10c)}$$

And, near the crack tip, the functions are,

$$\Phi_1'(z_1) = \frac{q\sqrt{a}}{2\sqrt{2r}} \frac{(\lambda_2\lambda_3 - 1)}{\Delta} \frac{1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} \tag{A.11a}$$

$$\Phi_2'(z_2) = \frac{q\sqrt{a}}{2\sqrt{2r}} \frac{(1 - \lambda_1\lambda_3)}{\Delta} \frac{1}{\sqrt{\cos\theta + \mu_2 \sin\theta}} \tag{A.11b}$$

$$\Phi_3'(z_3) = \frac{q\sqrt{a}}{2\sqrt{2r}} \frac{(\lambda_1 - \lambda_2)}{\Delta} \frac{1}{\sqrt{\cos\theta + \mu_3 \sin\theta}} \tag{A.11c}$$

where $- q\sqrt{\pi a}$ is equal to k_2, the isotropic stress intensity factor.

The stresses near the crack tip may be evaluated by substitution of Equations (A.11) into Equations (10) and are tabulated in Equations (28).

The displacements may be found by following the same procedure as used in the symmetric case and are given by Equations (29).

Appendix B

Evaluation of Constants in Concentrated Force Solution

(1) Evaluation of A_1

Consider the following set of simultaneous equations,

$$A_1 + A_2 - \bar{A}_1 - \bar{A}_2 = \frac{P_y}{2\pi i} \tag{B.1a}$$

$$\mu_1 A_1 + \mu_2 A_2 - \bar{\mu}_1 \bar{A}_1 - \bar{\mu}_2 \bar{A}_2 = -\frac{P_x}{2\pi i} \tag{B.1b}$$

$$p_1 A_1 + p_2 A_2 - \bar{p}_1 \bar{A}_1 - \bar{p}_2 \bar{A}_2 = 0 \tag{B.1c}$$

$$q_1 A_1 + q_2 A_2 - \bar{q}_1 \bar{A}_1 - \bar{q}_2 \bar{A}_2 = 0 \tag{B.1d}$$

The solution to these equations is found for A_1 in the following form,

$$A_1 = \frac{\Delta_1 + i\Delta_2}{\Delta} \tag{B.2}$$

where,

$$\Delta = \begin{vmatrix} 1 & 1 & -1 & -1 \\ \mu_1 & \mu_2 & -\bar{\mu}_1 & -\bar{\mu}_2 \\ p_1 & p_2 & -\bar{p}_1 & -\bar{p}_2 \\ q_1 & q_2 & -\bar{q}_1 & -\bar{q}_2 \end{vmatrix} \tag{B.3}$$

and,

$$\Delta_1 + i\Delta_2 = \begin{vmatrix} P_y/2\pi i & 1 & -1 & -1 \\ -P_x/2\pi i & \mu_2 & -\bar{\mu}_1 & -\bar{\mu}_2 \\ 0 & p_2 & -\bar{p}_1 & -\bar{p}_2 \\ 0 & q_2 & -\bar{q}_1 & -\bar{q}_2 \end{vmatrix} \qquad \text{(B.4)}$$

The solutions to Equations (B.3) and (B.4) are given in Equations (46), (47), and (48) where p_j and q_j are broken up into real and imaginary parts as follows:

$$p_j^{(1)} = a_{11}(\alpha_j^2 - \beta_j^2) - a_{16}\alpha_j + a_{12} \qquad \text{(B.5a)}$$

$$p_j^{(2)} = \beta_j(2a_{11}\alpha_j - a_{16}) \qquad \text{(B.5b)}$$

$$q_j^{(1)} = \alpha_j\left(a_{12} + \frac{a_{22}}{\alpha_j^2 + \beta_j^2}\right) - a_{26} \qquad \text{(B.5c)}$$

$$q_j^{(2)} = \beta_j\left(a_{12} - \frac{a_{22}}{\alpha_j^2 + \beta_j^2}\right) \qquad \text{(B.5d)}$$

(2) Evaluation of A_1 for the orthotropic case:

Case (i): $\mu_1 = \alpha_o + i\beta_o$ and $\mu_2 = -\alpha_o + i\beta_o$

In this case, Equation (46) reduces to,

$$\Delta = -\frac{64\,\alpha_o^2\beta_o^2 a_{11}a_{22}}{(\alpha_o^2 + \beta_o^2)} \qquad \text{(B.6)}$$

and Equations (47) and (48) become,

$$\Delta_1 = \frac{P_y}{\pi}\{-\frac{4\alpha_o\beta_o a_{22}}{(\alpha_o^2+\beta_o^2)}(a_{11}(\alpha_o^2-\beta_o^2)+a_{12})\}$$

$$+\frac{P_x}{\pi}\{-\frac{4\alpha_o^2\beta_o a_{11}}{(\alpha_o^2+\beta_o^2)}(a_{12}(\alpha_o^2+\beta_o^2)+a_{22})\} \qquad (B.7)$$

$$\Delta_2 = \frac{P_y}{\pi}\{8\alpha_o^2\beta_o^2 a_{11}a_{22}\}$$

$$+\frac{P_y}{\pi}\{4\beta_o^2\alpha_o a_{11}(a_{12}-\frac{a_{22}}{(\alpha_o^2+\beta_o^2)})\} \qquad (B.8)$$

A_1 may now be calculated by use of equation (B.2) and is given in Equation (53).

Case (ii): $\mu_1 = i\beta_1$ and $\mu_2 = i\beta_2$

For this case, Equation (46) reduces to,

$$\Delta = -\frac{4a_{11}a_{22}(\beta_2^2-\beta_1^2)^2}{\beta_1\beta_2} \qquad (B.9)$$

and the expressions for Δ_1 and Δ_2 are,

$$\Delta_1 = \frac{P_x}{\pi}\{a_{11}\beta_2(a_{12}-\frac{a_{22}}{\beta_2^2})(\beta_1^2-\beta_2^2)\} \qquad (B.10)$$

$$\Delta_2 = \frac{P_y}{\pi} \{- a_{11}\beta_2^2[\beta_1\beta_2(a_{12} - \frac{a_{22}}{\beta_1^2}) - \beta_1\beta_2(a_{12} - \frac{a_{22}}{\beta_2^2})]\}$$

$$= \frac{P_y}{\pi} \{a_{11}a_{22}\beta_1\beta_2^3 (\frac{\beta_2^2 - \beta_1^2}{\beta_1^2\beta_2^2})\} \qquad (B.11)$$

and, the resulting expression for A_1 is given by Equation (55).

Appendix C

Evaluation of Stress Intensity Factors for Case Of a Concentrated Force on The Surface Of a Crack in an Orthotropic Plate

From Equation (52) we get the relation,

$$k_1\mu_2 + k_2 = -\frac{4\sqrt{\pi}}{2\sqrt{a}}\{(\mu_1 - \mu_2)A_1 + \frac{P_x + \mu_2 P_y}{2\pi i}(\frac{1}{2} - \frac{i}{2}(\frac{a+b}{a-b})^{1/2})\}$$

(C.1)

If we first consider case (i) where $\mu_1 = \alpha_o + i\beta_o$ and $\mu_2 = -\alpha_o + i\beta_o$, then Equation (C.1) becomes, by use of Equation (53),

$$(k_2 - \alpha_o k_1) + i\beta_o k_1 = -\frac{2\sqrt{\pi}P_y}{\sqrt{a}}\{\frac{\alpha_o}{8\pi}(\frac{1}{\alpha_o\beta_o}[\frac{a_{12}}{a_{11}} + \{\alpha_o^2 - \beta_o^2\}] - 2i)$$

$$+ \frac{\beta_o}{4\pi} + \frac{\alpha_o}{4\pi}(\frac{a+b}{a-b})^{1/2} + i(\frac{\alpha_o}{4\pi} - \frac{\beta_o}{4\pi}(\frac{a+b}{a-b})^{1/2})\}$$

$$-\frac{2\sqrt{\pi}P_x}{\sqrt{a}}\{\frac{\alpha_o}{8\pi}(\frac{1}{\beta_o}[\frac{a_{12}}{a_{22}}(\alpha_o^2 + \beta_o^2) + 1] - \frac{i}{\alpha_o}[\frac{a_{12}}{a_{22}}(\alpha_o^2 + \beta_o^2) - 1])$$

$$-\frac{1}{4\pi}(\frac{a+b}{a-b})^{1/2} - \frac{i}{4\pi}\}$$

(C.2)

Equating the real and imaginary parts of (C.2),

$$k_1 = \frac{P_y}{2(\pi a)^{1/2}} \left(\frac{a+b}{a-b}\right)^{1/2} + \frac{P_x}{2(\pi a)^{1/2}} \left\{\frac{1}{2\beta_0} \left[\frac{a_{12}}{a_{22}} (\alpha_0^2 + \beta_0^2) + 1\right]\right\} \tag{C.3}$$

$$k_2 = -\frac{P_y}{2(\pi a)^{1/2}} \left\{\frac{1}{2\beta_0} \left[\frac{a_{12}}{a_{11}} + (\alpha_0^2 + \beta_0^2)\right]\right\} + \frac{P_x}{2(\pi a)^{1/2}} \left(\frac{a+b}{a-b}\right)^{1/2} \tag{C.4}$$

For case (ii), where $\mu_1 = i\beta_1$ and $\mu_2 = i\beta_2$, Equation (C.1) becomes,

$$k_2 + i\beta_2 k_1 = -\frac{2\sqrt{\pi}P_y}{\sqrt{a}} \left\{(\beta_1 - \beta_2) \frac{\beta_2}{4(\beta_2^2 - \beta_1^2)} + \frac{\beta_2}{4\pi} - \frac{i\beta_2}{4\pi} \left(\frac{a+b}{a-b}\right)^{1/2}\right\}$$
$$- \frac{2\sqrt{\pi}P_x}{\sqrt{a}} \left\{\frac{i(\beta_1 - \beta_2)\beta_1(a_{12}\beta_2^2 - a_{22})}{4\pi(\beta_2^2 - \beta_1^2)a_{22}} - \frac{i}{4\pi} - \frac{1}{4\pi} \left(\frac{a+b}{a-b}\right)^{1/2}\right\} \tag{C.5}$$

and, equating the real and imaginary parts of (C.5), k_1 and k_2 are seen to be,

$$k_1 = \frac{P_y}{2(\pi a)^{1/2}} \left(\frac{a+b}{a-b}\right)^{1/2} + \frac{P_x}{2(\pi a)^{1/2}} \left\{\frac{1}{\beta_2 + \beta_1} \left(\frac{a_{12}}{a_{22}} \beta_1\beta_2 + 1\right)\right\} \tag{C.6}$$

$$k_2 = -\frac{P_y}{2(\pi a)^{1/2}}\left(\frac{\beta_1\beta_2}{\beta_2+\beta_1}\right) + \frac{P_x}{2(\pi a)^{1/2}}\left(\frac{a+b}{a-b}\right)^{1/2} \tag{C.7}$$

Appendix D

Transformation of Elastic Constants For Orthotropic Body and Evaluation of Energy Expression

From Lekhnitskii we have the following transformation of elastic constants in an orthotropic body:

$$a'_{22} = a_{11} \sin^4 \phi + (a_{66} + 2a_{12}) \sin^2 \phi \cos^2 \phi + a_{22} \cos^4 \phi \qquad \text{(D.1)}$$

$$\mu'_k = \frac{\mu_k \cos \phi - \sin \phi}{\cos \phi + \mu_k \sin \phi} \qquad \text{(D.2)}$$

The unprimed constants are with respect to axes of orthotropy.

Consider now the following,

$$\mu'_1 + \mu'_2 = \frac{\mu_1 \cos \phi - \sin \phi}{\cos \phi + \mu_1 \sin \phi} + \frac{\mu_2 \cos \phi - \sin \phi}{\cos \phi + \mu_2 \sin \phi}$$

$$\mu'_1 + \mu'_2 = \frac{(\mu_1 + \mu_2) + 2(\mu_1\mu_2 - 1) \sin \phi \cos \phi - 2(\mu_1 + \mu_2) \sin^2\phi}{\cos^2 \phi + (\mu_1 + \mu_2) \sin \phi \cos \phi + \mu_1\mu_2 \sin^2 \phi} \qquad \text{(D.3)}$$

$$\mu'_1\mu'_2 = \frac{\mu_1\mu_2 \cos^2 \phi - (\mu_1 + \mu_2) \sin \phi \cos \phi + \sin^2 \phi}{\cos \phi + (\mu_1 + \mu_2) \sin \phi \cos \phi + \mu_1\mu_2 \sin^2 \phi} \qquad \text{(D.4)}$$

Therefore,

$$\frac{\mu_1' + \mu_2'}{\mu_1'\mu_2'} = \frac{(\mu_1 + \mu_2) + 2(\mu_1\mu_2 - 1)\sin\phi\cos\phi - 2(\mu_1 + \mu_2)\sin^2\phi}{\mu_1\mu_2 \cos^2\phi - (\mu_1 + \mu_2)\sin\phi\cos\phi + \sin^2\phi} \tag{D.5}$$

The complex numbers, μ_1 and μ_2, form the following real numbers.

$$\mu_1\mu_2 = -\sqrt{\frac{E_1}{E_2}} = -\sqrt{\frac{a_{22}}{a_{11}}} \tag{D.6a}$$

$$\mu_1^2 + \mu_2^2 = 2\nu_{12} - \frac{E_1}{G_{12}} = -\left(\frac{2a_{12}}{a_{11}} + \frac{a_{66}}{a_{11}}\right) \tag{D.6b}$$

$$-i(\mu_1 + \mu_2) = \sqrt{\frac{E_1}{G_{12}} - 2\nu_{12} + 2\sqrt{\frac{E_1}{E_2}}} = \sqrt{\frac{2a_{12}}{a_{11}} + \frac{a_{66}}{a_{11}} + 2\sqrt{\frac{a_{22}}{a_{11}}}} \tag{D.6c}$$

We can therefore substitute (D.6) into (D.5) to get,

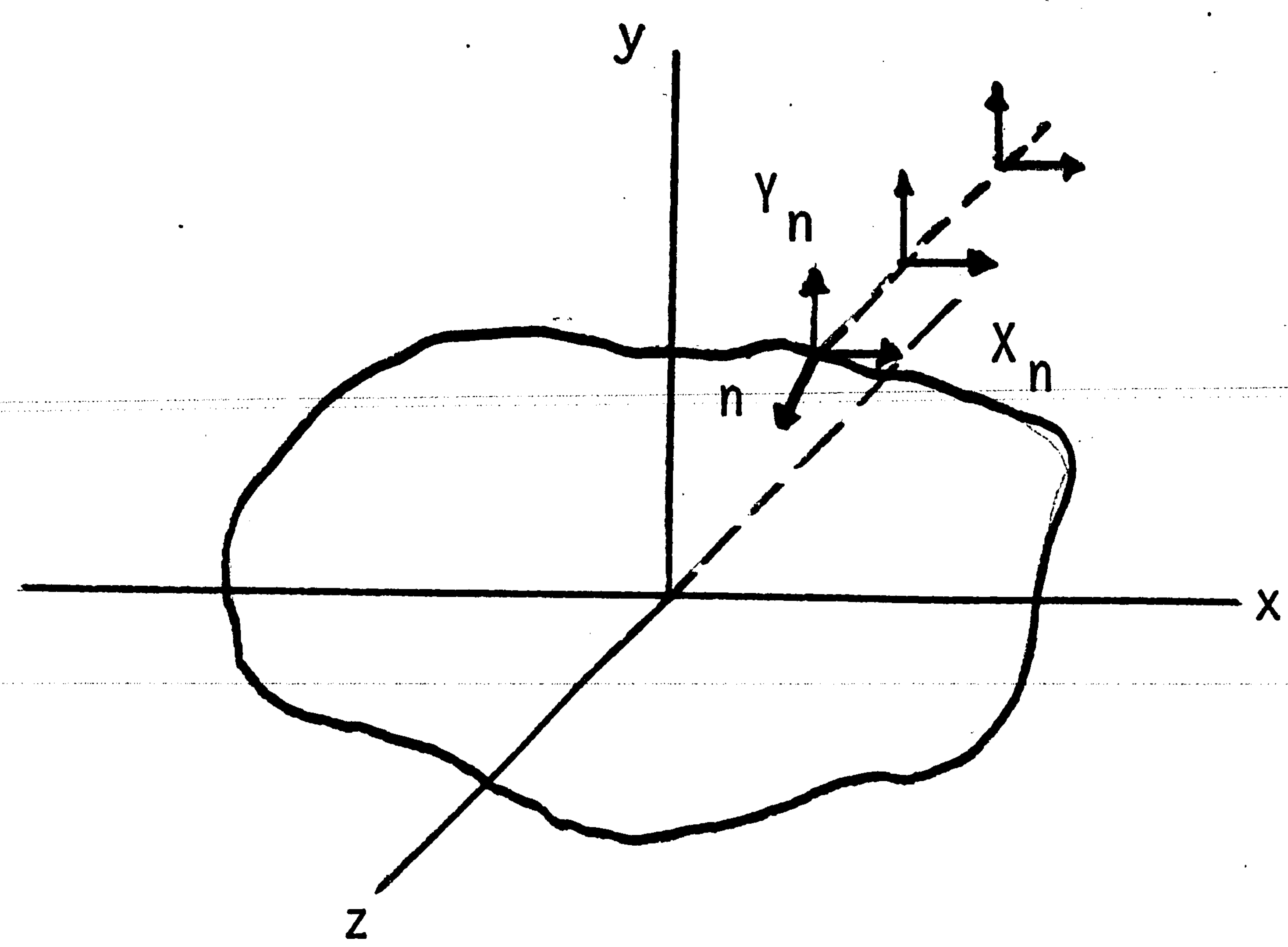
$$\frac{\mu_1' + \mu_2'}{\mu_1'\mu_2'} =$$

$$\frac{i\left(\sqrt{\frac{2a_{12}}{a_{11}} + \frac{a_{66}}{a_{11}} + 2\sqrt{\frac{a_{22}}{a_{11}}}}\right)(1 - 2\sin^2\phi) - 2\left(1 + \sqrt{\frac{a_{22}}{a_{11}}}\right)\sin\phi\cos\phi}{\left(-\sqrt{\frac{a_{22}}{a_{11}}}\cos^2\phi + \sin^2\phi\right) - i\left(\sqrt{\frac{2a_{12}}{a_{11}} + \frac{a_{66}}{a_{11}} + 2\sqrt{\frac{a_{22}}{a_{11}}}}\right)\sin\phi\cos\phi} \tag{D.7}$$

The imaginary part of (D.7) is,

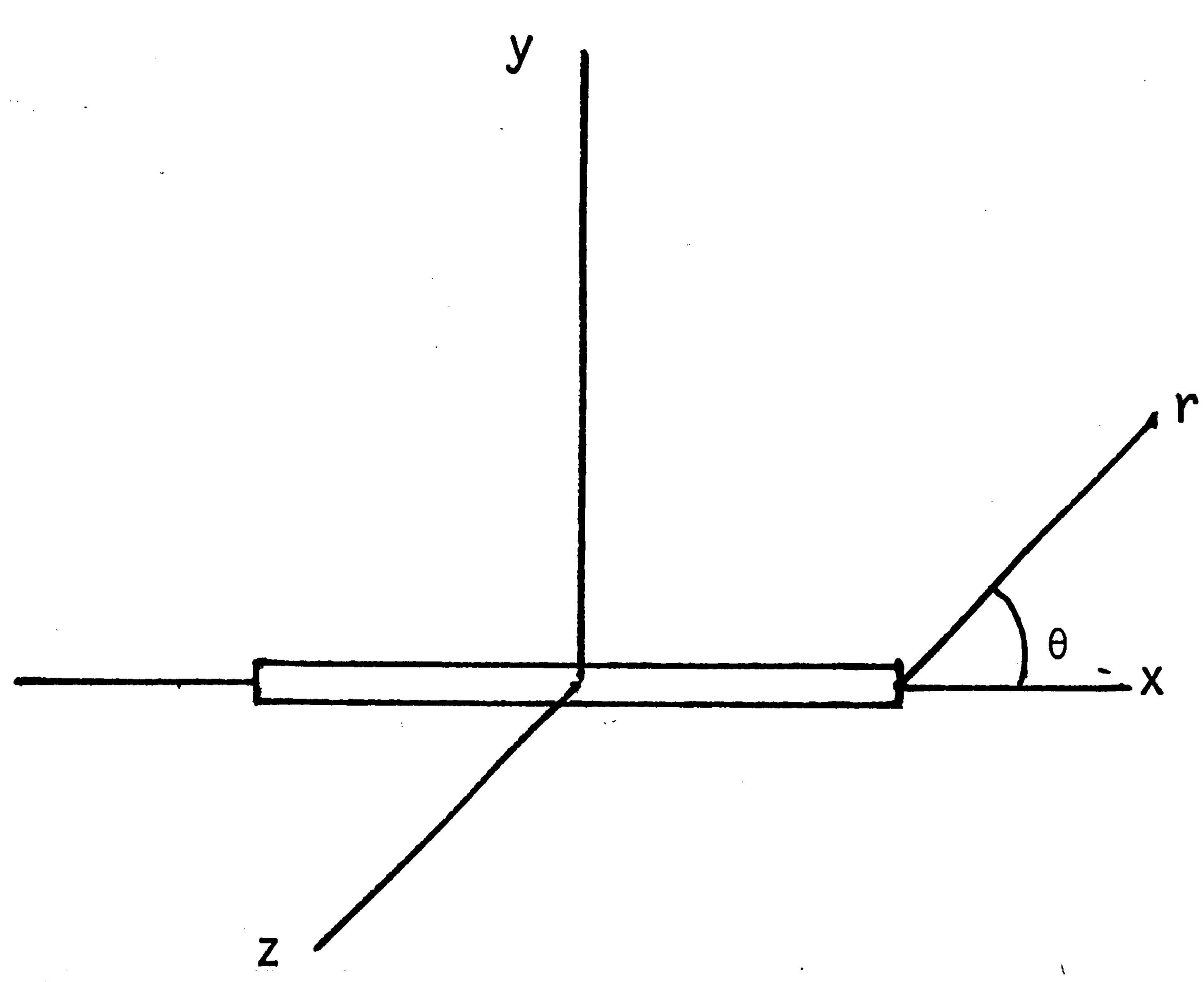
$$\text{Im}\,\frac{\mu_1' + \mu_2'}{\mu_1'\mu_2'} = \frac{-a_{11}\left(\frac{2a_{12}}{a_{11}} + \frac{a_{66}}{a_{11}} + 2\,\frac{a_{22}}{a_{11}}\right)\left(\frac{a_{22}}{a_{11}}\cos^2\phi + \sin^2\phi\right)}{a_{22}\cos^2\phi + a_{11}\sin^4\phi + (2a_{12} + a_{66})\sin^2\phi\cos^2\phi} \tag{D.8}$$

Hence, using Equation (D.8) and (D.1), the energy expression can be evaluated in terms of orthotropic elastic constants. The result is given by Equation (67).



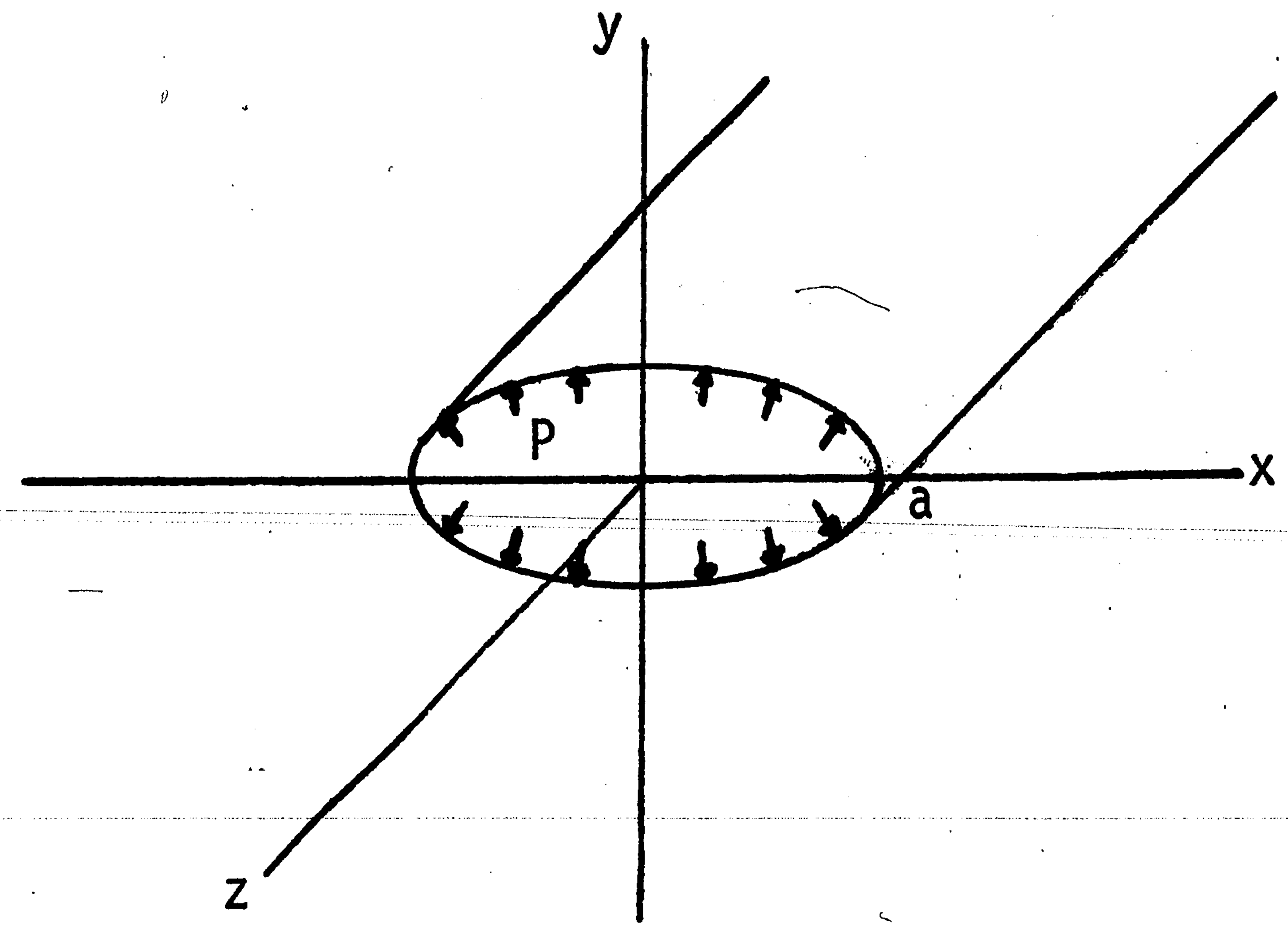
BODY IN GENERALIZED PLANE DEFORMATION WITH SURFACE TRACTIONS APPLIED TO CAVITY IN BODY

FIGURE I



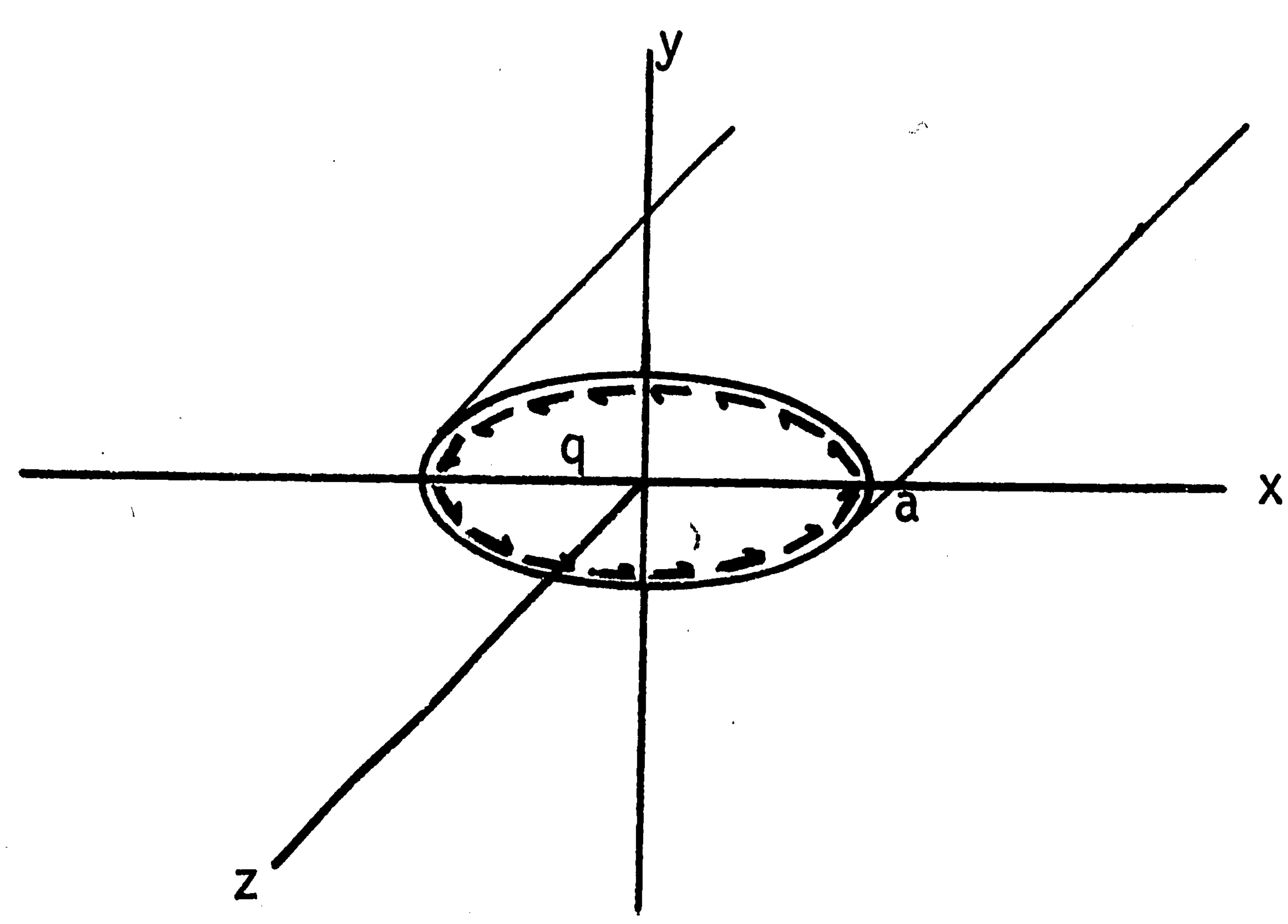
POLAR COORDINATE SYSTEM AT CRACK TIP

FIGURE II



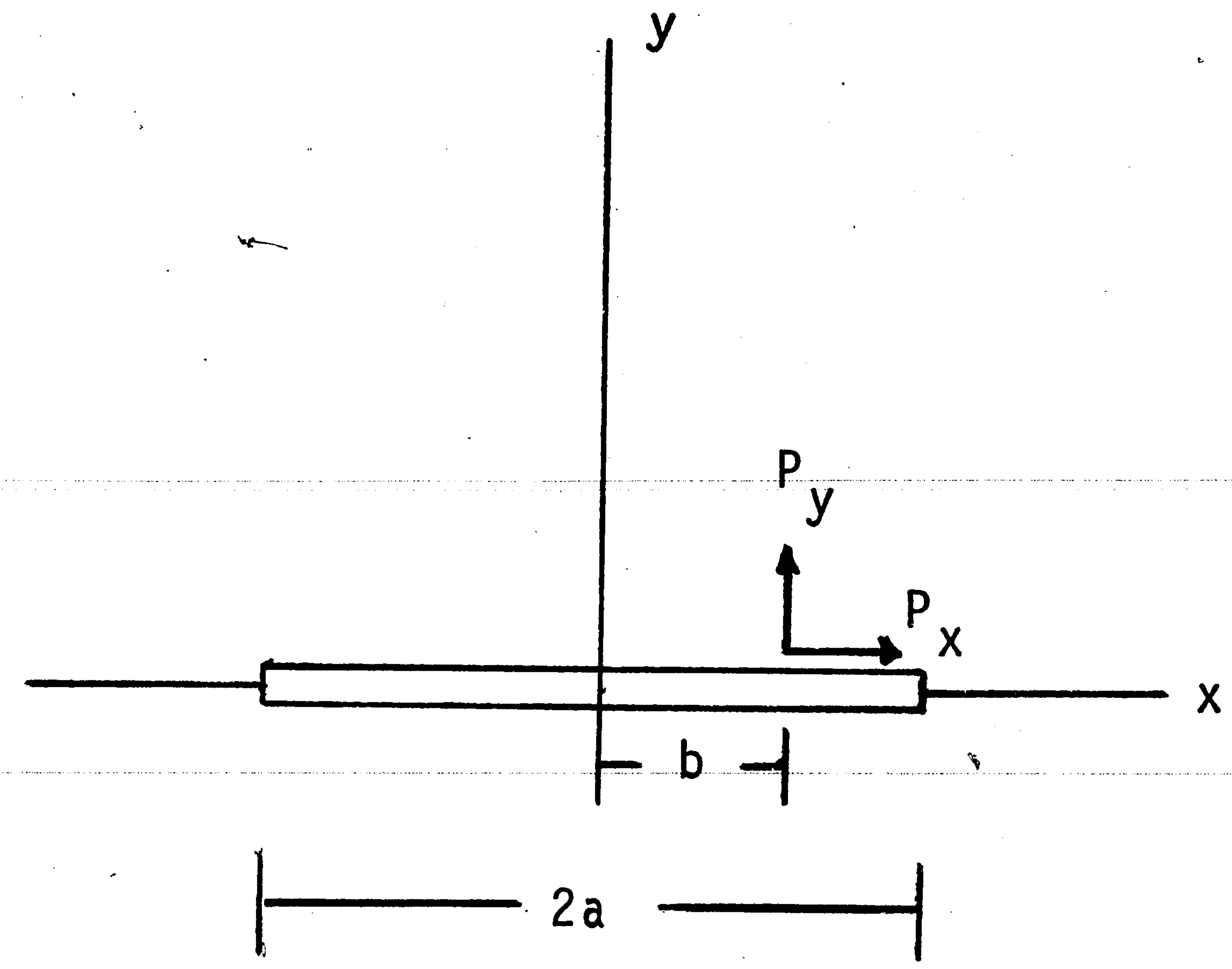
ELLIPTICAL CAVITY SUBJECT TO A NORMAL UNIFORM PRESSURE

FIGURE III



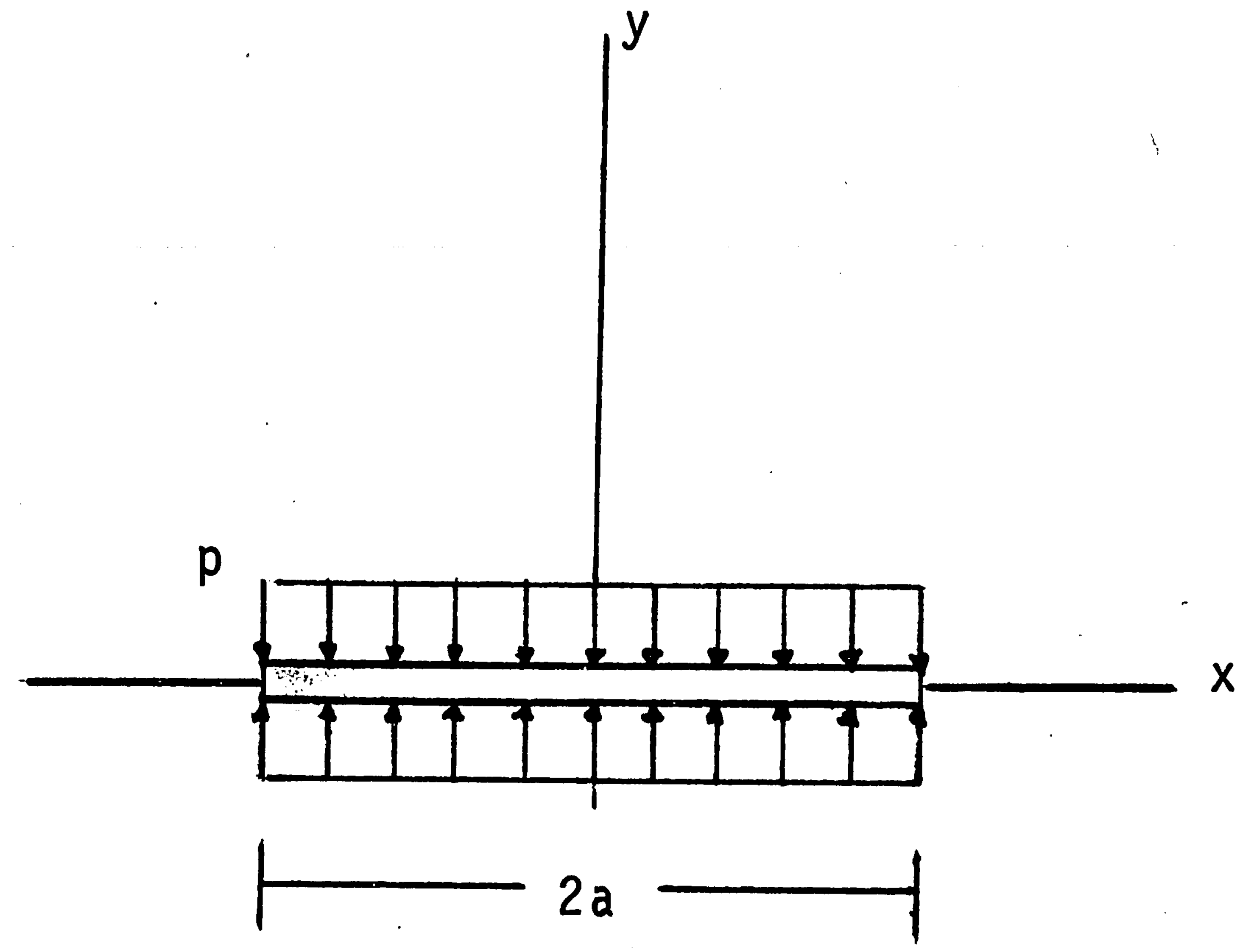
ELLIPTICAL CAVITY SUBJECT TO TANGENTIAL STRESS

FIGURE IV



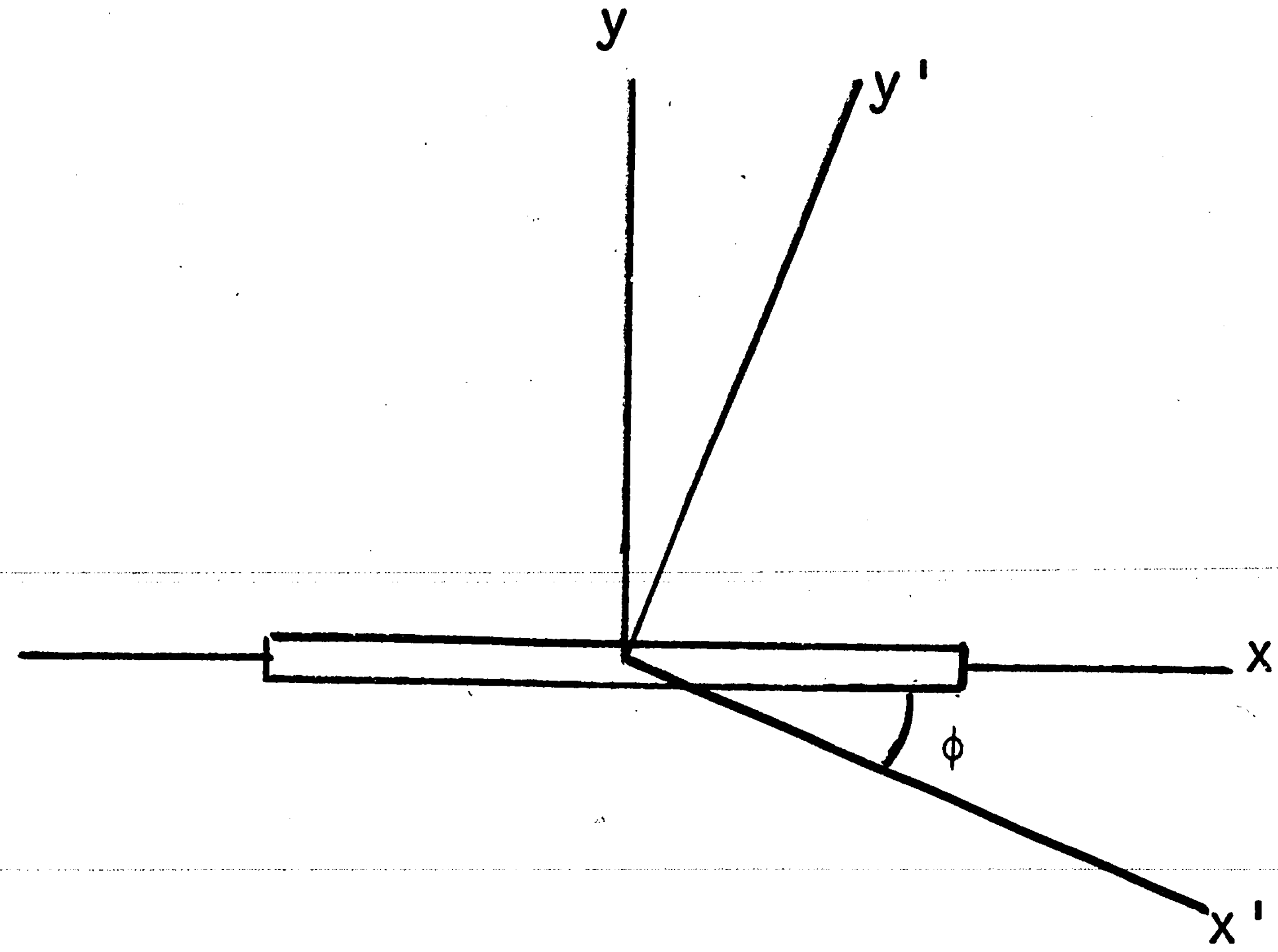
A CONCENTRATED FORCE ON THE SURFACE OF A CRACK IN AN INFINITE PLATE

FIGURE V



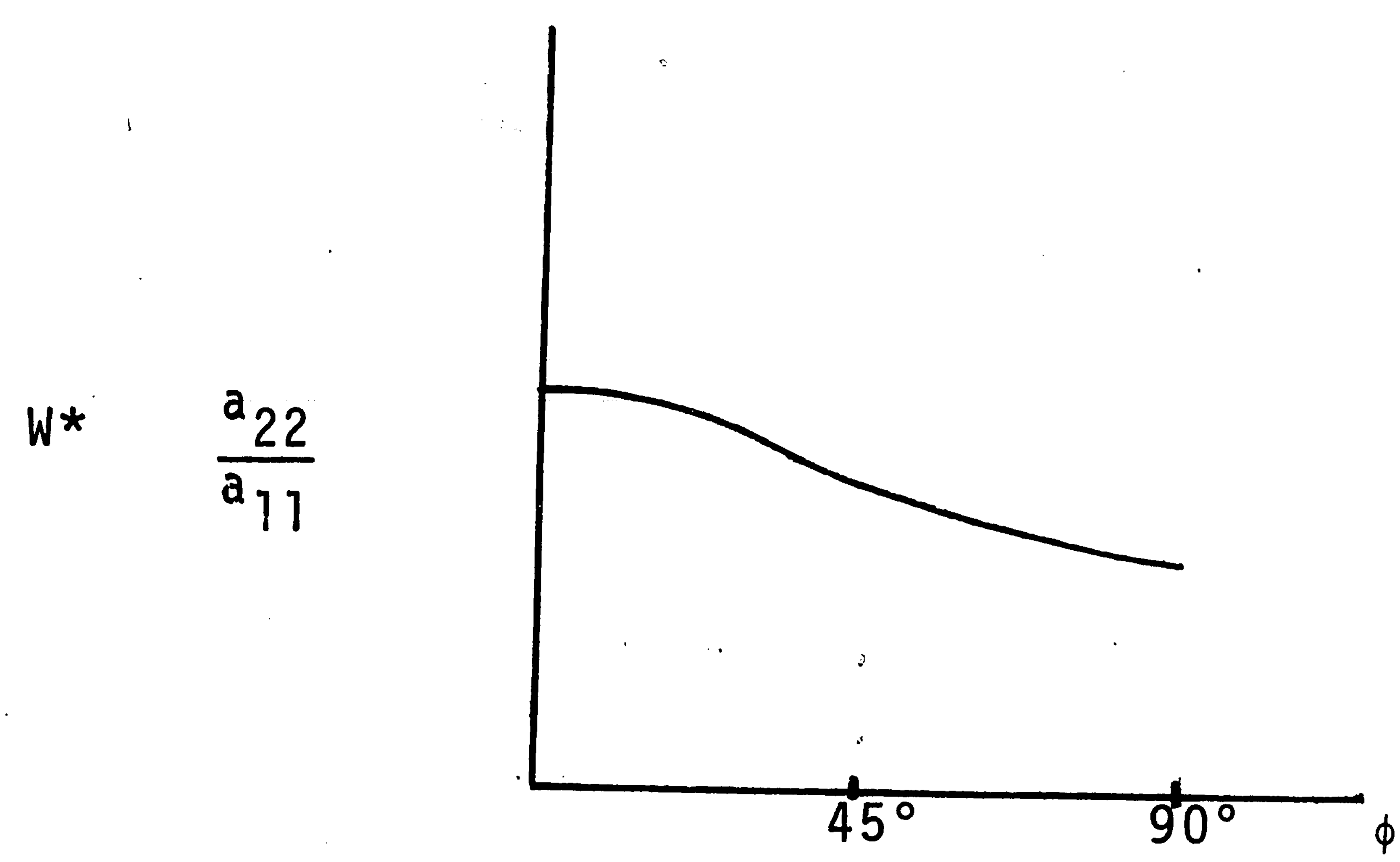
UNIFORM PRESSURE ON THE SURFACE OF A CRACK IN AN INFINITE PLATE

FIGURE VI



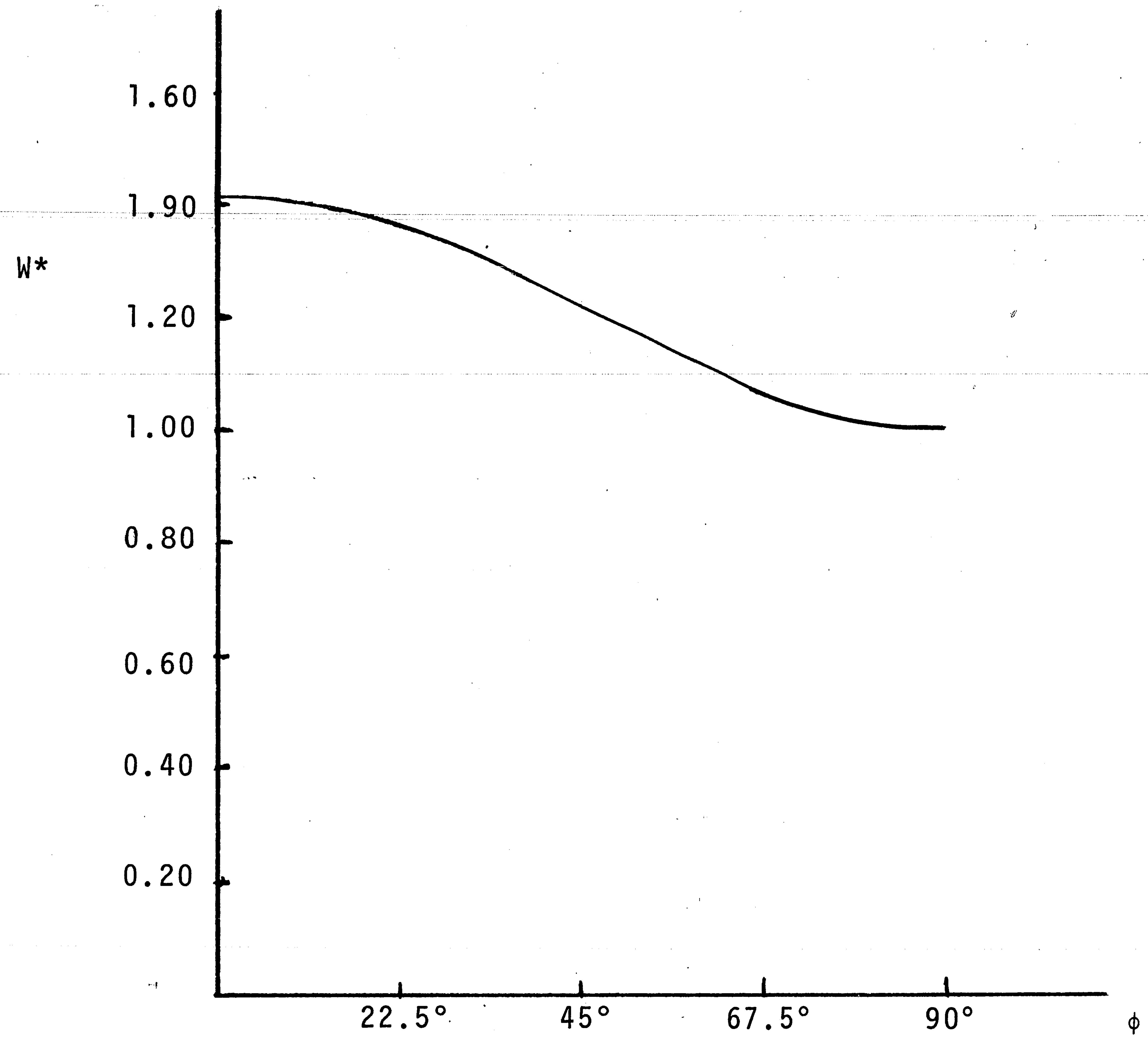
CRACK IN AN INFINITE ORTHOTROPIC PLATE WITH AXES OF SYMMETRY ORIENTED AT AN ANGLE ϕ WITH RESPECT TO THE CRACK

FIGURE VII



RELATIONSHIP BETWEEN W* AND ϕ FOR $a_{22} > a_{11}$

FIGURE VIII



VALUES OF W* FOR VARIOUS ANGLES ϕ FOR THE CASE OF A SHEET
OF THREE LAYERED BIRCH PLYWOOD GLUED
TO A BAKELITE LAYER

FIGURE IX

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Vita

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