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The crack and crack-contact problems for an orthotropic strip.

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THE CRACK AND CRACK-CONTACT PROBLEMS FOR AN ORTHOTROPIC STRIP

by

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

In this study the plane elastostatic problem for an infinite orthotropic strip containing a crack located arbitrarily parallel to the sides is considered. Fourier integral transformation technique is used to reduce the problem to two coupled singular integral equations which are subsequently solved numerically. The stress intensity factors are calculated for various crack geometries, crack locations and material parameters under various loading conditions.

In addition to the crack problem, the problem of wedge-loading by a frictionless rigid wedge pressed into the crack is considered. The resulting crack-contact problem is formulated by modifying the integral equation which is obtained for the crack problem. It is shown that for wedge lengths b less than a critical value b_{cr} the continuous contact along the wedge-crack interface is maintained. However, for $b > b_{cr}$ the crack surfaces separate from the wedge along a certain finite region. The problem is formulated and solved for both cases and numerical results for b_{cr}, distances determining the separation area, contact stresses, and stress intensity factors are given.

1. INTRODUCTION

In this study the linear fracture mechanics problem for an infinite orthotropic strip containing a crack parallel to the sides is considered. No assumption of symmetry about crack location is made. The problem is formulated in terms of a system of singular integral equations. In addition to this problem, the crack-contact problem for a frictionless rigid wedge pressed into crack is studied and the resulting problem is solved for both cases of continuous contact and interface separation.

In recent years the increasing use of multi-layered bonded plates in many engineering structures and especially in aerospace industry, has brought up the need for more intensive fracture analysis of anisotropic materials. Physically, it is obvious that any manufacturing flaw that exists would be either in the bonding layer or, perhaps more likely, on the interface. Thus, this flaw may be considered as an interface crack problem. The composite materials are combinations of various different materials and are, in general, anisotropic and non-homogeneous. However, mostly because of analytical expediency they are usually assumed to be orthotropic and homogeneous. What makes fiber composite materials so important is that, during the process of manufacturing, they may be strengthened in certain directions, which improves their structural resistance to unstable crack propagation. The practical importance of the problem under consideration lies in the fact that the results may be used in experimental strength characterization as well as in structural fracture studies. For example, the cracked infinite strip may approximate a long beam or plate clamped at one end and loaded at the other end. The crack may grow due to effect of the shear stress. The wedge loading of elastic materials is also used in practice mainly in certain fracture toughness characterization tests and in fracturing solids by wedge-splitting or cleaving. In fracturing of solids, of course, the geometry is bounded in both directions. However, the assumption of an infinitely long strip would not affect the character of the results.

In plane problems, for an infinite orthotropic medium containing a line crack, it has been shown that (taking limit as $H_1 \to \infty$ and $H_2 \to \infty$, Fig. 1) the orthotropy does not affect the stress intensity factors and the results are the same as those obtained from the isotropic case. However, for the bounded geometry, the stress intensity factors are highly dependent on the orthotropy of the material. We may refer to a number of previous works to study this dependence. For example, in [1] the problem of periodically arranged orthotropic strip containing cracks has been studied and in [2] an orthotropic strip containing an internal or edge crack is investigated for both material types I and II. The stress intensity factors are calculated and are compared with isotropic results. Recently the problem of an infinite orthotropic strip containing a crack normal to the sides of the strip is considered in [3] and results are compared with the isotropic case. The inclined internal crack problem for isotropic and orthotropic strips was studied in [4] and [5], respectively. The wedge loading of a semi-infinite strip with an edge crack is considered in [6]. It is formulated for the isotropic case and results are obtained for various wedge shapes. In formulating the problem under consideration, it is assumed that both shear and normal stresses are applied on the crack surface. The results for other loading conditions may be obtained by using the superposition technique. The results are obtained for various crack geometries, crack locations and material parameters, under various basic loading conditions. The results are obtained for plane stress case. The formulation of plane strain problem is identical to plane stress if we redefine the material parameters κ and δ (see appendix I).

In the second part of the study the problem of a frictionless rigid flat wedge pressed into crack is considered. It is assumed that crack is located in the middle of the strip. The resulting crack-contact problem is formulated for both continuous contact and interface separation cases by modifying the integral equation obtained for the crack problem. The numerical calculations for determination of critical wedge length, contact stresses, stress intensity factors and distances determining the separation area are given.

2. MATERIAL PARAMETERS AND VARIABLE TRANSFORMATIONS

2.1 DEFINITION OF MATERIAL PARAMETERS

In the plane theory of elasticity the Hook's law for generalized plane stress and orthotropic materials can be expressed as

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{12}}{E_{11}} & 0 \\ -\frac{\nu_{21}}{E_{22}} & \frac{1}{E_{22}} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \tag{2.1}$$

where

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \quad (i, j = 1, 2). \tag{2.2}$$

Let us define the following new constants [7]:

Effective stiffness: $E = (E_{11} E_{22})^{1/2}$

Effective Poisson's ratio: $\nu = (\nu_{12} \nu_{21})^{1/2}$

Stiffness ratio: $\delta^4 = \frac{E_{11}}{E_{22}} = \frac{\nu_{12}}{\nu_{21}}$

Shear Parameter: $\kappa = \frac{1}{2}(E_{11} E_{22})^{1/2}\left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right)$ (2.3a-d)

Using (2.3a-d), equation (2.1) becomes

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} \delta^{-2} & -\nu & 0 \\ -\nu & \delta^{2} & 0 \\ 0 & 0 & \kappa+\nu \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \tag{2.4}$$

2.2 VARIABLE TRANSFORMATIONS

The governing differential equation in the plane theory of elasticity for an orthotropic material is given by [8]

$$\frac{E_{11}}{E_{22}}\frac{\partial^4\phi}{\partial x_1^4}+\left(\frac{E_{11}}{G_{12}}-2\nu_{12}\right)\frac{\partial^4\phi}{\partial x_1^2\partial x_2^2}+\frac{\partial^4\phi}{\partial x_2^4}=0 \qquad (2.5)$$

where $\phi = \phi(x_1,x_2)$ is the airy stress function and stresses are given in terms of ϕ as follows:

$$\sigma_{11}=\frac{\partial^2\phi}{\partial x_2^2},\quad \sigma_{22}=\frac{\partial^2\phi}{\partial x_1^2},\quad \sigma_{12}=-\frac{\partial^2\phi}{\partial x_1\partial x_2} \qquad (2.6a\text{-}c)$$

The new variables x,y and components of the displacement vector are defined as

$$x=\frac{x_1}{\sqrt{\delta}},\quad y=\sqrt{\delta}\,x_2,$$
$$u=u_1\sqrt{\delta},\quad v=\frac{u_2}{\sqrt{\delta}}. \qquad (2.7a\text{-}d)$$

It follows from the equations (2.7a-d) that components of strains in terms of the new variables can be expressed as

$$\epsilon_{xx}=\frac{\partial u}{\partial x}=\delta\,\epsilon_{11}$$
$$\epsilon_{yy}=\frac{\partial v}{\partial y}=\frac{\epsilon_{22}}{\delta}$$
$$\epsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\epsilon_{12} \qquad (2.8a\text{-}c)$$

The stresses in transformed and real planes can be related by using equations (2.6) and (2.7a,b) as follows:

$$\sigma_{xx}=\frac{\partial^2\phi}{\partial y^2}=\frac{\sigma_{11}}{\delta}$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = \delta\,\sigma_{22}$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y} = \sigma_{12} \tag{2.9a-c}$$

Substituting the equations (2.8) and (2.9) into (2.4), and (2.3) and (2.7) into (2.5) we obtain

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & \kappa+\nu \end{bmatrix}\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \tag{2.10}$$

$$\frac{\partial^4\phi}{\partial x^4} + 2\kappa\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \tag{2.11}$$

where

$\phi = \phi(x,y)$ is the stress function. The equation (2.11) contains just one material parameter which is κ. In the isotropic case $\kappa = 1$ and equation (2.11) reduces to the well known form.

3. FORMULATION OF THE PROBLEM

Consider the orthotropic strip shown in Fig. 1(a). The problem may be formulated by expressing the field quantities as the sum of those for a homogeneous strip without a crack and those for an infinite plane with a crack and by satisfying all the boundary conditions for the actual cracked strip.

3.1 INFINITE STRIP WITHOUT CRACK

Applying the complex fourier integral transformation technique to solve the governing equation (2.11), the solution can be expressed as

$$\phi_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{j=1}^{4} C_j(\alpha) e^{S_j \alpha y - i\alpha x} d\alpha \tag{3.1}$$

where $S_j (j=1,4)$ are the roots of

$$S^4 - 2\kappa S^2 + 1 = 0 \tag{3.2}$$

From (3.2) we can write

$$S_1 = (\kappa + \sqrt{\kappa^2 - 1})^{1/2}, \quad S_3 = -S_1,$$

$$S_2 = (\kappa - \sqrt{\kappa^2 - 1})^{1/2}, \quad S_4 = -S_2 .$$

Examining the roots S_1 and S_2 it can be shown that they are either real or complex conjugates.

Material type I: $\kappa \geq 1$.

since $\kappa \geq 1$ thus S_1 and S_2 are real.

Material type II: $\kappa < 1$.

Case 1: $-1 < \kappa < 1$

$S_1 = w_1 + iw_2$, $S_2 = w_1 - iw_2$

Case 2: $\kappa \leq -1$

$S_1 = iw_3$, $S_2 = iw_4$

where w_1, w_2, w_3 and w_4 are real constants. In this study we will assume that the material is of type I. The results for type II materials may be obtained with slight modification in the analysis.

3.2 INFINITE PLANE WITH CRACK

The solution of (2.11) satisfying the regularity conditions at $y = \mp \infty$ as $y \to \mp \infty$, can be written as

$$\phi_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2} A_j(\alpha)\, e^{-S_j|\alpha|y - i\alpha x}\, d\alpha \quad \text{for } y>0,$$

$$\phi_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2} A_{j+2}(\alpha)\, e^{S_j|\alpha|y - i\alpha x}\, d\alpha \quad \text{for } y<0. \qquad (3.3a,b)$$

3.3 STRESS FUNCTION

The stress function ϕ is constructed in terms of ϕ_1 and ϕ_2 obtained in the previous section as

$$\phi(x,y) = \phi_1(x,y) + \phi_2(x,y)\ . \qquad (3.4)$$

The continuity conditions for the stress vector at $y = 0$ may be used to eliminate two of the constants. These conditions are

$$\sigma_{yy}(x,+0) = \sigma_{yy}(x,-0)\ ,\quad \sigma_{xy}(x,+0) = \sigma_{xy}(x,-0) \qquad (3.5a-b)$$

and may be shown to be identical to

$$\phi_2(x,+0) = \phi_2(x,-0)$$

$$\frac{\partial}{\partial y}\phi_2(x,+0) = \frac{\partial}{\partial y}\phi_2(x,-0)\ . \qquad (3.6a,b)$$

Using the equations (3.3a,b) and (3.6a,b) we obtain

$$\begin{aligned} A_1(\alpha) + A_2(\alpha) &= A_3(\alpha) + A_4(\alpha) \\ -S_1 A_1(\alpha) - S_2 A_2(\alpha) &= S_1 A_3(\alpha) + S_2 A_4(\alpha) \end{aligned} \qquad (3.7)$$

Solving $A_3(\alpha)$ and $A_4(\alpha)$ in terms of $A_1(\alpha)$ and $A_2(\alpha)$ from (3.7) we obtain

$$A_3(\alpha) = -\lambda_7 A_1(\alpha) - \lambda_6 A_2(\alpha)$$

$$A_4(\alpha) = \lambda_5 A_1(\alpha) + \lambda_7 A_2(\alpha) \tag{3.8a-b}$$

where

$$\lambda_5 = \frac{2S_1}{S_1 - S_2}, \quad \lambda_6 = \frac{2S_2}{S_1 - S_2}, \quad \lambda_7 = \frac{S_1 + S_2}{S_1 - S_2}. \tag{3.9a-c}$$

Defining the auxiliary functions and their Fourier transforms by

$$F_1(x) = \frac{\partial}{\partial x}\left[V(x,+0) - V(x,-0)\right],$$

$$F_2(x) = \frac{\partial}{\partial x}\left[U(x,+0) - U(x,-0)\right], \tag{3.10a-b}$$

$$\bar{F}_1(\alpha) = \int_{-\infty}^{\infty} F_1(t)\, e^{i\alpha t}\, dt,$$

$$\bar{F}_2(\alpha) = \int_{-\infty}^{\infty} F_2(t)\, e^{i\alpha t}\, dt. \tag{3.11a-b}$$

The functions $A_1(\alpha)$ and $A_2(\alpha)$ can be found in terms of $\bar{F}_1(\alpha)$ and $\bar{F}_2(\alpha)$ in the following way. From the equation (2.10) we have

$$\epsilon_{xx} = \frac{\partial U}{\partial x} = \frac{1}{E}\left(\sigma_{xx} - \nu\,\sigma_{yy}\right),$$

$$\epsilon_{yy} = \frac{\partial V}{\partial y} = \frac{1}{E}\left(\sigma_{yy} - \nu\,\sigma_{xx}\right),$$

$$2\epsilon_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \frac{2(K+\nu)}{E}\sigma_{xy}. \tag{3.12a-c}$$

Differentiating the equation (3.12C) respect to x and (3.12a) with respect to y and eliminating the terms $\frac{\partial^2 u}{\partial x \partial y}$ we obtain

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{E}\left[2(K+\nu)\frac{\partial \sigma_{xy}}{\partial x} - \frac{\partial \sigma_{xx}}{\partial y} + \nu\frac{\partial \sigma_{yy}}{\partial y}\right]$$

or in terms of the stress function

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{E}\left[-2(\kappa+\nu)\frac{\partial^3\phi}{\partial x^2\partial y} - \frac{\partial^3\phi}{\partial y^3} + \nu\frac{\partial^3\phi}{\partial x^2\partial y}\right] \tag{3.13}$$

Taking limiting values of the last equation as $y \to \mp 0$ and subtracting, we obtain

$$\frac{\partial^2}{\partial x^2}\left[v(x,+0)-v(x,-0)\right] = \frac{1}{E}\left[-2(\kappa+\nu)\frac{\partial^3}{\partial x^2\partial y} - \frac{\partial^3}{\partial y^3} + \nu\frac{\partial^3}{\partial x^2\partial y}\right]\left[\phi(x,+0)-\phi(x,-0)\right] \tag{3.14}$$

Observing that

$$\phi = \phi_1 + \phi_2\ ,\quad \frac{\partial^n}{\partial y^n}\left[\phi_1(x,+0)-\phi_1(x,-0)\right] = 0 \tag{3.15}$$

$$n = 0, 1, 2, \ldots$$

and using (3.6) we obtain

$$\frac{\partial^2}{\partial x^2}\left[v(x,+0)-v(x,-0)\right] = -\frac{1}{E}\frac{\partial^3}{\partial y^3}\left[\phi_2(x,+0)-\phi_2(x,-0)\right]. \tag{3.16}$$

From the equations (3.3) and (3.8), it follows that,

$$\frac{\partial^2}{\partial x^2}\left[v(x,+0)-v(x,-0)\right] = \frac{1}{2\pi E}\int_{-\infty}^{\infty}\left[\lambda_1 A_1(\alpha)+\lambda_2 A_2(\alpha)\right]\alpha^2|\alpha|e^{-i\alpha x}\,d\alpha \tag{3.17}$$

integrating with respect to x from (3.17) we obtain

$$\frac{\partial}{\partial x}\left[v(x,+0)-v(x,-0)\right] = \frac{1}{2\pi E}\int_{-\infty}^{\infty}\left[\lambda_1 A_1(\alpha)+\lambda_2 A_2(\alpha)\right]i\alpha|\alpha|e^{-i\alpha x} \tag{3.18}$$

where

$$\lambda_1 = (1-\lambda_7)S_1^3 + \lambda_5 S_2^3\ ,\quad \lambda_2 = (1+\lambda_7)S_2^3 - \lambda_6 S_1^3\ . \tag{3.19}$$

Taking inverse transform and using the equations (3.10) and (3.11) we obtain

$$i\alpha|\alpha|\left[\lambda_1 A_1(\alpha) + \lambda_2 A_2(\alpha)\right] = \bar{F}_1(\alpha) E . \tag{3.20}$$

Expressing the equation (3.12a) in terms of $\phi(x,y)$, it follows that,

$$\frac{\partial U}{\partial x} = \frac{1}{E}\left[\frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2}\right] . \tag{3.21}$$

Similarly, we can show that

$$\frac{\partial}{\partial x}\left[U(x,+0) - U(x,-0)\right] = \frac{1}{E} \frac{\partial^2}{\partial y^2}\left[\phi_2(x,+0) - \phi_2(x,-0)\right] . \tag{3.22}$$

using the equations (3.3) and (3.8) we obtain

$$\frac{\partial}{\partial x}\left[U(x,+0) - U(x,-0)\right] = \frac{1}{2\pi E}\int_{-\infty}^{\infty}\left[\lambda_3 A_1(\alpha) + \lambda_4 A_2(\alpha)\right]\alpha^2 e^{-i\alpha x} d\alpha \tag{3.23}$$

where

$$\lambda_3 = (1+\lambda_7)S_1^2 - \lambda_5 S_2^2 , \quad \lambda_4 = (1-\lambda_7)S_2^2 + \lambda_6 S_1^2 . \tag{3.24a-b}$$

Taking inverse transform of (3.23) and using the equations (3.10b) and (3.11b) we obtain

$$\alpha^2\left[\lambda_3 A_1(\alpha) + \lambda_4 A_2(\alpha)\right] = E \bar{F}_2(\alpha) . \tag{3.25}$$

The solution of equations (3.20) and (3.25) is straightforward and gives $A_1(\alpha)$ and $A_2(\alpha)$ as

$$A_1(\alpha) = E\left[\lambda_{11}\frac{\bar{F}_2(\alpha)}{\alpha^2} + i\lambda_{12}\frac{\bar{F}_1(\alpha)}{\alpha|\alpha|}\right],$$

$$A_2(\alpha) = E\left[\lambda_{13}\frac{\bar{F}_2(\alpha)}{\alpha^2} + i\lambda_{14}\frac{\bar{F}_1(\alpha)}{\alpha|\alpha|}\right] \tag{3.26a-b}$$

where

$$\lambda_{10} = \lambda_4\lambda_1 - \lambda_2\lambda_3 ,$$

$$\lambda_{11} = -\frac{\lambda_2}{\lambda_{10}}, \quad \lambda_{12} = -\frac{\lambda_4}{\lambda_{10}}, \tag{3.27}$$

$$\lambda_{13} = \frac{\lambda_1}{\lambda_{10}}, \quad \lambda_{14} = \frac{\lambda_3}{\lambda_{10}} .$$

Adding the equations (3.1) and (3.3a), we obtain the stress function $\phi(x,y)$ which is valid in the domain $y > 0$, $|x|<\infty$ as

$$\phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_{j=1}^{4} C_j(\alpha)e^{S_j\alpha y} + \sum_{j=1}^{2} A_j(\alpha)e^{-S_j|\alpha|y}\right]e^{-i\alpha x}d\alpha, \; y>0. \tag{3.28}$$

substitution of the equations (3.8) in to (3.3b) gives

$$\phi_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A_1(\alpha)\left(\lambda_5 e^{S_2|\alpha|y} - \lambda_7 e^{S_1|\alpha|y}\right) + A_2(\alpha)\left(\lambda_7 e^{S_2|\alpha|y} - \lambda_6 e^{S_1|\alpha|y}\right)\right]e^{-i\alpha x}d\alpha,$$

$$y < 0. \tag{3.29}$$

The stress function $\phi(x,y)$, in the domain $y<0$, $|x|<\infty$, can be obtained as

$$\phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A_1(\alpha)\left(\lambda_5 e^{S_2|\alpha|y} - \lambda_7 e^{S_1|\alpha|y}\right) + A_2(\alpha)\left(\lambda_7 e^{S_2|\alpha|y} - \lambda_6 e^{S_1|\alpha|y}\right)\right]e^{-i\alpha x}d\alpha$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4} C_j(\alpha)e^{S_j\alpha y - i\alpha x}d\alpha, \; y<0. \tag{3.30}$$

In the equations (3.28) and (3.30), $A_1(\alpha)$ and $A_2(\alpha)$ are given in terms of displacement derivatives by (3.26) and the only unknown functions are $c_j(\alpha)$ $(j=1,\ldots,4)$. They can be determined from the homogeneous boundary conditions which will be discussed later. The relevant stress components are given in terms of the stress function as

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}\,, \quad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y}$$

which may be expressed for y>0 and y<0 as follows:

$$\sigma_{yy} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\sum_{j=1}^{4} c_j(\alpha)\, e^{s_j\alpha y} + \sum_{j=1}^{2} A_j(\alpha)\, e^{-s_j|\alpha|y}\Big]\alpha^2 e^{-i\alpha x}\, d\alpha,$$

$$\sigma_{xy} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\Big[\sum_{j=1}^{4} c_j(\alpha)\, s_j\, e^{s_j\alpha y} - \sum_{j=1}^{2}\frac{|\alpha|}{\alpha} A_j(\alpha)\, s_j\, e^{-s_j|\alpha|y}\Big]\alpha^2 e^{-i\alpha x}\, d\alpha,$$

$$y>0. \qquad (3.31a\text{-}b)$$

$$\sigma_{yy} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4} c_j(\alpha)\,\alpha^2 e^{s_j\alpha y - i\alpha x}\, d\alpha - \frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha^2\Big[A_1(\alpha)\big(\lambda_5 e^{s_2|\alpha|y} - \lambda_7 e^{s_1|\alpha|y}\big) + A_2(\alpha)\big(\lambda_7 e^{s_2|\alpha|y} - \lambda_6 e^{s_1|\alpha|y}\big)\Big] e^{-i\alpha x}\, d\alpha,$$

$$\sigma_{xy} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4} c_j(\alpha)\,\alpha^2 s_j\, e^{s_j\alpha y - i\alpha x}\, d\alpha + \frac{i}{2\pi}\int_{-\infty}^{\infty}\alpha|\alpha|\Big[A_1(\alpha)\big(\lambda_5 s_2 e^{s_2|\alpha|y} - \lambda_7 s_1 e^{s_1|\alpha|y}\big) + A_2(\alpha)\big(\lambda_7 s_2 e^{s_2|\alpha|y} - \lambda_6 s_1 e^{s_1|\alpha|y}\big)\Big] e^{-i\alpha x}\, d\alpha,$$

$$y<0. \qquad (3.32a\text{-}b)$$

In the above expressions $S_3 = -S_1$ and $S_4 = -S_2$ have been used whenever needed.

3.4 BOUNDARY CONDITIONS

Referring to Fig. 1 and equation (2.7), the boundary conditions in the transformed plane can be written as

$$\sigma_{yy}(x, h_1) = 0 , \quad |x| < \infty,$$
$$\sigma_{xy}(x, h_1) = 0 , \quad |x| < \infty,$$
$$\sigma_{yy}(x, -h_2) = 0 , \quad |x| < \infty,$$
$$\sigma_{xy}(x, -h_2) = 0 , \quad |x| < \infty . \tag{3.33a-d}$$

$$\sigma_{yy}(x, 0) = -p_1(x) , \quad |x| < d,$$
$$V(x, +0) - V(x, -0) = 0 , \quad |x| > d . \tag{3.34}$$

$$\sigma_{xy}(x, 0) = -p_2(x) , \quad |x| < d$$
$$U(x, +0) - U(x, -0) = 0 , \quad |x| > d . \tag{3.35}$$

where

$$h_1 = \sqrt{\delta}\, H_1 , \quad h_2 = \sqrt{\delta}\, H_2 , \quad d = \frac{a}{\sqrt{\delta}} . \tag{3.36}$$

Using the first four homogeneous conditions we may evaluate $c_j(\alpha)$ $(j=1,\ldots,4)$ in terms of the displacement derivatives. The last two mixed conditions give a pair of singular integral equations to solve the displacement derivatives. The homogeneous boundary conditions give the following four linear algebraic equations for $c_j (j=1,\dots,4)$:

$$
\begin{aligned}
& e^{S_1\alpha h_1} C_1(\alpha) + e^{S_2\alpha h_1} C_2(\alpha) + e^{-S_1\alpha h_1} C_3(\alpha) + e^{-S_2\alpha h_1} C_4(\alpha) \\
& + e^{-S_1|\alpha| h_1} A_1(\alpha) + e^{-S_2|\alpha| h_1} A_2(\alpha) = 0 \;, \\
& e^{S_1\alpha h_1} S_1 C_1(\alpha) + e^{S_2\alpha h_1} S_2 C_2(\alpha) - e^{-S_1\alpha h_1} S_1 C_3(\alpha) - e^{-S_2\alpha h_1} S_2 C_4(\alpha) \\
& - \frac{|\alpha|}{\alpha} e^{-S_1|\alpha| h_1} S_1 A_1(\alpha) - \frac{|\alpha|}{\alpha} e^{-S_2|\alpha| h_1} S_2 A_2(\alpha) = 0 \;, \\
& e^{-S_1\alpha h_2} C_1(\alpha) + e^{-S_2\alpha h_2} C_2(\alpha) + e^{S_1\alpha h_2} C_3(\alpha) + e^{S_2\alpha h_2} C_4(\alpha) \\
& + \left(\lambda_5 e^{-S_2|\alpha| h_2} - \lambda_7 e^{-S_1|\alpha| h_2}\right) A_1(\alpha) + \left(\lambda_7 e^{-S_2|\alpha| h_2} - \lambda_6 e^{-S_1|\alpha| h_2}\right) A_2(\alpha) = 0 \;, \\
& S_1 e^{-S_1\alpha h_2} C_1(\alpha) + S_2 e^{-S_2\alpha h_2} C_2(\alpha) - S_1 e^{S_1\alpha h_2} C_3(\alpha) - S_2 e^{S_2\alpha h_2} C_4(\alpha) \\
& + \frac{|\alpha|}{\alpha} \left(\lambda_5 S_2 e^{-S_2|\alpha| h_2} - \lambda_7 S_1 e^{-S_1|\alpha| h_2}\right) A_1(\alpha) \\
& + \frac{|\alpha|}{\alpha} \left(\lambda_7 S_2 e^{-S_2|\alpha| h_2} - \lambda_6 S_1 e^{-S_1|\alpha| h_2}\right) A_2(\alpha) = 0 . \qquad (3.37a\text{-}d)
\end{aligned}
$$

Using the equation (3.26) and solving $c_j(\alpha)$ $(j=1,\dots,4)$ from the above expressions, after some lengthy manipulations we obtain

$$C_1(\alpha)=E\left[\frac{\lambda_{11}m_{10}(\alpha)+\lambda_{13}m_{12}(\alpha)}{m_9(\alpha)}\right]\frac{\bar{F}_2(\alpha)}{\alpha^2}+iE\left[\frac{\lambda_{12}m_{10}(\alpha)+\lambda_{14}m_{12}(\alpha)}{m_9(\alpha)}\right]\frac{\bar{F}_1(\alpha)}{|\alpha|\alpha}$$

$$C_2(\alpha)=E\left[\frac{\lambda_{11}m_{14}(\alpha)+\lambda_{13}m_{15}(\alpha)}{m_9(\alpha)}\right]\frac{\bar{F}_2(\alpha)}{\alpha^2}+iE\left[\frac{\lambda_{12}m_{14}(\alpha)+\lambda_{14}m_{15}(\alpha)}{m_9(\alpha)}\right]\frac{\bar{F}_1(\alpha)}{\alpha|\alpha|}$$

$$C_3(\alpha)=E\left[\frac{\lambda_{11}m_{11}(\alpha)+\lambda_{13}m_{13}(\alpha)}{m_9(\alpha)}\right]\frac{\bar{F}_2(\alpha)}{\alpha^2}+iE\left[\frac{\lambda_{12}m_{11}(\alpha)+\lambda_{14}m_{13}(\alpha)}{m_9(\alpha)}\right]\frac{\bar{F}_1(\alpha)}{\alpha|\alpha|}$$

$$C_4(\alpha)=E\left[\frac{\lambda_{11}m_{16}(\alpha)+\lambda_{13}m_{17}(\alpha)}{m_9(\alpha)}\right]\frac{\bar{F}_2(\alpha)}{\alpha^2}+iE\left[\frac{\lambda_{12}m_{16}(\alpha)+\lambda_{14}m_{17}(\alpha)}{m_9(\alpha)}\right]\frac{\bar{F}_1(\alpha)}{\alpha|\alpha|}\quad(3.38a\text{-}d)$$

where the functions $m_j(\alpha)$ are defined in the appendix II. It can be shown that $c_j(\alpha) \to 0$ $(j=1,\ldots,4)$ as $\alpha \to \mp\infty$.

3.5 SINGULAR INTEGRAL EQUATIONS

Substituting the stress expressions into (3.34) and (3.35), after some lengthy manipulations we obtain the integral equations in the following form (see Appendix III for derivation):

$$\int_{-d}^{d}\left(\frac{F_1(t)}{t-x}+\sum_{j=1}^{2}K_{1j}(t,x)F_j(t)\right)dt=-\frac{\pi}{E(\lambda_{12}+\lambda_{14})}P_1(x)$$

$$\int_{-d}^{d}\left(\frac{F_2(t)}{t-x}+\sum_{j=1}^{2}K_{2j}(t,x)F_j(t)\right)dt=-\frac{\pi}{E(S_1\lambda_{11}+S_2\lambda_{13})}P_2(x)\quad(3.39)$$

Two additional conditions expressing the continuity of the displacements outside the crack are expressed as

$$\int_{-d}^{d}F_1(t)dt=0\,,\quad\int_{-d}^{d}F_2(t)dt=0\,.\qquad(3.40a,b)$$

The kernels $K_{ij}(t,x)$ $(i,j = 1,2)$ are defined in Appendix II. The integral equations in the real plane may be obtained by changing the variables in (3.39) as follows

$$x = \frac{x_1}{\sqrt{\delta}}\,, \quad t = \frac{t_1}{\sqrt{\delta}}\,, \quad y = x_2\sqrt{\delta}\,. \qquad (3.41a\text{-}c)$$

where x, y, t are in the transformed plane and x_1, x_2, t_1 are in the real plane. Similarly

$$G_1(x_1) = \frac{\partial}{\partial x_1}\Big(U_2(x_1,+0) - U_2(x_1,-0)\Big)$$

$$G_2(x_1) = \frac{\partial}{\partial x_1}\Big(U_1(x_1,+0) - U_1(x_1,-0)\Big) \qquad (3.42a,b)$$

Using the equations (2.7a-d) and (3.10a,b) it can be shown that

$$F_1(x) = G_1(x_1)\,, \quad F_2(x) = \delta\, G_2(x_1)\,. \qquad (3.43a,b)$$

Using (3.41) and (3.43), the integral equations in the real plane become

$$\int_{-a}^{a}\left(\frac{G_1(t_1)}{t_1 - x_1} + \sum_{j=1}^{2}\delta^{j-1}\bar{K}_{1j}(t_1,x_1)\,G_j(t_1)\right)dt_1 = -\frac{\pi\delta}{E\lambda_{15}}\,p(x_1)$$

$$\int_{-a}^{a}\left(\frac{\delta\, G_2(t_1)}{t_1 - x_1} + \sum_{j=1}^{2}\delta^{j-1}\bar{K}_{2j}(t_1,x_1)\,G_j(t_1)\right)dt_1 = -\frac{\pi}{E\lambda_{16}}\,q(x_1) \qquad (3.44a,b)$$

$$|x_1| < a$$

$$\int_{-a}^{a} G_1(t_1)\,dt_1 = 0, \quad \int_{-a}^{a} G_2(t_1)\,dt_1 = 0. \qquad (3.45a,b)$$

where

$$\lambda_{15} = \lambda_{12} + \lambda_{14}\,, \quad \lambda_{16} = S_1\lambda_{11} + S_2\lambda_{13}\,,$$

$$p(x_1) = -\sigma_{22}(x_1, 0) ,$$

$$q(x_1) = -\sigma_{12}(x_1, 0) , \qquad (3.46a\text{-}d)$$

$$\bar{K}_{ij}(t_1, x_1) = \frac{1}{\sqrt{\delta}} K_{ij}(t, x) , \; (i, j = 1, 2) . \qquad (3.47)$$

and 2a is the crack length in the real plane. The expressions of $K_{ij}(t,x)$ $(i,j=1,2)$ are of the following form (see Appendix II):

$$K_{ij}(t, x) = \int_0^\infty H_{ij}\left(e^{\Gamma_1 \alpha}, e^{\Gamma_2 \alpha}, e^{\Gamma_3 \alpha}, e^{\Gamma_4 \alpha}\right) \begin{matrix} \sin \\ \cos \end{matrix} \alpha(t-x) d\alpha \qquad (3.48)$$

Using the equations (3.41) and (3.48), (3.47) becomes

$$\bar{K}_{ij}(t_1, x_1) = \frac{1}{\sqrt{\delta}} \int_0^\infty H_{ij}\left(e^{R_1 \alpha}, e^{R_2 \alpha}, e^{R_3 \alpha}, e^{R_4 \alpha}\right) \begin{matrix} \sin \\ \cos \end{matrix} \frac{\alpha}{\sqrt{\delta}}(t_1 - x_1) d\alpha \qquad (3.49)$$

where

$$R_1 = (S_1 + S_2) H_1 \sqrt{\delta} , \quad R_2 = (S_1 - S_2) H_1 \sqrt{\delta} ,$$

$$R_3 = (S_1 + S_2) H_2 \sqrt{\delta} , \quad R_4 = (S_1 - S_2) H_2 \sqrt{\delta} . \qquad (3.50a\text{-}d)$$

Replacing $\frac{\alpha}{\sqrt{\delta}}$ by α, the equation (3.49) becomes

$$\bar{K}_{ij}(t_1 - x_1) = \int_0^\infty H_{ij}\left(e^{\bar{\Gamma}_1 \alpha}, e^{\bar{\Gamma}_2 \alpha}, e^{\bar{\Gamma}_3 \alpha}, e^{\bar{\Gamma}_4 \alpha}\right) \begin{matrix} \sin \\ \cos \end{matrix} \alpha(t_1 - x_1) d\alpha \qquad (3.51)$$

where

$$\bar{\Gamma}_1 = (S_1 + S_2) H_1 \delta , \quad \bar{\Gamma}_2 = (S_1 - S_2) H_1 \delta ,$$

$$\bar{\Gamma}_3 = (S_1 + S_2) H_2 \delta , \quad \bar{\Gamma}_4 = (S_1 - S_2) H_2 \delta . \qquad (3.52a\text{-}d)$$

From the equation (3.48) and (3.51), we can conclude that $\bar{K}_{ij}(t_1,x_1)$ and $K_{ij}(t,x)$ $(i,j=1,2)$ have the same expressions if $r_1,\ldots,r_4$ are replaced by $\bar{r}_1,\ldots,\bar{r}_4$. If $x_2 = 0$ is a plane of symmetry, then $\bar{K}_{12}(t_1,x_1) = 0 = \bar{K}_{21}(t_1,x_1)$ and system of integral equations reduce to two uncoupled integral equations.

3.6 NORMALIZATION

Changing the variables as

$$\begin{aligned} t_1 &= ar, \quad -a < t_1 < a, \quad -1 < r < 1 \\ x_1 &= as, \quad -a < x_1 < a, \quad -1 < s < 1 \end{aligned} \tag{3.53}$$

After normalization integral equations take the form

$$\int_{-1}^{1}\left(\frac{1}{r-s} + k_{11}(r,s)\right)g_1(r)dr + \int_{-1}^{1}\delta\, k_{12}(r,s)g_2(r)dr = -\frac{\pi\delta}{E\lambda_{15}}\, p^*(s)$$

$$\delta\int_{-1}^{1}\left(\frac{1}{r-s} + k_{22}(r,s)\right)g_2(r)dr + \int_{-1}^{1}k_{21}(r,s)g_1(r)dr = -\frac{\pi}{E\lambda_{16}}\, q^*(s) \tag{3.54a,b}$$

$$\int_{-1}^{1}g_1(r)dr = 0, \quad \int_{-1}^{1}g_2(r)dr = 0. \tag{3.55a,b}$$

where

$$\begin{aligned} g_j(r) &= G_j(t_1), \quad k_{ij}(r,s) = \bar{K}_{ij}(t_1,x_1), \quad (i,j=1,2), \\ p^*(s) &= p(x_1), \\ q^*(s) &= q(x_1). \end{aligned} \tag{3.56}$$

Since the crack has integrable singularities at both ends, the solution will be sought in the form

$$g_j(r) = \phi_j^*(r)(1-r^2)^{-1/2}, \quad (j=1,2) \tag{3.57}$$

where $\phi_1^*(r)$ and $\phi_2^*(r)$ are Holder-continuous in the interval $-1 \le r \le 1$.

The singular integral equations (3.54a,b) subject to single-valuedness conditions (3.55a,b) are solved by using Gauss-Chebyshev integration formulas. Thus, equations (3.54a,b) and (3.55a,b) are, respectively, replaced by [10]

$$\sum_{j=1}^{n}\left[\left(\frac{1}{r_j - s_i} + k_{11}(r_j, s_i)\right)\phi_1^*(r_j) + \delta\, k_{12}(r_j, s_i)\phi_2^*(r_j)\right]W_j = -\frac{\pi\delta}{E\lambda_{15}} p^*(s_i)$$

$$\sum_{j=1}^{n}\left[\left(\frac{1}{r_j - s_i} + k_{22}(r_j, s_i)\right)\phi_2^*(r_j)\delta + k_{21}(r_j, s_i)\phi_1^*(r_j)\right]W_j = -\frac{\pi}{E\lambda_{16}} q^*(s_i)$$

$$i = 1, \ldots, n-1 \qquad (3.58a,b)$$

$$\sum_{j=1}^{n}\phi_1^*(r_j)W_j = 0, \quad \sum_{j=1}^{n}\phi_2^*(r_j)W_j = 0 \qquad (3.59a,b)$$

where

$$r_j = \cos\left(\pi\frac{j-1}{n-1}\right), \quad j = 1, \ldots, n \qquad (3.60)$$

$$s_i = \cos\left(\pi\frac{2i-1}{n-1}\right), \quad i = 1, \ldots, n-1 \qquad (3.61)$$

$$W_1 = W_n = \frac{\pi}{2(n-1)}, \quad W_j = \frac{\pi}{n-1}, \quad j = 2, \ldots n-1 \qquad (3.62)$$

The equations (3.58a,b) and (3.59a,b) provide 2n equations to solve the unknown functions $\phi_1^*(r_j)$ and $\phi_2^*(r_j)$ at 2n discrete points.

3.7 STRESS INTENSITY FACTORS

The stress intensity factors at the crack tips are defined as

$$\begin{matrix} k_{1A} \\ k_{2A} \end{matrix} = \lim_{x_1 \to -a} \sqrt{2(-a-x_1)} \; \begin{matrix} \sigma_{22}(x_1,0) \\ \sigma_{12}(x_1,0) \end{matrix} \qquad (3.63a)$$

$$\begin{matrix} k_{1B} \\ k_{2B} \end{matrix} = \lim_{x_1 \to a} \sqrt{2(x_1-a)} \; \begin{matrix} \sigma_{22}(x_1,0) \\ \sigma_{12}(x_1,0) \end{matrix} \qquad (3.63b)$$

Let us consider the following sectionally holomorphic function

$$F(z) = \frac{1}{\pi}\int_a^b \frac{\phi^*(t)dt}{t-z} \qquad (3.64)$$

$$\phi^*(t) = g(t)(b-t)^{\alpha}(t-a)^{\beta} = g(t)(t-b)^{\alpha}(t-a)^{\beta} e^{-\pi i\alpha}$$

where g(t) is Holder-continuous function in the interval $a \leq t \leq b$ and $-1 < Re(\alpha) < 0 \quad -1 < Re(\beta) < 0$. Examination of the singular behavior of F(z) around the end points was investigated by Muskhelishvili [9] and was shown to be in the following form

$$F(z) = -\frac{g(a)(b-a)^{\alpha} e^{-\pi i\beta}}{\sin \pi\beta}(z-a)^{\beta} + \frac{g(b)(b-a)^{\beta}}{\sin \pi\beta}(z-b)^{\alpha} + G(z) \qquad (3.65)$$

The function G(z) is bounded everywhere except possibly at the ends a, b where it may have the following behavior

$$|G(z)| < \frac{A}{|z-a|^{\rho}}, \qquad \rho < -Re(\beta)$$

$$|G(z)| < \frac{B}{|z-b|^{r}}, \qquad r < -Re(\alpha)$$

Using the equations (3.64) and (3.65), the singular behavior of stresses around the crack tips can be expressed as

$$\frac{\delta}{E\lambda_{15}}\sigma_{22}(x_1,0) = \frac{(2a)^{1/2}}{2}\left(\frac{\phi_1^*(-1)}{\sqrt{-a-x_1}} - \frac{\phi_1^*(1)}{\sqrt{x_1-a}}\right)$$

$$\frac{\sigma_{12}(x_1,0)}{\delta E\lambda_{16}} = \frac{(2a)^{1/2}}{2}\left(\frac{\phi_2^*(-1)}{\sqrt{-a-x_1}} - \frac{\phi_2^*(1)}{\sqrt{x_1-a}}\right) \qquad (3.66a,b)$$

Substitution of (3.66a,b) into (3.63a,b) gives the stress intensity factors as

$$k_{1B} = -\frac{E}{\delta}\lambda_{15}\sqrt{a}\,\phi_1^*(1), \quad k_{1A} = \frac{E}{\delta}\lambda_{15}\sqrt{a}\,\phi_1^*(-1), \qquad (3.67a,b)$$

$$k_{2B} = -E\lambda_{16}\delta\sqrt{a}\,\phi_2^*(1), \quad k_{2A} = E\lambda_{16}\delta\sqrt{a}\,\phi_2^*(-1).$$

In the system of linear equations, the kernels may be evaluated for given $\frac{\delta H_1}{a}$ and $\frac{\delta H_2}{a}$. In this case, the stress intensity factors can be found without replacing the value of δ by loading normal and shear stresses separately on the crack surface. The proof is given as follows:

CASE I: NO SHEAR STRESS ON THE CRACK SURFACE

$$\left(p^*(s) \neq 0,\ q^*(s) = 0\right)$$

Since $q^*(s) = 0$ and, the kernels can be evaluated for given $\delta H_1/a$ and $\delta H_2/a$. From the equations (3.58a,b) and (3.59a,b) we can solve $\phi_1^*(r_j)/\delta$ and $\phi_2^*(r_j)$ $(j=1,n)$. Then, using the equations (3.67a,b), it may be seen that for given fixed values of $\frac{\delta H_1}{a}$ and $\delta H_2/a$, k_1 does not depend on δ explicitly and k_2 is proportional to δ.

CASE II: NO PRESSURE ON THE CRACK SURFACE

$$\left(p^*(s) = 0,\ q^*(s) \neq 0\right)$$

Similarly, for a given $\delta H_1/a$ and $\delta H_2/a$, the $\phi_1^*(r_j)$ and $\delta\phi_2^*(r_j)$ can be solved from the system of linear equations. Using equations (3.67a,b), it may again be shown that k_1 is inversely proportional to δ and k_2 does not depend on δ explicitly. If $H_1 = H_2 = H$, then $k_2 = 0$ for case I and $k_1 = 0$ for case II, therefore, k_1 and k_2 depend on δ only through the combination $\delta H/a$ and not explicitly. From the definition of stress intensity factors, the stresses at the crack end may be expressed as

$$\sigma_{22}(x_1,0) = \frac{k_1}{\sqrt{2(x_1-a)}}, \qquad (x_1>a);$$

$$\sigma_{12}(x_1,0) = \frac{k_2}{\sqrt{2(x_1-a)}}, \qquad (x_1>a). \qquad (3.68a,b)$$

Using the equations (3.67a,b)

$$\phi_1^*(1) = -\frac{\delta k_1}{E\lambda_{15}\sqrt{a}} = \lim_{x_1\to a}\left(\phi_1^*\left(\frac{x_1}{a}\right)+O(a-x_1)\right),$$

$$(3.69a,b)$$

$$\phi_2^*(1) = -\frac{k_2}{E\delta\lambda_{16}\sqrt{a}} = \lim_{x_1\to a}\left(\phi_2^*\left(\frac{x_1}{a}\right)+O(a-x_1)\right); \; x_1<a.$$

since

$$\phi_1^*(s) = g_1(s)\sqrt{1-s^2}, \quad \phi_2^*(s) = g_2(s)\sqrt{1-s^2},$$

$$s = \frac{x_1}{a}, \quad g_1(s) = G_1(x_1), \quad g_2(s) = G_2(x_1)$$

and $G_1(x_1)$, $G_2(x_1)$ are displacement derivatives, they can be expressed as

$$G_1(x_1) = \frac{a\,\Phi_1^*\left(\frac{x_1}{a}\right)}{\sqrt{(a+x_1)(a-x_1)}} ,$$

$$G_2(x_1) = \frac{a\,\Phi_2^*\left(\frac{x_1}{a}\right)}{\sqrt{(a+x_1)(a-x_1)}} \qquad (3.70a,b)$$

where $x_1 < a$.

Using equations (3.69a,b), (3.70a,b) become

$$G_1(x_1) = -\frac{\delta k_1}{E\lambda_{15}}\left[\frac{1}{\sqrt{2(a-x_1)}} + O(a-x_1)\right], \; x_1 < a$$

$$(3.71a,b)$$

$$G_2(x_1) = -\frac{k_2}{E\delta\lambda_{16}}\left[\frac{1}{\sqrt{2(a-x_1)}} + O(a-x_1)\right], \; x_1 < a$$

Using equations (3.42a,b) and (3.71a,b), we obtain

$$U_2(x_1,+0) - U_2(x_1,-0) = \frac{\delta k_1}{E\lambda_{15}}\int_{x_1}^{a}\frac{dx_1}{\sqrt{2(a-x_1)}}$$

$$(3.72a,b)$$

$$U_1(x_1,+0) - U_1(x_1,-0) = \frac{k_2}{E\delta\lambda_{16}}\int_{x_1}^{a}\frac{dx_1}{\sqrt{2(a-x_1)}}$$

After evaluating the integrals, we get

$$U_2(x_1,+0) - U_2(x_1,-0) = \frac{\delta k_1}{E\lambda_{15}}\sqrt{2(a-x_1)}, \; x_1 < a$$

$$(3.73a,b)$$

$$U_1(x_1,+0) - U_1(x_1,-0) = \frac{k_2}{E\delta\lambda_{16}}\sqrt{2(a-x_1)}, \; x_1 < a$$

Now let the crack front advance parallel to itself by an amount da. The externally added (Δu) and internally released energy

(ΔV) per unit crack front may then be expressed as

$$\Delta U-\Delta V=\int_a^{a+da}\frac{1}{2}\Big[\sigma_{22}(x_1,0)\big(u_2(x_1-da,+0)-u_2(x_1-da,-0)\big)$$

$$+\sigma_{12}(x_1,0)\big(u_1(x_1-da,+0)-u_1(x_1-da,-0)\big)\Big]dx_1 \,. \qquad (3.74)$$

Substitution of (3.68a,b) and (3.73a,b) into (3.74) gives

$$\Delta U-\Delta V=\frac{1}{2}\left(\frac{\delta k_1^2}{E\lambda_{15}}+\frac{k_2^2}{E\delta\lambda_{16}}\right)\int_a^{a+da}\left(\frac{a+da-x_1}{a-x_1}\right)dx_1$$

or

$$\Delta U-\Delta V=\frac{\pi}{4E}\left(\frac{\delta k_1^2}{\lambda_{15}}+\frac{k_2^2}{\delta\lambda_{16}}\right) \qquad (3.75)$$

Since u-v is a "potential" and a is a "distance", consequently

$$G=\frac{d}{da}(U-V) \qquad (3.76)$$

has the dimension of "force" which is also known as the "crack extension force". From (3.75) and (3.76), we get

$$G=\frac{\pi}{4E}\left(\frac{\delta k_1^2}{\lambda_{15}}+\frac{k_2^2}{\delta\lambda_{16}}\right) \qquad (3.77)$$

The stress intensity factors are calculated for various crack geometries under various loading conditions. Therefore from equation (3.77) G can be found. For isotropic cases (plane stress and plane strain) $\kappa=1=\delta$, $S_1=1=S_2$. For this case, the value of λ_{15} and λ_{16} may be obtained as $\lambda_{15}=\frac{1}{2}=\lambda_{16}$. From equation (3.77) G becomes

$$G=\frac{\pi}{2E}\left(k_1^2+k_2^2\right) \qquad (3.78)$$

4. THE PROBLEM OF CONTINUOUS CONTACT

Referring to Figure 1(b), the problem of the frictionless full contact, can be solved with a slight modification in the integral equation (3.44a). In this study we will consider the symmetric problem (i.e. $x_1=0$ and $x_2=0$ are planes of symmetry), the equations (3.5a,b) imply that $x_2=0$ has to be a plane of symmetry (i.e. crack must be located in the middle of the strip) but $x_1=0$ may not be a plane of symmetry.

For symmetric problem $\bar{K}_{12}(t_1,x_1) = 0$, the integral equation (3.44a) becomes

$$\int_{-a}^{a}\left(\frac{1}{t_1-x_1}+\bar{K}_{11}(t_1,x_1)\right)G_1(t_1)dt_1 = -\frac{\pi\delta}{E\lambda_{15}}p(x_1). \quad (4.1)$$

Noting that $p(x_1) = 0$ for $b < |x_1| < a$ and assuming a flat wedge (i.e., $G_1(x_1) = 0$ for $|x_1| < b$) (4.1) becomes

$$\int_{-a}^{-b}\left(\frac{1}{t_1-x_1}+\bar{K}_{11}(t_1,x_1)\right)G_1(t_1)dt_1+\int_{b}^{a}\left(\frac{1}{t_1-x_1}+\bar{K}_{11}(t_1,x_1)\right)G_1(t_1)dt_1=0 \quad (4.2)$$

which, by using the conditions of symmetry with respect to $x_1 = 0$ plane, may be reduced to

$$\int_{b}^{a}\left(\frac{1}{t_1-x_1}+\frac{1}{t_1+x_1}+\bar{K}_1(t_1,x_1)\right)G_1(t_1)dt_1 = 0 \quad (4.3)$$

where

$$\bar{K}_1(t_1,x_1) = \bar{K}_{11}(t_1,x_1) - \bar{K}_{11}(-t_1,x_1)\,, \quad b<x_1<a.$$

Also, the pressure distribution along the contact area can be expressed as

$$-\frac{\pi\delta}{E\lambda_{15}}\,p(x_1)=\int_b^a\left(\frac{1}{t_1-x_1}+\frac{1}{t_1+x_1}+\bar{K}_1(t_1,x_1)\right)G_1(t_1)\,dt_1$$

$$-b<x_1<b \qquad (4.4)$$

In this problem if we assume that $G_1(x_1) = -G_1(-x_1)$ the single-valuedness condition is automatically satisfied and, from the physics of the problem the additional condition may be expressed as

$$\int_b^a G_1(t_1)\,dt_1=-V_0 \qquad (4.5)$$

where v_0 is the total thickness of the wedge.

To simplify the numerical analysis, the following dimensionless quantities are introduced

$$r=\frac{2t_1}{a-b}-\frac{a+b}{a-b}\,,\quad -1<r<1,\ b<t_1<a,$$

$$s_1=\frac{2x_1}{a-b}-\frac{a+b}{a-b}\,,\quad -1<s_1<1,\ b<x_1<a,$$

$$s_3=\frac{x_3}{b}\,,\quad -1<s_3<1,\ -b<x_3<b. \qquad (4.6a\text{-}c)$$

After normalization, the equations (4.3), (4.5) and (4.4) are, respectively, replaced by

$$\int_{-1}^{1}\left(\frac{1}{r-s}+\frac{1}{r+s+\frac{2(a+b)}{a-b}}+k_1(r,s)\right)g_1^*(r)\,dr=0\,,$$

$$\int_{-1}^{1}g_1^*(r)\,dr=-\frac{2V_0}{a-b}. \qquad (4.7a,b)$$

$$-\frac{\pi\delta}{E\lambda_{15}}\,p^*(s)=\int_{-1}^{1}\left(\frac{1}{r-\frac{2b}{a-b}s+\frac{a+b}{a-b}}+\frac{1}{r+\frac{2b}{a-b}+\frac{a+b}{a-b}}+k_2(r,s)\right)g_1^*(r)\,dr$$

$$(4.8)$$

where

$$g_1^*(r) = G_1(t_1) \ , \quad p^*(s) = p(x_1) \tag{4.9}$$

$$k_1(r, s_1) = \bar{K}_1(t_1, x_1) \ , \quad k_2(r, s_3) = \bar{K}_1(t_1, x_3)$$

and since the variables s_1 and s_3 vary between -1 and +1, the subscripts have been deleted.

Noting that the index of the singular integral equation (4.7a) is +1, its solution may be expressed as

$$g_1^*(r) = \frac{\phi^*(r)}{\sqrt{1-r^2}} \ , \quad -1 < r < 1 . \tag{4.10}$$

The equations (4.7a,b) may then be replaced by

$$\sum_{j=1}^{n} \left(\frac{1}{r_j - s_i} + \frac{1}{r_j + s_i + \frac{2(a+b)}{a-b}} + k_1(r_j, s_i) \right) W_j \phi^*(r_j) = 0 ,$$

$$i = 1, \cdot\cdot, n-1$$

$$\sum_{j=1}^{n} \phi^*(r_j) W_j = - \frac{2 V_0}{a-b} \tag{4.11a,b}$$

where r_j, s_i, and w_j are respectively given by equations (3.60), (3.61) and (3.62).

The unknowns $\overset{*}{\phi}(r_j)$ (j=1·,n) are determined from the system of equations (4.11), using the $\phi(r_j)$ and (4.8); the pressure at the various points can be evaluated as

$$-\frac{\pi \delta}{E \lambda_{15}} p^*(s_i) = \sum_{j=1}^{n} \left(\frac{1}{r_j - \frac{2b}{a-b} s_i + \frac{a+b}{a-b}} + \frac{1}{r_j + \frac{2b}{a-b} s_i + \frac{a+b}{a-b}} + k_2(r_j, s_i) \right) W_j \phi^*(r_j) \ , \quad i = 1, \cdot\cdot n-1 . \tag{4.12}$$

It should be observed that the integral equation is valid provided the contact stresses obtained from (4.12) is compressive everywhere, i.e. for $0<x_1<b$. To investigate the separation on the interface we need to know b_{cr} which can be determined from the condition that the contact pressure be zero at $x_1=0$. The value of b_{cr} may be found for a given crack geometry by iterating equations (4.11) and (4.12).

5. THE PROBLEM OF INTERFACE SEPARATION $b > b_{cr}$

Since the contact stress may be tensile, for $b > b_{cr}$ there would be separation in the neighborhood of $x_1=0$ on both sides of the wedge. Let the separation area be described by $-c<x_1<c$ (see Fig. 1(c)), where c is unknown and is a function of $\frac{H}{a}$, $\frac{b}{a}$ and material parameters.

In this case, too, the problem can be solved by slightly modifying the integral equation (4.1). In this problem we have

$$G_1(t_1) = \begin{cases} \bar{\Phi}_1(t_1) , & b < |t_1| < a \\ 0 , & c < |t_1| < b \\ \bar{\Phi}_2(t_2) & -c < t_1 < c \end{cases} \tag{5.1a}$$

$$\bar{\Phi}_1(t_1) = -\bar{\Phi}_1(-t_1) , \quad b < t_1 < a \tag{5.1b}$$

$$p(x_1) = 0 , \quad |x_1| < c , \quad b < |x_1| < a . \tag{5.1c}$$

Using equations (5.1a), the equation (4.1) becomes

$$\left(\int_{-a}^{-b}+\int_{b}^{a}\right)\left(\frac{1}{t_1-x_1}+\bar{K}_{11}(t_1,x_1)\right)\bar{\Phi}_1(t_1)dt_1+\int_{-c}^{c}\left(\frac{1}{t_1-x_1}\right.$$

$$\left.+\bar{K}_{11}(t_1,x_1)\right)\bar{\Phi}_2(t_1)dt_1=\begin{cases}-\dfrac{\pi\delta}{E\lambda_{15}}p(x_1), & c<|x_1|<b\\ 0, & |x_1|<c,\ b<|x_1|<a.\end{cases} \tag{5.2}$$

Using equation (5.1b), then (5.2) can be modified as

$$\int_{b}^{a}\left(\frac{1}{t_1-x_1}+\frac{1}{t_1+x_1}+\bar{K}_1(t_1,x_1)\right)\bar{\Phi}_1(t_1)dt_1+\int_{-c}^{c}\left(\frac{1}{t_1-x_1}\right.$$

$$\left.+\bar{K}_{11}(t_1,x_1)\right)\bar{\Phi}_2(t_1)dt_1=\begin{cases}0, & b<x_1<a & (5.3)\\ -\dfrac{\pi\delta}{E\lambda_{15}}p(x_1), & c<x_1<b & (5.4)\\ 0, & -c<x_1<c & (5.5)\end{cases}$$

where

$$\bar{K}_1(t_1,x_1)=\bar{K}_{11}(t_1,x_1)-\bar{K}_{11}(-t_1,x_1).$$

The equation (4.5) may be replaced by

$$\int_{b}^{a}\bar{\Phi}_1(t_1)dt_1=-V_c \tag{5.6}$$

Introducing the following dimensionless quantities as

$$r_1=\frac{2t_1}{a-b}-\frac{a+b}{a-b},\quad b<t_1<a,\ -1<r_1<1$$

$$r_3=\frac{t_3}{c},\quad -c<t_1<a,\ -1<r_3<1$$

$$s_1 = \frac{2x_1}{a-b} - \frac{a+b}{a-b}, \quad b < x_1 < a, \; -1 < s_1 < 1$$

$$s_3 = \frac{x_3}{c}, \quad -c < x_3 < c, \; -1 < s_3 < 1 \qquad (5.7)$$

$$s_4 = \frac{2x_4}{b-c} - \frac{b+c}{b-c}, \quad c < x_4 < b, \; -1 < s_4 < 1$$

After normalization, the equations (5.3), (5.5), (5.6) and (5.4) respectively become

$$\int_{-1}^{1}\left(\frac{1}{r-s} + \frac{1}{r+s+\frac{2(a+b)}{a-b}} + h_1(r,s)\right)\bar{g}_1(r)dr$$

$$+\int_{-1}^{1}\left(\frac{1}{r-\frac{a-b}{2c}s-\frac{a+b}{2c}} + h_2(r,s)\right)\bar{g}_2(r)dr = 0 \qquad (5.8)$$

$$\int_{-1}^{1}\left(\frac{1}{r-\frac{2c}{a-b}s+\frac{a+b}{a-b}} + \frac{1}{r+\frac{2c}{a-b}s+\frac{a+b}{a-b}} + h_3(r,s)\right)\bar{g}_1(r)dr$$

$$+\int_{-1}^{1}\left(\frac{1}{r-s} + h_4(r,s)\right)\bar{g}_2(r)dr = 0 \qquad (5.9)$$

$$\int_{-1}^{1}\bar{g}_1(r)dr = -\frac{2V_c}{a-b} \qquad (5.10)$$

$$\int_{-1}^{1}\left(\frac{1}{r+\frac{a+b}{a-b}-\frac{b-c}{a-b}s-\frac{b+c}{a-b}} + \frac{1}{r+\frac{a+b}{a-b}+\frac{b-c}{a-b}s+\frac{b+c}{a-b}}\right.$$

$$\left.+h_5(r,s)\right)\bar{g}_1(r)dr + \int_{-1}^{1}\left(\frac{1}{r-\frac{b-c}{2c}s-\frac{b+c}{2c}} + h_6(r,s)\right)\bar{g}_2(r)dr$$

$$= -\frac{\pi\delta}{E\lambda_{15}}\,p^*(s) \qquad (5.11)$$

where

$$g_1(r_1) = \bar{\Phi}_1(t_1)\,,\; g_2(r_3) = \bar{\Phi}_2(t_3)\,,\; p^*(s_4) = p(x_4)\,,$$
$$h_1(r_1, s_1) = \bar{K}_1(t_1, x_1)\,,\; h_2(r_3, s_1) = \bar{K}_{11}(t_3, x_1)\,,$$
$$h_3(r_1, s_3) = \bar{K}_1(t_1, x_3)\,,\; h_4(r_3, s_3) = \bar{K}_{11}(t_3, x_3)\,, \qquad (5.12)$$
$$h_5(r_1, s_4) = \bar{K}_1(t_1, x_4)\,,\; h_6(r_3, s_4) = \bar{K}_{11}(t_3, x_4)\,.$$

and since the variables r_1, r_3, s_1, s_3, s_4 all vary between -1 and +1, the subscripts have been deleted.

To solve the system of integral equations, in this problem it is more convenient to assume that (5.9) as well as (5.8) has an index +1 and let

$$\bar{g}_1(r) = \frac{\psi_1(r)}{\sqrt{1-r^2}}\,, \quad -1 < r < 1$$
$$\bar{g}_2(r) = \frac{\psi_2(r)}{\sqrt{1-r^2}}\,, \quad -1 < r < 1 \qquad (5.13)$$

To insure smooth contact at the end points of the separation area we then impose the following conditions on ψ_2

$$\psi_2(-1) = 0\,, \quad \psi_2(1) = 0\,. \qquad (5.14a,b)$$

Using now the Gauss-Chebyshev integration formula (5.8), (5.9) and (5.10) become

$$\sum_{j=1}^{n}\left(\frac{1}{r_j - s_i} + \frac{1}{r_j + s_i + \frac{2(a+b)}{a-b}} + h_1(r_j, s_i)\right) W_j\, \psi_1(r_j) +$$

$$\sum_{j=1}^{n}\left(\frac{1}{r_j-\frac{a-b}{2c}s_i-\frac{a+b}{a-b}}+h_2(r_j,s_i)\right)w_j\,\psi_2(r_j)=0 \qquad (5.15)$$

$$i=1,\cdots,n-1$$

$$\sum_{j=1}^{n}\left(\frac{1}{r_j-\frac{2c}{a-b}s_i+\frac{a+b}{a-b}}+\frac{1}{r_j+\frac{2c}{a-b}s_i+\frac{a+b}{a-b}}\right. \qquad (5.16)$$

$$\left.+h_3(r_j,s_i)\right)w_j\,\psi_1(r_j)+\sum_{j=1}^{n}\left(\frac{1}{r_j-s_i}+h_4(r_j,s_i)\right)w_j\,\psi_2(r_j)=0$$

$$i=1,\cdots,n-1$$

$$\sum_{j=1}^{n}\psi_1(r_j)\,w_j=-\frac{2V_0}{a-b} \qquad (5.17)$$

where r_j, s_i and w_j are respectively given by equations (3.60), (3.61) and (3.62). Thus (5.14)-(5.17) give 2n+1 algebraic equations to determine 2n+1 unknowns $\psi_1(r_j)$, $\psi_2(r_j)$ $i = 1,n$ and c. The system is nonlinear. However, the problem may be somewhat simplified by assuming that c is given. Thus, the linear system consisting of (5.14a,b), (5.15), (5.17) and n-2 equations from (5.16) may be solved for various values of c and the correct value of c may be determined from the last equation in (5.16) by using an iteration technique to satisfy that equation. It should be noted that from (5.11), the pressure distribution on the contact region may easily be evaluated as

$$-\frac{\pi\delta}{E\lambda_{15}}p^*(s_i)=\sum_{j=1}^{n}\left(\frac{1}{r_j+\frac{a+b}{a-b}-\frac{b-c}{a-b}s_i-\frac{b+c}{a-b}}\right.$$

$$\left.+\frac{1}{r_j+\frac{a+b}{a-b}+\frac{b-c}{a-b}s_i+\frac{b+c}{a-b}}+h_5(r_j,s_i)\right)W_j\,\psi_1(r_j)$$

$$+\sum_{j=1}^{n}\left(\frac{1}{r_j-\frac{b-c}{2c}s_i-\frac{b+c}{2c}}+h_6(r_j,s_i)\right)W_j\,\psi_2(r_j) \qquad i=1,\dots,n-1 \tag{5.18}$$

From the system of linear equation (5.14)-(5.17), we can determine $\psi_1(r_j)$, $\psi_2(r_j)$ and c. Substitution of these into (5.18) gives the value of pressures at various points.

6. RESULTS AND DISCUSSIONS

The singular integral equations (3.44a,b) and (3.45a,b) were solved numerically by first normalizing the interval (-a,a) to (-1,1) and then using the procedure outlined in part 3.6 for the middle locations of the crack. For values of $\frac{H_1}{H_2}$ close to one, no difficulty of convergence was encountered. However, for $\frac{H_2}{H_1} \gg 1$, especially for large relative crack lengths, more collocation points had to be used to improve the accuracy.

The stress intensity factors are usually calculated without choosing the value of δ, by specifying $\frac{\delta H}{a}$, and by loading through normal and shear stresses separately on the crack surface. The variation of the stress intensity factors with material parameters (i.e. stiffness ratio and shear parameter) and geometry of the crack under various loading conditions are then studied.

Figures 2 and 3 compare the stress intensity factors for orthotropic (birch, yellow) and isotropic materials subjected to uniform normal and shear tractions on the crack surface where $\frac{H}{a}$ has been kept constant ($\frac{H}{a}$ = 0.75) and crack location H_1/H has been changed. An interesting result can be seen for the unit shear stress case which is that k_2 goes through a minimum for a certain value of H_1 in $0<H_1/H<1$ rather than for $H_1=H$ as one might have expected.

Figures 4 and 5 give the variation of the stress intensity factors with $\frac{H}{a}$, for various stiffness ratios $\delta=(\frac{1}{3},1,3)$ or $\frac{E_{11}}{E_{22}} = (\frac{1}{81}, 1, 81)$ which covers almost the entire practical range for the orthotropic materials. The external loads are again uniform normal and shear tractions on the crack surface.

Table 1 shows the effect of the crack geometry on the stress intensity factors for various loading conditions. The considered crack locations and loads are

1-) $\frac{H_1}{H} = 1,\ 0.7,\ 0.4$ 2-) $p(x_1) = 1,\ x_1,\ x_1^2,\ \delta(x_1),\ q(x_1) = 0,$

and $p(x_1) = 0,\ q(x_1) = 1,\ x_1,\ x_1^2$

For every case, the geometry of the crack varied as $\frac{\delta H}{a}$ = 1.5, 0.6, 0.4, 0.25, 0.15, 0.10. The results for various combinations of the loadings may be obtained by using the superposition technique. In Table 1, the results are obtained for birch yellow which has shear parameter $\kappa = 1.2895$. Partial results giving the k_1 and k_2 for uniform normal and shear crack surface tractions are also displayed in Figure 6. Table 2 shows the effect of the material parameters on the stress intensity factors. The cases considered are:

1-) shear parameter effect: in this case two crack locations are considered $\frac{H_1}{H} = 1, 0.4$ and $p(x_1) = 1$, $\delta(x_1)$, $q(x_1) = 1$ are applied separately, on the crack surface. The shear parameter is varied as $\kappa = 1,2,4,8,12$.

2-) stiffness ratio effect: the symmetric case is considered and the same stresses are applied on the crack surface. The stiffness ratio has been varied as $\delta = 0.3, 0.4, 0.6, 0.8, 1.2, 1.5, 1.8, 2, 3, 10$. From Table 2 we can conclude that stress intensity factors are slightly varying with shear parameter but they are highly dependent on the stiffness ratio such that the kernels go to zero as $\delta \to \infty$ and become divergent as $\delta \to 0$ (i.e. from the equations (3.52a-d) it can be shown that $\delta \to \infty$ and $\delta \to 0$ are, respectively, identical to $H_1 \to \infty$, $H_2 \to \infty$ and $H_1 \to 0$, $H_2 \to 0$ for a fixed δ); consequently $k_1 \to 1$, $k_2 \to 1$ as $\delta \to \infty$ and $k_1 \to \infty$, $k_2 \to \infty$ as $\delta \to 0$.

The problem of full contact is solved by modifying the integral equation obtained for the crack problem. For a given $\frac{\delta H}{a}$, if we increase $\frac{b}{a}$, the pressure will decrease and becomes zero on the middle of the wedge for $b=b_{cr}$. Table 3 gives the critical values of $\frac{b}{a}$ for various $\frac{\delta H}{a}$. To simplify the numerical analysis the stress intensity factors and pressure are normalized with respect to $\frac{Ev_0}{a}$, where v_0 is the thickness of the wedge.

The integral equation obtained in part 4 is no longer valid if $b > b_{cr}$. In this case the interface separation problem has been solved by modifying the integral equation as outlined in part 5. The separation area is defined by 2c which is calculated by iteration. Figure 7 shows the variation of $\frac{c}{b}$ with $\frac{b}{a}$ for $\frac{\delta H}{a} = 0.6, 0.4, 0.2$. The pressure distributions, when $b \le b_{cr}$ and $b \ge b_{cr}$ can be seen in Figure 8 for $\frac{\delta H}{a} = 0.6$ and $\frac{b}{a} = 0.2$, 0.293, 0.33, 0.40, where $\frac{b}{a} = 0.293$, is critical value. For various values of $\frac{H}{a}$ and $\frac{b}{a}$, the pressure may be obtained from Table 4.

Finally, in Figures 9 and 10 the variation of the stress intensity factors with $\frac{b}{a}$ may be seen at the crack and wedge tips for $\frac{\delta H}{a} = 0.2, 0.4, 0.6$. For all the crack contact problems, it is assumed that $\kappa = 2$.

Table 1 a. The effect of crack geometry and loading conditions on the stress intensity factors for $H_1 = H$ and $\kappa = 1.2895$.

$\frac{\delta H}{a}$	$\sigma_{22} = p(x_1)$, $\sigma_{12} = 0$			$\sigma_{22} = 0$, $\sigma_{12} = q(x_1)$	
	$p(x_1)=px_1$	$p(x_1)=px_1^2$	$p(x_1)=p\delta(x_1)$	$q(x_1)=qx_1$	$q(x_1)=qx_1^2$
	$k_{1A}=-k_{1B}$	$k_{1A}=k_{1B}$	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$	$k_{2A}=k_{2B}$
	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$
1.5	0.535	0.603	0.613	0.516	0.548
1.0	0.597	0.693	0.880	0.542	0.582
0.6	0.773	0.902	1.517	0.605	0.648
0.4	1.049	1.205	2.470	0.685	0.725
0.25	1.646	1.837	4.514	0.813	0.848
0.15	2.955	3.180	9.091	1.008	1.030
0.10	4.923	5.197	16.177	1.210	1.222

Table 1b. Same as Table 1a, $H_1=0.7H$, $\sigma_{12}=0$, $\sigma_{22}=-p(x_1)$

$\frac{\delta H}{a}$	$p(x_1) = p$		$p(x_1) = px_1$		$p(x_1)=px_1^2$		$p(x_1)=p\delta(x_1)$	
	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$	$k_{1A}=-k_{1B}$	$k_{2A}=k_{2B}$	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$
	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{\delta p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{\delta p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{\delta p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{\delta p\sqrt{a}}$
1.5	1.545	-0.132	0.557	0.013	0.629	-0.028	0.681	-0.104
1.0	2.009	-0.273	0.641	0.010	0.735	-0.055	1.008	-0.207
0.6	3.105	-0.622	0.866	-0.025	0.983	-0.125	1.767	-0.472
0.4	4.717	-1.160	1.208	-0.101	1.341	-0.233	2.895	-0.869
0.25	8.156	-2.371	1.941	-0.297	2.089	-0.477	5.344	-1.777
0.15	15.704	-5.156	3.524	-0.774	3.691	-1.036	10.822	-3.859
0.10	27.050	-9.484	5.931	-1.587	6.077	-1.917	19.098	-7.069

Table 1c. Same as Table 1a, H_1=0.7H, σ_{22}=0, σ_{12}=$-\hat{q}(x_1)$

	$q(x_1) = q$		$q(x_1) = q x_1$		$q(x_1) = q x_1^2$	
	$k_{1A}=-k_{1B}$	$k_{2A}=k_{2B}$	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$	$k_{1A}=-k_{1B}$	$k_{2A}=k_{2B}$
$\frac{\delta H}{a}$	$\frac{\delta k_{1B}}{q\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$	$\frac{\delta k_{1B}}{q\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$	$\frac{\delta k_{1B}}{q\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$
1.5	0.102	1.190	-0.021	0.522	0.021	0.546
1.0	0.178	1.310	-0.032	0.553	0.035	0.576
0.6	0.306	1.526	-0.048	0.622	0.055	0.635
0.4	0.433	1.757	-0.063	0.706	0.075	0.703
0.25	0.612	2.106	-0.083	0.843	0.103	0.814
0.15	0.857	2.612	-0.111	1.046	0.141	0.983
0.10	1.095	3.127	-0.120	1.250	0.179	1.161

Table 1d. Same as Table 1a, H_1=0.4H, σ_{12}=0, σ_{22}=-p(x_1)

$\frac{\delta H}{a}$	$p(x_1) = p$		$p(x_1) = px_1$		$p(x_1) = px_1^2$		$p(x_1) = p\delta(x_1)$	
	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$	$k_{1A}=-k_{1B}$	$k_{2A}=k_{2B}$	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$
	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{\delta p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{\delta p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{\delta p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{\delta p\sqrt{a}}$
1.5	2.099	-0.470	0.678	0.007	0.757	-0.097	1.061	-0.353
1.0	2.938	-0.926	0.864	-0.041	0.949	-0.186	1.638	-0.700
0.6	4.934	-2.066	1.309	-0.211	1.394	-0.413	3.038	-1.555
0.4	7.907	-3.856	1.964	-0.511	2.040	-0.773	5.164	-2.875
0.25	14.392	-7.816	3.347	-1.238	3.414	-1.565	9.872	-5.864
0.15	26.351	-15.638	6.372	-3.065	6.156	-3.298	17.866	-11.193

Table 1e. Same as Table 1a, H_1=0.4H, σ_{22}=0, σ_{12}=-q(x_1)

$\frac{\delta H}{a}$	$q(x_1) = q$		$q(x_1) = qx_1$		$q(x_1) = qx_1^2$	
	$k_{1A}=-k_{1B}$	$k_{2A}=k_{2B}$	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$	$k_{1A}=-k_{1B}$	$k_{2A}=k_{2B}$
	$\frac{\delta k_{1B}}{q\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$	$\frac{\delta k_{1B}}{q\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$	$\frac{\delta k_{1B}}{q\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$
1.5	0.260	1.166	-0.055	0.556	0.049	0.544
1.0	0.390	1.239	-0.080	0.603	0.068	0.570
0.6	0.580	1.384	-0.116	0.695	0.092	0.623
0.4	0.757	1.553	-0.151	0.802	0.114	0.688
0.25	0.991	1.824	-0.196	0.966	0.140	0.798
0.15	1.320	2.234	-0.164	1.149	0.183	0.970

Table 2: The effect of the shear parameter κ on the stress intensity factors.

(a) $\frac{H_1}{H} = 1.0$, $\frac{\delta H}{a} = 0.35$

κ	$p(x_1)=p$, $q(x_1)=0$	$p(x_1)=p\delta(x_1)$, $q(x_1)=0$	$p(x_1)=0$, $q(x_1)=q$
	$k_{1A}=k_{1B}$	$k_{1A}=k_{1B}$	$k_{2A}=k_{2B}$
	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$
1	4.801	2.971	2.047
2	4.657	2.826	1.890
4	4.564	2.697	1.717
8	4.553	2.612	1.550
12	4.611	2.592	1.463
16	4.692	2.613	1.405

(b) $\frac{H_1}{H} = 0.4$, $\frac{\delta \tilde{H}}{a} = 0.35$

κ	$p(x_1)=p$, $q(x_1)=0$		$p(x_1)=p\delta(x_1)$, $q(x_1)=0$		$p(x_1)=0$, $q(x_1)=q$	
	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$	$k_{1A}=k_{1B}$	$k_{2A}=-k_{2B}$	$k_{1A}=-k_{1B}$	$k_{2A}=k_{2B}$
	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{\delta p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{\delta p\sqrt{a}}$	$\frac{\delta k_{1B}}{q\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$
1	9.519	-4.855	6.342	-3.646	0.846	1.658
2	8.986	-4.404	5.903	-3.295	0.761	1.553
4	8.447	-3.908	5.432	-2.905	0.666	1.441
8	7.968	-3.370	4.977	-2.487	0.563	1.336
12	7.753	-3.095	4.737	-2.266	0.508	1.283

Table 2c. Effect of the stiffness parameter δ on the stress intensity factors, $\frac{H_1}{a} = \frac{H_2}{a} = 0.35$, $\kappa = 2$.

δ	$\sigma_{22}(x_1,0)=p$	$\sigma_{22}(x_1,0)=p\delta(x_1)$	$\sigma_{12}(x_1,0)=q$
	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{1B}}{p\sqrt{a}}$	$\frac{k_{2B}}{q\sqrt{a}}$
0.3	20.350	14.218	3.231
0.4	13.880	9.426	2.829
0.6	8.348	5.436	2.353
0.8	5.954	3.732	2.075
1.2	3.851	2.261	1.755
1.5	3.103	1.745	1.610
1.8	2.637	1.427	1.507
2	2.415	1.279	1.453
3	1.797	0.858	1.282
10	1.112	0.393	1.043

Table 3: The critical values of $\frac{b}{a}$ for various values of H/a, shear parameter κ = 2.

$\frac{\delta H}{a}$	$(\frac{b}{a})_{cr}$	$\frac{\delta a}{Ev_o} \frac{k_{1A}}{\sqrt{a}}$	$\frac{\delta a}{Ev_o} \frac{k_{1B}}{\sqrt{a}}$
1.5	0.961	4.761×10^{-1}	-4.568×10^{-1}
1.0	0.665	1.569×10^{-1}	-1.476×10^{-1}
0.60	0.293	8.544×10^{-2}	-6.792×10^{-2}
0.40	0.161	5.836×10^{-2}	-3.874×10^{-2}
0.20	0.059	2.801×10^{-2}	-1.188×10^{-2}

Table 4. Pressure distribution for the wedge problem (Figures 1b and 1c), κ = 2 (The intervals (-b,b) and (c,b) normalized to (-1,1).

(a) $\frac{\delta H}{a} = 0.6$

$\frac{b}{a}$	$\frac{c}{b}$	S_i	$\frac{\delta a}{Ev_0} \frac{k_{1A}}{\sqrt{a}}$	$\frac{\delta a}{Ev_0} \frac{k_{1B}}{\sqrt{a}}$	$\frac{\delta a}{Ev_0} p(s_i)$
0.1	Full Contact	0.0 0.342 0.643 0.866	7.229×10^{-2}	-5.559×10^{-2}	1.313×10^{-1} 1.453×10^{-1} 1.959×10^{-1} 3.379×10^{-1}
0.5	0.688	-0.990 -0.756 -0.282 0.282 0.756 0.990	1.099×10^{-1}	-9.961×10^{-2}	6.253×10^{-2} 7.987×10^{-2} 1.267×10^{-1} 2.253×10^{-1} 4.707×10^{-1} 3.542
0.6	0.738	-0.990 -0.756 -0.282 0.282 0.756 0.990	1.288×10^{-1}	-1.208×10^{-1}	8.653×10^{-2} 1.070×10^{-1} 1.624×10^{-1} 2.797×10^{-1} 5.734×10^{-1} 3.998
0.8	0.766		2.016×10^{-1}	-1.931×10^{-1}	1.218×10^{-1} 1.504×10^{-1} 2.288×10^{-1} 3.978×10^{-1} 8.279×10^{-1} 4.863

Table 4: (cont.)

(b) $\frac{\delta H}{a} = 1.0$

$\frac{b}{a}$	s_i	$\frac{\delta a}{Ev_o} \frac{k_{1A}}{\sqrt{a}}$	$\frac{\delta a}{Ev_o} \frac{k_{1B}}{\sqrt{a}}$	$\frac{\delta a}{Ev_o} p(s_i)$
0.2	0.0 0.342 0.643 0.866	9.322×10^{-2}	-8.580×10^{-2}	1.416×10^{1} 1.571×10^{-1} 2.124×10^{-1} 3.601×10^{-1}
0.4	0.0 0.342 0.643 0.866	1.131×10^{-1}	-9.833×10^{-2}	4.940×10^{-2} 6.653×10^{-2} 1.238×10^{-1} 2.604×10^{-1}
0.6	0.0 0.342 0.643 0.866	1.425×10^{-1}	-1.315×10^{-1}	8.550×10^{-3} 2.993×10^{-2} 1.002×10^{-1} 2.611×10^{-1}
0.665	0.0 0.342 0.643 0.866	1.569×10^{-1}	-1.475×10^{-1}	0.0 2.268×10^{-2} 9.785×10^{-2} 2.708×10^{-1}

Table 4 (cont.)

(c) $\frac{\delta H}{a} = 1.5$

$\frac{b}{a}$	s_i	$\frac{\delta a}{Ev_o} \frac{k_{1A}}{\sqrt{a}}$	$\frac{\delta a}{Ev_o} \frac{k_{1B}}{\sqrt{a}}$	$\frac{\delta a}{Ev_o} p(s_i)$
0.3	0.0 0.342 0.643 0.866	1.043×10^{-1}	-1.088×10^{-1}	1.536×10^{-1} 1.691×10^{-1} 2.249×10^{-1} 3.750×10^{-1}
0.6	0.0 0.342 0.643 0.866	1.457×10^{-1}	-1.385×10^{-1}	6.900×10^{-2} 8.681×10^{-2} 1.479×10^{-1} 3.002×10^{-1}
0.8	0.0 0.342 0.643 0.866	2.072×10^{-1}	-2.011×10^{-1}	4.147×10^{-2} 6.303×10^{-2} 1.377×10^{-1} 3.307×10^{-1}
0.961	0.0 0.342 0.643 0.866	4.751×10^{-1}	-4.550×10^{-1}	0.0 1.920×10^{-2} 9.572×10^{-2} 3.633×10^{-1}

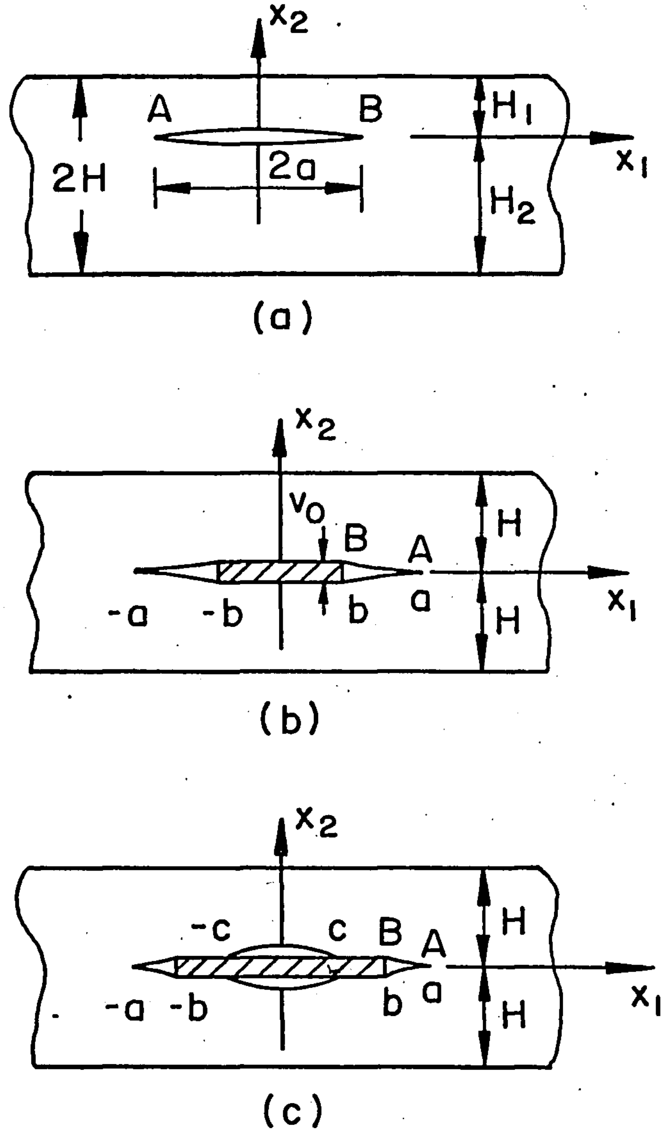

Figure 1: Geometry of the crack and crack-contact problems.

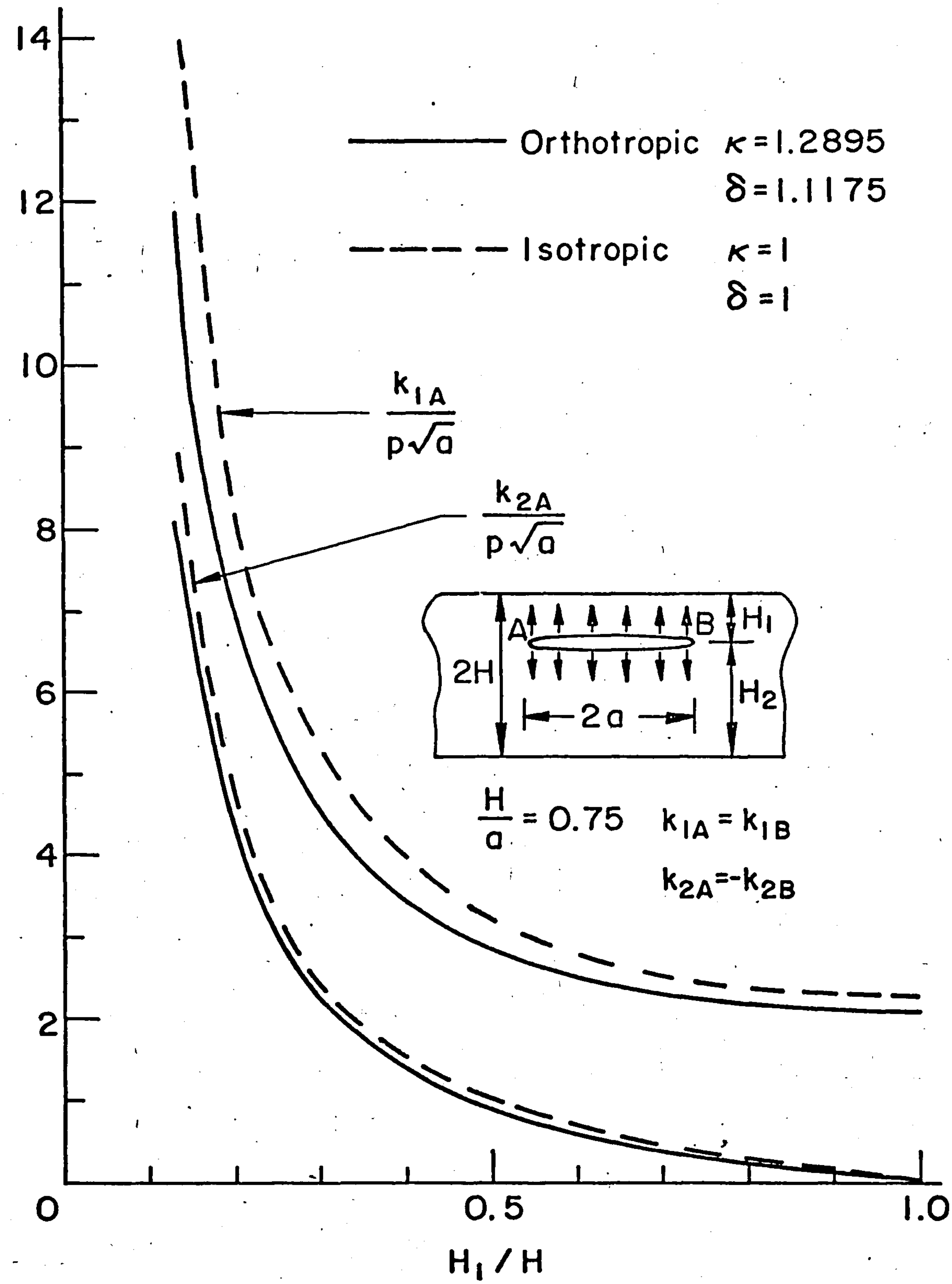

Figure 2: The effect of the crack location on the stress intensity factors for uniform surface pressure. $H = 0.75a$, $\delta = 1 = \kappa$ for the isotropic materials and $\delta = 1.1175$, $\kappa = 1.2895$ for the orthotropic material (yellow birch).

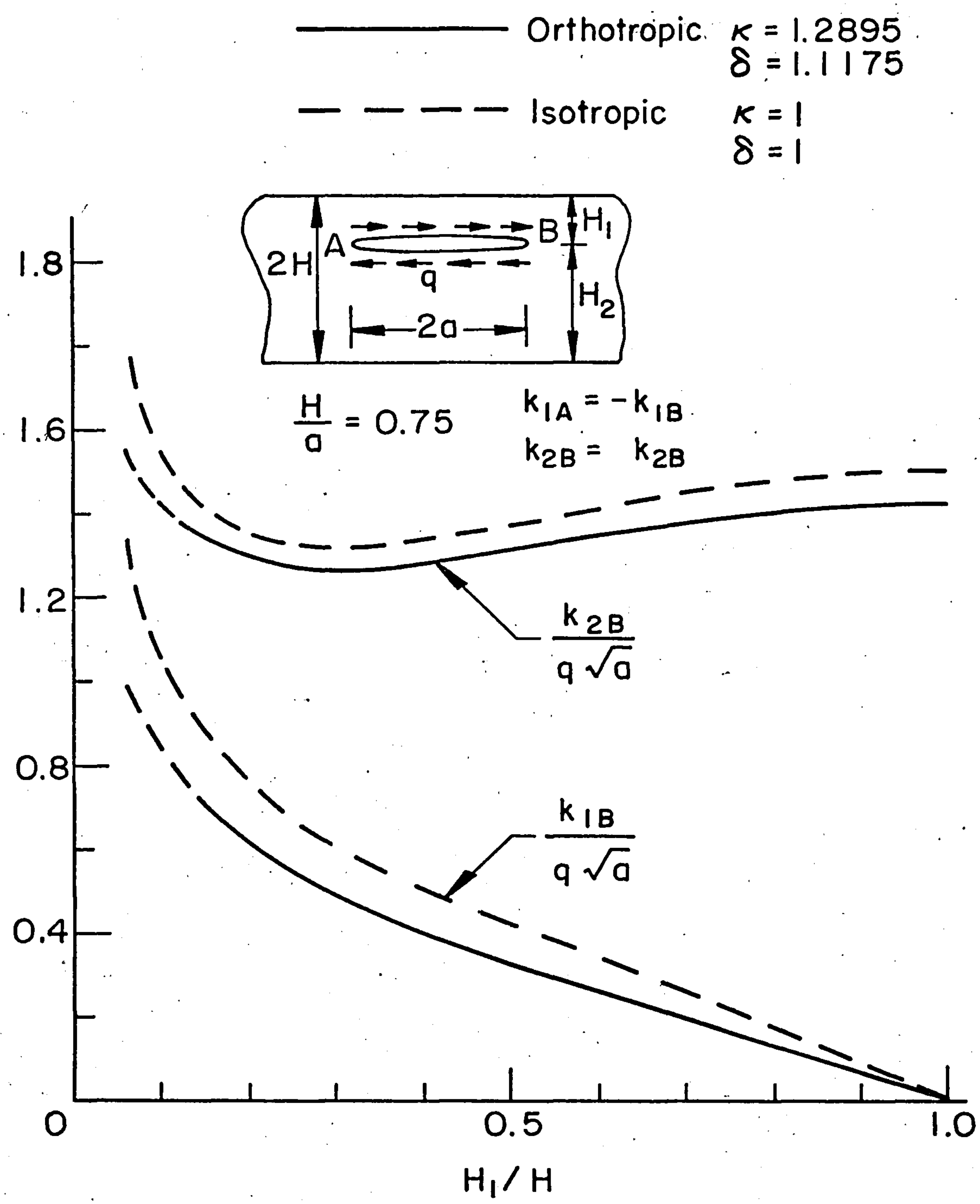

Figure 3: Same as figure 2 for uniform shear applied to the crack surface.

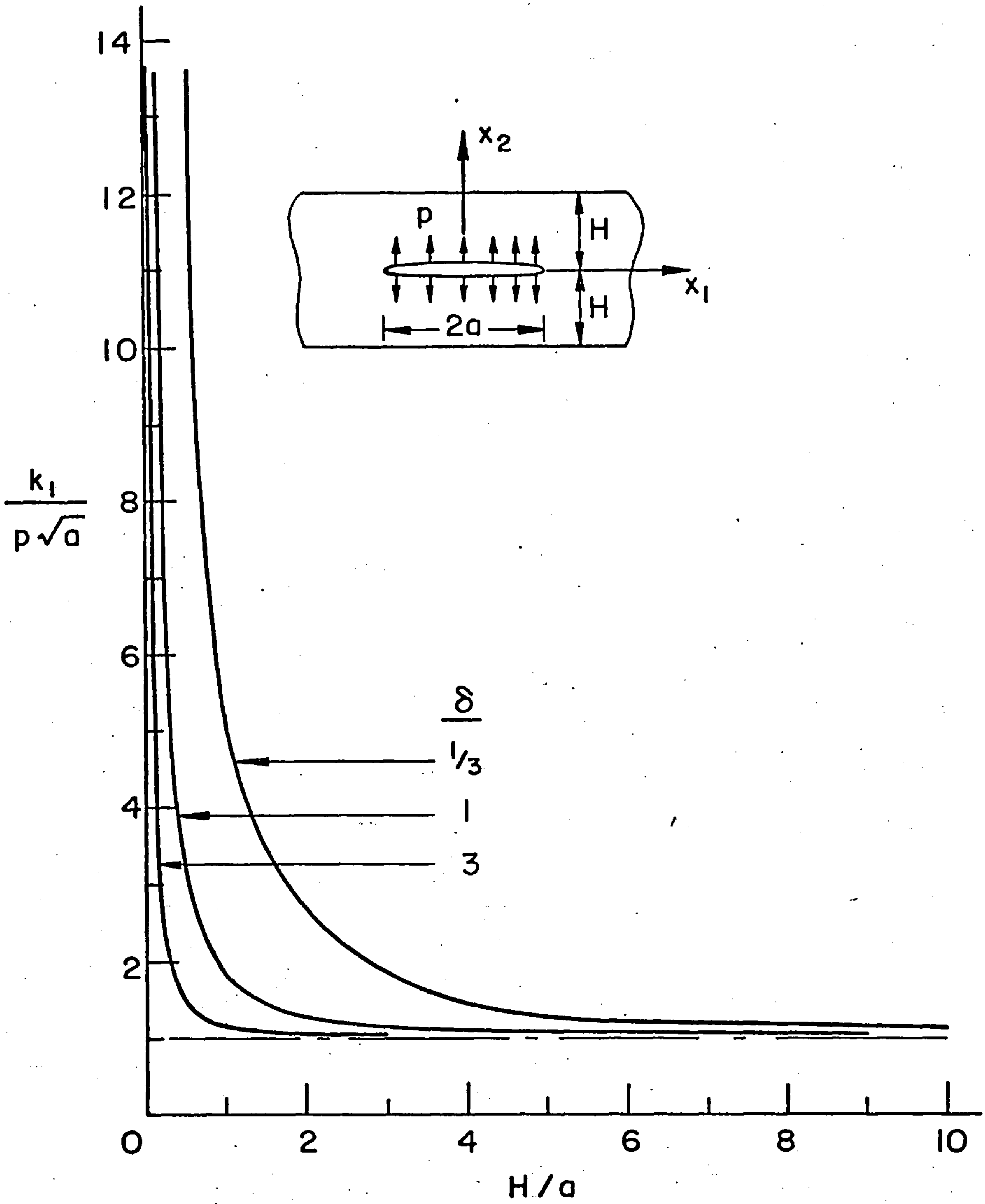

Figure 4: Effect of the crack length on the stress intensity factor for a symmetrically located crack under uniform pressure, κ = 1.

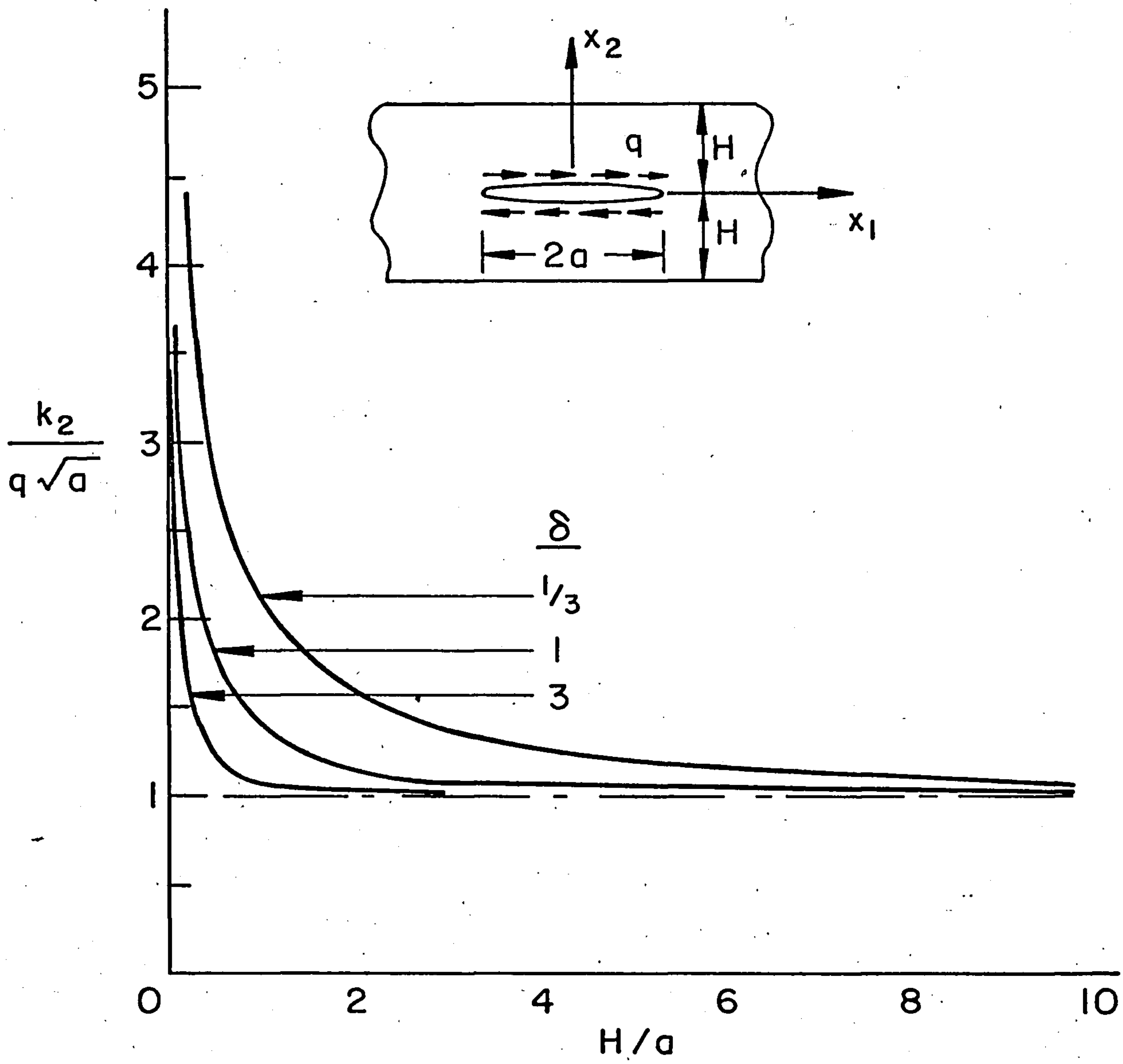

Figure 5: Same as figure 4 for uniform shear applied to crack surface.

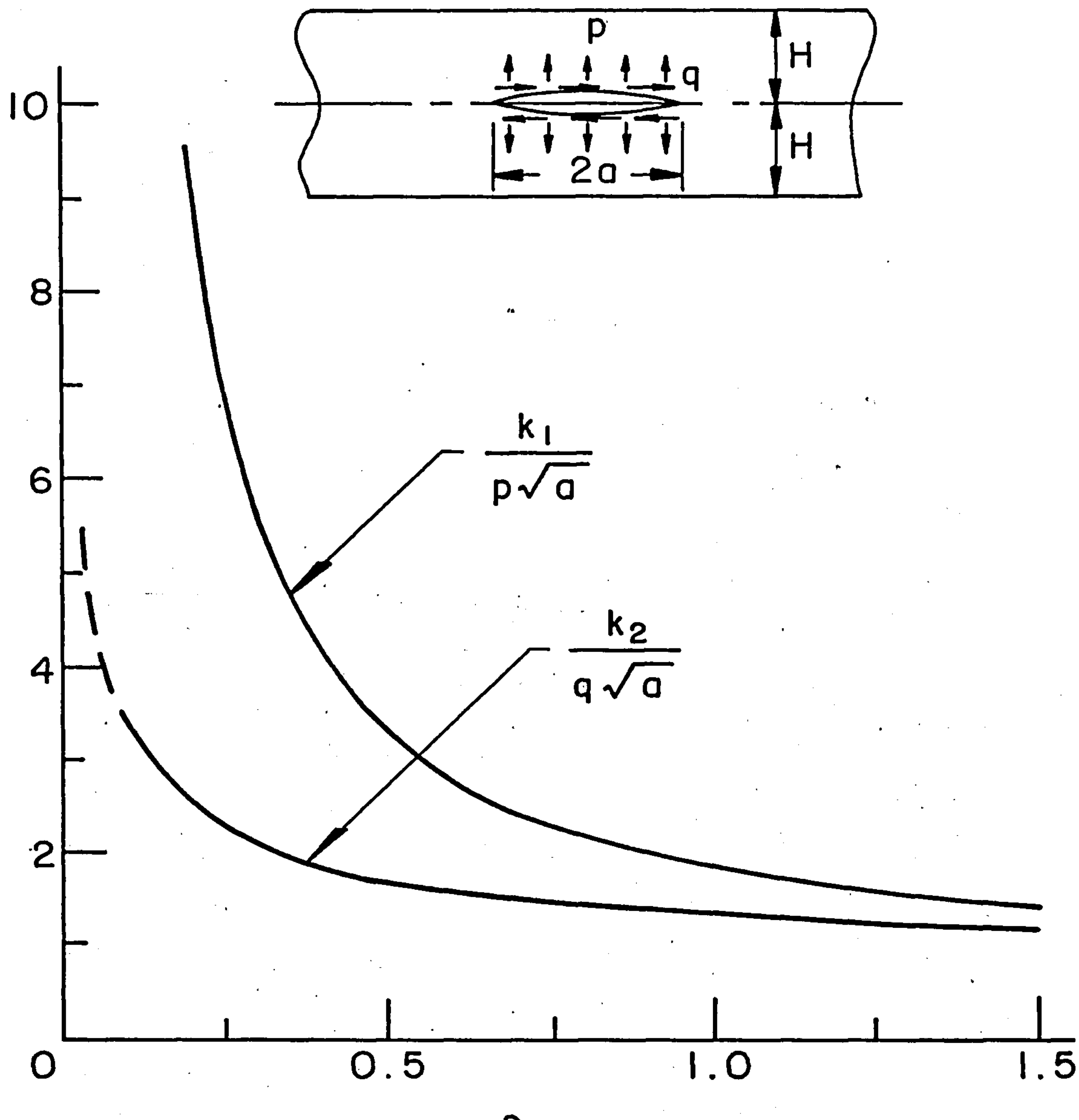

Figure 6: The effect of $\delta H/a$ on the stress intensity factors for a symmetrically located crack under uniform pressure or uniform shear, $\kappa = 1.2895$.

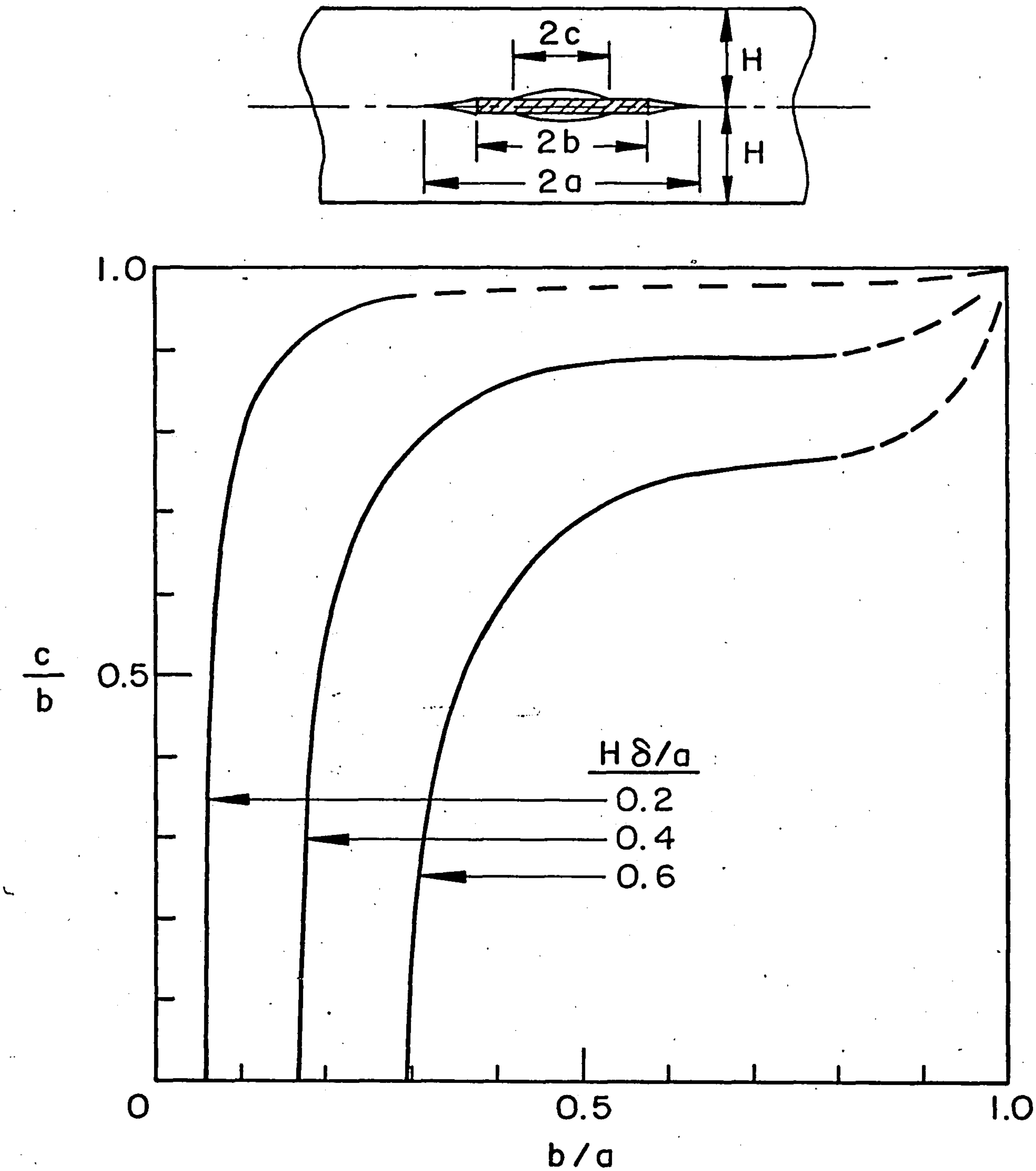

Figure 7: Separation length for the wedge problem, $\kappa = 2$.

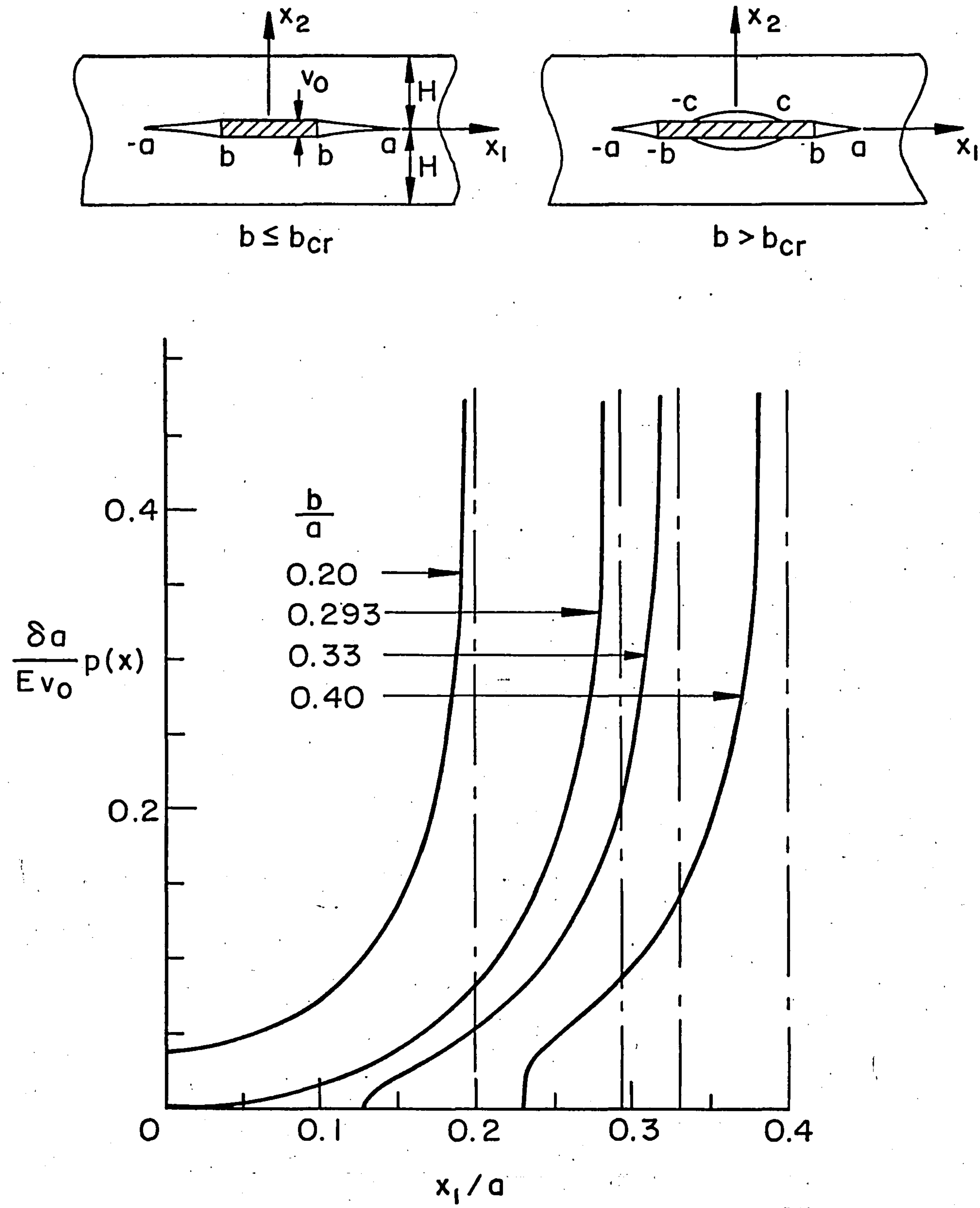

Figure 8: Pressure distribution for the wedge problem, $H = 0.6a$, $\kappa = 2$.

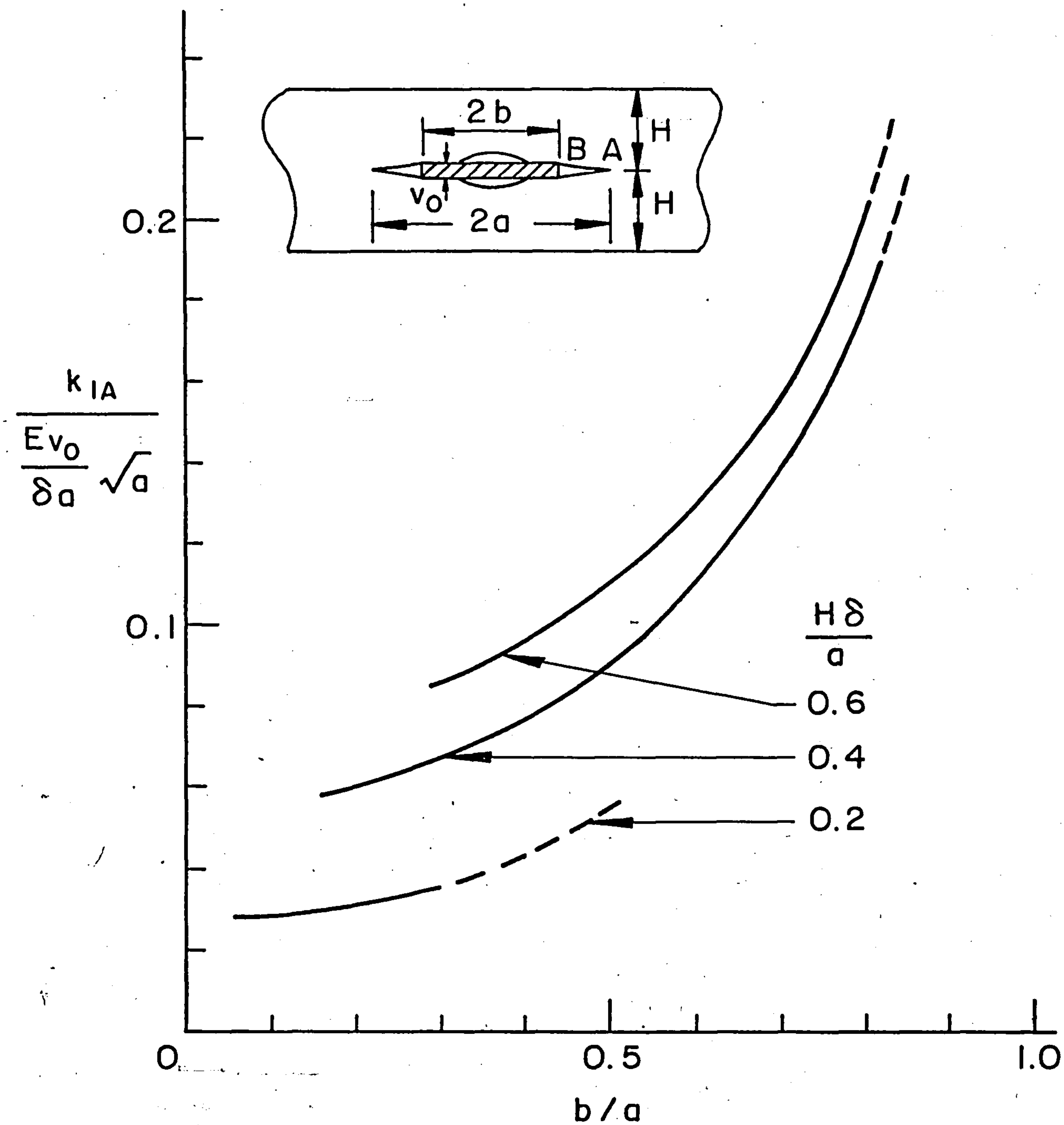

Figure 9: Crack tip stress intensity factor for the wedge problem, $\kappa = 2$.

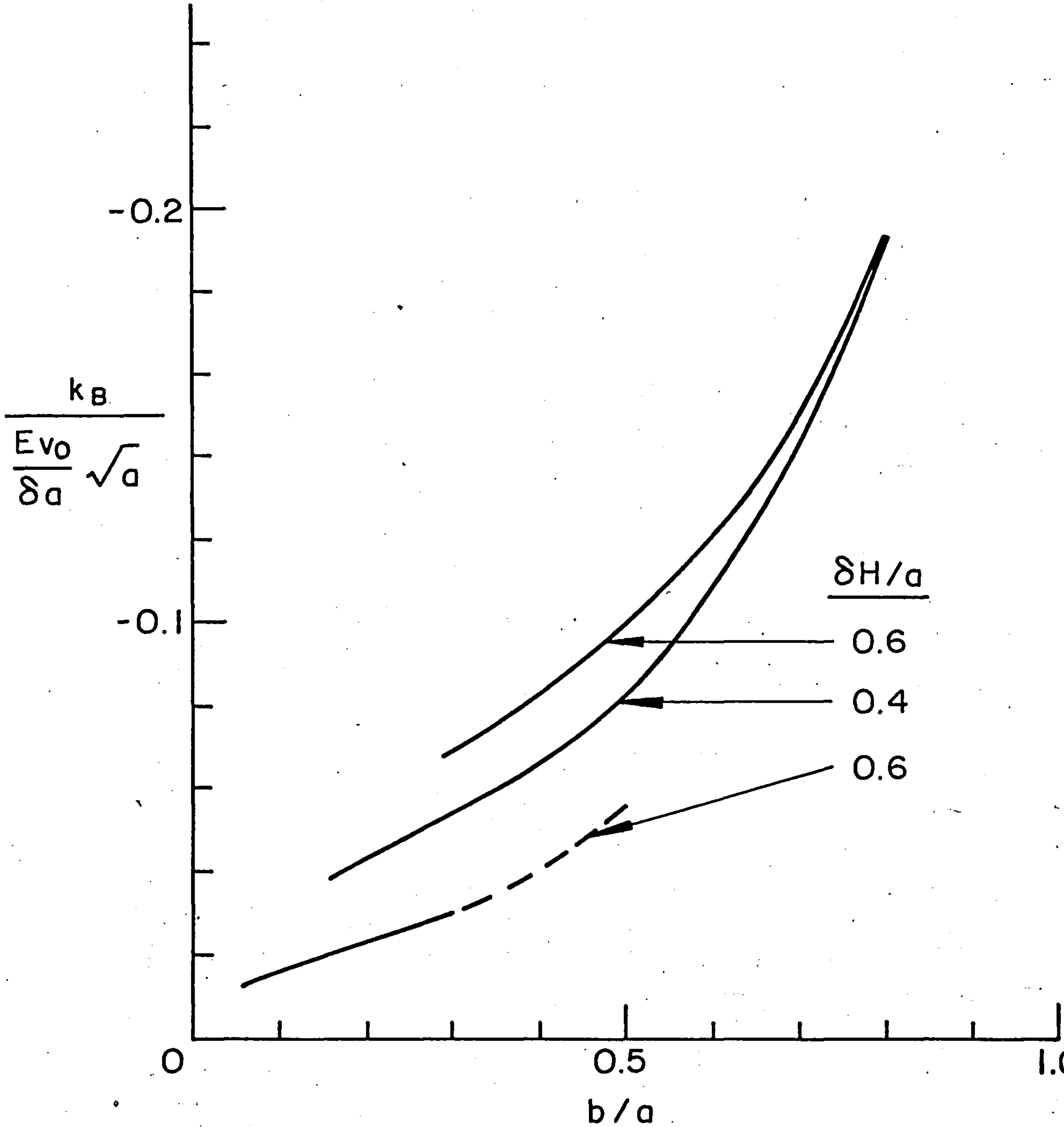

Figure 10: Wedge tip stress intensity factor for the crack contact problem, $\kappa = 2$.

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APPENDIX I

DEFINITION OF THE MATERIAL CONSTANTS

$$E = \left(E_{11} E_{22}\right)^{1/2}$$

$$\nu = \left(\nu_{21} \nu_{12}\right)^{1/2}$$

$$\delta^4 = \frac{E_{11}}{E_{22}} = \frac{\nu_{12}}{\nu_{21}}$$

$$K = \frac{1}{2}\left(E_{11} E_{22}\right)^{1/2}\left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right)$$

$$S_1 = \left(K + \sqrt{K^2 - 1}\right)^{1/2}, \quad S_3 = -S_1$$

$$S_2 = \left(K - \sqrt{K^2 - 1}\right)^{1/2}, \quad S_4 = -S_1$$

$$\lambda_5 = \frac{2 S_1}{S_1 - S_2}$$

$$\lambda_6 = \frac{2 S_2}{S_1 - S_2}$$

$$\lambda_7 = \frac{S_1 + S_2}{S_1 - S_2}$$

$$\lambda_1 = S_1^3 (1 - \lambda_7) + \lambda_5 S_2^3$$

$$\lambda_2 = S_2^3 (1 + \lambda_7) - \lambda_6 S_1^3$$

$$\lambda_3 = S_1^2 (1 + \lambda_7) - \lambda_6 S_2^2$$

$$\lambda_4 = S_2^2(1-\lambda_7) + \lambda_6 S_1^2$$

$$\lambda_8 = -\frac{S_1 + S_2}{2S_2}$$

$$\lambda_9 = \frac{1}{\lambda_6}$$

$$\lambda_{10} = \lambda_4 \lambda_1 - \lambda_2 \lambda_3$$

$$\lambda_{11} = -\frac{\lambda_2}{\lambda_{10}}$$

$$\lambda_{12} = -\frac{\lambda_4}{\lambda_{10}}$$

$$\lambda_{13} = \frac{\lambda_1}{\lambda_{10}}$$

$$\lambda_{14} = \frac{\lambda_3}{\lambda_{10}}$$

$$\lambda_{15} = \lambda_{12} + \lambda_{14}$$

$$\lambda_{16} = S_1 \lambda_{11} + S_2 \lambda_{13}$$

The problem is solved for plane stress. The results for plane strain case can be obtained by redefining material parameters κ and δ as

$$\delta^4 = \frac{E_{11}}{E_{22}} \frac{1-\nu_{23}\nu_{32}}{1-\nu_{13}\nu_{31}},$$

$$K = \frac{1}{2}(E_{11}E_{22})^{1/2}\left(\frac{1}{(1-\nu_{13}\nu_{31})(1-\nu_{23}\nu_{32})}\right)^{1/2}\left(\frac{1}{G_{12}} - \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_{22}} - \frac{\nu_{12}+\nu_{13}\nu_{32}}{E_{11}}\right)$$

APPENDIX II

THE FREDHOLM KERNELS

$$\Gamma_1 = (S_1 + S_2)\, h_1$$

$$\Gamma_2 = (S_1 - S_2)\, h_1$$

$$\Gamma_3 = (S_1 + S_2)\, h_2$$

$$\Gamma_4 = (S_1 - S_2)\, h_2$$

$$m_1(\alpha) = \lambda_8 \left(e^{\Gamma_2 \alpha} - e^{-\Gamma_4 \alpha} \right)$$

$$m_2(\alpha) = \lambda_9 \left(e^{-\Gamma_1 \alpha} - e^{\Gamma_3 \alpha} \right)$$

$$z = \frac{|\alpha|}{\alpha} = \begin{cases} 1, & \alpha > 0 \\ -1, & \alpha < 0 \end{cases}$$

$$m_3(\alpha) = \frac{S_1 z - S_2}{2 S_2}\, e^{-h_1 (S_2 + S_1 z)\alpha} + \lambda_5 \frac{1+z}{2}\, e^{S_2 h_2 (1-z)\alpha} - \lambda_7 \frac{S_2 + S_1 z}{2 S_2}\, e^{h_2 (S_2 - S_1 z)\alpha}$$

$$m_4(\alpha) = \frac{z-1}{2}\, e^{-S_2 h_1 (1+z)\alpha} + \lambda_7 \frac{1+z}{2}\, e^{S_2 h_2 (1-z)\alpha} - \lambda_6 \frac{S_2 + S_1 z}{2 S_2}\, e^{h_2 (S_2 - S_1 z)\alpha}$$

$$m_5(\alpha) = \lambda_9 \left(e^{-\Gamma_3 \alpha} - e^{\Gamma_1 \alpha} \right)$$

$$m_6(\alpha) = \lambda_8 \left(e^{\Gamma_4 \alpha} - e^{-\Gamma_2 \alpha} \right)$$

$$m_7(\alpha) = \lambda_5 \frac{z-1}{2} e^{-S_2 h_2 (1+z)\alpha} + \lambda_7 \frac{S_2 - S_1 z}{2 S_2} e^{-h_2 (S_2 + S_1 z)\alpha}$$

$$+ \frac{S_2 + S_1 z}{2 S_2} e^{h_1 (S_2 - S_1 z)\alpha}$$

$$m_8(\alpha) = \lambda_7 \frac{z-1}{2} e^{-S_2 h_2 (1+z)\alpha} + \lambda_6 \frac{S_2 - S_1 z}{2 S_2} e^{-h_2 (S_2 + S_1 z)\alpha}$$

$$+ \frac{1+z}{2} e^{S_2 h_1 (1-z)\alpha}$$

$$m_9(\alpha) = m_2(\alpha) m_5(\alpha) - m_1(\alpha) m_6(\alpha)$$

$$m_{10}(\alpha) = m_3(\alpha) m_6(\alpha) - m_7(\alpha) m_2(\alpha)$$

$$m_{11}(\alpha) = m_7(\alpha) m_1(\alpha) - m_3(\alpha) m_5(\alpha)$$

$$m_{12}(\alpha) = m_4(\alpha) m_6(\alpha) - m_8(\alpha) m_2(\alpha)$$

$$m_{13}(\alpha) = m_8(\alpha) m_1(\alpha) - m_4(\alpha) m_5(\alpha)$$

$$m_{14}(\alpha) = \lambda_8 e^{\Gamma_2 \alpha} m_{10}(\alpha) + \lambda_9 e^{-\Gamma_1 \alpha} m_{11}(\alpha) + \frac{S_1 z - S_2}{2 S_2} m_9(\alpha) e^{-h_1 (S_2 + S_1 z)\alpha}$$

$$m_{15}(\alpha) = \lambda_8 e^{\Gamma_2 \alpha} m_{12}(\alpha) + \lambda_9 e^{-\Gamma_1 \alpha} m_{13}(\alpha) + \frac{z-1}{2} m_9(\alpha) e^{-S_2 h_1 (1+z)\alpha}$$

$$m_{16}(\alpha) = \lambda_9 e^{\Gamma_1 \alpha} m_{10}(\alpha) + \lambda_8 e^{-\Gamma_2 \alpha} m_{11}(\alpha) - \frac{S_2 + S_1 z}{2 S_2} e^{h_1 (S_2 - S_1 z)\alpha} m_9(\alpha)$$

$$m_{17}(\alpha) = \lambda_9 e^{\Gamma_1 \alpha} m_{12}(\alpha) + \lambda_8 e^{-\Gamma_2 \alpha} m_{13}(\alpha) - \frac{1+z}{2} e^{S_2 h_1 (1-z)\alpha} m_9(\alpha)$$

$$m_{18}(\alpha) = m_{12}(\alpha) + m_{13}(\alpha) + m_{15}(\alpha) + m_{17}(\alpha)$$

$$m_{19}(\alpha) = m_{10}(\alpha) + m_{11}(\alpha) + m_{14}(\alpha) + m_{16}(\alpha)$$

$$m_{20}(\alpha) = m_{12}(\alpha) - m_{13}(\alpha)$$

$$m_{21}(\alpha) = m_{15}(\alpha) - m_{17}(\alpha)$$

$$m_{22}(\alpha) = m_{11}(\alpha) - m_{10}(\alpha)$$

$$m_{23}(\alpha) = m_{16}(\alpha) - m_{14}(\alpha)$$

$$H_{11}(\alpha) = \frac{\lambda_{14} m_{18}(\alpha) + \lambda_{12} m_{19}(\alpha)}{m_9(\alpha)}$$

$$H_{12}(\alpha) = \frac{\lambda_{13} m_{18}(\alpha) + \lambda_{11} m_{19}(\alpha)}{m_9(\alpha)}$$

$$H_{21}(\alpha) = \frac{S_1\lambda_{14} m_{20}(\alpha) + S_2\lambda_{14} m_{21}(\alpha) - S_1\lambda_{12} m_{22}(\alpha) - S_2\lambda_{12} m_{23}(\alpha)}{m_9(\alpha)}$$

$$H_{22}(\alpha) = \frac{S_1\lambda_{13} m_{20}(\alpha) + S_2\lambda_{13} m_{21}(\alpha) - S_1\lambda_{11} m_{22}(\alpha) - S_2\lambda_{11} m_{23}(\alpha)}{m_9(\alpha)}$$

$$K_{11}(t,x) = \frac{1}{\lambda_{15}} \int_0^\infty H_{11}(\alpha) \sin\alpha(t-x)\, d\alpha$$

$$K_{12}(t,x) = -\frac{1}{\lambda_{15}} \int_0^\infty H_{12}(\alpha) \cos\alpha(t-x)\, d\alpha$$

$$K_{21}(t,x) = -\frac{1}{\lambda_{16}} \int_0^\infty H_{21}(\alpha) \cos\alpha(t-x)\, d\alpha$$

$$K_{22}(t,x) = -\frac{1}{\lambda_{16}} \int_0^\infty H_{22}(\alpha) \sin\alpha(t-x)\, d\alpha$$

The kernels $\bar{K}_{ij}(t_1,x_1)$ and $K_{ij}(t,x)$ $(i,j = 1,2)$ if $r_1,\ldots,r_4$ are replaced by $\bar{r}_1,\ldots,\bar{r}_4$ given by

$$\bar{r}_1 = (S_1 + S_2) H_1 \delta \;, \quad \bar{r}_2 = (S_1 - S_2) H_1 \delta$$

$$\bar{r}_3 = (S_1 + S_2) H_2 \delta \;, \quad \bar{r}_4 = (S_1 - S_2) H_2 \delta$$

in the numerical integration $\alpha > 0$ or $z = 1$ some of the $m_j(\alpha)$ become

$$m_3(\alpha) = \lambda_5 + \lambda_9 e^{-r_1\alpha} + \lambda_7\lambda_8 e^{-r_4\alpha}$$

$$m_4(\alpha) = \lambda_7 + \lambda_6\lambda_8 e^{-r_4\alpha}$$

$$m_7(\alpha) = -\lambda_7\lambda_9 e^{-r_3\alpha} - \lambda_8 e^{-r_2\alpha}$$

$$m_8(\alpha) = 1 - \lambda_6\lambda_9 e^{-r_3\alpha}$$

$$m_{14}(\alpha) = \lambda_8 e^{r_2\alpha} m_{10}(\alpha) + \lambda_9 e^{-r_1\alpha} m_{11}(\alpha) + \lambda_9 e^{-r_1\alpha} m_9(\alpha)$$

$$m_{15}(\alpha) = \lambda_8 e^{r_2\alpha} m_{12}(\alpha) + \lambda_9 e^{-r_1\alpha} m_{13}(\alpha)$$

$$m_{16}(\alpha) = \lambda_9 e^{r_1\alpha} m_{10}(\alpha) + \lambda_8 e^{-r_2\alpha} m_{11}(\alpha) + \lambda_8 e^{-r_2\alpha} m_9(\alpha)$$

$$m_{17}(\alpha) = \lambda_9 e^{r_1\alpha} m_{12}(\alpha) + \lambda_8 e^{-r_2\alpha} m_{13}(\alpha) - m_9(\alpha)$$

APPENDIX III

DERIVATION OF INTEGRAL EQUATIONS

$$\sigma_{yy}(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{4} C_j(\alpha)\, e^{S_j \alpha y} + \sum_{j=1}^{2} A_j(\alpha)\, e^{-S_j|\alpha|y}\right)\alpha^2 e^{-i\alpha x}\, d\alpha$$

$$\sigma_{xy}(x,y) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{j=1}^{4} C_j(\alpha) S_j\, e^{S_j \alpha y} - \sum_{j=1}^{2}\frac{|\alpha|}{\alpha} A_j(\alpha) S_j\, e^{-S_j|\alpha|y}\right)\alpha^2 e^{-i\alpha x}\, d\alpha$$

Using the mixed boundary conditions

$$-P_1(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4} C_j(\alpha)\alpha^2 e^{-i\alpha x}\, d\alpha - \frac{1}{2\pi}\lim_{y\to 0}\left(\int_{-\infty}^{\infty}\sum_{j=1}^{2} A_j(\alpha)\alpha^2 e^{-S_j|\alpha|y - i\alpha x}\, d\alpha\right)$$

$$-P_2(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{4} C_j(\alpha) S_j \alpha^2 e^{-i\alpha x}\, d\alpha - \frac{i}{2\pi}\lim_{y\to 0}\left(\int_{-\infty}^{\infty}\sum_{j=1}^{2} S_j A_j(\alpha)\alpha|\alpha|\, e^{-S_j|\alpha|y - i\alpha x}\, d\alpha\right)$$

Substitution of (3.26 a,b) and (3.38 a-d) into above equations gives

$$-\frac{P_1(x)}{E} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} H_{12}(\alpha)\,\bar{F}_2(\alpha)\, e^{-i\alpha x}\, d\alpha - \frac{i}{2\pi}\int_{-\infty}^{\infty} H_{11}(\alpha)\bar{F}_1(\alpha)\frac{\alpha}{|\alpha|} e^{-i\alpha x}\, d\alpha$$

$$-\frac{1}{2\pi}\lim_{y\to 0}\int_{-\infty}^{\infty}\left(\lambda_{11} e^{-S_1|\alpha|y} + \lambda_{13} e^{-S_2|\alpha|y}\right)\bar{F}_2(\alpha)\, e^{-i\alpha x}\, d\alpha$$

$$-\frac{i}{2\pi}\lim_{y\to 0}\int_{-\infty}^{\infty}\left(\lambda_{12} e^{-S_1|\alpha|y} + \lambda_{14} e^{-S_2|\alpha|y}\right)\frac{\alpha}{|\alpha|}\bar{F}_1(\alpha)\, e^{-i\alpha x}\, d\alpha$$

$$-\frac{P_2(x)}{E} = \frac{i}{2\pi}\int_{-\infty}^{\infty} H_{22}(\alpha)\bar{F}_2(\alpha)e^{-i\alpha x}d\alpha - \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{21}(\alpha)\frac{\alpha}{|\alpha|}\bar{F}_1(\alpha)e^{-i\alpha x}d\alpha$$

$$-\frac{i}{2\pi}\lim_{y\to 0}\int_{-\infty}^{\infty}\Big[\Big(S_1\lambda_{11}e^{-S_1|\alpha|y} + S_2\lambda_{13}e^{-S_2|\alpha|y}\Big)\frac{|\alpha|}{\alpha}\bar{F}_2(\alpha)$$

$$+i\left(S_1\lambda_{12}e^{-S_1|\alpha|y} + S_2\lambda_{14}e^{-S_2|\alpha|y}\right)\bar{F}_1(\alpha)\Big]e^{-i\alpha x}d\alpha$$

where

$$H_{11}(\alpha) = \frac{\lambda_{14}m_{18}(\alpha) + \lambda_{12}m_{19}(\alpha)}{m_9(\alpha)}$$

$$H_{12}(\alpha) = \frac{\lambda_{13}m_{18}(\alpha) + \lambda_{11}m_{19}(\alpha)}{m_9(\alpha)}$$

$$H_{21}(\alpha) = \frac{S_1\lambda_{14}m_{20}(\alpha) + S_2\lambda_{14}m_{21}(\alpha) - S_1\lambda_{12}m_{22}(\alpha) - S_2\lambda_{12}m_{23}(\alpha)}{m_9(\alpha)}$$

$$H_{22}(\alpha) = \frac{S_1\lambda_{13}m_{20}(\alpha) + S_2\lambda_{13}m_{21}(\alpha) - S_1\lambda_{11}m_{22}(\alpha) - S_2\lambda_{11}m_{23}(\alpha)}{m_9(\alpha)}$$

using equations (3.11a,b) and continuity of displacements outside crack and changing the order of integration we obtain

$$-\frac{P_1(x)}{E} = \int_{-a}^{d}\Big[F_1(t)\,G_{11}(t,x) + F_2(t)\,G_{21}(t,x) + g_{11}(t,x)F_1(t)$$

$$+ g_{12}(t,x)F_2(t)\Big]dt$$

$$-\frac{P_2(x)}{E} = \int_{-d}^{d} F_2(t)\, G_{22}(t,x)\,dt + \int_{-d}^{d} F_1(t)\, G_{21}(t,x)\,dt$$

$$+ \int_{-d}^{d} g_{21}(t,x)\, F_1(t)\,dt + \int_{-d}^{d} g_{22}(t,x)\, F_2(t)\,dt$$

where

$$G_{11}(t,x) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} H_{11}(\alpha)\, \frac{\alpha}{|\alpha|}\, e^{i\alpha(t-x)}\, d\alpha$$

$$G_{12}(t,x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} H_{12}(\alpha)\, e^{i\alpha(t-x)}\, d\alpha$$

$$G_{21}(t,x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} H_{21}(\alpha)\, \frac{\alpha}{|\alpha|}\, e^{i\alpha(t-x)}\, d\alpha$$

$$G_{22}(t,x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} H_{22}(\alpha)\, e^{i\alpha(t-x)}\, d\alpha$$

$$g_{11}(t,x) = -\frac{i}{2\pi} \lim_{y\to 0} \int_{-\infty}^{\infty} \left(\lambda_{12} e^{-S_1|\alpha|y} + \lambda_{14} e^{-S_2|\alpha|y}\right) \frac{\alpha}{|\alpha|}\, e^{i\alpha(t-x)}\, d\alpha$$

$$g_{12}(t,x) = -\frac{1}{2\pi} \lim_{y\to c} \int_{-\infty}^{\infty} \left(\lambda_{11} e^{-S_1|\alpha|y} + \lambda_{13} e^{-S_2|\alpha|y}\right) e^{i\alpha(t-x)}\, d\alpha$$

$$g_{21}(t,x) = \frac{1}{2\pi} \lim_{y\to 0} \int_{-\infty}^{\infty} \left(S_1\lambda_{12} e^{-S_1|\alpha|y} + S_2\lambda_{14} e^{-S_2|\alpha|y}\right) e^{i\alpha(t-x)} d\alpha$$

$$g_{22}(t,x) = -\frac{i}{2\pi} \lim_{y\to 0} \int_{-\infty}^{\infty} \left(S_1\lambda_{11} e^{-S_1|\alpha|y} + S_2\lambda_{13} e^{-S_2|\alpha|y}\right) \frac{|\alpha|}{\alpha} e^{i\alpha(t-x)} d\alpha$$

we know that

$$\int_{-\infty}^{\infty} F(\alpha)\,d\alpha = \int_{0}^{\infty} \left[F(\alpha) + F(-\alpha)\right] d\alpha$$

$$e^{i\alpha(t-x)} + e^{-i\alpha(t-x)} = 2\cos\alpha(t-x)$$

$$e^{i\alpha(t-x)} - e^{-i\alpha(t-x)} = 2i\sin\alpha(t-x)$$

$$\lim_{y\to 0} \int_0^{\infty} e^{-y\alpha} \sin\alpha(t-x)\,d\alpha = \frac{1}{t-x}$$

$$\lim_{y\to 0} \int_0^{\infty} e^{-y\alpha} \cos\alpha(t-x)\,d\alpha = 0.$$

Using the above expressions we can show that

$$g_{21}(t,x) = 0 = g_{12}(t,x),$$

$$g_{11}(t,x) = \frac{\lambda_{12} + \lambda_{14}}{\pi(t-x)}, \quad g_{22}(t,x) = \frac{S_1\lambda_{11} + S_2\lambda_{13}}{\pi(t-x)}$$

In the same way we can simplify $G_{ij}(t,x)$ $(i,j = 1,2)$ as

$$G_{11}(t,x) = -\frac{i}{2\pi}\int_0^\infty \left(H_{11}(\alpha)\, e^{i\alpha(t-x)} - H_{11}(-\alpha)\, e^{-i\alpha(t-x)} \right) d\alpha$$

$$G_{12}(t,x) = -\frac{1}{2\pi}\int_0^\infty \left(H_{12}(\alpha)\, e^{i\alpha(t-x)} + H_{12}(-\alpha)\, e^{-i\alpha(t-x)} \right) d\alpha$$

$$G_{21}(t,x) = -\frac{1}{2\pi}\int_0^\infty \left(H_{21}(\alpha)\, e^{i\alpha(t-x)} - H_{21}(-\alpha)\, e^{-i\alpha(t-x)} \right) d\alpha$$

$$G_{22}(t,x) = \frac{i}{2\pi}\int_0^\infty \left(H_{22}(\alpha)\, e^{i\alpha(t-x)} + H_{22}(-\alpha)\, e^{-i\alpha(t-x)} \right) d\alpha$$

Using the definitions of $H_{ij}(\alpha)$ $(i,j=1,2)$ and $m_j(\alpha)$ $(j=1,23)$ we can show that

$$H_{11}(\alpha) = H_{11}(-\alpha)$$

$$H_{12}(\alpha) = H_{12}(-\alpha)$$

$$H_{21}(\alpha) = -H_{21}(-\alpha)$$

$$H_{22}(\alpha) = -H_{22}(-\alpha)$$

These conditions can be used to simplify the $G_{ij}(t,x)$ $(i,j = 1,2)$ in the following form

$$G_{11}(t,x) = \frac{1}{\pi}\int_0^\infty H_{11}(\alpha)\,\text{Sin}\,\alpha(t-x)\,d\alpha$$

$$G_{12}(t,x) = -\frac{1}{\pi}\int_0^\infty H_{12}(\alpha)\,\text{Cos}\,\alpha(t-x)\,d\alpha$$

$$G_{21}(t,x) = -\frac{1}{\pi}\int_0^\infty H_{21}(\alpha)\,\text{Cos}\,\alpha(t-x)\,d\alpha$$

$$G_{22}(t,x) = -\frac{1}{\pi}\int_0^\infty H_{22}(\alpha)\,\text{Sin}\,\alpha(t-x)\,d\alpha$$

Finally integral equations become

$$-\frac{\pi}{E\lambda_{15}}P_1(x) = \int_{-d}^{d}\left[\frac{F_1(t)}{t-x} + K_{11}(t,x)F_1(t) + K_{12}(t,x)F_2(t)\right]dt$$

$$-\frac{\pi}{E\lambda_{16}}P_2(x) = \int_{-d}^{d}\left[\frac{F_2(t)}{t-x} + K_{22}(t,x)F_2(t) + K_{21}(t,x)F_1(t)\right]dt$$

where

$$K_{1j}(t,x) = \frac{\pi}{\lambda_{15}}G_{1j}(t,x),$$

$$K_{2j}(t,x) = \frac{\pi}{\lambda_{16}}G_{2j}(t,x), \quad (j = 1,2).$$

VITA

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He attended high school in Cankiri, Turkey and received his high school diploma in June 1973. He continued his studies at Technical University of Istanbul, where he received the degree Bachelor of Science in Mechanical Engineering in June, 1978. He began work towards the M.S. degree in Mechanical Engineering and Mechanics at Lehigh University in the fall of 1978, having received a research assistantship to pursue this further study.