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Combinatorial aspects of Hecke algebra characters

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Combinatorial aspects of Hecke algebra characters

by

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Combinatorial aspects of Hecke algebra characters

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Abstract

Iwahori-Hecke algebras are deformations of Coxeter group algebras. Their origins lie in the theory of automorphic forms but they arise in the representation theory of Coxeter groups and Lie algebras and in quantum group theory. The Kazhdan-Lusztig bases of these algebras, originally introduced in the late 1970s in connection with representation-theoretic concerns, has turned out to have deep connections to Schubert varieties, intersection cohomology, and related topics.

Matrix immanants were originally introduced by Littlewood as a generalization of determinants and permanants. They remained obscure until the 1980s when their connections to symmetric function and representation theory as well as their surprising algebraic and combinatorial properties came to light. In particular, it was discovered that they have a fruitful connection to the theory of total positivity. More recently, a theory of quantum immamants was developed, providing a bridge to the quantum group theory.

In this paper we develop the theory of certain planar networks, which provide a unified combinatorial setting for these fields of study. In particular, we use these networks to evaluate certain characters of the symmetric group algebra. We give new combinatorial interpretations of the quantum induced sign and trivial characters of the type A Iwahori-Hecke algebras.

Introduction

The main topic of this paper is the representation theory of Iwahori-Hecke algebras, which are deformations of Coxeter group algebras. Their origins lie in the theory of automorphic forms but they arise in the representation theory of Coxeter groups and Lie algebras and in quantum group theory. From the latter point of view, they can be viewed as a quantization of the Weyl group of a Lie algebra; in particular, there is a quantum Schur-Weyl duality between the representation theory of certain Hecke algebras and the representation theory of corresponding quantum groups. In particular, Jimbo (see [32], [31]) showed that their representation theory could be used to generate solutions to the Yang-Baxter equation. The study of the representation theory of Hecke algebras led to the development of Kazhdan-Lusztig polynomials, introduced by Kazdhan and Lustig in [36], [37]. These polynomials arose as the structure constants for certain Hecke algebra bases. The coefficients of these polynomials encode a great deal of geometric and representation theoretic information, and their study has become a field of its own. For example, see [5], [6], [13], [8], [14], [12], [19], or [22]. An important recent result is the general interpretation of Kazhdan-Lustig coefficients, valid in all Coxeter systems, given by Elias and Williamson in [17], settling the 1979 positivity conjecture of Kazhdan and Lusztig.

Matrix immanants were originally introduced by Littlewood in [40]. They can be viewed as a family of matrix functions that interpolates between the permanent and the determinant. They remained mostly ignored until the 1985 paper of Merris and Watkins [42]. The combinatorial theory of matrix imminants was subsequently developed by Goulden-Jackson [25], Greene [26], Stembridge [51], [52], and Haiman

[28], among others. In particular, it was discovered that they have a fruitful connection to the theory of total positivity, whose connections with Lie theory were being developed around the same time (cf. [41]). An overview of the combinatorial study of positivity is given in [7]. It should be mentioned that a major source of motivation for the work in this dissertation was the conjectures given by Stembridge in [52].

The heart of this paper is the study of planar networks, which provide a unified combinatorial setting for these fields. Totally positive matrices arise as the path matrices of planar networks, and their immanants can be combinatorially evaluated using them. Many of the positivity phenomena regarding immanants (see for example [52], [26], [51]) can therefore be understood in this light. Planar networks are also closely connected to Kazhdan-Lusztig theory, where they give a combinatorial model for certain Kazhdan-Lusztig basis elements (cf. [46], [2]).

In the first chapter, we introduce the basic definitions that we will need from the representation theory of the symmetric group.

In the second chapter, we introduce two families of planar networks, the *descending star networks* and the *zig-zag networks*. We give some of their combinatorial properties and introduce their connection to the symmetric group. We then introduce F-tableaux, which are a generalization of Young tableaux to the planar network setting. These tableaux are the main tool which is used to obtain the results in this paper.

In the third chapter we will give combinatorial interpretations of the η, χ and ϕ characters evaluated at the combinatorial $\mathbf{C}[S_n]$ elements introduced in Chapter 2. None of these interpretations are wholly new, although the χ and ϕ interpretations are expressed in different terms than have appeared in the past. The main purpose of the section is to unify these results using the combinatorics of F-tableaux. In particular, we hope that the work on the combinatorics of the ϕ character will aid efforts to prove Stanley's e-positivity conjecture (see [47] and [28]). Recent work of Morales, Guay-Paquet, and Rowland [27] shows descending star networks are the only networks needed in this context.

In the fourth chapter we introduce quantizations of the algebraic and combinatorial constructions from Chapters 1 and 2. In particular, we define the Iwahori-Hecke algebras, quantum matrix algebras, Kazhdan-Lusztig bases, and quantum immanants. Finally, we give the two main results of this thesis, combinatorial interpretations of the quantum induced trivial and induced sign characters of the type A Iwahori-Hecke algbera.

Chapter 1

Representation theory of the symmetric group

1.1 The symmetric group

Probably the most fundamental realization of the symmetric group is as the group of automorphisms of the set $[n] := \{1, \ldots, n\}$. This is often referred to in the literature as the "group of permutations" of $[n]$ - a description that elides some ambiguity. Further ambiguity is introduced in the identification of this group with the abstract Coxeter group of type A_{n-1}. To avoid confusion, I will start by carefully defining the conventions that this document will use. An account of the symmetric group from a point of view similar to ours is given in [43]. A thorough treatment of the symmetric group from the point of view of permutations is given in [4].

We define the symmetric group S_n to be the group of automorphisms of $[n]$, with group operation given by composition of functions. We define a permutation of $[n]$ to be a sequence $p = [p_1, \ldots, p_n]$ so that each element of $[n]$ appears exactly once in p. We will sometimes omit the commas and/or braces in this notation. S_n acts from the left on permutations of $[n]$ by $w[p_1, \ldots, p_n] = [w(p_1), \ldots, w(p_n)]$, and from the right by $[p_1, \ldots, p_n]w = [p_{w(1)}, \ldots, p_{w(n)}]$. These actions agree on the identity permutation $[1, \ldots, n]$. For a symmetric group element $w \in S_n$, we define its *one-line notation* to be the permutation $w[1, \ldots, n] = [1, \ldots, n]w = [w(1), \ldots, w(n)]$. Clearly, this is a bijection between automorphisms and permutations of $[n]$; we will often conflate the permutation and its one line notation. Sometimes, we will use the notation $[w_1, \ldots, w_n]$ to refer to the one line notation of w.

Formally, the Coxeter group of type A_n is defined to be the group generated by elements $s_1, \ldots, s_n$ with relations:

$$\begin{aligned} s_i^2 &= e \\ s_i s_j &= s_j s_i & \qquad |i-j| > 1 \\ s_i s_j s_i &= s_j s_i s_j & \qquad |i-j| = 1 \end{aligned}$$

Note that here and throught this manuscript e refers to the identity element of the group under discussion. We will identify the Coxeter group of type A_n with the symmetric group S_{n+1} by associating each generator s_i to the transposition $[1, \ldots, i+1, i, \ldots, n+1]$ (notice that we have begun to conflate words and symmetric group elements, as promised). It is a standard fact that this identification is an isomorphism; we will often use it to conflate Coxeter groups with the corresponding symmetric groups.

It should be noted that expressions $w = s_{i_1} \cdots s_{i_n}$ are not unique (with respect to w). Define the *length* $l(w)$ of a Coxeter group element w to be the smallest number l so that there is an expression $w = s_{i_1} \cdots s_{i_l}$. We call such an expression *reduced*; reduced expressions still, in general, fail to be unique.

Given a permutation $p = [p_1, \ldots, p_n]$ we say that a pair $p_i = a, p_j = b$ is an *inversion* in p if $i < j$ and $a > b$. Define $\mathrm{inv}(p)$ to be the number of inversions in p. Then, conflating all our notions of the symmetric group, we have that $l(w) = \mathrm{inv}(w)$ for $w \in S_n$.

The Coxeter group structure also induces a poset structure on S_n known as the *Bruhat order*. This order is defined by $u < v$ if there is a reduced expression $s_{i_1} \cdots s_{i_l}$ for v with a subexpression $s_{i_{x_1}} \cdots s_{i_{x_m}}$ that is a reduced expression for u. Note that length is a grading of S_n with respect to this poset structure; that is, we have $u < v \implies l(u) < l(v)$.

We conclude our definition of the symmetric group by introducing cycle notation. We define the *cycle* $(x_1, \ldots, x_k)$ to be the S_n element w satisfying $w(x_i) = x_{i+1}$ for $i < k$, $w(x_k) = x_1$, and $w(j) = j$ for $j \notin \{x_1, \ldots, x_k\}$. We can the write any S_n element as a product of disjoint cycles; eg: $[5\,2\,4\,1\,3\,7\,6] = (1\,5\,3\,4)(2)(6\,7)$. Regarding two cycles as being identical if they represent the same permutation, such an expression is unique up to the order of the cycles. Let $\lambda_1, \lambda_2, \ldots$ be the lengths of the cycles in the disjoint cycle expression of w written in weakly decreasing order; then $\lambda = [\lambda_1, \lambda_2, \ldots]$ is called the *cycle type* of w. We say that a disjoint product of cycles is *canonical* if each cycle is written with its largest entry first, and the cycles are ordered from left to right by their largest elements, from smallest to largest.

We will illustrate our various conventions by completely working out the simplest nontrivial case, S_3, in Figure 1.1.

1.2 Representation theory of $\mathbf{C}[S_n]$

We will briefly review the neccesary concepts from representation theory; for more information, [43], [30], [20], or [18] are good references. In particular, [11] provides a comprehensive treatment which also introduces Iwahori-Hecke algebras, which will be a major topic of this paper. A *representation* (V, ρ) of an algebra A over a field k is a vector space V over k together with an algebra homomorphism $\rho : A \to \mathrm{End}_k V$.

If we have such a map we say that V is an A-module with the action given by $av = \rho(a)v$. We will often refer to representations by simply referencing either the space or the map; it is understood that both exist. All representations will be assumed to be finite-dimensional (that is, the underlying vector space is finite dimensional).

The choice of any basis for V gives a homomorphism $A \to \mathrm{Mat}_{dxd}(k)$, where d is the dimension of V and $\mathrm{Mat}_{d\mathrm{x}d}(k)$ denotes the algebra of d by d matrices over k. We will also refer to such a homomorphism as a representation. A morphism between two representations (V, ρ) and (W, σ) is a vector space morphism $f : V \to W$ that commutes with the representation maps; ie, $f(\rho(x)v) = \sigma(x)f(v)$ for all $x \in A$, $v \in$

w as a …			
function	permutation	reduced expression	Π of cycles
$1 \mapsto 1$ $2 \mapsto 2$ $3 \mapsto 3$	$[1\,2\,3]$	e	$(1)(2)(3)$
$1 \mapsto 2$ $2 \mapsto 1$ $3 \mapsto 3$	$[2\,1\,3]$	s_1	$(2\,1)(3)$
$1 \mapsto 1$ $2 \mapsto 3$ $3 \mapsto 2$	$[1\,3\,2]$	s_2	$(1)(3\,2)$
$1 \mapsto 2$ $2 \mapsto 3$ $3 \mapsto 1$	$[2\,3\,1]$	$s_1 s_2$	$(3\,1\,2)$
$1 \mapsto 3$ $2 \mapsto 1$ $3 \mapsto 2$	$[3\,1\,2]$	$s_2 s_1$	$(3\,2\,1)$
$1 \mapsto 3$ $2 \mapsto 2$ $3 \mapsto 1$	$[3\,2\,1]$	$s_1 s_2 s_1 = s_2 s_1 s_2$	$(2)(3\,1)$

Figure 1.1: Realizations of S_3

V. If f is a vector space isomorphism, then it is an isomorphism of representations. In matrix terms, an isomorphism between representations $\rho, \sigma : A \to \mathrm{Mat}_{d\mathrm{x}d}(k)$ is an algebra automorphism f of $\mathrm{Mat}_{d\mathrm{x}d}(k)$ satisfying $f(\rho(x)) = \sigma(x)$. Up to isomorphism, the two definitions of a representation are equivalent, justifying our abuse of the terminology.

A direct sum of representations $V \oplus W$ of A is a representation of A under the map $\rho \oplus \sigma$, the tensor product $V \otimes_k W$ is a representation under $\rho \otimes \sigma$. These operations give the (isomorphism classes of) representations of A a ring. A representation is *indecomposable* if it is not isomorphic (as a representation) to a direct sum of nontrivial representations and *irreducible* if it contains no nontrivial subspace that is closed under the action of A. Clearly, an irreducible representation is indecomposable. We say that A is *semisimple* (cf. [18, Prop. 2.16]) if these conditions are equivalent for all its representations; all of the algebras that we work with are semisimple.

1.2.1 Characters

Given a representation ρ of A, we can define a map $\tau : A \to k$ by taking its trace; that is, by setting $\tau(x) = \mathrm{Tr}(\rho(x))$. This map is called the *character* of ρ. For representations ρ, σ of A, we have $\mathrm{Tr}(\rho \oplus \sigma) = \mathrm{Tr}(\rho) + \mathrm{Tr}(\sigma)$ and $\mathrm{Tr}(\rho \otimes \sigma) = \mathrm{Tr}(\rho)\mathrm{Tr}(\sigma)$. Thus, positive $\mathbf{Z}$-linear combinations of A-characters are themselves A-characters. Call a function $f : A \to k$ a *trace* if it satisfies $f(xy) = f(yx)$ for all $x, y \in A$. Since the ordinary matrix trace (note that we will always use *the* trace to mean the ordinary trace, and *a* trace in the sense just defined) satisfies this condition, characters are a subset of traces.

1.2.2 Representations of $\mathbf{C}[S_n]$

Let $\mathbf{C}[S_n]$ be the *group algebra* of S_n, that is, the $\mathbf{C}$ algebra of formal $\mathbf{C}$-linear combinations of S_n elements. It is well known that the representation theory of $\mathbf{C}[S_n]$ (as an associative algebra) is equivalent to the representation theory of S_n as a group (this is true for group algebras generally). The fundamental result in the representation theory of $\mathbf{C}[S_n]$ is that it is semisimple and a complete list of its (isomorphism classes of) irreducible representations is given by the *Schur modules.* These are indexed by *partitions* of n. A partition λ of n is a sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0$ with $\sum_i \lambda_i = n$. We will use the notation $\lambda \vdash n$ to mean that lambda is a partition of n. Then let $\{S^\lambda \mid \lambda \vdash n\}$ be the set of irreducible representations of $\mathbf{C}[S_n]$. In fact there is a canonical association between partititions and irreducible S_n representations, which we will give shortly.

Let $\lambda = \lambda_1, \ldots, \lambda_d$ be a partition of n and set $c_i = \lambda_1 + \cdots + \lambda_i$ for $0 < i < d$, and $c_0 = 0$. Let $S_\lambda \subset S_n$ be the subgroup defined by requiring, for all $w \in S_\lambda$, that $w(x) \in [c_i, c_{i+1}]$ for any $x \in [c_i, c_{i+1}]$. Then S_λ is naturally isomorphic to $S_{\lambda_1} \times \cdots \times S_{\lambda_d}$. (In fact S_λ is a "parabolic subgroup" of S_n. For a more complete discussion of these subgroups see eg. [3, Section 2.4])

There are two important operations on group representations; induction and restriction. For a group G, given a $\mathbf{C}[G]$ representation (ρ, V) and a subgroup $H \subset G$, the *restriction* $\rho \downarrow_H$ is simply the restriction of the function ρ to the subalgebra $\mathbf{C}[H]$. *Induction* is the adjoint operation to restriction, and can be defined by $V \uparrow_H^G := \mathbf{C}[G] \oplus_{\mathbf{C}[H]} V$.

Denote the $\mathbf{C}[S_n]$ traces by R^n, and let $R := \oplus_i R^i$. R^n has a natural vector space structure given by pointwise addition of traces. Let σ, τ be the characters of representations V, W of S_m and S_n respectively. Then we can define a product by $\sigma \cdot \tau := \mathrm{Tr}((V \otimes W) \uparrow_{S_m \times s_n}^{S_{m+n}})$. This product is compatible with addition and with the grading of R. Since the characters are a spanning set for R, the product can be extended to give R the structure of a graded algebra, which we will call the *trace algebra*. A natural inner product on R^n is given by

$$\langle \tau, \sigma \rangle := \frac{1}{n!} \sum_{w \in S_n} \overline{\tau(w)} \sigma(w) \,.$$

We can now specify the association between partitions and irreducible representations. There are two one dimensional representations of S_n: the *sign representation* sgn which acts by $w \mapsto [(-1)^{l(w)}]$, and the *trivial representation* 1 which acts by $w \mapsto [1]$. Since these representations are one dimensional, we can conflate them with their characters. It turns out that for each $\lambda \vdash n$, there is a unique irreducible representation, which we will denote S^λ, whose character χ^λ satisfies both $\langle \chi^\lambda, 1 \uparrow_{S_\lambda}^{S_n} \rangle \neq 0$ and $\langle \chi^\lambda, \mathrm{sgn} \uparrow_{S_{\lambda^t}}^{S_n} \rangle \neq 0$ (here λ^t denotes the transpose of λ, see (2.2) for a precise definition). These representations can be explicitly realized using Young tableaux and are called the *Specht modules*.

Any $\mathbf{C}[S_n]$ representation is determined by its character. Let χ^λ denote the character of S^λ and call the set $\{\chi^\lambda \mid \lambda \vdash n\}$ the *irreducible characters* of $\mathbf{C}[S_n]$.

It turns out that this set forms a basis of the space of $\mathbf{C}[S_n]$ traces. In fact, it is an orthonormal basis. We will obtain several other natural bases by exploiting a connection between the character theory of $\mathbf{C}[S_n]$ and the theory of symmetric functions.

1.2.3 Symmetric functions

Let $\mathbf{x} = \{x_1, x_2, \dots\}$. Let $\Lambda(\mathbf{x})$ (or simply Λ if there is no ambiguity) denote the ring of *symmetric polynomials*, that is, polynomials in $\mathbf{C}[\mathbf{x}]$ that are invariant (for all n) under the S_n-action given by $w(x_i) := x_{w(i)}$. Let Λ^n denote the symmetric polynomials of homogenous degree n; then we have $\Lambda = \oplus_i \Lambda^i$. The most natural basis for Λ^n is the monomial basis $\{m_\lambda \mid \lambda \vdash n\}$ defined by setting, for $\lambda = \lambda_1, \dots, \lambda_d$,

$$m_\lambda = \sum_{i_1 \neq \cdots \neq i_d} x_{i_1}^{\lambda_1} \cdots x_{i_d}^{\lambda_d} \, .$$

where the sum is over sequences which yield distinct monomials of the given form. To be precise, we can take the sum over all choices of $i_1, \dots, i_d$ such that if $\lambda_a = \lambda_b$ and $a < b$, we have $i_a < i_b$.

We will use several other bases for Λ^n. Let

$$\begin{aligned} e_d &:= \sum_{i_1 < \cdots < i_d} x_{i_1} \cdots x_{i_d} \, , \\ h_d &:= \sum_{i_1 \leq \cdots \leq i_d} x_{i_1} \cdots x_{i_d} \, , \text{ and} \\ p_d &:= \sum_i x_i^d \, . \end{aligned}$$

Then the elementary, complete, and power sum symmetric functions, denoted e_λ, h_λ, and p_λ, respectively, are defined by $e_\lambda := e_{\lambda_1} \cdots e_{\lambda_d}$, $h_\lambda := h_{\lambda_1} \cdots h_{\lambda_d}$, and $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_d}$. Finally, we define the Schur function s_λ by

$$s_\lambda := \det(H) = \det(E)$$

where H is the d by d matrix whose i, j entry is $h_{\lambda_i - i+j}$ and E is the λ_1 by λ_1 matrix whose i, j entry is $e_{\lambda^t_i - i+j}$. Here we define $h_i = e_i = 0$ for $i < 0$, and λ^t to be the partition $\lambda^t := \#\{i \mid \lambda_i \geq \lambda_1\}, \#\{i \mid \lambda_i \geq \lambda_1 - 1\} \ldots, \#\{i \mid \lambda_i \geq 1\}$. That these two determinants are equal is a theorem due to Jacobi and Trudi. Each of these sets forms a basis for Λ^n. This can be seen explicitly therough the combinatorial study of the transition matrices between them, which are treated thoroughly in [1]. A more complete discussion of the basic combinatorial theory of symmetric functions can be found in [49, Ch. 7].

1.2.4 The characteristic map

For a vector space V, $\mathbf{C}[S_n]$ naturally acts on $V^{\otimes n}$ by

$$w(v_1 \otimes \cdots \otimes v_n) := w(v_1) \otimes \cdots \otimes w(v_n)\,,$$

and $\mathrm{GL}(V)$ acts by

$$X(v_1 \otimes \cdots \otimes v_n) := Xv_1 \otimes \cdots \otimes Xv_n\,.$$

It turns out that there is a decomposition

$$V^{\otimes n} = \oplus_{\lambda \vdash n} V_\lambda \otimes G_\lambda\,,$$

where G_λ are distinct irreducible representations of $\mathrm{GL}(V)$. Furthermore, the characters of the representations G_λ are given, for $\mathrm{GL}(V)$, by polynomials in the eigenvalues of the elements of $\mathrm{GL}(V)$. These polynomials are clearly symmetric, as the constructions are coordinate free. This gives a map from $\mathbf{C}[S_n]$ characters to symmetric polynomials called the characteristic map. We will explicitly define this map and give its important properties.

We also have the *Hall inner product* on Λ_n, which can be defined by setting

$$\langle h_\lambda, m_\mu \rangle := \delta_{\lambda\mu}\,.$$

Then Schur functions are an orthonormal basis with respect to this inner product. We can then define the *characteristic map* ch from $\mathbf{C}[S_n]$ traces to symmetric polynomials by $\mathrm{ch}(\chi^\lambda) := (s_\lambda)$. In addition to preserving the inner product, this map gives an algebra isomorphism between the trace algebra R and Λ.

Using ch, we obtain bases for the space R^n of $\mathbf{C}[S_n]$ traces corresponding to the bases $e_\lambda, h_\lambda, p_\lambda$, and m_λ. We denote these bases by ϵ^λ, η^λ, ψ^λ, and ϕ^λ, respectively. It turns out that we have

$$\begin{aligned} \eta_\lambda &= 1 \uparrow_{S_\lambda}^{S_n}, \\ \epsilon_\lambda &= \mathrm{sgn} \uparrow_{S_\lambda}^{S_n} \text{, and} \\ \psi\lambda(w) &= z_w \delta_{\mathrm{sh}(w),\lambda}, \end{aligned}$$

where $\mathrm{sh}(w)$ is the cycle type of w, z_λ is the order of the centralizer of w, and δ is the Kronecker delta. In general, neither ϕ_λ nor ψ_λ are characters.

Chapter 2

Planar Networks

2.1 Positivity and planar networks

For a matrix $A \in \text{Mat}_{nxn}$ given by $A = [a_{i,j}]$, $I, J \subset [n]$, let $A_{I,J}$ denote the submatrix consisting of all entries $a_{i,j}$ such that $i \in I$, $j \in J$. A matrix is said to be *totally positive* (respectively, *totally nonnegative*) if $\det(A_{I,J})$ is positive (nonnegative) for all $I, J \subset [n]$ with $|I| = |J|$. In this section we will introduce the notion of a planar network and outline the basic connection between planar networks and positivity theory. This theory has far reaching applications and generalizations. A survey of the history of total positivity with emphasis on its applications in Lie theory can be found in [41]. An overview of the combinatorial study of positivity is given in [7].

A *planar network* F of order n is an acyclic weighted directed planar graph embedded in a disk, with $2n$ boundry vertices labelled (in order) $s_1, \ldots, s_n, t_1, \ldots, t_n$. We call the vertices $s_1, \ldots, s_n$ *sources* and $t_1, \ldots, t_n$ *sinks.* By convention we require that all sources have indegree 0 and all sinks have outdegree 0. We can therefore infer the direction of the edges from context. We will also assume all edge weights are 1 unless otherwise indicated. An example of a planar network is given in Figure 2.1

The *weight* $\text{wt}(\pi)$ of a path π is the product of the weights of its edges. The *path*

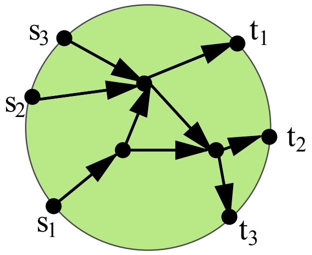


Figure 2.1: Example of a planar network

matrix $A(F)$ of a planar network F is the n by n matrix defined by

$$A(F) = [a_{i,j}],\ a_{i,j} = \sum_{\pi} \mathrm{wt}(\pi)$$

where the sum is over all paths from s_i to t_j for each $a_{i,j}$. A *path family* $\Pi = \{\pi_1, \ldots, \pi_n\}$ is a collection of source to sink paths satisfying that each source lies in exactly one path in Π, and each sink lies in exactly one path in Π. If π is a source to sink path in a planar network, we will use the notation $s(\pi)$ to denote the index of its source and $t(\pi)$ to denote the index of its sink. The *weight* of a path family, denoted $\mathrm{wt}(\Pi)$, is defined to be the product of the weights of its paths. The *type* $w(\Pi)$ of a path family Π is the permutation $w \in S_n$ so that Π contains a path from s_i to $t_{w(i)}$ for each $i \in [n]$. We say that Π is *nonintersecting* if the paths $\{\pi_1, \ldots, \pi_n\}$ are pairwise nonintersecting. A basic result in the theory of total positivity is the following theorem, discovered by Karlin and MacGregor [34] and Lindstrom [39].

Theorem 1. *(Lindstrom's Lemma) Let the weights of F lie in a commutative ring. Then we have*

$$\det(A(F)) = \sum_{\Pi} \mathrm{wt}(\Pi)$$

where the sum is over nonintersecting path families on F with type identity.

Lindstrom's Lemma gives one direction of the following result.

Theorem 2. *A matrix is totally nonnegative if and only if it is the path matrix of a planar network with nonnegative real weights.*

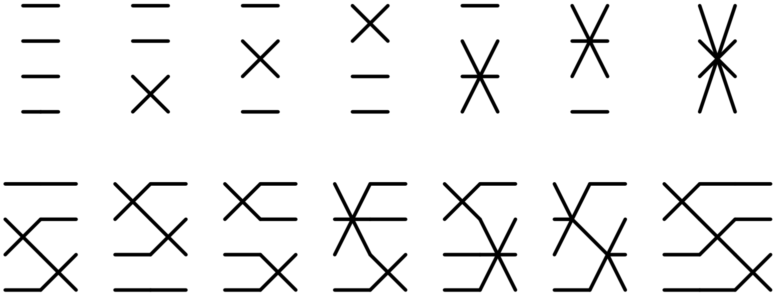


Figure 2.2: Desending star networks, $n = 4$

Proof. ($\Longleftarrow$) Any submatrix $A_{I,J}$ of the path matrix A of a planar network F is the path matrix of the planar network obtained from F by deleting vertices s_i, t_j for all $i \notin I$, $j \notin J$ (along with their incident edges). By Lindstrom's Lemma, the determinant of this matrix is nonnegative if F has nonnegative weights. $\square$

2.2 Descending star networks

An important combinatorial class of planar networks is the class of *descending star networks*. A *star* $F^*_{[i,j]}$ of order n is a planar network of order n consisting of a single internal vertex (call it v), an edge from s_l to v for $i \leq l \leq j$, an edge from v to t_l for $i \leq l \leq j$, and an edge from s_l to t_l for $l \notin [i,j]$. Throughout this paper, we will assume the left vertices of each planar network are the sources and the right vertices are the sinks, always labelled in ascending order from bottom to top. All edges will therefore be directed from left to right.

Define the *composition* $F \circ G$ of planar networks F and G to be the planar network given by the union of F and G, with the sinks of F identified with the corresponding sources of G and then unlabelled, so that the sources of $F \circ G$ are the sources of F, and the sinks of $F \circ G$ are the sinks of G. A *descending star network* is a composition $F^*_{[i_1,j_1]} \circ \cdots \circ F^*_{[i_m,j_m]}$, satisfying $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$, with redundant paths deleted so that there is at most one path between any two vertices. The composition and deletion processes are illustrated in Figure 2.3, and

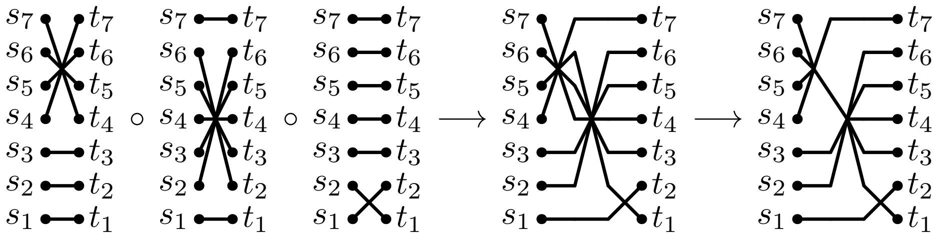


Figure 2.3: Descending star network construction

the descending star networks of order 4 are given in Figure 2.2. This gives us a planar network that is acyclic as an undirected graph. We call such planar networks *totally acyclic*.

In a totally acyclic planar network, there is at most one path from s_i to t_j. We will often denote this path $\pi_{i,j}$; this notation can always be assumed to refer to the unique s_i to t_j path in a totally acyclic network. The s_i to t_i paths have particular importance; we will sometimes shorten their label to π_i. Note in particular that in a descending star network, we always have paths π_i for $1 \leq i \leq n$.

There are C_n descending star networks of order n, where $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. We will show this by exhibiting a bijection between certain permutations and descending star networks.

In the following section we will require a couple of basic notions from the theory of posets. An *ideal* in a poset P is a set $I \subset P$ satisfying that for all $y \in P$, we have $x < y \implies x \in P$. An ideal is a *principal* ideal if it is of the form $\{x \mid x \leq t\}$ for some $t \in P$. We say I is the ideal *generated* by t.

For a totally acyclic planar network F, let $Q(F)$ denote the set $\{w(\Pi)\}$ where Π runs over all path families on F. Then we can we associate to F an element $\beta(F) \in \mathbf{C}[S_n]$ by

$$\beta(F) := \sum_{w \in Q(F)} w\,.$$

For any matrix $[a_{i,j}]$ and $v, w \in S_n$, let $a^{v,w}$ denote the product $a_{v_1,w_1} \cdots a_{v_n,w_n}$, where $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are the one line notations of v and w respectively.. We record the following simple observation.

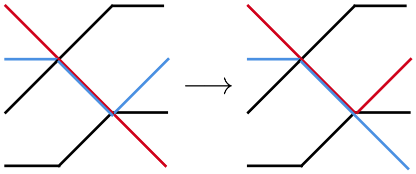


Figure 2.4: Replacing paths from s_i, s_j to $t_{w(i)}$, $t_{w(j)}$ by paths from s_i, s_j to $t_{v(i)}$, $t_{v(j)}$

Lemma 1. *For a totally acyclic planar network F with path matrix $A(F) = [a_{i,j}]$ we have:*

$$a^{e,w} = \begin{cases} 1 & \textit{if } w \in Q(F) \\ 0 & \textit{otherwise} \end{cases} . \tag{2.1}$$

Now we will give an important property of totally acyclic networks.

Lemma 2. *For a totally acyclic planar network F, $Q(F)$ is an ideal.*

Proof. Suppose that $v < w$ for some $w \in Q(F)$, and further that there is no z satisfying $v < z < w$ (in the language of posets, we say that w *covers* v). Then we have (cf. [4, Ch. 7]), for some $i < j$, $w_i > w_j$ and $v_i < v_j$. Since $w \in Q(F)$, there is a path family Π_w of type w on F; in particular, we have paths $\pi_{i,w(i)}, \pi_{j,w(j)} \in \Pi_w$. By the construction of F, $\pi_{i,w(i)}$ separates the disc of support of F, with j and $w(j)$ in different components. Thus, the $\pi_{j,w(j)}$ must intersect $\pi_{i,w(i)}$. Let x be the final vertex in $\pi_{j,w(j)} \cap \pi_{i,w(i)}$ Set $\pi_{i,x}, \pi_{x,w(i)}$ to be the sections of $\pi_{i,w(i)}$ from s_i to x and from x to $t_{w(i)}$, respectively. Set $\pi_{j,x}, \pi_{x,w(j)}$ to be the sections of $\pi_{j,w(j)}$ from s_j to x and from x to $t_{w(j)}$, respectively. Then $p_{i,j} = \pi_{i,x} \cup \pi_{x,w(j)}$ and $p_{j,i} = \pi_{j,x} \cup \pi_{x,w(i)}$ are paths from s_i to t_j and t_i to s_j, respectively. Replacing $\pi_{i,w(i)}$ and $\pi_{j,w(j)}$ by $p_{i,j}$ and $p_{j,i}$ does not change the union of the paths in Π_w, so $\Pi_w - \{\pi_{i,w(i)}, \pi_{j,w(j)}\} + \{p_{i,j}, p_{j,i}\}$ is a path family on F of type v and we have $v \in Q(F)$. For an illustration of this procedure, see Figure 2.4.

Since the poset in question is finite, it suffices to establish inclusion in the case of covering relations and thus we have the result. □

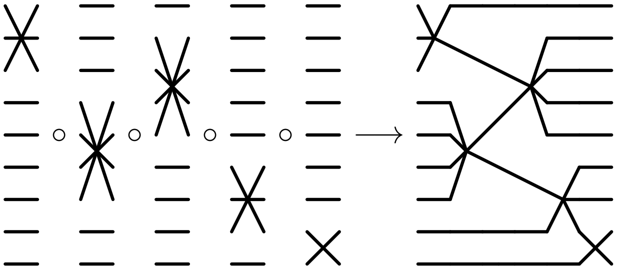


Figure 2.5: Zig-zag network example

2.3 Enumeration of networks

A *zig-zag* network is constructed in the same manner as a descending star network, except that instead of requiring our stars be chosen so that $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$, we impose the condition that for any indices $a < b < c$, if $[i_a, j_a] \cap [i_b, j_b] \cap [i_c, j_c] \neq \emptyset$, we have either $i_a < i_b < i_c$ and $j_a < j_b < j_c$, or $i_a > i_b > i_c$ and $j_a > j_b > j_c$. An example of such a network is given in Figure 2.5. Like descending star networks, zig-zag networks are totally acyclic and thus have paths uniquely determined by source and sink indices.

Lemma 3. *Any zig-zag network is totally acyclic.*

Proof. Let F be a zig-zag network. Let P and Q be a minimal-length pair of paths forming the bottom and top components of a cycle in F. Say that P and Q begin in the central vertex of star $S_1 = F^*_{[i_1,j_1]}$, and end at the central vertex of star $S_2 = F^*_{[i_2,j_2]}$. Then by construction, at least one of P or Q must have passed through at least one other star in between stars S_1 and S_2; say that it was path P (the argument is essentially identical either way). By the minimal-length property of the cycle, then, the intermediate star $S_3 = F^*_{[i_3,j_3]}$ must satisfy that $j_3 < j_1$ and that Q passes above S_3. But then by the defining property of a zig-zag network, all subsequent stars that P passes through must lie below S_3; in particular, they can never intersect Q, which is a contradiction $\square$

Given permutations $w = [w_1, \ldots, w_n]$, $v = [v_1, \ldots, v_m]$, we say that w *avoids* v

if no subsequence $w_{i_1}, \cdots, w_{i_m}$ satisfies $w_{i_a} < w_{i_b} \iff v_a < v_b$ for all a, b. In [46], Skandera proved the following result for zig-zag networks.

Lemma 4. *Let F be a zig-zag network. Then $Q(F)$ is the principal ideal generated by a $3142, 4231$ avoiding permutation w. This correpondence is a bijection between zig-zag networks and $3142, 4231$ permutations, up to reordering the stars in the zig-zag network.*

For any $3142, 4231$ avoiding permutation w, we will denote by F_w the corresponding zig-zag network. For descending star networks, we have the following refinement of the correspondence. Note that since 312 avoiding permutations are known to be counted by Catalan numbers (as noted in [50]), this proves the above enumeration of descending star networks.

Lemma 5. *Let F_w be a zig-zag network. Then F_w is a descending star network if and only if the permutation w is 312 avoiding.*

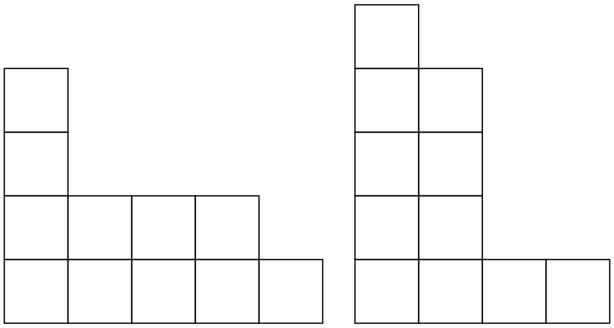
Proof. $(\Longleftarrow)$

Let $F_v = F = F^*_{[c_1,d_1]} \circ \cdots \circ F^*_{[c_k,d_k]}$ be a zig-zag network indexed by the permutation $v = v_1 \ldots v_n$. Suppose that $[c_i, d_i]$, $[c_j, d_j]$ is the first pair with $c_i < c_j$ and $c_j < d_i$. (If such a pair does not exist, then F is a descending star network.) Note that by rearranging the stars, we can assume $j = i + 1$.

Define the "straight line" paths in F to be the (unique) paths that connect s_i to t_{v_i} for some i. Let P',Q', and R' be the "straight line" source to sink paths in F which enter $F^*_{[c_i,d_i]}$ in position c_i, enter $F^*_{[c_i,d_i]}$ in position d_i, and leave $F^*_{[c_j,d_j]}$ in position c_j, respectively. Let P, Q, and R be the images of these paths in G and let $s(X)$ and $t(X)$ denote the source and sink of a path X.

Note that immediately to the left of $F^*_{[c_i,d_i]}$, P is below Q which is below R. Immediately to the right of $F^*_{[c_j,d_j]}$, P is above R which is above Q. Since P has intersected Q and R, none of these paths can have any other intersections (since a zig-zag network has no undirected cycles). Thus, $s(P) < s(Q) < s(R)$ and $t(P) > t(R) > t(Q)$. So $v_{s(P)}, v_{s(Q)}, v_{s(R)}$ is an occurrence of 312 in v.

$(\Longrightarrow)$



(a) $\lambda = 5411$ **(b)** $\lambda^t = 42221$

Figure 2.6: A Young diagram and its transpose

Set $F_v = F$. Suppose that v_i, v_j, v_k is an occurrence of 312 in v. Let P, Q, and R be images in F of the "straight line" paths in F starting at v_i, v_j, and v_k. P has to cross both Q and R. Since $v_j < v_K$, the three paths do not all intersect in a common star. Since F is a zig-zag, the two crossings must be the only intersections between the three paths. In particular, P intersects Q to the left of its intersection with R. This means that the star in which P intersects R is above, overlapping, and to the right of the star in which P intersects Q. So, F_v is not a descending star network.

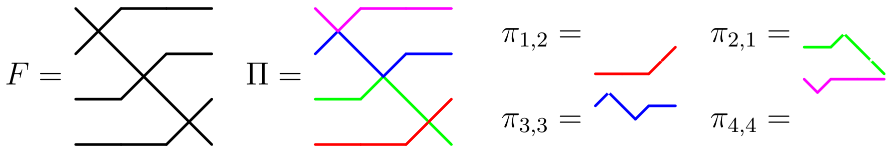
□

2.4 F-tableaux

A *Young diagram* of *shape* $\lambda \vdash n$ is simply an arrangement of "cells" into left-justified rows, with row i consisting of λ_i cells. Following the French convention, we will display our diagrams by placing each row above the previous one. An example of a Young diagram is given in Figure 2.6.

For a partition λ, we can define its *transpose* λ^t to be the shape of the transpose of the Young diagram. To be precise, we have

$$\lambda^t := [\#\{\lambda_i \mid \lambda_i \geq 1\}, \#\{\lambda_i \mid \lambda_i \geq 2\}, \ldots, \#\{\lambda_i \mid \lambda_i \geq \lambda_1\}]\,. \tag{2.2}$$



(a) A path family

$\pi_{1,2}$	$\pi_{2,1}$	$\pi_{3,3}$	$\pi_{4,4}$

$\pi_{1,2}$	$\pi_{2,1}$	$\pi_{4,4}$	$\pi_{3,3}$

$\pi_{2,1}$	$\pi_{1,2}$	$\pi_{4,4}$	$\pi_{3,3}$

$\cdots$

(b) Tableaux of shape 4, content Π

$\pi_{1,2}$		
$\pi_{2,1}$	$\pi_{3,3}$	$\pi_{4,4}$

$\pi_{1,2}$		
$\pi_{2,1}$	$\pi_{4,4}$	$\pi_{3,3}$

$\pi_{2,1}$		
$\pi_{1,2}$	$\pi_{4,4}$	$\pi_{3,3}$

$\cdots$

(c) Tableaux of shape 31, content Π

Figure 2.7: Examples of F-tableaux

In general, a "tableau" usually means a filling of a Young diagram with elements of some poset (that is, an assignment of one such element to each cell of the diagram). The best studied and most important of these are the *Young tableaux*. A Young tableaux (or $[n]$-tableaux) is a filling of a Young diagram with the letters $[n]$. We define the *shape* of a tableau to be the shape of the underlying diagram.

The development of Young tableaux was originally stimulated by their importance in the representation theory of S_n and GL_n. Subsequently, there has been extensive research in Young tableaux combinatorics as well as their applications. An overview of the theory of Young tableaux is given by Fulton in [21]. We will label the elements of a tableau by row, column indices. Note that although this is almost the same convention as is used for matrix entries, the rows appear in the opposite order as is generally used with matrices. For example, in Figure 2.7, the last tableau T has entries $t_{1,1} = \pi_{1,2}$, $t_{1,2} = \pi_{4,4}$, $t_{1,3} = \pi_{3,3}$, $t_{2,1} = \pi_{2,1}$.

A tableau $T = [t_{i,j}]$ of shape λ is *row-semistrict* if $t_{i,j} \not> t_{i,j+1}$ for all i, $j < \lambda_i$. It is *row-strict* if $t_{i,j} < t_{i,j+1}$ for all i, $j < \lambda_i$. It is *column-strict* if its transpose is row-strict, and *column-semistrict* if its transpose is row-semistrict.

Given a planar network F of order n, we will define an F-tableau of shape $\lambda \vdash n$ to be a filling of a Young diagram of shape λ by the paths in a path family on F. Here we define a partial ordering on the set of source to sink paths in F by $p < q$ if p does not intersect q and $s(p) < s(q)$ (it is easy to see that we could equivalently require $t(p) < t(q)$). For a planar network F, we will call the set of F-tableaux $\mathcal{T}(F)$. Figure 2.7 gives some examples of F tableaux. We define the *type* of an F-tableau to be the type of the path family that it contains. (See Section 2.1 for the definition of the type of a path family.)

Call the multiset of elements contained by a tableau its *content*. It is natural to associate to any F-tableau $T = T_{i,j}$ of shape λ two Young tableaux, the source tableau $s(T)$ defined by $s(T)_{i,j} = s(T_{i,j})$ and the sink tableau $t(T)$ defined by $t(T)_{i,j} = t(T_{i,j})$. Because the content of F is a path family, $s(T)$ and $t(T)$ both have content $[n]$.

Chapter 3

Combinatorics of classical characters

In this section we give a useful map on F-tableaux, the *drop map.* This map gives a bijective proof of the equivalence of different combinatorial interpretations of the classical ($q = 1$) characters for the algebra elements associated to descending star networks. This is an important special case. In particular, there is a connection between these character evaluations and Stanley's e-positivity conjecture (see [47] and [28]). Recent work of Morales, Guay-Paquet, and Rowland [27] shows descending star networks are the only networks needed in this context.

3.1 The drop map

We say that an F-tableau T is *row-closed* if each row of $t(T)$ is a rearrangement of the corresponding row of $s(T)$. We say that an F tableau T is *canonical* if $s(T)$ is row strict. Let $\mathcal{T}_\lambda(F)$ be the set of canonical row-closed F-tableaux, and $\mathcal{T}^\circ_\lambda(F) \subset \mathcal{T}_\lambda(F)$ be the subset of those that are row-strict. These sets provide combinatorial interpretations of the induced trivial and sign characters:

$$\eta^\lambda(\beta(F)) = |\mathcal{T}_\lambda(F)|\,, \tag{3.1}$$

$$\epsilon^\lambda(\beta(F)) = |\mathcal{T}^\circ_\lambda(F)|\,. \tag{3.2}$$

These are proved in full generality later in this paper (Theorems 4, 5) with the above formulas being obtained by setting $q = 1$. Note that all tableaux in the set $\mathcal{T}^{\circ}_{\lambda}(F)$ have type e. From this point of view we can characterize it as the set of all row-strict F-tableaux of shape λ and type e, an interpretation which is more natural from the point of view of chromatic symmetric functions (cf. [44], [10], [47], [23]). We can give a similar interpretation of the induced trivial character by means of a “parentheses dropping” bijection.

$$U = \begin{array}{|c|c|c|c|c|}\hline \pi_{1,1} & \pi_{2,5} & \pi_{3,3} & \pi_{4,2} & \pi_{5,4} \\ \hline \end{array}$$

$$\begin{aligned} w(U) &= [1,4,3,5,2] \\ &= (1)(2,5,4)(3) \\ w^{-1} &= (1)(4,5,2)(3) \\ &= (1)(3)(5,2,4) \end{aligned}$$

$$\mathrm{drop}(U) = \begin{array}{|c|c|c|c|c|}\hline \pi_1 & \pi_3 & \pi_5 & \pi_2 & \pi_4 \\ \hline \end{array}$$

Figure 3.1: Example of drop

Let F be a descending star network, and let $RSST_{\lambda}(F)$ denote the row-semistrict F tableaux of type e and shape λ. Our goal is to define a bijection $\mathcal{T}_{\lambda}(F) \rightarrow RSST_{\lambda}(F)$. We will begin with the case where $\lambda = [m]$. Let U be a row-closed F tableau of shape $[m]$. Then the content of U forms a path family of type w for some w. We define $\mathrm{drop}(U)$ to be the tableau obtained by writing the permutation w in canonical cycle notation, dropping the parentheses, and recording the type-e paths corresponding to the word. An example is given in Figure 3.1.

To see that the map $\mathrm{drop} : \mathcal{T}_m(F) \rightarrow RSST_m(F)$ is a bijection, we construct its inverse. Given a tableau $V = [\rho_{x_1}, \ldots, \rho_{x_m}]$ of type e in $\mathcal{T}_m(F)$, let $w \in S_m$ be the permutation given in cycle notation by

$$w = (x_1, \dots, x_{i_1-1})(x_{i_1}, \dots, x_{i_2-1}) \cdots (x_{i_k}, \dots, x_m)$$

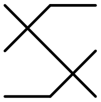
where $i_1, \dots, i_k$ are the positions of the records of the word $x_1, \dots, x_r$. Then write $w^{-1} = w_1^{-1}, \dots, w_r^{-1}$ in one line notation. Finally, let $V' \in \mathcal{T}_m(F)$ be the tableau whose ith entry is the unique path in F_u from i to w_i^{-1}. To show that the map $V \mapsto V'$ is well defined we need to verify that there actually are such paths for all i.

To do this, consider any cycle $(x_j, \dots, x_{j+l})$ of w. We want to verify that there is a path from x_{j+s+1} to x_{j+s} for each s, and a path from x_j to x_{j+l}. There must be paths from x_{j+1} to x_j and from x_j to x_{j+1} as otherwise, we would have $\rho_{x_j} > \rho_{x_{j+1}}$. Now assume that there is a path from x_j to x_{j+s}. If $x_{j+s+1} < x_{j+s}$, then there must be a path from x_{j+s+1} to x_{j+s} by the same reasoning as for x_j and x_{j+1}; otherwise, such a path exists because $x_j > x_{j+s+1} > x_{j+s}$ and there is a path from x_j to x_{j+s}. This implies that there is also a path from x_j to x_{j+s+1} and we can therefore proceed inductively to get paths from each x_{j+s+1} to each x_{j+s} as well as paths from x_j to all the others. In particular there is a path from x_j to x_{j+l} and so we have verified that all the neccesary paths exist.

It is clear that this map is inverse to drop, so they are bijections.

We can then extend this map to all row-closed tableaux by applying it row by row. For $I \subset [n]$, let $\mathcal{T}_\lambda(F|I)$ denote the canonical row bijective shape-λ F-tableaux with source and sink index set I, and let $RSST_\lambda(F|I)$ be the row-semistrict shape-λ F-tableaux of type e with source and sink index set I. By the same argument as above, drop gives a bijection from $\mathcal{T}_{[|I|]}(F|I)$ to $RSST_{[|I|]}(F|I)$ for any I. For $T \in \mathcal{T}_\lambda(F)$, $T = [r_1, \cdots, r_k]$, define $\text{drop}(T)$ to be the tableau $[\text{drop}(r_1), \cdots, \text{drop}(r_k)]$. Then since tableaux in $\mathcal{T}_\lambda(F)$ and $RSST_\lambda(F)$ are both determined by an arbitrary choice of an index set I for each row together with rows using the given indices, drop is a bijection $\mathcal{T}_\lambda(F) \to RSST_\lambda(F)$. An example of this bijection is given in Figure 3.2

We can now give another interpretation of η^λ. Equation (3.1) gives a formula for η^λ in terms of the canonical row-bijetive F tableaux. By applying the drop bijection to this formula, we obtain the following.

$F =$ 

U	drop(U)
$\begin{array}{ll}\pi_{1,1} & \\ \pi_{2,2} & \pi_{3,3}\end{array}$	$\begin{array}{ll}\pi_{1} & \\ \pi_{2} & \pi_{3}\end{array}$
$\begin{array}{ll}\pi_{1,1} & \\ \pi_{2,3} & \pi_{3,2}\end{array}$	$\begin{array}{ll}\pi_{1} & \\ \pi_{3} & \pi_{2}\end{array}$
$\begin{array}{ll}\pi_{3,3} & \\ \pi_{1,1} & \pi_{2,2}\end{array}$	$\begin{array}{ll}\pi_{3} & \\ \pi_{1} & \pi_{2}\end{array}$
$\begin{array}{ll}\pi_{3,3} & \\ \pi_{1,2} & \pi_{2,1}\end{array}$	$\begin{array}{ll}\pi_{3} & \\ \pi_{2} & \pi_{1}\end{array}$

U	drop(U)
$\begin{array}{lll}\pi_{1,1} & \pi_{2,2} & \pi_{3,3}\end{array}$	$\begin{array}{lll}\pi_{1} & \pi_{2} & \pi_{3}\end{array}$
$\begin{array}{lll}\pi_{1,1} & \pi_{2,3} & \pi_{3,2}\end{array}$	$\begin{array}{lll}\pi_{1} & \pi_{3} & \pi_{2}\end{array}$
$\begin{array}{lll}\pi_{1,2} & \pi_{2,1} & \pi_{3,3}\end{array}$	$\begin{array}{lll}\pi_{2} & \pi_{1} & \pi_{3}\end{array}$
$\begin{array}{lll}\pi_{1,2} & \pi_{2,3} & \pi_{3,1}\end{array}$	$\begin{array}{lll}\pi_{3} & \pi_{2} & \pi_{1}\end{array}$

Figure 3.2: Bijection for $F_{[231]}$, $\lambda = [2,1]$ and $\lambda = [3]$

Theorem 3. *Let F be a descending star network, and $\lambda \vdash n$. Then we have*

$$\eta^{\lambda}(\beta(F)) = |RSST_{\lambda}(F)|\,.$$

□

Chapter 4

Combinatorics of quantum characters

4.1 q-Analogs

The use of "q-analogs" in the study of special functions goes back almost a century, and their importance in algebraic combinatorics is pervasive. Roughly speaking, there are many situations in which replacing integers with formal expressions in a variable q that evaluate to the original numerical formula at $q = 1$ turns out to be fruitful, and many of these situations are combinatorially connected with each other. An excellent account of the combinatorial study of these phenomena can be found in [48, Ch. 1].

Recently the theory of quantum groups has provided a unified viewpoint on many of these phenomena; in particular, all of the q-analogs that we use in this paper can be understood in this context (See, e.g., [35]). We will use this theory to introduce quantum analogs of the classical representation-theoretic and combinatorial objects the we treated in previous sections.

4.1.1 Hecke algebras

Hecke algebras have appeared over the last 60 years in many fields of mathematics, having been studied in the context of automorphic forms (cf. [9]), representation theory (cf. [15], [16]), knot theory (cf. [33]), and quantum groups (cf. [32], [31]) to name a few settings. From the latter point of view, they can be viewed as a quantization of the Weyl group of a Lie algebra; in particular, [31] establishes a quantum Schur-Weyl duality between the representation theory of certain Hecke algebras and the representation theory of corresponding quantum groups.

The combinatorial study of Hecke algebras led to the development of *Kazhdan-Lusztig* polynomials, introduced by Kazdhan and Lustig in [36]. These polynomials arose as the structure constants for certain Hecke algebra bases. The coefficients of these polynomials encode a great deal of geometric and representation theoretic information, and their study has become a field of its own. Introductions to Kazhdan-Lusztig theory can be found in [3, Ch. 5-6], or [29, Ch. 7]. For more information on the combinatorial theory, a good place to start is [13]. An important recent result is the general interpretation of Kazhdan-Lustig coefficients, valid in all Coxeter systems given by Elias and Williamson in [17], settling the 1979 positivity conjecture given by Kazhdan and Lusztig in [36].

Our results concern the combinatorial theory of Hecke algebra characters. We will give the basic definitions and facts that will be required for our results. For a more thorough discussion of the material, our standard reference is [24].

Let $R = \mathbf{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Then we define a quantum analog H_n of the symmetric group algebra, called the *generic 1-parameter Iwahori-Hecke Algebra*, to be the algebra generated over R by elements $T_1, \dots, T_{n-1}$ subject to the relations:

$$\begin{aligned} T_i^2 &= q + (q-1)T_i & & \\ T_iT_j &= T_jT_i & & |i-j| > 1 \\ T_iT_jT_i &= T_jT_iT_j & & |i-j| = 1 \end{aligned}$$

For $w \in S_n$, set $T_w = T_{i_1} \cdots T_{i_l}$ where $s_{i_1} \dots s_{i_l} = w$ is any reduced expression. It is a fact (a proof is found in [24, Thm. 4.4.6]) that this gives a well defined map

$w \mapsto T_w$. It turns out that the set $\{T_w \mid w \in S_n\}$ is a basis for H_n, called the *natural basis.*

There is related basis of H_n that turns out to be useful, sometimes called the *modified natural basis.* It can be defined by $\tilde{T}_i := q^{\frac{-1}{2}} T_i$. We then have the following relations:

$$\begin{aligned} \tilde{T}_i^2 &= 1 + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\tilde{T}_i & & \\ \tilde{T}_i\tilde{T}_j &= \tilde{T}_j\tilde{T}_i & & |i-j| > 1 \\ \tilde{T}_i\tilde{T}_j\tilde{T}_i &= \tilde{T}_j\tilde{T}_i\tilde{T}_j & & |i-j| = 1 \end{aligned}$$

H_n is a q-analog of $\mathbf{C}[S_n]$ in the sense that setting $q = 1$ recovers the classical algebra. (This can be seen from the presentations of H_n and S_n, which are identical except for the first relation. This relation, in turn, becomes identical when we set $q = 1$.) It should be noted that the parameter q is often taken to be a unit in some ring, typically in connection with the theory of p-adic groups. The $q = 0$ specialization is also important in applications and has a rich combinatorial theory of its own. On the other hand, a generic choice of q gives a representation theory isomorphic to the $q = 1$ case (cf. [24, Thm. 8.1.5]). For our purposes, however, q will simply be an indeterminate.

As a result, the character theory of H_n is closely related to the character theory of $\mathbf{C}[S_n]$. In particular, it can be shown using Tits's deformation theorem that the irreducible characters of H_n are in bijection with those of $\mathbf{C}[S_n]$ (cf. [24, Thm. 8.1.7]). We can thus carry over the $e_\lambda, h_\lambda, p_\lambda, m_\lambda$, and s_λ bases of the space of $\mathbf{C}[S_n]$ traces to bases of the space of H_n traces. We label the corresponding H_n traces ϵ_q^λ, η_q^λ, ψ_q^λ, ϕ_q^λ, and χ_q^λ. The matrices relating these bases to one another are identical to the matrices relating the $\mathbf{C}[S_n]$ traces to one another, and thus to the matrices relating various bases of Λ.

The quantum matrix algebra $\mathcal{A}_n$ arises in quantum group theory in the construction of the coordinate ring of quantum $SL_n(\mathbf{C})$. It is a quantization of the polynomial algebra $\mathbf{C}[x_{1,1}, \ldots, x_{n,n}]$ in the entries of a matrix (and thus the coordinate ring of quantum SL_n can be obtained by localizing it at the "quantum determinant"). Abstractly, it is possible to view H_n and A_n as arising from the same quantization process, in which affine space is replaced by quantum affine space (for a detailed exposition see [35, Ch. 4]), groups are replaced by Hopf algebras, and so forth (here H_n enters the picture via quantum Schur-Weyl duality, as explained in [32]). Thus we can expect classical relationships between the two algebras that are sufficiently abstract (essentially, those that can be expressed in terms of commutative diagrams) to carry though to the quantum setting. From our point of view, though, the main point of $\mathcal{A}_n$ is that by using certain canonical bases, we can satisfactorily quantize the classical immanants, as developed in [38].

In this section we will define the quantum matrix algebra, and give the basic constructions that we will use it for.

For convenience, we will set $\Delta q := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. Define the *quantum matrix algebra* $\mathcal{A}_n$ to be the R-algebra generated by n^2 elements $\{x_{i,j} \mid i, j \in [n]\}$ subject to the relations:

$$x_{i,l}x_{i,k} = q^{\frac{1}{2}} x_{i,k}x_{i,l} \,, \tag{4.1}$$

$$x_{j,k}x_{i,k} = q^{\frac{1}{2}} x_{i,k}x_{j,k} \,, \tag{4.2}$$

$$x_{j,k}x_{i,l} = x_{i,l}x_{j,k} \,, \tag{4.3}$$

$$x_{j,l}x_{i,k} = x_{i,k}x_{j,l} + (\Delta q)x_{i,l}x_{j,k} \tag{4.4}$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$. Note that by applying the relations 1-4, we can sort any monomial in $\mathcal{A}_n$ into lexicographic order on the subscripts (possibly we will end up with a sum of sorted monomials, by relation 4). We call such a monomial, that is, a monomial $x_{i_1,j_1} \cdots x_{i_l,j_l}$ satisfying that $i_a \leq i_b$ for $a < b$ and $j_a \leq j_b$ for $a < b, i_a = i_b$, *standard.* Then we have (restating the above discussion) that the standard monomials form a basis for $\mathcal{A}_n$.

For permutations u, v, define $x^{u,v} := x_{u_1,v_1} \cdots x_{u_n,v_n}$. Let $\mathcal{A}_n^\circ \subset \mathcal{A}_n$ be the span of the elements $\{x^{u,v} \mid u, v \in S_n\}$. This subspace is sometimes called the "immanant space" and from this point on, essentially all our computations will take place inside it. Notice that since, for any monomial $x_{i_1,j_1} \cdots x_{i_l,j_l}$ the sets $\{i_1, \ldots, i_l\}$ and $\{j_1, \ldots, j_l\}$ are preserved by relations 1-4, a basis for $\mathcal{A}_n^\circ$ is given by the standard monomials in $\mathcal{A}_n^\circ$, that is, by the set $\{x^{e,w} \mid w \in S_n\}$.

4.1.2 Quantum Immanants

Matrix immanants were originally introduced by Littlewood in [40]. They can be viewed as a family of matrix functions that interpolates between the permanent and the determinant. They remained mostly ignored until the 1985 paper of Merris and Watkins [42]. The combinatorial theory of matrix imminants was subsequently developed by Goulden-Jackson [25], Greene [26], Stembridge [51], [52], and Haiman [28], among others. In particular, much of the work in this dissertation was motivated by conjectures given by Stembridge in [52].

The classical immanants can be defined as follows. Given a $\mathbf{C}[S_n]$ trace f, define the *f-immanant* $\mathrm{Imm}_f(x)$ to be the element of $\mathbf{C}[x_{1,1}, \ldots, x_{n,n}]$ given by

$$\mathrm{Imm}_f(x) = \sum_{w \in S_n} f(w) x^{e,w}$$

(here $\mathbf{C}[x_{1,1}, \ldots, x_{n,n}]$ is an ordinary ring of commutative polynomials in n^2 variables.) Note that the imminant was originally defined to be, in our terminology, $\mathrm{Imm}_{\chi^\lambda}$. Also note that the imminant does indeed interpolate between the permanent and the determinant in the sense that $\mathrm{Imm}_{\chi^{1^n}}(x) = \mathrm{Imm}_{\mathrm{sgn}}(x) = \det(x)$ and $\mathrm{Imm}_{\chi^n}(x) = \mathrm{Imm}_1(x) = \mathrm{perm}(x)$

Of course we usually think of the ring $\mathbf{C}[x_{1,1}, \ldots, x_{n,n}]$ as the ring of polynomials in the entries of an n by n matrix, and we have the natural evaluation map given by $x_{i,j}(A) = a_{i,j}$ where $A = [a_{i,j}]$.

Let $q_{u,v} := q^{\frac{1}{2}(l(v)-l(u))}$. Given an H_n trace f, define the *quantum f-immanant* $\mathrm{Imm}_f(x)$ to be the element of $\mathcal{A}_n$ given by the formula

$$\mathrm{Imm}_f(x) = \sum_{w \in S_n} f(\tilde{T}_w) x^{e,w} = \sum_{w \in S_n} (q_{e,w})^{-1} f(T_w) x^{e,w} . \tag{4.5}$$

In the classical case, we clearly can recover a trace f from its immanant $\mathrm{Imm}_f(x)$ by evaluating $\mathrm{Imm}_f(x)$ on permutation matrices: $f(w) = \mathrm{Imm}_f(P(w))$, where $P(w) = [\delta(i, w_j)]$ is the permutation matrix corresponding to w. This is also true in the quantum case, provided that we appropriately define what it means to apply a quantum polynomial to a matrix. We define the following evaluation of $\mathcal{A}_n^\circ$-elements on matrices. For $p(x) \in \mathcal{A}_n^\circ$, let $\{p_w \mid w \in S_n\} \subset R$ be the coefficients of p with respect to the standard basis; i.e., the coefficients so that we have $p = \sum_{w \in S_n} p_w x^{e,w}$. Given an n by n matrix $A = [a_{i,j}]$, let

$$\sigma_{A,e}(p) := \sum_{w \in S_n} q_{e,w} p_w a^{e,w} \, .$$

We can then recover character evaluation with the formula

$$f(\tilde{T}_w) = \sigma_{P(w),e}(\mathrm{Imm} f(x)) \, . \tag{4.6}$$

4.1.3 Planar networks and Kazhdan-Lusztig elements

We need to generalize our planar network machinery to the quantum setting. For a planar network F, $\beta_q(F)$ is defined to be the Hecke algebra element $\sum_{w \in Q(F)} T_w$. These elements are closely connected to the so-called *signless Kazhdan-Lusztig basis* $\{C'_w(q) \mid w \in S_n\}$ for H_n.

In particular, recall that a permutation $w = w_1, \ldots, w_n$ is v-avoiding for a permutation $v = v_1, \ldots, v_m$ if there is no substring $w_{i_1}, \ldots, w_{i_m}$ satisfying that $w_{i_j} < w_{i_k}$ if and only if $v_{i_j} < v_{i_k}$. Call a permutation w *smooth* if it is 3412-avoiding and 4231-avoiding. Then for smooth permutations w, it is known that

$$C'_w(q) = q^{-l(w)/2} \sum_{v \leq w} T_v \, .$$

Moreover, using so-called "reversal factorizations", Skandera constructed [46, Lem. 5.3] for each smooth w a totally acyclic planar network F satisfying

$$\beta_q(F) = \sum_{v \leq w} T_v = q^{l(w)/2} C'_w(q) \, .$$

Therefore, for an H_n trace f and w smooth, evaluating f on $C'_w(q)$ is equivalent to evaluating f on $\beta_q(F)$ for some totally acyclic planar network F.

It should be noted that additional Kazhdan-Lusztig basis elements have been shown to be combinatorially realizable by Billey and Warrington in [2]. However certain combintorial difficulties prevent the proofs in this paper from carrying over directly to these elements. Another more general family of planar networks that would be interesting to consider would be those that correspond to the extremal rays of the cone $\mathcal{C}(\Pi)$ defined by Stembridge in [52, Sec. 5].

4.2 Interpretations of q-characters

In this section we will give two of the main results of this dissertation: combinatorial interpretations of the quantum induced trivial and sign characters on the H_n elements corresponding to zig-zag networks. The interpretation of the induced trivial character is novel, while the interpretation of the induced sign character was originally proposed by B. Shelton in [45]. The proof given here is original. We begin by proving some combinatorial facts.

4.2.1 Combinatorial lemmas

To prove our main results, we will need to establish several facts concerning the combinatorics of planar networks.

Claim 1. *Given a planar network F with paths π of type $(i \to j)$ and π' of type $(i' \to j')$, if $\pi \cap \pi' \neq \emptyset$, there exist paths ρ, ρ' in F of types $(i \to j')$ and $(i' \to j)$.*

Proof. Choose a vertex v in the intersection of π and π'. Then ρ can be constructed by taking the union of the segment of π joining i and v and the segment of π' joining v and j'. We can construct ρ' similarly. □

Claim 2. *Let F be totally acyclic. Then for $i < j$ and $k < l$, if F contains paths of types $(i \to k)$, $(j \to l)$, $(i \to l)$, and $(j \to k)$, the paths $\pi_{i,k}$ of type $(i \to k)$ and $\pi_{j,l}$ of type $(j \to l)$ intersect.*

Proof. Since $i < j$ and $k < l$, the paths $\pi_{i,l}$ and $\pi_{j,k}$ cross and thus intersect. Applying the construction in the proof of the previous claim, we get paths of type $(i \to k)$ and $(j \to l)$ which also intersect. Since this F is totally acyclic, those must be the unique paths $\pi_{i,k}$ and $\pi_{j,l}$. □

Claim 3. *Let F be totally acyclic with intersecting paths p of type $(i \to j)$ and p' of type $(i' \to j')$, with $i < i'$. Then for any path q of type $(k \to l)$ with $i < k < i'$, q intersects p' if and only if q intersects the path $\pi_{i',j}$ of type $(i' \to j)$.*

Proof. Let v be the leftmost vertex in $p \cap p'$. If q intersects the path joining $s_{i'}$ and v (including v), we are done (since p' and $\pi_{i',j}$ both contain this path).

Otherwise, since $i < k < i'$, q has to intersect the path joining s_i and v, say at v'. Suppose q intersects p' and let w be a vertex in $p' \cap q$. Then there is a path joining v' and w given by the portion of p connecting v' and v, and the portion of p' connecting v and w. Since q also contains both of these vertices, and F is totally acyclic, q must contain this path. In particular, q contains v and thus intersects $\pi_{i',j}$.

By exactly the same reasoning, if q intersects $\pi_{i',j}$, we again have that q contains v and thus intersects p'. $\square$

4.2.2 Interpretation of $\sigma_{A,e}(q_{u,v}x^{u,v})$

The heart of the proofs of our formulas for the evaluation of the quantum characters is the combinatorial interpretation of expressions of the form $\sigma_{A,e}(q_{u,v}x^{u,v})$. In this section we give this interpretation, which we will later apply to the character evaluations (by making use of some algebraic techniques).

We will begin by establishing some notation. Let F be a zig-zag network. Given an integer tableau T, the *row word* $\mathrm{rw}(T)$ is the permutation which consists of the entries of T read left to right, first row to last row. For a permutation u, set $T(u)$ to be the Young tableau with a single row and row word u. For permutations u, v set $T(u, v)$ to be the F-tableau with source and sink tableaux $T(u)$ and $T(v)$, respectively, if such a tableau exists. The existence of the tableau depends on whether a certain permutation lies in $Q(F)$.

Claim 4. *For permutations u, v, $T(u, v)$ is well defined if and only if $vu^{-1} \in Q(F)$.*

Proof. The tableau exists if and only if there is an appropriate path for each of its cells. Denoting the one-line notations of u and v by $u_1 \cdots u_n$ and $v_1 \cdots v_n$, respectively, the ith entry in $T(u, v)$ is a path from source u_i to sink v_i, in other words, a path from source u_i to sink $(vu^{-1})(u_i)$. So the tableau exists if and only if there are paths from source u_i to sink $(vu^{-1})(u_i)$ for all $i \in [n]$, or equivalently from source j to sink $(vu^{-1})(j)$ for all $j \in [n]$. This is precisely the statement that $vu^{-1} \in Q(F)$. □

Call a pair of paths π, π' in a tableau T an *inversion* if π appears to the left of π' in T, π has a greater sink than π', and $\pi \cap \pi' \neq \emptyset$. Set $\mathrm{inv}(T)$ to be the number of inversions in T.

We can now state the main combinatorial formula that we will prove. Let F be a totally acyclic planar network with path matrix $A = [a_{i,j}]$ and fix this notation through Claim 9. We want to establish the following formula for $w < v \in S_n$.

$$\sigma_{A,e}(q_{w,v}x^{w,v}) = \begin{cases} q^{\mathrm{inv}(T(w,v))} & \text{if } vw^{-1} \in Q(F) \\ 0 & \text{otherwise} \end{cases}. \tag{4.7}$$

First, we show that it holds when $w = e$.

Claim 5. *For $v \in S_n$, we have*

$$\sigma_{A,e}(q_{e,v}x^{e,v}) = \begin{cases} q^{\mathrm{inv}(T(e,v))} & \textit{if } v \in Q(F) \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. Recall that Equation (2.1) states that $a^{e,v} = 1$ if $v \in Q(F)$ and 0 otherwise. If $v \in Q(F)$, then $T(e, v)$ is well defined and for every pair of indices appearing out of order in $t(T(e, v))$, the corresponding paths in F cross and thus certainly intersect. These pairs are the inversions of v so we have $q^{l(v)} = q^{\mathrm{inv}(T(e,v))}$. Thus, we have

$$\begin{aligned}\sigma_{A,e}(q_{e,v}x^{e,v}) &= q^{\frac{1}{2}l(v)}\sigma_{A,e}(x^{e,v})\\ &= q^{\frac{1}{2}l(v)}q^{\frac{1}{2}l(v)}a^{e,v}\\ &= q^{l(v)}a^{e,v}\\ &= q^{\mathrm{inv}(T(e,v))}a^{e,v}\\ &= \begin{cases} q^{\mathrm{inv}(T(e,v))} & \text{if } v \in Q(F)\\ 0 & \text{otherwise}\end{cases}.\end{aligned}$$

□

Now we show the general case. We will proceed by induction on the length of w, but first we will prove two lemmas that constitute the core of the argument.

Claim 6. *Fix $u, v, s_k \in S_n$ so that $0 < l(u) < l(v)$, $us_k < u$, and $vs_k > v$. If Equation (4.7) holds when (w, v) is replaced by (us_k, vs_k) then it also holds with (w, v) replaced by (u, v).*

Proof. By the defining equation (4.3) of the quantum matrix bialgebra, we have $\sigma_{A,e}(q_{u,v}x^{u,v}) = \sigma_{A,e}(q_{u,v}x^{us_k,vs_k})$. Also, note that we have

$$\begin{aligned}q_{us_k,vs_k} &= q^{\frac{1}{2}(l(vs_k)-l(us_k))}\\ &= q^{\frac{1}{2}((l(v)+1-(l(us_k)+1))}\\ &= q^{\frac{1}{2}((l(v)-l(us_k)+2)}\\ &= (q)q_{u,v}\,.\end{aligned}$$

We have $vs_k(us_k)^{-1} = vs_ks_k^{-1}u^{-1} = vu^{-1}$, so if $vu^{-1} \in Q(F)$, then the tableaux $T(u, v)$ and $T(us_k, vs_k)$ are well defined and contain the same sets of paths. In fact they are identical except for the pair of paths π_{u_k,v_k} and $\pi_{u_{k+1},v_{k+1}}$. This pair appears in order with respect to sinks in $T(u, v)$ and out of order in $T(us_k, vs_k)$; since the paths cross they certainly form an inversion in the latter tableau. Thus,

$q^{-1}q^{\mathrm{inv}(T(us_k,vs_k))} = q^{\mathrm{inv}(T(u,v))}$ and we have

$$\begin{aligned}
\sigma_{A,e}(q_{u,v}x^{u,v}) &= \sigma_{A,e}(q_{u,v}x^{us_k,vs_k}) \\
&= \sigma_{A,e}(q^{-1}q_{us_k,vs_k}x^{us_k,vs_k}) \\
&= q^{-1}\begin{cases} q^{\mathrm{inv}(T(us_k,vs_k))} & \text{if } vs_k(us_k)^{-1} \in Q(F) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} q^{-1}q^{\mathrm{inv}(T(us_k,vs_k))} & \text{if } vu^{-1} \in Q(F) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} q^{\mathrm{inv}(T(u,v))} & \text{if } vu^{-1} \in Q(F) \\ 0 & \text{otherwise} \end{cases} .
\end{aligned}$$

□

Claim 7. *Fix $u, v, s_k \in S_n$ so that $0 < l(u) < l(v)$, $us_k < u$, and $vs_k < v$. If Equation (4.7) holds with (w, v) replaced by (us_k, vs_k) and with (w, v) replaced by (us_k, v) then it also holds with (w, v) replaced by (u, v).*

Proof. By arguments almost identical to the ones above, in this case we have

$$\begin{aligned} q_{us_k,vs_k} &= q_{u,v} , \\ q_{us_k,v} &= (q^{\frac{1}{2}})q_{u,v} . \end{aligned} \tag{4.8}$$

Again by the defining relations of the quantum matrix bialgebra, we have

$$\sigma_{A,e}(q_{u,v}x^{u,v}) = \sigma_{A,e}(q_{u,v}x^{us_k,vs_k}) + \sigma_{A,e}((\Delta q)q_{u,v}x^{us_k,v}) . \tag{4.9}$$

As before, $T(u, v)$ is well defined if and only if $T(us_k, vs_k)$ is well defined, equivalently, if and only if $vu^{-1} \in Q(F)$. Furthermore, Claim 1 implies that if $v(us_k)^{-1} \notin Q(F)$, (implying that there is no tableau $T(us_k, v)$), then the pair of paths π_{u_k,v_k} and $\pi_{u_{k+1},v_{k+1}}$ do not intersect. Since the order in which these paths appear is the only difference between $T(us_k, vs_k)$ and $T(u, v)$, we have that if $v(us_k)^{-1} \notin Q(F)$, then

$$\mathrm{inv}(T(us_k, vs_k)) = \mathrm{inv}(T(u, v)) . \tag{4.10}$$

On the other hand if $v(us_k)^{-1} \in Q(F)$ (meaning that $T(us_k, v)$ is well defined), then Claim 2 implies that π_{u_k,v_k} and $\pi_{u_{k+1},v_{k+1}}$ do intersect. Thus, these paths are an inversion in $T(u, v)$. Also, π_{u_{k+1},v_k} and $\pi_{u_k,v_{k+1}}$ cross and are thus an inversion in $T(us_k, v)$. Finally, π_{u_k,v_k} and $\pi_{u_{k+1},v_{k+1}}$ appear in order in $T(us_k, vs_k)$, and so cannot be an inversion. Since these three tableaux are all identical everywhere else, we have, if $v(us_k)^{-1} \in Q(F)$

$$\operatorname{inv}(T(us_k, v)) = \operatorname{inv}(T(u, v)) = \operatorname{inv}(T(us_k, vs_k)) + 1\,. \tag{4.11}$$

Now we will establish the claim. By Equation (4.9), we have

$$\sigma_{A,e}(q_{u,v}x^{u,v}) = \sigma_{A,e}(q_{u,v}x^{us_k,vs_k}) + \sigma_{A,e}((\Delta q)q_{u,v}x^{us_k,v})\,. \tag{4.12}$$

Applying Equation (4.8) to the right hand side of Equation (4.12), we obtain

$$\begin{aligned} \sigma_{A,e}(q_{u,v}x^{u,v}) &= \sigma_{A,e}(q_{us_k,vs_k}x^{us_k,vs_k}) + \sigma_{A,e}((\Delta q)(q^{-\frac{1}{2}})q_{us_k,v}x^{us_k,v}) \\ &= \sigma_{A,e}(q_{us_k,vs_k}x^{us_k,vs_k}) + \sigma_{A,e}(1 - q^{-1})q_{us_k,v}x^{us_k,v})\,. \end{aligned} \tag{4.13}$$

Appling Equation (4.7) to the right hand side of Equation (4.13) gives

$$\sigma_{A,e}(q_{u,v}x^{u,v}) = \begin{cases} q^{\operatorname{inv}(T(us_k,vs_k))} + (1 - q^{-1})q^{\operatorname{inv}(T(us_k,v))} & \text{if } vu^{-1}, v(us_k)^{-1} \in Q(F) \\ q^{\operatorname{inv}(T(us_k,vs_k))} & \text{if } vu^{-1} \in Q(F),\ v(us_k)^{-1} \notin Q(F) \\ 0 & \text{otherwise} \end{cases} \,. \tag{4.14}$$

Applying Equation (4.10) to the first case in Equation (4.14) and Equation (4.11) to the second case gives

$$\sigma_{A,e}(q_{u,v}x^{u,v}) = \begin{cases} (q^{-1})q^{\mathrm{inv}(T(u,v))} + (1-q^{-1})q^{\mathrm{inv}(T(u,v))} & \text{if } vu^{-1}, v(us_k)^{-1} \in Q(F) \\ q^{\mathrm{inv}(T(u,v))} & \text{if } vu^{-1} \in Q(F),\ v(us_k)^{-1} \notin Q(F) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} q^{\mathrm{inv}(T(u,v))} & \text{if } vu^{-1}, v(us_k)^{-1} \in Q(F) \\ q^{\mathrm{inv}(T(u,v))} & \text{if } vu^{-1} \in Q(F),\ v(us_k)^{-1} \notin Q(F) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} q^{\mathrm{inv}(T(u,v))} & \text{if } vu^{-1} \in Q(F) \\ 0 & \text{otherwise} \end{cases} .$$

This completes the proof. □

Claim 8. *Fix $u \in S_n$ and suppose that for each $w \in S_n$ with $l(w) < l(u)$, Equation (4.7) holds for all $v \in S_n$ with $l(v) > l(w)$. Then it holds for $w = u$, for all $v \in S_n$ with $l(v) > l(u)$.*

Proof. We have already checked $u = e$ (Claim 5); assume $l(v) > l(u) > 0$. We can therefore choose k so that $us_k < u$. This implies that $l(vs_k)$ and $l(v)$ are both greater than $l(us_k)$ (cf. [3, Prop. 2.2.7]), so by the hypotheses of the claim, we have that Equation (4.7) holds for both $w = us_k$, $v = v$ and $w = us_k$, $v = vs_k$. Thus, if $vs_k > v$ the claim is given by Claim 6, and if $vs_k < v$ it is given by Claim 7. □

We are now ready to prove the section's main claim.

Claim 9. *Given $w, v \in S_n$, with $w < v$, we have*

$$\sigma_{A,e}(q_{w,v}x^{w,v}) = \begin{cases} q^{\mathrm{inv}(T(w,v))} & \textit{if } vw^{-1} \in Q(F) \\ 0 & \textit{otherwise} \end{cases} .$$

Proof. By Claim 8, if the equation holds for all pairs w', v with $l(w') < l(w)$, $l(v) > l(w')$, then it holds for all $v \in S_n$ with $l(v) > l(w)$. By Claim 5, the equation holds for $w = e$ and all v. Thus, by induction on $l(w)$, the claim holds in general. □

4.2.3 Interpretation of $\eta_q^\lambda(\beta_q(F))$

For any subset $X = \{x_1, \dots, x_m\}$ of the positive integers, labelled so that

$$x_1 < \cdots < x_m \, ,$$

we denote by S_X the group of automorphisms of X. As with S_n, we can represent elements of S_X as linear orderings of the letters $\{x_1, \dots, x_m\}$. In particular, the one-line notation of an automorphism $w \in S_X$ is given by $[w(x_1), \dots, w(x_m)]$. Much of our discussion of permutation notation carries over to this situation and will be used without further comment. In particular, the length $l(w)$ of a permutation $w \in S_X$ is the number of pairs $0 < i < j \leq m$ satisfying $w(x_i) > w(x_j)$.

Given $X \subset Y$, $w \in S_X, v \in S_Y$, we say that w is the restriction of v to X, and write $v|_X = w$, if w is the restriction of v as functions. For $X_1, \dots, X_k$ disjoint with $X = \bigcup_i X_i$, there is a natural embedding $S_{X_1} \times \cdots \times S_{X_k} \hookrightarrow S_X$ which associates to a tuple $[w_1 \times \cdots \times w_k] \in S_{X_1} \times \cdots \times S_{X_k}$ the unique automorphism $w \in S_X$ satisfying $w|_{X_i} = w_i$ for $0 < i \leq k$. We will use this embedding to regard $S_{X_1} \times \cdots \times S_{X_k}$ as a subgroup of S_X, and write $S_{X_1} \times \cdots \times S_{X_k} \subset S_X$.

If λ is a partition of n, we say that an ordered set partition $I = I_1, \dots, I_k$ of $[n]$ has *shape* λ if $|I_j| = \lambda_j$ for all $j \in [k]$. Define $S_I \subset S_n$ by $S_I = S_{I_1} \times \cdots \times S_{I_j}$. Let w_{I_j} denote the identity element of S_{I_j}. Let w_I be the permutation whose one-line notation is given by the concatenation $w_{I_1} \cdots w_{I_k}$ of the one-line notations of the identity elements of the blocks, in order. For example, if

$$I = [\{1, 4, 7\}, \{2, 6\}, \{3, 5\}]$$

then w_I is the permutation given in one-line notation by

$$w_I = [1\,4\,7\,2\,6\,3\,5] \, .$$

For an ordered set partition I of $[n]$, and a permutation $w \in S_I$, the number of inversions in the S_n-element ww_I can be related to the number of inversions in the restrictions of w to each block of I.

Claim 10. *Given an ordered set partition $I = I_1, \ldots, I_k$ of $[n]$, permutations $w_1, \ldots, w_k$ with $w_j \in S_{I_j}$ for all j, and $w \in S_I$ satisfying that w restricts to w_j on each I_j, we have*

$$\sum_{j \in [k]} l(w_j) = l(ww_I) - l(w_I)\,.$$

Proof. In one-line notation, ww_I is just the concatenation of the one-line notations of the w_i's. The inversions within each block are counted by the sum on the left hand side, and the inversions among blocks are counted by $l(w_I)$ (since the one-line notation of w_I consists of the same blocks, each written in increasing order). Since this is all the inversions of ww_I we have the claim. □

Using Claim 10, we can simplify certain expressions in $\mathcal{A}_n$ that will appear in the expansions of our immanants.

Claim 11. *Fix an ordered set partition $I = I_1, \ldots, I_k$ of $[n]$. Then we have*

$$\prod_{j \leq k} \sum_{u \in S_{I_j}} q^{l(u)} x^{w_{I_j}, u} = \sum_{w \in S_I} q^{l(ww_I) - l(w_I)} x^{w_I, ww_I}\,. \tag{4.15}$$

Proof. Expanding the left hand side, we see that each term in the sum is determined by choosing $u_j \in S_{I_j}$ for each j. Applying Claim 10 we have

$$\begin{aligned}\prod_{j \leq k} \sum_{u \in S_{Ij}} q^{l(u)} x^{w_{I_j}, u} &= \sum_{u_1, \ldots, u_k \in S_{I_1}, \ldots, S_{I_k}} q^{l(u_1)} \cdots q^{l(u_k)} x^{w_{I_1}, u_1} \cdots x^{w_{I_k}, u_k} \\ &= \sum_{w \in S_I} q^{l(ww_I) - l(w_I)} x^{w_{I_1}, w|_{I_1}} \cdots x^{w_{I_k}, w|_{I_k}} \\ &= \sum_{w \in S_I} q^{l(ww_I) - l(w_I)} x^{w_I, ww_I}\,.\end{aligned}$$

□

Recall that an F-tableau T is *row-closed* if each row of $t(T)$ is a rearrangement of the corresponding row of $s(T)$, and *canonical* if it satisfies $s(T)$ is row strict. Let $\mathrm{flat}(T)$ be the single row tableau obtained from T by composing the rows of T, first to last.

We are now in a position to give a combinatorial formula for certain evaluations of η_q^λ.

Fix a partition λ of n. Let $\mathcal{O}(\lambda)$ denote the set of ordered set partitions of $[n]$ with shape λ. Let F be a totally acyclic planar network of order n with path matrix $A = [a_{i,j}]$. Let $\mathcal{T}_\lambda(F)$ denote the set of canonical row-closed F-tableaux of shape λ.

Theorem 4. *Let $v \in S_n$ be a 3412, 4231-avoiding permutation with planar network F_v, and $\lambda \vdash n$. Then we have*

$$\eta_q^\lambda(\beta_q(F)) = \sum_{T \in \mathcal{T}_\lambda(F)} q^{\mathrm{inv}(\mathrm{flat}(T))}\,.$$

Proof. By a result of Konvalinka and Skandera [38, Thm. 5.4], we have

$$\mathrm{Imm}_{\eta_q^\lambda}(x) = \sum_{I \in \mathcal{O}(\lambda)} per_q(x_{I_1,I_1}) \cdots per_q(x_{I_k,I_k})\,. \tag{4.16}$$

We can use this result to apply our previous combinatorial formula to the evaluation of $\eta_q^\lambda(\beta_q(F))$. First, we apply the definition of β_q and perform a couple simple algebraic manipulations. This gives us

$$\begin{aligned}\eta_q^\lambda(\beta_q(F)) &= \eta_q^\lambda(\sum_{w \in Q(F)} T_w)\\ &= \sum_{w \in Q(F)} \eta_q^\lambda(T_w)\\ &= \sum_{w \in Q(F)} q_{e,w}(q_{e,w})^{-1}\eta_q^\lambda(T_w)\,.\end{aligned} \tag{4.17}$$

By Equation (2.1) we can take the sum in the last expression of Equation (4.17) to be over all of S_n, using the path matrix of F to cancel the extra terms. This gives

$$\begin{aligned}\eta_q^\lambda(\beta_q(F)) &= \sum_{w \in S_n} q_{e,w}(q_{e,w})^{-1}\eta_q^\lambda(T_w)a^{e,w}\\ &= \sigma_{A,e}(\sum_{w \in S_n}(q_{e,w})^{-1}\eta_q^\lambda(T_w)x^{e,w})\,.\end{aligned} \tag{4.18}$$

Applying Equation (4.5) to the final expression of Equation (4.18), we obtain

$$\eta_q^\lambda(\beta_q(F)) = \sigma_{A,e}(\mathrm{Imm}_{\eta_q^\lambda}(x))\,. \tag{4.19}$$

Now we can use Konvalinka and Skandera's formula. Applying Equation (4.16) to the right hand side of Equation (4.19) yields

$$\eta_q^\lambda(\beta_q(F)) = \sigma_{A,e}(\sum_{I\in\mathcal{O}(\lambda)} per_q(x_{I_1,I_1})\cdots per_q(x_{I_k,I_k}))\,. \tag{4.20}$$

Applying the definition of the quantum permanant and Claim 11, with the substitution of $q^{\frac{1}{2}}$ for q in Equation (4.15), gives the following transformation of the right hand side of Equation (4.20)

$$\begin{aligned}\sum_{I\in\mathcal{O}(\lambda)} \sigma_{A,e}(per_q(x_{I_1,I_1})\cdots per_q(x_{I_k,I_k})) &= \sigma_{A,e}(\sum_{I\in\mathcal{O}(\lambda)}\prod_{j\leq k}\sum_{u\in S_{I_j}} q^{\frac{1}{2}l(u)}x^{w_{I_j},u}) \\ &= \sum_{I\in\mathcal{O}(\lambda)}\sum_{w\in S_I}\sigma_{A,e}(q_{w_I,ww_I}x^{w_I,ww_I})\,.\end{aligned} \tag{4.21}$$

Finally, we apply Equation (4.7) to the right hand side of Equation (4.21) to get

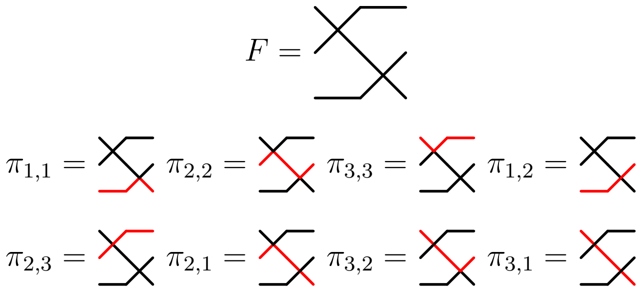
$$\eta_q^\lambda(\beta_q(F)) = \sum_{I\in\mathcal{O}(\lambda)}\sum_{w\in S_I\cap Q(F)} q^{\mathrm{inv}(T(w_I,ww_I))}\,.$$

A canonical row-closed tableau T of shape λ is determined by a choice of an ordered set partition of $[n]$ of shape λ to be the source/sink set of each row, and a choice of paths using the given sources and sinks in each row. The row word of $s(T)$ is then w_I, and the type of T is an element of S_I. This correspondence can be reversed, and furthermore flat$(T) = T(w_I, ww_I)$, so we have the claim. □

An example of this formula in action is given in Figure 4.1. A planar network F is given Figure 4.1a, and its eight source to sink paths are illustrated and labelled. For the given network, we have $\beta_q(F) = T_e + T_{[213]} + T_{[132]} + T_{[231]}$. Thus, we can calculate character evaluations of this Hecke algebra element using F-tableaux.

Figure 4.1b gives such an evaluation. There are four canonical row-closed F-tableaux of shape 3; these are pictured. Note that flattening a one-row tableau is trivial so for each tableau T of shape 3 we have inv(flat(T)) = inv(T). In this case the tableaux have 0, 1, 1, and 2 inversions (these inversion numbers are given in the figure). Thus, we have

$$\begin{aligned}\eta_q^3(T_e + T_{[213]} + T_{[132]} + T_{[231]}) &= q^0 + q^1 + q^1 + q^2 \\ &= q^2 + 2q + 1\,.\end{aligned}$$



$$\beta_q(F) = T_e + T_{[213]} + T_{[132]} + T_{[231]}$$

(a) F and its paths

T:	$\pi_{1,1}$ $\pi_{2,2}$ $\pi_{3,3}$	$\pi_{1,1}$ $\pi_{2,3}$ $\pi_{3,2}$	$\pi_{1,2}$ $\pi_{2,1}$ $\pi_{3,3}$	$\pi_{1,2}$ $\pi_{2,3}$ $\pi_{3,1}$
inv(flat(T)):	0	1	1	2

$\eta_q^3(\beta_q(F)) = q^2 + 2q + 1$

(b) $\lambda = 3$

T:	$\pi_{1,1}$	$\pi_{1,1}$	$\pi_{2,2}$	$\pi_{3,3}$	$\pi_{3,3}$
	$\pi_{2,2}$ $\pi_{3,3}$	$\pi_{2,3}$ $\pi_{3,2}$	$\pi_{1,1}$ $\pi_{3,3}$	$\pi_{1,1}$ $\pi_{2,2}$	$\pi_{1,2}$ $\pi_{2,1}$
inv(flat(T)):	1	2	1	0	1

$\eta_q^{21}(\beta_q(F)) = q^2 + 3q + 1$

(c) $\lambda = 21$

Figure 4.1: Example computations of η_q^λ

Similarly, Figure 4.1c gives the evaluation of $\eta_q^{21}(\beta_q(F))$ for the given network F. In this case there are five F-tableau of shape 21, which are pictured. For each tableau T, inv(flat(T)) is given; note that here the flattening is not vaccuous. The tableaux have 0, 1, 1, 1, and 2 inversions and thus we have $\eta_q^{21}(\beta_q(F)) = q^2 + 3q + 1$ for the given F.

4.2.4 Interpretation of $\epsilon_q^\lambda(\beta_q(F))$

In this subsection, we extend our interpretation of the induced trivial characters to the induced sign characters. We will do this by means of the tools we have developed in previous sections, with the main new ingredient being a certain involution on F-tableau. We will begin by developing the combinatorics neccesary to obtain our result.

Let F be a zig-zag network. It will be useful to consider a slightly modified inversion statistic on tableaux. For an F-tableau T, call a pair of paths π, π' in T a *strict inversion* if π and π' appear in the same row of T, π appears to the left of π' in T, π has a greater sink than π', and $\pi \cap \pi' \neq \emptyset$. Set $\mathrm{inv}^-(T)$ to be the number of strict inversions in T.

Similarly, let a strict inversion in an integer tableau U be a pair $i < j$ appearing in the same row of U with i to the right of j, and set $\mathrm{inv}^-(U)$ to be the number of strict inversions in U.

We state a simple identity involving strict inversions.

Claim 12. *For any F-tableau T, we have* $\mathrm{inv}(\mathrm{flat}(T)) = \mathrm{inv}^-(T) + \mathrm{inv}(T^\top)$.

Proof. The first term is the number of inversions within each row of T, the second is the number of inversions among the rows. □

In order to obtain the desired interpretation of ϵ_q^λ, we will slightly modify Equation (4.7). For $w \in S_I$, let $T(I, w)$ be the unique F-tableau with shape λ and type w satisfying that the row word of $s(T(I, w))$ is w_I, if such a tableau exists. While this notation is very similar to $T(u, v)$, introduced in Subsection 4.2.2, the type of the first parameter can be used to distinguish the two.

Note that $T(w_I, ww_I) = \mathrm{flat}(T(I, w))$, and so it follows from Claim 4 that both $T(w_I, ww_I)$, $T(I, w)$ are well defined if and only if $w \in Q(F)$.

Claim 13. *Let F be a descending star network with path matrix $A = [a_{i,j}]$. Let I be an ordered set partition of shape λ. Then for $w \in S_I$, we have*

$$\sigma_{A,e}((q_{w_I,ww_I})^{-1} x^{w_I,ww_I}) = \begin{cases} q^{\mathrm{inv}(T(I,w)^\top)} & \textit{if } w \in Q(F) \\ 0 & \textit{otherwise} \end{cases}. \tag{4.22}$$

Proof. We begin by trivially reorganizing the left hand side of (Equation 4.22) to obtain

$$\sigma_{A,e}((q_{w_I,ww_I})^{-1}x^{w_I,ww_I}) = (q_{w_I,ww_I})^{-2}\sigma_{A,e}(q_{w_I,ww_I}x^{w_I,ww_I})\,. \tag{4.23}$$

To improve this expression, we will examine the strict inversions of $T(I,w)$. Since there are no inversions within the rows of $s(T(I,w))$, any potential strict inversion - a pair of paths whose sinks are out of order - must be a pair of crossing paths, and thus a pair of intersecting paths. Therefore, we have

$$\text{inv}^-(T(I,w)) = \text{inv}^-(t(T(I,w)))\,.$$

By the construction of $T(I,w)$ and Claim 10, we can rewrite the right hand side of this equation to obtain

$$\text{inv}^-(T(I,w)) = l(ww_I) - l(w_I)\,.$$

Applying this identity to the right hand side of Equation (4.23) gives us

$$\sigma_{A,e}((q_{w_I,ww_I})^{-1}x^{w_I,ww_I}) = q^{-\text{inv}^-(T(I,w))}\sigma_{A,e}(q_{w_I,ww_I}x^{w_I,ww_I})\,. \tag{4.24}$$

Now we can apply Equation (4.7) to the right hand side of Equation (4.24) to obtain

$$\begin{aligned}\sigma_{A,e}((q_{w_I,ww_I})^{-1}x^{w_I,ww_I}) &= \begin{cases} q^{-\text{inv}^-(T(I,w))}q^{\text{inv}(T(w_I,ww_I))} & \text{if } ww_I(w_I)^{-1} \in Q(F) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} q^{\text{inv}(T(w_I,ww_I))-\text{inv}^-(T(I,w))} & \text{if } w \in Q(F) \\ 0 & \text{otherwise} \end{cases}\,. \end{aligned} \tag{4.25}$$

Using the fact that $T(w_I,ww_I) = \text{flat}(T(I,w))$ and applying Claim 12 to the last expression of Equation (4.25) we have

$$\sigma_{A,e}((q_{w_I,ww_I})^{-1}x^{w_I,ww_I}) = \begin{cases} q^{\text{inv}(T(I,w)^\top)} & \text{if } w \in Q(F) \\ 0 & \text{otherwise} \end{cases}\,.$$

□

Fix a partition λ of n, and let F be a zig-zag network of order n. As we have already seen in the proof of Theorem 4, the set $\mathcal{T}_\lambda$ consists entirely of tableaux of the form $T(I, w)$.

Claim 14. $\mathcal{T}_\lambda = \{T(I, w) \mid I \in \mathcal{O}(\lambda), w \in S_I \cap Q(F)\}$.

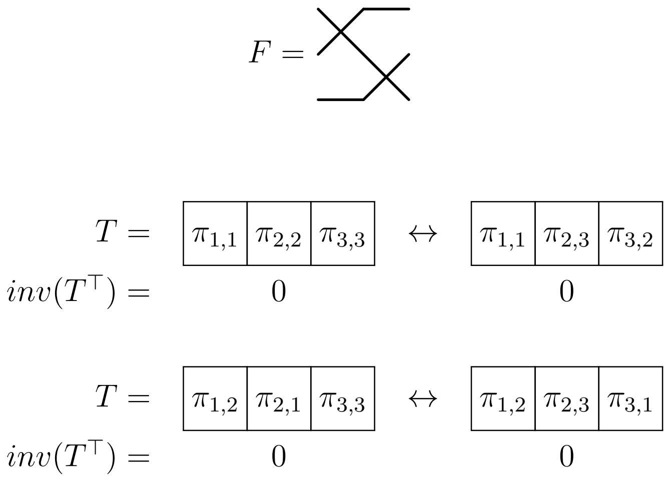
Now, let $\mathcal{T}_\lambda^\circ \subset \mathcal{T}_\lambda$ denote the set of canonical row-closed row-strict F-tableaux of shape λ. As with η, we have an expression for the ϵ immanant, which is similar to Equation (4.16) except with determinants rather than permanants. This, of course, introduces signs; to get a combinatorial interpretation of ϵ, we will need a sign reversing involution (in a sense we will soon define) on $\mathcal{T}_\lambda \setminus \mathcal{T}_\lambda^\circ$ that preserves inv (again, in a particular sense). Such a map is easy to construct.

Given an F-tableau $T = t_{i,j}$, let (i, j') and (i, j) (for $j < j'$) be the maximal pair of indices such that $t_{i,j}$ intersects $t_{i,j'}$. By maximal we mean that (i, j', j) is the (left to right) lexicographically maximal triple such that $t_{i,j}$ intersects $t_{i,j'}$. Suppose $t_{i,j}$ has type $(m \to n)$ and $t_{i,j'}$ has type $(m' \to n')$. Then by Claim 1, and the uniqueness of paths in a descending star network, there is an F-tableau U that is identical to T except that $u_{i,j}$ has type $(m \to n')$ and $u_{i,j'}$ has type $(m' \to n)$. Let $\iota : \mathcal{T}_\lambda \to \mathcal{T}_\lambda$ be the map that assigns to each tableau T that has a pair of intersecting paths in some row this unique U, and fixes all other tableaux.

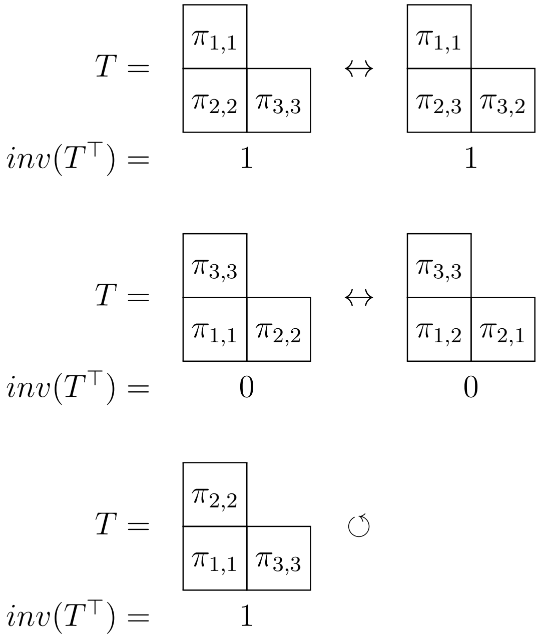
As an example, consider again the descending star network given in Figure 4.1. Figure 4.2a shows the four F-tableaux of shape 3, and Figure 4.2b shows the five F-tableaux of shape 21. The number of inversions in these tableaux's transposes is also given, along with arrows indicating the action of the map ι. It can be seen that for this choice of F, ι is an involution that fixes the inversion number. This is in fact true in general.

Claim 15. *Fix $\lambda \vdash n$. The map ι is an involution on $\mathcal{T}_\lambda$ which fixes $\mathcal{T}_\lambda^\circ$.*

Proof. Let T be a tableau in $\mathcal{T}_\lambda$. If there are no intersecting paths in any row of T, then we have $\iota(\iota(T)) = \iota(T) = T$. Otherwise, let paths p of type $(m \to n)$ and p' of type $(m' \to n')$ be the unique pair of paths in T that are modified by ι. Suppose they are in positions (i, j) and (i, j'), respectively, with $j' > j$. Then $m' > m$ (since



(a) $\lambda = 3$



(b) $\lambda = 21$

Figure 4.2: Example of ι involution. The paths in F are illustrated in Figure 4.1.

T is canonical). By our choice of j and j', no path to the right of p' in T intersects either p or p', and thus, no such path intersects $\pi_{m,n'}$ or $\pi_{m',n}$. Again by our choice of j and j', no path lying between p and p' in T intersects p', and thus by Claim 3 (which applies since T is canonical) no such path intersects $\pi_{m',n}$. Since all other cells in $\iota(T)$ are identical to those of T, we have that (i, j', j) is the lexicographically maximal triple of indices for which $(\iota(T))_{i,j}$ and $(\iota(T))_{i,j'}$ intersect. So, applying ι again will switch back the same pair of paths, and we have $\iota(\iota(T)) = T$.

The tableaux which are fixed by ι are precisely those which have no intersecting paths in any row. In particular, for such a tableau T, no adjacent pair of paths in a row of T intersects. Since T is canonical, this is equivalent to the condition that T is row-strict, i.e., $T \in \mathcal{T}_\lambda^\circ$. □

We will now show that the map ι preserves the number of inversions in the transpose of a tableau.

Claim 16. *For any $T \in \mathcal{T}_\lambda$, we have* $\operatorname{inv}(\iota(T)^\top) = \operatorname{inv}(T^\top)$.

Proof. If $T \in \mathcal{T}_\lambda^\circ$, the claim is trivial. Otherwise, let (i, j) and (i, j') be the indices of the paths that get switched by ι. Let $\iota(p)$ and $\iota(p')$ be the paths in positions (i, j) and (i, j') of $\iota(T)$, respectively. Let v_1 and v_2 be the the leftmost and rightmost vertices, respectively, in the intersection of p and p'. Let A, B, C, D, and E be the paths connecting m to v_1, m' to v_1, v_2 to n, v_2 to n', and v_1 to v_2, respectively. Let $q = \pi_{a,b}$ be a path that forms a weak inversion with p or p' in $T^\top$. Then (by the definition of a weak inversion) q is in a different column of $T^\top$ than that occupied by both p and p' and therefore a different row of T from both p and p'; in particular, q is unchanged by ι.

If q intersects E, then it intersects p, p', $\iota(p)$, and $\iota(p')$. So (q, p) is an inversion if and only if $(q, \iota(p'))$ is, and likewise for (q, p') and $(q, \iota(p))$. Thus, the number of weak inversions in the transpose involving q is unchanged if q intersects E

If q does not intersect E, then since F is acyclic as an undirected graph, q can intersect at most one of A, B, C, and D. Suppose that q intersects only A. Then q does not intersect either p' or $\iota(p')$. The only possible arrangement of sinks is

$b < n < n'$, so q forms an inversion with p if and only if it does with $\iota(p)$. The analysis of the other three cases (q intersects B, C, or D) is almost identical. Thus, the number of weak inversions in the transpose involving q is unchanged if q does not intersect E.

Since the number of inversions in the transpose involving any particular path is unchanged by ι, we have the claim. $\square$

The purpose of the ι map is to associate tableaux that contribute negatively in certain expressions (which will we see later) for the evaluation of ϵ_q^λ with tableaux that contribute positively in these expressions. The following claim will provide this property.

Claim 17. *For $T \in \mathcal{T}_\lambda \setminus \mathcal{T}_\lambda^\circ$, we have $(-1)^{l(\mathrm{rw}(t(T)))} = -(-1)^{l(\mathrm{rw}(t(\iota(T))))}$*

Proof. Sign, i.e., the map $w \mapsto (-1)^{l(w)}$, is a homomorphism. Applying ι to a tableau T amounts to multiplying $\mathrm{rw}(t(T))$ by a (not necessarily adjacent) transposition, all of which have sign -1. $\square$

Finally, we have that ι is an involution.

Claim 18. $\iota(\iota(T)) = \iota(T)$

Proof. This is by the definition of ι. $\square$

We have everything we need to prove the following formula, originally proposed by B. Shelton.

Theorem 5. *Let $v \in S_n$ be a $3412, 4231$-avoiding permutation with planar network F_v, and $\lambda \vdash n$, and let $\mathcal{T}_\lambda^\circ$ be the set of canonical row-closed row-strict F_v tableaux. Then we have*

$$\epsilon_q^\lambda(\beta_q(F_v)) = \sum_{T \in \mathcal{T}_\lambda^\circ} q^{\mathrm{inv}(T^\top)}\,. \tag{4.26}$$

Proof. Again by a result of Konvalinka and Skandera [38, Thm. 5.4], we have

$$\epsilon_q^\lambda(\beta_q(F)) = \sum_{I \in \mathcal{O}(\lambda)} \sigma_{A,e}(det_q(x_{I_1,I_1}) \cdots det_q(x_{I_k,I_k})\,.$$

Applying the definition of the quantum determinant and Claim 11, with the substitution of $q^{\frac{1}{2}}$ for q in Equation (4.15), we obtain

$$\begin{aligned}\sum_{I\in\mathcal{O}(\lambda)} &\sigma_{A,e}(det_q(x_{I_1,I_1})\cdots det_q(x_{I_k,I_k})\\ &= \sigma_{A,e}(\sum_{I\in\mathcal{O}(\lambda)}\prod_{j\le k}\sum_{u\in S_{I_j}}(-1)^{l(u)}q^{-\frac{1}{2}l(u)}x^{w_{I_j},u}) \qquad (4.27)\\ &= \sum_{I\in\mathcal{O}(\lambda)}\sum_{w\in S_I}\sigma_{A,e}((-1)^{l(ww_I)-l(w_I)}(q_{w_I,ww_I})^{-1}x^{w_I,ww_I})\,.\end{aligned}$$

Applying Equation (4.22) to the final expression in Equation (4.27) gives

$$\sum_{I\in\mathcal{O}(\lambda)}\sigma_{A,e}(det_q(x_{I_1,I_1})\cdots det_q(x_{I_k,I_k}) = \sum_{I\in\mathcal{O}(\lambda)}\sum_{w\in S_I\cap Q(F)}(-1)^{l(ww_I)-l(w_I)}q^{\mathrm{inv}(T(I,w)^\top)}\,. \tag{4.28}$$

Transforming the right hand side of Equation (4.28) using the construction of $T(I,w)$ and then applying Claim 14, gives

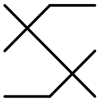
$$\begin{aligned}\sum_{I\in\mathcal{O}(\lambda)} &\sigma_{A,e}(det_q(x_{I_1,I_1})\cdots det_q(x_{I_k,I_k})\\ &= \sum_{I\in\mathcal{O}(\lambda)}\sum_{w\in S_I\cap Q(F)}(-1)^{l(\mathrm{rw}(t(T(I,w))))-l(\mathrm{rw}(s(T(I,w))))}q^{\mathrm{inv}(T(I,w)^\top)}\\ &= \sum_{T\in\mathcal{T}_\lambda}(-1)^{l(\mathrm{rw}(t(T)))-l(\mathrm{rw}(s(T)))}q^{\mathrm{inv}(T^\top)}\,.\end{aligned}$$

Let us examine the terms of this sum. For any $T\in\mathcal{T}_\lambda\setminus\mathcal{T}_\lambda^\circ$, we have by Claims 15, 16, 17, and 18 that

$$(-1)^{l(\mathrm{rw}(t(T)))-l(\mathrm{rw}(s(T)))}q^{\mathrm{inv}(T^\top)} = (-1)(-1)^{l(\mathrm{rw}(t(\iota(T))))}(-1)^{l(\mathrm{rw}(s(\iota(T))))}q^{\mathrm{inv}(\iota(T)^\top)}\,.$$

Thus, the ι involution pairs all the terms of the sum into canceling pairs except those that come from $\mathcal{T}_\lambda^\circ$. Now, for $T\in\mathcal{T}_\lambda^\circ$, we have $t(T)=s(T)$ so $(-1)^{l(\mathrm{rw}(t(T)))-l(\mathrm{rw}(s(T)))}=1$ and each of these terms contribute positively. The desired formula follows. □

Consider again the planar network F illustrated in Figure 4.1. Figure 4.3 gives the full computation of $\epsilon_q^\lambda(\beta_q(F))$ for this network for $\lambda=3$ and $\lambda=21$. Note that

$F =$ 

$\mathcal{T}_3^\circ = \emptyset$

$\epsilon_q^3(\beta_q(F)) = 0$

$$\mathcal{T}_{21}^\circ = \{T = \begin{array}{|c|c}\hline \pi_{2,2} & \\ \hline \pi_{1,1} & \multicolumn{1}{|c|}{\pi_{3,3}} \\ \hline \end{array}\},\ \mathrm{inv}(T^\top) = 1$$

$\beta_q(F) = T_e + T_{[213]} + T_{[132]} + T_{[231]}$

$\epsilon_q^{21}(\beta_q(F)) = q$

Figure 4.3: Example computations of ϵ_q.

in the case $\lambda = 3$, there are no canonical row-bijective row-semistrict F-tableaux so in this case, the right hand side of Equation (4.26) is an empty sum and is thus equal to 0.

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