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# On a fundamental question in hydrodynamic lubrication theory.

Hesham A. Ezzat

# ON A FUNDAMENTAL QUESTION IN HYDRODYNAMIC LUBRICATION THEORY

by

Hesham A. Ezzat

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# CERTIFICATE OF APPROVAL

This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

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Professor in Charge

Chairman of Department

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## ABSTRACT

The subject of lubrication is of fundamental importance in practical mechanics. One of the early investigations in this field was carried out by Mr. Beauchamp Tower; his first report was published in November, 1883.

Consider a flat surface moving relative to a fixed inclined plane with the two bodies separated by a viscous fluid. Mr. Tower discovered that the moving surface drags the viscous fluid into the gap between the two bodies causing a continuous fluid film to be maintained.

Osborne Reynolds [1] followed on the theory of lubrication and its application to Mr. Tower's experiments. Reynolds, upon reading Tower's report, thought that a mathematical description of the maintenance of the fluid film could be obtained from the equations of hydrodynamics. In doing this he formulated the fundamental equation, which bears his name, of Lubrication Theory.

Following Reynolds, in the early 1900's, Lord Rayleigh [2] had his own research in the field of lubrication. Rayleigh introduced the step bearing which appears to be the form which must be approached if we wish to maximize the load carrying capacity of the bearing.

Our object in the present thesis is to derive in detail and from first principles the basic Reynolds' equation in one dimension and hence consider different geometries of the fixed surface, while the moving surface is kept plane, in search of the optimum profile that provides a maximum load carrying capacity. The numerical values appearing throughout this thesis as well as the graphs describing the behaviour of the load with the change in the maximum to minimum film thickness ratio have been obtained on the computer.

In all the questions raised in this study we may anticipate that our calculations correspond pretty closely with what actually happens in practical application.

## SECTION I

## BASIC DEFINITIONS

(1) Shear stress ($\tau$)

A straining action wherein tangentially applied forces produce a sliding or skewing type of deformation is always denoted as shearing.

A shearing force acts parallel to a plane as distinguished from tensile or compressive forces which act normal to a plane.

Shearing stress is the intensity of distributed tangential force expressed as force per unit area.

Examples of force systems producing shearing action are forces transmitted from one plate to another by a rivet, that tend to shear the rivet (Figure 1).

With the aid of the latter example it is easier to extend the concept of shearing forces to fluids, considering the forces transmitted from one plate to another by a fluid film that tend to shear the film. (Figure 2).

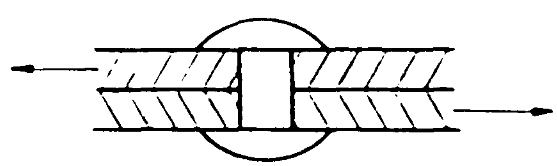


fig. 1

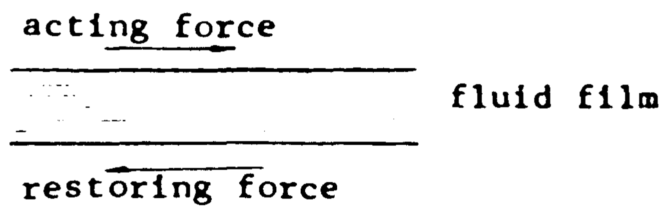



fig. 2

(2) Viscosity ($\mu$)

Viscosity is the property of fluids which causes them to resist variations in velocity that occur across a section of a flowing fluid; by causing shearing stresses between adjacent layers of fluid moving relatively to each other.

It was Newton who first stated that the shearing stress in a fluid is directly proportional to the velocity gradient normal to the flow, that is, the rate at which the velocity varies across a section. The coefficient of viscosity ($\mu$) is defined as the stress (force per unit area) which in a given fluid results from unit velocity gradient. In otherwords it is the constant of proportionality in the relation

$$\tau = \mu \frac{du}{dy} \ldots \qquad (I.1)$$

where

$\tau$ = shear stress,

$u$ = velocity in the x-direction,

$\frac{du}{dy}$ = velocity gradient normal to the direction of the flow.

SECTION II

DERIVATION OF THE ONE DIMENSIONAL REYNOLDS' EQUATION .

To derive the Reynolds' equation in one dimension, we shall examine the case of two parallel plates moving relative to each other with a constant velocity of U.

This case, often called the plane slider, is shown in Figure (3).

It is noteworthy to mention that our calculations will be carried out for an infinite bearing; that is, the dimension of the bearing in the z-direction is infinite. This is to avoid the complications resulting from leakage.

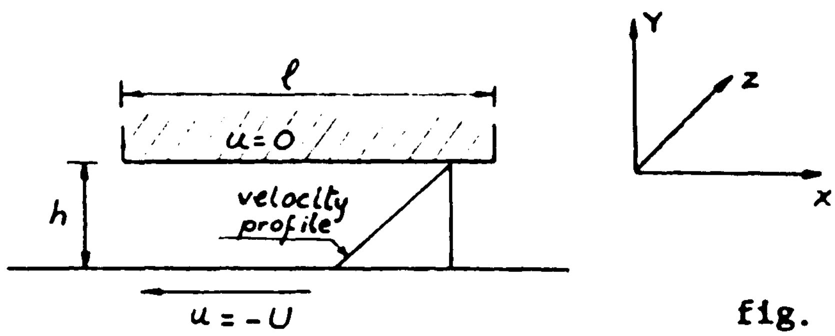



fig. 3

(i) Assumptions.

1) Flow is laminar i.e. particles of fluid move in straight lines parallel to the boundary, that is motion is everywhere parallel to x-direction (no velocity in y-direction).

2) Body forces are neglected, i.e. no extra fields of forces acting on the fluid.

3) No slip at the boundary, i.e. velocity of the fluid layer adjacent to the boundary is the same as that of the boundary.

4) Fluid is Newtonian; i.e., obeys the relation

$$\tau = \mu \frac{du}{dy}$$

as discussed earlier.

5) We assume here for <u>simplicity</u> that viscosity is constant.

6) We are assuming a very thin fluid film; in otherwords, we are assuming that $\frac{h}{\ell} << 1$. Consequently the pressure gradient across the film is insignificant compared to the pressure gradient along the film. According to D. Dawson ["Generalized Reynolds' Eq$^n$ for fluid film lubrication", 1961] it can be shown that $\frac{dp}{dy}$ is $(\frac{h}{\ell})$ times $\frac{dp}{dx}$.

7) The whole flow of fluid is regarded as incompressible between zero and h, i.e. the density is constant.

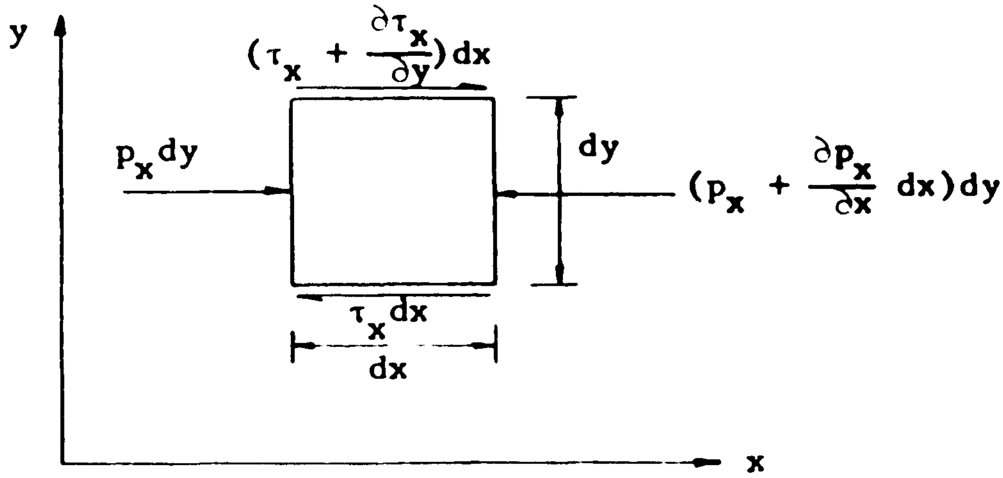



fig. 4

(ii) Equilibrium of a rectangular element

Consider a general element of fluid (dxdy) as shown in Figure (4) where

$p_x$ = pressure along the film $\{x\text{-dir}^{\underline{n}}\}$.

and

$\tau_x$ = shear stress along the film $\{x\text{-dir}^{\underline{n}}\}$.

According to assumptions (1), (2) and (6) the above is the only system of forces acting on the element.

For the equilibrium of forces in the x-direction

$$p_x dy + \tau_x dx + \frac{\partial \tau_x}{\partial y} dydx$$
$$= \tau_x dx + p_x dy + \frac{\partial p_x}{\partial x} dxdy.$$

Hence,

$$\frac{\partial \tau_x}{\partial y} = \frac{\partial p_x}{\partial x} \quad .$$

According to assumption (4) the fluid is Newtonian and thus it obeys the relation

$$\tau = \mu \frac{du}{dy} \ .$$

Since we assumed that $\mu$ is constant, we have

$$\frac{\partial \tau_x}{\partial y} = \mu \frac{d^2u}{dy^2} \quad .$$

Hence, $\mu \dfrac{d^2u}{dy^2} = \dfrac{\partial p_x}{\partial x}$ .

Let $p_x$ be $p(x)$. Then,

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} \ . \qquad \text{(II.1)}$$

Equation (II.1) is directly integrable and yields

$$\frac{du}{dy} = \frac{1}{\mu} \frac{dp}{dx} y + A$$

and

$$u = \frac{1}{2\mu} \frac{dp}{dx} y^2 + Ay + B \qquad \text{(II.2)}$$

where A and B are constants of integration obtained from the boundary conditions as follows.

From assumption (3)

$$u = -U \quad \text{at} \quad y = 0$$

and

$$u = 0 \quad \text{at} \quad y = h \ .$$

Hence $B = -U$ and $A = \frac{U}{h} - \frac{1}{2\mu}\frac{dp}{dx}h$ .

Now (II.2) becomes,

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + \frac{U}{h}y - \frac{1}{2\mu}\frac{dp}{dx}hy - U$$

or,

$$u = \frac{y^2 - hy}{2\mu}\frac{dp}{dx} - U(1 - \frac{y}{h}) \qquad \text{(II.3)}$$

The volume rate of fluid flow through an element of thickness (dy) is given by

$$q_x = u\,dy \ .$$

Accordingly the volume rate of fluid flow through the entire film of thickness (h) would be given by

$$Q_x = \int_0^h u\,dy \ .$$

Therefore,

$$Q_x = \int_0^h u\,dy = \frac{dp}{dx}\int_0^h \frac{y^2 - hy}{2\mu}dy - U\int_0^h \frac{(1-y/h)}{h}dy$$

or,

$$Q_x = -\frac{dp}{dx}\frac{h^3}{12\mu} - \frac{Uh}{2} \qquad \text{(II.4)}$$

According to assumption (7) the fluid flow is incompressible between 0 and h; hence the density is constant.

Taking the fluid mass balance in the x direction across a section of infinitismal length dx as shown in Figure (5).

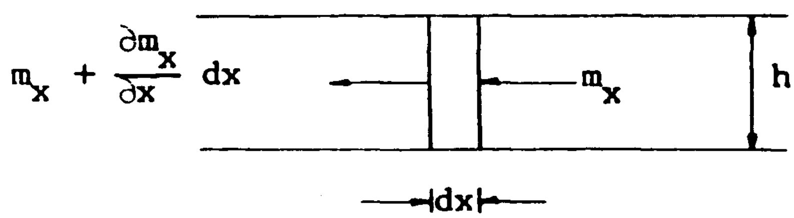



fig. 5

The mass of fluid entering the section, denoted by $m_x$, is given by

$$m_x = \rho Q_x$$

where $\rho$ is the density.

The mass of fluid leaving the section will be

$$m_x + \frac{\partial m_x}{\partial x} dx .$$

where $\frac{\partial m_x}{\partial x}$ is the rate of change of mass in the x direction and dx is small enough to treat $\frac{\partial m_x}{\partial x}$ as linear.

But, the mass entering the section is equal to the mass leaving the section. Hence

$$m_x = m_x + \frac{\partial m_x}{\partial x} dx .$$

Thus

$$\frac{\partial m_x}{\partial x} = 0 \quad \text{or} \quad \frac{\partial}{\partial x} \rho Q_x = 0 .$$

But $\rho$ is constant and not zero. Therefore

$$\frac{\partial Q_x}{\partial x} = 0$$

The last equation tells us that $Q_x$ is constant. Now let

$$Q_x = -Q .$$

Then, Equation (II.4) becomes

$$-Q = - \frac{dp}{dx} \frac{h^3}{12\mu} - \frac{Uh}{2}$$

or,

$$\frac{dp}{dx} = \frac{12\mu}{h^3} (Q - \frac{Uh}{2}) . \qquad \text{(II.5)}$$

Introducing a new constant H given by

$$H = \frac{2Q}{U} .$$

Equation (II.5) reads

$$\frac{dp}{dx} = \frac{6\mu U}{h^3} \left\{ H - h \right\}$$

or, (II.6)

$$\frac{dp}{dx} = - \frac{6\mu U}{h^3} (h - H)$$

Equations (II.6) represent the well known form of Reynolds' Equation in one dimension.

Equation (II.6) is correct within a constant H which we introduced earlier and which happens to have a physical significance. H is the thickness of the fluid film for which $\frac{dp}{dx} = 0$.

We notice that in the case of a plane slider the thickness h of the film is the same everywhere; in otherwords, the right hand side of the Reynolds' equation will remain constant. Therefore

$$\frac{dp}{dx} = \text{constant.}$$

Hence we can expect the pressure distribution to be linear in the x-direction.

Now the choice of the pressure at both entry and exit to the slider is arbitrary since we are always considering gauge pressure.

To simplify the mathematics we choose, without loss of generality, the pressure at both ends of the slider to be zero. Therefore $p(x) = 0$ and

$$\frac{dp}{dx} = 0$$

in which case, from equation (II.6),

$$-\frac{6\mu U}{h^3}(h-H) = 0.$$

This implies that $h = H$ which justifies our previous explanation for the constant $H$.

## SECTION III

## ANALYSIS OF LORD RAYLEIGH'S STEP BEARING

Reynolds' Equation in one dimension is

$$\frac{dp}{dx} = -6\mu U \left\{\frac{h-H}{h^3}\right\}$$

where H, as we stated before, is a constant that actually has physical significance. Let us consider the entry and exit to be a and b respectively. Furthermore let p(a) and p(b) be zero.

With these conditions we can evaluate H from the equation

$$\int_a^b dp = -6\mu U \left\{\int_a^b \frac{dx}{h^2} - H \int_a^b \frac{dx}{h^3}\right\} ;$$

i.e.

$$H = \frac{\int_a^b dx/h^2}{\int_a^b dx/h^3}$$

Using the notation

$$I_{mn} = \int_a^b \frac{x^m}{h^n} dx$$

we may write, $H = \frac{I_{02}}{I_{03}}$ .

With this definition of H we integrate Reynolds' Equation from a to x to obtain

$$p(x) = 6\mu U\left[\int_a^x \frac{dt}{h^2} - \frac{I_{02}}{I_{03}}\int_a^x \frac{dt}{h^3}\right] \qquad \text{(III.1)}$$

Let us look now at Rayleigh's step bearing shown in Figure (6)

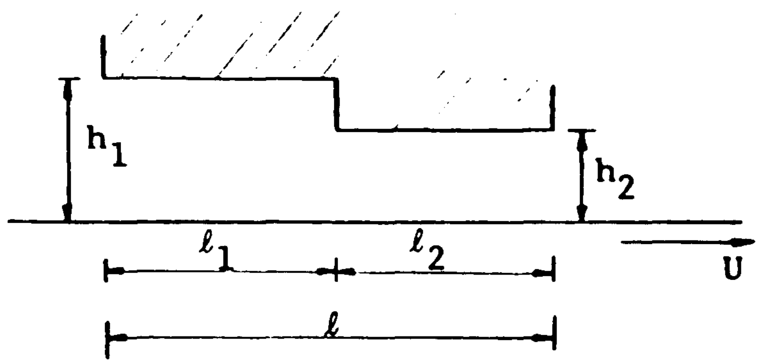



fig. 6

Let $\lambda = \frac{h_1}{h_2}$, $\beta = \frac{\ell_1}{\ell}$ and let $a = 0$ and $b = \ell$. In this case

$$I_{mn} = \frac{1}{h_1^n}\int_0^{\ell_1} x^m dx + \frac{1}{h_2^n}\int_{\ell_1}^{\ell} x^m dx$$

$$= \frac{1}{m+1}\frac{\ell^{m+1}}{h_1^n}\left[\beta^{m+1}+\lambda^n(1-\beta^{m+1})\right]$$

$$= \frac{1}{m+1}\frac{\ell^{m+1}}{h_1^n}\tilde{I}_{mn}$$

where $\tilde{I}_{mn} = \left[\beta^{m+1}+\lambda^n(1-\beta^{m+1})\right]$.

The pressure distribution given by equation (III.1) will be

$$p(x) = \begin{cases} \frac{6\mu U}{h_1^2}\left(1 - \frac{I_{02}}{I_{03}h_1}\right)x & 0 \le x \le \ell_1 \\ \frac{6\mu U}{h_1^2}\left[\ell_1\left(1 - \frac{I_{02}}{I_{03}h_1}\right)+\lambda^2\left(1 - \frac{I_{02}}{I_{03}h_2}\right)(x-\ell_1)\right] & \ell_1 \le x \le \ell \end{cases}$$

(III.2)

where $I_{02} = \frac{\ell}{h_1^2}[\beta+\lambda^2(1-\beta)]$ and $I_{03} = \frac{\ell}{h_1^3}[\beta+\lambda^3(1-\beta)]$.

The pressure distribution is sketched in Figure (7)

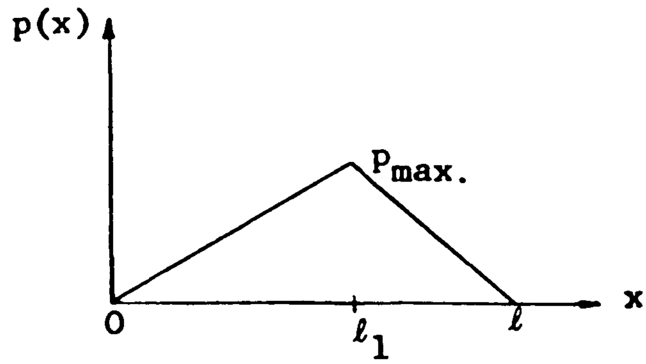



fig. 7

The load per unit width, $\frac{W}{L}$, shall in this case be given by the area under the triangle shown in Figure (7); i.e.,

$$\frac{W}{L} = \frac{1}{2} P_{max} \ell$$

with $P_{max} = P(\ell_1) = \frac{6\mu U}{h_1^2}\left\{1 - \frac{I_{02}}{I_{03}h_1}\right\} \ell_1$. Thus,

$$\frac{W}{L} = \frac{3\mu U}{h_1^2}\left\{1 - \frac{I_{02}}{I_{03}h_1}\right\} \ell_1 \ell .$$

This can be written as,

$$\frac{W}{L} = \frac{3\mu U \ell^2}{h_2^2} \left\{ \frac{\beta(1-\beta)(\lambda-1)}{\beta+\lambda^3(1-\beta)} \right\} \qquad \text{(III.3)}$$

According to Lord Rayleigh the centre of pressure is the position through which the load acts. It is given by taking moments about the origin.

Assume the distance of the centre of pressure from the origin to be $\bar{x}$. Then

$$\frac{W}{L}\bar{x} = \int_0^{\ell} px \; dx \; .$$

Thus,

$$\frac{3\mu U \ell^2}{h_2^2} \left\{ \frac{\beta(1-\beta)(\lambda-1)}{\beta+\lambda^3(1-\beta)} \right\} \bar{x} = \int_0^{\ell_1} \frac{6\mu U}{h_1^2} \left\{ 1 - \frac{I_{02}}{I_{03}h_1} \right\} x^2 dx$$

$$+ \int_{\ell_1}^{\ell} \frac{6\mu U}{h_1^2} \left\{ \ell_1 (1 - \frac{I_{02}}{I_{03}h_1}) \right.$$

$$\left. + \lambda^2 (1 - \frac{I_{02}}{I_{03}h_2})(x-\ell_1) \right\} x \; dx$$

$$\frac{\ell^2}{h_2^2} \left\{ \frac{\beta(1-\beta)(\lambda-1)}{\beta+\lambda^3(1-\beta)} \right\} \bar{x} = \frac{2}{3h_1^2} [\, \ell_1^3 (1 - \frac{I_{02}}{I_{03}h_1})]$$

$$+ \frac{2}{h_1^2} \left\{ \frac{\ell_1 \ell^2}{2} [1 - \frac{I_{02}}{I_{03}h_1}] - \frac{\ell_1^3}{2} [1 - \frac{I_{02}}{I_{03}h_1}] \right.$$

$$+ \lambda^2 \frac{\ell^3}{3} [1 - \frac{I_{02}}{I_{03}h_2}] - \frac{\lambda^2 \ell_1^3}{3} [1 - \frac{I_{02}}{I_{03}h_2}]$$

$$- \frac{\lambda^2 \ell_1 \ell^2}{2} [1 - \frac{I_{02}}{I_{03}h_2}] + \frac{\lambda^2 \ell_1^3}{2} [1 - \frac{I_{02}}{I_{03}h_2}] \Bigg\} .$$

After some algebraic simplifications we have,

$$\bar{x} = \ell \frac{(1+\beta)}{3} \qquad \text{(III.4)}$$

## SECTION IV

## CASE OF INCLINED SLIDER

We now consider the case in which the slider is still flat but inclined at a very small angle to the first surface; see Figure (8).

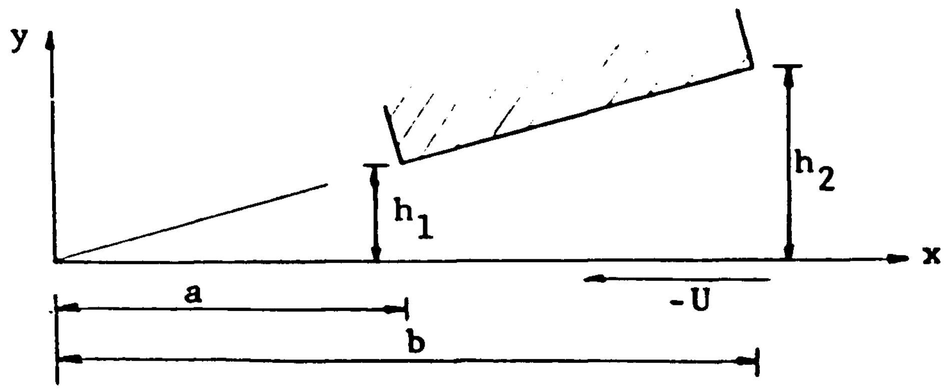



fig. 8

We take in this case

$$h(x) = mx$$

where m the slope of the slider; i.e.

$$m = \frac{h_1}{a} = \frac{h_2}{b} .$$

Let $b - a = c$ and define $\frac{h_2}{h_1} = k$. From similar triangles

$$\frac{h_2}{h_1} = \frac{b}{a} = k .$$

Therefore $m = \frac{(k-1)h_1}{c}$ .

Hence, $h = mx = \frac{(k-1)h_1}{c} x$. If we substitute this form of h in the Reynold's Equation we find

$$\frac{dp}{dx} = -\frac{6\mu Uc^3}{h_1^3(k-1)^3} \left\{\frac{(k-1)h_1}{c} x - H\right\} \frac{1}{x^3}$$

$$= -\frac{6\mu Uc^3}{h_1^3(k-1)^3} \left\{\frac{(k-1)h_1}{cx^2} - \frac{H}{x^3}\right\} .$$

Performing the integration, taking into consideration that $p(a) = p(b) = 0$, we get

$$p(x) = -\frac{6\mu Uc^3}{h_1^3(k-1)^3} \left\{-\frac{(k-1)h_1}{cx} + \frac{H}{2x^2} + \lambda\right\}$$

where $\lambda$ is a new constant of integration.

But at $x = a$, $p(a) = 0$. Hence

$$0 = -\frac{(k-1)h_1}{ca} + \frac{H}{2a^2} + \lambda .$$

Therefore,

$$\lambda = \frac{(k-1)h_1}{ca} - \frac{H}{2a^2} \qquad \text{(i)}$$

Also at $x = b$, $p(b) = 0$. Hence,

$$0 = -\frac{(k-1)h_1}{cb} + \frac{H}{2b^2} + \lambda ;$$

i.e.

$$\lambda = \frac{(k-1)h_1}{cb} - \frac{H}{2b^2} \qquad \text{(ii)}$$

Solving equations (i) and (ii) simultaneously for H we find,

$$\frac{H}{2} = \left\{\frac{(k-1)h_1}{ca} - \frac{(k-1)h_1}{cb}\right\} \Big/ \frac{b^2-a^2}{a^2b^2}$$

$$\frac{H}{2} = (k-1)h_1 \Big/ \left\{\frac{(b-a)(b+a)}{a^2k}\right\}$$

$$= (k-1)h_1 \Big/ \frac{a^2(k-1)(k+1)}{a^2k}$$

or

$$H = \frac{2kh_1}{k+1} .$$

From this

$$\lambda = \frac{(k-1)h_1}{cb} - \frac{2kh_1}{2b^2(k+1)}$$

$$= \frac{h_1}{b} \left\{\frac{(k-1)}{c} - \frac{k}{b(k+1)}\right\} = \frac{h_1}{a^2(k+1)} .$$

The pressure distribution then becomes

$$p(x) = \frac{-6\mu Uc^3}{h_1^3(k-1)^3}\left\{-\frac{(k-1)h_1}{cx} + \frac{kh_1}{(k+1)x^2} + \frac{h_1}{a^2(k+1)}\right\} \quad \text{(IV.1)}$$

The total load, W/L, is defined by

$$\frac{W}{L} = \int_a^b p(x)dx .$$

Therefore

$$\frac{W}{L} = - \frac{6\mu Uc^3}{h_1^3(k-1)^3} \int_a^b \left(- \frac{(k-1)h_1}{cx} + \frac{kh_1}{(k+1)x^2} + \frac{h_1}{a^2(k+1)}\right) dx \; .$$

$$= \frac{6\mu Uc^3}{h_1^3(k-1)^3}\left[\frac{(k-1)h_1}{c} \; \ell n \; x \; \Big|_a^b + \frac{kh_1}{(k+1)x} \; \Big|_a^b - \frac{h_1}{a^2(k+1)} \; x \; \Big|_a^b \; \right]$$

$$= \frac{6\mu Uc^3}{h_1^3(k-1)^3}\left[\frac{(k-1)h_1}{c} \; \ell n \; k + \frac{kh_1}{(k+1)}\left(\frac{1}{b} - \frac{1}{a}\right) - \frac{h_1}{a^2(k+1)}(b-a)\right]$$

$$= \frac{6\mu Uc^3}{h_1^3(k-1)^3}\left[\frac{(k-1)}{c} \; h_1 \; \ell n \; k - \frac{kh_1}{(k+1)} \; \frac{c}{a^2k} - \frac{h_1c}{a^2(k+1)}\right]$$

$$= \frac{6\mu Uc^2}{h_1^2(k-1)^2} \left[ \ell n \; k - \frac{c^2}{a^2(k^2-1)} - \frac{c^2}{a^2(k^2-1)}\right]$$

$$= \frac{6\mu Uc^2}{h_1^2(k-1)^2} \left[ \ell n \; k - \frac{2c^2}{a^2(k^2-1)}\right]$$

Hence

$$\frac{W}{L} \; \frac{1}{6\mu U} = \frac{c^2}{h_1^2(k-1)^2} \left[ \ell n \; k - \frac{2(k-1)}{k+1}\right] \qquad \text{(IV.2)}$$

Now to find the centre of the load. $\overline{x}$ we take moment about the origin, i.e.

$$\frac{W}{L} \; \overline{x} = \int_a^b p(x) \cdot x \; dx \; .$$

Hence

$$\frac{6\mu Uc^2}{h_1^2(k-1)^2}\left\{\ell n\ k - \frac{2(k-1)}{k+1}\right\}\bar{x} = \frac{6\mu Uc^3}{h_1^3(k-1)^3}\int_a^b$$

$$\left\{\frac{(k-1)h_1}{c} - \frac{kh_1}{(k+1)x} - \frac{h_1 x}{a^2(k+1)}\right\} dx\ .$$

$$\left\{\ell n\ k - \frac{2(k-1)}{k+1}\right\}\bar{x} = \frac{c}{h_1(k-1)}\left\{\frac{(k-1)h_1 c}{c} - \frac{kh_1}{(k+1)}\ \ell n\ k\right.$$

$$\left. - \frac{h_1}{2a^2(k+1)}\ (b^2-a^2)\right\}$$

$$= \frac{c}{h_1(k-1)}\left\{\frac{(k-1)}{c}\ h_1 c - \frac{kh_1}{k+1}\ \ell n\ k - \frac{h_1(k-1)}{2}\right\}$$

$$= \left\{c - \frac{kc}{(k-1)(k+1)}\ \ell n\ k - \frac{c}{2}\right\}\ .$$

$$\left\{\frac{(k+1)\ \ell n\ k\ -\ 2(k-1)}{k+1}\right\}\bar{x} = \frac{c(k-1)(k+1) - kc\ \ \ell n\ k - (c/2)(k-1)(k+1)}{(k-1)(k+1)}$$

$$\bar{x} = \frac{2c(k^2-1) - 2ck\ \ \ell n\ k\ -\ c(k^2-1)}{2(k-1)((k+1)\ \ \ell n\ k\ -\ 2(k-1))}$$

$$\frac{2\bar{x}}{c} = \frac{2(k^2-1)\ -\ 2k\ \ \ell n\ k\ -\ (k^2-1)}{(k^2-1)\ \ \ell n\ k\ -\ 2(k-1)^2}$$

or

$$\frac{\bar{x}}{c/2} = \frac{k^2\ -\ 1\ -\ 2k\ \ \ell n\ k}{(k^2-1)\ \ \ell n\ k\ -\ 2(k-1)^2} \qquad \text{(IV.3)}$$

Focusing our attention on the expression of the load,

$$\frac{W}{L} = \frac{6\mu U c^2}{h_1^2 (k-1)^2} \left\{ \ln k - \frac{2(k-1)}{k+1} \right\} .$$

We find that U being positive the sign of $\frac{W}{L}$ is that of $\left\{ \ln k - \frac{2(k-1)}{k+1} \right\}$. If $k > 1$, that is $h_2 > h_1$, this quantity is positive. The derivative is also positive. In order that a load may be sustained, the layer must be thicker where the liquid enters.

So far the value of k is left open. Reynolds examined that value of k for which $\frac{W}{L}$ becomes a maximum and he found it to be 2.2. Lord Rayleigh agrees with this value and accordingly, from the expression of the load, finds the maximum load to be;

$$\frac{W}{L} = 0.1602\mu \frac{Uc^2}{h_1^2} .$$

It is noteworthy that whatever value k assumes the load $\frac{W}{L}$ varies with the square of $\frac{c}{h_1}$. With the above value of k,

$$H = 1.27h_1 .$$

Thus fixing the position at which $\frac{dp}{dx} = 0$. With the same value of k we examine the distance of centre of pressure from the trailing edge; i.e.

$$\bar{x} - a = \frac{c}{2} \quad \frac{k^2-1 - 2k \ln k}{(k^2-1)\ln k - 2(k-1)^2} - \frac{c}{k-1} .$$

With $k = 2 \cdot 2$ we find that

$$\bar{x} - a = 0.42276c .$$

If we take the limit of $(\bar{x}-a)$ as $k$ tends to infinity we find that $(\bar{x}-a)$ tends to zero. This means that as $k$ becomes very large the centre of pressure approaches the trailing edge. As $k$ approaches unity we get

$$\lim_{k \to 1} (\bar{x}-a) = \frac{c}{2} .$$

From the above, whatever the value of $k$, the centre of pressure is always nearer the narrower end of the fluid film.

Numerical Analysis of $W/L$ vs. $k$.

We have that

$$W/L = \frac{6\mu Uc^2}{h_1^2(k-1)^2} \left\{ \ln k - \frac{2(k-1)}{(k+1)} \right\} .$$

Let $W^* = \dfrac{h_1^2 W/L}{6\mu Uc^2}$ . Then

$$W^* = \frac{1}{(k-1)^2} \left\{ \ell n\ k - \frac{2(k-1)}{k+1} \right\} .$$

But,

$$\ell n\ k = (k-1) - \frac{(k-1)^2}{2} + \frac{(k-1)^3}{3} \quad \ldots$$

Therefore,

$$W^* = \frac{1}{(k-1)^2} \left\{ (k-1) - \frac{(k-1)^2}{2} + \frac{(k-1)^3}{3} \ldots - \frac{2(k-1)}{k+1} \right\}$$

$$= \frac{1}{(k-1)^2} \left\{ \frac{(k-1)^2}{(k+1)} - \frac{(k-1)^2}{2} + \frac{(k-1)^3}{3} \ldots \right\}$$

$$= \left\{ \frac{1}{k+1} - \frac{1}{2} + \frac{k-1}{3} - \frac{(k-1)^2}{4} + \ldots \right\}$$

$$= \frac{1}{(k+1)} \left\{ 1 - \frac{k+1}{2} + \frac{(k+1)(k-1)}{3} - \frac{(k+1)(k-1)^2}{4} \ldots \right\} .$$

$$= \frac{1}{k+1} \left\{ - \frac{(k-1)}{2} + \frac{(k+1)(k-1)}{3} - \frac{(k+1)(k-1)^2}{4} \ldots \right\}$$

$$= \frac{(k-1)}{(k+1)} \left\{ - \frac{1}{2} + \frac{k+1}{3} - \frac{(k+1)(k-1)}{4} + \ldots \right\}$$

or

$$W^* = \frac{(k-1)}{6(k+1)} \left\{ \frac{1}{2} + 2k - \frac{3}{2} k^2 + \ldots \right\}$$

given that

$$W^* = \frac{1}{(k-1)^2} \left\{ \ell n\ k - \frac{2(k-1)}{k+1} \right\}$$

we have

$$\frac{dW^*}{dx} = \left\{\ell n\ k - \frac{2(k-1)}{k+1}\right\}\left(\frac{-2}{(k-1)^3}\right) + \left\{\frac{1}{k} - \frac{4}{(k+1)^2}\right\} \frac{1}{(k-1)^2} = 0.$$

Therefore

$$-\frac{2\ \ell n\ k}{k-1} + \frac{4}{(k+1)} + \frac{1}{k} - \frac{4}{(k+1)^2} = 0;$$

i.e.

$$-\frac{2\ \ell n\ k}{k-1} + \frac{4(k+1)k + (k+1)^2 - 4k}{k(k+1)^2} = 0$$

or,

$$\ell n\ k = \frac{(5k^2+2k+1)(k-1)}{2(k(k+1)^2)} \quad .$$

Hence $k = 2.18873361$. Now

$$\frac{d^2W^*}{dk^2} = -\frac{4}{(k-1)^3}\left\{\frac{1}{k} - \frac{4}{(k+1)^2}\right\} + \frac{6}{(k-1)^4}\left\{\ell n\ k - \frac{2(k-1)}{(k+1)}\right\}$$

$$-\frac{1}{(k-1)^2}\left\{\frac{1}{k^2} - \frac{8}{(k+1)^3}\right\}$$

and substituting this value of $k$ gives

$$\frac{d^2W^*}{dk^2} \cong -0.0058 \ .$$

Since the second derivative is negative, this asserts the fact that the maximum load is at this value of $k = 2.18873361$ which, when rounded, agrees with Reynolds' value of $k$; i.e., $k = 2.2$.

## SECTION V

## ANALYSIS OF THE CASE $h = mx^n$

In this section we examine the case when the $h$ varies with $x$ in a more general form. The profile of the slider in this case being given by $h = mx^n$, where $n$ is any number.

A diagramitic sketch of such a profile is shown in Figure (9).

As in the earlier parts we set

$$c = b - a, \quad p(a) = p(b) = 0,$$

and

$$k = \frac{h_2}{h_1}$$

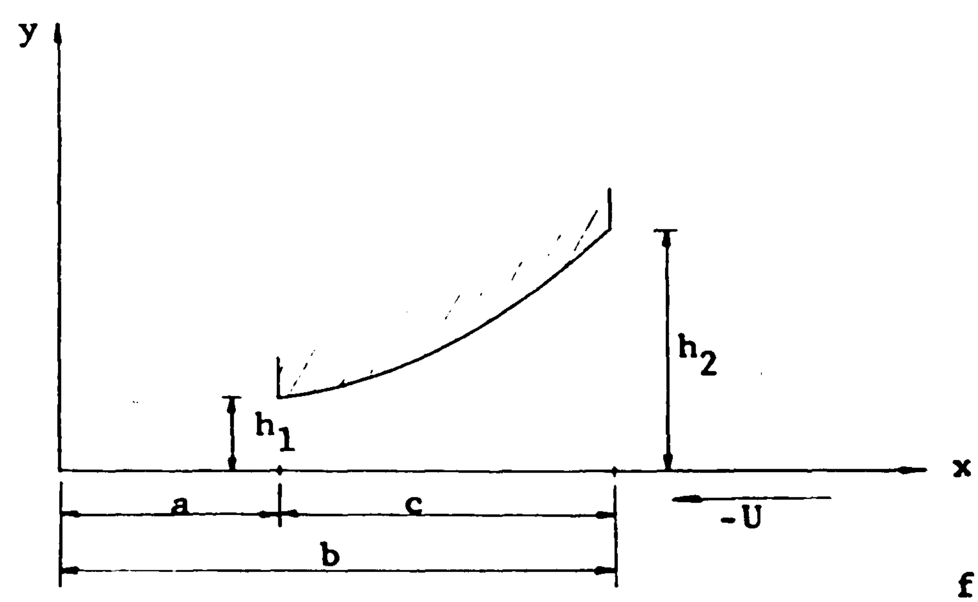



fig. 9

Since $h_1 = ma^n$ and $h_2 = mb^n$,

$$\frac{h_2}{h_1} = \frac{mb^n}{ma^n} = \left(\frac{b}{a}\right)^n = k$$

Thus

$$k^{1/n} = \frac{b}{a} ,$$

$$k^{1/n} - 1 = \frac{b-a}{a} = \frac{c}{a} ,$$

and

$$a = c/(k^{1/n}-1).$$

Since $m = \dfrac{h_1}{a^n}$ ,

$$m = \frac{h_1(k^{1/n}-1)^n}{c^n} .$$

From Equation (II.6) we have,

$$dp/dx = \frac{-6\mu U}{h^3} \left\{h-H\right\}$$

where $H$ is a constant yet to be evaluated.

Integrating once gives

$$p(x) = -6\mu U \left[\int \frac{dx}{h^2} - \int \frac{Hdx}{h^3}\right]$$

$$= -6\mu U \left[\int \frac{dx}{m^2x^{2n}} - H\int \frac{dx}{m^3x^{3n}}\right]$$

$$= -6\mu U \left[\frac{x^{-2n+1}}{m^2(-2n+1)} - \frac{Hx^{-3n+1}}{m^3(-3n+1)} + \lambda\right]$$

where $\lambda$ is the new integration constant. **Applying** $p(a) = 0$,

$$0 = \frac{a^{-2n+1}}{m^2(-2n+1)} - H\frac{a^{-3n+1}}{m^3(-3n+1)} + \lambda$$

and applying $p(b) = 0$,

$$0 = \frac{b^{-2n+1}}{m^2(-2n+1)} - H\frac{b^{-3n+1}}{m^3(-3n+1)} + \lambda\ .$$

Solving both equations for $H$, therefore

$$\left\{\frac{b^{-2n+1}-a^{-2n+1}}{m^2(-2n+1)}\right\} - H\left\{\frac{b^{-3n+1}-a^{-3n+1}}{m^3(-3n+1)}\right\} = 0$$

or

$$H\left\{\frac{a^{(-3n+1)}\left\{\left(\frac{b}{a}\right)^{-3n+1}-1\right\}}{m^3(-3n+1)}\right\} = \frac{a^{-2n+1}\left\{\left(\frac{b}{a}\right)^{-2n+1}-1\right\}}{m^2(-2n+1)}$$

or

$$H = \frac{ma^n(-3n+1)\left\{\left(\frac{b}{a}\right)^{-2n+1}-1\right\}}{(-2n+1)\left\{\left(\frac{b}{a}\right)^{-3n+1}-1\right\}}\ .$$

Putting $\frac{b}{a} = k^{1/n}$, $m = \frac{h_1(k^{1/n}-1)^n}{c^n}$, and $a^n = c^n/(k^{1/n}-1)^n$ we write

$$H = \frac{h_1(k^{1/n}-1)^n c^n\left\{k^{-2+1/n}-1\right\}(3n-1)}{c^n(k^{1/n}-1)^n(2n-1)\left\{k^{-3+1/n}-1\right\}}$$

$$= \frac{h_1\left\{k^{-2+1/n}-1\right\}(3n-1)}{(2n-1)\left\{k^{-3+1/n}-1\right\}}$$

Thus

$$\lambda = \frac{c}{h_1^2(k^{1/n}-1)(2n-1)} \left\{1 - \frac{(k^{-2+1/n}-1)}{(k^{-3+1/n}-1)}\right\}$$

The pressure distribution then becomes

$$p(x) = 6\mu U\left\{\frac{x^{-2n+1}}{m^2(2n-1)} - \frac{Hx^{-3n+1}}{m^3(3n-1)} - \lambda\right\} \qquad \textbf{(V.1)}$$

where $H$ and $\lambda$ are constants computed above.

For the load per unit width,

$$\frac{W}{L} = \int_a^b p(x)\, dx \ ,$$

we get

$$\frac{W}{L} = \int_a^b 6\mu U\left(\frac{x^{-2n+1}}{m^2(2n-1)} - \frac{Hx^{-3n+1}}{m^3(3n-1)} - \lambda\right) dx \ ;$$

i.e.

$$\frac{1}{6\mu U}\frac{W}{L} = \frac{-x^{-2n+2}}{m^2(2n-1)(2n-2)}\Bigg|_a^b + \frac{H\ x^{-3n+2}}{m^3(3n-1)(3n-2)}\Bigg|_a^b - \lambda x\Bigg|_a^b \ .$$

Calculating each term on the R.H.S. gives

$$-\frac{x^{-2n+2}}{m^2(2n-1)(2n-2)}\Bigg|_a^b = -\left\{\frac{a^{-2n+2}\left\{k^{-2+2/n}-1\right\}a^{2n}}{h_1^2(2n-1)(2n-2)}\right\}$$

$$= -\frac{c^2\ (k^{-2+2/n}-1)}{h_1^2(k^{1/n}-1)^2(2n-1)(2n-2)} \ ,$$

$$\frac{H\,x^{-3n+2}}{m^3(3n-1)(3n-2)}\Bigg|_a^b = \frac{h_1\left\{k^{-2+1/n}-1\right\}(3n-1)a^{-3n+2}\left\{k^{-3+2/n}-1\right\}a^{3n}}{(2n-1)(k^{-3+1/n}-1)h_1^3(3n-1)(3n-2)}$$

$$= \frac{c^2(k^{-2+1/n}-1)(k^{-3+2/n}-1)}{h_1^2(k^{1/n}-1)^2(2n-1)(3n-2)(k^{-3+1/n}-1)} ,$$

and

$$-\lambda x\Bigg|_a^b = -\lambda c = \frac{-c^2}{h_1^2(k^{1/n}-1)(2n-1)}\left\{1 - \frac{(k^{-2+1/n}-1)}{(k^{-3+1/n}-1)}\right\} .$$

Therefore

$$\frac{1}{6\mu U}\ \frac{W}{L} = \frac{-c^2(k^{-2+2/n}-1)}{h_1^2(k^{1/n}-1)^2(2n-1)(2n-2)}$$

$$+ \frac{c^2(k^{-2+1/n}-1)(k^{-3+2/n}-1)}{h_1^2(k^{1/n}-1)^2(2n-1)(3n-2)(k^{-3+1/n}-1)}$$

$$- \frac{c^2}{h_1^2(k^{1/n}-1)(2n-1)}\ (1 - \frac{(k^{-2+1/n}-1)}{(k^{-3+1/n}-1)})$$

$$= \frac{c^2}{h_1^2(k^{1/n}-1)^2(2n-1)}\ \left\{\frac{-(k^{-2+2/n}-1)}{(2n-2)}\right.$$

$$+ \frac{(k^{-2+1/n}-1)(k^{-3+2/n}-1)}{(3n-2)(k^{-3+1/n}-1)}$$

$$\left. - (k^{1/n}-1)[1 - \frac{(k^{-2+1/n}-1)}{(k^{-3+1/n}-1)}]\right\}$$

$$= \frac{c^2}{h_1^2(k^{1/n}-1)^2} \left\{ \frac{(3n-1)}{(2n-1)(3n-2)} \quad \frac{(k^{-2+1/n}-1)(k^{-3+2/n}-1)}{(k^{-3+1/n}-1)} \right.$$

$$\left. - \frac{(k^{(-2+2/n)}-1)}{2n-2} \right\} \tag{V.2}$$

Define $W^* = \frac{1}{6\mu Uc^2} \frac{W}{L} h_1^2$. Then

$$W^* = \frac{1}{(2n-1)(k^{1/n}-1)^2} \left\{ \frac{(k^{-2+1/n}-1)(k^{-3+2/n}-1)}{(3n-2)(k^{-3+1/n}-1)} \right.$$

$$\left. - \frac{(k^{-2+2/n}-1)}{(2n-2)} - (k^{1/n}-1)[1 - \frac{(k^{-2+1/n}-1)}{(k^{-3+1/n}-1)}] \right\}$$

$$W^* = \frac{1}{(2n-1)(3n-2)} \left\{ \frac{(k^{-2+1/n}-1)(k^{-3+2/n}-1)}{(k^{1/n}-1)^2(k^{-3+1/n}-1)} \right\}$$

$$- \frac{1}{(2n-1)(2n-2)} \left\{ \frac{(k^{-2+2/n}-1)}{(k^{1/n}-1)^2} \right\} - \frac{1}{(2n-1)(k^{1/n}-1)}$$

$$+ \frac{(k^{-2+1/n}-1)}{(2n-1)(k^{1/n}-1)(k^{-3+1/n}-1)}$$

Now

$$\frac{d}{dk} \left\{ \frac{(k^{-2+1/n}-1)(k^{-3+2/n}-1)}{(2n-1)(3n-2)(k^{1/n}-1)^2(k^{-3+1/n}-1)} \right\} = \frac{1}{(2n-1)(3n-2)}$$

$$\left\{ \frac{(k^{1/n}-1)(k^{-3+1/n}-1)\{(k^{-2+1/n}-1)(-3+2/n)k^{-4+2/n}}{(k^{1/n}-1)^3(k^{-3+1/n}-1)^2} \right.$$

$$\frac{\left.+(k^{-3+2/n}-1)(-2+1/n)k^{-3+1/n}\right\}}{(k^{1/n}-1)^3(k^{-3+1/n}-1)^2}$$

$$\frac{-(k^{-2+1/n}-1)(k^{-3+2/n}-1)\left\{(k^{1/n}-1)(-3+1/n)k^{-4+1/n}\right.}{(k^{1/n}-1)^3(k^{-3+1/n}-1)^2}$$

$$\frac{\left.+(2/n)(k^{-3+1/n}-1)k^{-1+1/n}\right\}}{(k^{1/n}-1)^3(k^{-3+1/n}-1)^2} .$$

$$\frac{d}{dk}\left\{\frac{(k^{-2+2/n}-1)}{(2n-1)(2n-2)(k^{1/n}-1)^2}\right\} = \frac{1}{(2n-1)(2n-2)}$$

$$\left\{\frac{(k^{1/n}-1)(-2+2/n)k^{-3+2/n}-(2/n)(k^{-2+2/n}-1)k^{-1+1/n}}{(k^{1/n}-1)^3}\right\} ,$$

$$\frac{d}{dk}\frac{1}{(2n-1)(k^{1/n}-1)} = \frac{(1/n)k^{-1+1/n}}{(2n-1)(k^{1/n}-1)} ,$$

and

$$\frac{d}{dk}\frac{(k^{-2+1/n}-1)}{(2n-1)(k^{1/n}-1)(k^{-3+1/n}-1)} = \frac{1}{(2n-1)}$$

$$\left\{\frac{(k^{1/n}-1)(k^{-3+1/n}-1)(-2+1/n)k^{-3+1/n}-(k^{-2+1/n}-1)}{(k^{1/n}-1)^2(k^{-3+1/n}-1)^2}\right.$$

$$\left.\frac{\left\{(k^{1/n}-1)(-3+1/n)k^{-4+1/n}+(k^{-3+1/n}-1)(1/n)k^{-1+1/n}\right\}}{(k^{1/n}-1)^2(k^{-3+1/n}-1)^2}\right\}$$

Thus to maximize W* we add the above terms and equate the sum to zero; i.e.

$$\frac{dW^*}{dk} = 0 = -k^{-1+1/n}(k^{-2+1/n}-1) - (\frac{2n-1}{3n-2})(k^{-3+2/n}-1)$$

$$+ k^{1/n}(k^{-3+1/n}-1) + \frac{3n-1}{3n-2}\,\frac{(k^{-2+1/n}-1)(k^{-3+2/n}-1)}{k(k^{-3+1/n}-1)}$$

$$- \frac{2k^2(k^{-3+2/n}-1)(k^{-2+1/n}-1)}{(3n-2)(k^{1/n}-1)}$$

$$+ \frac{k^2(k^{-2+2/n}-1)(k^{-3+1/n}-1)}{(n-1)(k^{1/n}-1)} + k^2(k^{-3+1/n}-1)$$

$$- (2n-1)(k^{1/n}-1) + \frac{(3n-1)(k^{1/n}-1)(k^{-2+1/n}-1)}{k(k^{-3+1/n}-1)}$$

$$- k^2(k^{-2+1/n}-1). \qquad \text{(V.3)}$$

It is noteworthy that the previous Equation will yield the expression $\frac{dW^*}{dk}$ in the case of $h = mx$ by placing $n = 1$. However the sixth term should be taken by evaluating the limit since, by direct substitution, the term becomes indeterminate.

In Table I we give the maximizing values and the maximum values of $W(k)/L$ as $n$ ranges from 0.01 to 100. These values were obtained by finding a zero of $W'(k)$. Such a zero would be an absolute maximum if $W(k)$ were concave downward. As we can not prove that we give in Figures 10, 11 and 12 selected graphs of $W(k)$ to suggest that $W(k)$ is indeed concave downward.

Tabulated values of maximum load $W^* = \frac{W/L\ h_1^2}{6\mu U c^2}$ and $k^*$ (ratio $\frac{h_2}{h_1}$ giving maximum load) at various values of $n$ in $h = mx^n$.

Table (I)

| $n$ | $k^* = (\frac{h_2}{h_1})^*$ | $W^*_{max} = (\frac{W/L}{6\mu U}\ \frac{h_1^2}{c^2})_{max}$ |
|---|---|---|
| 0.01 | 1.90 | 0.00072466 |
| 0.02 | 1.90 | 0.00151773 |
| 0.03 | 1.90 | 0.00238589 |
| 0.04 | 1.90 | 0.00333656 |
| 0.05 | 1.90 | 0.00437699 |
| 0.06 | 1.90 | 0.00550975 |
| 0.07 | 1.90 | 0.00672540 |
| 0.08 | 1.90 | 0.00799983 |
| 0.09 | 1.90 | 0.00929904 |
| 0.10 | 1.90 | 0.01058724 |
| 0.11 | 1.90 | 0.01183355 |
| 0.12 | 1.90 | 0.01301521 |
| 0.13 | 1.90 | 0.01411801 |
| 0.14 | 1.90 | 0.01513507 |
| 0.15 | 1.90 | 0.01606504 |
| 0.16 | 1.90 | 0.01691032 |
| 0.17 | 1.90 | 0.01767561 |
| 0.18 | 1.90 | 0.01836687 |
| 0.19 | 1.90 | 0.01899057 |
| 0.20 | 1.90 | 0.01955318 |
| 0.21 | 1.90 | 0.02006093 |
| 0.22 | 1.90 | 0.02051959 |
| 0.23 | 1.90 | 0.02093446 |
| 0.24 | 1.90 | 0.02131031 |
| 0.25 | 1.90 | 0.02165141 |
| 0.26 | 1.90 | 0.02196155 |
| 0.27 | 1.90 | 0.02224409 |
| 0.28 | 1.90 | 0.02250200 |
| 0.29 | 1.90 | 0.02273789 |
| 0.30 | 1.90 | 0.02295408 |
| 0.31 | 1.90 | 0.02315259 |
| 0.32 | 1.90 | 0.02333523 |

| n | k* | $W^*_{max}$ |
|---|---|---|
| 0.33 | 1.91069389 | 0.02350473 |
| 0.34 | 1.92237116 | 0.02366401 |
| 0.35 | 1.93349994 | 0.02381384 |
| 0.36 | 1.94410964 | 0.02395493 |
| 0.37 | 1.95422840 | 0.02408789 |
| 0.38 | 1.96388305 | 0.02421331 |
| 0.39 | 1.97309910 | 0.02433173 |
| 0.40 | 1.98190074 | 0.02444364 |
| 0.41 | 1.99031088 | 0.02454949 |
| 0.42 | 1.99835116 | 0.02464969 |
| 0.43 | 2.00604200 | 0.02474463 |
| 0.44 | 2.01340264 | 0.02483466 |
| 0.45 | 2.02045121 | 0.02492011 |
| 0.46 | 2.02720475 | 0.02500127 |
| 0.47 | 2.03367930 | 0.02507843 |
| 0.48 | 2.03988993 | 0.02515183 |
| 0.49 | 2.04585078 | 0.02522172 |
| 0.50 | 2.05182072 | 0.02530178 |
| 0.51 | 2.05707554 | 0.02535180 |
| 0.52 | 2.06236366 | 0.02541239 |
| 0.53 | 2.06745053 | 0.02547025 |
| 0.54 | 2.07234650 | 0.02552554 |
| 0.55 | 2.07706128 | 0.02557841 |
| 0.56 | 2.08160400 | 0.02562899 |
| 0.57 | 2.08598325 | 0.02567743 |
| 0.58 | 2.09020711 | 0.02572384 |
| 0.59 | 2.09428316 | 0.02576833 |
| 0.60 | 2.09821855 | 0.02581101 |
| 0.61 | 2.10202000 | 0.02585197 |
| 0.62 | 2.10569385 | 0.02589131 |
| 0.63 | 2.10924607 | 0.02592912 |
| 0.64 | 2.11268228 | 0.02596547 |
| 0.65 | 2.11600778 | 0.02600044 |
| 0.66 | 2.11922758 | 0.02603410 |
| 0.67 | 2.12234642 | 0.02606652 |
| 0.68 | 2.12536874 | 0.02609775 |
| 0.69 | 2.12829877 | 0.02612786 |
| 0.70 | 2.13114050 | 0.02615690 |
| 0.71 | 2.13389771 | 0.02618492 |
| 0.72 | 2.13657395 | 0.02621197 |
| 0.73 | 2.13917261 | 0.02623810 |
| 0.74 | 2.14169690 | 0.02626335 |
| 0.75 | 2.14414986 | 0.02628775 |
| 0.76 | 2.14653436 | 0.02631135 |
| 0.77 | 2.14885314 | 0.02633418 |

| n | k* | $W^*_{max}$ |
|---|---|---|
| 0.78 | 2.15110879 | 0.02635628 |
| 0.79 | 2.15330379 | 0.02637768 |
| 0.80 | 2.15544046 | 0.02639840 |
| 0.81 | 2.15752105 | 0.02641849 |
| 0.82 | 2.15954767 | 0.02643796 |
| 0.83 | 2.16152234 | 0.02645684 |
| 0.84 | 2.16344698 | 0.02647515 |
| 0.85 | 2.16532342 | 0.02649293 |
| 0.86 | 2.16715341 | 0.02651018 |
| 0.87 | 2.16893860 | 0.02652694 |
| 0.88 | 2.17068060 | 0.02654321 |
| 0.89 | 2.17238090 | 0.02655903 |
| 0.90 | 2.17404097 | 0.02657440 |
| 0.91 | 2.17566218 | 0.02658935 |
| 0.92 | 2.17724586 | 0.02660389 |
| 0.93 | 2.17879326 | 0.02661804 |
| 0.94 | 2.18030561 | 0.02663180 |
| 0.95 | 2.18178404 | 0.02664520 |
| 0.96 | 2.18322968 | 0.02665826 |
| 0.97 | 2.18464357 | 0.02667097 |
| 0.98 | 2.18602675 | 0.02668335 |
| 0.99 | 2.18738018 | 0.02669542 |
| 1 | 2.18873361 | 0.02670719 |
| 2 | 2.25192894 | 0.02720732 |
| 3 | 2.27191770 | 0.02733571 |
| 4 | 2.28170422 | 0.02739231 |
| 5 | 2.28751025 | 0.02742380 |
| 6 | 2.29135372 | 0.02744376 |
| 7 | 2.29408582 | 0.02745751 |
| 8 | 2.29612770 | 0.02746754 |
| 9 | 2.29771159 | 0.02747517 |
| 10 | 2.29897603 | 0.02748118 |
| 11 | 2.30000882 | 0.02748602 |
| 12 | 2.30086827 | 0.02749001 |
| 13 | 2.30159464 | 0.02749335 |
| 14 | 2.30221663 | 0.02749619 |
| 15 | 2.30275522 | 0.02749864 |
| 16 | 2.30322613 | 0.02750076 |
| 17 | 2.30364137 | 0.02750262 |
| 18 | 2.30401026 | 0.02750427 |
| 19 | 2.30434015 | 0.02750573 |
| 20 | 2.30463691 | 0.02750705 |
| 21 | 2.30490529 | 0.02750823 |

| n | k* | $W^*_{max}$ |
|---|---|---|
| 22 | 2.30514918 | 0.02750931 |
| 23 | 2.30537179 | 0.02751028 |
| 24 | 2.30557578 | 0.02751118 |
| 25 | 2.30576340 | 0.02751200 |
| 26 | 2.30593654 | 0.02751275 |
| 27 | 2.30609682 | 0.02751345 |
| 28 | 2.30624561 | 0.02751409 |
| 29 | 2.30638411 | 0.02751469 |
| 30 | 2.30651335 | 0.02751525 |
| 31 | 2.30663424 | 0.02751577 |
| 32 | 2.30674754 | 0.02751626 |
| 33 | 2.30685397 | 0.02751672 |
| 34 | 2.30695411 | 0.02751715 |
| 35 | 2.30704852 | 0.02751756 |
| 36 | 2.30713768 | 0.02751794 |
| 37 | 2.30722200 | 0.02751830 |
| 38 | 2.30730188 | 0.02751864 |
| 39 | 2.30737765 | 0.02751897 |
| 40 | 2.30744962 | 0.02751928 |
| 41 | 2.30751808 | 0.02751957 |
| 42 | 2.30758327 | 0.02751985 |
| 43 | 2.30764542 | 0.02752011 |
| 44 | 2.30770474 | 0.02752036 |
| 45 | 2.30776142 | 0.02752060 |
| 46 | 2.30781564 | 0.02752084 |
| 47 | 2.30786754 | 0.02752106 |
| 48 | 2.30791727 | 0.02752127 |
| 49 | 2.30796497 | 0.02752147 |
| 50 | 2.30801076 | 0.02752166 |
| 51 | 2.30805475 | 0.02752185 |
| 52 | 2.30809705 | 0.02752203 |
| 53 | 2.30813775 | 0.02752220 |
| 54 | 2.30817693 | 0.02752237 |
| 55 | 2.30821470 | 0.02752253 |
| 56 | 2.30825111 | 0.02752268 |
| 57 | 2.30828624 | 0.02752283 |
| 58 | 2.30832015 | 0.02752297 |
| 59 | 2.30835292 | 0.02752311 |
| 60 | 2.30838459 | 0.02752325 |
| 61 | 2.30841523 | 0.02752337 |
| 62 | 2.30844487 | 0.02752350 |
| 63 | 2.30847357 | 0.02752362 |

| n | k* | $W^*_{max}$ |
|---|---|---|
| 64 | 2.30850137 | 0.02752374 |
| 65 | 2.30852832 | 0.02752385 |
| 66 | 2.30855445 | 0.02752396 |
| 67 | 2.30857980 | 0.02752407 |
| 68 | 2.30860440 | 0.02752417 |
| 69 | 2.30862828 | 0.02752427 |
| 70 | 2.30865149 | 0.02752437 |
| 71 | 2.30867404 | 0.02752447 |
| 72 | 2.30869596 | 0.02752456 |
| 73 | 2.30871728 | 0.02752465 |
| 74 | 2.30873803 | 0.02752473 |
| 75 | 2.30875822 | 0.02752482 |
| 76 | 2.30877788 | 0.02752490 |
| 77 | 2.30879702 | 0.02752498 |
| 78 | 2.30881568 | 0.02752506 |
| 79 | 2.30883386 | 0.02752514 |
| 80 | 2.30885159 | 0.02752521 |
| 81 | 2.30886888 | 0.02752528 |
| 82 | 2.30888575 | 0.02752535 |
| 83 | 2.30890221 | 0.02752542 |
| 84 | 2.30891828 | 0.02752549 |
| 85 | 2.30893397 | 0.02752556 |
| 86 | 2.30894929 | 0.02752562 |
| 87 | 2.30896427 | 0.02752568 |
| 88 | 2.30897890 | 0.02752574 |
| 89 | 2.30899320 | 0.02752580 |
| 90 | 2.30900719 | 0.02752586 |
| 91 | 2.30902086 | 0.02752592 |
| 92 | 2.30903424 | 0.02752598 |
| 93 | 2.30904733 | 0.02752603 |
| 94 | 2.30906015 | 0.02752608 |
| 95 | 2.30907269 | 0.02752614 |
| 96 | 2.30908497 | 0.02752619 |
| 97 | 2.30909700 | 0.02752624 |
| 98 | 2.30910878 | 0.02752629 |
| 99 | 2.30912032 | 0.02752634 |
| 100 | 2.30913164 | 0.02752638 |

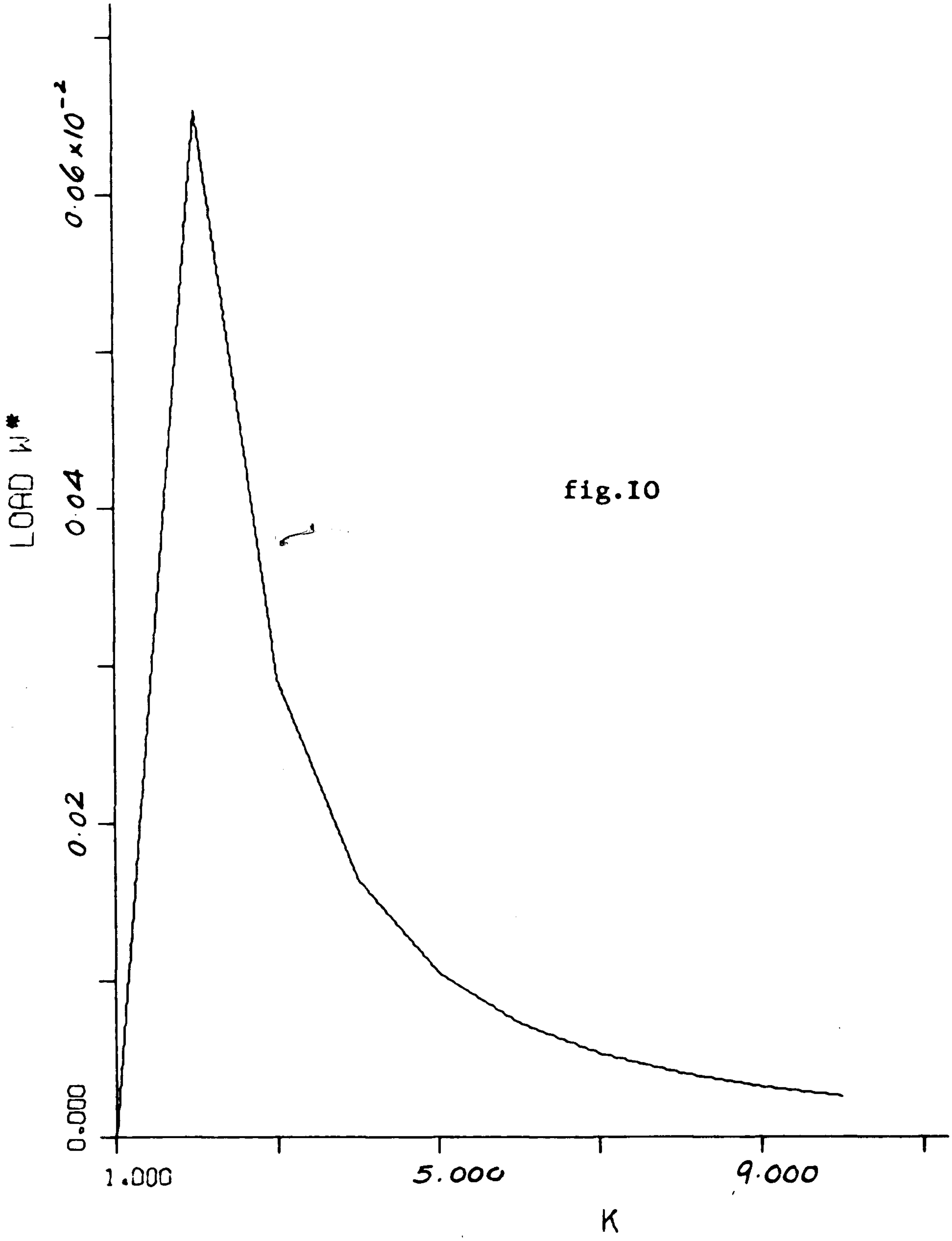
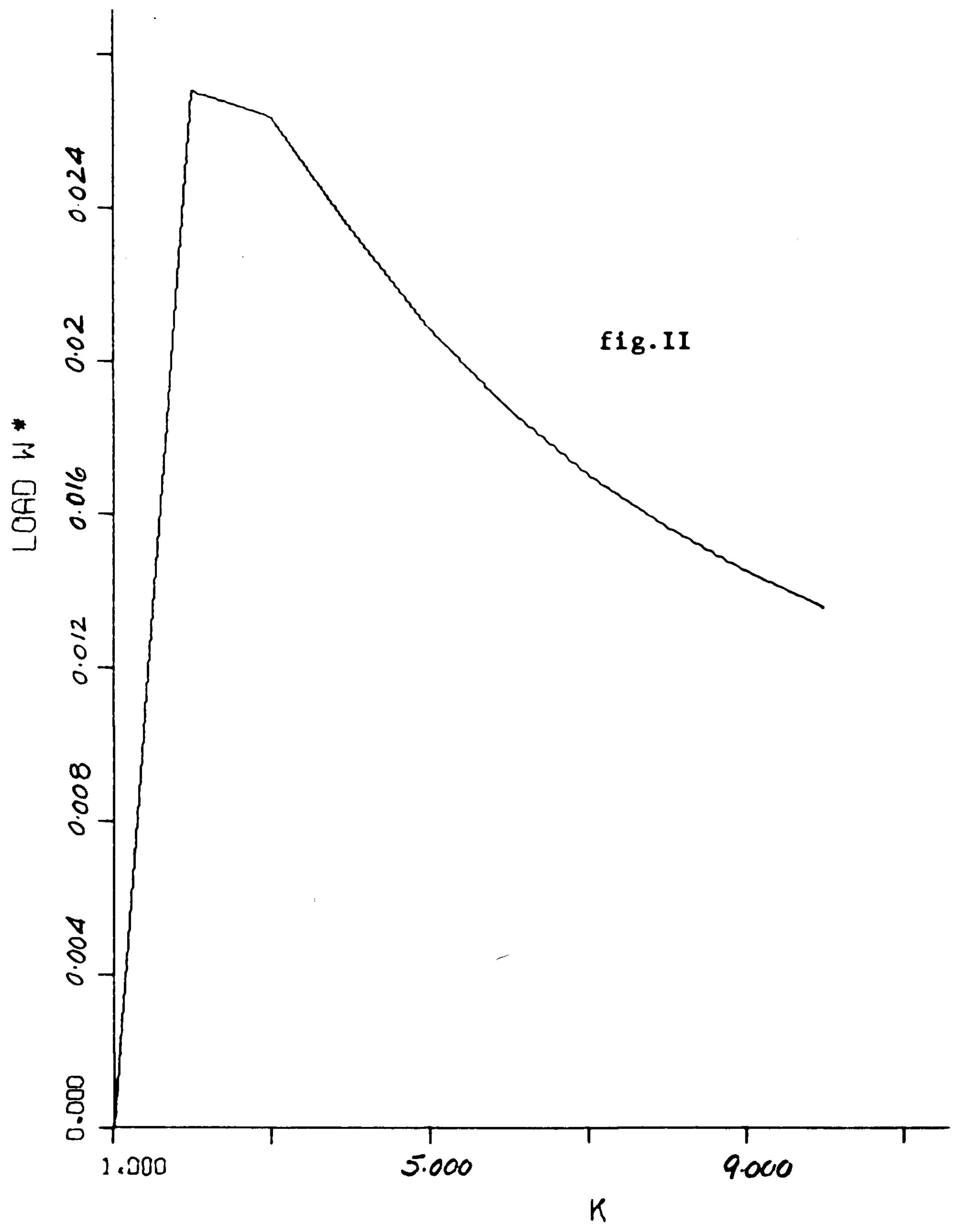
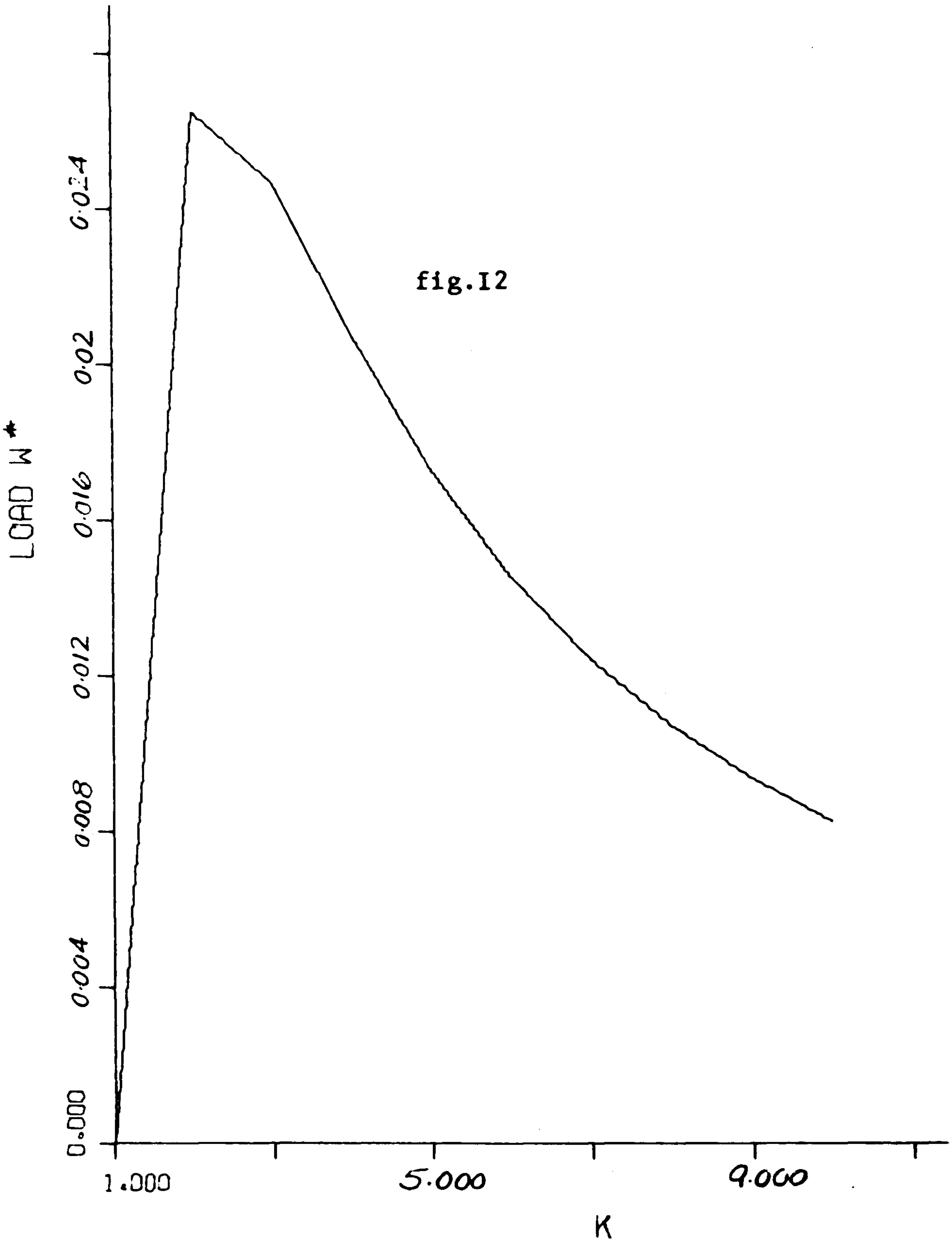



fig.IO

LOAD VS. H2/H1 N=0.01




LOAD VS. H2/H1 N=100.




N= 1.00 LOAD VS. RATIO H2/H1

## SECTION VI

## ANALYSIS OF CASE WITH $h = e^{\beta x}$

In this section we further generalize the profile to be of the form $h = e^{\beta x}$ where $\beta$ is any real number. Let

$$k = \frac{h_2}{h_1}$$

and

$$c = b - a \quad .$$

Since $h = e^{\beta x}$, $h_1 = e^{\beta a}$ , $h_2 = e^{\beta b}$ , and

$$k = \frac{e^{\beta b}}{e^{\beta a}} = e^{\beta (b-a)} = e^{\beta c} \quad .$$

Thus $\beta c = \ell n\ k$.

By direct integration of (II.6) we have

$$p(x) = -6\mu U \left\{ \int \frac{dx}{e^{2\beta x}} - H \int \frac{dx}{e^{3\beta x}} + \lambda \right\}$$

$$= -6\mu U \left\{ - \frac{e^{-2\beta x}}{2\beta} + \frac{He^{-3\beta x}}{3\beta} + \lambda \right\} .$$

Since $p(a) = p(b) = 0$ ,

$$\lambda = \frac{e^{-2\beta a}}{2\beta} - \frac{He^{-3\beta a}}{3\beta}$$

and

$$\lambda = \frac{e^{-2\beta b}}{2\beta} - \frac{He^{-3\beta b}}{3\beta} \quad .$$

Thus

$$-\frac{H}{3\beta}\left\{e^{-3\beta b}-e^{-3\beta a}\right\} = -\frac{1}{2\beta}\left\{e^{-2\beta b}-e^{-2\beta a}\right\}$$

and

$$H\ \frac{e^{-3\beta a}\left\{e^{-3\beta b}\ e^{+3\beta a}-1\right\}}{3\beta} = \frac{e^{-2\beta a}\left\{e^{-2\beta b}\ e^{2\beta a}-1\right\}}{2\beta}$$

Since $\ell n\ k = \beta c$ and $h_1 = e^{\beta a}$,

$$H = \frac{3h_1}{2}\ \frac{k^3\left\{1-k^2\right\}}{\left\{1-k^3\right\}\ k^2} = \frac{3}{2}\ \frac{h_1 k(k^2-1)}{(k^3-1)}$$

Accordingly,

$$\lambda = \frac{1}{2\beta h_1^2}\left\{\frac{1}{k^2+k+1}\right\} = \frac{1}{2\beta h_1^2}\left\{\frac{(k-1)}{(k^3-1)}\right\}$$

Hence, the pressure distribution becomes

$$p(x) = -6\mu U\left\{-\frac{e^{-2\beta x}}{2\beta} + \frac{3h_1 k(k^2-1)}{6\beta(k^3-1)}\ e^{-3\beta x} + \frac{1}{2\beta h_1^2}\left\{\frac{1}{k^2+k+1}\right\}\right\}. \qquad \text{(VI.1)}$$

Now the load per unit width is given by

$$\frac{W}{L} = \int_a^b p\ dx\ .$$

Therefore

$$\frac{1}{6\mu U}\frac{W}{L} = \frac{1}{2\beta}\int_a^b e^{-2\beta x}dx - \frac{h_1 k(k^2-1)}{2\beta(k^3-1)}\int_a^b e^{-3\beta x}dx$$

$$- \frac{x}{2\beta h_1^2}\left\{\frac{k-1}{k^3-1}\right\}\Big|_a^b$$

or

$$\frac{1}{6\mu U}\frac{W}{L} = \frac{(k^2-1)}{4\beta^2 h_1^2 k^2} - \frac{(k^2-1)}{6\beta^2 h_1^2 k^2} - \frac{(k-1)c}{2\beta h_1^2(k^3-1)}$$

$$= \frac{1}{2\beta^2 h_1^2 k^2}\left\{\frac{(k^2-1)}{2} - \frac{(k^2-1)}{3} - \beta k^2\,\frac{(k-1)c}{(k^3-1)}\right\}$$

$$= \frac{1}{2\beta^2 h_1^2 k^3}\left\{\frac{(k^2-1)}{6} + \frac{\beta(k^2-k^3)c}{(k^3-1)}\right\} .$$

Since $\ell n\ k = \beta c$ (or $\beta = \frac{\ell n\ k}{c}$), we obtain by substituting this in the above expression,

$$\frac{W}{L} = \frac{3\mu Uc^2}{h_1^2 k^2(\ell n\ k)^2}\left\{\frac{k^2-1}{6} - \frac{k^2(k-1)\ell n\ k}{(k^3-1)}\right\} \qquad \text{(VI.2)}$$

Let us try to determine the value of $k$ giving maximum value to $\frac{W}{L}$ .

Define $W^* = \frac{h_1^2 W/L}{3\mu Uc^2}$ . Then

$$W^* = \frac{1}{k^2(\ell n\ k)^2}\left\{\frac{k^2-1}{6} - \frac{k^2(k-1)\ell n\ k}{k^3-1}\right\}$$

and

$$\frac{dW^*}{dk} = \frac{d}{dk}\left\{\frac{k^2-1}{6(k\ \ell n\ k)^2} - \frac{(k-1)}{(\ell n\ k)(k^3-1)}\right\}$$

$$= \frac{1}{6}\left\{\frac{2k(k\ \ell n\ k)^2-(k^2-1)\ 2(k\ \ell n\ k)(1+\ell n\ k)}{(k\ \ell n\ k)^4}\right\}$$

$$- \left\{\frac{k\ \ell n\ k(k^3-1)-(k-1)(k^3-1)-3k^3(k-1)\ell n\ k}{k(\ell n\ k)^2(k^3-1)^2}\right\}$$

$$= \frac{k\ \ell n\ k-(k^2-1)-(k^2-1)\ell n\ k}{3k^3(\ell n\ k)^3}$$

$$- \frac{k(\ell n\ k)(k^3-1)-(k-1)(k^3-1)-3k^3(k-1)\ell n\ k}{k(k^3-1)^2(\ell n\ k)^2}$$

$$= \frac{k(k^3-1)^2\ell n\ k-(k^2-1)(k^3-1)^2-(k^2-1)(k^3-1)^2\ell n\ k}{3k^3(k^3-1)^2(\ell n\ k)^3}$$

$$\frac{-3k^3(k^3-1)(\ell n\ k)^2+3k^2(k-1)(k^3-1)\ell n\ k}{3k^3(k^3-1)^2(\ell n\ k)^3}$$

$$\frac{+9k^5(k-1)(\ell n\ k)^2}{3k^3(k^3-1)^2(\ell n\ k)^3}\ .$$

Equating to zero and solving for $k$ we have,

$$k = 2.31023078.$$

Substituting this in $W^*$, where,

$$W^* = \frac{Wh_1^2}{3\mu Uc^2L} = \frac{1}{k^2(\ell n\ k)^2}\left\{\frac{k^2-1}{6} - \frac{k^2(k-1)\ell n\ k}{k^3-1}\right\} ,$$

gives

$$W^* = 0.05506206 .$$

Thus

$$\frac{W}{L} = 0.16518618 \ \frac{\mu Uc^2}{h_1^2} .$$

## SECTION VII

## A SUCCESSION OF INCLINED LINES, (r REPETITIONS) .

After examining the various cases, namely the step bearing, plane inclined slider, profile with general equation $h = mx^n$, and finally the case $h = e^{\beta x}$, we find that there was some improvement though not too much.

It is of our interest now to investigate the case of r repetitions of the same curve, mainly what kind of improvement is to be gained regarding the maximum load per unit width.

A schematic sketch of the aggregate is shown in Figure 13.

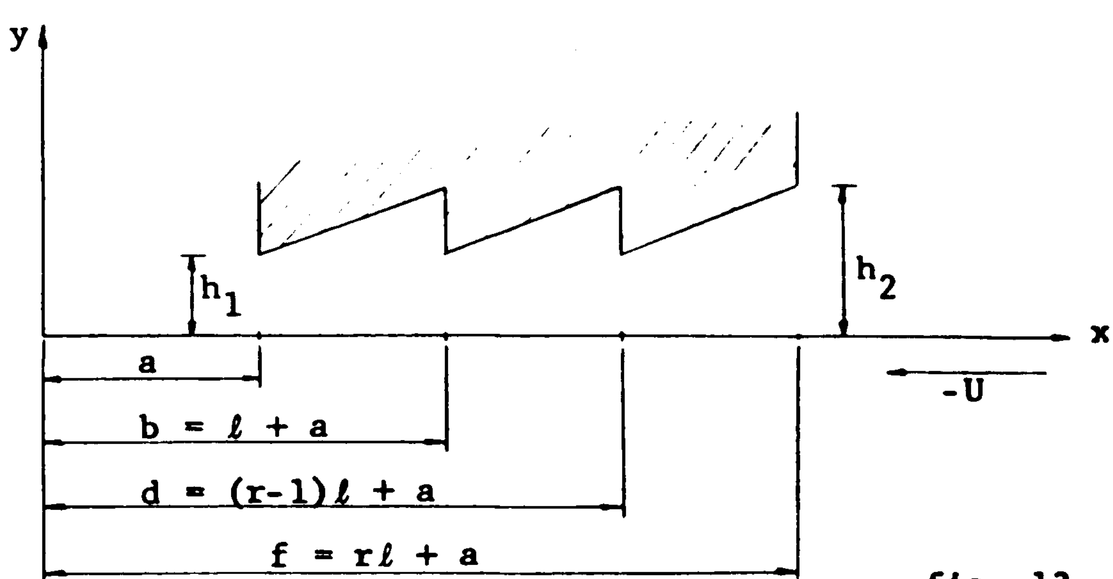



fig. 13

Let $\ell$ = length of a single member, and define

$$k = \frac{h_2}{h_1} \quad .$$

Let the first member be represented by $h = mx$. The other members will have the same slope $m$ yet with different equations. Since $m = \frac{(k-1)h_1}{\ell}$, from discussion of inclined slider, the general equation for the $r^{th}$ member shall be

$$h = mx - m\ell(r-1) \quad .$$

Notice that for $r = 1$ (single pad) $h = mx$.

From the Reynolds' equation

$$\frac{dp}{dx} = -6\mu U\left\{\frac{1}{h^2} - \frac{H}{h^3}\right\}$$

we get

$$\int_a^b dp + \int_b^d dp + \int_d^f dp = -6\mu U\Big[ \int_a^b \frac{(h-H)}{h^3}\,dx$$

$$+ \int_b^d \left(\frac{h-H}{h^3}\right)dx + \int_d^f \left(\frac{h-H}{h^3}\right)\,dx \Big].$$

Since $p(b-0) = p(b+0)$ and $p(d-0) = p(d+0)$, the $\ell$.h.s. becomes (remembering that inlet and exit pressures are taken to be zero)

$$\int_a^f dp = 0 \quad .$$

The above equation becomes

$$\int_a^b \left(\frac{1}{h^2} - \frac{H}{h^3}\right)dx + \int_b^d \left(\frac{1}{h^2} - \frac{H}{h^3}\right)dx$$

$$+ \int_d^f \left(\frac{1}{h^2} - \frac{H}{h^3}\right)dx = 0 .$$

From this equation we calculate $H$. For mathematical convenience let $r = 3$. Then the equations of the inclined lines shall be

$$h = mx,$$

$$h = m(x-\ell) ,$$

and

$$h = m(x-2\ell) .$$

Performing the integration

$$\int_a^b \frac{dx}{m^2x^2} - H\int_a^b \frac{dx}{m^3x^3} + \int_b^d \frac{dx}{m^2(x-\ell)^2}$$

$$- H\int_b^d \frac{dx}{m^3(x-\ell)^3} + \int_d^f \frac{dx}{m^2(x-2\ell)^2} - H\int_d^f \frac{dx}{m^3(x-2\ell)^3}$$

$$= 0 ,$$

we get

$$m\left[\left\{\frac{1}{a}-\frac{1}{b}\right\}+\left\{\frac{1}{(b-\ell)}-\frac{1}{(d-\ell)}\right\}+\left\{\frac{1}{(d-2\ell)}-\frac{1}{(f-2\ell)}\right\}\right]$$

$$=\frac{H}{2}\left[\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\left(\frac{1}{(b-\ell)^2}-\frac{1}{(d-\ell)^2}\right)+\left(\frac{1}{(d-2\ell)^2}-\frac{1}{(f-2\ell)^2}\right)\right].$$

But, using Figure 13,

$$m\left\{\left(\frac{b-a}{ab}+2\left[\frac{a+\ell-a}{a(a+\ell)}\right]\right\}=\frac{H}{2}\left[\frac{b^2-a^2}{a^2b^2}+2\left[\frac{(a+\ell)^2-a^2}{a^2(a+\ell)^2}\right]\right],$$

$$m\left\{\frac{a(k-1)}{a^2k}+\frac{2(k-1)}{ak}\right\}=\frac{H}{2}\left[\frac{(k^2-1)}{a^2k^2}+\frac{2(k^2-1)}{a^2k^2}\right],$$

and

$$3m\left\{\frac{(k-1)}{ak}\right\}=\frac{3H}{2}\left(\frac{k^2-1}{a^2k^2}\right).$$

Hence $H=\frac{2mak}{(k+1)}$ with $ma=h_1$; i.e.

$$H=\frac{2kh_1}{k+1}.$$

We notice that this value of $H$ is the same as the value for a single member.

The above tells us that since the geometry of the members is identical and since $H$ is the same for every member of the aggregate, then from Reynolds' Equation we have $dp/dx$ is the same for all members.

Now, since the load is given by

$$W/L = \int_a^b p\ dx = -\int_a^b x\ dp/dx\ dx,$$

$$-\int_0^{r\ell} x\ dp/dx\ dx = \frac{W}{L} = -\left\{ \int_0^{\ell} x\ dp/dx + \int_{\ell}^{2\ell} x\ dp/dx\ dx + \ldots + \int_{(r-1)\ell}^{r\ell} x\ dp/dx\ dx \right\} .$$

But

$$\int_0^{\ell} dx = \int_{\ell}^{2\ell} dx = \ldots = \int_{(r-1)\ell}^{r\ell} dx .$$

Hence

$$W/L = -r\int_0^{\ell} x\ dp/dx\ dx .$$

This shows that the load for the whole aggregate is $r$ times the load for one single member.

Now, if we imagine that the inclined lines are spread over the entire length we obtain a single inclined line of length $r\ell$. Recalling that the load $W/L$ is proportional to the square of the pad length, we find that the load now will be $r^2$ times the load for an inclined plane of length $\ell$.

Comparing the latter result with the one obtained for $r$ repetitions we find that repetitions of inclined lines is highly unfavorable for loads.

It must be noticed that we are dealing with two dimensions. If we were to extend our study to three dimensions, that is, considering the width of the pad as well, we may find an increase in the length of the pad inadequate.

Up till now we have considered cases where the variation of $h$ with $x$ was rather mathematically convenient. It remains open to find a form which according to Reynolds' equation will maximize the load, subject to the conditions of a given pad length and a given minimum film thickness $h_1$.

# SECTION VIII

## STATIONARIZING THE LOAD

Let us examine the problem of finding the profile that provides maximum load from the variational calculus point of view.

When we first considered Lord Rayleigh's step bearing we derived an expression for the constant H. This expression reads

$$0 = \int_a^b \frac{dx}{h^2} - H \int_a^b \frac{dx}{h^3}$$

Let $h(x)$ represent the profile that provides maximum load. Examine the case that $h$ becomes $(h+\epsilon\,\delta h)$ where $\delta h$ is an infinitismal variation in $h$ and $\epsilon$ is any real number; notice that for $\epsilon = 0$ we regain the original profile $h(x)$.

With these considerations the above expression becomes,

$$0 = \int_a^b \frac{dx}{(h+\epsilon\,\delta h)^2} - H \int_a^b \frac{dx}{(h+\epsilon\,\delta h)^3}$$

Considering all integrations being always over the length we can dispense with using any specified limits at this moment.

Differentiating with respect to $\epsilon$ and then setting $\epsilon = 0$ we have,

$$0 = 2\int \frac{\delta h}{h^3}\,dx - 3H\int \frac{\delta h}{h^4}\,dx + \delta H\int \frac{dx}{h^3} .$$

Hence,

$$\delta H = \int \frac{(-2h+3H)\,\delta h}{h^4}\,dx \Big/ \int \frac{dx}{h^3} .$$

Similarly, the load W/L was given by

$$W/L = -\int_a^b x\; dp/dx\; dx$$

Hence,

$$\frac{W/L}{6\mu U} = \int_a^b \frac{x\;dx}{h^2} - H\int_a^b \frac{x\;dx}{h^3} .$$

Letting h become $(h+\epsilon\,\delta h)$ we get

$$\frac{W/L}{6\mu U} = \int \frac{x\;dx}{(h+\epsilon\,\delta h)^2} - H\int \frac{x\;dx}{(h+\epsilon\,\delta h)^3} .$$

Again by differentiating with respect to $\epsilon$ then setting $\epsilon = 0$ we have,

$$\frac{\delta(W/L)}{6\mu U} = \int \frac{\delta h}{h^4}\,(-2h+3H)x\;dx - \delta H\int \frac{x\;dx}{h^3} .$$

Substituting $\delta H$ in the above expression gives

$$\frac{\delta(W/L)}{12\mu U} = -\int \frac{\delta h}{h^4} \left\{ x - \frac{\int xh^{-3}dx}{\int h^{-3}\, dx} \right\} (h - \tfrac{3}{2} H)dx \qquad \text{(VIII.1)}$$

For the variation in the load to vanish regardless of any variation in $h$ we demand that over the whole range either,

$$x = \frac{\int h^{-3}x\ dx}{\int h^{-3}\ dx}$$

or

$$h = 3/2\ H\ .$$

But this is not the requirement postulated. It suffices here to have the coefficient of $\delta h$, on the right hand side of the equation, vanish over the part where $h > h_1$ and that it be negative when $h = h_1$ so that a positive $\delta h$ in this region will result in a decrease in W/L, a negative $\delta h$ being excluded apriori.

The above conditions may be satisfied if we make $h = h_1$ from $x = 0$ to $x = \ell_1$ where $\ell > \ell_1 > 0$, and $h = 3/2\ H$ over the remainder of the length that is from $x = \ell_1$ to $x = \ell_2$ where $\ell_1 + \ell_2 = \ell$ is the whole length concerned. See Figure 14.

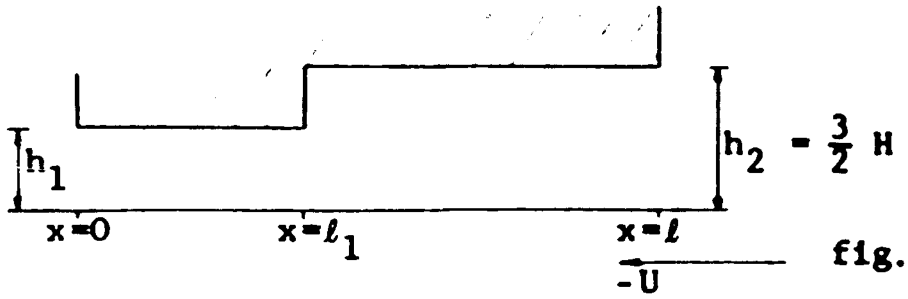



fig. 14

From the first condition, and recalling the value of H for a step bearing,

$$\frac{2}{3} h_2 = H = \frac{\int_0^{\ell_1} dx/h_1^2 + \int_{\ell_1}^{\ell} dx/h_2^2}{\int_0^{\ell_1} dx/h_1^3 + \int_{\ell_1}^{\ell} dx/h_2^3} .$$

Hence

$$\frac{2}{3} h_2 = \frac{\ell_1/h_1^2 + \ell/h_2^2 - \ell_1/h_2^2}{\ell_1/h_1^3 + \ell/h_2^3 - \ell_1/h_2^3} = \frac{\ell_1/h_1^2 + \ell_2/h_2^2}{\ell_1/h_1^3 + \ell_2/h_2^3}$$

Letting $k = h_2/h_1$ gives

$$\ell_2/\ell_1 = k^2(2k-3) .$$

To satisfy this relation is to insure that $h = 3/2\ H$ over the range where $h = h_2$. When $h = h_1$, $h - \frac{3H}{2}$ is negative and the second condition over the range $[0, \ell_1]$ requires that

$$\left\{ \frac{\int x h^{-3} dx}{\int h^{-3} dx} - x \right\} > 0$$

with $\ell_1$ the largest possible value of $x$; i.e.

$$\int h^{-3} x \, dx - \ell_1 \int h^{-3} \, dx > 0$$

or

$$\int h^{-3} x \, dx > \ell_1 \int h^{-3} \, dx \quad .$$

Hence

$$\frac{1}{h_1^3} \int_0^{\ell_1} x \, dx + \frac{1}{h_2^3} \int_{\ell_1}^{\ell} x \, dx$$

$$> \ell_1 \left\{ \frac{1}{h_1^3} \int_0^{\ell_1} dx + \frac{1}{h_2^3} \int_{\ell_1}^{\ell} dx \right\} .$$

This gives

$$\frac{\ell_1^2}{2h_1^3} + \frac{(\ell_1+\ell_2)^3}{2h_2^3} - \frac{\ell_1^2}{2h_2^3} > \frac{\ell_1^2}{h_1^3} + \frac{\ell_2\ell_1}{h_2^3} + \frac{\ell_1^2}{h_2^3} - \frac{\ell_1^2}{h_2^3} ,$$

$$\frac{\ell_1^2}{2h_2^3} + \frac{2\ell_1\ell_2}{2h_2^3} + \frac{\ell_2^2}{2h_2^3} - \frac{\ell_1^2}{2h_2^3} > \frac{\ell_1^2}{2h_1^3} + \frac{\ell_2\ell_1}{h_2^3} \quad ,$$

or

$$\frac{\ell_2^2}{2h_2^3} > \frac{\ell_1^2}{2h_1^3} \quad .$$

Hence

$$\left(\frac{\ell_2}{\ell_1}\right)^2 > \left(\frac{h_2}{h_1}\right)^3$$

or

$$k^3 < (\ell_2/\ell_1)^2 .$$

But we have found before that

$$\ell_2/\ell_1 = k^2(2k-3) .$$

Therefore the condition is

$$k(2k-3)^2 > 1 .$$

If $k$ is chosen to satisfy this condition, every admissible variation in $h$ diminishes $W/L$ .

Notice that $\ell$ is fixed but the ratio $\ell_2/\ell_1$ is still at our disposal (within limits).

In terms of $\ell$ and $k$ we write

$$\ell_1 = \frac{\ell}{1+2k^3-3k^2}$$

$$\ell_2 = \frac{\ell(2k^3-3k^2)}{1+2k^3-3k^2} .$$

Recalling that the load $W/L$ for a step bearing is given by

$$W/L = \frac{3\mu U\ell^2}{h_1^3} \left\{\frac{\beta(1-\beta)(\lambda-1)}{\beta+\lambda^3 \ (1-\beta)}\right\}$$

where $\beta = \ell_2/\ell_1+\ell_2 = \ell_2/\ell$ and $\lambda = k$ we get

$$W/L \ \frac{1}{\mu U} = \frac{3\ell^2}{h_1^2}\left\{\frac{\ell_2/\ell\left(\frac{\ell-\ell_2}{\ell}\right)(k-1)}{\ell_2/\ell + k^3\left(\frac{\ell-\ell_2}{\ell}\right)}\right\}$$

$$= \frac{3\ell^2}{h_1^2}\left\{\frac{\ell_1\ell_2(k-1)}{\ell(k^3\ell_1+\ell_2)}\right\}$$

Substituting the values for $\ell_1$ and $\ell_2$ gives

$$\frac{W/L}{\mu U} = \frac{\ell^2}{h_1^2} \quad \frac{3(k-1)\cdot\ell^2(2k^3-3k^2)}{(1+2k^3-3k^2)^2\cdot\ell\left\{\frac{k^3\ell+\ell(2k^3-3k^2)}{(1+2k^3-3k^2)}\right\}}$$

$$= \frac{\ell^2}{h_1^2} \quad \frac{(k-1)3\ell^2(2k^3-3k^2)}{(1+2k^3-3k^2)\ell^2 3k^2(k-1)}$$

$$= \frac{\ell^2}{h_1^2}\left\{\frac{2k-3}{1+2k^3-3k^2}\right\} = \frac{\ell^2}{h_1^2} \quad f(k)$$

where $f(k) = \frac{2k-3}{(1+2k^3-3k^2)}$ . Now we find the value of $k$ at which $f(k)$ is maximum. From

$$f'(k) = \frac{2(1+2k^3-3k^2)-(2k-3)(6k^2-6k)}{(1+2k^3-3k^2)^2} = 0$$

we get

$$(k-1)(4k^2-8k+1) = 0$$

which has the roots $k = 1$, $k = 1.87$, and $k = 0.134$. Since

$$f''(k) = \frac{-(12k^2-24k+9)(1+2k^3-3k^2)^2}{(1+2k^3-3k^2)^4}$$

$$+\frac{2(1+2k^3-3k^2)(6k^2-6k)(4k^3-12k^3+9k-1)}{(1+2k^3-3k^2)^4}$$

the values $k = 1$ and $k = 0.134$ are not going to give a maximum. Thus the maximum of $f(k)$ occurs at $k = 1.87$ with $f(k) = 0.20626737$. Moreover, $k = 1.87$ satisfies the requirement,

$$k(2k-3)^2 > 1 .$$

Notice that 1.87 is a critical value since it maximizes the load and satisfies the above inequality, the following Table shows some neighboring values which are of interest for comparison:

| k | f(k) | $k(2k-3)^2$ |
|---|---|---|
| 1.86 | 0.20624 | 0.964 |
| * 1.87 | 0.20626 | 1.024 |
| 1.88 | 0.20617 | 1.086 |

Finally,

$$W/L = 0.20626 \ \frac{\mu U c^2}{h_1^2}$$

and the ratio

$$\ell_2/\ell_1 = 2.588 \ .$$

This defines the form of the slider which gives the maximum load per unit width of the slider when the minimum thickness and total length are prescribed.

The distance $\bar{x}$ of the centre of pressure from the narrow end is given by

$$\bar{x} = \frac{\ell(1+\beta)}{3} = \frac{\ell(1+\ell_1/\ell)}{3} = \frac{\ell+\ell_1}{3} \ .$$

But

$$\ell_1 = \frac{\ell}{1+2k^3-3k^2}$$

Hence

$$\bar{x} = \ell \ \frac{2+2k^3-3k^2}{3(1+2k^3-3k^2)}$$

$$= 0.42625\ell \quad \text{for} \quad k = 1.87 \ .$$

# APPENDIX A

## THE RATIO OF THE TOTAL FRICTION (F) TO THE LOAD W/L

In our previous discussions we have sought the profile that would maximize the load carrying capacity of the bearing. Up to this point we have not discussed friction on the surface, yet, frictional forces are one of the major factors in the design of a bearing.

Our objective in this section is to derive an expression for the total friction at the surface in terms of the same parameters used in load expressions; and hence obtain the ratio of the total friction to the load $W/L$ for two specific profiles, namely, the inclined plane slider and Lord Rayleigh's step bearing.

It should be noticed that one could seek the profile that would minimize the friction to load ratio rather than maximize the load. The former approach will not be discussed in detail, but it is worthwhile saying that it does not appear to make much practical difference.

Derivation of the total friction expression.

When we were deriving the Reynolds' Equation we obtained the expression

$$u = \frac{y^2-hy}{2\mu}\ dp/dx - U(1 - \frac{y}{n}) .$$

Hence

$$\frac{du}{dy} = \frac{1}{2\mu}\ dp/dx\ 2y - \frac{h}{2\mu}\ dp/dx + \frac{U}{h} .$$

and at $y = 0$

$$\frac{du}{dy} = - \frac{h}{2\mu}\ dp/dx + \frac{U}{h} .$$

In which case the shear stress on the surface will be given by

$$\tau = \mu \frac{du}{dy} = - \frac{h}{2}\ dp/dx + \frac{\mu U}{h} .$$

(Naturally if we put $y = h$ we receive some expression for $\tau$ with a different sign which is a characteristic of shear stresses). Since

$$dp/dx = - \frac{6\mu U}{h^3}(h-H) ,$$

we have

$$\tau = + \frac{3\mu U}{h^2}(h-H) + \frac{\mu U}{h} ;$$

i.e.

$$\tau = \frac{3\mu U(h-H)+\mu Uh}{h^2} = \frac{3\mu Uh-3\mu UH+\mu Uh}{h^2}$$

or

$$\tau = \mu U\left(\frac{4h-3H}{h^2}\right) .$$

The total friction accordingly will be given by

$$F = \int \tau \, dx$$

or

$$\frac{F}{\mu U} = 4 \int \frac{dx}{h} - 3H \int \frac{dx}{h^2} .$$

Now we analyze these for different geometries.

a) <u>For an inclined plane slider with $h = mx$</u>

Hence we have

$$\frac{F}{\mu U} = 4 \int_a^b \frac{dx}{mx} - 3H \int_a^b \frac{dx}{m^2x^2}$$

$$= \frac{4}{m} \left\{ \ell n \; b/a \right\} - \frac{3H}{m^2} \left\{ \frac{b-a}{ab} \right\} .$$

Letting

$$m = \frac{(k-1)h_1}{c} \quad , \quad H = \frac{2kh_1}{k+1} \quad \text{and}$$

$$k = h_2/h_1 = b/a$$

we may write

$$\frac{F}{\mu U} = \frac{4a}{h_1} \ln b/a - \frac{3Hc}{h_1 h_2}$$

$$= \frac{4c \ln k}{(k-1)h_1} - \frac{6kh_1 c}{(k+1)kh_1^2}$$

$$= \frac{c}{h_1} \left\{\frac{4 \ln k}{k-1} - \frac{6}{k+1}\right\}$$

$$= \frac{c}{h_1} \left\{\frac{4(k+1)\ln k - 6(k-1)}{(k^2-1)}\right\} .$$

The ratio of friction to load W/L will be

$$F/W/L = \frac{c/h_1 \left\{\frac{4(k+1)\ln k - 6(k-1)}{(k^2-1)}\right\}}{6\ c^2/h_1^2 \left\{\frac{\ln k}{(k-1)^2} - \frac{2}{k^2-1}\right\}}$$

$$= \frac{h_1}{c}\frac{1}{6} \left\{\frac{4(k+1)\ln k - 6(k-1)}{\frac{(k+1)\ln k}{k-1} - 2}\right\}$$

$$= \frac{h_1}{c} \left\{\frac{2(k^2-1)\ln k - 3(k-1)^2}{3(k+1)\ln k - 6(k-1)}\right\}$$

For $k = 2.1877$ the ratio of the total friction at the surface to the maximum load is given by

$$F/W/L = 4.7063\ h_1/c\ .$$

b) <u>The Case of Lord Rayleigh's step bearing</u>

Here

$$\frac{F}{\mu U} = 4\int_0^{\ell_1} \frac{dx}{h_1} - 3H\int_0^{\ell_1} \frac{dx}{h_1^2} + 4\int_{\ell_1}^{\ell} \frac{dx}{h_2} - 3H\int_{\ell_1}^{\ell} \frac{dx}{h_2^2}$$

$$= \frac{4}{h_1}\,\ell_1 - \frac{3H}{h_1^2}\,\ell_1 + \frac{4}{h_2}\left\{\ell - \ell_1\right\} - \frac{3H}{h_2^2}\left\{\ell - \ell_1\right\}$$

$$= [\frac{4h_1 - 3H}{h_1^2}]\ell_1 + [\frac{4h_2 - 3H}{h_2^2}]\ell_2 \ .$$

But

$$\ell_1 = \frac{\ell}{1+2k^3-3k^2} \quad , \quad \ell_2 = \frac{\ell(2k^3-3k^2)}{1+2k^3-3k^2}$$

$$H = \frac{2}{3}\,h_2 \quad \text{and} \quad kh_1 = h_2$$

we have

$$\frac{F}{\mu U} = \frac{2\ell}{1+2k^3-3k^2}\left\{\frac{(2-k)}{h_1} + \frac{k^2(2k-3)}{h_2}\right\}$$

$$= \frac{2\ell}{h_1(1+2k^3-3k^2)}\left\{2-k+2k^2-3k\right\}$$

$$= \frac{4\ell}{h_1}\ \frac{(k-1)^2}{(k-1)^2(2k+1)} = \frac{4\ell}{(2k+1)h_1}$$

Hence

$$F/W/L = \frac{4\ell/h_1(k-1)^2(2k+1)}{\ell^2/h_1^2(2k-3)(2k+1)}$$

$$= \frac{4h_1(k-1)^2}{\ell(2k-3)}$$

For $k = 1.87$

$$F/W/L = 4.091\ h_1/\ell\ .$$

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# VITA

Hesham A. Ezzat, son of Dr. Ahmed M. Ezzat was born in Cairo, Egypt on January 9, 1953. He received a B.Sc. degree in Mechanical Engineering from Cairo University in 1974.