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# BOOLEAN VALUES FOR FUZZY SETS

by

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A Thesis

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in

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This thesis is accepted and approved in partial fulfillment of the requirements for the degree of Master of Science.

May 6, 1976
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## ABSTRACT

In the past decade there has been some exploration in using truth values other than the usual {T,F} or {0,1} values. In particular this was done by Dana Scott constructing Boolean Valued Models of Set Theory, Richard Mansfield resulting in Boolean Ultrapowers, and L. A. Zadeh developing Fuzzy Sets.

Boolean Valued Models for Set Theory generalizes the Boolean algebra of {0,1} to any complete Boolean algebra. Boolean ultrapowers investigate the generalization of the Boolean algebra $2^I$ to an arbitrary complete Boolean algebra. Fuzzy sets take values in the interval [0,1] instead of the usual {0,1} values.

In this paper we briefly look at each of these three concepts, and then compare the three, keeping in mind the possibility of using values of an arbitrary Boolean algebra for Fuzzy Sets instead of the interval [0,1].

# CHAPTER 1

## BOOLEAN VALUED MODELS OF SET THEORY

When we formulate the axioms of set theory, we have in mind an intuitive idea of "sets". Mathematicians have discovered that certain objects and relations (i.e. point, line, betweenness, group, set, etc.) are best regarded as undefined notions with specified properties.

These properties or axioms should be specified in such a manner that no paradoxes or inconsistencies result. A theory is inconsistent if it contains a formula $\Phi$ such that both $\Phi$ and $\sim\Phi$ are theorems. If the theory contains no formula $\Phi$ such that both $\Phi$ and $\sim\Phi$ are theorems, it is consistent. Once a consistent set of axioms is written down, we can freely study any structure for which the axioms hold. This usually includes a much broader scope of structures than the original notions for which the axioms were specified.

Axiomatic set theory is incomplete; that is, there are some statements in set theory which can neither be proved nor disproved. In particular Godel's Incompleteness Theorem implies that using the axioms of set theory, we cannot prove that set theory is inconsistent.

There are certain properties which we feel are true in our intuitive idea of set theory but which are not theorems of the formal set theory. To make the theory more like our intuitive concept, we add more axioms to those which we already have. Naturally, the new axioms must be consistent with the old ones.

When we add an additional axiom, A, to the axioms we started with, we want to know if the theory now says more than it did before. Do any new theorems result? In other words, is A independent of the other axioms? One way to show A is independent is to find a model satisfying the original axioms and the negation of A; that is, to find an interpretation in which the original axioms and $\sim A$ are true and which is consistent. In logic this is known as the Completeness Theorem: A set of axions $\Gamma$ is consistent if and only if $\Gamma$ has a model.

We will consider models of set theory in which the predicate $\varepsilon$ will take values in any Boolean algebra, instead of the 2-valued Boolean algebra, $\{0,1\}$. The use of these Boolean-valued models for set theory is in constructing independence proofs.

We now attempt to give the reader a brief background. We first develop the idea of a Boolean algebra. A set X is a partially ordered set if it satisfies the following properties:

a) Reflexivity-- for all $x \varepsilon X$, $x \leq x$.
b) Antisymmetry-- for all x, $y \varepsilon X$, if $x \leq y$ and $y \leq x$, then $x = y$.
c) Transitivity-- for all x, y, $z \varepsilon X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

A Lattice is a partially ordered set in which each two element subset, $\{x,y\}$, has both a supremum denoted $x \vee y$ and an infimum denoted $x \wedge y$. For any lattice the following identities hold:

a) $x \vee y = y \vee x$ $\qquad$ $x \wedge y = y \wedge x$
b) $x \vee (y \vee z) = (x \vee y) \vee z$ $\qquad$ $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
c) $(x \vee y) \wedge y = y$ $\qquad$ $(x \wedge y) \vee y = y$

A lattice L is complemented if it has a maximum element, 1, a minimum element, 0, and if for each $x \in L$ there is an element $y \in L$ satisfying: $x \vee y = 1$ and $x \wedge y = 0$. A lattice L is distributive if for all x, y, $z \in L$ the following identities hold:

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$
$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$$

A Boolean algebra is a complemented distributive lattice with at least two elements.

We also want our Boolean algebra to be complete, that is, each set of any number of elements has both an infimum and a supremum. We desire this so that we can find values of statements involving quantifiers, such as $\exists x \Phi x$. To avoid confusing symbols of Boolean algebra with logical symbols, we will use the following notation for Boolean algebra:

- $+$ for $\vee$, the supremum of two elements.
- $\cdot$ for $\wedge$, the infimum of two elements.
- $\Sigma$ for the supremum of more than two elements.
- $\Pi$ for the infimum of more than two elements,

A model is an interpretation of the predicates, constants, and free variables of a language which results in each sentence being given a value. The values given are usually either 0 (false) or 1 (true). For example, consider the sentence $\sigma$ , which is $\forall y_o\ M(v_o)$. If we interpret M as the predicate "is an Asian," then $\sigma$ is true if we interpret the variables as ranging over all Vietnamese, and false if we interpret the variables as ranging over all physicists. However, if we interpret M to be "is a scientist," then $\sigma$ is true if we interpret the variables as ranging over all physicists, and false if we interpret the variables as ranging over all Vietnamese.

A model of Set Theory is an interpretation of constants, variables, and the predicate $\varepsilon$ in such a way that the sentences which are axioms of set theory are true. In our model a true sentence has the value 1 in the Boolean algebra.

Set theory starts with two concepts:

A Universe $\mathfrak{U}$ ----a structure satisfying the axioms of set theory. $\mathfrak{U}$ is basically a collection of objects called sets.

A single binary relation $\varepsilon$ (called a membership relation).

The axioms of set theory (the properties we want $\varepsilon$ and = to have) are:

Extensionality---no two distinct sets in $\mathfrak{U}$ have the same elements.
$\forall x \forall y \; [\forall z (z \varepsilon x \leftrightarrow z \varepsilon y) \rightarrow x = y]$

Union------------to every set A there corresponds a set B whose members are precisely the members of the members of A.
$\forall x \exists y \forall z [z \varepsilon y \leftrightarrow \exists t (t \varepsilon x \wedge z \varepsilon t)]$

Replacement------suppose a formula $E(x, y, a_0, \ldots, a_{k-1})$ defines a singulary functional relation with parameters $a_0, \ldots, a_{k-1}$. Let A be any set. The universe $\mathfrak{U}$ contains a set B whose elements are the images of this functional relation for those elements of A in its domain.
$\forall x_0 \ldots \forall x_{k-1} [\forall x \forall y \forall y' [E(x, y, x_0, \ldots, x_{k-1}) \wedge E(x, y', x_0, \ldots, x_{k-1}) \rightarrow y = y'] \rightarrow \forall t \exists w \forall v [v \varepsilon w \leftrightarrow \exists u [u \varepsilon t \wedge E(u, v, x_0, \ldots, x_{k-1})]]]$

Power Set--------given a set A there exists a set B whose members are just A's subsets.
$\forall x \exists y \forall z [z \varepsilon y \leftrightarrow z \subseteq x]$

Infinity---------there is an ordinal which is not finite.
$\exists x [0 \varepsilon x \wedge \forall y (y \varepsilon x \rightarrow y \cup \{y\} \varepsilon x)]$

Foundation---------every non-empty set has an element which is disjoint from it.

$\forall x[x\neq\emptyset\rightarrow\exists y(y\varepsilon x\wedge y\cap x=\emptyset]$

Choice------------If a is a set of pairwise disjoint but non-empty sets, then there is a set whose intersections with elements of a are always singletons.

$\forall a\ [[\forall x(x\varepsilon a\rightarrow x\neq\emptyset)\ \wedge\ \forall x\forall y(x\varepsilon a\wedge y\varepsilon a\rightarrow (x=y\ \forall x\cap y=\emptyset))]\rightarrow\exists b\forall x\exists u(x\varepsilon a\rightarrow b\cap x=\{u\}\ )]$

Shortly, we will proceed to define a universe $V^{(\beta)}$ and predicates $\varepsilon$ and $=$. $V^{(\beta)}$, $\varepsilon$, and $=$ will be defined in such a way that for any two objects of $V^{(\beta)}$, a and b, values P and Q of a Boolean algebra will be associated with $a\varepsilon b$ and $a=b$ respectively. These will be called the "Boolean values" of $a\varepsilon b$ and $a=b$ denoted by $||a\varepsilon b||$ and $||a=b||$.

With our intuitive concept of $\varepsilon$, given two sets a and b, either a is an element of b, or a is not an element of b. That is, $a\ \varepsilon b$ takes on the value of 0 or 1. By taking values from a Boolean algebra other than the usual 2-valued Boolean algebra, we can allow for other possibilities of truth values. If a is likely to be an element of b with probability P say, then we will assign $||a\varepsilon b|| = P$, where P is a Boolean value.

Example: Suppose we have a set C of 20 coins $(C_1,\ldots,C_{20})$, and we define a subset of C as follows:

b = the set of all coins in C which when flipped, land heads up.

Let $C_1$ be a two-headed coin, then $||C_1\ \varepsilon b|| = 1$. If $C_2$ is a two-tailed coin, $||C_2\ \varepsilon b|| = 0$. Now suppose $C_3$ is a symmetric coin, and is as likely to land heads up as tails up. We will assign $||C_3\varepsilon b||$ a truth value $B_1$ which is a Boolean value, where $B_1$ can be thought of as the value assigned to the statement "a symmetric coin will land heads up." If a coin $C_{10}$ were loaded so that it would come up heads 30 percent of the time, $||C_{10} \varepsilon b||$ would be assigned a value $B_2$. $B_2$ is also a Boolean value.

More complex statements can be made from members of $V^{(B)}$, $\varepsilon$, and = using the following rules:

a) Each member of $V^{(B)}$ is a term.
b) If p and q are terms, then $(p=q)$ and $(p \varepsilon q)$ are sentences.
c) If x is a variable and X is a sentence, then $\forall x X$ is a sentence.
d) If X and Y are sentences, $\sim X$ and $X \wedge Y$ are sentences.[1]

Once $p \varepsilon q$ and $p=q$ have been given values, all sentences can be assigned values in a natural way.

$$||f|| = 0$$
$$||T|| = 1$$
$$||\Phi \vee \Psi|| = ||\Phi|| + ||\Psi||$$
$$||\Phi \wedge \Psi|| = ||\Phi|| \cdot ||\Psi||$$
$$||\sim\Phi|| \equiv -||\Phi||$$
$$||\exists u \Phi(u)|| = \sum_{u \varepsilon V^{(B)}} ||\Phi(u)||$$
$$||\forall u \Phi(u)|| = \prod_{u \varepsilon V^{(B)}} ||\Phi(u)||$$

Taking a look at membership, $a \varepsilon b$ thought of in the usual sense takes on the value of the characteristic function corresponding to b, $u_b(a)$.

$u_b(a) = 1$ if $a \varepsilon b$
$u_b(a) = 0$ if $\sim(a \varepsilon b)$

---

1. J. B. Rosser, Simplified Independence Proofs, p. 35.

We can dispense with b, and replace it by u obtaining a function u with values 0 and 1 rather than a set b with members and nonmembers. Then $a\varepsilon u$ is a statement having truth value $u(a)$.

Then the standard universe $V = \{V_\alpha : \alpha$ is an ordinal number$\}$ where $V_\alpha = \{x : \exists\xi[(\xi<\alpha)\wedge[x<V_\xi]]\}$ can be thought of as $V^{(2)} = \{V_\alpha^{(2)} : \alpha$ is an ordinal number$\}$ where $V^{(2)} = \{u\varepsilon 2^{dom(u)} : \exists\xi[(\xi<\alpha)\wedge[dom(u)\varepsilon V^{(2)}]]\}$. The Boolean valued analog in which we are interested is $V^{(\beta)} = \{V^{(\beta)} : \alpha$ is an ordinal number$\}$ where $V_\alpha^{(\beta)} = \{u\varepsilon\beta^{dom(u)} : \exists\xi[[\xi<\alpha]\wedge[dom(u)\varepsilon V_\alpha^{(\beta)}]]\}$. A "set" of level $\alpha$ will be a function u whose values are elements of the Boolean algebra and whose domain (dom(u)) is included in some earlier level. The first few levels are:

$$V_0^{(\beta)} = 0$$
$$V_1^{(\beta)} = \{0\}$$
$$V_2^{(\beta)} = \{0\}\ \{\{\langle 0,b\rangle\} : b\varepsilon\beta\}$$

We will say that a formula $\Phi$ is $\beta$-valid in $V^{(\beta)}$ iff $||\Phi|| = 1$ and we see that

THEOREM: All the rules and axioms of propositional calculus are $\beta$-valid, and all the rules and axioms of predicate calculus are $\beta$-valid.[2]

Now we want the axioms of set theory to hold in the model. So we make the following definition keeping the axiom of extensionality in mind.

For all $u, v\varepsilon V^{(\beta)}$

$$||u\varepsilon v|| = \sum_{y\varepsilon dom(v)} (||(y\varepsilon v)\wedge(u=y)||)$$

---

[2]. All definitions and theorems of Boolean Valued Models for Set Theory will be from D. Scott and R. Solovay "Lectures on Boolean Valued Models for Set Theory", pp. 4, 10-13, 20.

which is

$$||u\varepsilon v|| = \sum_{y \varepsilon dom(v)} (v(y) \cdot ||u=y||)$$

$$||u=v|| = \prod_{x\varepsilon dom(u)} (||(x\varepsilon u) \to (x\varepsilon v)||) \cdot \prod_{y\varepsilon dom(v)} (||(y\varepsilon v) \to (y\varepsilon u)||)$$

$$||u=v|| = \prod_{x \varepsilon dom(u)} (u(x) \Rightarrow ||x\varepsilon v||) \cdot \prod_{y\varepsilon dom(v)} (v(y) \Rightarrow ||y\varepsilon u||)$$

We see that

(i) the predecessors of $u\varepsilon v$ are the $u=y$ for $y\varepsilon dom(v)$.
(ii) the predecessors of $u=v$ are the $x\varepsilon v$ for $x\varepsilon dom(u)$ together with $y\varepsilon u$ for $y\varepsilon dom(v)$. Since $x\varepsilon dom(u)$ is well founded, the relation is well founded.

From this definition we can verify

THEOREM: All the axioms of equality are $\beta$-valid; indeed for all $u, v, w\varepsilon V^{(\beta)}$

(i) $||u=u|| = 1$
(ii) $u(x) \leq ||x\varepsilon u||$ for $x\varepsilon dom(u)$
(iii) $||u=v|| = v=u$
(iv) $||u=v|| \cdot ||v=w|| \leq ||u=w||$
(v) $||u=u'|| \cdot ||u\varepsilon v|| \leq ||u'\varepsilon v||$
(vi) $||v=v'|| \cdot ||u\varepsilon v|| \leq ||u \varepsilon v'||$

COROLLARY: For any formula $\Phi(u)$ and all $u, v\varepsilon V^{(\beta)}$
$||u=v|| \cdot ||\Phi(u)|| \leq ||\Phi(v)||$

COROLLARY: For any formula $\Phi(x)$ and all $u\varepsilon V^{(\beta)}$

$$|| x[(x\varepsilon u) \wedge \Phi(x)]|| = \prod_{x\varepsilon dom(u)} (u(x) \cdot ||\Phi(x)||)$$

$$|| x[(x\varepsilon u) \to \Phi(x)]|| = \prod_{x\varepsilon dom(u)} (u(x) \Rightarrow ||\Phi(x)||)$$

The axioms of set theory can be shown to be valid in the model.

Once again they are:

| | | | |
|---|---|---|---|
| I | Extensionality | V | Infinity |
| II | Union | VI | Foundation |
| III | Power Set | VII | Choice |
| IV | Replacement | | |

I will go through a proof of the Foundation axiom:

$\forall x[\forall y[(y\varepsilon x)\rightarrow\Phi(y)]\rightarrow\Phi(x)]\rightarrow\forall x\Phi(x)$

<u>Proof</u>

Let $b=||\forall x\forall y[(y\varepsilon x)\rightarrow\Phi(y)]\rightarrow\Phi(x)]||$. We show by induction that $x\varepsilon V^{(\beta)}$ implies $b\leq||\Phi(x)||$. Assume for $y\varepsilon dom(x)$ that $b\leq||\Phi(y)||$ then

$$b\leq \prod_{y\varepsilon dom(x)} [x(y)\Rightarrow||\Phi(y)||]=||\forall y[(y\varepsilon x)\rightarrow\Phi(y)]||$$

but $\quad b\leq||\forall y[(y\varepsilon x)\rightarrow\Phi(y)]||\Rightarrow||\Phi(x)||$

so $\quad b\leq||\Phi(x)||$

As we can see, the Boolean valued interpretation of $\langle \mathcal{U}, \varepsilon, =\rangle$ is a model of set theory. Also the Boolean algebra {0.1}. the usual notion of truth, is just a specific instance of the general Boolean values Models of Set Theory which we have defined. Now by producing a Boolean valued model in which the continuum hypothesis, $\aleph=2^{\aleph}$, is false we can demonstrate independence of the Continuum Hypothesis. Similarly by constructing appropriate models we can show the independence of V=L and the independence of the axiom of choice from the other axioms of set theory. Anyone wishing to pursue these models should consult <u>Simplified Independence Proofs</u> by Rosser.

## CHAPTER 2

## BOOLEAN ULTRAPOWERS

The idea of using an arbitrary Boolean algebra instead of the usual Boolean algebra is also used in the theory of Ultrapowers. As before, we will give some background into the subject.

A filter in a lattice L is a non-empty subset F of L which satisfies:

a) for all $x,y\varepsilon F$, $x\wedge y\varepsilon F$
b) for all $x\varepsilon F$ and $y\varepsilon L$, if $x\leq y$ then $y\varepsilon F$

Filters in a Boolean algebra $\beta$ are subsets of $\beta$ with certain properties. They can be ordered by set inclusion. An Ultrafilter is a maximal filter with respect to this ordering. A more useful way to look at ultrafilters is: If F is a filter in a Boolean algebra $\beta$, then F is an ultrafilter if for each $x\varepsilon\beta$ either $x\varepsilon F$ or $x^*\varepsilon F$ but not both. (Here $x^*$ denotes the complement of x.)

A relational structure, $\mathfrak{U}$, is written $\mathfrak{U} = \langle A, R_\xi\rangle$ where the $R_\xi$ are relations on A. The $R_\xi$ can be thought of as functions from powers of A into $\{0,1\}$. All relational structures will be of the same type. Let I be an index set, and for $i\varepsilon I$ let $\mathfrak{U}_i = \langle A_i, R_i\rangle$ be a relational structure. $\Pi_{i\,\varepsilon\, I}A_i$ is the cartesian product of the sets $A_i$. Let f, g be elements of $\Pi A_i$. We denote the ith coordinate of f by $f(i)$.

Let F be an ultrafilter on the power set Boolean algebra of I, $\langle P(I),\subseteq\rangle$. We define the equivalence relation $\sim_F$ on $\Pi A_i$ by $f\sim_F g$ if and only if $\{i\varepsilon I\colon f(i) = g(i)\}\varepsilon F$. For each $f\varepsilon\Pi\, A_i$ we let $f/F$ be the equivalence class to which f belongs under the relation $\sim_F$, and we let $\Pi A_i/F = \{f/F: f \in \Pi A_i\}$ . $\Pi \mathfrak{U}_i/F = \langle \Pi A_i/F, R_F \rangle$ is called an ultraproduct. If for each $i \in I$, $\mathfrak{U}_i = \mathfrak{U}$, the ultraproduct is denoted by $\mathfrak{U}^I/F$ and is called an ultrapower.[3]

Let $\mathfrak{U} = \langle A, R_\xi \rangle$ and $\tau = \langle B, S_\xi \rangle$ be two relational structures. $\mathfrak{U}$ is a substructure of $\tau$, and $\tau$ is an extension of $\mathfrak{U}$ (written $\mathfrak{U} \subseteq \tau$) if $A \subseteq B$ and each of the relations of $\mathfrak{U}$ is the restriction of the corresponding relation of $\tau$ to A. $\mathfrak{U}$ is elementarily equivalent to $\tau$, $\mathfrak{U} \equiv \tau$, if each sentence of the language which is true in $\mathfrak{U}$ is also true in $\tau$. In other words, $\mathfrak{U} \equiv \tau$ if for each sentence $\sigma$, $\mathfrak{U} \models \sigma$ iff $\tau \models \sigma$. $\mathfrak{U}$ is an elementary substructure of $\tau$, and $\tau$ is an elementary extension of $\mathfrak{U}$ (written $\mathfrak{U} \prec \tau$) if $\mathfrak{U} \subseteq \tau$ and for any formula $\Phi(v_0, \ldots, v_n)$ of the language and any $a_0, \ldots, a_n$ in A, $\mathfrak{U} \models \Phi[a_0, \ldots, a_n]$ iff $\tau \models \Phi[a_0, \ldots, a_n]$. An embedding h of $\mathfrak{U}$ into $\tau$ is said to be an elementary embedding of $\mathfrak{U}$ into $\tau$ if for each formula $\Phi(v_0, \ldots, v_n)$ of the language and any $a_0, \ldots, a_n$ in A we have $\mathfrak{U} \models \Phi[a_0, \ldots, a_n]$ iff $\tau \models \Phi[h(a_0), \ldots, h(a_n)]$. Thus if $\mathfrak{U} \subseteq \tau$, $\mathfrak{U} \prec \tau$ iff the injection of $\mathfrak{U}$ into $\tau$ is an elementary embedding.[4]

To construct a Boolean ultrapower we must develop a Boolean-valued counterpart to $\mathfrak{U}^I$. A first order Boolean-valued model consists of a set M together with a collection of functions from various finite powers of M into a complete Boolean algebra, $\beta$. These functions can be considered $\beta$-valued relations on M. The truth value for sentences, $\|\Phi\|$, is the same as in Boolean-valued models for set theory. That is

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[3]. J. L. Bell and A. B. Slomson, Models and Ultraproducts, pp. 87-89.
[4]. Ibid, pp. 73-75.

$$||R(m_1 \ldots m_n)|| = R(m_1 \ldots m_n)$$

$$||\Phi \vee \Psi|| = ||\Phi|| \; ||\Psi||$$

$$||\sim\Phi|| = \sim||\Phi||$$

$$\exists x \Phi(x) = \prod_{m \in M} ||\Phi(m)||$$

We now let $\mathfrak{A} = \langle A, R \rangle$ be an arbitrary two-valued structure and construct a $\beta$-valued elementary extension $\mathfrak{A}^{(\beta)} = \langle A^{(\beta)}, R_\xi \rangle$. The base set of this elementary extension is the set of all functions from A into $\beta$ whose ranges partition $\beta$, $\{f \epsilon \beta^A : \forall n, m \epsilon A [m \neq n \rightarrow f(n) + f(m) = 0] \wedge \prod_{m \epsilon A} f(m) = 1\}$. If R is an n-place relation on A, we extend it to a $\beta$-valued relation on $A^{(\beta)}$ by $R(f_1, \ldots, f_n) = \Pi\{\sum_{i=1}^{n} f_i(m_i) : \langle m_i \rangle \epsilon A^n \wedge R(m_1 \ldots m_n)\} = \prod_{R(m_1 \ldots m_n)} \sum_{i=1}^{n} f_i(m_i)$.[5]

For example, the equality relation is $||f = g|| = \prod_{m=n} f(m) + g(n) = \prod_m f(m) + g(m)$. We can see, as was true with Boolean valued models of set theory, the properties of general Boolean truth values are the same as the special case of the 2-valued Boolean truth.

THEOREM: For a formula $\Phi$ in the language of $\mathfrak{A}$,
$||\Phi(f_1, \ldots, f_n)|| = \prod_{(m_1, \ldots, m_n)} \sum_{i=1}^{n} f_i(m_i)$.

For a fixed m in A we can define a characteristic function m* in $A^{(\beta)}$:

$$m^*(x) = \begin{cases} 1 & \text{if } x = m \\ 0 & \text{if } x \neq m \end{cases}$$

From this we get:

---

[5]. All theorems, definitions, and equations of Boolean ultraproducts are from Richard Mansfield, "The Theory of Boolean Ultraproducts", Anals of Mathematical Logic 2, pp. 298-305.

COROLLARY: The map $m \mapsto m^*$ is an elementary embedding of $\mathfrak{U}$ into $\mathfrak{U}^{(\beta)}$. This means that a sentence $\Phi(m_1^*,\ldots,m_k^*)$ has value one in $\mathfrak{U}^{(\beta)}$ iff $\Phi(m_1,\ldots,m_k)$ is true in $\mathfrak{U}$, and $\Phi(m_1^*,\ldots,m_n^*)$ has value zero in $\mathfrak{U}^{(\beta)}$ iff $\Phi(m_1,\ldots,m_n)$ is false in $\mathfrak{U}$.

We can reason directly from the corollary that the equality axioms are $\beta$-valid in $\mathfrak{U}^{(\beta)}$.

$$||f=f|| = 1$$
$$||f=g|| = ||g=f||$$
$$||f=g|| + ||g=h|| \le ||f=h||$$
$$||f=g|| + ||\Phi(g)|| \le ||\Phi(f)||$$

Also $||f=m^*|| = f(m)$

Now, for $f\varepsilon A^{(\beta)}$ the condition that $\prod_m ||f(m)|| = 1$ translates to $m[m\varepsilon A \wedge f = m]$ which is really $f\varepsilon A$. So $A^{(\beta)}$ is the set A with the objects f such that $||f\varepsilon A|| = 1$. We see that $A^{(\beta)}$ is very large, and so we cannot extend it by the same procedure to something bigger.

THEOREM: If $\{b_i\}_{i\varepsilon I}$ is a pairwise disjoint collection from $\beta$ and $\{f_i\}_{i\varepsilon I}$ is any collection from $\mathfrak{U}^{(\beta)}$ there is an f in $\mathfrak{U}^{(\beta)}$ with $||f = f_i|| \ge b_i$. If in addition $\prod_i b_i = 1$ this f is unique.

We also note that:

For any formula $\Phi(x)$ there is an f in $A^{(\beta)}$ with

$$||\exists x\Phi(x)|| = ||\Phi(f)||$$

Now we are ready to define a Boolean ultrapower. For $\nu$ an arbitrary ultrafilter on $\beta$ we can define a two valued model $\mathfrak{U}^{(\beta)}/\nu$, called the $(\beta,\nu)$ ultrapower of $\mathfrak{U}^{(\beta)}$. This is done by factoring $\beta$ valued relations on $A^{(\beta)}$ by the ultrafilter $\nu$. Symbolically this is written

$\mathfrak{A}^{(\beta)}/\nu \models R(f_1,\ldots,f_n)$ iff $R(f_1,\ldots,f_n)\varepsilon\nu$. Immediately we see from the definition that a formula $\Phi(f_1,\ldots,f_n)$ is true in $\mathfrak{A}^{(\beta)}/\nu$ iff $||\Phi(f_1,\ldots,f_n)||\varepsilon\nu$, and that $\mathfrak{A}^{(\beta)}/\nu$ is an elementary extension of $\mathfrak{A}$.

Finally we note that the case of Boolean untrapower $\beta = 2^I$ corresponds to the normal concept of ultrapowers. There is an isomorphism which takes an f in $\mathfrak{A}^I$ to the map $m \mapsto f^{-1}\{m\}$ in $\mathfrak{A}^{(2^I)}$, and hence $\mathfrak{A}^{(2^I)}/\nu \simeq \mathfrak{A}^I/\nu$. So we see that Boolean ultrapowers are a generalization of ultrapowers. We further observe that $V^{(\beta)}$ of Boolean Valued Models for Set Theory is a special case of $A^{(\beta)}$ as dealt with by Mansfield. Some work on $V^{(\beta)}$ as an ultrapower was done by Scott.[6]

[6]. D. Scott, Boolean Valued Models for Set Theory, pp. 49-53.

# CHAPTER 3

## FUZZY SETS

In deductive logic, it is assumed that every proposition is either true or false. This is formally expressed in the axiomatic propositional calculus by the law of the excluded middle. This two-valued logic fits nicely into the design of switching systems because of its simplicity and the fact that basic switching modules in common use are two propositional. The difficulty has come in that, although simple and convenient, this approach does not mesh with most real world problems. Real life situations are often ambiguous, and variables usually have values other than truth or falsehood. For example, the set of all beautiful women is a set having loosely (fuzzily) defined attributes, and the concept of "belonging to a set" does not take on the usual two-valued, true-false logic. The classification of a woman as a beautiful woman is subjective in nature and depends on the person doing the classification, his background, his mood, etc.

A few more examples:

1) The set of all poor people. (Do graduate students belong to this set? Do professors?)
2) The set of all large numbers. (Is 10,000 a member of this set?)
3) The set of all tall men. (You would probably not belong to this set if you lived among the Watusi tribe of Africa. However, you would most likely qualify for membership among a pigmi tribe in South America. Do you feel you are a member?)

We need a way to deal with fuzzily defined sets, whose members do not possess sharply defined attributes. L. A. Zadeh proposed a "fuzzy set" in order to deal with the problem.[7] A fuzzy set is a class in which there may be a continous infinity of grades of membership, with the grade of membership of an object x in a fuzzy set A represented by a number $\mu_A(x)$ in the interval [0,1].

For example, consider the set of large numbers, A, and ascribe to 0, 5, 101, 1050, and 100,000 the membership grades of 0, 0, 0.134, 0.314, and 1.0 respectively. Thus we see $\mu_A(5)=0$, $\mu_A(1050)=0.314$, and $\mu_A(100{,}000)=1.0$. Such an assignment is subjective, but is precise and well-defined once it has been made.

Some definitions relating to fuzzy sets follow:[8]

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| EQUALITY | Two fuzzy sets A and B in a space X are equal, A=B, iff $\mu_A(x)=\mu_B(x)$ for all x in X. (We shall write $\mu_A=\mu_B$, suppressing the x.) |
| CONTAINMENT | A fuzzy set A is contained in a fuzzy set B, $A\subseteq B$, iff $\mu_A\leq\mu_B$. That is, for all x in X, $\mu_A(x)\leq\mu_B(x)$ iff $A\subseteq B$. |
| COMPLEMENTATION | A fuzzy set A' is the complement of a fuzzy set A iff $\mu_{A'}=1-\mu_A$. |
| UNION | The union of two fuzzy sets A and B, $A\cup B$, is defined as the smallest fuzzy set containing both A and B. From this we get the membership function of $A\cup B$. $\mu_{A\cup B}(x)=\mathrm{Max}\{\mu_A(x),\mu_B(x)\}$. |
| INTERSECTION | The intersection of two fuzzy sets A and B, $A\cap B$, is defined as the largest fuzzy set contained in both A and B. We know the membership |

7. L. A. Zadeh, "Fuzzy Sets," Information and Control 8, p.338.
8. Ibid, pp. 341-345, 350.

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| | function of A B to be $\mu_{A\cap B}(x)$=Min $\{u_A(x),\mu_B(x)\}$ |
| SHADOW | Consider a fuzzy set A in $E^n$ with membership function $\mu_A(x_1,\ldots,x_n)$ with $x=(x_1,\ldots,x_n)$. Let H be a hyperplane in $E^n$. Let L be a line orthogonal to H and let h be its point of intersection with H. The shadow of A on H, $S_H(A)$, is a fuzzy set with $\mu_{S_H(A)}(h)=\text{Sup}_{x\epsilon L}\ \mu_A(x)$ and $\mu_{S_H(A)}(x)=0$ for $x\epsilon H$. |
| FUZZY RELATION | A fuzzy relation in X is a fuzzy set in the product space X*X. For example, the relation denoted by $x \gg y$, $x, y \epsilon R$ may be regarded as a fuzzy set A in $R^2$ with a membership function $f_A(x,y)$. |

Using the definitions of Union, Intersection, and Complementation, we can develop a fuzzy logic. We will call a fuzzy variable the membership grade of a variable in a set. For example, let $x_1=\mu_A(V_1)$. $x_1$ is a fuzzy variable. We will use the operator + to denote the maximum of the membership grades, * to denote the minimum of the membership grades, and $^-$ to denote 1- the membership grade.

Fuzzy formulas are defined recursively, generated by fuzzy variables $x_1$, $x_2$, ... $x_n$ as follows:

1) A variable $x_1$ is a fuzzy formula. 0 and 1 are fuzzy formulas.
2) If $\Phi$ is a fuzzy formula, then $\overline{\Phi}$ is a fuzzy formula.
3) If $\Phi$ and $\Psi$ are fuzzy formulas, then Max $\{\Phi,\Psi\}$ and Min $\{\Phi,\Psi\}$ are fuzzy formulas.
4) The above are the only fuzzy formulas.

When we examine the algebra on fuzzy sets, we see that we have a complete distributive lattice with V =Max and $\Lambda$ =Min. The lattice is not a complemented and hence not a Boolean algebra.

Zadeh's fuzzy sets with membership function on the interval [0,1] can be generalized to an L-fuzzy set with membership function in some complete distributive lattice, L. An L-fuzzy set A on a set X is a function $A: X \to L$ where L is a complete distributive lattice. When L is the lattice with two elements, {0,1} we get something analogous to set theory, where A can be thought of as the relation $\varepsilon C$, where $C \subset X$. Thus, the crisp or sharply defined case, although it is the qualitative opposite of fuzziness, is technically a special case of an L-fuzzy set, the case L={0,1}. The lattice [0,1] of fuzzy sets is also a special case of an L-fuzzy set.

To make fuzzy sets more useful in decision processes, the ideas of fuzzy restrictions and fuzzy algorithms were proposed. A fuzzy restriction is a fuzzy relation which acts as an elastic constraint on the values that may be assigned to a variable.[9] Some examples of fuzzy restrictions are,

The soup is hot.
Mark is short.
Susan is a blonde.
The car is fast.
Dzidra is a very beautiful woman.

Essentially a fuzzy restriction limits the possible values of an object by placing it in a fuzzy subset of some class, either fuzzy or nonfuzzy, of which we are talking. Thus Dzidra is in a fuzzy subset of the fuzzy set of beautiful women, namely the set of very beautiful women. Also the fuzzy set of beautiful women is a fuzzy subset of the set of women.

Fuzzy algorithms are fuzzy instructions that deal with fuzzy sets. One such instruction is, "If x is large, increase y by several

9. L. A. Zadeh, "Calculus of Fuzzy Restrictions", Fuzzy Sets and Their Application to Cognitive and Decision Processes, p. 2.

units." A nonfuzzy, or crisp, version of this instruction could be, "If x >1000, increase y by three units." In the crisp case we can have several instructions covering possible values of x:

If $x \geq 1000$, increase y by 5 units.
If $100 \leq x < 1000$, increase y by 3 units.
If $0 \leq x < 100$, increase y by 2 units.

The fuzzy case can parallel this with a set of fuzzy instructions $C_i$, where i is an index that ranges over a nonfuzzy set.

If x is very large, increase y by a few units.
If x is large, increase y by many units.
If x is small, increase y by a little.
If x is very small, increase y by several units.

Fuzzy sets and fuzzy algorithms will have practical use because most realistic problems tend to be so complex as to be algorithmically unsolvable or solvable in theory but not solvable practically. A move in chess, although it theoretically possesses an ideal solution, is not feasible due to the large number of potential moves which must be considered. As a result, fuzzy, short-range goals are introduced which make calculation of a move feasible. The military version of chess, troop deployment and movement, is more complicated and has more variables involved such as terrain, psychological advantage of surprise, weather, etc. Military strategists have used fuzzy goals in their thinking for centuries. Pattern recognition is another area where fuzzy sets and fuzzy algorithms may be applicable. Fuzzy criteria may be set up to decide if a letter is the letter "B" or whether it is another letter. Fuzzy sets can be used in programming a robot or machine to do certain jobs. In psychology fuzzy sets would be useful in the study of learning and memory.

# CHAPTER 4

## CONCLUSIONS

Although the concept of Fuzzy Sets was developed for different reasons and has different applications than either Boolean-valued models of Set Theory or Boolean-valued ultrapowers, the concepts do have some connections. The immediate similarity is that a complete Boolean algebra is a complete distributive lattice. The difference is that two elements of a Boolean algebra have supremum and infimum operators, while the operators in fuzzy set theory are maximum and minimum. The theories are different generalizations of the case of the 2-valued Boolean algebra $\{0,1\}$ when maximum and minimum are respectively equivalent to supremum and infimum.

The fact that the lattice $[0,1]$ of fuzzy sets is not complemented does have advantages and is a very useful way of looking at an object in relation to certain criteria of membership which are not clearly defined. For example,

Let A = the numbers much larger than 1000
B = the numbers much smaller than 1000
C = the numbers much larger than 100.

We can say from this that A contains C. Also as we look at a number X, we see that intuitively Max $\{\mu_A(X), \mu_B(X)\}$ is $\mu_{A\cup B}(X)$. Fuzzy sets prove to be useful when we relate the sets themselves to one another.

It seems, however, that Zadeh's fuzzy sets, because of their simplicity, are cumbersome for dealing with two or more objects and their respective membership functions in a certain fuzzy set. This is evident by the fact that there have been several papers on the simplification or minimization of combinations of membership functions. Also our intuition seems to want a stronger relationship between objects. For example, suppose that sitting before you are three dinners:

Dinner X is roast beef.
Dinner Y is lobster tail.
Dinner Z is burnt black toast.

We want to consider the fuzzy set A, the set of "delicious meals". We will assign the values $\mu_A(x) = 0.85$, $\mu_A(y) = 0.9$, and $\mu_A(Z) = 0.005$. Suppose now that our best friend is having dinner and we want to serve him a delicious meal. We want to carry two of the meals out to him and let him choose which he wishes to eat. The value given "Y is a delicious meal or Z is a delicious meal" is Max $\{0.9, 0.005\} = 0.9$, and likewise the value given "X is a delicious meal or Y is a delicious meal" is Max $\{0.85, 0.9\} = 0.9$. According to our values there should be no difference between a menu of lobster and burnt black toast, and a menu of lobster and roast beef. Intuitively this seems distorted. Also our friend is more likely to have a delicious meal if given the choice between roast beef and lobster as opposed to simply getting lobster.

One possibility, which to our knowledge has not been adequately pursued, is that of letting the membership function take on values of a Boolean Algebra. This would be assigning $x \varepsilon A$ a Boolean value, as done in Boolean-Valued Models of Set Theory. Or it could be thought of as assigning functions defined on some set Boolean values, as done with Boolean Ultrapowers. A boolean-valued membership function would alleviate the above problems of dealing with two or more objects. There are situations in which we would wnat some fuzzy sets to be complemented. Suppose we are trying to recognize an alphabetic character. We know, or assume that the object was intended to be a character. It may not be clear, however, which character it is, and so we get the fuzzy subsets "is a B", "is a P", and so forth. The character might be "B" or "P", but it is surely not both. Also Boolean-valued assignments might help us deal with the context in which the character was found.

One disadvantage of a Boolean valued membership function is that it is more complicated, and we must be careful in our assignment of values to objects. The reason for care is that supremums and infimums which relate objects or sets to one another must be taken into account during value assignment. One alternative which seems very feasible is to use a Boolean-valued membership function, and where and when desirable, converting to a membership function on [0,1] by taking a measure. This procedure would take advantage of both concepts.

The concept of L-fuzzy sets, which has the same lattice structure as Boolean valued fuzzy sets with the exception of complementation, should have a great deal of applications. However, most applied work has been done with fuzzy sets, and work on L-fuzzy sets has not been in the area of applied problems. Boolean valued fuzzy sets can be viewed as a specific part of L-fuzzy sets.

The idea of Boolean valued Fuzzy Sets may have some use in certain situations, and as the subject of fuzzy sets is pursued by its growing number of researchers, these applications may become clearer.

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## VITA

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