1972

Interaction of a non-linear gravity wave with shear flows containing a vortex layer

Kai-Nan An
Lehigh University

INTERACTION OF A NON-LINEAR GRAVITY WAVE WITH SHEAR FLOWS CONTAINING A VORTEX LAYER

by

Kai-Nan An

Abstract

The interaction of a large amplitude progressing wave with a piecewise linear shear flow is described. It is assumed that a gravity wave propagates over a horizontal bed into a region of uniform depth. In the undisturbed region ahead of the wave the velocity is at rest for a fixed distance from the bottom, then increases rapidly in a thin layer of constant but large vorticity to a uniform value which it retains up to the free surface.

No restriction is placed on the magnitude of the disturbance, so the governing equations are non-linear. The only assumptions are those of classical shallow water theory, i.e., that the fluid is inviscid and that the hydrostatic pressure law may be used.

INTERACTION OF A NON-LINEAR GRAVITY WAVE WITH SHEAR FLOWS CONTAINING A VORTEX LAYER

by

Kai-Nan An

A Thesis

Presented to the Graduate Committee

of Lehigh University

in Candidacy for the Degree of

Master of Science in Mechanics

Lehigh University

1972

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

September 11, 1972
(date)

Eric P. Salathé
Professor in Charge

Frederick F. Ling [signature]
Chairman of the Department

Acknowledgment

I am very grateful to Professor E. P. Salathe who directed this work and to Professor E. Varley for his help. I would also like to thank Mrs. Fern Sotzing for her assistance with the preparation of the manuscript.

The results presented in this thesis were obtained in the course of research sponsored by Department of Defense project THEMIS under Contract No. DAAD-69-C-0053 and monitored by the Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland.

Table of Contents

List of Figures

Abstract

The interaction of a large amplitude progressing wave with a piecewise linear shear flow is described. It is assumed that a gravity wave propagates over a horizontal bed into a region of uniform depth. In the undisturbed region ahead of the wave the velocity is at rest for a fixed distance from the bottom, then increases rapidly in a thin layer of constant but large vorticity to a uniform value which it retains up to the free surface.

No restriction is placed on the magnitude of the disturbance, so the governing equations are non-linear. The only assumptions are those of classical shallow water theory, i.e., that the fluid is inviscid and that the hydrostatic pressure law may be used.

I. Introduction

In a recent report, Blythe, Kazakia and Varley [1] have shown how to analyze the behavior of gravity waves propagating into a region where the flow is sheared in a vertical direction. No restriction was placed on the amplitude of the wave, the only assumptions being those of classical shallow water theory. That is, the fluid is taken to be inviscid, and the pressure is given by the hydrostatic pressure law.

The theory was applied by the authors to a number of specific problems involving ambient shear profiles of a simple nature in order to illustrate the special features present in these flows. In this paper the method is applied to a more complex shear profile, chosen because it models certain flows of interest which occur in the atmosphere.

It is assumed that a gravity wave is propagating over a horizontal bed into a region of uniform depth. In the undisturbed region ahead of the wave the velocity is at rest for fixed distance from the bottom, then increases rapidly in a thin layer of constant but large vorticity, to a uniform value which it retains up to the free surface. This shear profile is illustrated in figure (1).

The flow which results when the wave interacts with this shear profile should provide a qualitative description of a pressure front in the atmosphere propagating into a region where two masses of air moving at uniform but different velocities are separated by a thin layer of high vorticity.

In the next section the theory of Blythe, Kazakia and Varley will be briefly outlined for the two dimensional case only. For the details the original report should be consulted.

2. Theory of gravity waves on shear flows

We consider the two dimensional motion of fluid bounded from below by the plane $y=0$ and above by the free surface $y=H(x)$, where (x,y) are the axes of a Cartesian coordinate system, and $H(x)$ is the depth of the fluid. The basic assumption of the theory is that the fluid momentum in the y direction can be neglected so that the fluid pressure is given by the hydrostatic pressure law. This is the well-known shallow water theory of water waves.

The equations governing the motion are the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \qquad (2.1)$$

the momentum equation in the horizontal plane

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{Du}{Dt} \qquad (2.2)$$

and the pressure law

$$p = p_0 + \rho g(H-y) \qquad (2.3)$$

where the pressure p_0 at the free surface is assumed constant. Here u and v are the components of fluid velocity in the x and y directions respectively, ρ is the fluid density, assumed constant, and g is the acceleration due to gravity. These equations must be solved subject to the boundary conditions that

$$v = 0 \quad \text{on} \quad y = 0$$
$$v = \frac{DH}{Dt} \quad \text{on} \quad y = H \, . \qquad (2.4)$$

We are interested in describing the behavior of a large amplitude plane wave propagation in a direction of increasing x into a region of constant depth in which there is a specified shear flow. We therefore take as initial conditions that

$$\text{at} \quad t = 0 \, , \qquad x > 0 : \quad H = H_0 \, ,$$
$$u = u_0(y) \qquad (2.5)$$
$$x < 0 : \quad H = \tilde{H}(x) \, .$$

The behavior of the wave is therefore described in terms of the ambient shear flow and the elevation of the free surface at some initial time.

The essential feature in the solution of this problem is the realization that there exist, as natural extensions to the simple waves of the classical theory, similarity solutions which describe progressing waves. In terms of the independent variables (x,z,t), where z is defined by

$$y = zH \qquad (2.6)$$

and the dependent variables (u,w,H), where w is defined by

$$w = \frac{Dz}{Dt} \tag{2.7}$$

these solutions have the forms

$$\begin{aligned} u &= U(H,z) \\ w &= \frac{\partial H}{\partial x} W(H,z) \quad . \end{aligned} \tag{2.8}$$

The elevation of the free surface, $H(x,t)$, is determined from the initial profile $\tilde{H}(x)$ by the simple wave relation

$$\begin{aligned} H &= \tilde{H}(\alpha) \\ x &= \alpha + c(H)t \quad . \end{aligned} \tag{2.9}$$

Substituting these similarity forms (2.8) into the governing equations yields a set of relations for the determination of the wave speed $c(H)$ and the fluid velocities $U(H,z)$, $W(H,z)$ in terms of the ambient shear flow. Using as independent variables (H,ψ) instead of (H,z), where ψ is constant at a particle and $\psi = y/H_0$ when $H=H_0$, the horizontal component of fluid velocity relative to the local wave speed,

$$\bar{U}(H,\psi) = u - c(H) \tag{2.10}$$

is related to $c(H)$ by the equation

$$\bar{U}\frac{\partial \bar{U}}{\partial H} + \bar{U}c'(H) + g = 0 \tag{2.11}$$

and the condition that at fixed H

$$\int_0^1 \frac{\partial \bar{U}}{\partial \psi} \frac{d\psi}{u_0'(\psi)} = \frac{H}{H_0} . \tag{2.12}$$

Equations (2.11) and (2.12) are solved subject to the boundary condition that

$$\text{when } H = H_0 : \quad \bar{U} = u_0(\psi) - c_0 \tag{2.13}$$

where $c_0 = c(H_0)$ is determined from

$$gH_0 \int_0^1 \frac{d\psi}{[u_0(\psi) - c_0]^2} = 1 . \tag{2.14}$$

When $\bar{U}(H,\psi)$ and $c(H)$ have been determined, the vertical height of the particle ψ at a point where the height of the free surface is H is given by

$$y = H_0 \int_0^\psi \frac{\partial \bar{U}}{\partial \psi} \frac{d\psi}{u_0'(\psi)} \tag{2.15}$$

and the vertical component of fluid velocity is given by

$$v = gH_0 \frac{\partial H}{\partial x} \bar{U} \int_0^\psi \bar{U}^{-2} \frac{\partial \bar{U}}{\partial \psi} \frac{d\psi}{u_0'(\psi)} \tag{2.16}$$

For the details of the derivation of these results the reader is referred to [1].

3. Interaction of a progressing wave with a shear flow containing a thin vortex layer

We turn now to the solution of the equations obtained in the last section for the case when the ambient shear flow is given by

$$u_0(\psi) = \begin{cases} 0 & 0 \le \psi \le r \\ \frac{1}{\varepsilon}(\psi - r) & r \le \psi \le r+\varepsilon \\ 1 & r+\varepsilon \le \psi \le 1 \end{cases} \quad . \tag{3.1}$$

In (3.1) and in what follows all velocities are regarded as normalized with respect to U_A^* , the velocity at the free surface in the undisturbed shear flow. Also, all distances are normalized with respect to H_0 , the height of free surface in the undisturbed region.

When the ambient shear profile (3.1) is used in equation (2.14), we obtain for the wave speed at the front

$$F^2 = \frac{c_0^2 - 2c_0 r - c_0 \varepsilon + r}{c_0^2 (1-c_0)^2} \tag{3.2}$$

where

$$F = \frac{U_A^*}{(gH_0)^{\frac{1}{2}}}$$

is the Froude number of the shear flow.

In order to apply the formulae obtained in section 2, we assume that the ambient velocity profile has the form

$$u_0(\psi) = \begin{cases} \varepsilon_1 \psi & 0 \le \psi \le r \\ \varepsilon_1 r + \frac{1}{\varepsilon}(\psi - r) & r \le \psi \le r+\varepsilon \\ \varepsilon_1 r + 1 + \varepsilon_2(\psi - r - \varepsilon) & r+\varepsilon \le \psi \le 1 \end{cases} \quad (3.3)$$

and then take the limit $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$. It can be seen immediately from (2.15) that if the ambient shear profile is piecewise linear, so that $u_0'(\psi)$ is piecewise constant, the horizontal velocity profile in the wave will also be piecewise linear. It also follows that the interfaces which separates these different regions will be stream surfaces. In this case they correspond to $\psi = r$ and $\psi = r+\varepsilon$.

If $y = I_1(H)$ and $y = I_2(H)$ denote the height of the interfaces at a location where the free surface elevation is H, and if $\bar{U}_A(H)$, $\bar{U}_B(H)$, $\bar{U}_{I_1}(H)$ and $\bar{U}_{I_2}(H)$ denote the values of $\bar{U}$ at the free surface, the bottom, and the two interfaces, respectively, then it follows immediately from equation (2.15) that

$$\bar{U}(y,H) = \begin{cases} \bar{U}_B + \varepsilon_1 y & 0 \le y \le I_1 \\ \bar{U}_{I_1} + \frac{y}{\varepsilon} - \frac{1}{\varepsilon\varepsilon_1}[\bar{U}_{I_1} - \bar{U}_B] & I_1 \le y \le I_2 \\ \bar{U}_{I_2}(1-\varepsilon\varepsilon_2) + \varepsilon_2 y + \varepsilon\varepsilon_2 \bar{U}_{I_1} - \frac{\varepsilon_2}{\varepsilon_1}[\bar{U}_{I_1} - \bar{U}_B] & I_2 \le y \le H \end{cases} \quad (3.4)$$

Since $\bar{U}_A$, $\bar{U}_B$, $\bar{U}_{I_1}$ and $\bar{U}_{I_2}$ are obtained from $\bar{U}(H,\psi)$ for specific choices of ψ, they each satisfy equation (2.11)

$$\bar{U}_j \frac{d\bar{U}_j}{dH} + \bar{U}_j \frac{dc}{dH} + \frac{1}{F^2} = 0 \qquad (3.5)$$

where $\bar{U}_j$ is any one of the four velocities. Additional relations are obtained by noting that

$$\text{at} \quad y = I_1 : \quad \bar{U}_B + \varepsilon_1 I_1 = \bar{U}_{I_1} \qquad (3.6)$$

$$\text{at} \quad y = I_2 : \quad \bar{U}_{I_2} = \bar{U}_{I_1} + \frac{I_2}{\varepsilon} - \frac{1}{\varepsilon\varepsilon_1}[\bar{U}_{I_1} - \bar{U}_B] \qquad (3.7)$$

and

$$\text{at} \quad y = H : \quad \bar{U}_A = \bar{U}_{I_2} - \varepsilon_2 (I_2 - H) \quad . \qquad (3.8)$$

Equations (3.5)-(3.8) represent seven equations for the variation of $\bar{U}_A$, $\bar{U}_B$, $\bar{U}_{I_1}$, $\bar{U}_{I_2}$, I_1, I_2 and c as functions of H .

In the limit $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ we obtain from (3.6)

$$\bar{U}_B = \bar{U}_{I_1} \qquad (3.9)$$

$$\frac{\bar{U}_{I_1} - \bar{U}_B}{\varepsilon} = I_1 \qquad (3.10)$$

from (3.8)

$$\bar{U}_A = \bar{U}_{I_2} \qquad (3.11)$$

$$\frac{\bar{U}_{I_2} - \bar{U}_A}{\varepsilon_2} = I_2 - H \qquad (3.12)$$

and from (3.7)

$$\bar{U}_{I_2} - \bar{U}_{I_1} = \frac{1}{\varepsilon}(I_2 - I_1) \qquad (3.13)$$

Substracting the equations (3.5) for $\bar{U}_A$ and $\bar{U}_{I_2}$, and similarly these equations for $\bar{U}_{I_1}$ and $\bar{U}_B$, and using (3.10) and (3.12) respectively, yields

$$\frac{dI_1}{dH} - \frac{I_1}{F^2\bar{U}_B^2} = 0 \tag{3.14}$$

$$\frac{dI_2}{dH} + \frac{H-I_2}{F^2\bar{U}_A^2} = 1 . \tag{3.15}$$

Equation (3.14) and (3.15) together with equations (3.5) for $\bar{U}_A$ and $\bar{U}_B$, i.e.,

$$\frac{d\bar{U}_A}{dH} + \frac{dc}{dH} + \frac{1}{F^2\bar{U}_A} = 0 \tag{3.16}$$

$$\frac{d\bar{U}_B}{dH} + \frac{dc}{dH} + \frac{1}{F^2\bar{U}_B} = 0 \tag{3.17}$$

and (3.13), which becomes

$$\bar{U}_A - \bar{U}_B = \frac{1}{\varepsilon}(I_2 - I_1) \tag{3.18}$$

form a complete set for the determination of

$$\bar{U}_A, \bar{U}_B, I_1, I_2 \text{ and } c .$$

Once these unknown functions are determined, equation (3.4) gives $\bar{U}(x_3,H)$. In the limit $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, these reduce to

$$\bar{U}(y,H) = \begin{cases} \bar{U}_B & 0 \le y \le I_1 \\ \bar{U}_B + \frac{1}{\varepsilon}(y - I_1) & I_1 \le y \le I_2 \\ \bar{U}_A & I_2 \le y \le H \end{cases} \quad . \tag{3.19}$$

The vertical components of fluid velocity is obtained from (2.16), which can be rewritten in the form (see [1])

$$v = \frac{1}{F^2} \left(\frac{\partial H}{\partial x}\right) (\bar{U}) \int_0^y \frac{d\tilde{y}}{\bar{U}^2} \tag{3.20}$$

and the expression (3.19) for $\bar{U}$. The result, in normalized variables, is

$$F^2 v / \left(\frac{\partial H}{\partial x}\right) = \begin{cases} \dfrac{y}{\bar{U}_B} & 0 \le y \le I_1 \\ \dfrac{I_1 (y - I_1)/\varepsilon + \bar{U}_B y}{\bar{U}_B^2} & I_1 \le y \le I_2 \\ \dfrac{I_1 (\bar{U}_A^2 - \bar{U}_B^2) + y \bar{U}_B^2 + \varepsilon (\bar{U}_A - \bar{U}_B)^2 \bar{U}_B}{\bar{U}_A \bar{U}_B^2} & I_2 \le y \le H \end{cases} \quad . \tag{3.21}$$

The variation of x_3 with H at a particle is given by the expression (see [1])

$$\frac{Dy}{DH} = \frac{v}{\bar{U}} / \left(\frac{\partial H}{\partial x}\right) \quad . \tag{3.22}$$

so that we obtain

$$\frac{Dy}{DH} = \begin{cases} \dfrac{y}{F^2\bar{U}_B^2} & 0\le y\le I_1 \\ \dfrac{I_1(y-I_1)/\varepsilon+\bar{U}_B y}{F^2\bar{U}_B^2(\bar{U}_B+\frac{1}{\varepsilon}(x_3-I_1))} & I_1\le y\le I_2 \\ \dfrac{I_1(\bar{U}_A^2-\bar{U}_B^2)+y\bar{U}_B^2+\varepsilon(\bar{U}_A-\bar{U}_B)^2\bar{U}_B}{F^2\bar{U}_A^2\bar{U}_B^2} & I_2\le y\le H \end{cases} \qquad (3.23)$$

At a stream line, however, viewed from a reference frame moving with respect to the front,

$$\frac{dy}{dH} = \frac{v}{u-c_0} \Big/ \left(\frac{\partial H}{\partial x}\right)$$

$$= \begin{cases} \dfrac{y}{F^2\bar{U}_B(\bar{U}_B+c-c_0)} & 0\le y\le I_1 \\ \dfrac{I_1(y-I_1)/\varepsilon+\bar{U}_B y}{F^2\bar{U}_B^2[\bar{U}_B+\frac{1}{\varepsilon}(y-I_1)+c-c_0]} & I_1\le y\le I_2 \\ \dfrac{I_1(\bar{U}_A^2-\bar{U}_B^2)+y\bar{U}_B^2+\varepsilon(\bar{U}_A-\bar{U}_B)^2\bar{U}_B}{F^2\bar{U}_A\bar{U}_B^2(\bar{U}_A+c-c_0)} & I_2\le y\le H \end{cases} \qquad (3.24)$$

4. Solution for $\varepsilon\to 0$

The equations obtained above can be solved in the limit as $\varepsilon\to 0$ by expanding the variables according to

$$
\begin{aligned}
U_A &= U_A^{(0)} + \varepsilon U_A^{(1)} + \dots \\
U_B &= U_B^{(0)} + \varepsilon U_B^{(1)} + \dots \\
I_1 &= I_1^{(0)} + \varepsilon I_1^{(1)} + \dots \\
I_2 &= I_2^{(0)} + \varepsilon I_2^{(1)} + \dots \\
c &= c^{(0)} + \varepsilon c^{(1)} + \dots \quad .
\end{aligned}
$$

From equation (3.13) we see that $I_1^{(0)}=I_2^{(0)}=I$. Dropping the superscript zero, we obtain for the lowest order terms the equations

$$\frac{d\bar{U}_A}{dH} + \frac{dc}{dH} + \frac{1}{F^2\bar{U}_A} = 0 \tag{4.1}$$

$$\frac{d\bar{U}_B}{dH} + \frac{dc}{dH} + \frac{1}{F^2\bar{U}_B} = 0 \tag{4.2}$$

$$\frac{dI}{dH} - \frac{I}{F^2\bar{U}_B^2} = 0 \tag{4.3}$$

$$\frac{dI}{dH} + \frac{H-I}{F^2\bar{U}_A^2} = 1 \ . \tag{4.4}$$

Substituting these expansions into (3.19), (3.21), (3.23) and (3.24) gives, to lowest order,

$$\bar{U}(y,H) = \begin{cases} \bar{U}_B & 0 \le y \le I_1 \\ \bar{U}_B + \frac{1}{\varepsilon}(y-I_1) & 0 \le \frac{1}{\varepsilon}(y-I_1) \le \bar{U}_A - \bar{U}_B \\ \bar{U}_A & I_2 \le y \le H \end{cases} \tag{4.5}$$

$$F^2 v \Big/ \frac{\partial H}{\partial x} = \begin{cases} y/\bar{U}_B & 0 \le y \le I_1 \\ \frac{I_1}{\bar{U}_B} + \frac{1}{\varepsilon}(y-I_1)\frac{I_1}{\bar{U}_B^2} & 0 \le \frac{1}{\varepsilon}(y-I_1) \le \bar{U}_A - \bar{U}_B \\ \frac{I_1(\bar{U}_A^2-\bar{U}_B^2)+y\bar{U}_B^2}{\bar{U}_A\bar{U}_B^2} & I_2 \le y \le H \end{cases} \tag{4.6}$$

$$\frac{Dy}{DH} = \begin{cases} \frac{y}{F^2\bar{U}_B^2} & 0 \le y \le I_1 \\ \frac{I_1\frac{y-I_1}{\varepsilon}+\bar{U}_B I_1}{F^2\bar{U}_B^2(\bar{U}_B+\frac{y-I_1}{\varepsilon})} & 0 \le \frac{y-I_1}{\varepsilon} \le \bar{U}_A - \bar{U}_B \\ \frac{I_1(\bar{U}_A^2-\bar{U}_B^2)+y\bar{U}_B^2}{F^2\bar{U}_A^2\bar{U}_B^2} & I_2 \le y \le H \end{cases} \tag{4.7}$$

and

$$
\frac{dy}{dH} = \begin{cases} \dfrac{y}{F^2\bar{U}_B(\bar{U}_B+c-c_0)} & 0 \le y \le I_1 \\[2ex] \dfrac{I_1\dfrac{(y-I_1)}{\varepsilon}+\bar{U}_B I_1}{F^2\bar{U}_B^2[\bar{U}_B+\frac{y-I_1}{\varepsilon}+c-c_0]} & 0 \le \dfrac{y-I_1}{\varepsilon} \le \bar{U}_A - \bar{U}_B \\[2ex] \dfrac{I_1(\bar{U}_A^2-\bar{U}_B^2)+y\bar{U}_B^2}{F^2\bar{U}_A\bar{U}_B^2(\bar{U}_A+c-c_0)} & I_2 \le y \le H \end{cases} \tag{4.8}
$$

Clearly, the solution in the upper and lower region, which are independent of ε to lowest order, can be regarded as describing the flow for an initially discontinuous shear profile. The solution in the center region, whose thickness is given to lowest order by

$$
I_2 - I_1 = \varepsilon(\bar{U}_A-\bar{U}_B) \tag{4.9}
$$

provides the transition between these two layers when the shear profile is given as in (3.1).

Eliminating $\frac{dI}{dH}$ from equations (4.3) and (4.4) gives

$$
I = \frac{F^2\bar{U}_A^2-H}{\bar{U}_A^2-\bar{U}_B^2}\,\bar{U}_B^2 \quad . \tag{4.10}
$$

Similarly, equations (4.3) and (4.4) can be solved for $\frac{dI}{dH}$. Equating this to the derivative of (4.10) yields an equation for $\bar{U}_A$ and $\bar{U}_B$ alone. A second equation for $\bar{U}_A$ and $\bar{U}_B$ is obtained by eliminating c from (4.1) and (4.2). Defining

$$\bar{U}_A = \frac{A\sqrt{H}}{F}$$

$$\bar{U}_B = \frac{B\sqrt{H}}{F} \qquad (4.11)$$

we obtain

$$\frac{dA}{d\ln H} = \frac{-A^2B(A^2-AB+B^2-1)-(3A-B)(A^2+B^2-1)+2AB^2}{(2AB)(A^2-AB+B^2-1)} \qquad (4.12)$$

$$\frac{dB}{d\ln H} = \frac{-AB^2(A^2-AB+B^2-1)+(A-B)(A^2+B^2-1)+2B(1-B^2)}{(2AB)(A^2-AB+B^2-1)} \qquad (4.13)$$

Dividing these two equations gives a first order non-linear differential equation for A and B. Introducing the parameter τ by

$$A = B\tau$$

results in a simplification of the algebra, and we obtain

$$\frac{dB}{d\tau} = \frac{B^5(\tau^3-\tau^2+\tau)-B^3(\tau^3-\tau^2+2\tau-3)+(\tau-3)B}{B^2(\tau^2-1)(\tau+1)^2-(\tau^2-1)} . \qquad (4.14)$$

The integral curves of (4.14) are shown in figure (2). There are fourteen singular points, located at

$$\begin{cases}B=0\\ \tau=1,\end{cases}\quad \begin{cases}B=0\\ \tau=-1,\end{cases}\quad \begin{cases}B=1\\ \tau=1,\end{cases}\quad \begin{cases}B=-1\\ \tau=1\ ,\end{cases}\quad \begin{cases}B=\sqrt{4/3}\\ \tau=-1\ ,\end{cases}\quad \begin{cases}B=-\sqrt{4/3}\\ \tau=-1\ ,\end{cases}\quad \begin{cases}B=1\\ \tau=-1,\end{cases}$$

$$\begin{cases}B=-1\\ \tau=-1,\end{cases}\quad \begin{cases}B=1\\ \tau=0,\end{cases}\quad \begin{cases}B=-1\\ \tau=0\ ,\end{cases}\quad \begin{cases}B=2\\ \tau=-\frac{1}{2},\end{cases}\quad \begin{cases}B=-2\\ \tau=-\frac{1}{2},\end{cases}\quad \begin{cases}B=1\\ \tau=-2,\end{cases}\quad \begin{cases}B=-1\\ \tau=-2\end{cases}\ .$$

In terms of B and τ, equation (4.10) for I is

$$\frac{I}{H} = \frac{\tau^2 B^2 - 1}{\tau^2 - 1}\ . \qquad (4.15)$$

From the requirement that the interface lies between the bottom and the free surface, that is $0 \leq \frac{I}{H} \leq 1$, it can be seen from (4.15) that the region of physical interest in the B-τ plane is bounded by the curves $B=\pm 1$ and $B\tau=\pm 1$. These are indicated in the figure by the dot-dash line.

It is readily seen that equation (4.10) for I and (3.2) for the wave speed c_0 in the ambient state are equivalent. Choosing some point (A_0, B_0) in the acceptable portion of the phase plane as the reference state, so that $H=1$ there and $\bar{U}_A = A^*/F = 1-c_0$, $\bar{U}_B = -B^*/F = -c_0$ gives for this flow $F = A^*+B^*$ and $c_0 = B^*/(A^*+B^*)$. Equation (4.10) then becomes

$$I = \frac{F^2 c_0^2 (1-c_0)^2 - c_0^2}{1-2c_0}$$

which is identical with (3.2) since $I=r$ in the undisturbed region.

The acceptable portion of the phase plane is divided into six regions, as shown in the figure. Regions I and II, in the first quadrant, correspond to what is termed the slow wave, because the wave speed c lies between U_A and U_B, i.e. $\bar{U}_B=-\sqrt{H}B/F_0<0$ and $\bar{U}_A=\sqrt{H}A/F_0>0$. Similarly, the second quadrant, regions III and IV, corresponds to fast waves, $U_B<c$, $U_A<c$, and the third quadrant, regions V and VI, corresponds to backward waves, because $c<0$ when viewed in a reference frame for which $U_B=0$ (i.e. $\bar{U}_A>0$, $\bar{U}_B>0$).

Typical flow profiles in each of these six regions are sketched in figure (3), as well as the profile which characterizes the transition from I to II (i.e. the singular point ($\tau=1, B=1$). As the singular point which separates regions III and IV is approached, $\bar{U}_A/\bar{U}_B \to 1$. However it will be shown that $H \to 0$ at this singular point, so it also follows that

$$\bar{U}_A \to 0 , \quad \bar{U}_B \to 0 .$$

The same is true in region V and VI as the singular point ($\tau=-1$, $B=-1$) is approached. We exclude regions IV and VI because they correspond to states in which $U_A-U_B<0$, and only flows for which the undisturbed shear profile satisfies $U_A-U_B>0$ are considered (e.g. equation (3.1)). Although one may start from an initial point in region III, say, and follow an integral curve through the singular point $(\tau=-1, B=1)$, into region IV, it has already been pointed out that the singular point corresponds to $H=0$, and so this possibility will not be considered. The same is true for transition from region V into region VI. The entire fourth quadrant is excluded as it also corresponds to profile for which $U_A-U_B<0$.

Flows which correspond to integral curves in the slow wave region contain critical layers. These are curves in the physical plane along which $\bar{U}=0$. All integral curves in I and II can be regarded as originating from the singular point $(\tau=1, B=1)$. Those in region II cross the τ axis, where $\bar{U}_A=U_A-c=0$, and then correspond to fast waves. They converge to the singular point $(B=2, \tau=-\frac{1}{2})$. In region I, it can be shown that along the integral curves $\tau\to\infty$, $B\to 0$ and $A=B\tau$ approaches a constant which depends on the particular curve.

Integral curves in region III, in addition to those crossing $\tau=0$ from region II, originate from $(\tau=-1, B=1)$ and converge to the singular point $(B=2, \tau=-\frac{1}{2})$. The two types of integral curves in region III are separated by a line from the saddle point at $(B=1, \tau=0)$ to the point $(B=2, \tau=-1)$. This curve is shown in the figure.

The value of H at any point is obtained by integrating (4.13) along an integral curve, and is given by

$$\ln \frac{H}{H^*} = \int_{\tau^*}^{\tau} \frac{2B^2\tau(B^2\tau^2-B^2\tau+B^2-1)}{(\tau^2-1)[1-B^2(\tau+1)^2]}\, d\tau \ . \tag{4.16}$$

Here H^* is the height of the free surface at the point (τ^*,B^*) . We may start the integration at any point and regard this as the ambient state, where, because of the normalization, $H^*=1$.

With this schematic picture of the phase plane, the numerical procedure for integrating the equation is clear. We start at one of the singular points, using asymptotic approximations valid in that region, integrate a short distance along the integral curve, and then use one of the standard numerical techniques. Once B has been determined in terms of τ along a given integral curve in this manner H can be obtained from (4.16).

The wave speed $c(H)$ is found from

$$\frac{d\bar{U}_B}{dH} + \frac{dc}{dH} + \frac{1}{F^2}\,\frac{1}{\bar{U}_B^2} = 0 \ .$$

Defining $\hat{c}=F_0 c$, we have

$$\hat{c}(H) = \hat{c}^* = \sqrt{H}B - \sqrt{H^*}B^* - \int_{H^*}^{H} \frac{1}{\sqrt{H}B}\, dH \ . \tag{4.17}$$

Again, the * refers to some reference state which can be arbitrarily chosen as the ambient state with $H^*=1$, and $\hat{c}^*=B^*$.

Slow wave

All integral curves corresponding to the slow waves originate at $(\tau=1, B=1)$. Around this point

$$(B-1) \sim \lambda(\tau-1) \qquad (4.18)$$

where λ is the slope of the integral curve at the singular point and $-1<\lambda<0$. Using (4.18) in the neighborhood of the singular point , with $\tau<1$, and continuing the integration of (4.14) numerically, through its transition into a fast wave and into the singular point $(\tau=-1, B=2)$, we obtain the integral curves for various choices of λ . Similarly, for $\tau>1$, the integral curves in the region I can be obtained.

With B as a known function of τ for various λ , equation (4.16) can be used to determine H along different integral curves. Since the integrand is not defined at $(\tau=1, B=1)$, we use L'Hospital's rule and (4.18) to obtain its value as $-\frac{2}{3}\lambda - \frac{1}{3}$ at that point. The value of the integrand is defined at all other points on the integral curve, so the integration can now be carried out in a straightforward manner. As the integral curves approach $B=2$, $\tau=-\frac{1}{2}$, $H\to\infty$. This can be seen from equation (4.16) and the asymptotic form $B - 2 \sim 4(\tau + \frac{1}{2})$ valid near that point.

In region I, since $B=0(1/\tau)$ as $\tau \to \infty$, it follows from (4.16) that H approaches a finite limiting value. Similarly, (4.17) shows that $\hat{c}$ is bounded in this limit, and from (4.15) $I \to 0$. Therefore, as $\tau \to \infty$ along integral curves in region I, the flow approaches a uniform profile of finite height and velocity, the values of which depend on the integral curve chosen. At the singular point $(\tau=1, B=1)$ I/H can be determined from (4.11) by using the asymptotic form (4.18), and is given by

$$\frac{I}{H} = 1 + \lambda . \tag{4.19}$$

Figure (4) shows the curves $\bar{U}_A(H)$, $\bar{U}_B(H)$ computed in this manner, starting at the singular point $\tau=1$, $B=1$ and integrating for decreasing τ across $\tau=0$, into the fast wave region. At the transtion into a fast wave $H = H_{min} = 0.285$.

<u>Fast wave</u>

All the integral curves in region III except those which come from region II originate at the singular point $(B=1, \tau=-1)$. However, unlike the case of the slow waves, this singular point cannot be used as a reference state since it corresponds to $H=0$. This can readily be seen from (4.16) as the limit $\tau \to -1$, is approached, using the asymptotic expressions $(B-1) \sim \lambda(\tau+1)$, valid near this point. Also, as the singular point $(B=2,\ \tau=-\frac{1}{2})$ is approached, $H \to \infty$, as already pointed out. Therefore, in this region some convenient point away from the singular points should be taken as the reference state, and the integration performed numerically along an integral curve in the direction of one of the singular points. For example, the one parameter family of integral curves intersecting the line $\tau=-\frac{3}{4}$ completely describe the flow in region III.

Backward wave

The situation in region V, for the backward wave is analogous to that of the fast wave. The singular points $(\tau=-1,\ B=-1)$ and $(\tau=-2,\ B=-1)$ correspond to $H=0$ and $H=\infty$ respectively. Convenient reference states can be chosen as the one parameter family of states along the line $\tau=-\frac{3}{2}$, for example.

A more detailed sketch of the phase plane for regions I, II and III is shown in figure (5). The dotted line which runs from the $B=1,\ \tau=0$ through the singular point $B=1,\ \tau=1$, and out to infinity in region I separates regions I and II into a lower and an upper portion. In region I $\partial H/\partial\tau>0$, $\partial c/\partial H<0$ in the lower portion, and

$\partial H/\partial\tau<0$, $\partial c/\partial H>0$ in the upper portion. The reverse is true for region II. Only integral curves for which $\lambda<-\frac{1}{2}$ enter the lower portion of region I, and they all eventually cross over to the upper portion as $\tau\to\infty$. Similarly, only integral curves with $\lambda>-\frac{1}{2}$ enter the lower portion of region II, and these all enter the upper region as τ decreases.

The upper and lower dotted lines drawn from $\tau=-\frac{1}{2}$, $B=2$ in region III are the locus of the points for which $dB/d\tau=0$ and ∞ , respectively, along the integral curves. All the integral curves in region III cross the upper line, and all those originating from $\tau=-1$, $B=1$ cross both lines. Below the lower dotted line, $\partial H/\partial\tau>0$, $\partial c/\partial H<0$, while $\partial H/\partial\tau<0$, $\partial c/\partial H>0$ above this line.

5. Example: the centered expansion wave

As a specific example we consider a centered expansion wave, which satisfy the relation

$$\frac{x}{t} = c(H) \ . \qquad (5.1)$$

Various possible free surface shapes, $H(x)$, for these waves are illustrated in figure (6) corresponding to different integral curves. These curves have been normalized so that $H=1$ corresponds to the singular point $(B=1, \tau=1)$. Therefore, for these flows $F=A^*+B^*=2$, $c_0=B^*/F=0.5$, and $x/t=c=0.5\hat{c}$. The portion of the curves for $x/t<0.5$ correspond to region I and approach the limiting values H_{lim}, C_{lim} as $\tau\to+\infty$. For $\lambda<-0.5$ they initially increase as x/t decreases but reach a maximum corresponds to crossing from the lower to upper portion of region I, and all the curves decrease to their limiting value. The locus of these maxima are shown by the dotted line in the figure. As x/t increase from 0.5 those curves corresponding to $\lambda>-0.5$ initially increase, since they lie in the lower portion of region II. They reach a maximum when they cross into the upper portion, and the locus of these maxima are also shown in the figure. All curves reach a minimum at the transition to a fast wave, and then $H\to\infty$ as $x/t\to\infty$, which corresponds to approaching the singular point $B=2$,

$\tau=-\frac{1}{2}$. The locus of these minima are shown in the figure.

Any point in the acceptable portion of the phase space can serve as an initial state. Figure (7) shows the streamline pattern, as observed from a reference frame moving with respect to the front, when the initial state corresponds to the point ($B=5.1356$, $\tau=0.1$) in the slow wave region. The integration is carried out along the integral curve in the direction of increasing τ . The thin center region is not shown in figure (7), so this flow may also be regarded as due to an initially discontinuous shear profile. The interface height I is represented by the dotted line in the figure. Below the interface, where the fluid velocity is less than the local wave speed, particles move with a negative relative speed with respect to the front, i.e. $U_B-c_0 < 0$. Above the interface, the fluid velocity is greater than the local wave speed. However, the fluid velocity is greater than the front speed $U_A-c_0 > 0$ only in a finite region adjacent to the front. Behind that region, $U_A-c_0 < 0$, and the fluid particles move away from the front.

The flow in the center region is illustrated in figure (8) by means of a modified streamline diagram. The vertical scale is stretched by $\frac{1}{\varepsilon}$ and is taken as $\frac{1}{\varepsilon}(y-I_1)$. Therefore, the lower interface is the horizontal axis of the figure and the upper interface is at $\bar{U}_A-\bar{U}_B$. The lines in the figure give the true direction of the fluid velocity at a given value of x and at a given distance from the lower interface at that position.

The locus of points along which either the horizontal or vertical velocity component vanishes are indicated by broken lines. They intersect at a saddle point in the interior of the region, and divide the flow field into four parts. In the lower part, which is adjacent to the region below the interface I_1 the fluid particles move up and away from the front while in the upper part, which borders on the region above the I_2 interface, the particles move down and toward the front. This part terminates at a finite distance from the front, at the location where the velocity in the region above it is the vertical direction, and the particles then move down and away from the front. This part terminates at a finite distance from the front, at the location where the velocity in the region above it is the vertical direction, and the particles then move down and away from the front. In the remaining region, bordering the front, the particles move up and toward the front.

A final illustration is provided by figure (9), in which the initial state is taken as the point (B=4.79,

$\tau=-0.2$) in the fast wave region. This wave is followed through its transition into a slow wave and through the singular point ($\tau=1$, $B=1$) where it is continued into region I along the integral curve which has the same slope at the singular point, until it reaches its limiting value as $\tau\to\infty$, $B\to 0$. The interface height $I(H)$ is indicated in the figure by the broken line. The transition into a slow wave occurs at $x/t=0.485$ where $I=H$ and the velocity is a uniform horizontal flow. The singular point ($\tau=1$, $B=1$) occurs at $x/t=0.155$ and the maximum H occurs at $x/t=0.0818$ corresponding to $\tau=1.8$, $B=0.642$. The limiting height occurs as $\tau\to\infty$, $B\to 0$, and is $H=0.35$, $x/t=0.05$ at which point $I=0$. Since the streamlines are drawn with respect to an observer moving with respect to the front, all the streamlines move away from the front, even in the slow wave portion above the interface.

Any two sections can be taken to represent an initial and final state and the portion of the figure between them describes the transition between these states. But then of course the reference frame from which this particular figure is drawn becomes artificial.

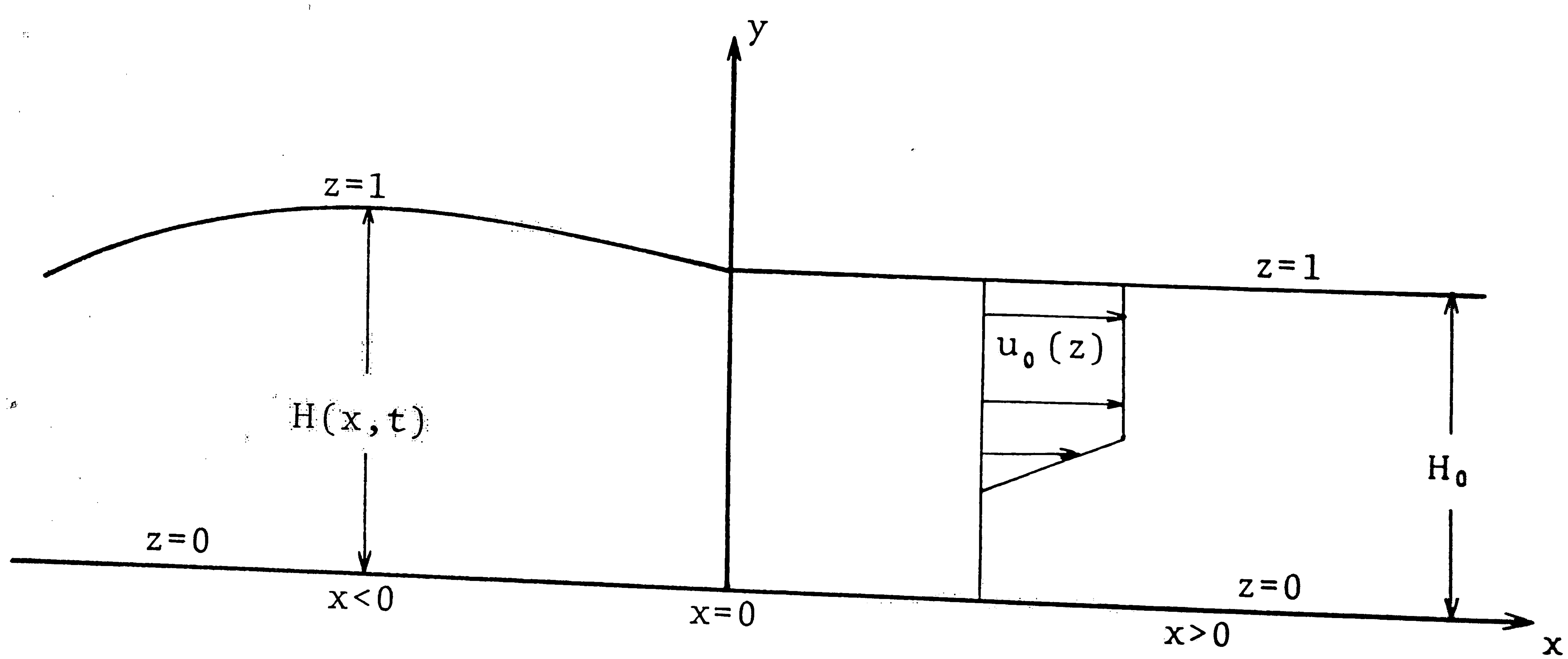

FIGURE 1.

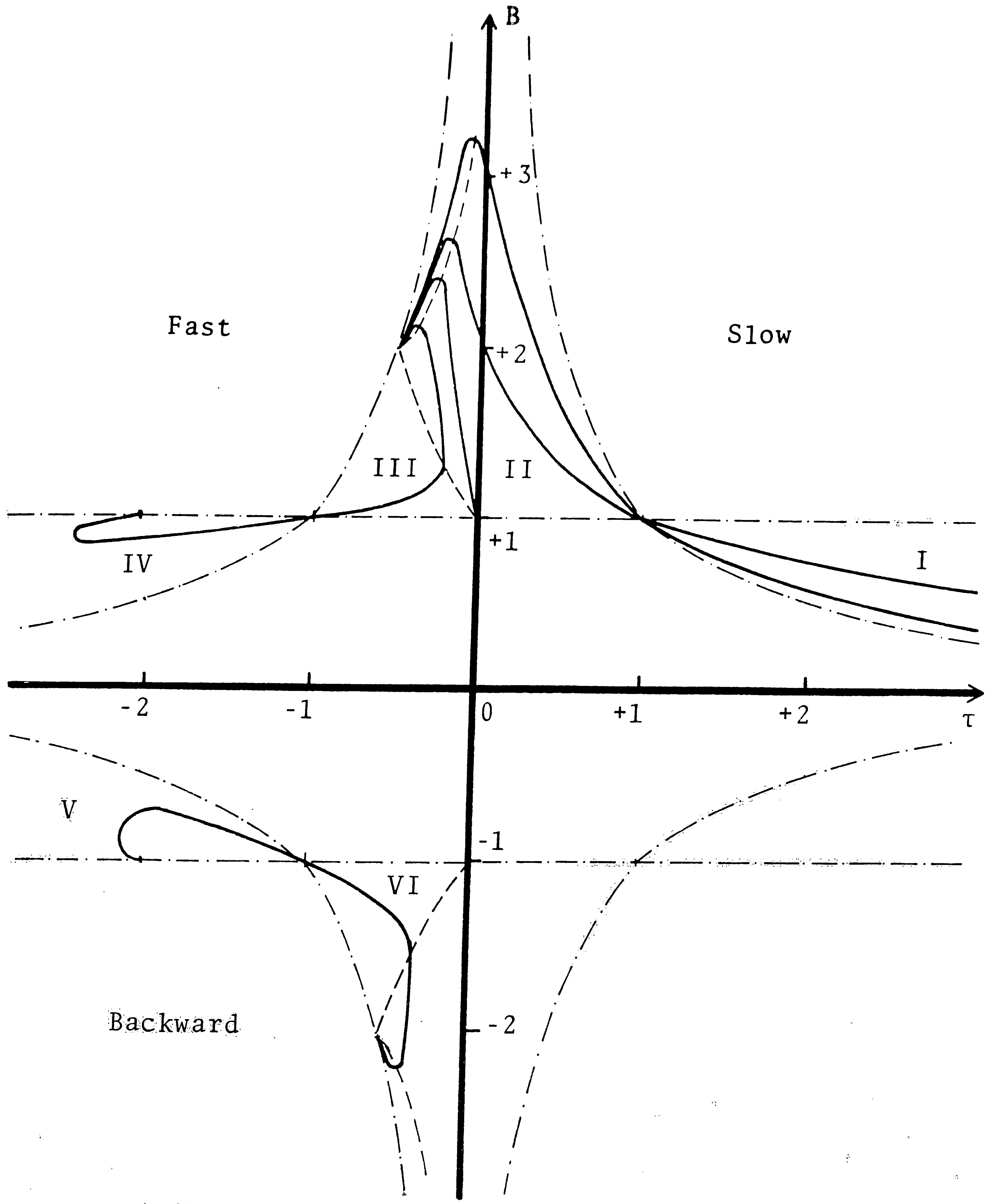

FIGURE 2. Integral curves in the phase plane. The broken lines represent either infinite slope or zero slope of integral curves. The dot-dash lines represent the boundary of acceptable region.

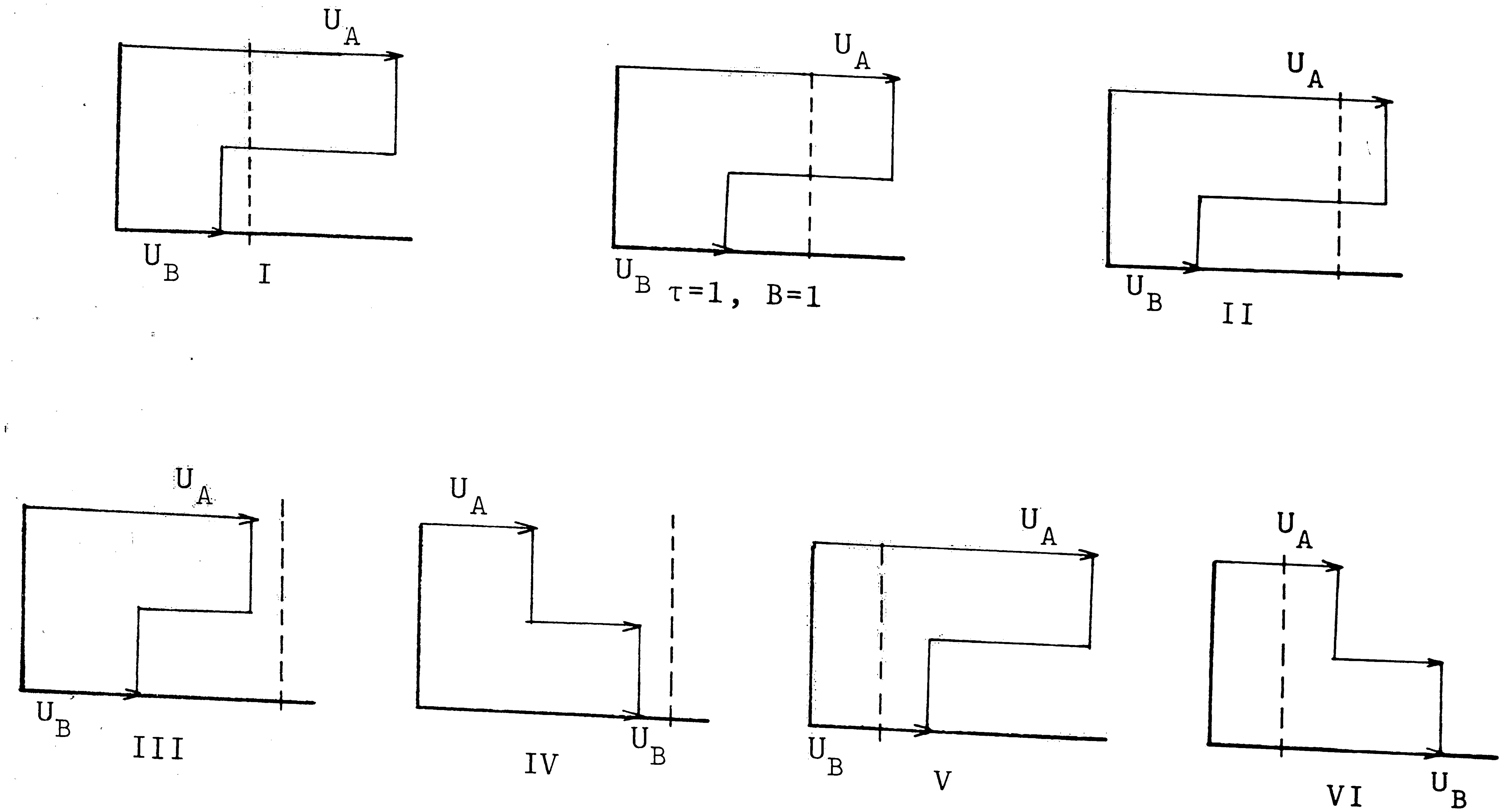

FIGURE 3. Characteristic velocity profiles for different regions and at singular point (τ=1, B=1) . The broken lines represent the local wave speed, c .

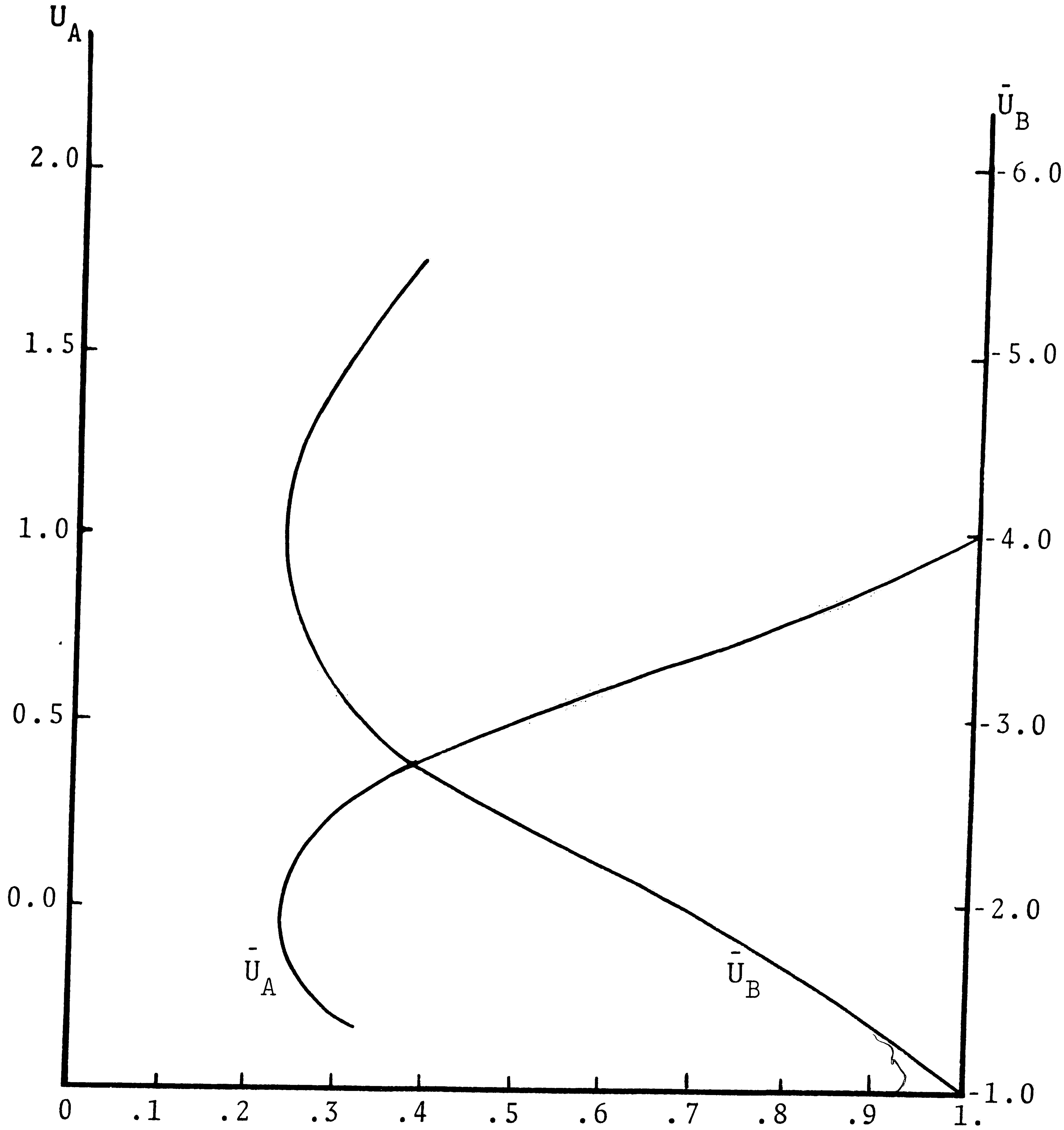

FIGURE 4. $\bar{U}_A$ and $\bar{U}_B$ at different H for slow wave

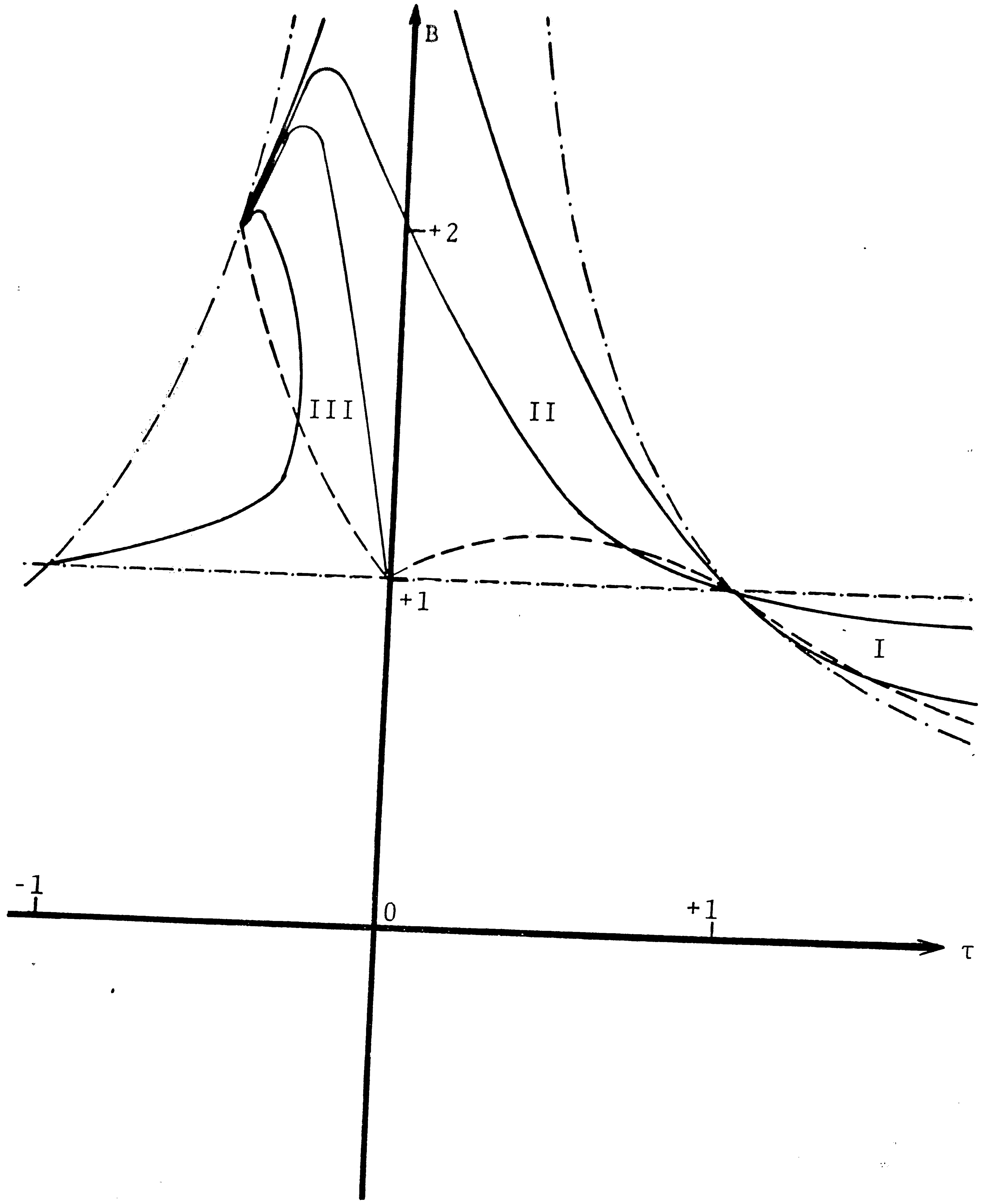

FIGURE 5. Enlarged view of portion of phase plane showing regions where $\frac{\partial c}{\partial \tau}$ and $\frac{\partial H}{\partial \tau}$ are positive or negative

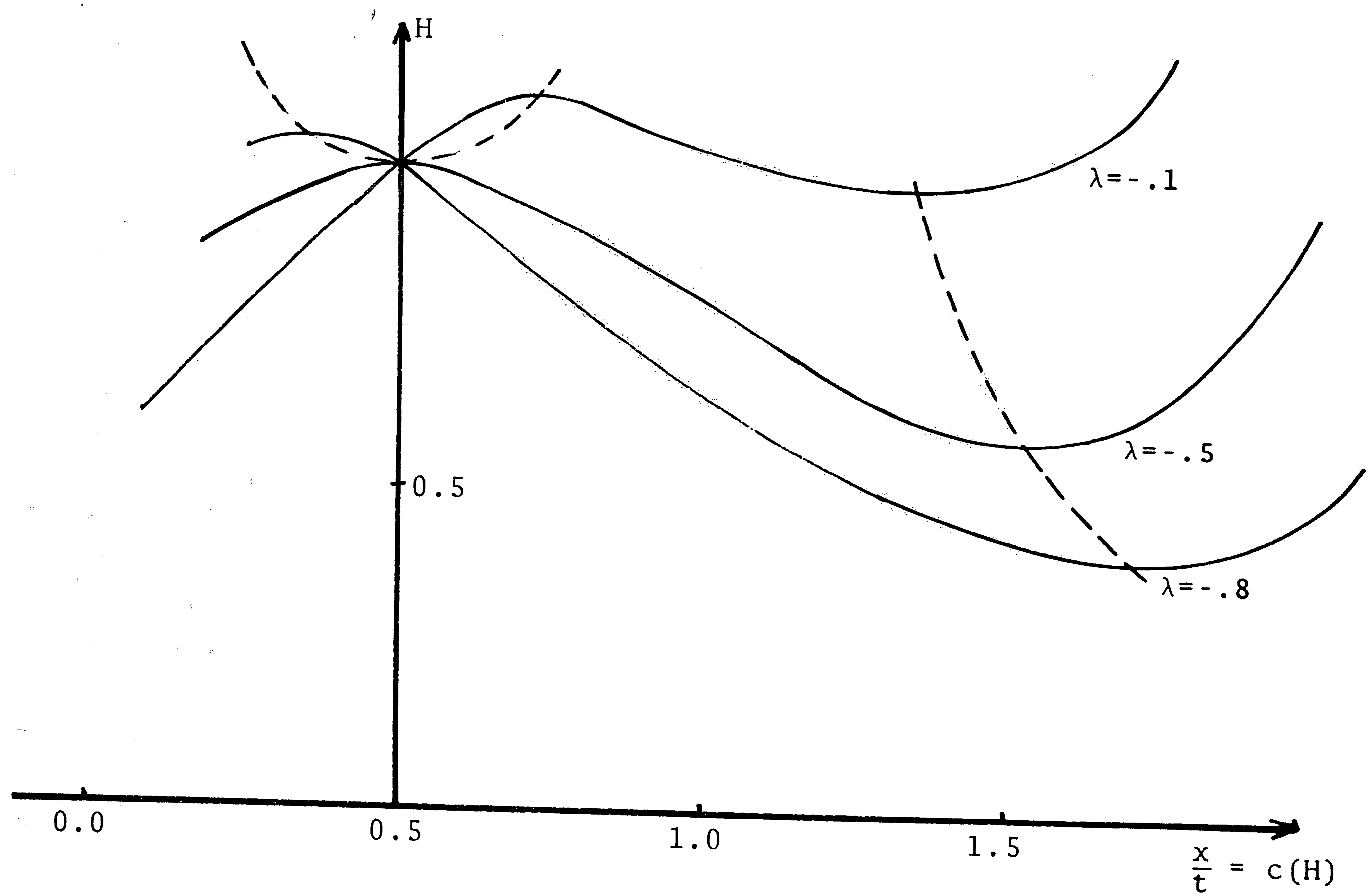

FIGURE 6. Various possible free surface shapes, $H(x)$, for the centered wave. The broken lines indicate locus of either maximum or minimum free surface height.

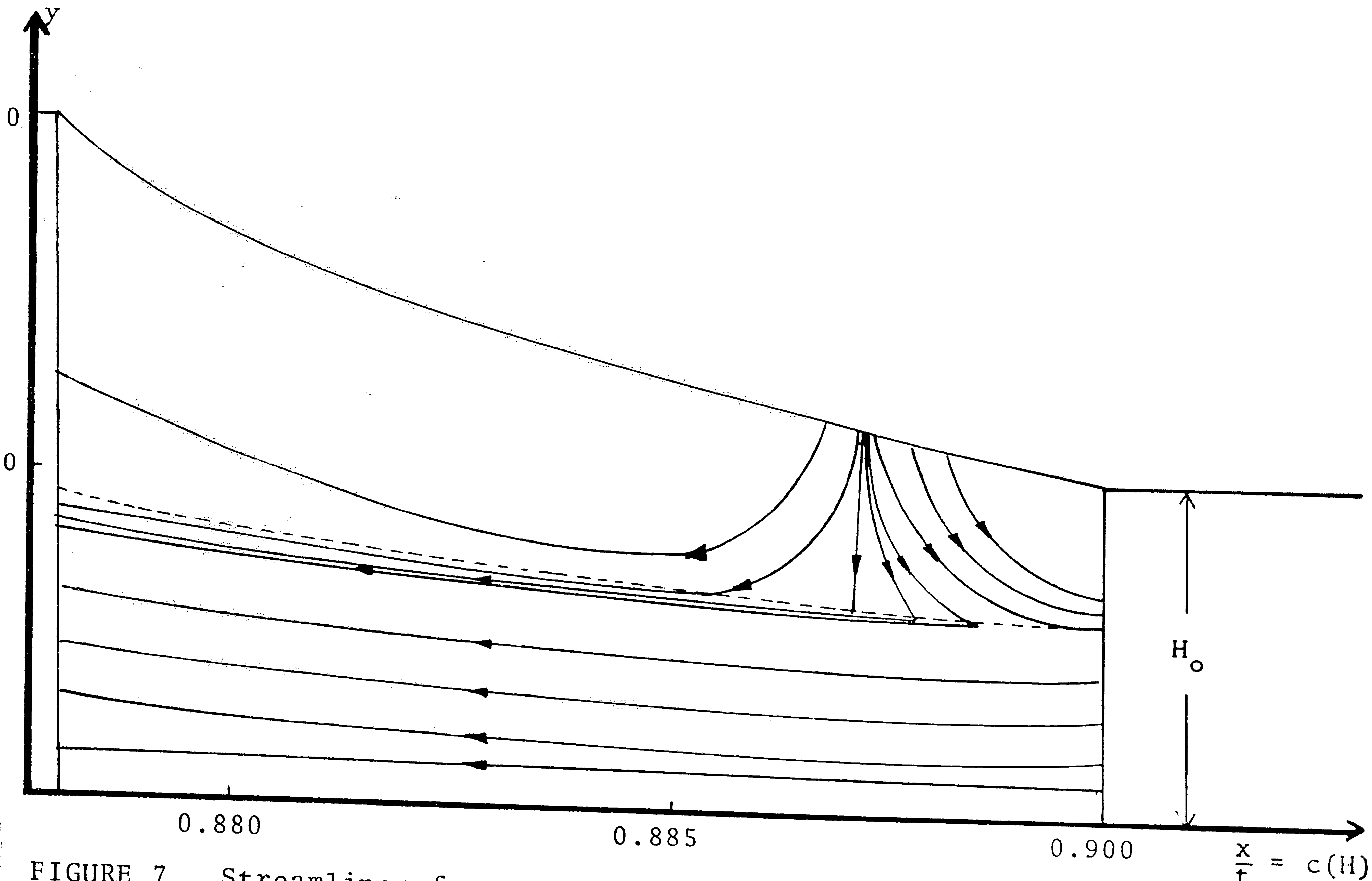

FIGURE 7. Streamlines for a centered wave observed from a reference frame moving with the front. Entire flow corresponds to the slow wave region. The broken line indicates the interface.

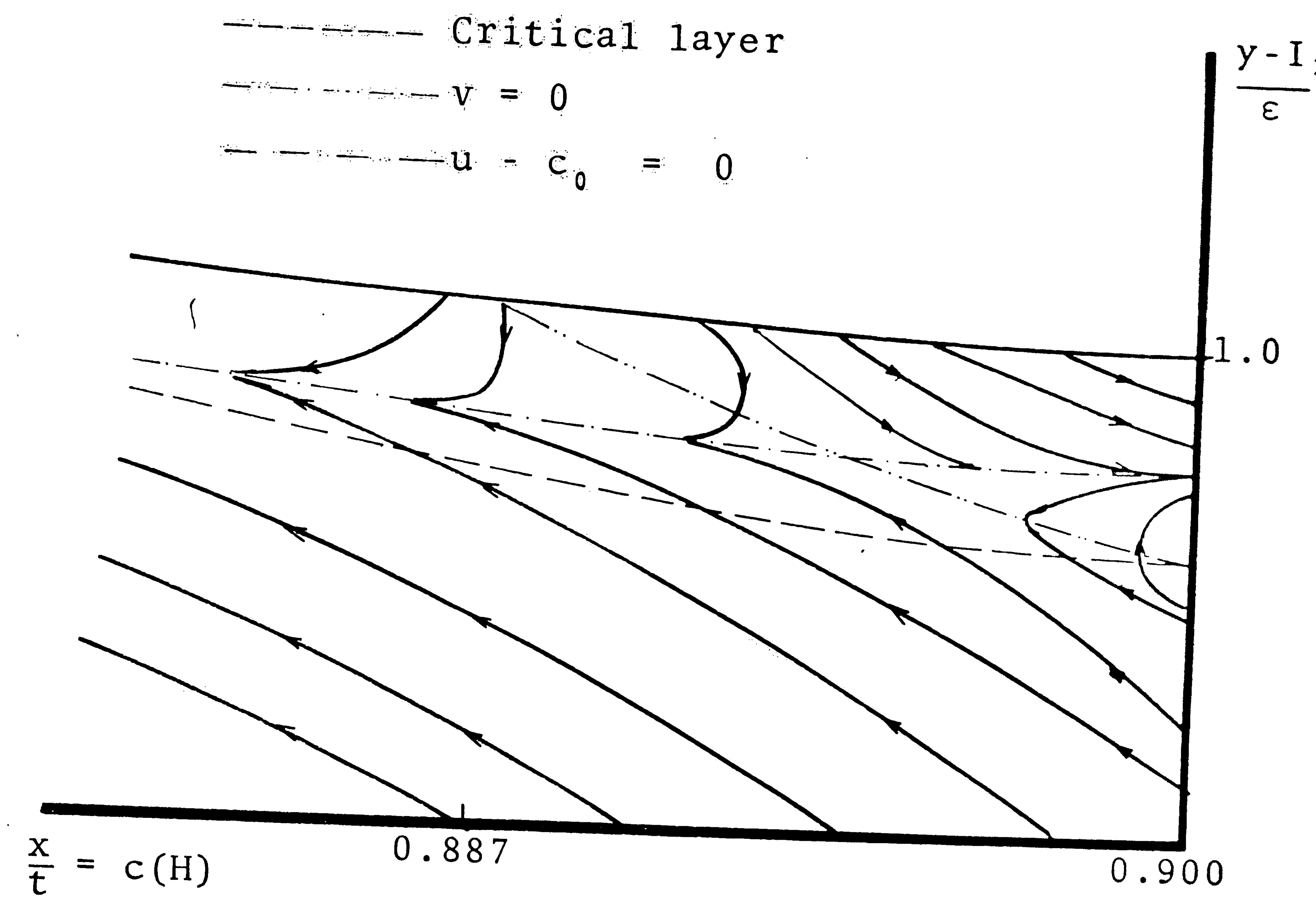

FIGURE 8. Modified streamlines illustrating the flow in the thin central region, corresponding to the wave of figure 7.

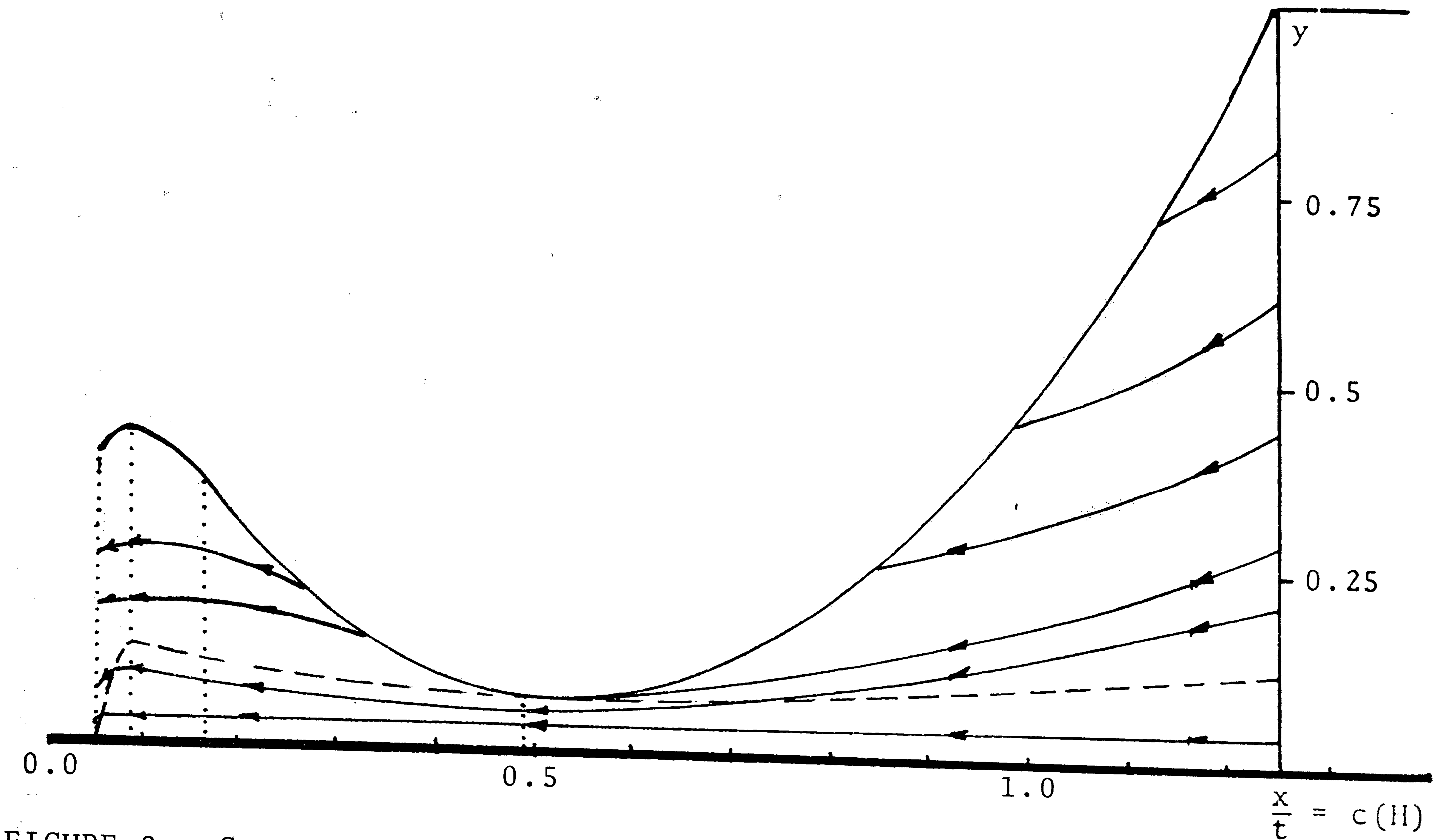

FIGURE 9. Streamlines for a centered wave observed from a frame moving with the front. The wave speed near the front corresponds to the fast wave, but changes to a slow wave at $x/t = 0.485$.

Reference

[1] Blythe, P. A., Kazakia, Y. and Varley, E. 1971
The interaction of large amplitude shallow water waves with an ambient shear flow. (Pending publication).

Résumé

Kai-Nan An was born on August 1, 1947 in Nanking, China.

After finishing six years elementary education at a local primary shcool and six years middle education at Taiwan First High School, he attended the Department of Mechanical Engineering at National Cheng-Kung University, Taiwan from July 1965 until July 1969. He was awarded a B.S. degree. After graduation he served in the Chinese Air Force for one year. Following his military service he began graduate work in the Mechanics Department at Lehigh University in September 1970 and currently holds a Research Assistantship in the Center for the Application of Mathematics. He expects to be awarded the M.S. degree in October, 1972.