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Frobenius groups and frobenius kernels

Neil Steven Wetcher
Lehigh University

FROBENIUS GROUPS AND FROBENIUS KERNELS

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Neil Steven Wetcher

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Albert W. Otto
Professor in Charge

Ernest Fitcher
Head of the Department

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ABSTRACT

The concept of Frobenius groups and Frobenius kernels arises from the theorem: If H is a non-$\{e\}$ subgroup of G, where $xHx^{-1} \cap H = \{e\}$ for each x in $G-H$, and if M is the set of elements of G not in any conjugate of $H^{\#}$, then M is a normal subgroup of G. The G and M discussed in the theorem are respectively, a Frobenius group and a Frobenius kernel. The proof of the theorem requires a knowledge of group representations and characters, and therefore included in this paper is a short summary of the basic results of representation and character theory.

Two major properties of Frobenius kernels that are presented in this paper are that Frobenius kernels are unique and that they are nilpotent. Once establishing the main properties of Frobenius groups and their kernels, I will give applications of Frobenius groups and their kernels in the field of character theory and general group theory.

INTRODUCTION

The concept of Frobenius groups and Frobenius kernels arises from a famous theorem due to Frobenius. The proof of this theorem requires a knowledge of group representations and characters. Therefore included in this paper is a short summary of the basic results of representation and character theory. The reader who is familiar with these topics is advised to read through these results in order to learn the notation that will be used in the rest of the thesis.

The chapters will be arranged as follows.

Chapter I will contain a summary of basic properties of characters. Theorems will be stated without proof. Those who wish to see a more rigorous treatment of this topic are advised to see Ribenboim, Scott and Feit.

Chapter II will begin with a theorem of Frobenius, which is crucial to this paper. This result will motivate the definition of Frobenius groups and their kernels. Also included in this chapter will be various characterizations of Frobenius groups. It will be necessary to discuss the concept of Hall P-subgroups and their properties which will be given without proof (reader is referred to Scott for proofs).

Chapter III will be concerned with basic properties of Frobenius groups and their kernels. Included in this chapter is a discussion of Thompson subgroups. Those who wish a more rigorous treatment of this topic are advised to see Feit and Passman.

Chapter IV will deal with applications of Frobenius groups and Frobenius kernels in character and general group theory.

I believe it necessary to note that for the most part the results proven in this paper can be found in Scott. The reader will notice that I have changed the order of presentation. Furthermore, with the aid of other texts, I have hopefully in some cases clarified Scott's proof and in other cases used different proofs.

CHAPTER I

Representations and Characters

Let G be a finite group, F be a field, and V be a vector space over F of finite dimension n.

A linear representation of G, of degree n over F, with representation space V is a mapping $R:G \to L_F(V)$ such that

1) $eR = i$ (the identity transformation) and
2) $(g_1g_2)R = (g_1R)(g_2R)$.

Notice that gR is necessarily invertible.

Two linear representations R_1 and R_2 of G over F with representation spaces V_1 and V_2, both of degree n, are said to be equivalent whenever there exists an F-isomorphism θ from V_1 to V_2 such that

$$(x(gR_1))\theta = (x\theta)(gR_2)$$

for every $x \in V_1$ and $g \in G$.

Since V is a vector space of dimension n, and the group of invertible linear transformations of V can be shown to be isomorphic (in many ways) to the group of invertible $n \times n$ matrices over F, we could have just as well defined the representation R as a mapping

$$R:G \to M_{n\times n}(F) \text{ such that}$$

1) $eR = I$ (identity matrix) and

2) $(g_1g_2)R = (g_1R)(g_2R)$ (matrix multiplication)

with equivalence of two representations R_1 and R_2 of G over F with degree n meaning there exists an invertible matrix $T \in M_{n\times n}(F)$ such that

$$(gR_1)T = T(gR_2) \quad \text{for all} \quad g \in G.$$

It will often be convenient to think of a representation in this way.

The <u>character</u> of a representation R is a map $X_R:G \to F$ defined by $X_R(g) = Tr(gR)$ for all g in G. When the representation is obvious (due to context), X_R will be denoted by X.

Examples.

1) The principal character.

The representation R of G of degree 1 where gR is the identity linear transformation for every $g \in G$, called the one-representation, has character ζ, called the principal character of G, where $\zeta(g) = 1$ for every $g \in G$.

2) Right-regular representation and character.

The group ring FG can be considered as an F-space with scalar multiplication defined by $b(\sum_{g \in G} a_g g) = \sum_{g \in G} (ba_g)g$ for $b, a_g \in F$ and $g \in G$ with basis the elements of G and dimension equal to the order of G.

Consider the representation R of G over F with representation space FG such that gR is right multiplication on the elements of FG; that is, $(\sum_{g \in G} a_g g)(g'R) = \sum_{g \in G} a_g (gg')$ for all g' in G. R is called the right-regular representation of G.

The Cayley Jordan theorem tells us that the set of right multiplications of G by elements of G form a group of permutations. Thus relative to the basis $\{g | g \in G\}$ of FG $g'R$ considered as a matrix is merely a permutation matrix. Specifically, eR is the identity matrix while gR has a null diagonal for every $g \neq e$.

Suppose the character of the right regular representation of G is P. By the preceding paragraph

$$P(e) = O(G) \quad \text{and}$$

$$P(g) = 0 \text{ for } g \text{ in } G, g \neq e \ .$$

Two relatively simple, but important results of character theory are as follows.

1.1 Theorem. Equivalent representations have equal characters.

1.2 Theorem. If R is a representation of G and X_R is its character, then X_R is constant on each conjugate class of G.

Some further terms are needed to continue a discussion of character theory.

A representation R of G over F, with representation space V of degree n, is said to be reducible if and only if there is a subspace W, $\{0\} < W < V$, which is invariant under (G)R. Otherwise, R is said to be irreducible. Furthermore, R is said to be completely reducible if and only if V is the direct sum of irreducible (G)R-invariant subspaces. A character X_R will be called irreducible if and only if it is the character of an irreducible representation R.

There is a very important relationship between the irreducible characters of representations. In order to state it, we must introduce the following notion.

Let X and Y be functions from G into F and suppose char(F) + o(G). Consider the expression (X,Y) $= \frac{1}{o(G)} \sum_{g \text{ in } G} X(g)Y(g^{-1}) \in F$. It is easy to verify that (,) is a symimetric bilinear form; that is,

1) $(X,Y) = (Y,X)$,
2) $(X_1 + X_2, Y) = (X_1, Y) + (X_2, Y)$, and
3) $(aX, Y) = a(X,Y)$ for a in F.

From this point on, unless explicit mention is made, we will assume F is the field C of complex numbers.

1.3 Theorem. (Orthogonality of irreducible characters) If R and S are irreducible representations of G with characters X and Y respectively, then

$$(X,Y) = \begin{cases} 1 & \text{if } R \text{ and } S \text{ are equivalent and} \\ 0 & \text{if } R \text{ and } S \text{ are inequivalent.} \end{cases}$$

1.4 Theorem. Two representations of a finite group are equivalent if and only if they have equal characters.

1.5 Theorem. If R is a representation with character X, then R is irreducible if and only if $(X,X) = 1$.

1.6 <u>Theorem</u>. The number of nonequivalent irreducible characters of G equals the number of conjugate classes of G.

Suppose $C_1,\dots,C_s$ are the conjugate classes of G, and $X_1,\dots,X_s$ are the nonequivalent irreducible characters of G. Consider the $s \times s$ matrix M over C in which the (i,j)-th place is $X_i(C_j)$, X_1 is the principle character and $C_1 = \{e\}$. M is called the <u>character matrix</u> of G.

If T is a representation of G over C with representation space V, then it can be shown that V is a direct sum of n (GT)-invariant subspaces for some n. If $\{W_i\}_{i=1}^{n}$ are the direct summands of V, let T_i be the induced representation of T on W_i. T can then be defined as the <u>direct sum</u> of the T_i's. In particular, it can be shown that any representation T over C is a direct sum of irreducible representations. If $T = \sum T_i$, then the T_i's are unique up to equivalence and the characters satisfy the relation $X_T = \sum X_i$ where $X_i = X_{T_i}$. Theorem 1.6 tells us that there are only a finite number of nonequivalent irreducible characters, say $\{X_1,\dots,X_s\}$. Therefore, every character X of G over C is a sum of members of this set.

Let G be a group (finite) and let H be a subgroup of G of index n. If R is a representation of G over

C of degree m, then R/H is a representation of H over C of degree m. If X is the character of R, we will denote the character of R/H by $(X)_H$. We would like to reverse the process. That is, we would like to go from a representation of H to one of G.

Suppose R is a representation of H over C of degree d expressed in terms of matrices. We first extend R to G in the trivial fashion by contructing $\tilde{R}: G \to M_{d \times d}(C)$ where

$$g \to \begin{cases} g^R & \text{for } g \text{ in } H \text{ and} \\ 0 & \text{for } g \text{ not in } H. \end{cases}$$

Suppose $\{x_1, \ldots, x_n\}$ is a set of coset representatives of H in G. We now construct $R^*: G \to M_{nd \times nd}(C)$ such that

$$g \to ((x_i g x_j^{-1})\tilde{R})_{i,j} \quad \text{block}$$

R^* is called the representation of G <u>induced</u> by the representation R of H. Furthermore, R^* up to equivalence is independent of the choices of, or the order of, the coset representatives of H. Let X and X^* be the characters of R and R^* respectively and let $\tilde{X}$ be the trivial extension of X to G.

1.7 <u>Lemma</u>. For g in G, $X^*(g) = \frac{1}{o(H)} \sum_{u \text{ in } G} \tilde{X}(ugu^{-1})$.

1.8 <u>Theorem</u>. (Frobenius Reciprocity Theorem) If H is a subgroup of G, $\{Y_1, \ldots, Y_s\}$ is the set of irreducible characters of H over C, $\{X_1, \ldots, X_r\}$ is the set of irreducible characters of G over C, then if for each j, $1 \leq j \leq s$, $(Y_j)^* = \sum_{i=1}^{r} a_{ij}X_i$ where a_{ij} are non-negative integers, then for each i, $1 \leq i \leq r$, $(X_i)_H = \sum_{j=1}^{s} a_{ij}Y_j$.

1.9 Corollary. With the same hypothesis as above,

$$((Y_j)^*, X)_G = ((X_i)_H, Y_j)_H .$$

We can extend the notion of induced characters to that of induced class functions as follows. Suppose, as before, that $H \subset G$ with $\{x_1, \ldots, x_n\}$ a set of coset representatives. If f is a function from H to C which is constant on the conjugacy classes of H, we first define $\tilde{f}$ to be the trivial extension of f to G and then define

$$f^*(y) = \sum_{i=1}^{n} \tilde{f}(x_i y x_i^{-1}) \quad \text{for } y \text{ in } G.$$

As before, f^* is independent of the choice of coset representatives. It can be shown that the operation $*$ is linear; that is, if f, f_1 and f_2 are functions from H to C which are constant on the conjugacy classes of H, then $(f_1 + f_2)^* = f_1^* + f_2^*$, and $(cf)^* = c(f^*)$ for c in C.

A generalized character of a group G is a function X of the form $X = \sum_i n_i X_i$ where each n_i is an integer and $\{i\}$ is an indexing set for the irreducible characters of G. If multiplication is suitably defined, namely coordinatewise multiplication, the generalized characters form a ring denoted by ch(G) and called the <u>character ring</u> of G over C.

1.10 <u>Theorem</u>. If $X = \sum_i n_i X_i$ is in ch(G), then

$$(X,X) = \sum_i n_i^2 .$$

1.11 <u>Theorem</u>. Suppose H is a subgroup of G. If Y is in ch(H), then Y* is in ch(G).

1.12 <u>Lemma</u>. If H is a non-$\{e\}$ subgroup of G, where $xHx^{-1} \cap H = \{e\}$ for each x in G-H, and if X is a generalized character of H, then

1) $X^*(h) = X(h)$ for h in $H^{\#}(= H-\{e\})$ and

2) if $X(e) = 0$, then $X^*(e) = 0$ and $(X,X)_H = (X^*,X^*)_G$.

If X is a character of G, then the <u>kernel of X</u>, written ker(X), is $\{g \in G : X(g) = X(e)\}$. If $X = X_R$, then ker(R) = ker(X). Therefore, ker(X) $\triangle$ G .

1.13 <u>Theorem</u>. If $x \in G^{\#}$, then there is an irreducible character X of G such that $x \notin \ker(X)$.

Suppose R is a representation of G over a field F (not necessarily C), then R is also a representation of G over any field K containing F. We will say R is <u>absolutely irreducible</u> over F if and only if it is irreducible over every field K of finite degree over F.

1.14 <u>Theorem</u>. If R is a representation of G over F, and the characteristic of F does not divide the order of G, then there is a finite extension field K of F such that $R = \sum R_i$ where each R_i is an absolutely irreducible representation of G over K.

1.15 <u>Theorem</u>. If R is an absolutely irreducible representation of a finite Abelian group G over a field F whose characteristic does not divide the order of G, then the degree of R is 1.

CHAPTER II

Frobenius Groups and Frobenius Kernels: Definitions.

With the basic knowledge of group representations and characters presented in the first chapter, we will be able to study the concept of Frobenius groups and their kernels.

2.1 Theorem. (Frobenius) If H is a non-$\{e\}$ subgroup of G, where $xHx^{-1} \cap H = \{e\}$ for each x in $G-H$, and if M is the set of elements of G not in any conjugate of $H^{\#}$, then M is a normal subgroup of G.

Proof:

Suppose $X_1,\ldots,X_r$ are the irreducible characters of G over C.

Suppose $Y_1,\ldots,Y_s$ are the irreducible characters of H over C.

Wlog, assume X_1 is the principal character of G over C and Y_1 is the principal character of H over C. Note that $(X_1)_H = Y_1$.

By the corollary to the Frobenius Reciprocity Theorem for $i \geq 1$ we have

$$(Y_i^*,X_1)_G = (Y_i,(X_1)_H)_H = (Y_i,Y_1)_H = \delta_{i_1} = \begin{cases} 0 & \text{if } i \neq 1 \text{ and} \\ 1 & \text{if } i = 1 . \end{cases}$$

Consider i, where $2 \leq i \leq s$, and suppose $\theta_i = Y_i(e)Y_1 - Y_i$. Then θ_i is a generalized character of H.

Then $\theta_i(e) = Y_i(e) - Y_i(e) = Y_i(e) - Y_i(e) = 0$.

Also $\theta_i^* = (Y_i(e)Y_1 - Y_i)^* = Y_i(e)Y_1^* - Y_i^*$ which is a generalized character of G. Furthermore,

$$\begin{aligned}(\theta_i^*, X_1) &= (Y_i(e)Y_1^* - Y_i^*, X_1) \\ &= Y_i(e)(Y_1^*, X_1) - (Y_i^*, X_1) \\ &= Y_i(e) \cdot 1 - 0 \\ &= Y_i(e)\ .\end{aligned}$$

Thus in the representation of θ_i^* as a Z-linear combination of $X_1, \ldots, X_r$, X_1 appears with the coefficient $Y_i(e)$ and so there is a Z_i in ch(G) such that $\theta_i^* = Y_i(e)X_1 + Z_i$ and X_1 does not appear in Z_i; that is, $(X_1, Z_i) = 0$.

Since θ_i is a generalized character of H over C and $\theta_i(e) = 0$, by Lemma 1.12 $(\theta_i^*, \theta_i^*)_G = (\theta_i, \theta_i)_H$. But

$$\begin{aligned}(\theta_i^*, \theta_i^*)_G &= (Y_i(e)X_1 + Z_1,\ Y_i(e)X_1 + Z_i)_G \\ &= (Y_i(e))^2(X_1, X_1)_G + Y_i(e)(X_1, Z_i)_G \\ &\quad + Y_i(E)(Z_i, X_1)_G + (Z_i, Z_i)_G \\ &= (Y_i(e))^2 + (Z_i, Z_i)_G.\end{aligned}$$

Also, $$\begin{aligned}(\theta_i, \theta_i)_H &= (Y_i(e)Y_1 - Y_i, Y_i(e)Y_1 - Y_i)_H \\ &= (Y_i(e))^2(Y_1, Y_1)_H + Y_i(e)(Y_1, Y_2)_H \\ &= Y_i(e)(Y_i, Y_1)_H + (Y_i, Y_i)_H \\ &= (Y_i(e))^2 + 1.\end{aligned}$$

Therefore, $(Y_i(e))^2 + (Z_i,Z_i)_G = (Y_i(e))^2 + 1$ and so $(Z_i,Z_i)_G = 1$. Therefore, there is a $j(i)$, $2 \leq j(i) \leq r$, such that $Z_i = X_{j(i)}$ or $Z_i = -X_{j(i)}$.

Recall that $\theta_i(e) = 0$ implies that $\theta_i^*(e) = 0$ and thus $0 = \theta_i^*(e) = Y_i(e)X_1(e) + Z_i(e) = Y_i(e) + Z_i(e)$ where $Y_i(e)$ and $Z_i(e)$ are integers (see definition of characters). Since $Y_i(e)$ is the degree of the irreducible representation of which Y_i is the character, $Y_i(e) > 0$. Therefore $Z_i(e) < 0$. Since $X_{j(i)}(e) > 0$, $Z_i(e) \neq X_{j(i)}(e)$ and so $Z_i = -X_{j(i)}$.

We now have $\theta_i^* = Y_i(e)X_1 - X_{j(i)}$. Since $0 = \theta_i^*(e)$ $Y_i(e)X_1(e) - X_{j(i)}(e) = Y_i(e) - X_{j(i)}(e)$, $Y_i(e) = X_{j(i)}(e)$.

Suppose x is in $H^{\#}$. Then

$$
\begin{aligned}
Y_i(e) - Y_i(x) &= Y_i(e)Y_1(x) - Y_i(x) \\
&= \theta_i(x) \\
&= \theta_i^*(x) \\
&= Y_i(e)X_1(x) - X_{j(i)}(x) \\
&= Y_i(e) - X_{j(i)}(x) .
\end{aligned}
$$

Thus $Y_i(x) = X_{j(i)}(x)$.

Combining the last two paragraphs we have for each h in H, $Y_i(h) = X_{j(i)}(h)$.

Suppose y is in $M^{\#}$. Then y is not in any conjugate of H. Thus by Lemma 1.7 $0 = \theta_i^*(y) = Y_i(e)X_1(y) - X_{j(i)}(y)$

$$= Y_i(e) - X_{j(i)}(y)$$

$$= X_{j(i)}(e) - X_{j(i)}(y) .$$

Thus, $X_{j(i)}(y) = X_{j(i)}(e)$ or in other words, y is in $\mathrm{Ker}(X_{j(i)})$. Therefore, each member of $M^{\#}$ is in $\mathrm{Ker}(X_{j(i)})$ for i, $2 \leq i \leq s$, and so $M \subseteq \bigcap_{i=2}^{s} \mathrm{Ker}(X_{j(i)})$.

Suppose y is in $H^{\#}$. By Theorem 1.13 there is an i, $1 \leq i \leq s$, where y is not in $\mathrm{Ker}(Y_i)$. Since Y_1 is is the principal character of H, its kernel is H. Therefore, $i \geq 2$ and y not in $\mathrm{Ker}(Y_i)$ implies $Y_i(y) \neq Y_i(e)$ and so $X_{j(i)}(y) \neq X_{j(i)}(e)$. Thus, y is not in $\mathrm{Ker}(X_{j(i)})$ for some $i \geq 2$.

Since $\mathrm{Ker}(X_{j(i)})$ is normal in G, $x^{-1}yx$ is not in $\mathrm{Ker}(X_{j(i)})$ for each x in G. Therefore, for each x in G, $x^{-1}yx$ is not in $\bigcap_{i=2}^{s} \mathrm{Ker}(X_{j(i)})$. So no member of a conjugate of $H^{\#}$ is in $\bigcap_{i=2}^{s} \mathrm{Ker}(X_{j(i)})$. Thus $\bigcap_{i=2}^{s} \mathrm{Ker}(X_{j(i)}) \subseteq M$. Therefore, $M = \bigcap_{i=2}^{s} \mathrm{Ker}(X_{j(i)})$. So M is a normal subgroup of G.//

Another form of the theorem of Frobenius is the following theorem about permutation groups.

2.2 <u>Theorem</u>. If G is a transitive permutation group on a finite set Ω and x in $G^{\#}$ implies $\mathrm{ch}(x) = 0$ or 1, then $M = \{x \text{ in } G : \mathrm{ch}(x) = 0 \text{ or } x = e\}$ is a regular normal subgroup of G.

Proof:

Suppose a is in Ω. Since G is transitive on Ω, the conjugates of G_a are G_b for b in Ω, where $G_a = \{x \in G \mid ax = a\}$, and G_b is similarly defined. Notice that for c and d in Ω, $c \neq d$, $G_c \cap G_d = \{e\}$ since only the identity fixes more than one element.

Suppose $G_a = \{e\}$. Then $G_b = \{e\}$ for each b in Ω. Therefore, if x is in $G^{\#}$, $ch(x) = 0$ and hence G is regular. Also, $M = G$ and therefore M is a regular normal subgroup of G.

We now consider the important case when $G_a \neq \{e\}$. Suppose g is in $G-G_a$. Since g is not in G_a, $ag \neq a$ and so $G_{ag} = g^{-1}G_a g$, $gG_a g^{-1} \cap G_a = \{e\}$. If x is not in any conjugate of $G_a^{\#}$, then x is not in $G_b^{\#}$ for any b in Ω, and so $ch(x) = 0$ or $x = e$. Therefore x is in M, and conversely. Thus M is the set of elements not in any conjugate of $G_a^{\#}$ and by Theorem 2.1, M is a normal subgroup of G. Also, G is the disjoint union of M and the $G_b^{\#}$'s for b in Ω, and so, $o(G) = o(M) + o(\Omega)(o(G_a)-1)$. Since g in $G-G_a$ implies $g\,G_{ag}{-1} \cap G_a = \{e\}$, $N_G(G_a) = G_a$ and therefore G has $[G:G_a]$ conjugates; in other words, $[G:G_a] = o(\Omega)$. Thus

$$o(G) = o(M) + [G:G_a]\cdot o(G_a) - [G:G_a]$$
$$= o(M) + o(G) - o(\Omega) \quad \text{and hence}$$

$o(M) = o(\Omega)$. If θ is an orbit of M where a is in θ, then $o(M) = o(M_a) = o(\theta)$. If g is in M_a, then g fixes a, which implies $ch(g) \geq 1$ and thus $g = e$. So $M_a = \{e\}$ and $o(M_a) = 1$ which implies $o(M) = o(\theta)$ and $\Omega = \theta$. Therefore, M is transitive on Ω, and so M is regular on Ω. //

Theorem 2.2 gives us a class of transitive permutation groups in which only the identity fixes more than one element and with a nontrivial subgroup fixing an element. Groups of this type are known as Frobenius groups after the person who first studied them. In the proof of Theorem 2.2 we notice that Theorem 2.1 abstracts the group theoretic properties of Frobenius groups. The subgroup M discussed in this theorem will be called the Frobenius kernel. Nevertheless, we will define Frobenius groups and Frobenius kernels in a more useful manner and will later show the equivalence of the two definitions.

A finite group G is a Frobenius group with Frobenius kernel M if and only if

1) $M \triangle G$, $\{e\} = E \neq M \neq G$ and
2) $m \in M^{\#}$ implies $C_G(m) \subseteq M$.

Before continuing our discussion on Frobenius groups and their kernels it will be necessary to briefly discuss the concept of Hall P-subgroups.

Let P be a set of primes. A group G is a <u>P-group</u> if and only if G is periodic and $g \in G$ implies $o(g) \in P$. Suppose P' is the set of primes not in P. A subgroup H of a finite group G is called a <u>Hall P-subgroup</u> of G, written $H \in Hall_P(G)$, if and only if H is a P-group and $[G:H] \in P'$. It should be obvious that $\gcd(o(H),[G:H]) = 1$.

2.3 <u>Lemma</u>. Subgroups and factor groups of P-groups are P-groups.

2.4 <u>Lemma</u>. If H and G/H are P-groups, then so is G.

2.5 <u>Theorem</u>. If $H \in Hall_P(G)$ and $A \triangle G$, then $A \cap H \in Hall_P(A)$.

2.6 <u>Theorem</u>. If $H \in Hall_P(G)$ and A is a subnormal subgroup of G, then $A \cap H \in Hall_P(A)$.

2.7 <u>Theorem</u>. (Schur's Splitting Theorem) If H is a normal Hall P-subgroup of a finite group G, then H has a complement.

A Hall subgroup is a Hall P-subgroup for some set P of primes.

2.8 <u>Lemma</u>. If G is a Frobenius group with Frobenius kernel M, then $M \in Hall_Q(G)$ for some set Q of Primes.

<u>Proof:</u>

Suppose P_o is a Sylow p-subgroup of M and P is a Sylow p-subgroup of G with $E \neq P_o \leq P$. I first wish to show that $Z(P) \subseteq C_G(P_o) \subseteq M$. Suppose $x \in Z(P)$. Then $x \in P$ and $xq = qx$ for all q in P. Since $P \leq G$ and

$P_o \lesssim P$, $x \in G$ and $xq = qx$ for all q in P_o. This tells us that $x \in C_G(P_o)$, which implies $x \in \bigcap_{p_o \in P_o} C_G(P_o) \subseteq \bigcap_{p_o \in P_o^{\#}} C_G(P_o)$. Since $C_G(P_o) \subseteq M$ for all $p_o \in P_o^{\#}$, $x \in M$. I now wish to show that $C_G(Z(P)) \subseteq M$. Suppose $x \in C_G(Z(P))$. $P \neq E$ implies that $Z(P) \neq E$ and so there is a z in $Z(P)$ where $z \neq e$. Then $xz = zx$ and so x is in $C_G(z)$. z in $Z(P)$ implies that z is in M, and so x is in $C_G(z)$ M. Thus $C_G(Z(P)) \subseteq M$. Since q in P implies that $xq = qx$ for all x in $Z(P)$, $P \subseteq C_G(Z(P)$. Therefore P $C_G(Z(P)) \subseteq M$ and so $P = P_o$. Therefore, M is a Hall subgroup of G. //

2.9 <u>Theorem</u>. If G is finite and M is a subgroup of G with $E < M < G$, then G is Frobenius with Frobenius kernel M iff there is an $H \in \text{Hall}(G)$ such that $G = MH$, $M \cap H = E$ and $H \cap H^x = E$ for all $x \in G-H$. Moreover, for such an H, M is the set of elements of G not in any conjugate of $H^{\#}$.

<u>Proof</u>:

Suppose G is a Frobenius group with Frobenius kernel M. The Schur Splitting theorem guarantees the existence of a subgroup H of G such that $G = MH$ and $M \cap H = E$. Since $M \in \text{Hall}(G)$ and $o(H) = [G:M]$, $\gcd([G:H], o(H)) = 1$ and so we have $H \in \text{Hall}\ (G)$. Let $x \in G-H$. By way of contradiction, suppose $H \cap H^x \neq E$. Since $G = HM$, we can assume $x \in M^{\#}$. Let $e \neq h \in H \cap H^x$. Then $xhx^{-1} \in H$ and so $(xhx^{-1})h^{-1} \in H$. $M \triangle G$ implies $hx^{-1}h^{-1} \in M$ and so $x(hx^{-1}h^{-1}) \in M$. Therefore, $xhx^{-1}h^{-1} \in H \cap M = E$. So $xhx^{-1}h^{-1} = e$, or in other words, $h \in C_G(x) \subseteq M$, a contradication. Therefore $H \cap H^x = E$ for $x \in G-H$.

Conversely, suppose that $G = MH$, $M \cap H = E$ and $H \cap H^x = E$ for all x in $G-H$. First, suppose that $y \in M$ is in a conjugate of $H^{\#}$. Then $y = x^{-1}hx$ for some h in $H^{\#}$ and some x in G. Suppose $x = h_1m_1$ where $h_1 \in H$ and $m_1 \in M$. Then $y = m_1^{-1}hh_1m_1 \in M$. Therefore $h_1^{-1}hh_1 \in M$. But $h_1^{-1}hh_1 \in H^{\#}$ and we have a contradiction. Thus M only contains elements not in any conjugate of $H^{\#}$; that is, $M \subseteq G - \bigcup_{g \in G} g^{-1}H^{\#}g$. By a similar argument as in the proof of Theorem 2.2, $o(\bigcup_{g \in G} g^{-1}H^{\#}g) = (o(H)-1)(G:H) = o(G) - (G:H)$. Therefore $o(G - \bigcup_{g \in G} g^{-1}H^{\#}g) = o(G:H) = o(M)$. Thus M contains all the elements not in any conjugate of $H^{\#}$. By Theorem 2.1 the set M is a normal subgroup of G. Let $x \in M^{\#}$ and suppose that $C_G(x) \subseteq M$. There is an element $y \neq e$ in $C_G(x)$ which is in some conjugate of H. Therefore $y^{-1}xy = x$ and also $y \in H^{h'm} = H^m$ for some $h' \in H$ and $m \in M$. Thus $y = m^{-1}hm$ for some h in $H^{\#}$ and so $(m^{-1}h^{-1}m)x(m^{-1}hm) = x$ which tells us that

$$h^{-1}(mxm^{-1})h = mxm^{-1}.$$

If we let $z = mxm^{-1} \in M^{\#}$, $h \in C_G(z)$ and $h = zhz^{-1}$. Therefore, $H \cap H^z \neq E$, a contradiction. So $C_G(x) \subseteq M$ for all $x \in M^{\#}$, and G is a Frobenius group with Frobenius kernel M. //

2.10 <u>Theorem</u>. G is a Frobenius group with Frobenius kernel M iff these are integers $m > 1$ and $n > 1$ with $\gcd(m,n) = 1$ such that $o(G) = mn$, $o(M) = m$, $M \triangle G$, and if $g \in G$ then either $g^m = e$ or $g^n = e$, and also $M = \{g \mid g^m = e\}$.

<u>Proof:</u>

Suppose G is Frobenius with Frobenius kernel M. We know that M is a Hall subgroup of G, and therefore, if $m = o(M)$ and $[G:M] = n$, then $m > 1$, $n > 1$, $o(G) = mn$ and $\gcd(m,n) = 1$. Since $M \triangle G$ and is a Hall subgroup of order m, $M = \{g \mid g^m = e\}$. Let $g \in G$ and suppose $g^n = e$. Then $(g^n)^m = e$ and so $g^n \in M^{\#}$. By assumption, $g \in C_G(g^n) \subseteq M$. Therefore $g^m = e$.

Conversely, suppose $M \triangle G$, $o(G) = mn$, $o(M)$ 7 m, $\gcd(m,n) = 1$, $m > 1$ and $n > 1$, $g \in G$ implies $g^m = e$ or $g^n = e$ and also $M = \{g \mid g^m = e\}$. M is obviously a normal Hall subgroup of G. By Schur's Splitting Theorem, there is a complement H of M. H is also a Hall subgroup of G of order n. Suppose there is a g in G-H such that $H \cap H^g \neq E$. Since $G = MH$, we can assume that $g \in M^{\#}$. Since $H \cap H^g \neq E$, there is an $h \in H^{\#}$ such that $ghg^{-1} \in H$ and so

$ghg^{-1}h^{-1} \in H$. Since $M \triangle G$, $ghg^{-1}h^{-1} \in M$ and so $ghg^{-1}h^{-1} \in H \cap M = E$. Therefore, $gh = hg$. Let $o(h) = n$, and $o(g) = m$. Where $n_1 | n$ and $m_1 | m$. Since $(m_1 n_1) = 1$, $o(hg) = m_1 n_1$. By our assumption, $n_1 = 1$ or $m_1 = 1$ and so $h = e$ or $g = e$, a contradiction. Thus $H \cap H^g = E$ for $g \in G-H$ and by Theorem 2.9, G is Frobenius with kernel M. //

The following theorem allows us to use inductive methods in our study of the properties of Frobenius groups.

2.11 <u>Theorem.</u> Let G be a Frobenius group with Frobenius kernel M and let H be a complement of M. Then

(i) $E \neq H_1 < H$ implies $H_1 M$ is Frobenius with kernel M,

(ii) $E \neq K < M$ and $K \triangle G$ implies G/K is Frobenius with Frobenius kernel M/K, and

(iii) $E \neq K \quad M$, $E \neq H_1 \subseteq H$, $H_1 \quad N_G(k)$ implies $H_1 K$ is Frobenius with Frobenius kernel K.

<u>Proof</u>:

(i) Since $M \triangle G$ and $H_1 < G = N_G(M)$, (i) is true if (iii) is true.

(ii) Since $K \triangle G$ and $E = K < M \triangle G$, $E < M/K \triangle G/K$. By Theorem 2.9, we know that if $o(M) = m$, then there is a positive integer $n > 1$ such that $o(G) = mn$ and $\gcd(m,n) = 1$. Suppose $o(K) = q$. Then $o(M) = qr = m$ for some integer $r > 1$, and

$\gcd(r,n) = 1$. Notice that $o(G/K) = rm$ and $o(M/K) = r$. Since M is a normal Hall subgroup of G, M/K is a normal Hall subgroup of G/K of order r. Thus $M/K = \{g : g^r = e\}$. By the same argument as in the first part of Theorem 2.10, $g \in G/K$ implies $g^r = e$ or $g^n = e$. Thus G/K is Frobenius with Frobenius kernel M/K.

(iii) H_1K is a subgroup of G. Suppose $x \in K^{\#} \subset M^{\#}$. $C_{H_1K}(x) \subseteq C_G(x)$ M. Therefore, $C_{H_1K}(x) \subseteq M \cap H_1K$. Suppose $y \in H_1K \cap M$. Then $y \in H_1K$. Since $H_1 \cap M$ $H \cap M = E$, $y \in k$ and so $H_1K \cap M \subseteq K$. Suppose $y \in K$. Then $y \in M$ and $y \in e \cdot K$ and so $y \in H_1K \cap M$. Thus, $H_1K \cap M = K$. Therefore $C_{H_1K}(x) \subseteq K$, and since $H_1 \subseteq N(K)$, $K \triangle H_1K$ and $E < K < H_1K$. //

CHAPTER III

Frobenius Groups and Frobenius Kernels: Properties

3.1 Theorem. If G is a Frobenius group, then there is a Frobenius subgroup B with an elementary abelian p-group K for a Frobenius kernel (p a prime) and a complement L of K of prime order.

Proof:

Without loss of generality we can assume that G has no proper Frobenius subgroup. Therefore, we must show that G satisfies the conditions of B in the theorem.

Let M be a Frobenius kernel of G with complement H. If $E < A < H$, then by Theorem 2.11, MA is a proper Frobenius subgroup of G. Thus H is of prime order.

Suppose $E \neq P \in Syl(M)$. Then $G = N(P)M$. Since H has prime order, $o(H) | o(N(P))$. Therefore, since $H \in Syl(G)$, some conjugate of H, say H_1, is contained in $N(P)$. H_1 is again a complement of M. By Theorem 2.11 iii, PH_1 is Frobenius and therefore $PH_1 = G$ and $P = M$; that is, M is a p-group for some prime p. If M has a characteristic subgroup $Q \neq E$, then $Q \triangle G$ and again by Theorem 2.11, QH would be Frobenius and $M = Q$. Thus M is characteristically simple. Since $Z(M)$ is a non-E characteristic subgroup of M, $M = Z(M)$. Now, the set of all elements of M of order 1 or p form a non-E characteristic subgroup of M and therefore is M. Thus M is an elementary abelian p-group as desired. //

3.2 <u>Theorem</u>. If G is a Frobenius group with Frobenius kernel M, and $A \vartriangle G$, then either $A \subseteq M$ or $M \subseteq A$.

<u>Proof</u>:

Let H be a complement of M, and suppose that x is in $A-M$. Then $C_G(x) \cap M = E$, for otherwise $x \in M$, a contradiction. It follows that $o(C_G(x)) | o(H)$ and so that $o(M) = [G:H] | o(c\ell(x))$ where $c\ell(x)$ is the set of conjugates of x in G. Since $c\ell(x) \subseteq A$, $o(M) | o(A)$. But M is a normal Hall subgroup of G and therefore every Sylow p-subgroup of A such that $p | o(M)$ is contained in M. Hence $M \subseteq A$. //

3.3 <u>Theorem</u>. If G is a Frobenius group with Frobenius kernel M, and if H is a complement of M, then H does not contain a Frobenius group.

<u>Proof</u>:

By way of contradiction, assume G is a group of minimal order which makes the theorem false. Thus H itself is a Frobenius group with no proper Frobenius subgroups, for if there was a proper Frobenius subgroup H' of H, then $H'M$ would be a smaller counterexample. By Theorem 3.1, H has an elementary abelian Frobenius kernel Q with a complement K of prime order.

Let $E < P \in Syl(M)$. So $G = MN(P)$. Thus $o(H) | o(N(P))$. Also since $M \vartriangle G$, $N(P) \cap M \vartriangle N(P)$. Since H $G/M = MN(P)/M$ $N(P)/N(P) \cap M$, $N(P) \cap M \in Hall\ (N(P))$. By Schur's Splitting theorem, there is a subgroup S of G such that $N(P) = (N(P) \cap M)S$ and $(N(P) \cap M) \cap S = E$. Notice that $S \approx N(P)/N(P) \cap M$ and therefore $S \approx H$, and $MS = G$. Since $S \subseteq N(P)$, PS is a subgroup of G. By Theorem 2.11(iii) PS is Frobenius with Frobenius kernel P, and also $S \approx H$ is a Frobenius group. By our assumption $G = PS$. Therefore, $M = P$; that is, M is a p-group for some prime p. As in the proof of Theorem 3.1, M is characteristically simple and therefore is an elementary abelian p-group. Moreover, if M' is a non-E proper subgroup of M such that $M' \triangle G$, $M'H < MH$ which is impossible by our assumption. Thus M is a minimal, normal, non-E subgroup of G.

M is a vector space over the field of integers mod p, (J_p). Consider the map $A : H \rightarrow L_{J_p}(M)$ where $m(hA) = h^{-1}m$ $h \in M$. A is a J_p linear representation of G.

Suppose $h \in H$ and $m(hA) = h^{-1}mh = m$ for all m in M. Then $h \in C_G(M) \subseteq M$ and so $h = e$. Thus A is faithful. Suppose M has a J_p submodule M' which is invariant under A. Then $h^{-1}M'h \subseteq M'$ for all h in H, and so $H \subseteq N(M')$. Since $M \subseteq N(M')$ also, $HM = G \subseteq N(M')$. Thus, $G = N(M')$ which tells us that $M' \triangle G$. Thus $M' = \{e\}$ or M and therefore A is irreducible.

We now know that A is a faithful, irreducible representation of H over J_p with representation space M. By Theorem 1.14 there is a finite field extension F of J_p

such that A is a direct sum of absolutely irreducible representations of H over F. Let B be one of these absolutely irreducible representations and let V be the vector space being operated upon; that is, $B : H \to L_F(V)$ is absolutely irreducible. We will say an element h of H acts on V to mean that hB is an F-linear transformation of V and sometimes will denote its acting on v by vh. Consider B/Q. WLOG, we have chose F large enough such that $V = V_1+\ldots+V_n$ where each V_i is an irreducible Q-space. By Theorem 1.15, $\dim_Q(V_i) = 1$. Therefore xB is a scalar on each V_i for every $x \in Q$. Suppose by combining the V_i's properly, we get $V = W_1+\ldots+W_r$ where each W_i is a maximal subspace such that xB is scalar on W_i for all x in Q. Recall, K is of prime order and hence cyclic. Suppose $K = \langle y \rangle$, $v \in W_i$ and $x \in Q$.

$$\begin{aligned} vyx &= vyxy^{-1}y \\ &= [v((yxy^{-1}))B)]y \\ &= cvy \end{aligned}$$

since $yxy^{-1} \in Q$ and thus acts on v as a scalar. Also since $vy \in W_iy$, xB/W_iy is scalar. Thus $W_iy \subseteq W_j$ for some j. Suppose $W_iy < W_j$. Then, $Vy < V$ which is impossible since yB is invertible. Thus $W_iy = W_j$. Suppose S is the linear span of this set. Since S is invariant under y and Q, and since $H = \langle y, Q \rangle$, S is invariant under H. But V is irreducible under H and so $V = S$. Also no two members of Ω are the same since K is of prime order. Thus $s = r$ and $V = W_i + W_i y + \ldots + W_i y^{r-1}$.

Suppose $w \in W_i^{\#}$. Then $v = w + wy + \ldots + wy^{r-1}$ is invariant under y and is non zero. Thus $v(yB) = v$, and yB has 1 as one of its characteristic values. Therefore every B in the characterization of A as a direct sum of absolutely irreducible representations has 1 as a characteristic value of yB. Therefore yA has a characteristic value of 1. So there is a characteristic vector $z \in M^{\#}$ of yA such that $z(yA) = z$ or $z = y^{-1}zy$. Thus $y \in C(z) \subseteq M$, a contradiction. Therefore the theorem is proved. //

3.4 <u>Theorem</u>. A Frobenius group has a unique Frobenius kernel.

<u>Proof</u>:

By way of contradiction, suppose G is a Frobenius group with two distinct Frobenius kernels M and M'. By Theorem 3.2 we can suppose that $M < M'$. Let H be a complement of M. Therefore $M' = MK$, where $K = H \cap M' \triangle H$ and $E < K < H$.

Suppose $x \in K^{\#}$. Since $x \in M'$ and M' is a Frobenius kernel of G, $C_H(x)$ $C_G(x) \subseteq M'$. Also $C_H(x)$ H. Therefore $C_H(x)$ $H \cap M' = K$ for $x \in K^{\#}$. But K is normal in

H and so H is Frobenius with Frobenius kernel K. This contradicts Theorem 3.3 and therefore a Frobenius group has a unique Frobenius kernel. //

Before proving that Frobenius kernels are nilpotent, it is necessary to define a Thompson subgroup and to state a nontrivial result concerning it.

Let P be a p-group. If A is an abelian subgroup of P, let m(A) be the minimum number of generators of A and d(P) be the maximum of m(A) over all abelian subgroups A of P. The Thompson Subgroup of P, denoted by T(P), is the subgroup of P generated by all abelian subgroups A of P with m(A) = d(P). T(P) is clearly characteristic in P.

3.5 <u>Theorem</u>. (Thompson) Let p be an odd prime and G_p be a Sylow p-subgroup of G. If $C_G(Z(G_p))$ and $N_G(T(G_p))$ have normal p-complements, then G has a normal p-complement.

3.6 <u>Theorem.</u> (Thompson) Frobenius Kernels are nilpotent.

<u>Proof</u>:

Suppose G is a Frobenius group with Frobenius kernel M. The proof will be by induction on the order of G. Without loss of generality, by Theorem 2.11, we may assume that G = MH, where o(H) = p, a prime.

Suppose $Z(M) \neq E$. If $Z(M) = M$, then M is certainly nilpotent. If $Z(M) \neq M$, then $Z(M) \triangle G$ and by Theorem 2.11

$G/Z(M)$ is Frobenius with Frobenius kernel $M/Z(M)$. The induction hypothesis then tells us that $M/Z(M)$ is nilpotent. Therefore M is nilpotent.

If $Z(M) = E$, then M is not nilpotent. Therefore, there is normal q-complement. Suppose first that $q \neq 2$. Recall that $p \nmid o(M)$ and H permutes the Sylow q-subgroups of M by conjugation. Since the number of Sylow q-subgroups of M divides $o(M)$, and thus is relatively prime to p, there is a Sylow q-subgroup Q of M which is normalized by H. Let $Q_o = Z(Q)$ or $T(Q)$ so that Q_o is characteristic in Q and hence normalized by H. Suppose $n \in N_M(Q_o)$, $q_o \in Q_o$ and $h \in H$. Then,

$$(h^{-1}nh)^{-1}q_o(h^{-1}nh) = h^{-1}n^{-1}hq_oh^{-1}nh.$$

Suppose $q_o' = hq_oh^{-1}$. q_o' is in Q_o since H normalizes Q_o. If we let $q_o'' = n^{-1}q_o'n$, then q_o'' is in Q_o since $n \in N_M(Q_o)$. Finally, $h^{-1}q_o''h$ is in Q_o since $n \in N_M(Q_o)$. Therefore $h^{-1}n^{-1}hq_oh^{-1}nh \in Q_o$, and so, $h^{-1}nh \in N_M(Q_o)$. Thus H normalizes $N_M(Q_o)$

Suppose $Q_o = T(Q)$. Then by Theorem 2.11 iii, $K = HN_M(T(Q))$ is a Frobenius group with Frobenius kernel $N_M(T(Q))$. If $N_M(T(Q)) = M$, then $T(Q) \triangle M$. Since H also normalizes $T(Q)$, $T(Q) \triangle G$. Now suppose that $N_M(T(Q)) < M$. By the induction hypothesis, $N_M(T(Q))$ is nilpotent and therefore has a normal q-complement. By Theorem 3.5,

$C_M(Z(Q))$ does not have a normal q-complement and thus is not nilpotent. This implies that $N_M(Z(Q))$ is not nilpotent. Recall that H-$N_M(Z(Q))$ is Frobenius. Therefore $N_M(Z(Q)) = M$ and thus $Z(Q) \triangle M$. Since H normalizes $Z(Q)$, $Z(Q) \triangle G$.

Suppose that we cannot find and odd q to satisfy our requirements; we will show that M has a normal Sylow 2-subgroup R. If $O(M) = 2^{\alpha} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, let P_i be a normal complement of p_i. Since $o(P_i) = 2^{\alpha} p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_{i+1}^{\alpha_{i+1}} \cdots p_r^{\alpha_r}$, $\bigcap_{i=1}^{r} P_i = R$ is a normal subgroup of M of order 2^{α} and so is a normal Sylow 2-subgroup of M. Since R is also characteristic in M, $R \triangle G$ and therefore $Z(R)$ and $T(R)$ are characteristic in R and hence normal in G.

In any case, we have a non-E normal subgroup of G of order power of q contained in M. Let's call it Q_o. $Z(Q_o)$ and $C_G(Z(Q_o))$ are normal in G. Since M is the Frobenius kernel of G and $Z(Q_o)$ M, then $C_G(Z(Q_o))$ M. If G acts on $Z(Q)$ by conjugation, the kernel of this action is $C_G(Z(Q_o))$ and so $G/C_G(Z(Q_o))$ acts faithfully on $Z(Q_o)$. Suppose $\bar{h} = hC_G(Z(Q_o)) \in G/C_G(Z(Q_o))$ where $h \in H^{\#}$. Then $\bar{h}$ has order p and when acting on $Z(Q)$ of order a power of q (where $(p,q) = 1$), induces an automorphism on $Z(Q)$ of order p. Thus $\bar{h}$ fixes a non-identity element; that is, $\bar{h}$ centralizes an element v of $Z(Q)^{\#}$. In other words, $h \in C_G(v)$ M, a contradication.

CHAPTER IV

Frobenius Groups and Frobenius Kernels: Applications

Before presenting a major result in character theory dealing with Frobenius groups and their kernels, the following lemma must be proved.

4.1 Lemma. Suppose S is a finite indexing set and D is a nonsingular $S \times S$ matrix. If G is a group of permutations of $S \times S$ such that $g \in G$ implies that there are g_R and $g_c \in Sym(S)$ such that for every (i,j), $D((i,j)g) = D(ig_R, j) = D(i, jg_c)$, then

(i) $G_R = \{g_R | g \in G\}$ is a group and similarly for $G_c = \{g_c | g \in G\}$,

(ii) the number of orbits of G_R equals the number of orbits of G_c, and

(iii) if G is cyclic, then $Ch(G_R) = Ch(G_c)$.

Proof:

Suppose we are given $g \in Sym(S \times S)$ and two distinct g_R and g_R' satisfying the above equality. Then the two rows ig_R and ig_R' of D where $ig_R \neq ig_R'$ are equal which contradicts the fact that D is nonsingular. Similarly, there is a unique g_c for any $g \in G$.

Suppose $A(g)$ is the $S \times S$ permutation matrix corresponding to g_R and then $D_g = A(g)D$. Then,

$$\begin{aligned} A(gh)D &= Dgh \\ &= (Dg)h \\ &= (A(g)D)h \\ &= A(h)A(g)D \end{aligned}$$

Since D is nonsingular, $A(gh) = A(h)A(g)$. Then, since the function which maps a permutation onto the corresponding permutation matrix is one-to-one and reverses products, G_R is a group. Moreover, the mapping $g \to A^T(g)$ is a representation of G. For later use, suppose this representation of G has character X. Similarly, one finds that if $B(g)$ is the permutation matrix corresponding to g_c and so $D_g = DB(g)$, then G_c is a group and the map $g \to B(g)$ is a representation of G. Suppose the character of this representation is Y.

By assumption $A(g)D = DB(g)$. Thus $A(g)$ is similar to $B(g)$. Therefore, $Tr(A^T(g)) = Tr(B(g))$ and so $X(g) = Y(g)$ for all g. Since $X(g) = Ch(g_R)$, $\Sigma_{g \text{ in } G} X(g) = o(G_R)$ (the number of orbits of G_R). Similarly, since $Y(g) = Ch(g_c)$, $\Sigma_{g \text{ in } G} Y(g) = o(G_c)$ (the number of orbits of G_c). Thus (ii) holds, because $o(G_c) = o(G) = o(G_R)$.

Finally, suppose G is cyclic. Then $G = \langle g \rangle$ and so, $G_R = \langle g_R \rangle$ and $G_c = \langle g_c \rangle$. Thus

$$\begin{aligned} Ch(G_R) &= Ch(g_R) \\ &= X(g) \\ &= Y(g) \\ &= Ch(G_c). \quad // \end{aligned}$$

4.2 Theorem. If G is a Frobenius group with Frobenius kernel M, Y is an irreducible character of M other than the principal character, and X is an irreducible character of G, then

(i) Y^* is an irreducible character of G, and

(ii) either $M \subseteq \mathrm{Ker}(X)$ or $X = W^*$ for some irreducible character W of M.

Proof:

Suppose H is a complement of M in G. For $x \in H$ define $T_x : M \to M$ by $mT_x = x^{-1}mx$. Recall that T_x is an automorphism of M. Let $\{C_j\}_{j=1}^{r}$ be the conjugate classes of M with $C_1 = \{e\}$ and $\{Y_j\}_{j=1}^{r}$ be the set of irreducible characters of M with Y_1 the principal character. Finally, let D_i be an irreducible representation of M with character Y_i.

If μ_j is a representative of C_j, then

(*) $\mathrm{Tr}(\mu_j^x D_i) = \mathrm{Tr}(x^{-1}\mu j x) D_i) = \mathrm{Tr}(\mu_j (TxD_i)$ for x in H.

Now suppose x is in H. We observe that $T_x D_i$ is an irreducible representation of M. Also, suppose xA is the permutation of $\{1,\ldots,r\}$ such that $C_{j(xB)} = C_j^x = C_j T_x$. By (*), $Y_{i(xA)}(C_j) = \mathrm{Tr}(C_j(T_x D_i) = \mathrm{Tr}(C_{j(xB)} D_i) = Y_i(C_{j(xB)})$. Consider the character matrix R where the (i,j)-th element is $Y_i(C_j)$. R is nonsingular, and if we define $(i,j)x = (i,j(xB))$ for x in H, then $R(i,j)x) = R(i,j(xB))$, and the hypothesis of Lemma 4.1 is satisfied.

Suppose (BWOC) $x \in H^{\#}$, $C_j \neq \{e\}$ and $C_j^x = C_j$. Then there is a $y \in C_j$ and a $z \in M$ such that $x^{-1}yx = z^{-1}yz$. In other words, $zx^{-1}yxz^{-1} = y$, or $xz^{-1} \in C(y) \subseteq M$, which implies $x \in M$, a contradiction. Therefore, $C_j^x \neq C_j$ for all x in $H^{\#}$ when $C_j \neq \{e\}$. Thus $Ch(xB) = 1$ if $x \in H^{\#}$. By Lemma 4.1, $Ch(xA) = 1$ for all x in $H^{\#}$. In fact, xA fixes i' if and only if $Y_{i'}$ is the principal character. Since $Y = Y_{i_o}$ for some i_o, $i_o \geq 2$, we will let $Y(xA)$ denote $Y_{i_o(xA)}$ and we will denote D_{i_o} by D.

Suppose $Y(xA) = Y(yA)$ for $x, y \in H$. Then $i_o(xA) = i_o(yA)$ and so $i_o(y^{-1}xA) = i_o$ where $i_o \geq 2$. Thus $x = y$, and so if x and y are distinct elements of H, $Y(xA) \neq Y(yA)$ and $(Y(xA), Y(yA) = 0$.

Recall that $G = \cup\{Mx \mid x \in H\}$. Therefore, for z in M we have

$$Y^*(z) = \sum_{x \in H} \tilde{Y}(x^{-1}zx)$$

$$= \sum_{x \in H} Y(x^{-1}zx)$$

$$= \sum_{x \in H} Y(zT_x)$$

$$= \sum_{x \in H} tr(z(T_x D))$$

$$= \sum_{x \in H} (Y(xA)(z))$$

$$= (\sum_{x \in H} Y(xA))(z))$$

Also, for y in G-M, $Y^*(y) = \sum_{x \in H} \tilde{Y}(x^{-1}yx) = 0$ Thus Y^*/M $\sum_{x \in H} Y(xA)$. Therefore,

$$(Y^*, Y^*)_G = \frac{1}{o(G)} \sum_{y \in G} Y^*(y) Y^*(y^{-1})$$

$$= \frac{1}{o(G)} \sum_{y \in M} Y^*(y) Y^*(y^{-1})$$

$$= \frac{1}{o(G)} \sum_{y \in M} \sum_{x \in H} \sum_{u \in H} Y(xA)(y) \cdot (uA)(y^{-1})$$

$$= \frac{1}{o(G)} \sum_{x \in H} \sum_{u \in H} (\sum_{y \in M} Y(xA)(y) \cdot Y(uA)(y^{-1}))$$

$$= \frac{o(M)}{o(G)} \sum_{x \in H} \sum_{u \in H} (Y(xA), Y(uA))$$

$$= \frac{o(M)}{o(G)} \sum_{x \in H} 1$$

$$= \frac{o(M) \cdot o(H)}{o(G)} = 1$$

Therefore Y^* is irreducible.

Suppose $X/M = \sum_{i=1}^{r} a_i Y_i$. If some $a_i \neq 0$, where $i \geq 2$, then by the reciprocity theorem $(Y_i^*, X) = a_i \neq 0$ and therefore $Y_i^* = X$ since Y_i^* and X are both irreducible. If $a_i = 0$ for all $i \geq 2$, then $X/M = a_1 Y_1$ and $X(x) = a_1 = X(e)$ for all $x \in M$. Hence $M \subseteq \mathrm{Ker}(X)$. //

4.3 <u>Corollary</u>. If G is Frobenius with Frobenius kernel M, H is a complement of M, X is an irreducible character of G, $M \subseteq \mathrm{Ker}\, X$ and Y is the character of the regular representation of H, then $X/H = nY$.

<u>Proof</u>:

By the theorem there is an irreducible character U of M such that $X = U^*$. We note that $U^*(x) = \sum_{h \in H} \tilde{U}(h^{-1}xh) = 0$ for $x \in H^{\#}$ and also that $U^*(e) = \sum_{g \in H} \tilde{U}(g^{-1}eg) = \sum_{g \in H} \tilde{U}(e)$ $= \sum_{g \in H} U(e) = o(H)U(e) = U(e)Y(e)$. Thus,

$$X(y) = U^*(y) = \begin{cases} U(e)Y(e) & \text{if } y = e \text{ and} \\ 0 & \text{if } y \in H^{\#}. \end{cases}$$

Therefore, $X/H = nY$ where $n = U(e)$ is a positive integer. //

4.4 <u>Lemma</u>. If G is a noncyclic group of order pq and p and q are primes with $p < q$, then G is a Frobenius group, with Frobenius kernel $M \in Syl_q(G)$ and a complement $H \in Syl_p(G)$.

<u>Proof</u>:

Let M be a Sylow q-subgroup of G. Since the number of Sylow q-subgroups of G is the form $1 + kq$, with $k > 0$, which must divide p, M is the only Sylow q-subgroup of G and hence is normal in G. Let x be in $M^{\#}$. Since M is of order q and is thus abelian, $M \subseteq C_G(x)$. If $C_G(x) = G$, then x and hence M is contained in $Z(G)$ in which case G/M is cyclic with $M \subseteq Z(G)$ which is impossible in a non-abelian group. So $M = C_G(x)$ and therefore G is Frobenius with kernel M. Also, a complement H of M in G is of order $[G : M] = p$ and thus is in $Syl_p(G)$. //

4.5 <u>Theorem</u>. If G is Frobenius with Frobenius kernel M, H is a complement of M, and p and q are distinct primes, then any subgroup of H of order p^2 or pq is cyclic.

<u>Proof</u>:

Suppose G is a counterexample of minimal order. Then $o(H) = p^2$ or pq and M is a minimal normal subgroup of G (see Theorem 2.11). If $o(H) = pq$, then since H is not cyclic, it is Frobenius (see Lemma 4.4) which contradicts Theorem 3.3. Hence H is an abelian noncyclic group of order p^2. As in the proof of Theorem 3.1, M is an elementary abelian r-group, where r is a prime. Again as in Theorem 3.1 there is an irreducible representation A of H over J_r the field with r elements) with M as the J_r-space. By Theorems 1.14 and 1.15 there is a finite extension field F of J_r over which A is the direct sum of one dimensional representations. Therefore each matrix A(h) is a diagonal matrix over F.

Since each element h of H has order dividing p, the entries of A(h) on the diagonal are p^{th} roots of unity in F. Since there are at most p p^{th} roots of unity in F and $o(H) = p^2$, there are distinct elements $x, y \in H$ with the top row of A(x) and A(y) the same. Thus the diagonal matrix $A(xy^{-1})$ over F has 1 in the top row, and hence 1 is a characteristic value. Therefore the linear representation $A(xy^{-1})$ over J_r has characteristic value 1. In other words, xy^{-1} in $H^{\#}$ centralizes an element of $M^{\#}$, a contradiction. //

4.6 <u>Theorem</u>. If G is Frobenius with kernel M and complement H and $E < P \in Syl_p(H)$, then

(i) $p \neq 2$ implies P is cyclic and

(ii) $p = 2$ implies P is cyclic or a generalized quarternion.

Proof:

By Theorem 4.5 P contains no noncyclic subgroup of order p^2. There is a subgroup K contained in the center of P with $o(K) = p$. If p contains another subgroup L of order p, then KL is a noncyclic group of order p^2, a contradiction. Hence P has just one subgroup of order p. Therefore, by a classic result P is cyclic or is a generalized quarternion. //

4.7 Theorem. If G is a Frobenius group with Frobenius kernel M and $[G : M]$ is even, then M is abelian.

Proof:

Since a complement H of M has an element of order 2, there is an automorphism J of M of order 2. Since no element in $H^{\#}$ centralizes an element of $M^{\#}$, J has no fixed points but e. Suppose $h_1^{-1}(h_1J) = h_2^{-1}(h_2J)$ for h_1, h_2 in M. Then, $h_2h_1^{-1} = (h_2h_1^{-1})J$. Thus $h_1 = h_2$. Therefore, there are $o(M)$ distinct elements of the form $h^{-1}(hJ)$ in M, and so for $g \in M$, $g = h^{-1}(hJ)$ for some $h \in M$. Thus $gJ = (h^{-1}(hJ))J = (h^{-1}J)h = g^{-1}$. Therefore, for $g_1, g_2 \in M$,

$$\begin{aligned} g_1g_2 &= (g_2^{-1}g_1^{-1})J \\ &= (g_2^{-1}J)(g_1^{-1}J) \\ &= g_2g_1 . \quad // \end{aligned}$$

<u>Appendix</u>: Notation

A B	A implies B
(BWOC)	By way of contradiction
$C_G(H)$, $C(H)$	Centralizer of H in G
Ch	Character of a permutation or a group of permutations
char(F)	Characteristic of the field F
[G : H]	Index of H in G
gcd(m,n)	Greatest common divisor of m and n
$H \triangle G$	H is a normal subgroup of G
$\mathrm{Hall}_p(G)$	Set of Hall P-subgroups of G
Hall(G)	Set of Hall P-subgroups for some set P of primes
$L_F(V)$	Ring of linear transformations of the vector space V over the field F
$M_{n \times n}(F)$	Ring of $n \times n$ matrices over F
$N_G(H)$, $N(H)$	Normalizer of H in G
o(G)	Order of G
$R/\wedge G$	Map R restricted to G
$\mathrm{Syl}_p(G)$	Set of Sylow p-subgroups of G
Syl(G)	Set of Sylow subgroups of G
Tr	Trace
U	Disjoint union
Wlog	Without loss of generality
Z	Ring of integers
Z(G)	Center of F

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VITA

Neil S. Wetcher, son of Mr. and Mrs. Harry Wetcher, was born on December 7, 1945 in Brooklyn, New York. He attended Brooklyn Technical High School from 1959 to 1963 and then proceeded to receive his Bachelor of Science degree from Brooklyn College, New York City, in 1967. In February 1966 he became a charter member of Pi Mu Epsilon, New York Gamma Chapter. He was married to Miss Judith A. Asen in New York on November 22, 1967. Presently, Mr. Wetcher is employed by the Bethlehem Area School District and is completing his requirements for a Master of Science in mathematics at Lehigh University.