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Duality between measure and Baire category.

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DUALITY BETWEEN MEASURE AND BAIRE CATEGORY

by

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AN ABSTRACT

of

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Let X be a second countable Baire metric space, and μ the completion of a regular non-atomic Borel measure with support X. I prove a duality theorem showing that statements about sets of first category are equivalent to the dual statements about sets of measure zero (generalizing the result of Seirpinski for the reals). This duality theorem is then used to prove some measure theoretic results from the dual results of Baire category. I show that a more general duality principle between measurable sets and sets with the property of Baire is not valid.

I consider category measure spaces (i.e. where "measure zero" coincides with "first category"), obtaining some general r results about them, and exhibiting some examples.

INTRODUCTION

I will consider in my thesis the duality between measure and Baire category.

Several fundamental definitions I will use include: a Borel measure is a measure on the Borel sets of a topological space. A measure μ on a topological space is called a regular measure if for every measurable set A

$$\mu(A) = \sup\{\mu(F) \ / \ F \subset A \text{ and } F \text{ is a closed set}\} = \inf\{\mu(G) \ / \ G \supset A \text{ and } G \text{ is an open set}\} .$$

We will also assume the continuum hypothesis, i.e. there does not exist any cardinal number between the cardinal number of the natural numbers and the cardinal number of the reals.

Unless otherwise stated X is a second countable Baire metric space and μ is the completion of a regular measure on the Borel sets such that points have measure zero and $\mu(G) > 0$ for all open sets G in X.

I will now look more closely at the hypothesis on X. The condition $\mu(G) > 0$ for all open sets G besides eliminating any isolated points in X [since points have measure zero] and assuming that $\mu(X)$ is nontrivial, also ensures that the topology of X is not detached from the measure μ.

Suppose X is any second category topological space. Let $\{O_\alpha\}$ be the collection of all open sets of first category and let $G = \cup O_\alpha$. Then by the Banach Category Theorem G is of first category. Now $X = (X-G) \cup G$, therefore X is the disjoint union of an open set of first category and a nontrivial Baire space. Hence we do not sacrifice any generality by assuming that a second category space is a Baire space.

It is easy to prove (see [14]) that the cardinality of any second countable T_1 space is at most c(i.e. the cardinality of the real numbers). Therefore X has cardinality at most c. But since the measure on X is strictly greater than zero it is clear that the cardinality of X is c.

The main results of this thesis are:

1) a duality theorem for X between the sets of measure zero and the sets of first category generalizing a result by Seirpinski [16] ,

2) I generalize a result by Szpilrajn [18] , proving that there exists no duality theorem for X between the measurable sets and the sets having the property of Baire,

3) my construction of category measure spaces via a density topology on X is a generalization of material in [10] and [6], also I derive category measure spaces via Boolean measure spaces, and stonian spaces with a finite normal measure whose support is the entire space,

4) the set-theoretic equality of Boolean measure spaces to stonian spaces with a finite normal measure with support the entire space.

5) the construction of stonian spaces with a finite normal

measure with support the entire space by applying the Gelfand-Naimark representation theorem to the L^∞ spaces of arbitrary finite measure spaces.

Much of the material of (1) and (2) done for the real line is found in Oxtoby's book [10].

CHAPTER 1

A DUALITY THEOREM

The main purpose of this section is to establish a duality principle between sets of first category and nullsets. In 1934 Sierpinski [16] proved the following theorem. Assuming the continuum hypothesis, there exists a one-to-one mapping f of the line, i.e. the real numbers, onto itself such that $f(E)$ is a nullset if and only if E is of first category. It is not known whether Sierpinski's Theorem can be proved without the continuum hypothesis. Sierpinski asked whether a stronger version of his theorem was true, in particular, does there exist a mapping f that maps each of the two classes onto the other simutaneously? This question was answered in the affirmative by Erdős [4] in 1943. Namely Erdős proved the following. Assuming the continuum hypothesis, there exists a one-to-one mapping f of the line onto itself such that

$$f = f^{-1} \quad \text{and such that} \quad f(E)$$

is a nullset if and only if E is of first category. (It follows from these properties that $f(E)$ is of first category if and only if E is a nullset.) His proof required a relatively small refinement of Sierpinski's proof. In this chapter I will prove that Erdős's theorem holds in a more general setting, i.e. for certain measures on a particular class of metric spaces (see Theorem 1.17 below). I will prove this by making use of several set--theoretic theorems found in [10]. These set--theoretic results were proved and used by Erdős for the specific classes of nullsets and sets of first category. Unless otherwise stated X will be a second countable Baire metric space, and μ will be the completion of a regular measure on the Borel sets of X such that points have measure zero and

$$\mu(G) > 0$$

for all nonempty open sets G in X.

I now state two trivial dual results which we will use later in the proof of the duality theorem.

Definition 1.1: A class of sets C is a σ--ideal if every countable union of sets from C is in C, and every subset of a set in C is in C.

Lemma 1.2: i) In any topological space the class of sets of first category is a σ--ideal. ii) In any complete measure space the class of sets of measure zero is a σ--ideal.

Proof i) clear by the definition of first category. ii) clear by the countable additivity of the measure and by the definition of a complete measure.

Remark 1.3: Lemma 1.2 holds for X.

Lemma 1.4: i) In any topological space every set of first category is contained in an F_σ of first category. ii) In any topological space which has a regular measure on its Borel sets, any nullset is contained in a G_δ nullset.

Proof i) this is true since the closure of a nowhere dense set is nowhere dense and the closure of every set contains the set. ii) this result follows easily from the definition of a regular measure.

Example 1.5: I will show that this result need not hold for a non-regular measure. Consider the reals, $\mathbb{R}$, with the euclidean topology on it and let μ be the completion of a measure defined for all the Borel sets by $\mu(E) = 0$ if E is of first category, and $\mu(E) = \infty$ if E is of second category. Clearly μ is a non-regular measure. Now $\mu(Q) = 0$ where Q is the set of rational numbers, but there does not exist a G_δ containing Q of zero measure [[since a dense G_δ is residual and X is Baire]] .

Remark 1.6: Lemma 1.4 holds for X. The following Lemmas and corallary are generalizations of results found in [10] .

Lemma 1.7: X can be decomposed into two complementary sets A and B such that A is of first category and B is of measure zero.

Proof Since X is second countable it is separable. Let $\{a_i\}$ be any countable dense set of X. Let 0_{ij} be an open set containing a_i and $\mu(0_{ij}) < \frac{1}{2^{i+j}}$ [[this is possible because points have measure zero and μ is regular]] . Let

$$G_j = \bigcup_{i=1}^{\infty} 0_{ij} \qquad \text{for } j = 1, 2, 3, \ldots \text{ and } B = \bigcap_{j=1}^{\infty} G_j.$$

Now for any $\epsilon > 0$ we can choose j so that $\frac{1}{2^j} < \epsilon$.

Then $B \subset \bigcap_{i=1}^{\infty} O_{ij}$ and $\sum_{i=1}^{\infty} \mu(O_{ij}) < \sum_{i=1}^{\infty} \frac{1}{2^{i+j}} = \frac{1}{2^j} < \epsilon.$

Hence B is a nullset (i.e. a set of measure zero). On the other hand, G_j is a dense open subset of X, since it is the union of open subsets containing a dense set. Therefore its complement G_j' is nowhere dense, and $A = B' = \bigcup_{j=1}^{\infty} G_j'$ is of first category.

Corollary 1.8: Every subset of X can be represented as the union of a nullset and a set of first category.

Remark 1.9: We will now consider two examples to show that the conclusion of Lemma 1.7 does not hold without our assumptions on X. First of all observe that this result does not hold for Example 1.5 since $\mathbb{R}$ is of second category in itself.

Example 1.10: Consider $\{a,b\}$ with the discrete topology on it and the regular measure μ on $\{a,b\}$ such that

$$\mu(\{a\}) = \mu(\{b\}) = 1.$$

Then $\{a,b\}$ cannot be decomposed into two disjoint sets, one of first category and the other of measure zero [since only the empty set has measure zero and X is not of first category].

Example 1.11: Now assume

$$X = \mathbb{R}^2 - \{0\}, \text{ and } d(x,y) = d_e(x,0) + d_e(y,0)$$

for all $x,y \in X$ where d_e is the euclidean distance function for the plane. Then it follows easily that X is a nonseparable, topologically complete (hence a Baire space) metric space [X is topologically complete because it is a G_δ in $(\mathbb{R}^2, d_e)$ which is a complete metric space]. For any measurable subset E of a line $y = mx$ we define $\mu(F)$ to be equal to Lebesque measure on the line. If E is any Borel subset of X, then $\mu(E) = \infty$ if E intersects uncoutably many lines $y = mx$. If E is a Borel subset of X which intersects only countably many lines $y = mx$, then $\mu(E)$ = the sum of its "sectional" measure on each line. It is easy to check that μ is a regular measure on the Borel sets of X. Let $\bar{\mu}$ be the completion of μ, then X cannot be decomposed into two disjoint sets, one of $\bar{\mu}$-measure zero and the other of first category [[a set is of first category in X if and only if it intersects each line $(y = mx)$ in a set of first category. Hence the complement of a set of first category has infinite measure]].

<u>Lemma 1.12</u>: Any uncountable G_δ subset of X contains a nowhere dense closed set C of measure zero that can be mapped onto $[0,1]$, i.e. onto a set of cardinality c.

<u>Proof</u> Let $E = \bigcap_{m=1}^{\infty} G_m$, G_m open, and E is an uncountable G_δ set. Let F denote the set of all condensation points of E that belong to E, that is, all points x in E such that every neighborhood of x contains uncountably many points of E. F is nonempty; otherwise, if $\{B_i\}_{i=1}^{\infty}$ is a base for X and $\{B_i'\} \subset \{B_i\}$ is a subclass of $\{B_i\}$ such that $\{B_i'\}$ contains only countably many points of E, then this subclass would cover E and E would be countable. Similar reasoning shows that F has no isolated points. Let

$$F(0) \quad \text{and} \quad F(1)$$

be two disjoint closed sets of measure at most 1/3 whose interiors meet F and whose union is contained in G_1. Proceeding inductively, if 2^m disjoint closed sets $F(i_1, \ldots, i_m)$ ($i_K = 0$ or 1) whose interiors all meet in F and whose union is contained in G_m have been defined, let

$$F(i_1, \ldots, i_{m+1}) \quad (i_{m+1} = 0 \text{ or } 1)$$

be disjoint closed sets of measure at most $\frac{1}{3^{m+1}}$ contained in $G_{m+1} \cap F(i_1, \ldots, i_m)$ whose interiors meet F. From the fact that F has no isolated points and that $F \subset G_{m+1}$ it is clear that such sets exist. Thus a family of sets $F(i_1, \ldots, i_m)$ having the stated properties can be defined. Let

$$C = \bigcap_{m=1}^{\infty} \bigcup_{i_1, \ldots, i_m} F(i_1, \ldots, i_m).$$

Then C is a closed nowhere dense subset of E. C has measure zero for the same reason as the Cantor set [[for a discussion of the Cantor set see page 4 of [10]]]. Now for each $x \in C$ there is a unique sequence $\{i_m\}$ $i_n = 0$ or 1, such that

$$x \in F(i_1, \ldots, i_m)$$

for every n, and every such sequence corresponds to some point of C. Let f(x) be the real number having binary development. $.i_1 i_2 i_3 i_4 i_5 \ldots\ldots$ Then f maps C onto $[0,1]$. Hence C has cardinality c.

Lemma 1.13: The complement of any nullset of X contains a nullset of cardinality c. The complement of any first category set in X contains a first category set of cardinality c.

Proof Clearly since μ is regular the complement of a nullset contains an uncountable closed set. By Theorem 1.1.2 in [12] this closed set is a G_δ. Hence by Lemma 1.12 it contains a closed nullset of cardinality c. The complement of a set of first category contains an uncountable G_δ set [by Lemma 1.4 and since X is a Baire space]. By Lemma 1.12 this set contains a nowhere dense set of cardinality c.

Remark 1.14: The category part of the above theorem does not hold if X is not a Baire space. For example, if C = the cantor set, then

$$X = C \cup ((1,2) \cap Q) \text{ where } Q \text{ is the rationals,}$$

is a second countable metric space of cardinality c. Now C is a set of first category, in fact it is a nowhere dense set in X, but

$$X - C = (1,2) \cap Q$$

does not contain a first category set of cardinality c.

The following two theorems are found on page 76 in [10].

Theorem 1.15: Let X be a set of cardinality c, and let K be a class of subsets of X with the following properties:

(a) K is a σ-ideal,

(b) the union of K is X,

(c) K has a subclass G of cardinality $\leq$ c with the property that each member of K is contained in some member of G,

(d) the complement of each member of K contains a set of cardinality c that belongs to K.

Then X can be decomposed into c disjoint sets X_α, each of power c, such that a subset E of X belongs to K if and only if E is contained in a countable union of sets X_α.

Proof Let $A = \{\alpha : \alpha \leq 0 < \Omega\}$ be the set of ordinals of first or second class, that is, all ordinals less than the first ordinal, Ω, that has uncountably many predecessors. Then A has cardinality c, and there exists a mapping $\alpha \to G$ of A onto G. For each $\alpha \in A$ define

$$H_\alpha = \bigcup_{\beta \leq \alpha} G_\beta \text{ and } K_\alpha = H_\alpha - \bigcup_{\beta < \alpha} H_\beta .$$

Put $B = \{\alpha \in A : K_\alpha \text{ is uncountable}\}$. Properties (a), (c) and (d), imply that B has no upper bound in A. Therefore there exists a one-to-one order-preserving map ϕ of A onto B. For each α in A, define

$$X_\alpha = H_{\phi(\alpha)} - \bigcup_{\beta < \alpha} H_{\phi(\beta)} .$$

By construction and property (a), the sets X_α are disjoint and belong to K. Since $X_\alpha \supset K_{\phi(\alpha)}$, each of the sets X_α has cardinality c. For any $\beta \in A$, we have $\beta \in \phi(\alpha)$ for some $\alpha \in A$, and therefore

$$G_\beta \subset H_\beta \subset H_{\phi(\alpha)} = \bigcup_{\gamma \leq \alpha} X_\gamma .$$

Hence, by (c), each member of K is contained in a countable union of the sets X . Using (b), it follows that

$$X = \bigcup K \subset \bigcup_{\alpha \in A} X_\alpha .$$

Thus $\{X_\alpha : \alpha \in A\}$ is a decomposition of X with the required properties.

Theorem 1.16: Let X be a set of cardinality c. Let K and L be two classes of subsets of X each of which has properties (a) to (d) of Theorem 1.15. Suppose further that X is the union of two complementary sets M and N, with $M \in K$ and $N \in L$. Then there exists a one-to-one mapping f of X onto itself such that $f = f^{-1}$ and such that $f(E) \in L$ if and only if $E \in K$.

Proof. Let X_α $(0 \leq \alpha < \Omega)$ be a decomposition of X corresponding to K, as constructed in the proof of Theorem 1.15. We may assume that M belongs to the generating class G, and that G_0 is taken equal to M. Then $X_0 = M$, because M cannot be countable. Similarly, let Y_α $(0 \leq \alpha < \Omega)$ be a decomposition of X corresponding to L, with $Y_0 = N$.

Then

$$M = \bigcup_{0<\alpha<\Omega} Y_\alpha \quad \text{and} \quad N = \bigcup_{0<\alpha<\Omega} X_\alpha .$$

The sets X_α and Y_α, for $0<\alpha<\Omega$, constitute a decomposition of X into sets of cardinality c. For each $0<\alpha<\Omega$, let f_α be a one-to-one mapping of X_α onto Y_α. Define f equal to f_α on X_α, and equal to f_α^{-1} on Y_α, for $0<\alpha<\Omega$. Then f is a one-to-one mapping of X onto itself, f is equal to f^{-1}, and $f(X_\alpha) = Y_\alpha$ for all $0<\alpha<\Omega$. Since

$$X_0 = \bigcup_{0<\alpha<\Omega} Y_\alpha \text{ and } Y_0 = \bigcup_{0<\alpha<\Omega} X_\alpha ,$$

we have also $f(X_0) = Y_0$. Thus

$$f(X_\alpha) = Y_\alpha \text{ for all } 0 \leq \alpha < \Omega .$$

From the properties of X_α and Y_α stated in Theorem 1.15 it follows that $f(E) \in L$ if and only if $E \in K$.

The following is the main theorem of this discussion.

Theorem 1.17: There exists a one-to-one mapping f of X onto itself such that $f = f^{-1}$ and such that f(E) is a nullset if and only if E is of first category (It follows that f(E) is of first category if and only if E is a nullset).

<u>Proof</u> This theorem follows immediately from Theorem 1.16. Let K be the class of sets of first category, and let L be the class of nullsets. K is generated by the class of F_σ sets of first category, and L by the class of G_δ nullsets. Each of these generating classes has cardinality c [[since X is second countable]]. Condition (c) of Theorem 1.15 is therefore satisfied. Condition (d) is implied by Lemma 1.13 and condition (a) is from Lemma 1.2. While condition (b) holds since points have measure zero and X has no isolated points. For the sets M and N we may take the sets A and B of Lemma 1.7.

The interest of this theorem is that it establishes a strong form of duality, which may be stated as follows.

<u>Theorem 1.17</u>: (Duality Principle). Let P be any proposition involving solely the notions of measure zero, first category, and notions of pure set theory. Let P* be the Proposition obtained from P by interchanging the terms "nullset" and "set of first category" wherever they appear. Then each of the propositions P and P* implies the other.

CHAPTER 2

APPLICATIONS OF THE DUALITY THEOREM

In this section I will prove a number of results in Baire category theory and then apply the principle of duality which we established in the first chapter in order to get the corresponding dual results in measure theory. The first four dual theorems were proved for the real numbers by Sierpinski in [16]. Proofs also appear in [10]. We will generalize these results using techniques in [10].

This first theorem is a generalization of a theorem proved by Lusin for the real numbers in 1914 [15].

Theorem 2.1: Any set E of second category in X has a subset N of power c such that every uncountable subset of N is of second category.

Proof Let $\{X_\alpha : \alpha < \Omega\}$ be the decomposition of X corresponding to the class K of first category sets in the proof of Theorem 1.15. Let N be a set obtained by selecting just one point from each non-empty set of the form $E \cap X_\alpha$. Since E is of second category, N is uncountable and therefore of power c. No uncountable subset of N can be covered by countably many of the sets X_α. Hence no uncountable subset of N is of first category.

Definition 2.2: An uncountable set with the property that every uncountable subset is of second category is called a Lusin set.

The dual of Theorem 2.1 was proved for the real line by Sierpinski in 1924 [15].

Theorem 2.3 (Dual of Theorem 2.1): Any subset E of X of positive outer measure (defined in terms of the measure) has a subset N of power c such that every uncountable subset of N has positive outer measure.

Remark 2.4: The outer measure of any subset A of X is defined as

$$\mu^*(A) = \inf\left\{\mu(E) \,\middle|\, A\subset E \text{ and } E \text{ is open}\right\}.$$

Theorem 2.5: There exists a one-to-one mapping f of X onto a subset of itself such that f(E) is of second category whenever E is uncountable.

Proof Let f be any one-to-one mapping of X onto a Lusin set.

Theorem 2.6 (Dual of Theorem 2.5): There exists a one-to-one mapping f of X onto a subset of itself such that f(E) has positive outer measure whenever E is uncountable.

Theorem 2.7: Any subset E of X of second category contains c disjoint subsets of second category.

Proof Let f be a one-to-one mapping of X onto a Lusin set contained in E. Now let $\{X_\alpha : \alpha<\Omega\}$ be the decomposition of X corresponding to the class K of first category sets (or to the class of nullsets either will do). Then from Theorem 1.15 we know that each X_α is uncountable, therefore, $f(X_\alpha)$ is a second category subset of E for each $\alpha < \Omega$. The result follows since the cardinality of the collection

$$\{Y_\alpha : \alpha < \Omega\} \text{ is c.}$$

Theorem 2.8 (Dual of Theorem 2.7): Any subset E of X of positive outer measure contains c disjoint sets of positive outer measure.

Corollary 2.9: i) X can be decomposed into c disjoint subsets each of second category. ii) X can be decomposed into c disjoint subsets each of positive outer measure.

Proof i) and ii) follow directly from Theorem 2.7 and Theorem 2.8 since X is of second category and has positive outer measure.

Remark 2.10: The dual theorems considered so far do not involve measure and category simultaneously. However Lemma 1.7 in Chapter 1 says that X can be decomposed into two complementary sets, one of first category, the other of measure zero. This proposition is self-dual. Another result is that a subset of X is a nullset if its intersection with every set of first category is countable. The dual is : A subset of X is of first category if its intersection with every nullset is countable. Both of these results are corollaries of Lemma 1.7. (Observe that the continuum hypothesis is not needed for Lemma 1.7).

The following is a generalization of one of Sierpinski's propositions [35].

Theorem 2.11: For any class K of invertible nullset preserving transformations of X, with T^{-1} also nullset preserving and cardinality of K equal to c, there exists a subset E of X of first category and cardinality c such that $TE \Delta E$ is a countable set, for each T in K.

Proof Index the elements of K and X so that

$$K = \{ T_\alpha : \alpha < \Omega \} \quad \text{and} \quad X = \{ p_\alpha : \alpha < \Omega \}.$$

Let A be a nullset such that X - A is of first category ⟦by Lemma 1.7⟧. For $0 < \alpha < \Omega$, let G_α be the group generated by the transformations T_β with $\beta < \alpha$. Then G_α consists of all products of the form

$$T_{\beta_1}^{K_1} \, T_{\beta_2}^{K_2} \cdots T_{\beta_n}^{K_n} \quad \text{where } \beta_i < \alpha \text{ and } K_i = \pm 1 \ (i = 1,2,\ldots,n)$$

and n is any positive integer. Hence G_α is countable, and each T in G_α is nullset-preserving. For each T in G_α, the set TA is a nullset. Hence

$$A_\alpha = \bigcup \{ TA : T \in G_\alpha \}$$

is a nullset. Let $x_0 = p_0$. Assuming that the points x_β in X have been defined for all $\beta < \alpha$, put

$$B_\alpha = \{ Tx_\beta : \beta < \alpha, \ T \in G_\alpha \}.$$

Then B_α is a countable set and $A_\alpha \cup B_\alpha$ is a nullset ⟦since points have measure zero⟧. Let x_α be the first element in the well ordering of X such that x_α is not in $A_\alpha \cup B_\alpha$. Put

$$E_\alpha = \{ Tx_\alpha : T \in G_\alpha \}, \quad \text{and define} \quad E = \bigcup_{0 < \alpha < \Omega} E_\alpha.$$

Then E_α is countable, and E is uncountable. Moreover, E is a subset of X - A. Hence E is of first category. For any $\beta < \alpha < \Omega$, we have $T_\beta E_\alpha = E_\alpha$. Hence

$$T_\beta E \Delta E \subset \bigcup_{\alpha \leq \beta} (E_\alpha \cup T_\beta E_\alpha)$$

This shows that $TE \Delta E$ is countable, for each T in K.

Theorem 2.12(Dual of Theorem 2.11): For any class K of invertible category-preserving transformations of X, with T^{-1} also category-preserving and cardinality of X equal to c, there exists a subset E of X of measure zero and cardinality c such that $TE\Delta E$ is a countable set, for each T in K.

The following two theorems are vacuously true for X since every nonempty open set of X is of second category ⟦since X is a Baire space⟧, and has positive measure. Therefore for the next theorem we will assume that X is an arbitrary topological space. Also the duality theorem of Chapter 1 does not apply since open set is not a set theoretic concept. The following theorem is found in [10].

Theorem 2.13 (Banach Category Theorem): In any topological space X, the union of any family of open sets of first category is of first category.

Proof Let G be the union of a family $\mathcal{G}$ of non-empty open sets of first category. Let

$$\mathcal{F} = \{ U_\alpha / \alpha \in A \}$$

be a maximal family of disjoint non-empty open sets with the property that each is contained in some member of $\mathcal{G}$. Then the closed set $\overline{G} - \cup\mathcal{F}$ is nowhere dense.⟦otherwise $\mathcal{F}$ would not be maximal.⟧ Each set U_α can be represented as a countable union of nowhere dense sets, say $U_\alpha = \bigcup_{m=1}^{\infty} N_{\alpha,m}$. Put

$$N_m = \bigcup_{\alpha\in A} N_{\alpha,m}.$$

If an open set U meets N_m, then it meets some $N_{\alpha,m}$ and there exists a non-empty open set

$$V \subset (U \cap U_\alpha) - N_{\alpha,m}.$$

Hence $V \subset U - N_m$, and so N_m is nowhere dense. Therefore

$$G \subset (G - \cup\mathcal{F}) \cup \bigcup_{\alpha\in A} U_\alpha = (G - \cup\mathcal{F}) \cup \bigcup_{m=1}^{\infty} N_m$$

is of first category.

In this next theorem we will assume that X is second countable.

Theorem 2.14: For any second countable topological space, the union of any family of open sets of measure zero is of measure zero (provided the measure is defined for all open sets).

Proof Let $\mathcal{G} = \{O_\alpha : \alpha \in A\}$ be any family of open sets each of measure zero and let $\mathcal{B} = \{B_m\}_{m=1}^{\infty}$ be a base for X and μ be the measure. Now since X is Lindelof for each O_α in $\mathcal{G}$, there exists a collection $\{B_{\alpha_K}\} \subset \mathcal{B}$ such that

$$O_\alpha \subset \bigcup_{K=1}^{\infty} B_{\alpha_K}.$$

Since O_α is open we may assume that each B_{α_K} is contained in O_α. B_{α_K} contained in O_α implies that

$$\mu(B_{\alpha_K}) = 0 \text{ for all k.}$$

Now since X is second countable

$$\{B_{\alpha_K} / k = 1, 2, \ldots, \alpha\in A\}$$

is countable. Therefore

$$\mu(\bigcup_{\alpha\in A} O_\alpha) \le \mu(\bigcup_{\alpha\in A} (\bigcup_{K=1}^{\infty} B_{\alpha_K})) \le \sum \mu(B_{\alpha_K}) = 0.$$

Hence $\bigcup O_\alpha$ is a nullset.

Remark 2.15: Second countability cannot be omitted from the hypothesis of the above theorem. For if X is the plane with the discrete topology on it (hence not second countable) and the product measure of Lebesque measure on $\mathbb{R}$, then for each $0<x<1$, $E_x = \{(x,y) / 0<y<1\}$ is an open set of X and $\mu(E_x) = 0$. Put $\bigcup_{0<x<1} E_x = I \times I$ where I is the open unit interval in $\mathbb{R}$ and we have $\mu(I \times I) = 1 = \mu(\bigcup E_x) \neq 0$. (Observe also that is not a regular measure.)

CHAPTER 3

NON-MEASURABLE SETS

In this chapter we will discuss non-measurable sets in X. (Where X is as in Chapter 1, i.e., X is a second countable Baire metric space, and μ will be the completion of a regular measure on the Borel sets of X such that points have measure zero and

$$\mu(G) > 0$$

for all nonempty open sets G in X.) The final theorem of this chapter is an important result due to Ulam (1930).

To start we will prove two lemmas. The following lemma is found in [14].

Lemma 3.1: The class of all closed (open) sets in a topological space with a countable basis has cardinality $\leq$ c.

Proof Let K denote a topological space with a countable base, say $\{U_n\}_{n=1}^{\infty}$, and let U be an open set of K. Denote by $N(U)$ the set of natural numbers n for which $U_n \subset U$. It is evident from the definition of a basis that

$$U = \bigcup_{n \in N(U)} U_n.$$

Hence every open set $U \subset K$ is uniquely determined by a set of natural numbers i.e., by $N(U)$; consequently the class of all open sets of K has cardinality $\leq$ c. The theorem is thus proved for open sets and, since a closed set is the complement of an open set, the class of all closed sets has the same cardinal number as that of all open sets.

The following lemma is proved for the reals in [10].

Lemma 3.2: The class of uncountable closed sets of X has cardinality c.

Proof: Let x,y X such that $x \neq y$. Then there exists open sets O_x, and O_y such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$ [[since X is Hausdorff]]. Now since X is a Baire space O_x and O_y are of second category, hence O_x and O_y are uncountable. Thus O_x' (the complement of O_x) is an uncountable closed set [[since $O_x' \supset O_y$]]. Now points in X are closed since X is Hausdorff and there are an uncountable number of points in $(O_x')' = O_x$. So

$$\{\{z\} \cup O_x' / z \in O_x\}$$

is a collection of uncountable closed subsets of X. Clearly this collection has cardinality c. Therefore it follows from Lemma 3.1 that the class of uncountable closed sets has cardinality c.

Remark 3.3: In the above theorem we cannot drop the assumption of second countablity. For instance if X is any set of cardinality c with the discrete topology on it, then X is a Baire metric space which is not second countable. It is easy to see that the class of uncountable closed sets is of cardinality 2^c [[since the class of closed sets is of cardinality 2^c and the class of countable closed sets is of cardinality c.]]

The next three theorems are generalizations to X of theorems found in [10] for the real numbers. These theorems are due to

F. Bernstein (1908)[9].

Theorem 3.4: There exists a subset B of X such that both B and B' meet every uncountable closed subset of X.

Proof By the well ordering principle and Lemma 3.2, the class $\mathcal{F}$ of uncountable closed subsets of X can be indexed by the ordinal numbers less than ω_c here ω_c is the first ordinal having c predecessors, say

$$\mathcal{F} = \{F_\alpha : \alpha < \omega_c\}.$$

(Since we are assuming the continuum hypothesis, $\omega_c = \Omega$.) We may assume that X, and therefore each member of $\mathcal{F}$, i.e. each of the uncountable closed subsets, has been well ordered. Note that each member of $\mathcal{F}$ has power c. Let p_1 and q_1 be the first two members of F_1. Let p_2 and q_2 be the first two members of F_2 different from p_1 and q_1. If $1 < \alpha < \omega_c$ and if p_β and q_β have been defined for all $\beta < \alpha$, let p_α and q_α be the first two elements of $F_\alpha - \bigcup_{\beta<\alpha}\{p_\beta, q_\beta\}$. This set is non-empty (it has cardinality c) for each α, and so p_α and q_α are defined for all $\alpha < \omega_c$. Put

$$B = \{p_\alpha \mid \alpha < \omega_c\}$$

Since $p_\alpha \in B \cap F_\alpha$ and $q_\alpha \in B' \cap F_\alpha$ for each $\alpha < \omega_c$, the set B has the property that both it and its complement meet every uncountable closed set.

Definition 3.5: Any set B in a topological space K having the property that both it and its complement meet every uncountable closed set is called a Bernstein set.

Definition 3.6: A subset E of any topological space X is said to have the property of Baire if $E = G \Delta N$ where G is open and N is a first category set of X.

Theorem 3.7: Any Bernstein set B in X is non-measurable and lacks the property of Baire. Indeed, every measurable subset of either B or B' is a nullset, and any subset of B or B' that has the property of Baire is of first category.

Proof Let A be any measurable subset of B. Any closed subset F in A must be countable ⟦since every uncountable closed set meets B'⟧ , hence $\mu(F) = 0$ ⟦since points have measure zero⟧. Therefore $\mu(A) = 0$ since μ is regular. Similarly, if A is a subset of B having the property of Baire, then $A = E \cup P$, where E is a G_δ and P is of first category. The set E must be countable, since every uncountable G_δ set contains an uncountable closed set ⟦by Lemma 1.12⟧ , and therefore meets B'. Hence A is of first category. The same reasoning applies to B'. The first part of the theorem follows from Lemma 1.7.

Example 3.8: Now consider the following example. Let X = the real numbers with the euclidean topology on it and having the counting measure on its Borel sets. Then vanishes only on the empty set and is not regular. And it is easy to see that Lemma 3.1, Lemma 3.2, and Theorem 3.4 (which are all purely topological results) hold for X. However, every measurable subset of any Bernstein set B need not be a nullset ⟦any non-empty measurable subset of B has positive measure and there are measurable subsets for any Bernstein set of X]]. But this contradicts part of Theorem 3.7.

Theorem 3.9: Any subset of X with positive outer measure has a non-measurable subset. Any subset of X of second category has a subset that lacks the property of Baire.

Proof If A has positive outer measure and B is a Bernstein set, Theorem 3.7 shows that the subset $A \cap B$ and $A \cap B'$ cannot both be measurable. If A is of second category, these two subsets cannot both have the property of Baire.

We will now weaken some of the conditions on X for the following theorem.

Theorem 3.10: Let X be a separable topological space such that the class of Borel sets of X is not equal to the power set of X. And let μ be a regular measure defined on the Borel sets of X such that

$$\mu(\{x\}) = 0$$

for all $x \in X$. Then there exists subsets A, B of X such that A has the property of Baire but is non-measurable, and B is measurable but lacks the property of Baire. Thus in X neither of these two σ-algebras includes the other.

Proof Observe that the proof of Lemma 1.7 uses only the properties that the space is separable, its points have measure zero, and the measure is regular. Therefore we can apply this result to X to decompose it into two complementary sets, one of measure zero and the other of first category. Also we clearly have any subset of X being the disjoint union of two sets, one of measure zero and the other of first category. Let $V \subset X$, but V not a Borel set, i.e. V is not measurable with respect to the Borel sets. Then

$$V = A \cup B,$$

where $A \cap B = \emptyset$ and A is of first category and $\mu(B) = 0$. So A is non-measurable, but has the property of Baire [[$A = \emptyset \Delta A$ so A has the property of Baire; if A were measurable, then V would be measurable]]. While B is measurable, but lacks the property of Baire [[analogous reason]].

In the preceding discussion we assumed that the power set of X was not equal to the Borel sets of X. In general if X is any set of cardinality c and μ is any finite measure defined for all the subsets of X with $\mu(\{x\}) = 0$ for all $x \in X$, then the following theorem due to Ulam (1930), and found in [19] shows that $\mu(X) = 0$, i.e. that μ is identically zero on X.

Theorem 3.11 (Ulam): A finite measure μ defined for all subsets of a set X of cardinality c vanishes identically if it is equal to zero for every one - element subset.

Proof Since we are assuming the continuum hypothesis c is equal to the first uncountable ordinal. By hypothesis, there exists a well ordering of X such that for each y in X the set

$$\{x \mid x < y\}$$

is countable. Let $f(x,y)$ be a one-to-one mapping of this set onto a subset of the positive integers. Then f is an integer-valued function defined for all pairs (x,y) of elements of X for which $x < y$. It has the property

$$x < x' < y \text{ implies } f(x,y) \neq f(x',y) \qquad (1)$$

For each x in X and each positive integer n, define

$$F_x^n = \{ y \mid x < y,\ f(x,y) = n \}.$$

We may picture these sets as arranged in an array

$$\begin{array}{cccccccc}
F^1_{x_1} & F^1_{x_2} & F^1_{x_3} & \cdots & F^1_x & \cdots \\
F^2_{x_1} & F^2_{x_2} & F^2_{x_3} & \cdots & F^2_x & \cdots \\
\cdot & \cdot & \cdot & \cdots & & \cdots \\
\cdot & \cdot & \cdot & \cdots & & \cdots \\
F^n_{x_1} & F^n_{x_2} & F^n_{x_3} & \cdots & F^n_x & \cdots \\
\cdot & \cdot & \cdot & \cdots & & \cdots \\
\cdot & \cdot & \cdot & \cdots & & \cdots \\
\cdot & \cdot & \cdot & \cdots & & \cdots
\end{array}$$

with a countable number of rows and c columns. This array has the following properties:

(2) The sets in any row are mutually disjoint

(3) The union of the sets in any column is equal to X minus a countable set.

To verify (2), suppose $y \in F_x^n \cap F_{x'}^n$, for some n and some y, x, and x' with $x \leq x'$. Then $x < y$, $x' < y$, and $f(x,y) = f(x',y) = n$. Hence $x = x'$ by (1). Therefore, for any fixed n, the sets

$$F_x^n \quad (x \in X)$$

are disjoint.

To verify (3), observe that if $x < y$, then y belongs to one of the sets F^n_{x}, namely that one for which $n = f(x,y)$. Hence the union of the sets F^n_x $(n = 1,2,\ldots)$ differs from X by the countable set $\{y \,/\, y \leq x\}$.

By (2), in any row there can be at most countably many sets for which $\mu(F^n_x) > 0$ (since $\mu(X)$ is finite). Therefore there can be at most countably many such sets in the whole array. Since there are uncountably many columns, it follows that there exists an element x in X such that

$$\mu(F^n_x) = 0$$

for every n. The union of the sets of this column has measure zero, and the complementary countable set also has measure zero. Therefore $\mu(X) = 0$, and so μ is identically zero.

Remark 3.12: This theorem implies that Lebesgue measure cannot be extended to all subsets of the real numbers $[\![$ since Lebesgue measure is σ-finite and since every measure is countably additive $]\!]$.

CHAPTER 4

EXTENDED DUALITY

In Chapter 1 and 2 we have looked at the duality between sets of first category and sets of measure zero. Now we will try to extend this duality by comparing measurable sets to sets that have the property of Baire. Throughout this chapter we will take X to be a second countable Baire metric space, and will be the completion of a regular measure on the Borel sets of X such that points have measure zero and $\mu(G) > 0$ for all nonempty open sets G in X. Where the results are more generally applicable we will note this.

The material of 4.1 to 4.4 is from [10].

<u>Definition 4.1</u>: A subset A of any topological space is said to have the property of Baire if it can be represented in the form $A = G \triangle P$, where G is open and P is of first category.

<u>Lemma 4.2</u>: A set A has the property of Baire if and only if it can be represented in the form $A = F \triangle Q$, where F is closed and Q is of first category.

<u>Proof</u> If $A = G \triangle P$, G open and P of first category, then

$$N = \bar{G} - G$$

is a nowhere dense closed set, and $Q = N \triangle P$ is of first category. Let $F = \bar{G}$. Then

$$A = G \triangle P = (\bar{G} \triangle N) \triangle P = \bar{G} \triangle (N \triangle P) = F \triangle Q.$$

Conversely, if $A = F \triangle Q$, where F is closed and Q is of first category, let G be the interior of F. Then $N = F - G$ is nowhere dense $P = N \triangle Q$ is of first category, and

$$A = F \triangle Q = (G \triangle N) \triangle Q = G \triangle (N \triangle Q) = G \triangle P.$$

Lemma 4.3: If A has the property of Baire, then so does its complement.

Proof For any two sets A and B we have $(A \triangle B)' = A' \triangle B$. Hence if

$$A = G \triangle P, \text{ then } A' = G' \triangle P,$$

and the conclusion follows from Lemma 4.2.

Theorem 4.4: The class of sets having the property of Baire is a σ-algebra. It is the σ-algebra generated by the open sets together with the sets of first category.

Proof Let $A_i = G_i \triangle P_i$ $(i = 1,2,\ldots)$ be any sequence of sets having the property of Baire. Put $G = \cup G_i$, $P = \cup P_i$, and $A = \cup A_i$. Then G is open, P is of first category, and

$$G - P \subset A \subset G \cup P.$$

Hence $G \triangle A \subset P$ is of first category, and $A = G \triangle (G \triangle A)$ has the property of Baire. This result together with Lemma 4.3, shows that the class in question is a σ-algebra. It is evidently the smallest σ-algebra that includes all open sets and all sets of first category.

Theorem 4.5 (dual to Theorem 4.4): The measure μ for X (which is defined to be the completion of a Borel measure on X)

is definėd on the σ-algebra generated by the open sets and the nullsets.

Proof This result follows trivially from the preliminary lemma in Chapter 1.

Theorem 4.6: A set has the property of Baire if and only if it can be represented as a G_δ set plus a set of first category (or as an F_σ set minus a set of first category).

Proof Since the closure of any nowhere dense set is nowhere dense, any set of first category is contained in an F_σ set of first category. If G is open and P is of first category, let Q be an F_σ set of first category that contains P. Then the set $E = G - Q$ is a G_δ, and we have

$$G \triangle P = [(G - Q) \triangle (G \cap Q)] \triangle (P \triangle Q) = E \triangle [(G \triangle P) \cap Q].$$

The set $(G \triangle P) \cap Q$ is of first category and disjoint to E. Hence any set having the property of Baire can be represented as the disjoint union of a G_δ set and a set of first category. Conversely, any set that can be so represented belongs to the σ-algebra generated by the open sets and the sets of first category; it therefore has the property of Baire. The parenthetical statement follows by complementation, with the aid of Lemma 4.3.

Remark 4.7: Notice that Theorems 4.4 and 4.6 hold for arbitrary topological spaces.

Theorem 4.8 (Dual of Theorem 4.6): A subset E of X is measurable if and only if it can be represented as an F_σ set plus a nullset (or as a G_δ set minus a nullset).

Proof If A is measurable, then since μ is regular we have for each n a closed set F_n and an open set G_n such that

$$F_n \subset E \subset G_n \quad \text{and} \quad \mu(G_n - F_n) < \frac{1}{n}.$$

Put $\qquad A = \cup F_n$ and $N = E - A.$

Then A is an F_σ set. N is a nullset, since

$$N \subset G_n - F_n \;,\; \mu(G_n - F_n) < \frac{1}{n}$$

for every n, and μ is complete. E is the disjoint union of A and N. It follows by complementation that E can also be represented as a G_δ set minus a nullset. Conversely, any set that can be so represented is measurable, since every nullset is measurable and since the measurable sets form a σ-algebra.

Definition 4.9: A real-valued function f on any topological space is said to have the property of Baire if $f^{-1}(U)$ has the property of Baire for every open set U in the reals.

Theorem 4.10: There exists a subset of X which lacks the property of Baire, and a real-valued function on X which lacks the property of Baire.

Proof Since X is of second category and by Theorem 3.9 in Chapter 3 we have that X contains a set, say A which lacks the property of Baire. Let χ_A be the characteristic function of A i.e. $\chi_A = 1$ for all $x \in A$, $\chi_A = 0$ for all $x \notin A$. Clearly χ_A lacks the property of Baire.

Theorem 4.11 (Dual to Theorem 4.10): There exists a subset of X which is nonmeasurable, and a real-valued function on X which is nonmeasurable.

Proof Since X has positive measure and by Theorem 3.9 in Chapter 3 we have that X contains a set, say A which is nonmeasurable. If χ_A is the characteristic function of A, then clearly χ_A is a nonmeasurable function.

The following theorem and its dual are proved for X = the real numbers in [10].

Theorem 4.12: A real-valued function f on X has the property of Baire if and only if there exists a set P of first category such that the restriction of f to X - P is continuous.

Proof Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be a countable base for the topology of $\mathbb{R}$ (the reals), for example, the open intervals with rational endpoints. If f has the property of Baire, then $f^{-1}(\mathcal{U}_i) = G_i \Delta P_i$, where G_i is open and P_i is of first category. Put

$$P = \bigcup_{i=1}^{\infty} P_i .$$

Then P is of first category. The restriction g of f to X - P is continuous, since

$$g^{-1}(\mathcal{U}_i) = f^{-1}(\mathcal{U}_i) - P = (G_i \Delta P_i) - P = G_i - P$$

is open relative to X - P for each i, and therefore so is $g^{-1}(\mathcal{U})$ for every open set $\mathcal{U}$.

Conversely, if the restriction g of f to the complement of some set P of first category is continuous, then for any open set $\mathcal{U}$, $g^{-1}(\mathcal{U}) = G - P$ for some open set G. Since

$$g^{-1}(\mathcal{U}) \subset f^{-1}(\mathcal{U}) \subset g^{-1}(\mathcal{U}) \cup P,$$

we have

$$G - P \subset f^{-1}(\mathcal{U}) \subset G \cup P,$$

Therefore $\qquad f^{-1}(\mathcal{U}) = G \triangle Q$ for some set $Q \subset P$.

Thus f has the property of Baire.

Remark 4.13: Observe that in this theorem we didn't use any specific topological properties of X, i.e. this theorem holds for any topological space.

Theorem 4.14 (Dual of Theorem 4.12): A real-valued function f on X is measurable if and only if for each $\epsilon > 0$ there exists a set E with $\mu(E) < \epsilon$ such that the restriction of f to X - E is continuous.

Proof Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be a countable base for the topology of $\mathbb{R}$. If f is measurable, then for each i there exists a closed set F_i an open set G_i such that

$$F_i \subset f^{-1}(\mathcal{U}_i) \subset G_i \quad \text{and} \quad \mu(G_i - F_i) < \frac{\epsilon}{2^i}$$

[[Since μ is a regular measure]].

Put $E = \bigcup_{i=1}^{\infty}(G_i - F_i)$. Then $\mu(E) < \epsilon$. If g denotes the restriction of f to X - E, then

$$g^{-1}(\mathcal{U}_i) = f^{-1}(\mathcal{U}_i) - E = G_i - E.$$

Hence $g^{-1}(\mathcal{U}_i)$ is open relative to X - E, and therefore g is continuous.

Conversely, if f has the stated property there is a sequence of sets $\{E_i\}$ with $\mu(E_i) < \frac{1}{i}$ such that the restriction f_i of f to $X-E_i$ is continuous. For any open set $\mathcal{U}$ there are open sets G_i such that

$$f_i^{-1}(\mathcal{U}) = G_i - E_i \quad (i = 1,2,\ldots).$$

Putting $E = \bigcap_{i=1}^{\infty} E_i$, we have

$$f^{-1}(\mathcal{U}) - E = \bigcup_{i=1}^{\infty} (f^{-1}(\mathcal{U}) - E_i) = \bigcup_{i=1}^{\infty} f_i^{-1}(\mathcal{U}).$$

Consequently,

$$f^{-1}(\mathcal{U}) = \left[f^{-1}(\mathcal{U}) \cap E\right] \cup \left[\bigcup_{i=1}^{\infty} (G_i - F_i)\right].$$

All of these sets are measurable, since $\mu(E) = 0$, and therefore f is a measurable function $\left[\!\left[E \cap f^{-1}(\mathcal{U}) \right.\right.$ is measurable since X is a complete measure space$\left.\left.\right]\!\right]$.

Remark 4.15: The above theorem holds if X is an arbitrary topological space with a complete regular measure μ such that μ is defined for all the open sets of X.

Theorem 4.16: In X every set of second category is the union of c disjoint sets each which lacks the property of Baire.

Proof If E is a set of second category, then by Theorem 2.7 in Chapter 2 we see that $E = \bigcup_{\alpha \in A} E_\alpha$ where each E_α is of second category and the cardinality of A is c. Now each E_α contains a set which lacks the property of Baire [[by Theorem 3.9 in Chapter 3]]. Hence E contains a family of disjoint subsets of cardinality c such that each member lacks the property of Baire.

Define in the obvious way a partial ordering on all such families. Then it is easy to see that we can apply Zorn's Lemma to this collection to get a maximal element, i.e. a maximal family of disjoint subsets of E of cardinality c each of which lacks the property of Baire. The complement of the union of this maximal family (with respect to E) is a set of first category [[otherwise the maximality of the family would be contradicted]].

Since every first category set has the property of Baire $[\![A = A \triangle \emptyset$ for each A a first category set$]\!]$ and since the sets with the property of Baire form a σ-algebra we have that the disjoint union of any first category set with a set which lacks the property of Baire will also lack the property of Baire. Hence E is the disjoint union of a family of sets each lacking the property of Baire and the cardinality of the family being c $[\![$take an element from the maximal family of sets lacking the property of Baire and adjoin to it the complement of the union of this maximal family$]\!]$.

Theorem 4.17 as stated below is a generalization of a discussion in [10] .

<u>Theorem 4.17 (Dual of Theorem 4.16)</u>: In X every set of positive outer measure is the union of c disjoint non-measurable sets.

<u>Proof</u> In Theorem 3.10 of Chapter 3 we showed that any set with positive outer measure contains a non-measurable set. And in Chapter 2, Theorem 2.8 we saw that any set of positive outer measure was the disjoint union of c sets each of positive outer measure. Therefore if E is a set of positive outer measure $E = \bigcup_{\alpha \in A} E_\alpha$ where each E_α has positive outer measure and card A = C. Also each $E_\alpha \supset A_\alpha$ where A_α is non-measurable. Hence E contains c disjoint non-measurable subsets. By Zorn's lemma, this family is contained in a maximal disjoint class of non-measurable subsets of E. The complement of the union of such a family must have measure zero. By adjoining it to one of the members of the family we obtain a decomposition of E into c disjoint non-measurable

subsets.

Corollary 4.18: X can be decomposed into c disjoint subsets each lacking the property of Baire.

Proof follows directly from Theorem 4.16.

Corollary 4.19: X can be decomposed into c disjoint non-measurable subsets.

Proof follows directly from Theorem 4.17.

Definition 4.20 as well as Lemma 4.21 and Lemma 4.22 are found in [9] .

Definition 4.20: Let X be any topological space and $E \subset X$, then E is said to be of the first category at a point $p \in X$, if there exists a neighborhood G of p such that the set $E \cap G$ is of the first category. The set of points where E is not of the first category (the points where E is of the second category) will be denoted by $D(E)$.

Lemma 4.21: If B and E are subsets of a topological space, then:

1) $D(E - D(E)) = \emptyset$

2) $[D(B) = \emptyset] \Rightarrow [D(E \cup B) = D(E) = D(E-B)]$

Proof The proofs are elementary. They can be found in [ ] pages 84-85.

Lemma 4.22: If X is any topological space and $E \subset X$, then

$$E = E \cap \left[\overline{(E-D(E))}\right] \cup \left[E-\overline{(E-D(E))}\right] = \left[E- \mathrm{Int}(D(E))\right] \cup \left[E \cap \mathrm{Int}(D(E))\right]$$

is a decomposition of E into two disjoint parts such that the first is of the first category and the second one is not of the first category at any of its points. Also the second member (in each equation) of the union is open relative to E.

Proof Clearly this is a disjoint decomposition of E such that the second part of the union is open relative to E [Since the interior of any set is open]. Now by formula (1), the set E - D(E) is of the first category; hence by (2),

$$D(E \cap D(E)) = D(E)$$

and

$$E \cap D(E) \subset D(E) = D(E \cap D(E)),$$

which shows that the set $E \cap D(E)$ is not of the first category at any of its points.

On the other hand, the set $E \cap (\overline{E-D(E)})$ is of the first category, as a union of two sets

$$E \cap D(E) \cap \overline{E-D(E)}$$

and

$$\left[E-D(E)\right] \cap \left[\overline{E-D(E)}\right],$$

the first one being nowhere dense as a subset of a nowhere dense set $D(E) \cap (\overline{X-D(E)})$,

and the second one being of the first category as a subset of the set E - D(E).

Since the set $E \cap \overline{E-D(E)}$ is of the first category, it follows from (2) that

$$D(E-(\overline{E-D(E)})) = D(E)$$

and we have $E-(\overline{E-D(E)}) \subset E-(E-D(E)) = E \cap D(E) \subset D(E) = D\left[E-(\overline{E-D(E)})\right]$

which proves that the set $E-(\overline{E-D(E)})$ is not of the first category at any of its points.

The following theorem is a generalization of a theorem found on page 112 in [15].

Theorem 4.23: Every subset Q of X which is of second category contains a subset which is not the intersection of Q with a set having the property of Baire.

Proof Suppose not; that is, suppose every subset of Q is the intersection of Q with a set having the property of Baire. Now by Theorem 2.7 in Chapter 2 we can decompose Q into c disjoint subsets, i.e. $Q = \bigcup_{\alpha \in A} Q_\alpha$ with each Q_α of second category. Now by the above lemma and since each Q_α is of second category, there exists for each $\alpha \in A$, an O_α open in X such that $O_\alpha \cap Q_\alpha \neq \emptyset$ and $O_\alpha \cap Q_\alpha$ is of second category at each of its points. Clearly we can pick these O_α's to be basic open sets from a countable base for X [[since X is second countable]]. Now since the Q_α's are disjoint and uncountable, but the O_α's are countable there exists $\alpha_0 \in A$ such that $O_{\alpha_0} \cap Q_\gamma$ and $O_{\alpha_0} \cap Q_\beta$ are sets having the above property and $\gamma \neq \beta$. Since Q_γ is a subset of Q, $Q_\gamma = Q \cap E$ where E has the property of Baire. Now $O_{\alpha_0} - E$ is of first category [[since Q_γ is of second category everywhere in O_{α_0}]].

Also $Q_\gamma = E \cap Q$, $Q_\beta \subset Q$ and $Q_\gamma \cap Q_\beta = \emptyset$ all imply $E \cap Q_\beta = \emptyset$ and

$O_{\alpha_0} - E \supset (O_{\alpha_0} - E) \cap Q_\gamma = (O_{\alpha_0} \cap Q_\beta) - (E - Q_\beta) = O_{\alpha_0} \cap Q_\beta$.

But then $0_{\alpha_c} - E \supset 0_{\alpha_0} \cap Q_\beta$ which is of second category. Hence we have a contradiction!

Therefore there exists $B \subset Q$ such that $B \neq Q \cap E$ for all $E \subset X$ having the property of Baire.

Recall that for any $A \subset X$, the outer measure of A, denoted by
$$\mu^*(A) = \inf \{\mu(G) / A \subset G \text{ and } G \text{ is open}\}.$$
We will need the following lemmas in the proof of Theorem 4.26. Lemma 4.24 is found in [13].

<u>Lemma 4.24</u>: Let A be any set, and $E_1, \ldots, E_n$ a finite sequence of disjoint measurable sets. Then
$$\mu^*(A \cap [\bigcup_{i=1}^{n} E_i]) = \sum_{i=1}^{n} \mu^*(A \cap E_i).$$

<u>Proof</u> We prove the lemma by induction on n. It is clear for n = 1, and we assume it is true if we have n-1 sets E_i. Since the E_i are disjoint sets, we have
$$A \cap [\bigcup_{i=1}^{n} E_i] \cap E_n = A \cap E_n$$
and
$$A \cap [\bigcup_{i=1}^{n} E_i] \cap E_n' = A \cap [\bigcup_{i=1}^{n-1} E_i].$$
Hence since E_n is measurable
$$\mu^*(A \cap [\bigcup_{i=1}^{n} E_i]) = \mu^*(A \cap E_n) + \mu^*(A \cap [\bigcup_{i=1}^{n-1} E_i]) =$$
$$\mu^*(A \cap E_n) + \sum_{i=1}^{n-1} \mu^*(A \cap E_i)$$
by our assumption.

<u>Lemma 4.25</u>: Let $\{E_i\}$ be a sequence of disjoint measurable sets and A any set. Then
$$\mu^*(A \cap [\bigcup_{i=1}^{\infty} E_i]) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i).$$

<u>Proof</u> The set function * is subadditive on P(X), i.e. for every sequence of sets $\{A_i\}$ $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Therefore $\mu^*(A \cap [\bigcup_{i=1}^{\infty} E_i]) = \mu^*(\bigcup_{i=1}^{\infty}(A \cap E_i)) \leq \sum_{i=1}^{\infty} \mu^*(A \cap E_i)$.

Let n be any integer, then by the above lemma

$$\mu^*(A \cap [\bigcup_{i=1}^{\infty} E_i]) = \sum_{i=1}^{n} \mu^*(A \cap E_i) \leq \sum_{i=1}^{\infty} \mu^*(A \cap E_i).$$

Since n is arbitrary we have

$$\mu^*(A \cap [\bigcup_{i=1}^{\infty} E_i]) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i).$$

The following theorems are generalization of a theorem found on page 110 in [15].

Theorem 4.26 (Dual of Theorem 4.23): Every subset Q of X of positive outer measure contains a subset which is not measurable relative to Q, i.e. which is not the intersection of Q and a measurable set.

Proof Suppose not, i.e. suppose every subset of Q is the intersection of a measurable set with Q. Now $\mu^*(\{x\}) = 0$ for each $x \in Q$ since μ vanishes at points and all points are measurable in X. Also $\mu^* \geq 0$ for each $E \subset Q$. Let $\{E_n\}$ be any sequence of subsets of Q, then by assumption $E_n = Q \cap A_n$ where A_n is measurable for all n.

$$\begin{aligned}\mu^*(\bigcup_{n=1}^{\infty} \mu^* E_n) &= \mu^*(\bigcup_{n=1}^{\infty}[Q \cap A_n]) = \mu^*(Q \cap [\bigcup_{n=1}^{\infty} A_n]) \\ &= \sum_{n=1}^{\infty} \mu^*(Q \cap A_n) \quad [[\text{by the above lemma}]] \\ &= \sum_{n=1}^{\infty} \mu^* E_n\end{aligned}$$

therefore μ^* meets the hypothesis of Theorem 3.11 in Chapter 3 implying $\mu^* Q = 0$ which is a contradiction! Therefore there exists $E_o \subset Q$ such that $E_o \neq Q \cap A$ for all measurable sets A in X.

Theorem 4.27: For any second category subset Q of X, there exists a real valued function defined on it which does not admit an extension to a real-valued function on X having the property of Baire.

Proof By Theorem 4.23, there exists $M \subset Q$ such that $M \neq Q \cap A$ where A is any subset of X having the property of Baire. Define φ on Q by $\varphi(x) = 1$ for all $x \in M$ and $\varphi(x) = 0$ for all $x \in Q - M$. Suppose φ can be extended to X such that the extension has the property of Baire. Let the extension of φ be denoted by f, then $f^{-1}\{(0,\infty)\} = \{x / f(x) > 0\}$ is a set having the property of Baire. But

$$f^{-1}\{(0,\infty)\} \cap Q = M$$

which is a contradiction. Hence there exists a real-valued function defined on Q which does not admit an extension to a real-valued function on X having the property of Baire.

Theorem 4.28 (Dual of Theorem 4.27): For any subset Q of X of positive outer measure, there exists a real-valued function defined on it and admitting no extension to a measurable function on X.

Proof By Theorem 4.26 there exists $N \subset Q$ such that $N \neq Q \cap A$ for every measurable subset of X. Let ψ be a real-valued function defined on Q by $\psi(x) = 1$ for all $x \in N$ and $\psi(x) = 0$ for all $x \in Q - N$. Suppose ψ can be extended to X such that the extension is measurable. Let the extension of ψ be denoted by g, then

$$g^{-1}\{(0,\infty)\} = \{x / g(x) > 0\}$$

is a measurable subset of X. But $g^{-1}\{(0,\infty)\} \cap Q = N$ which is

Let N be a set of positive measure, i.e. $\mu(N) = \epsilon > 0$, and let $E = f^{-1}(N)$. Then E has the property of Baire. Let $x_1, x_2, \ldots$ be a countable dense subset of E, [[exists since X is second countable and any second countable space is hereditarily separable]] and let O_i be an open set containing x_i such that $\mu(f(O_i) \cap N) < \frac{\epsilon}{2^{i+1}}$. Put $G = \bigcup_{i=1}^{\infty} O_i$. Then G is an open set and $E \subset \bar{G}$. Hence

$$E \subset \left[(G \cap E) \cup (\bar{G} - G)\right].$$

Therefore

$$N = f(E) \subset f(G \cap E) \cup f(\bar{G} - G) \subset \bigcup_{i=1}^{\infty}\left[f(O_i) \cap N\right] \cup f(\bar{G} - G).$$

Since $\bar{G} - G$ is nowhere dense, $f(\bar{G} - G)$ is a nullset, and so

$$\mu(N) \leq \sum_{i=1}^{\infty} \epsilon 2^{-i-1} = \epsilon/2 .$$

But $\epsilon = \mu(N) \leq \epsilon/2$ is a contradiction. Hence no such function can exist.

In the following theorem let X be as before but now assume that the measure on X is σ-finite.

<u>Theorem 4.31</u>: Let E_{ij} be a double sequence of measurable sets such that $E_{ij} \supset E_{i,j+1}$ for all positive integers i and j, and such that $\bigcap_j E_{ij}$ is a nullset for each i. Then there exists a sequence of mappings $m_k(i)$ of the set of positive integers into itself such that $\bigcap_k \bigcup_i E_{i, m_k(i)}$ is a nullset. Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of measurable sets each of finite measure such that $E_{k+1} \supset E_k$ for all k and $\bigcup_{k=1}^{\infty} E_k = X$ [[possible since μ is σ-finite]] . For each i and k there is a positive integer $n_k(i)$ such that $\mu(E_{i,n_k(i)} \cap E_k) < \frac{1}{k2^i}$. Hence

$$\mu\left(\bigcup_i E_{i,n_k(i)} \cap E_k\right) < 1/k .$$

a contradiction. Hence there exists a real-valued function defined on Q and admitting no extension to a measurable function on X.

Theorem 4.29: i) If every subset of a set $E \subset X$ is measurable, then E is a nullset. ii) If every subset of E has the property of Baire, then E is of first category.

Proof follows directly from Theorem 3.9.

In this section we have shown eight examples where the property of Baire has played a role analogous to measurability. In Chapter 1 we proved a duality theorem between the sets of first category and the nullsets of X. After seeing the dual results stated in this section between the property of Baire and measurability it is natural to ask: can the principle of duality be extended to include measurability and the property of Baire as dual notions? That is, is there a one-to-one mapping f of X onto itself such that f(E) is measurable if and only if E has the property of Baire, and such that f(E) is a nullset if and only if E is of first category? (This second property is a consequence of the first, and by Theorem 4.29 and its converse.) It was shown by Szpilrajn [18] that such a mapping is impossible for $\mathbb{R}$. A proof appears in [10]. We extend Oxtoby's argument to prove:

Theorem 4.30: There does not exist a one-to-one mapping f of X onto itself such that f(E) is measurable if and only if E has the property of Baire.

Proof Suppose f is a one-to-one mapping of X onto itself such that f(E) is measurable if and only if E has the property of Baire.

Put $E = \bigcap_k \bigcup_i E_{i,n_k(i)}$. Now let A be any closed subset of E such that $\mu(A) < \infty$. Then there exists N such that $A \subset E_n$ for all $n \geq N$. Then

$$(E \cap A) \subset \bigcup_i E_{i, n_k(i)} \cap E_K .$$

Hence $\mu(E \cap A) < 1/k$ for all sufficiently large k. Thus $E \cap A$ is a nullset for every A a closed subset of E with finite measure. Therefore, by the regularity of μ we have $\mu(E) = 0$.

Theorem 4.32: It is not in general true that: If E_{ij} is a double sequence of sets having the property of Baire such that $E_{ij} \supset E_{i\,j+1}$ for all positive integers i and j, and such that $\bigcap_j E_{ij}$ is of first category for each i, then there exists a sequence of mappings $n_K(i)$ of the set of positive integers into itself such that $\bigcap_k \bigcup_i E_{i, n_k(i)}$ is of first category. That is, the dual of Theorem 4.31 is false.

Proof Let r_i be an enumeration of all rational numbers, and let $E_{ij} = (r_i - 1/j \, , \, r_i + 1/j \,)$.

This double sequence satisfies the hypothesis of the proposition in question. For any mapping n(i) of the positive integers into positive integers, the set $\bigcup_i E_{i, n(i)}$ is a dense open set. For any sequence of such mappings $n_K(i)$, the set $\bigcap_K \bigcup_i E_{i, n_K(i)}$ is residual, i.e. its complement is of first category. But this is contrary to the stated conclusion.

Remark 4.33: Although we have shown that the extended principle of duality is not valid as a general principle, it has a c certain heuristic value. For example, many properties of measure depend only on properties of the class of measurable sets that are shared by the class of sets having the property of Baire. In such cases the principle may suggest (even though it does not prove) a valid dual. One then seeks an abstract theorem that includes both concepts.

CHAPTER 5

PRODUCT SPACES

In this chapter we will examine to some extent what duality exists between category and measure in product spaces. To start with we shall need several results from Real Analysis. The following three lemmas, as well as a development of product measure spaces, are found in Chapter Twelve of [13].

Lemma 5.1: Let X be any set. Let μ be a measure on an algebra $\mathcal{A}$ of subsets of X, μ^* the outer measure induced by μ, and E any set. Then for $\epsilon > 0$, there is a set $A \in \mathcal{A}_\sigma$ (those sets which are countable unions of sets of $\mathcal{A}$) with $E \subset A$ and $\mu^* A = \mu^* E + \epsilon$. There is also a set $B \in \mathcal{A}_{\sigma\delta}$ (sets which are countable intersections of sets in $\mathcal{A}_\sigma$) with $E \subset B$ and $\mu^* E = \mu^* B$.

Proof See [13].

Definition 5.2: Let X be a topological space. A collection $\mathcal{B}$ of nonempty open sets is called a pseudobase for X if every nonempty open set includes a member of $\mathcal{B}$.

For the rest of this chapter X and Y will be topological spaces with countable pseudobases and complete measures μ and ν respectively on arbitrary σ-algebras , on X and Y respectively. Also $(X \times Y, \mathcal{G}, \lambda)$ will be the complete product measure space of $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ as defined in [13].

Definition 5.3: The collection of all sets $A \times B$ such that

$$A \in \mathcal{A}, \; B \in \mathcal{B}$$

is denoted by $\mathcal{R}$ and each set of this form is called a measurable rectangle.

Remark 5.4: It is shown in [13] that $\mathcal{R}$ is a semialgebra which generates the σ-algebra $\mathcal{S}$ in the product measure space $(X \times Y, \mathcal{S}, \lambda)$.

Lemma 5.5: Let E be a set in $\mathcal{R}_{\sigma\delta}$ with $\lambda(E) < \infty$. Then the function g defined by $g(x) = \nu(E_x)$ is a measurable function of x and

$$\int g \, d\mu = \lambda(E).$$

Proof See [13].

Now we will prove four dual results between measure and Baire category on product spaces.

The following theorem is taken from [13].

Theorem 5.6: Let E be a subset of $X \times Y$ such that

$$\lambda(E) = 0.$$

Then $\nu(E_x) = 0$ except on a set of measure zero.

Proof By Lemma 5.1 there is a set F in $\mathcal{R}_{\sigma\delta}$ such that

$$E \subset F \quad \text{and} \quad \lambda(F) = 0.$$

It follows from Lemma 5.5 that $\nu(F_x) = 0$ except on a set of measure zero. But $E_x \subset F_x$ and so $\nu(E_x) = 0$ except on a set of measure zero since ν is complete.

The following theorem was proved by Kuratowski and Ulam in 1932 [9].

Theorem 5.7 (Dual of Theorem 5.6): If E is a subset of $X \times Y$ which is of first category, then E_x is a set of first category in Y for all x except a set of first category in X. If E is a nowhere dense subset of $X \times Y$, then E_x is a nowhere dense subset of Y for all x except a set of first category in X.

Proof The two statements are essentially equivalent. For if $E = \bigcup E_i$, then

$$E_x = \bigcup (E_i)_x.$$

Hence the first statement follows from the second. If E is nowhere dense, so is $\bar{E}$, and E_x is nowhere dense whenever $(\bar{E})_x$ is of first category. Hence the second statement follows from the first. It is therefore sufficient to prove the second statement for any nowhere dense closed set E.

Let $\{V_n\}$ be a countable pseudobase for Y, and put

$$G = (X \times Y) - E.$$

Then G is a dense open subset of $X \times Y$. For each positive integer n, let G_n be the projection of

$$G \cap (X \times V_n) \text{ in } X,$$

that is,

$$G_n = \{x \mid (x,y) \in G \text{ for some } y \in V_n\}.$$

Let $x \in G_n$ and $y \in V_n$ be such that $(x,y) \in G$. Since G is open, there exists open sets U and V in X and Y respectively such that $x \in U$, $y \in V \subset V_n$, and $U \times V \subset G$. It follows that $U \subset G_n$. Hence G_n is an open subset of X. For any nonempty open set U, the set $G \cap (U \times V_n)$ is non-empty, since G is dense in $X \times Y$. Hence G_n

contains points of U. Therefore G_n is a dense open subset of X, for each n. Consequently, the set $\bigcap G_n$ is the complement of a set of first category in X. For any $x \in \bigcap G_n$, the section G_x contains points of V_n for every n. Hence G_x is a dense open subset of Y and therefore $E_x = Y - G_x$ is nowhere dense. This shows that for all x except a set of first category, E_x is nowhere dense.

Theorem 5.8 and Lemma 5.12 are found in [13].

<u>Theorem 5.8</u>: Let E be a measurable subset of XxY such that $\lambda(E)$ is finite. Then except on a set of measure zero the set E_x is a measurable subset of Y.

<u>Proof</u> By Lemma 5.1 there is a set F in $\mathcal{R}_{\sigma\delta}$ such that

$$E \subset F \quad \text{and} \quad \lambda(F) = \lambda(E).$$

Let G = F−E. Since E and F are measurable, so is G, and

$$\lambda(F) = \lambda(E) + \lambda(G)$$

Since $\lambda(E)$ is finite and equal to $\lambda(F)$, we have $\lambda(G) = 0$. Thus by Theorem 5.6 we have $\nu(G_x) = 0$ except on a set of measure zero.

The following three theorems on Baire category are from [10].

<u>Theorem 5.9: (Dual of Theorem 5.8)</u>: If E is a subset of XxY with the property of Baire, then E_x has the property of Baire for all x except a set of first category in X.

<u>Proof</u> Let $E = G \Delta P$, where G is open and P is of first category. Then

$$E_x = G_x \Delta P_x, \text{ for all } x.$$

Every section of an open set is open, hence E_x has the property of Baire whenever B_x is of first category. By Theorem 5.6, this is the case for all x except a set of first category.

<u>Theorem 5.10</u>: A product set AxB is of measure zero in XxY if and only if at least one of the sets A or B is of measure zero.

<u>Proof</u> If $\lambda(A \times B) = 0 = \mu(A) \cdot \nu(B)$, then either

$$\mu(A) = 0 \text{ or } \nu(B) = 0.$$

The converse is trivial.

<u>Theorem 5.11 (Dual of Theorem 5.10)</u>: A product set AxB is of first category in XxY if and only if at least one of the sets A or B is of first category.

<u>Proof</u> If G is a dense open subset of X, then GxY is a dense open subset of XxY. Hence AxB is nowhere dense in XxY whenever A is nowhere dense in X. Since

$$(\bigcup A_i) \times B = \bigcup (A_i \times B),$$

it follows that AxB is of first category whenever A is of first category. Similar reasoning applies to B.

Conversely, if AxB is of first category and A is not, then by Theorem 5.6 there exists a point x in A such that $(A \times B)_x$ is of first category. Since

$$(A \times B)_x = B \text{ for all } x \text{ in } A,$$

it follows that B is of first category.

The following lemma is needed to prove Theorem 5.13.

<u>Lemma 5.12</u>: Let E be a measurable set of finite measure in XxY. Then the function g defined by $g(x) = \nu(E_x)$ is a measurable function defined except on a set of measure zero and

$$\int g\,d\mu = \lambda(E)$$

Proof Follows directly from Lemma 5.5 and Theorem 5.10.

Theorem 5.13: If E is a measurable subset of XxY, and if E_x is of measure zero for all x except a set of measure zero, then E is of measure zero.

Proof Define $g(x) = \nu(E_x)$ for all $x \in X$. Then by Lemma 5.12 g(x) is a measurable function defined except on a set of measure zero and $0 = \int g\,d\mu = \lambda(E)$ [[since g(x) = 0 except on a set of measure zero]].

The following theorem is a partial converse of Theorem 5.7.

Theorem 5.14 (Dual of Theorem 5.13): If E is a subset of XxY that has the property of Baire, and if E_x is of first category for all x except a set of first category, then E is of first category.

Proof Suppose the contrary. Then $E = G \triangle P$, where P is of first category and G is an open set of second category. There exist open sets U and V such that $U \times V \subset G$ and $U \times V$ is of second category [[this follows from Theorem 2.13]]. By Theorem 5.11, both U and V are of second category. For all x in U, $E_x \supset V - P_x$. By Theorem 5.7, P_x is of first category for all x except a set of first category. Therefore E_x is of second category for all x in U except a set of first category. This implies that E_x is of second category for all x in a set of second category, contrary to hypothesis.

CHAPTER 6

CATEGORY MEASURE SPACES

A category measure space is a regular Hausdoiff Topological space with a finite measure μ defined on the σ-algebra $\mathcal{S}$ of sets having the property of Baire, and such that $\mu(E) = 0$ if and only if E is of first category. In such a space the extended principle of duality is not only valid, it is a tautology (See Chapter 4 of this paper for a discussion of the extended principle of duality.)

In this chapter, after some general results about category measure spaces I will consider three ways of generating category measure spaces. First I shall show how to define a topology (<u>the density topology</u>) on certain metric spaces in terms of a measure, to make the measure a category measure. Secondly I will discuss category measure spaces obtained from Boolean measure spaces, that is, spaces obtained from finite measure algebras by means of the Stone representation theorem. The third class of category measures is obtained by means of the Gelfaud-Naimark representation theorem on the structure space of L^{∞} of a finite measure space. The later two provide examples of compact Hausdoiff spaces that admit a category measure.

A. <u>General Considerations</u>

I will begin by stating several needed definitions and look at some results stating when sets of first category are necessarily nowhere dense. The following material is taken from [11].

<u>Definition 6.1</u>: A topological space is called <u>category measurable</u> if it admits a category measure not identically zero.

<u>Definition 6.2</u>: A topological space will be called <u>regularly category measurable</u> if it admits a regular category measure.

<u>Definition 6.3</u>: A topological space is quasi-regular if for each non-empty open set $\mathcal{U}$ there exists a non-empty open set $\mathcal{V}$ such that $\overline{\mathcal{V}} \subset \mathcal{U}$. (Here $\overline{\mathcal{V}}$ stands for the closure of $\mathcal{V}$.)

<u>Theorem 6.4</u>: Let X be a category measurable space and let $\mathcal{S}$ be the union of all open sets of first category in X. The following assertions concerning X are equivalent:

1) some category measure in X is regular;

2) every category measure in X is regular;

3) $X - \overline{\mathcal{S}}$ is a quasi-regular subspace of X.

In particular, a category measurable Baire space is regularly category measurable if and only if it is quasi-regular.

<u>Proof</u> Suppose that μ is a regular category measure in X. Let $\mathcal{U}$ be any non-empty open set contained in $X - \overline{\mathcal{S}}$. Then $\mathcal{U}$ is of second category and $\mu(\mathcal{U}) > 0$. Let $F = X - \mathcal{U}$. Then $\mu(F) < \mu(X)$. Since μ is regular there exists an open set $\mathcal{W}$ such that $F \subset W$ and $\mu(\mathcal{W}) < \mu(X)$. Put $\mathcal{V} = X - \overline{\mathcal{W}}$. Then $\mathcal{V}$ is an open set, $\overline{\mathcal{V}} \subset X - \mathcal{W} \subset \mathcal{U}$,

and $\mu(V) = \mu(X) - \mu(\overline{W}) = \mu(X) - \mu(W) > 0.$

Hence V is a non-empty open subset of U and $\overline{V} \subset U$. Thus 1) implies 3).

Suppose that $Y = X - \mathcal{S}$ is a quasi-regular subspace of X. Let μ be any category measure in X, and let F be any closed subset of X. If F does not contain Y let $\mathcal{F}$ be a maximal disjoint family of non-empty open sets G such that $\overline{G} \subset Y - F$. The family $\mathcal{F}$ must be countable, say $\mathcal{F} = \{G_i\}$. The maximality of $\{G_i\}$ and the quasi-regularity of Y imply that $\bigcup_{i=1}^{\infty} G_i$ is dense in Y - F. Hence $\bigcup_{i=1}^{\infty} G_i$ differs from Y - F by a nowhere dense set, and

$$\mu(X-F) = \mu(Y-F) = \sum_{i=1}^{\infty} \mu(G_i) .$$

For any $\epsilon > 0$ there is a positive integer n such that (G

$$\sum_{i=1}^{n} \mu(G_i) > \mu(X-F) - \epsilon,$$

Hence $\qquad G = \bigcap_{i=1}^{n} (X - \overline{G}_i)$ is an open set containing F and,

$$\mu(G) = \mu(X) - \mu(\bigcup_{i=1}^{n} \overline{G}_i) \leq \mu(F) + \mu(X-F) - \sum_{i=1}^{n} \mu(G_i) < \mu(F) + \epsilon,$$

On the other hand, if F contains Y, then X itself is an open set containing F, and $\mu(X) = \mu(F)$. Thus 3) implies 2). Obviously 2) implies 1).

Theorem 6.5: If X is a quasi-regular category measurable Baire space then every set of first category in X is nowhere dense.

Proof By Theorem 6.4 there exists a regular category measure μ in X. Let $\{N_i\}$ be any sequence of nowhere dense sets, with $P = \bigcup N_i$. Then $\mu(\overline{N_i}) = 0$, and for any $\epsilon > 0$ there existts a sequence $\{G_i\}$ of open sets such that $\overline{N}_i \subset G_i$ and $\mu(G_i) \leq \epsilon/2^i$ for each i. Let $G = \bigcup G_i$. Then G is open and $\mu(\overline{G}) = \mu(G) \leq \epsilon$.

Since $\overline{P} \subset \overline{G}$ it follows that $\mu(\overline{P}) \leq \epsilon$ for every $\epsilon > 0$. Hence $\mu(\overline{P}) = 0$, $\mu(\overline{P}^{\prime-\prime}) = 0$, and therefore $\overline{P}^{\prime-\prime}$ is an open set of first category. Because X is a Baire space it follows that $\overline{P}^{\prime-\prime}$ is empty, that is, P is nowhere dense.

Theorem 6.6: The following assertions concerning a category measurable Baire space X are equivalent:

1) every set of first category in X is nowhere dense;

2) $\mu(E) = \mu(\overline{E}) = \mu(E^{\prime-\prime})$ for every category measure μ and for every set E having the property of Baire;

3) $\mu(E) = \mu(\overline{E})$ (or $\mu(E) = \mu(E^{\prime-\prime})$) for some category measure μ and for every set E having the property of Baire.

Proof Assume 1) Then any set E having the property of Baire is of the form $G \triangle N$, where G is open and N is nowhere dense. Hence $G - \overline{N} \subset E^{\prime-\prime} \subset E \subset \overline{E} \subset \overline{G} \cup \overline{N}$.

Since $\mu(G - \overline{N}) = \mu(\overline{G} \cup \overline{N})$ for any category measure μ it follows that 1) implies 2). Obviously 2) implies 3).

Assume 3). Since $\mu(E) = \mu(\overline{E})$ if and only if $\mu(E') = \mu(\overline{E}')$, either version of 3) implies that $\mu(E) = \mu(\overline{E}) = \mu(E^{\prime-\prime})$ for every set E having the property of Baire. In particular, if P is any set of first category then $\mu(P) = \mu(\overline{P}) = \mu(\overline{P}^{\prime-\prime}) = 0$. Hence $\overline{P}^{\prime-\prime}$ is an open set of first category, therefore empty, and P is nowhere dense. Thus 3) implies 1).

The following examples show that in a category measurable Baire space that is not quasi-regular it may or may not be true that every set of first category is nowhere dense.

Example 6.7: Let X be an uncountable set. In the first example let X have the cofinite topology (i.e. the class of closed sets consists of X and its finite subsets). In the second example let X have the cocountable topology (i.e. the class of closed sets consists of X and its countable subsets). Clearly both of these topologies are not quasi-regular since the only closed set a non-empty open set is contained in is X. Also in either case X is a T_1-space, but not Hausdoiff, and the sets of first category are the countable sets, the sets having the property of Baire are the countable sets and their complements, and every non-empty open set is of second category. Hence in either case X is a Baire space. If we define $\mu(E) = 0$ or one according as E or X - E is countable. Then is a category measure in either space, but not regular. In the first example the nowhere dense sets constitute a proper subclass of the class of sets of first category. In the second example every set of first category is nowhere dense.

The following discussion results in a necessary and sufficient condition for a metric space to be a category measurable space.

First observe that it follows from theorem that in any metrizable category measurable Baire space every set of first category is nowhere dense.

Theorem 6.8: In a metric space X every set of first category is nowhere dense if and only if the set D of isolated points of X is dense in X.

Proof Suppose the open set $X - \overline{D}$ is non-empty. For each positive integer n let E_n be a maximal subset of $X - \overline{D}$ with the property that the distance between any two points of E_n is at least equal to $\frac{1}{n}$. Then E_n is closed. Since $X - \overline{D}$ contains no isolated points, each of the sets E_n is nowhere dense. Hence the set $P = \cup E_n$ is of first category in X. The maximality of the sets E_n implies that P is dense in $X - \overline{D}$. Hence $X - \overline{D}$ is a non-empty open set contained in $\overline{P}$, and P is not nowhere dense, Since this contradicts that each first category set is nowhere dense it follows that if every set of first category is nowhere dense then the set D of isolated points of X is dense in X. (In case D is empty, the same reasoning shows that a metric space contains a dense set of first category if and only if the space is dense in itself.)

Conversely assume that D is dense in X. Let $F = \bigcup_{i=1}^{\infty} N_i$ where F is a set of first category and each N_i is nowhere dense. To show F nowhere dense is equivalent to showing that F does not meet D. But each $\overline{N_i}$ does not meet D, therefore, each N_i does not meet D. Hence F does not meet D. Therefore F is nowhere dense.

The following theorem follows easily from Theorem 6.5 and Theorem 6.8.

Theorem 6.9: A metrizable space is a category measurable Baire space if and only if the set D of isolated points of X is a countable dense subset of X. In this case the category measures in X are those and only those measures that are positive for each point of D and vanishes on X - D.

Corollary 6.10: Let X be the space defined in Chapter 1, i.e., X is a second countable Baire metric space, and μ will be the completion of a regular measure on the Borel sets of X such that points have measure zero and $\mu(G) > 0$ for all nonempty open sets G in X. Then X is not a category measurable Baire space.

B. Example: Density Topology

For our discussion of a density topology we will take X to be a second countable; Baire metric space and μ a non-atomic (i.e. points have measure zero) completion of a regular measure on the Borel sets such that $\mu(G)>0$ for each nonempty open set G in X. We will also impose two additional conditions on X. Let

$$N(x,r) = \{y/\ d(x,y) < r\} \text{ and}$$
$$D(x,r) = \{y/\ d(x,y) \leq r\}$$

where d is the distance function for X. Then we will assume:

i) for all $\epsilon>0$, there exists $\delta>0$ such that $\mu(D(x,\delta)) \leq \epsilon$ for all $x \in X$,

ii) there exists K, a positive real number, such that $\mu(D(x,3n)) \leq K\,\mu(D(x,n))$ for all n, and for all $x \in X$.

Definition 6.11: A measurable set $E \subset X$ is said to have density d at x if

$$\lim_{h \to 0} \frac{\mu\{E \cap D(x,h)\}}{\mu\{D(x,h)\}}$$

exists and is equal to d. We will denote the set of points of X at which E has density 1 by $\phi(E)$.

Remark 6.12: Observe that the density (if it exists) of any measurable set at any point is a real number between zero and one.

Now we will prove a generalization of the Vitali covering theorem. My proof will make use of the techniques found on page 109 of [21].

Theorem 6.13 (Generalized Vitali Covering Theorem): Let $A \subset X$, and $\{D_\alpha\}$ be a family of closed cells covering A in the sense of Vitali, i.e., for every $x \in A$, x is contained in a D_α of arbitrarily small measure. Then there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of disjoint closed cells of $\{D_\alpha\}$ with

$$\mu(A - \bigcup_{m=1}^{\infty} D_m) = 0.$$

Proof Observe that μ is σ-finite since X is Lindelöf X is second countable and by assumption (i) above. Therefore we may assume that A and all D_α are contained in a set of finite measure. Also we may take this set of finite measure to be a closed cell, therefore we may assume the $\sup\{\, r / D_\alpha(x,r) \in \{D_\alpha\}\} < \infty$. Let D_1 be arbitrary and assume we have picked $D_1, \ldots, D_m$. If $A \subset D_1 \cup \ldots \cup D_m$ we are done. Otherwise there is a point of A not in this union, and since $\{D_\alpha\}$ is a Vitali covering of A, there is a member of $\{D_\alpha\}$ which is disjoint from the union. Define $d_m = \sup\{\mu(D_\alpha) / D_\alpha$ is disjoint from $\bigcup_{j=1}^{m} D_j\}$. Now choose $D_{m+1}(x_{m+1}, r_{m+1})$ to be any D_α disjoint from $\bigcup_{j=1}^{m} D_j$ and such that

$$\mu(D_{m+1}(x_{m+1}, r_{m+1})) \geq \tfrac{1}{2} d_m$$

and if $D_{\alpha_0}(y,s)$ is any other member of $\{D_\alpha\}$ disjoint from $\bigcup_{j=1}^{m} D_j$ with $\mu(D_{\alpha_0}(y,s)) \geq \frac{1}{2} d_m$, then $r \geq s$. We shall show that $\mu^*(A - \bigcup_{j=1}^{\infty} D_j) = 0$. If $D_m = D_m(x_m, r_m)$, then define

$$D'_n = D'_n\ (x_n,\ 3r_n).$$

Hence by assumption (ii) there exists a positive number L such that $\mu(D'_n) \leq L\,\mu(D_n)$ for all n. Since the D_j's are disjoint and contained in a set of finite measure we have $\sum_{j=1}^{\infty} \mu(D_j) < \infty$ hence $\sum_{j=1}^{\infty} \mu(D'_j) < \infty$.

Now if $\mu^*(A - \bigcup_{j=1}^{\infty} D_j) > 0$, then for some integer ν

$$\sum_{\nu+1}^{\infty} \mu(D'_j) < \mu^*(A - \bigcup_{j=1}^{\infty} D_j).$$

It follows that there is an $x \in A - \bigcup_{j=1}^{\infty} D_j$, but not in $\bigcup_{\nu+1}^{\infty} D'_j$. Since $x \notin \bigcup_{i=1}^{\nu} D_i$ the Vitali covering property implies that there is a $D \in \{D_\alpha\}$ with $x \in D$ and $D \cap (\bigcup_{i=1}^{\nu} D_i) = \emptyset$. Now suppose D were disjoint from $D_1 \cup \ldots \cup D_n$. Then $d_n \geq \mu(D)$, and therefore $\mu(D_{n+1}) \geq \frac{1}{2}\mu(D)$. Thus if D were disjoint from all the D_n, we would have

$$\mu(D_n) \geq \tfrac{1}{2}\,\mu(D) > 0$$

for all n $[\![\mu(D) > 0$ since open sets have positive measure$]\!]$. and so $\sum_{j=1}^{\infty} \mu(D_j) = \infty$, a contradiction. Thus D is disjoint from $\bigcup_{j=1}^{\nu} D_j$, but is not disjoint from $\bigcup_{j=1}^{\infty} D_j$. Let n_0 be the first index with $D \cap D_{n_0} \neq \emptyset$. Of course $n_0 > \nu$. By our choice of D_{n_0}, $\mu(D_{n_0}) \geq \frac{1}{2} d_{n_0-1}$. But $D \cap D_n = \emptyset$ for $n = 1, 2, \ldots, n_0 - 1$, so $d_{n_0-1} \geq \mu(D)$; therefore the radius of $D \leq$ the radius of D_{n_0}. But by the definition of $\{D_n\}$ and $\{D'_n\}$ it follows that $D'_{n_0} \supset D$ and hence $x \in D'_{n_0}$. Since $n_0 > \nu$ and we had assumed that $x \notin \bigcup_{\nu+1}^{\infty} D'_i$ we have a contradiction.

The following theorems are generalizations of theorems in [10].

Theorem 6.14 (Lebesgue Density Theorem): For any measurable set $E \subset X$,

$$\mu(E \,\Delta\, \phi(E)) = 0$$

Proof It is sufficient to show that $E - \phi(E)$ is a nullset, since

$$\phi(E) - E \subset E' - \phi(E')$$

and E' is measurable. Since μ is σ-finite we may assume that $\mu(E) < \infty$. Furthermore,

$$E - \phi(E) = \bigcup_{m=1}^{\infty} A_m$$

where $A_m = \left\{ x \in E \,/\, \liminf_{h \to 0} \frac{\mu\{E \cap D(x,h)\}}{\mu\{D(x,h)\}} < 1 - \frac{1}{m} \right\}$.

Hence it is sufficient to show that A_m is a nullset for every positive integer m. Putting $A = A_m$ we shall obtain a contradiction from the supposition that $\mu^*(A) > 0$. (Here μ^* stands for the outer measure generated by μ as defined in Chapter 2).

If $\mu^*(A) > 0$, there exists an open set G of finite measure containing A such that $\mu(G) < \frac{\mu^*(A)}{1 - \frac{1}{m}}$. Let $\mathcal{E}$ denote the class of all closed cells $D(x,r)$ such that $D(x,r) \subset G$, and

$$\mu\{E \cap D(x,r)\} \leq (1 - \tfrac{1}{m})\, \mu\{D(x,r)\}.$$

Observe that (i) $\mathcal{E}$ includes closed cells of arbitrarily small measure about each point of A, and (ii) for any disjoint sequence $\{D_n(x_n, r_n)\}$ of members of $\mathcal{E}$, we have

$$\mu^*\{A - \cup D_n(x_n, r_n)\} > 0.$$

Property (ii) follows from the fact that

$$\mu^*\{A \cap \bigcup_{n=1}^{\infty} D_n(x_n, r_n)\} \leq \sum_{n=1}^{\infty} \mu\{E \cap D_n(x_n, r_n)\}$$
$$\leq (1 - \tfrac{1}{m}) \sum \mu\{D_n(x_n, r_n)\} \leq (1 - \tfrac{1}{m})\mu(G) < \mu^*(A).$$

But property (ii) contradicts Theorem 6.13.

Let us write $A \sim B$ when $\mu(A \triangle B) = 0$. This is an equivalence relation in the class $\mathcal{S}$ of measurable sets in X. The following theorems states that the mapping $\phi : \mathcal{S} \to \mathcal{S}$ may be regarded as a function that selects one member from each equivalence class. Moreover, it does so in such a way that the selected sets constitute a class that includes the empty set, the whole space, and is closed under intersection.

Theorem 6.15: For any measurable set A, let $\phi(A)$ denote the set of points of X where A has density 1. Then ϕ has the following properties where $A \sim B$ means that $A \triangle B$ is a nullset:

1) $\phi(A) \sim A$,

2) $A \sim B$ implies $\phi(A) = \phi(B)$,

3) $\phi(\emptyset) = \emptyset$ and $\phi(X) = X$,

4) $\phi(A \cap B) = \phi(A) \cap \phi(B)$,

5) $A \subset B$ implies $\phi(A) \subset \phi(B)$.

Proof The first assertion is just Theorem 6.14. The second and third are immediate consequences of the definition of ϕ. To prove 4), note that for any closed cell D we have

$$D - (A \cap B) = (D - A) \cup (D - B).$$

Hence $\mu(D) - \mu(D \cap A \cap B) \leq \mu(D) - \mu(D \cap A) + \mu(D) - \mu(D \cap B)$.

Therefore $$\frac{\mu(D \cap A)}{\mu(D)} + \frac{\mu(D \cap B)}{\mu(D)} - 1 \leq \frac{\mu(D \cap A \cap B)}{\mu(D)}.$$

Taking $D = D(x,h)$ and letting $h \to 0$ it follows that $\phi(A) \cap \phi(B) \subset \phi(A \cap B)$. The opposite inclusion is obvious. Property 5) is a consequence of 4).

Definition 6.16: Let $\mathcal{N}$ be the class of μ-nullsets in X and let $\mathcal{S}$ be the σ-algebra of measurable sets in X, and for every $E \in \mathcal{S}$ let $\phi(E)$ be the set of points of X at which E has density 1, then we define.

$$\mathcal{T} = \{\phi(A) - N / A \in \mathcal{S},\ N \in \mathcal{N}\}.$$

Theorem 6.17: $\mathcal{T}$ is a topology in X.

Proof Let Θ denote the nullset. Since $\Theta \in \mathcal{N}$ property 3) of Theorem 6.15 implies that $X = \phi(X) - \Theta$ and $\Theta = \phi(\Theta) - \Theta$ both belong to $\mathcal{T}$. By 4) in Theorem 6.5 we have

$$[\phi(A_1) - N_1] \cap [\phi(A_2) - N_2] = \phi[A_1 \cap A_2] - [N_1 \cup N_2].$$

Hence $\mathcal{T}$ is closed under finite intersections. To show that $\mathcal{T}$ is closed under arbitrary union, let

$$\mathcal{F} = \{\phi(A_\alpha) - N_\alpha : \alpha \in \Gamma\},\ A_\alpha \in \mathcal{S},\ N_\alpha \in \mathcal{N},$$

be any subfamily of $\mathcal{T}$. Let b denote the least upper bound of the measures of finite unions of members of $\mathcal{F}$, and choose a sequence $\{\alpha_m\}$ such that $\mu(\bigcup_{m=1}^{\infty} A_{\alpha_m}) = b$. (Note: b may be equal to ∞). Put $A = \bigcup_{m=1}^{\infty} A_{\alpha_m}$. Then $A \in \mathcal{S}$ and the definition of b implies that $A_\alpha - A \in \mathcal{N}$ for every $\alpha \in \Gamma$. Since $A_\alpha - (A_\alpha - A) \subset A$, it follows from 2) and 5) of Theorem 6.15 that

$$\phi(A_\alpha) \subset \phi(A) \quad \text{for every } \alpha.$$

Putting $\quad N_0 = \bigcup_{m=1}^{\infty} [N_{\alpha_m} \cup (A_{\alpha_m} - \phi(A_{\alpha_m}))]$, we have $N_0 \in \mathcal{N}$

and $\quad A - N_0 \subset \bigcup_{m=1}^{\infty} [\phi(A_{\alpha_m}) - N_{\alpha_m}] \subset \bigcup_{\alpha \in \Gamma} [\phi(A_\alpha) - N_\alpha] \subset \phi(A).$

The extremes differ by a nullset, and therefore

$$\bigcup_{\alpha \in \Gamma} [\phi(A_\alpha) - N_\alpha] = \phi(A) - N$$

for some $N \in \mathcal{N}$, by the completeness of μ.

Definition 6.18: The topology $\mathcal{T}$ will be called the density topology for X. We will now look at some of its properties.

Theorem 6.19: A set $N \subset X$ is nowhere dense relative to if and only if $N \in \mathcal{N}$. Every nowhere dense set is closed.

Proof If $N \in \mathcal{N}$, then $X-N = \phi(X)-N \in \mathcal{T}$, hence each member of $\mathcal{N}$ is closed. If $N \in \mathcal{N}$ and $\phi(A_1) - N_1 \subset N$ for some $A_1 \in \mathcal{S}'$ and $N_1 \in \mathcal{N}$, then $\phi(A_1) \in \mathcal{N}$ and so $\phi(A_1) = \emptyset$ by 2) and 3) in Theorem 6.15. Hence $\phi(A_1)-N_1 = \emptyset$, and therefore N is nowhere dense. Conversely, if F is closed and nowhere dense, then $X-F = \phi(A)-N$ for some $A \in \mathcal{S}'$ and $N \in \mathcal{N}$, hence F belongs to $\mathcal{S}'$. Since

$$F \supset \phi(F) - \left[\phi(F) - F\right] \in \mathcal{T},$$

the nowhere denseness of F implies that $\phi(F) \subset \phi(F) - F$. Hence $\phi(F) = \emptyset$, by 1), 2), and 3) of Theorem 6.15. Therefore $F \sim \emptyset$, that is $F \in \mathcal{N}$. Thus, $\mathcal{N}$ is identical with the class of closed nowhere dense sets. Since every nowhere dense set is contained in a closed nowhere dense set, and every subset of a member of $\mathcal{N}$ belongs to $\mathcal{N}$, it follows that every nowhere dense set is closed.

Theorem 6.20: A set $A \subset X$ has the property of Baire if and only if $A \in \mathcal{S}'$.

Proof If $A \in \mathcal{S}'$, then $A = \phi(A) \triangle (\phi(A) \triangle A)$. Since $\phi(A) \in \mathcal{T}$, and $\phi(A) \triangle A \in \mathcal{N}$, it follows from Theorem 6.9 that A has the property of Baire. Conversely, if A has the property of Baire, then $A = \left[\phi B)-N\right] \triangle M$ for some $B \in \mathcal{S}'$, some $N \in \mathcal{N}$, and some set M of first category. By Theorem 6.19, M belongs to

$\mathcal{N}$, and therefore $A \in \mathcal{S}'$.

Definition 6.21: Regular open set is a set that is equal to the interior of its closure. Any set of the form $\bar{A}'^{-\prime}$ is regular open where $\bar{A}$ and A' represents the closure and complement of A respectively.

Theorem 6.22: A set $G \subset X$ is regular open if and only if $G = \phi(A)$ for some $A \in \mathcal{S}'$.

Proof If $A \in \mathcal{S}'$, then $\phi(A)$ is open, and the closure of $\phi(A)$ is of the form $\phi(A) \cup N$ for some $N \in \mathcal{N}$, by Theorem 6.19. Let $\phi(A_1) - N_1$ be any open subset of $\phi(A) \cup N$. Then

$$\phi(A_1) - N_1 \subset \phi(A_1) = \phi(\phi(A_1) - N_1) \subset \phi(\phi(A) \cup N) \subset \phi(A).$$

Thus $\phi(A)$ is the largest open subset of $\phi(A) \cup N$. This shows that $\phi(A)$ is equal to the interior of its closure, that is, $\phi(A)$ is regular open. Conversely, if G is regular open, then $G = \phi(A) - \bar{N}$ for some $A \in \mathcal{S}'$ and $N \in \mathcal{N}$. Since $\phi(A) \triangle [\phi(A) - \bar{N}]$ is contained in $\bar{N}$, we have $\phi(A) \sim [\phi(A) - \bar{N}] = G$. Since G and $\phi(A)$ differ by a nowhere dense set, and both are regular open, it follows that $G = \phi(A)$.

Theorem 6.23: $\mathcal{T}$ is a Hausdorff topology.

Proof Observe that $\mathcal{T}$ consists of all measurable sets A such that A has density 1 at each of its points. Hence $\mathcal{T}$ includes all sets that are open in the ordinary topology, consequently it is Hausdorff.

Theorem 6.24: The density topology in X is regular.

Proof Let x be a point of a set $A \in \mathcal{T}$. Then A has density 1 at x. For each positive integer n, let F_n be an ordinary-closed subset of $D(x, \frac{1}{n}) \cap A$ such that

$$\mu(F_n) > \left\{1 - \mu\left(D(x, \tfrac{1}{n})\right)\right\} \mu\left[D(x, \tfrac{1}{n}) \cap A\right].$$

If $F = \{x\} \cup \left[\bigcup_{n=1}^{\infty} F_n\right]$, then $\phi(F) \subset F \subset A$. Since A has density 1 at x,

$$\frac{\mu\{D(x, \frac{1}{n}) \cap F\}}{\mu\{D(x, \frac{1}{n})\}} \geq \frac{\mu\{F_n\}}{\mu\{D(x, \frac{1}{n})\}} \rightarrow 1.$$

Therefore F has density 1 at x, and so $x \in \phi(F)$. Thus $\phi(F)$ is a $\mathcal{T}$-neighborhood of x whose $\mathcal{T}$-closure is contained in F, and therefore in A.

Remark 6.25: Hence μ is a category measure when restricted to any open cell of finite measure. Relative to this density topology, the extended principle is valid, and it is no longer possible to decompose X into a nullset and a set of first category [[Lemma 1]].

There are two more aspects of density topology which I wish to consider. First I would like to consider the class of approximately continuous functions on X. Secondly I will show that X with its density topology is not normal, hence it is not metrizable [[See Theorem 4.3.3 in [20]]].

Definition 6.26: A function f from X to a topological space is said to be approximately continuous at a point p if, for every open set G containing f(p), the set $f^{-1}(G)$ has metric density 1 at p.

Theorem 6.27: The set of real-valued functions which are continuous in the density topology is precisely the set of approximately continuous functions on X (with respect to its original topology.)

Proof Clear.

Remark 6.28: Nishiura has shown [6] that the density topology for Euclidean m space, E_m, is the coarsest topology for which its approximately continuous real-valued functions are continuous.

This is done by showing, that the density topology for E_m is completely regular, and hence coincides with the weak topology induced by its real-valued continuous functions [[See page 115 of [20]]].

I do not know whether the more general density topology defined above is completely regular, nor whether Nishiura's result is valid here.

I want to now show that the density topology for X is not normal. In order to do this we will first need to state some definitions and to prove several lemmas.

Definition 6.29: Let X, Y be any topological space. Then the family $\mathcal{B}F(X)$ of Baire functions is the smallest family of functions $f: X \rightarrow Y$ that contains all continuous functions and all pointwise limits of pointwise convergent sequences of functions of $\mathcal{B}F(X)$. A function is said to be of Baire class 1 if it is the limit function of a sequence of continuous functions.

The following theorem is found in Henstock's book Linear Analysis on page 89 [8].

Theorem 6.30: If $(X_1, \mathcal{T}_1)$ is a Baire space, $(X_2, \mathcal{T}_2)$ is a pseudometric space, and $f \in \mathcal{BA}(X_1)$, there is a $Z \subset X_1$ of the first category in $\mathcal{T}_1$, such that f is continuous in $X_1 - Z$.

The following corollary is a special case of the above theorem.

Corollary 6.31: Let X be defined as usual. Then if f is a real-valued function on X of Baire class 1, then f is continuous except at a set of points of first category.

Definition 6.32: Let f be any real-valued function on X. For any open cell N(x,r) in X, the quantity

$$\omega(N(x,r)) = \sup_{y \in N(x,r)} f(y) - \inf_{y \in N(x,r)} f(y)$$

is called the oscillation of f on N(x,r). For any fixed x, the function $\omega(N(x,r))$ decreases with r and approaches a limit

$$\omega(x) = \lim_{r \to 0} \omega(N(x,r)),$$

called the oscillation of f at x.

Remark 6.33: Observe that $\omega(x)$ is an extended real-valued function on X such that f is continuous at $x \in X$ if and only if $\omega(x) = 0$.

The following lemma is a generalization of a result of Goffman and Waterman [5].

Lemma 6.34: An approximately continuous function f from X to a metric space is of Baire class 1.

Proof Suppose f is not of Baire class 1. Then there is a nonempty perfect set R such that, at every point of R, f is discontinuous relative to R. R is a Baire space. For every m, let R_m be the subset of R at which the oscillation of f is not less than $\frac{1}{m}$. At least one R_m contains an open cell K in R. Then the oscillation of f relative to K is not less than $2\alpha = \frac{1}{m}$ at every point of K. Since the closure of any open subspace of K is perfect, the space K is of the second category.

Cover f(X) by a countable set of open cells of radius $\alpha/3$. The intersection of f(X) with one of these, having center γ, has inverse T_1 dense in a perfect subset $P \subset K$. Let T_2 be the set of $p \in X$ for which $d(f(p), \gamma) > \frac{2\alpha}{3}$. Then T_2 is also dense in P since the oscillation of f relative to P is not less than 2α at every point of P and T_1 is dense in P.

Let $\{p_n\}$ be a countable dense subset of T_1 which is dense in P and let $\{\epsilon_n\}$ be any sequence of positive numbers converging to zero. Since T_1 has density one at each of its points, there is a sequence of open cells, $\{N_n\}$, $p_n \in N_n$, $\lim_{n \to \infty} \mu(N_n) = 0$, such that the relative measure of T_1 in Q_n exceeds $1 - \epsilon_n$. The set V_1 of points belonging to infinitely many Q_n is residual relative to P. In the same fashion we can construct another residual set V_2 corresponding to T_2.

The set $V = V_1 \cap V_2$ is residual relative to P. The upper metric densities (here the upper metric density of a measurable set $S \subset X$ at a point p is $\lim_{n \to \infty} \sup_{\ell(p,r)} \left[\frac{\mu(S \cap N(p,r))}{\mu(N(p,r))} \Big| \mu(N(p,r)) < \frac{1}{n} \right]$) of T_1 and T_2 are equal to one at every point of V. Thus for any $p \in V$ we have simultaneously

$$d(f(p), \gamma) \geq \frac{2\alpha}{3} \text{ and } d(f(p), \gamma) \leq \alpha/3,$$

and so V is empty. This contradiction establishes our result.

We now prove that X with the density topology is not normal. Our proof will be a generalization of the one found in [6].

Theorem 6.35: X with the density topology is not a normal topological space.

Proof Let A, B be disjoint subsets of X which are dense in the usual topology of X, and such that $\mu(A) = \mu(B) = 0$ [[2nd count.]]. Then A, B are closed in the density topology for X. Suppose the density topology for X is normal. Then there is $f \in F_0$ such that $f: X \to [0,1]$ and $f(A) = 0$, $f(B) = 1$. But then f is discontinuous everywhere (in the usual topology for X). But this contradicts that f is of Baire class 1.

C. Example: Boolean Measure Spaces and Normal Measures on Stonian Spaces

I now show that a Boolean space gives rise to a category measure space when its dual algebra is a measure algebra. The following results are found in [7].

Definition 6.36: A Boolean space is a totally disconnected (i.e. the closed-open sets constitute a base) compact Hausdorff space.

Definition 6.37: The algebra of all closed-open sets in a Boolean space X is called the dual algebra of X.

Definition 6.38: A Boolean space is a Boolean σ-space if the closure of every open Baire set is open.

Definition 6.39: A Boolean measure space is a Boolean σ-space X together with a normalized measure ($\mu(X) = 1$) on the σ-algebra of Borel sets in X, such that non-empty open sets have positive measure and nowhere dense Borel sets have measure zero.

Remark 6.40: The completion of a Boolean measure space is a category measure space.

Definition 6.41: A measure algebra is a Boolean σ-algebra A together with a positive (i.e. the empty set is the only element at which μ takes the value zero), normalized measure μ on A.

Lemma 6.42: Every measure algebra is complete (i.e. every subset has a supremum).

Proof See page 67 of [7].

Lemma 6.43: The dual algebra A of a Boolean space X is complete if and only if X is complete.

Proof See page 92 of [7].

Lemma 6.44: Let f be a Boolean σ-epimorphism from a σ-algebra B to a σ-algebra A, and let μ be a normalized measure on A. If $\nu(q) = \mu(f(q))$ for every q in B, then ν is a normalized measure on B. The kernel of f is included in the set of all those elements q of B for which $\nu(q) = 0$; the kernel coincides with that set if and only if the measure μ is positive.

Proof The proofs of all the assertions of the lemma are immediate from the definitions.

Lemma 6.45: If μ is a positive, normalized measure on A, then f maps B onto A. If $\nu(S) = \mu(f(S))$ for every S in B, then ν

is a normalized measure on B such that non-empty open sets have positive measure and such that the sets of measure zero are exactly the sets of first category.

Proof The algebra A together with the measure μ is a measure algebra, and therefore complete by Lemma 6.42 It follows that the space X is complete by Lemma 6.43, and hence that every regular open set in X is both closed and open. This proves the first sentence of the lemma. The second sentence is an immediate consequence of Lemma 6.44.

Our main theorem is an immediate corollary of Lemma 6.45.

Theorem 6.46: The dual algebra A of a Boolean space X is a measure algebra if and only if X is a Boolean measure space.

We will now consider another method for constructing category measure spaces. Then I will show that this method and Boolean measure spaces are really just different views of the same concept.

I will prove a category result whose measure-theoretic analog follows trivially as a corollary. The following definitions, remark, and three theorems are found in [2] .

Definition 6.47: A compact Hausdorff space X is called stonian (or extremely disconnected) if disjoint open sets in X have disjoint closures.

Remark 6.48: X is stonian if and only if U open implies $\overline{U}$ is also open.

Proof Let X be stonian and U be open. Then the disjoint open sets U and $(\overline{U})'$ have disjoint closures. But the equalities

$$\overline{U} \cap (\overline{U})' = \emptyset \text{ and } \overline{U} \cap \overline{(\overline{U})'} = \emptyset$$

implies $(\overline{U})' = \overline{(\overline{U})'}$. Hence $\overline{U}$ is open. Conversely assume that open sets have open closures. Let U_1, U_2 be open and disjoint. Since U_1 is open $U_1 \cap \overline{U_2} = \emptyset$. But by assumption $\overline{U_2}$ is open, hence $\overline{U_1} \cap \overline{U_2} = \emptyset$.

Definition 6.49: A family $\{f_\alpha\}$ of functions from $C(X)$ where X is stonian is said to be *bounded above* if there exists $f_0 \in C(X)$ such that $f_\alpha \leq f_0$ for all α. We call f_0 an *upper bound* for the family. If $f_0 \leq g_0$ whenever g_0 is an upper bound we call f_0 the *least upper bound* and write $f_0 = \vee f$. One defines bounded below and greatest lower bound similarly. The lattice C(X) is said to be complete if every family of functions which is bounded above has a least upper bound. An equivalent definition could of course be given in terms of lower bounds.

The following theorem is due to M.H. Stone [17].

Theorem 6.50: Let X be a compact Hausdorff space. Then X is stonian if and only if the space C(X) of all real-valued continuous functions is a complete lattice.

Definition 6.51: A regular measure μ on a stonian space X is *normal* if for each bounded monotone increasing net $\{f_\alpha\}$ of real-valued functions in C(X) we have

$$\lim_\alpha \int_X f_\alpha \, d\mu = \int_X f_0 \, d\mu \qquad \text{where } f_0 = \vee f_\alpha .$$

Definition 6.52: The support of a measure μ is the complement of the largest open set of μ-measure zero. I will denote the support of μ by supp(μ).

Theorem 6.53: A regular measure μ is normal if and only if it vanishes on all nowhere dense Borel sets.

Proof See [2].

Theorem 6.54: The support of a normal measure is both open and closed.

Proof Let F be the support of μ and U = interior of F. Since F is closed $\overline{U} \subset F$ and since X is stonian $\overline{U}$ is open. Hence $\overline{U} \subset U$, so $U = \overline{U}$. Now $F - U$ is nowhere dense, so by Theorem 6.53

$$\mu(X - U) = \mu(X - F) + \mu(F - U) = 0$$

Thus $F = U$ by the definition of the support.

Corollary 6.55: If μ is a normal measure and supp (μ) = X then $\mu(A) = 0$ if and only if A is nowhere dense.

Proof Follows directly from Theorem 6.53 and Theorem 6.54.

Remark 6.56: The completion of finite normal measure μ on a stonian space with supp(μ) = X is a category measure space.

Theorem 6.57: Let X be a stonian space. If f is a bounded Borel measurable function on X, then there exists a unique continuous function g such that

$$\{ x : |f(x) - g(x)| > 0 \}$$

is of first category.

Proof See page 104 of [2].

The following is the measure-theoretic analog of the above theorem.

Corollary 6.58: Let X be a stonian space and μ a normal measure on the Borel sets of X. If f is a bounded Borel measurable function on X, then there exists a unique continuous function g such that

$$\mu(\{x: |f(x) - g(x)| > 0\}) = 0.$$

Proof By the above theorem, there exists a unique continuous function g such that $\{x: |f(x) - g(x)| > 0\}$ is of first category. But then this set is of measure zero by Theorem 6.53.

In Halmos's book Lectures on Boolean Algebras he discusses the properties of a Boolean measure space. Whereas in W. G. Bode's book The Banach Space $C(S)$, Bade discusses the properties of a stonian space with a normal measure μ on its Borel sets. If in addition $\mu(X) = 1$ and supp$(\mu) = X$, then both of these spaces yield category measure spaces if we consider their unique completions [[[13] Prop. 12.4 , and [10] Theorem 4.3]]. We will compare these two approaches (for constructing category measure spaces) and show that they are the same. First of all observe that an extremely disconnected Hausdorff space is totally disconnected [[#104 in Sec. 14.1 [20]]]. Also a Boolean measure space is extremely disconnected [[Lemma 6.42, Lemma 6.43]]. Therefore a Boolean measure space is topologically the same as a stonian space.

We will now show that a Boolean measure space and a stonian space with a normal measure on its Borel sets such that $\text{supp}(\mu) = X$ and $\mu(X) = 1$ are equivalent concepts. Let μ be the measure in the definition of a Boolean measure space X, then μ is a normal measure [[Theorem 6.53]]. The fact that $\mu(G) > 0$ for all nonempty open sets G in X implies that $\text{supp}(\mu) = X$ and by definition $\mu(X) = 1$.

Conversely assume X is a stonian space and μ is a normal measure on the Borel sets of X such that $\text{supp}(\mu) = X$ and $\mu(X) = 1$. Then $\mu(A) = 0$ if and only if A is a nowhere dense Borel set [[Thm. 6.53, Cor. 6.55]]. Hence $\mu(G) > 0$ for all nonempty open sets G in X. Therefore the concepts of a Boolean measure space and a stonian space with a normal measure μ such that $\mu(X) = 1$ and $\text{supp}(\mu) = X$ are really the same.

I will now show that Stonian spaces with normal measures such that the support of the measure is the whole space (i.e. a Boolean measure space) arise from L^∞ of any finite measure space.

<u>Theorem 6.59</u>: Let $(X, \mathcal{S}, \mu)$ be any finite measure space, then there exists a category measure space $(\Omega, \mathcal{M}, \nu)$ such that Ω is a compact Hausdorff space and

$$L^\infty(X, \mathcal{S}, \mu) \cong C(\Omega)$$

where $\cong$ is a isometric $*$-isomorphism. (See [3] for the definitions concerning B*-algebras.)

<u>Proof</u> $L^\infty(X, \mathcal{S}, \mu) \cong C(\Omega)$ where Ω is a compact Hausdorff space by the Gelfand-Naimark Theorem. (See [3]). We will first show that Ω is stonian. Let $\{f_\alpha\}$ be a bounded monotone net in $L^\infty(X, \mathcal{S}, \mu)$, then $\{f_\alpha\}$ has a weak* convergent

subnet [[immediate corollary to Banach-Alaoglu Theorem (see [13]) and by Problem 211 on page 133 in [20]]]. But this implies that $\{f_\alpha\}$ is weak* convergent [[since $\{f_\alpha\}$ is monotone]], i.e. $\{f_\alpha\} \to f \in L^\infty(X, \mathcal{S}, \mu)$. However it follows easily [[since μ is finite]] that $f = \vee f_\alpha$. Therefore Λ is Stonian [[by Theorem 6.50 and since $L^\infty(X, \mathcal{S}, \mu) \cong C(\Lambda)$ is an order-preserving isomorphism]]. If $f \in L^\infty(X, \mathcal{S}, \mu)$, let f' be the image of f under the above isomorphism. Then $I(f') = \int f d\mu$ is a positive linear functional on $C(\Lambda)$. Therefore there exists a unique finite regular measure $\hat{\mu}$ on the Borel sets, $\mathcal{B}$ of Λ such that $I(f') = \int f' d\hat{\mu}$ for all $f' \in C(\Lambda)$ [[See the Riesz Representation Theorem on page 182 of [1]]]. Observe that $f_\alpha \to f$ in the weak* topology of $L^\infty(X, \mathcal{S}, \mu)$ implies $\int f_\alpha g \, d\mu \to \int fg \, d\mu$ for all $g \in L^\infty(X, \mathcal{S}, \mu)$ [[by Riesz Representation Theorem in Chapter 11 of [13]]]. Now since $1 \in L^1(X, \mathcal{S}, \mu)$ we have $\int f_\alpha d\mu \to \int f d\mu$. Therefore $\hat{\mu}$ is a normal measure [[since $L^\infty(X, \mathcal{S}, \mu) \cong C(\Lambda)$]]. I claim that $\text{supp}(\hat{\mu}) = \Lambda$. [[$\text{supp}(\hat{\mu}) = \Lambda$ if and only if for all $f' \in C(\Lambda)$, $f' \geq 0$, $f' \neq 0$, $\int f' d\hat{\mu} > 0$. But if $\int f' d\hat{\mu} = 0$, i.e. $\text{supp}(\hat{\mu}) \neq \Lambda$, then $f = 0$.]]

Therefore if we let $(\Lambda, \mathcal{Q}, \nu)$ be the unique completion of $(\Lambda, \mathcal{B}, \hat{\mu})$ [[see Chapter 11 of [13]]] then $(\Lambda, \mathcal{Q}, \nu)$ is a category measure space.

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