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Overconvergence for a normal family of analytical functions

John J. Swetits
Lehigh University

OVERCONVERGENCE FOR A NORMAL FAMILY OF ANALYTIC FUNCTIONS

by

John J. Swetits

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J. P. King
Professor in charge

Everett Pitcher
Chairman of the Department

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ABSTRACT

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by

John J. Swetits

In the first part of this paper we provide some basic properties of power series and their analytic continuation which are essential in a study of function theory. Then, following E. Hille's Analytic Function Theory, we develop the theory of holomorphy preserving operators. This theory is a useful tool in establishing the classical Hadamard Gap Theorem. Holomorphy preserving operators also prove to be useful in establishing several theorems giving sufficient conditions for overconvergence. We then state and prove several theorems concerning necessary conditions of overconvergence.

Recently G. Johnson has defined the concepts of regular point, domain of regularity, and overconvergence for a normal family of analytic functions. We prove several theorems giving sufficient conditions and necessary conditions for a family of this type to be overconvergent. These theorems are essentially generalizations of the theorems on overconvergence given in the previous sections of this paper.

In addition, we provide a formula for computing the radius of regularity of a normal family of analytic functions.

1. Introduction

Let $f(z)$ be analytic in a domain $D = \{z : |z| < R\}$. Then $f(z)$ can be represented as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < R \tag{1.1}$$

If $|z_0| = R$, where z_0 is a regular point of $f(z)$, then (1.1) may or may not converge at z_0. We wish to provide conditions under which (1.1) will overconverge at z_0. To do this we develop the theory of holomorphy preserving operators, and we pay special attention to the two functions discussed in Section 4 of this paper. Much of this material may be found in [3].

Finally, we consider a normal family of analytic functions. The material on normality may be found in [5]. After making suitable definitions, we prove several theorems giving conditions for the overconvergence of this family. In doing this we follow the work of Johnson, [4], and Wilson, [6].

2. Singularities and Analytic Continuation

The results of this section are elementary in nature and the proofs will be omitted. Proofs can be found in [3].

Definition 2.1: Let ρ denote the radius of the largest circle for which the terms $|a_n z^n|$ of (1.1) are uniformly bounded for $|z| < \rho$. We call ρ the radius of convergence of (1.1).

<u>Theorem 2.2</u>: The series (1.1) converges absolutely for every z with $|z| < \rho$, and uniformly with respect to z for $|z| \leq r < \rho$.

Let $f(z;0)$ denote a power series about the origin with radius of convergence $\rho > 0$. Let b be a point such that $|b| < \rho$. Then we can rearrange the series $f(z;0)$ about the point b, yielding a series $f(z;b)$ with radius of convergence ρ_b. Let D(0) denote the interior of the circle of convergence of $f(z;0)$, and D(b) denote the interior of the circle of convergence of $f(z;b)$.

<u>Definition 2.3</u>: If $f(z)$ is defined by the series (1.1) with $\rho > 0$, if $|z_0| = \rho$, and if for all t, $0 < t < 1$, $\rho_b = (1-t)\rho$ where $b = tz_0$, then $z = z_0$ is called a singular point of $f(z)$.

<u>Definition 2.4</u>: If every z such that $|z| = \rho$ is a singular point of $f(z)$, then $\{z : |z| = \rho\}$ is called a natural boundary of $f(z)$.

<u>Theorem 2.5</u>: $\rho - |b| \leq \rho_b \leq \rho + |b|$.

If $\rho_b > \rho - |b|$, then D(0) and D(b) have a non-empty intersection. In this case we state the following:

<u>Definition 2.6</u>: $f(z;b)$ is called the analytic continuation of $f(z)$ in $D(b) - [D(b) \cap D(0)]$.

Lemma 2.7: If $\rho_b = \rho - |b|$, then z_0 is a singular point of $f(z)$, where z_0 is the common point on the boundaries of $D(0)$ and $D(b)$.

Theorem 2.8: There is at least one signular point on the circle of convergence of the function defined by (1.1).

3. Holomorphy Preserving Operators

In this section we wish to consider an operator of the type

$$G(\tfrac{d}{dz})\ [f] = \sum_{k=0}^{\infty} \frac{g_k}{k!} f^{(k)}(z) \tag{3.1}$$

where

$$G(w) = \sum_{k=0}^{\infty} \frac{g_k}{k!} w^k \tag{3.2}$$

is an entire function which satisfies $\lim_{k\to\infty} |g_k|^{1/k} = 0$.

Definition 3.3: Let $f(z;0)$ be a power series about the origin with radius of convergence ρ. We define the principal star to be the set of all points, a, such that $f(z;0)$ can be continued analytically along the line segment $[0,a]$. We denote the principal star by $A[f]$.

Definition 3.4: A sequence $\{\alpha_n : n = 0, 1, 2, \ldots\}$ is a holomorphy preserving factor sequence if, for every function $f(z)$ defined by a power series $f(z;0) = \sum_{n=0}^{\infty} a_n z^n$, the principal star of the transform $T[f]$ of $f(z)$ defined by

$\sum_{n=0}^{\infty} \alpha_n a_n z^n$ contains the principal star of $f(z)$.

We now consider the operator $\theta \equiv z(d/dz)$.

Define $\theta^0[f] = f,$

$$\theta^1[f] = \theta[f],$$

$$\vdots$$

$$\theta^n[f] = \theta[\theta^{n-1}[f]].$$

Lemma 3.5: The equations

$$\theta^n \left[\frac{1}{a-z}\right] = \frac{P_n(z,a)}{(a-z)^{n+1}}, \quad n = 0,\ 1,\ 2,\ \ldots \tag{3.6}$$

define functions $P_n(z,a)$ which satisfy

(i) $P_n(z,a)$ is a polynomial of degree n in z and $n-1$ in a, $n = 1, 2, 3, \ldots,$

$$P_n(z,a) = z \sum_{k=0}^{n-1} \alpha_{k,n} a^{n-k-1} z^k, \tag{3.7}$$

and

(ii) the coefficients $\alpha_{k,n}$ are positive integers such that

$$\sum_{k=0}^{n-1} \alpha_{k,n} = n!$$

Proof:

(i) We proceed by induction on n. It is clear that $P_1(z,a) = z$. Suppose $P_{n-1}(z,a)$ has the desired properties. Applying θ to $\dfrac{P_{n-1}(z,a)}{(a,z)^n}$ we find that

$$P_n(z,a) = z[nP_{n-1}(z,a) + (a-z)P'_{n-1}(z,a)]. \tag{3.8}$$

Since $P_{n-1}(z,a)$ is of degree n-1 in z and $P'_{n-1}(z,a)$ is of degree n-2 in a, we see that $P_n(z,a)$ is of degree n in z and of degree n-1 in a.

(ii) From (3.8) we find that the following hold:

$$\alpha_{0,n} = \alpha_{0,n-1}$$

$$\alpha_{k,n} = (k+1)\alpha_{k,n-1} + (n-k)\alpha_{k-1,n-1}, k=1,2,\ldots$$

$$\alpha_{n-1,n} = \alpha_{n-2,n-1}.$$

Since $P_1(z,a) = z$, we have $\alpha_{0,1} = 1$. Thus, by induction, if the coefficients of $P_{n-1}(z,a)$ are positive integers, we see by the recurrence relations that the coefficients of $P_n(z,a)$ are all positive integers.

Now, by (3.8), $P_n(a,a) = na\,P_{n-1}(a,a)$

$$= n(n-1)a^2P_{n-2}(a,a)$$

$$= n!a^{n-1}P_1(a,a)$$

$$= n!a^n.$$

Thus $\quad P_n(a,a) = a\sum_{k=0}^{n-1} \alpha_{k,n}a^{n-k-1}a^k$

$$= a\sum_{k=0}^{n-1} \alpha_{k,n}a^{n-1}$$

$$= a^n \sum_{k=0}^{n-1} \alpha_{k,n}$$

$$= n! a^n.$$

Thus $\sum_{k=0}^{n-1} \alpha_{k,n} = n!$

Q.E.D.

Corollary 3.9: If $r = \max(|z|, |a|)$, then $|P_n(z,a)| \le n!\, r^n$.

Proof: $|P_n(z,a)| = |z \sum_{k=0}^{n-1} \alpha_{k,n} a^{n-k-1} z^k|$

$$\le |z| \sum_{k=0}^{n-1} \alpha_{k,n} |a|^{n-k-1} |z|^k$$

$$\le r \sum_{k=0}^{n-1} \alpha_{k,n} r^{n-1}$$

$$= r^n \sum_{k=0}^{n-1} \alpha_{k,n}$$

$$= n!\, r^n.$$

Q.E.D.

Definition 3.10: Suppose $f(z)$ is analytic in a domain D. We say that an operator of the type (3.1), $G(\theta)$, applies to $f(z)$ in D if

$$G(\theta)[f] = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{g_k}{k!} \theta^k[f] \qquad (3.11)$$

exists for z in D, and uniformly on compact subsets of D.

Theorem 3.12: If $f(z)$ is analytic in a domain D, then $G(\theta)$ applies to $f(z)$ in D and $G(\theta)[f]$ is analytic in D. The domain of analyticity of $G(\theta)[f]$ contains that of $f(z)$.

Finally, $\{G(n)\}$ is a holomorphy preserving factor sequence.

Proof: Let Δ be a compact subset of D. Let C be a curve in D made up of a finite number of simple closed rectifiable curves with distance $\gamma > 0$ from Δ. Since Δ is compact it is closed and bounded. Thus there is a positive distance from Δ to the boundary of D. Because Δ is compact, it can be covered by a finite number of discs. Thus the curve C exists. By the Cauchy integral formula, we have

$$f(z) = 1/2\pi i \int_C f(t)/t-z \; dt, \quad z\varepsilon\Delta.$$

Since differentiation is allowed under the integral sign in Cauchy's formula, we can apply the operator θ under the integral sign. By Lemma 3.5

$$\theta^{(n)}[f(z)] = \frac{1}{2\pi i} \int_C \frac{P_n(z,t)}{(t-z)^{n+1}} f(t) \, dt$$

Then $\sum_{k=m}^{n} \frac{gk}{k!} \theta^{(k)}[f(z)] = \frac{1}{2\pi i} \int_C \left(\sum_{k=m}^{n} \frac{g_k \; P_k(z,t)}{k! \; (t-z)^k}\right) \frac{f(t)}{t-z} \, dt.$

Since Δ is compact we can find a positive number ρ such that $\max(|z|, |t|) \leq \rho$ whenever $z\varepsilon\Delta$, $t\varepsilon C$. Since $\lim_{k\to\infty} |g_k|^{1/k} = 0$ we can find $M = M(\gamma,\rho)$ such that $|g_k| < (\frac{\gamma}{2\rho})^k M$ for all k sufficiently large. Using this and Corollary 3.9 we have

$$\left|\sum_{k=m}^{n} \frac{g_k}{k!}\, \theta^{(k)}[f(z)]\right| \le \frac{1}{2\pi}\int_C \left(\sum_{k=m}^{n} \frac{|g_k|\,|P_k(z,t)|}{k!\,|t-z|^k}\left|\frac{f(t)}{t-z}\, dt\right|\right.$$

$$\le \frac{1}{2\pi}\int_C \left(\sum_{k=m}^{n} \left(\frac{\gamma}{2\rho}\right)^k M(\rho/\gamma)^k (\max(f;c/\gamma)(dt)\right.$$

$$= M(\max(f;c))\, \frac{\ell(c)}{\gamma} \sum_{k=m}^{n} \frac{1}{2^k}$$

where $\max\,(f;c) = \max\limits_{z\varepsilon c} |f(z)|$ and $\ell(c)$ = length of C.

Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, $\sum_{k=m}^{n} \frac{1}{2^k}$ can be made arbitrarily small.

Thus $\sum_{k=1}^{\infty} \frac{g_k}{k!}\, \theta^{(k)}\, [f(z)]$ converges uniformly on compact subsets of D. Therefore $G(\theta)$ applies to $f(z)$ in D and $G(\theta)[f]$ is analytic in D.

Now let $f(z) = \sum_{n=0}^{\infty} c_n z^n$, $|z| < R$. Then, by the definition of θ we have

$$G(\theta)[f(z)] = \sum_{k=0}^{\infty} \frac{g_k}{k!}\, \theta^{(k)}[f(z)]$$

$$= \sum_{n=0}^{\infty} \frac{g_k}{k!}\, \theta^{(k)}\left[\sum_{n=0}^{\infty} c_n z^n\right]$$

$$= \sum_{k=0}^{\infty} \frac{g_k}{k!} \sum_{n=0}^{\infty} c_n n^k z^n$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{g_k}{k!}\, n^k c_n z^n$$

$$= \sum_{n=0}^{\infty} G(n)\, c_n z^n.$$

Since $G(\theta)$ applies to $f(z)$, this series converges.

Because $G(\theta)$ is applicable to $f(z)$ in D, it follows that if $f(z)$ can be continued analytically, so can $\sum_{n=0}^{\infty} G(n) c_n z^n$. Thus $\{G(n)\}$ is a holomorphy preserving factor sequence.

Q.E.D.

4. Two Special Functions

In this section we develop some properties of the following two functions.

$$G(w) = \prod_{n=1}^{\infty} \left(1 - \frac{w}{a_n}\right), \quad \sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty$$

and $$H(w) = \prod_{n=1}^{\infty} \left(1 - \frac{w^2}{a_n^2}\right), \quad \frac{a_n}{n} \to \infty \text{ as } n \to \infty.$$

Lemma 4.1: $G(w)$ satisfies the properties of (3.1).

Proof: Since $\sum_{n=1}^{\infty} \frac{1}{|a_n|}$ converges, it follows that $G(w)$ is an entire function. [1]. We have

$$|G(w)| \leq \prod_{n=1}^{\infty} \left(1 + \left|\frac{w}{a_n}\right|\right)$$

$$\leq e^{|w|M}$$

where $M = \sum_{n=1}^{\infty} \frac{1}{|a_n|}$.

By the Cauchy integral formula,

$$|G^{(k)}(0)| = \frac{1}{2\pi} \left| \int_C \frac{G(t)}{t^{k+1}} dt \right|$$

where $C = \{t : |t| = \frac{k}{M}\}$.

We have $$|G^{(k)}(0)| \le \frac{e^k M^k}{k^k}$$

Therefore
$$|G^{(k)}(0)|^{1/k} \le \frac{eM}{k} .$$

Thus $$\lim_{k\to\infty} |G^{(k)}(0)|^{1/k} = 0.$$

But $G^{(k)}(0)$ is just g_k in the series expansion for

$G(w) = \sum_{k=1}^{\infty} \frac{g_k}{k!} w^k$.

Therefore $$\lim_{k\to\infty} |g_k|^{1/k} = 0.$$

Q.E.D.

<u>Lemma 4.2</u>: H(w) satisfies the properties of (3.1).

<u>Proof</u>: Since $\frac{a_n}{n} \to \infty$ as $n \to \infty$, we have $|a_n| > n$ for all n sufficiently large. Thus $\frac{1}{|a_n|^2} < \frac{1}{n^2}$ for all n sufficiently large. It follows that $\sum_{n=1}^{\infty} \frac{1}{|a_n|^2}$ converges, and so H(w) is an entire function [1]. Let $M = \sum_{n=1}^{\infty} \frac{1}{|a_n|^2}$.

Then $$|H(w)| \le \prod_{n=1}^{\infty} (1 + |\frac{w^2}{a_n^2}|)$$

$$\le e^{|w|^2 M} .$$

Using the same method as in Lemma 4.1 we have

$$\lim_{k\to\infty} |g_k|^{1/k} = 0.$$

Q.E.D.

5. Hadamard Gap Series

<u>Definition 5.1</u>: The series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ will be called a gap series if, for infinitely many n, $a_n = 0$.

<u>Definition 5.2</u>: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and if there are infinitely many k such that $n_{k+1} - n_k > n_k\,\theta$ for some $\theta > 0$, and if $a_n = 0$ for $n \neq n_k$, then we say that f(z) is a Hadamard gap series.

<u>Theorem 5.3</u>: (Hadamard Gap Theorem).

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $\{n_k\}$ be a subsequence of $\{n\}$. Suppose $a_n = 0$ for $n \neq n_{kn_k}$ and $a_{n_k} \neq 0$. Then $\sum_{k=0}^{\infty} a_{n_k} z$ has its circle of convergence as a natural boundary provided there exists a fixed $\lambda > 1$ such that $n_{k+1}/n_k \geq \lambda$ for all k.

<u>Proof</u>: The proof of this theorem will appear later as a corollary to Theorem 6.2.

<u>Theorem 5.4</u>: The series $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$, $a_k \neq 0$, has its circle of convergence as a natural boundary if $\lim_{k\to\infty} k/n_k = 0$.

<u>Proof</u>: We can assume that the circle of convergence of the series has radius 1.

We break up the sequence $\{n_k\}$ into two complementary subsequences, $\{m_k\}$ and $\{p_k\}$, so that

$$f(z) = \sum_{k=1}^{\infty} h_k z^{m_k} + \sum_{k=1}^{\infty} g_k z^{p_k} = h(z) + g(z).$$

Choose the sequence $\{m_k\}$ so that it satisfies the following two conditions:

$$\lim_{k\to\infty} |h_k|^{1/m_k} = 1 \tag{5.5}$$

$$m_{k+1}/m_k \geq 2 \tag{5.6}$$

The condition (5.5) is possible since $\limsup |a_k|^{1/n_k} = 1$. It is clear that (5.6) is possible. The series $h(z)$ has radius of convergence = 1.

Let $G(w) = \prod_{k=1}^{\infty} (1 - w^2/p_k^2)$. By hypothesis $\lim_{k\to\infty} k/p_k = 0$. Thus, by Lemma 4.2 and Theorem 3.12, $G(w)$ represents a holomorphy preserving operator.

We have the following condition:

$$\lim_{n\to\infty} (1/m_n) \log|G(m_n)| = 0 \ . \tag{5.7}$$

The proof can be found in [3].

Applying the operator $G(\theta)$ to $f(z)$, we have

$$G(\theta)[f(z)] = G(\theta)[h(z)] + G(\theta)[g(z)]$$

$$= \sum_{k=1}^{\infty} h_k \; G(m_k) z^{m_k} + \sum_{k=1}^{\infty} g_k \; G(p_k) z^{p_k}.$$

But $G(p_k) = 0$.

Thus

$$G(\theta)[f(z)] = \sum_{k=1}^{\infty} G(m_k)h_k z^{m_k}.$$

By (5.5) and (5.7) the radius of convergence of this series is 1.

By Theorem 5.3, the series has its circle of convergence as a natural boundary. Since $G(\theta)$ is a holomorphy preserving operator, $f(z)$ cannot be analytic where $G(\theta)[f(z)]$ is not.

Thus $f(z)$ has its circle of convergence as a natural boundary.

Q.E.D.

6. Sufficient Conditions for Overconvergence

In this section we provide some sufficient conditions for a series to be overconvergent and we give a proof of the Hadamard Gap Theorem.

Definition 6.1: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and let $\{S_n(z)\}$ be the sequence of partial sums of $f(z)$. Let $f(z)$ have radius of convergence R and let z_0 be a point such that $|z_0| = R$. if $\exists$ a neighborhood, N, about z_0 and a subsequence of partial sums, $\{S_{n_k}(z)\}$, such that $\{S_{n_k}(z)\}$ converges to $f(z)$ at all points of N, then we say that $f(z)$ is overconvergent at z_0.

Theorem 6.2: Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ have radius of convergence = 1. Let $\{m_k\}$ and $\{p_k\}$ be two sequences of positive integers such that $(1+\eta)m_k < p_k$, where $\eta > 0$ is fixed, and $m_k < p_k \leq m_{k+1}$. Assume $c_n = 0$ for $m_k < n < p_k$. Then the sequence of partial sums, $\{S_{m_k}(z)\}$, of f(z) converge in a full neighborhood of each point of the unit circle at which f(z) is analytic.

Proof: Let $\delta > 0$ be arbitrarily small.

Let

$$D_\delta = \{z: |z|<1-\delta,\ \theta_2<\arg z<\theta_1+2\pi\} \cup \{z: |z|<R, \theta_1<\arg z<\theta_2\}$$

where $R > 1$. Let C_δ denote the boundary of D_δ. Assume f(z) is analytic in the closure of D_δ.

Let $z \varepsilon D_\delta$. Consider

$$f(z)-S_{m_k}(z) = \frac{1}{2\pi i} \int_{C_\delta} \frac{f(t)}{t-z} \left(\frac{z}{t}\right)^{m_k+1} dt.$$

By hypothesis, at least $[\eta m_k] \equiv p$ terms are missing after the m_kth term. ($[x]$ is the greatest integer $\leq x$). So, we have

$$(f(z)-S_{m_k}(z))/z^{m_k+1} = \frac{1}{2\pi i} \int_{C_\delta} f(t)/t^{m_k+1} \left(\frac{1}{t-z} - \sum_{j=1}^{p} \alpha_j/t^j\right) dt.$$

where the α_j are arbitrary.

We can rewrite this as

$$(f(z)-S_{m_k}(z)/z^{m_k+1} = \frac{1}{2\pi i} \int_{C_\delta} f(t)/t^{m_k+2} \left(1/1-\frac{z}{t} - P\left(\frac{1}{t}\right)\right) dt \quad (6.3)$$

where $P(\frac{1}{t})$ is a polynomial of degree p or less.

Let B be a domain interior to C_δ in which

$a < |1-\frac{z}{t}| < b$, $z\varepsilon B$, $t\varepsilon C_\delta$. a and b are fixed positive numbers. By the Weierstrass approximation theorem, [3], we can find a polynomial $R(\frac{1}{t})$ such that

$$|1/1-\frac{z}{t} - R(\frac{1}{t})| < \frac{1}{2b}, \quad z\varepsilon B, \quad t\varepsilon C_\delta.$$

Let v be the degree of R and let

$$P(\frac{1}{t}) = (1-[1-(1-\frac{z}{t})R(\frac{1}{t})]^m)/1-\frac{z}{t}$$

where m is the largest integer such that $m(v+1) \leq p$. This is a polynomial in $\frac{1}{t}$ of degree $\leq p$, and it is to be used in (6.3).

We have

$$|1/1-\frac{z}{t} - P(\frac{1}{t})| \leq \frac{1}{a2^m}, \quad z\varepsilon B, \quad t\varepsilon C_\delta.$$

Then, from (6.3), we obtain the following estimate for $|f(z) - S_{m_k}(z)|$:

$$|f(z)-S_{m_k}(z)| \leq MR/a2^m(|z|/1-\delta)^{m_k+1}. \tag{6.4}$$

Take the m_kth root of both sides of (6.4). Note that $m/m_k \to n/v+1 \equiv \mu$. Take $\limsup_{k\to\infty} |f(z)-S_{m_k}(z)|^{1/m_k}$. By (6.4) we see that this does not exceed $(1-\delta)^{-1}|z|2^{-}$. But this holds for every $\delta > 0$. So we have

$$\limsup_{k\to\infty} |f(z)-S_{m_k}(z)|^{1/m_k} \leq |z|2^{-\mu} \tag{6.5}$$

for $z\varepsilon B$.

We now make a specific choice for the domain B. Let α, β be two positive numbers with $\beta > \alpha$. Let $1 < R_1 < R$, $0 < \varepsilon < \frac{1}{2}(\beta-\alpha)$. Let $B = \{z: |z| \leq R_1, \alpha+\varepsilon \leq \arg z \leq \beta-\varepsilon\}$. Since $2^\mu > 1$, we can find θ, $0 < \theta < 1$, such that $\theta 2^\mu > 1$. Restrict z so that $|z| \leq \min(\theta 2^\mu, R_1)$. Then (6.5) does not exceed θ. This means $\limsup_{k\to\infty} |f(z) - S_{m_k}(z)| = 0$. Thus $\lim_{k\to\infty} |f(z) - S_{m_k}(z)| = 0$. This limit is uniform with respect to z.

Q.E.D.

<u>Corollary 6.6</u>: (Hadamard Gap Theorem). If $p_k = m_{k+1}$, then the unit circle is the natural boundary of f(z).

<u>Proof</u>: In this case, the sequence $\{S_{m_k}(z)\}$ coincides with the sequence of all partial sums of f(z). But the sequence of all partial sums converges only inside the circle of convergence. Thus every point on the dircle of convergence is singular.

Q.E.D.

The next theorem will be stated without proof, since the proof lies outside the scope of this paper. The proof can be found in [3].

<u>Theorem 6.7</u>: Let f(z) be analytic and bounded in a domain D which is bounded by a finite number of simple closed curves, C. Assume $|f(z)| \leq M$ in D. Suppose $\exists$ a subarc Γ of C such that when z approaches any point of Γ we have

$\limsup |f(z)| \leq m < M$. Then a function $\lambda(z)$, $0 < \lambda(z) < 1$, such that for each $z \varepsilon D$

$$\log |f(z)| \leq \lambda(z) \log m + (1-\lambda(z))\log M.$$

Moreover on compact subsets of D, $\lambda(z)$ is bounded away from 0 and 1.

<u>Theorem 6.8</u>: Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ have radius of convergence = 1. Let $\{m_k\}$ and $\{p_k\}$ be two sequences of positive integers such that $p_k/m_k \to \infty$. Assume $C_n = 0$ for $m_k < n < p_k$. Then the sequence, $\{S_{m_k}(z)\}$, converges uniformly to f(z) in a neighborhood of any regular point of f(z). The partial sums converge in the complete domain of existence, D[f], of f(z), and f(z) is single valued in D[f].

<u>Proof</u>: Let D_0 be a bounded subdomain of D[f] with $\overline{D}_0 \subset D[f]$. Let $R > 1$ and suppose $D_0 \subset \{z:|z| < R\}$. Assume $\{z:|z| < r < 1\} \subset D_0$. Suppose that an arc Γ of $\{z:|z| = r\}$ forms part of the boundary of D_0.

Let

$$F_k(z) = f(z) - S_{m_k}(z), \quad z\varepsilon D_0.$$

Since $S_{m_k}(z)$ is a polynomial of degree m_k, we have $|S_{m_k}(z)| < AR^{m_k}$. Furthermore f(z) is bounded. Assume $|f(z)| < M$. Therefore, for $R_1 > R$, we have

$$|F_k(z)| \le |f(z)| + |S_{m_k}(z)|$$
$$\le M + AR^{m_k}$$
$$< R_1^{m_k}.$$

Now, for $|z| = r$ and $C_1 = \{t:|t| = r_1,\ r < r_1 < 1\}$, we have

$$F_k(z) = \frac{1}{2\pi i}\int_{C_1}\left(\frac{z}{t}\right)^{m_k+1}(f(t)/t-z)\,dt$$
$$= \frac{1}{2\pi i}\int_{C_1}\left(\frac{z}{t}\right)^{m_k+1} f(t)(1/t-z\ -\sum_{j=0}^{p_k-m_k-2} z^j/t^{j+1})\,dt.$$

The terms inserted represent coefficients which are equal to 0, and so they add nothing to the integral. Then the above is equal to

$$\frac{1}{2\pi i}\int_{C_1}\left(\frac{z}{t}\right)^{p_k}(f(t)|t-z)\,dt.$$

Since Γ is enclosed by C_1, we have for $z\varepsilon\Gamma$:

$$|F_k(z)| \le (r_1 M(r_1)/r_1-r)\left(\frac{r}{r_1}\right)^{p_k}$$
$$\equiv A(1-\delta)^{p_k}.$$

Therefore we have $M \le R_1^{m_k}$, $m \le A(1-\delta)^{p_k} \le M$, where we are using the notation of Theorem 6.7. By Theorem 6.7:

$$\log |F_k(z)| \le \lambda(z)[\log A+p_k \log(1-\delta)]+[1-\lambda(z)]m_k \log R_1$$

where $z\varepsilon D_0$.

Since $p_k/m_k \to \infty$ and $\log(1-\delta) < 0$, we have

$$\lim_{k\to\infty} \log |F_k(z)| = -\infty.$$

Thus

$$\lim_{k\to\infty} F_k(z) = 0.$$

This limit is uniform on compact subsets of D_0 because $\lambda(z)$ is bounded away from 0 and 1 on compact subsets of D_0.

If $f(z)$ can be continued analytically along a path leading from $z = 0$ to $z = a$, then we can imbed this path in a domain, D_0, of the type given above. Then the sequence, $\{S_{m_k}(z)\}$, converges to $f(z)$ along this path. It follows that $\{S_{m_k}(z)\}$ converges to $f(z)$ in the domain of existence, $D[f]$, of $f(z)$.

Since a polynomial is single valued, it follows that the limit function of a sequence of polynomials is single valued.

Q.E.D.

7. Necessary Conditions for Overconvergence.

We assume that a series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence = 1 and that $f(z)$ is overconvergent. We show that $f(z)$ can be written as a sum of two series, one having radius of convergence greater than 1 and the other a Hadamard gap series.

Definition 7.1: We say that $F(z) = \sum_{n=0}^{\infty} A_n z^n$, $A_n > 0$, is a dominant of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ if $|a_n| \le A_n$ for all n. We write this as $f(z) << F(z)$.

Definition 7.2: Let $u = f(z)$ map a domain D to a domain D'. If f(z) is single valued and one to one in D, then we say that f(z) is a univalent mapping.

The following theorem is due to Bieberbach [2].

Theorem 7.3: If $f(z) = z+a_2z^2+a_3z^3+\ldots$ is convergent and univalent in $\{z:|z| < 1\}$, then $|a_n| \le 5n^2$ for $n \ge 3$ and $|a_2| \le 2$.

Lemma 7.4: Under the hypotheses of Theorem 7.3, f(z) is dominated by $\phi(z) = z/1-\frac{bz}{r}$ where $b \gg$ and $0 < r < 1$.

Proof: If $b \ge$, then for $n \ge 3$, $5n^2 < b^{n-1}$. We also have $2 < b$ and $1 = b^0$. Then

$$|f(z)| \le |z|+2|z|^2 + 5.3^2|z|^3+\ldots+5n^2|z|^n+\ldots$$

$$\le |z|+2r|z|^2/r+\ldots+5n^2r^{n-1}|z|^n/r^{n-1}+\ldots$$

Replacing the coefficient of $|z|^n/r^{n-1}$ by b^{n-1} we have

$$|f(z)| \le |z|+2r|z|^2/r+\ldots+b^{n-1}|z|^n/r^{n-1}+\ldots$$

Except for the second term this is the series expansion for $\phi(z)$. Thus $f(z) << \phi(z)$.

Q.E.D.

Theorem 7.5: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a subsequence of partial sums, $\{S_{n_k}(z)\}$, overconverging uniformly in a neighborhood of a regular point on the circle of convergence, there is a $\theta > 0$ and a ρ, $0 < \rho < 1$, such that for sufficiently large k, we have $|a_n| < \rho^n$ for $n_k \leq n < (1+\theta)n_k$.

Proof: Consider a function $\phi(z) = a_n z^n + \ldots$ such that $|\phi(z)| \leq 1$ for $z \varepsilon \Delta$, where Δ is a simply connected domain. If we map Δ on $\{u : |u| < r\}$, $\phi(z)$ is transformed into $\bar{\phi}(u) = b_n u^n + \ldots$ We have $|\bar{\phi}(u)| \leq 1$ for $|u| < r$. By Cauchy's inequality, we have $|b_m| \leq r^{-m}$. Thus $\bar{\phi}(u)$ is dominated by $u^n/r^n(1-\frac{u}{r})$. By Lemma 7.4, u is dominated by $h(z) = z/1-\frac{bz}{r}$ where $b \geq$

So

$$\phi(z) << [z^n/r^n(1-\frac{bz}{r})^n][1/1-z/r(1-\frac{bz}{r})].$$

Thus

$$\phi(z) << z^n/r^n(1-\frac{2bz}{r})^n$$

From this we have

$$|a_{n+p}| \leq 2^p b^p C_p^{n+p}/r^{n+p}$$

where $C_p^{n+P} = \binom{n+p}{p}$.

Let $\sigma = p/n$. Let n+p tend to infinity in such a way that $\sigma < \theta$ where θ is a small positive number independent of n and p.

Since $p/n+p = \sigma/1+\sigma \leq \sigma$, we have

$$(|a_{n+p}|)^{1/n+p} \leq 2^{\sigma} b^{\sigma}/r \, [C_p^{n+p}]^{1/n+p} \tag{7.6}$$

For $p = 0$ and letting $n \to \infty$, the right hand side of (7.6) approaches $\frac{1}{r}$. For $p \geq 1$, we note that $C_p^{n+p} < (1 + \frac{p}{n})^{n+p} / (\frac{p}{n})^p$, or in terms of σ, $[C_p^{n+p}]^{1/n+p} < (1+\sigma)/\sigma^{\sigma/1+\sigma}$. When $\sigma \to 0$, the right hand side of the above inequality approaches 1. Thus, for $p \geq 0$, when $\sigma \to 0$, the right hand side of (7.6) approaches $\frac{1}{r}$.

We have

$$(|a_{n+p}|)^{1/n+p} \leq g(\sigma)/r$$

where $g(\sigma) \to 1$ as $\sigma \to 0$.

If $r > 1$, there is a $\theta > 0$ and a ρ, $0 < \rho < 1$, such that $g(\sigma)/r < \rho < 1$ when $0 \leq \rho < \theta$.

Consider the sequence $\{r_k(z)\}$ where $r_k(z) = f(z) - S_{n_k}(z)$. By hypothesis, $\{S_{n_k}(z)\}$ is uniformly overconvergent in a neighborhood of a point z_0 at which $f(z)$ is analytic and $|z_0| = 1$. Let the domain Δ be the union of this neighborhood together with the interior of the unit disc. Then for sufficiently large k, $|r_k(z)| \leq 1$ for all $z \epsilon \Delta$.

From the above we obtain constants ρ, $0 < \rho < 1$, and $\theta > 0$ such that $(|a_{n+p}|)\ 1/n+p < \rho$ for $0 \leq p/n_k < \theta$. For $p = 0$, $n = n_k$ and we have $p < n\theta$. So $|a_n| < \rho^n$ for $n_k \leq n \leq n_k + p$ which gives $|a_n| < \rho^n$ for $n_k \leq n < n_k(1+\theta)$.

Q.E.D.

Theorem 7.7: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a subsequence of partial sums, $\{S_{n_k}(z)\}$, overconverging uniformly in a neighborhood of a regular point on the circle of convergence, then $\sum_{n=0}^{\infty} a_n z^n$ is the sum of two power series, the first of which has radius of convergence greater than 1, and the coefficients of the second possess Hadamard gaps.

Proof: By Theorem 7.5 we have

$$|a_n| < \rho^n \quad \text{for } n_k \leq n < (1+\theta)n_k$$

where $0 < \rho < 1$ and $\theta < 0$.

Thus there exists $\varepsilon > 1$ such that $|a_n|\varepsilon^n < \rho^n$ for all n such that $n_k \leq n < (1+\theta)n_k$. The series $\sum a_n z^n$, $n_k \leq n < (1+\theta)n_k$, has radius of convergence greater than 1. The series formed from the remaining coefficients is a Hadamard gap series.

Q.E.D.

8. Overconvergence for a Normal Family of Analytic Functions.

In this final section we provide generalizations of the theorems of sections 6 and 7 to a normal family of analytic functions.

Definition 8.1: Let F be a family of functions. F is called normal if every sequence, $\{f_n\}$ $f_n \varepsilon F$, has a subsequence which either converges uniformly or diverges uniformly on compact sets.

Let R denote the radius of the greatest circle about the origin in which each $f \epsilon F$ is analytic and F is normal. Assume that the family is uniformly bounded on compact subsets of $\{z: |z| < R\}$.

Definition 8.2: R is called the radius of regularity for F.

Lemma 8.3: Let $f \epsilon F$, $f(z) = \sum_{n=0}^{\infty} a_n(f) z^n$. Each of the quantities, $\sup_{f \epsilon F} |a_n(f)|$, $n = 0, 1, 2, \ldots$, is finite.

Proof: Let $r < R$. Let $C = \{t: |t| = r\}$.

Then

$$a_n(f) = \frac{1}{2\pi i} \int_C f(t)/t^{n+1}\, dt.$$

Thus

$$|a_n(f)| \leq \max(f,c)/r^n$$

$$\leq M/r^n$$

where $\operatorname{Max}(f,c) = \max_{z \epsilon c} |f(z)|$ and M is the uniform bound for F. Therefore we have

$$\sup_{f \epsilon F} |a_n(f)| \leq M/r^n$$

So $\sup_{f \epsilon F} |a_n(f)|$, $n = 0, 1, 2, \ldots$, is finite

Q.E.D.

Lemma 8.4: $\limsup \sup_{f \epsilon F} |a_n(f)|^{1/n} = 1/R$

Proof: Let $r < R$. Then $|a_n(f) z^n|$ is bounded by $K(f)$.

We have

$$|a_n(f)| \leq K(f)/r^n$$

$$\leq M/r^n$$

where M is the uniform bound for F.

Therefore

$$|a_n(f)|^{1/n} \leq (M)^{1/n}/r$$

So

$$\sup_{f\varepsilon F} |a_n(f)|^{1/n} \leq (M)^{1/n}/r$$

It follows that

$$\lim_{n\to\infty} \sup \sup_{f\varepsilon F} |a_n(f)|^{1/n} \leq 1/r.$$

This holds for every $r < R$. Thus we have

$$\lim_{n\to\infty} \sup \sup_{f\varepsilon F} |a_n(f)|^{1/n} \leq 1/R.$$

Now assume $r > R$. Assume each $f\varepsilon F$ is analytic in $\{z:|z| < R\}$. Then F is no longer uniformly bounded. So we can find a point z_1 and a function $f_1 \varepsilon F$ such that $|f_1(z_1)| > 1$. Then we can find $z_2 \neq z_1$ and $f_2 \neq f_1$, $f_2 \varepsilon F$, such that $|f_2(z_2)| > 2$. Continuing this process inductively we can find z_n and $f_n \varepsilon F$ such that $|f_n(z_n)| > N$. We have $\lim_{n\to\infty} |f_n(z_n)| = \infty$.

Now the sequence, $\{z_n\}$, is bounded. Therefore it has a limit point z_0. It follows that $\lim_{n\to\infty} |f_n(z_0)| = \infty$.

By the properties of F, $\{f_n\}$ converges uniformly on compact subsets of $\{z:|z| < R\}$, and the limit function is analytic. Furthermore this function belongs to F. Let $f(z) = \lim_{n\to\infty} f_n(z)$ where $|z| < R$. By assumption, $f(z)$ is analytic in $\{z:|z| < r\}$. Since $f(z)$ agrees with

$\lim_{n\to\infty} f_n(z)$ for $|z| < R$, it agrees with $\lim_{n\to\infty} f_n(z)$ for $|z| < r$. [1]. But $\lim_{n\to\infty} f_n(z_0) = \infty$. This is a contradiction.

Therefore, given $r > R$, we can find $f\varepsilon F$ such that f has a singularity in $\{z:|z| < r\}$. It follows that the sequence, $\{|a_n(f)|r^n\}$, is unbounded. Thus for n sufficiently large,

$$|a_n(f)|r^n > 1.$$

This gives

$$|a_n(f)|^{1/n} > 1/r$$

which yields

$$\sup_{f\varepsilon F} |a_n(f)|^{1/n} > \frac{1}{r}.$$

It follows that

$$\limsup_{n\to\infty} \sup_{f\varepsilon F} |a_n(f)|^{1/n} > 1/r.$$

But this holds for every $r > R$. Thus we have

$$\limsup_{n\to\infty} \sup_{f\varepsilon F} |a_n(f)|^{1/n} \geq 1/R.$$

Q.E.D.

Definition 8.5: D will be called a domain of regularity of F if each $f\varepsilon F$ is analytic in D, F is normal in D, and F is uniformly bounded on compact subsets of D.

Definition 8.6: A point z on the boundary of D will be called a regular point of F if there is a neighborhood about z such that the union of the neighborhood and D is a domain of regularity.

Definition 8.7: A point z on the boundary of D will be called a singular point of F if and only if z is not a regular point of F.

Definition 8.8: Let $D = \{z: |z| < R\}$. If no point of $\{z: |z| = R\}$ is a regular point of F, $\{z: |z| = R\}$ is called a cut for F.

Lemma 8.9: Let $z = (w^q + w^{q+1})/2$ where q is a positive integer. If $|w| \leq 1$, $w \neq 1$, then $|z| < 1$.

Proof: It is clear that if $|w| < 1$, then $|z| < 1$. Suppose $|w| = 1$ and $|z| = 1$. Then we have

$$|1 + w| = 2.$$

Let $w = u + iv$. We obtain

$$(1+u)^2 + v^2 = 4$$

This is the equation of a circle about $u = -1$, $v = 0$, of radius 2. But the only point on this circle with modulus = 1 from the origin in $w = 1$. This is a contradiction. Thus $|z| < 1$ for $|w| \leq 1$, $w \neq 1$.

Q.E.D.

Lemma 8.10: Suppose $\lim_{n\to\infty} a_n = a > 0$ and $\limsup_{n\to\infty} b_n = b > 0$. Then $\limsup_{n\to\infty} a_n b_n = ab$.

Proof: Let $\varepsilon > 0$ be given. Then, for sufficiently large n we have

$$a_n b_n < (a+\varepsilon)(b+\varepsilon)$$

Since $\lim_{n\to\infty} a_n = a$, $a_n > a-\varepsilon$ for sufficiently large n. By the definition of lim sup, $b_n > b - \varepsilon$ for infinitely many n. Thus

$$(a-\varepsilon)(b-\varepsilon) < a_n b_n$$

for infinitely many n.

Therefore

$$\limsup_{n\to\infty} a_n \, b_n = ab.$$

Q.E.D.

<u>Theorem 8.11</u>: Let F be a family of functions. Let $f(z) = \sum_{n=0}^{\infty} a_n(f)z^n$, where $f(z)\varepsilon F$. Suppose F has radius of regularity $R = 1$ and F is uniformly bounded on compact subsets of $\{z:|z| < 1\}$. Suppose that to each function there corresponds a sequence of integers, $\{\lambda_p(f)\}$, such that $a_n(f) = 0$ if $n \neq \lambda_p(f)$. Also assume $\lambda_{p+1}(f)/\lambda_p(f) \geq \lambda > 1$ where λ is a constant independent of f. Then $\{z:|z| = 1\}$ is a cut for F.

<u>Proof</u>: Let q be a positive integer such that $q > 1/(\lambda-1)$. Let $z = (w^q+w^{q+1})/2$. Define a family of functions, G, by

$$g(u) = f(w^q+w^{q+1}/2) = \sum_{n=0}^{\infty} a_n(f)(w^q+w^{q+1}/2)^n$$

$$= \sum_{m=0}^{\infty} b_m(g)w^m.$$

By Lemma 8.9, $|w| \leq 1$, $w \neq 1$, implies $|z| < 1$. This means that each point of $\{w:|w| = 1\}$ is a regular point of G except possibly $w = 1$. Therefore, $R(G) \geq 1$. We wish to show that R(G) = 1, implying that w = 1 is a singular point of G. Then z = 1 is a singular point of F.

Now consider the polynomials

$$(w^q+w^{q+1}/2)^{\lambda_\ell(f)} \quad \text{and} \quad (w^q+w^{q+1}/2)^{\lambda_k(f)}$$

where $\ell \neq k$. They have no common powers of w. So we can express each coefficient, $b_m(g)$, in terms of a single $a_n(f)$. We have

$$b_{q_n+j} = (a_n(f)/2^n)C_{j,n}, \qquad 0 \le j \le n$$

where $C_{j,n}$ is the binomial coefficient $\binom{n}{j}$.

Let $m_n = q_n + [n/2]$, where [n/2] denotes the greatest integer less than or equal to n/2. We have

$$b_{m_n}(g) = (a_n(f)/2^n)C_{[n/2],n}.$$

Since R(F) = 1,

$$\limsup_{n\to\infty} \sup_{f\in F} |a_n(f)|^{1/n} = 1$$

It is also true that

$$\lim_{n\to\infty} [C_{[n/2],n}/2^n]^{1/m_n} = 1.$$

Then, by Lemma 8.10 we have

$$\limsup_{n\to\infty} \sup_{g\in G} |b_{m_n}(g)|^{1/m_n} = 1.$$

Thus

$$1/R(G) = \lim_{m\to\infty} \sup \sup_{g\in G} |b_m(g)|^{1/m}$$

$$\geq 1.$$

Therefore $R(G) \leq 1$, and so we have $R(G) = 1$. This means $w = 1$ is a singular point of G, and so $z = 1$ is a singular point of F.

If we consider the family of functions defined by

$$f(t) = \sum_{n=0}^{\infty} a_n(e^{i\theta}t)^n$$

where $z = e^{i\theta}t$, we see that this family satisfies the hypotheses of the theorem. But $t = 1$ is a singular point of this family, and so $z = e^{i\theta}$ is a singular point of F.

Therefore $\{z:|z| = 1\}$ is a cut for F.

Q.E.D.

We set $S_n(f,z) = \sum_{k=0}^{n} a_k(f)z^k$, and we call $\sum = \{S_n(f,z)\}$ the complete family of partial sums of F. By a subfamily of $\sum$ we mean a subset of $\sum$ determined by a family of sequences of integers, $\{m_n(f)\}$, where, for every f, $0 \leq m_1(f) < m_2(f) < \ldots$, $m_n(f) \to \infty$. We denote $\{S_{m_k(f)}(f,z)\}$ by $\sum[m_k(f)]$ and $\sum[n] = \sum$.

Definition 8.12: The subfamily of partial sums, $\sum[m_n(f)]$, converges uniformly to F on a set Δ if, given $\varepsilon > 0$, there is an $N(\varepsilon,\Delta)$, independent of f, such that $|S_{m_n(f)}(f,z)-f(z)|<\varepsilon$ for $n > N$, $f\varepsilon F$, $z\varepsilon\Delta$.

Lemma 8.13: If F has positive radius of regularity R, then every subfamily of partial sums, $\sum[m_n(f)]$, converges uniformly to F on every compact subset of $\{z:|z| < R\}$.

Proof: Suppose $\exists$ a compact subset Δ and a subfamily of partial sums, $\sum[m_n(f)]$, which does not converge uniformly to F on Δ.

Then $\exists$ $\varepsilon > 0$, $z\varepsilon\Delta$, and a sequence of functions $\{f_k(z)\}$ such that

$$|f_k(z) - S_{m_n(f_k)}(f_k,z)| \geq \varepsilon.$$

Since F is normal in Δ, there is a subsequence, $\{f_{k_j}(z)\}$, of $\{f_k(z)\}$ which converges uniformly on Δ to $f(z)$, $f(z)$ an analytic function.

We have

$$|f_k(z) - S_{m_n(f_k)}(f_k,z)| \leq |f_k(z)-f(z)|$$

$$+ |f(z) - S_n(f,z)| + |S_n(f,z) - S_{m_n(f_k)}(f_k,z)|$$

where $S_n(f,z)$ is the n^{th} partial sum of $f(z)$. Each term on the right can be made arbitrarily small. This is a contradiction.

Q.E.D.

Definition 8.14: Let F have radius of regularity R. We say that a subfamily $\sum[m_n(f)]$ of the partial sums of F is overconvergent if $\{z:|z| = R\}$ is not a cut for F, and if there is a simply connected domain D containing

$\{z:|z| < R\}$ and the points of $\{z:|z| = R$ where F is regular such that $\sum[m_n(f)]$ converges uniformly to F on every compact subset of D.

Lemma 8.15: Let F have radius of regularity R. Then the complete family of partial sums is never overconvergent.

Proof: We assume $\{z:|z| = R\}$ is not a cut for F, and that $\sum$ is overconvergent. By definition 8.14 $\exists$ z_0, $|z_0| > R$, where F is uniformly bounded. Assume $|f(z_0)| \leq M$ for all $f \varepsilon F$. By definition 8.12 there is an $N(1,z_0)$ such that $|S_n(f,z_0)| \leq M+1$ for $n > N$, for all $f \varepsilon F$. Then for $n > N$, and for all $f \varepsilon F$ we have

$$|a_n(f) z_0{}^n| = |S_n(f,z_0) - S_{n-1}(f,z_0)|$$

$$\leq 2(M+1).$$

Therefore

$$\sup_{f \varepsilon F} |a_n(f)|^{1/n} |z_0| \leq [2(M+1)]^{1/n}$$

Thus

$$\limsup_{n\to\infty} \sup_{f \varepsilon F} |a_n(f)|^{1/n} |z_0| \leq 1$$

This gives

$$(1/R)|z_0| \leq 1.$$

But this is a contradiction.

Q.E.D.

Definition 8.16: We say that F has gaps of Hadamard type if there is a positive number θ and, for each $f \epsilon F$, a pair of sequences of positive integers, $\{\lambda_n(f)\}$, $\{\mu_n(f)\}$, such that $\lambda_n(f) \leq \mu_n(f)$, $a_k(f) \neq 0$ implies $\lambda_n(f) \leq k \leq \mu_n(f)$ for some n and $\lambda_{n+1}(f) \geq (1+\theta)\mu_n(f)$.

Theorem 8.17: Suppose F has radius of regularity R = 1 and that $\{z:|z| = 1\}$ is not a cut for F. Then if F has gaps of Hadamard type, the subfamily of partial sums, $\sum[\mu_n(f)]$, is overconvergent.

Proof: Let D be a simply connected domain containing $\{z:|z| < 1\}$ and the points of $\{z:|z| = 1\}$ at which F is regular. We assume $1 \epsilon D$, and we show that $\sum[\mu_n(f)]$ converges uniformly to F on $\{z:|z-1| \leq \alpha\}$ for some positive α. Then the rotation used in Theorem 8.11 suffices to show over-convergence at each point $z = e^{i\theta}$ where F is regular.

Let p be a positive integer such that $1/p < \theta$. Let $z = w^p+w^{p+1}/2 = \phi(w)$. We define a family of functions G by

$$g(w) = f(\phi(w))$$

$$= \sum_{n=0}^{\infty} a_n(f)(\phi(w))^n$$

$$= \sum_{n=0}^{\infty} b_n(g)w^n.$$

By Lemma 8.9, we have $|w| \leq 1$, $w \neq$, implies $|\phi(w)| < 1$. Since $\phi(1) = 1 \epsilon D$, there is an $\varepsilon>0$ such that G is normal and uniformly bounded on compact subsets of $\{w:|w| < 1+\varepsilon\}$.

By Lemma 8.13 the subfamily of partial sums of G determined by the family of sequences, $\{(p+1)\mu_n(f)\}$, converges uniformly to G on $\{w:|w| \le 1+\varepsilon/2\}$. The image of $\{w:|w| \le 1+\varepsilon/2\}$ under $z = \phi(w)$ contains a disc about $z = 1$. Let this disc be $\{z:|z-1| \le \alpha\}$ where $\alpha > 0$. We have

$$\sum_{k=0}^{\mu_n(f)} a_k(f)\left(w^p+w^{p+1}/2\right)^k = \sum_{k=0}^{(p+1)\mu_n(f)} b_k(g)w^k.$$

The family on the right converges uniformly to G on $\{w:|w| \le 1+\varepsilon/2$. Therefore the family on the left converges uniformly to F on $\{z:|z-1| \le \alpha\}$. So the family is overconvergent at $z = 1$. By the remark at the beginning of the proof, the family is overconvergent at each point of $\{z:|z| = 1\}$ where F is regular.

Q.E.D.

<u>Theorem 8.18</u>: Let F have radius of regularity $R = 1$. Suppose that the subfamily, $\sum[m_n(f)]$, of partial sums of F is overconvergent. Then every $f\varepsilon F$ can be written as the sum of two power series, $f(z) = g(z) + h(z)$, where the family $\{g(z)\}$ has gaps of Hadamard type and the family $\{h(z)\}$ has radius of regularity greater than 1.

Proof: Let D be a simply connected domain containing the $\{z:|z| < 1\}$ and the points of $\{z:|z| = 1\}$ at which F is regular. Let Δ be a compact subset of D which also contains $\{z:|z| < 1\}$ and the points of $\{z:|z| = 1\}$ at which F is regular. By hypothesis there exists $N(1,\Delta)$ such that

$$|f(z)-S_{m_n(f)}(f,z)| \leq 1$$

for $n > N$ and for all $f \varepsilon F$.

The analysis in the proof of Theorem 7.5 applies uniformly to F. Therefore we have constants ρ, $0 < \rho < 1$, and $\theta > 0$ such that

$$|h_n(f)| < \rho^n$$

for $m_n(f) < n < (1+\theta)m_n(f)$, for every $f \varepsilon F$ and $n > N$.

We can find $\varepsilon > 1$ such that

$$|h_n(f)|\varepsilon^n < \rho^n$$

for every $f \varepsilon F$.

The family $\{h(z)\}$, where $h(z) = \sum h_n(f)z^n$, $m_n(f) \leq n < (1+\theta)m_n(f)$, is a family of functions each of which is analytic in a disc of radius greater than 1. Since $\rho < 1$, $\sum \rho^n$ is convergent and this series provides a uniform bound for $\{h(z)\}$. Therefore $\{h(z)\}$ has radius of regularity greater than 1.

The family $\{g(z)\}$ formed from the remaining coefficients has gaps of Hadamard type, where we take the sequences $\{\lambda_n(f)\}$ and $\{\mu_n(f)\}$ to be identical and equal to $\{m_n(f)\}$.

Q.E.D.

BIBLIOGRAPHY

[1] Ahlfors, L. V., Complex Analysis. New York: McGraw-Hill, Inc., 1953.

[2] Dienes, P., The Taylor Series. London: Oxford University Press, 1931.

[3] Hille, E., Analytic Function Theory. Vol. II. New York: Ginn and Co., 1962.

[4] Johnson, G., "Collective Singularities of a Family of Analytic Functions," Proc. Amer. Math. Soc. (7), (1956), pp. 653-655.

[5] Nehari, Z., Conformal Mapping. New York: McGraw-Hill, Inc., 1952.

[6] Wilson, A., "The Overconvergence of Bounded Normal Families of Analytic Functions," Proc. Amer. Math. Soc. (12), (1961), pp. 498-501.

VITA

The author was born on October 1, 1942 in Passaic, New Jersey to parents John and Helen. He entered Fordham University in 1960 and was graduated in 1964 with a B. S. in mathematics. Since 1964 the author has been attending Lehigh University under a National Defense Education Act Fellowship and is currently working towards the degree of Doctor of Philosophy there. He is a member of the Mathematical Association of America and the American Mathematical Society.