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A Cauchy-Gaussian Mixture Model For Basel-Compliant Value-At-Risk Estimation In Financial Risk Management

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A CAUCHY-GAUSSIAN MIXTURE MODEL FOR BASEL-COMPLIANT VALUE-AT-RISK ESTIMATION IN FINANCIAL RISK MANAGEMENT

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Abstract

The Basel II accords require banks to manage market risk by using Value-at-Risk (VaR) models. The assumption of the underlying return distribution plays an important role for the quality of VaR calculations. In practice, the most popular distribution used by banks is the Normal (or Gaussian) distribution, but real-life returns data exhibits fatter tails than what the Normal model predicts. Practitioners also consider the Cauchy distribution, which has very fat tails but leads to over-protection against downside risk. After the recent financial crisis, more and more risk managers realized that Normal and Cauchy distributions are not good choices for fitting stock returns because the Normal distribution tends to underestimate market risk while the Cauchy distribution often overestimates it.

In this thesis, we first investigate the goodness of fit for these two distributions using real-life stock returns and perform backtesting for the corresponding two VaR models under Basel II. Next, after we identify the weaknesses of the Normal and Cauchy distributions in quantifying market risk, we combine both models by fitting a new Cauchy-Normal mixture distribution to the historical data in a rolling time window. The method of Maximum Likelihood Estimate (MLE) is used to estimate the density function for this mixture distribution. Through a goodness of fit test and backtesting, we find that this mixture model exhibits a good fit to the data, improves the accuracy of VaR prediction, possesses more flexibility, and can avoid serious violations when a financial crisis occurs.

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Literature Review

1.1 Value at Risk

VaR represents the maximum loss (or worst loss) over a target horizon at a given confidence level. According to Jorion [13], the greatest advantage of Value at Risk (VaR) is that it summarizes the downside risk of an institution due to financial market variables in a single, easy-to-understand number. This commonly used risk measure can be applied to just about any asset class and takes into account many variables, including diversification, leverage and volatility, that make up the kind of market risk that traders and firms face every day (Nocera [22]).

Mathematically, VaR is defined as (Fabozzi [11]):

$$VaR_{1-\epsilon}(R_p) = \min\{R|P(-R_p \geq R) \leq \epsilon\}. \tag{1.1}$$

In Eq. 1.1, $VaR_{1-\epsilon}(R_p)$ is the value R such that the probability of the possible portfolio loss ($-R_p$) exceeding this value R is at most some small number ϵ such as 1%, 5%, or 10%.

1.1.1 Calculation of VaR

There are three methods for calculating VaR:

- Variance-Covariance
- Historical simulation

- Monte Carlo simulation

Variance-Covariance

The Variance-Covariance method assumes that the returns of the assets are Normally distributed with a mean of zero, which is reasonable because the expected change in portfolio value over a short holding period is almost always close to zero (Linsmeier [17]).

Therefore, the profit and loss distribution can be expressed as (Cho [6]):

$$P\&L \sim N(0, W^T \Sigma W), \tag{1.2}$$

where W is the vector of the amount of each asset in the portfolio and $W^T \Sigma W$ is the variance. Given the confidence level of (1-α), we can thus calculate VaR as:

$$VAR = z_{(1-\alpha)} \sqrt{W^T \Sigma W}, \tag{1.3}$$

where $z_{1-\alpha}$ is the corresponding percentile of the standard normal distribution.

The advantages of Variance-Covariance method are: (i) The methodology is based on well-known techniques (Munniksma [20]), (ii) The traditional mean-variance analysis is directly applied to VaR-based portfolio optimization, since VaR is a scalar multiple of the standard deviation of loss when the underlying distribution is Normal (Yamai and Yoshiba [27]).

The disadvantages of Variance-Covariance method are: (i) the portfolio is composed of assets whose changes are linear, (ii) the assumption that the asset returns are Normally distributed is rarely true (Munniksma [20]).

Historical Simulation

The fundamental assumption of the Historical Simulation methodology is that the recent past will reproduce itself in the near future. This assumption may be incorrect in very volatile markets or in periods of crisis (Berry [2]). The Historical Simulation (HS) approach generates the P&L distribution for VaR estimation from historical samples and does not rely on any statistical distribution or random process. According to JP Morgan, there are four steps in calculating Historical Simulation VaR:

- Calculate the returns (or price changes) of all the assets in the portfolio in each time interval,

- Apply the price changes calculated to the current mark-to-market value of the assets and re-value the portfolio,

- Sort the series of the portfolio-simulated P&L from the lowest to the highest value,
- Read the simulated value that corresponds to the desired confidence level.

The advantages of Historical Simulation are: (i) The method is simple to implement, (ii) it is non-parametric. In other words, it does not require a specific distribution (Munniksma [20]), (iii) it captures fat tails (rare events) in price change distribution (Berkowitz and OBrien [1]).

The disadvantages of Historical Simulation are: (i) it is difficult to optimize simulation-based VaR (Mausser and Rosen [19]), (ii) the simulation is computationally intensive (Munniksma [20]).

Monte Carlo Simulation

The Monte-Carlo method is based on the generation of a large number of possible future prices using simulation. The resulting changes in the portfolio value are then analyzed to arrive at a single VaR number (Cassidy and Gizycki [5]).

According to JP Morgan (Berry [3]), there are five steps in the application of Monte Carlo simulation:

- Determine the length T of the analysis horizon and divide it equally into a large number N of small time increments Δt (i.e. $\Delta t = T/N$),
- Draw a random number from a random number generator and update the price of the asset at the end of the first time increment,
- Repeat Step 2 until the end of the analysis horizon T is reached by walking along the N time intervals,
- Repeat Steps 2 and 3 a large number M of times to generate M different paths for the stock over T,
- Rank the M terminal stock prices from the smallest to the largest, read the simulated value in this series that corresponds to the desired $(1-\alpha)\%$ confidence level (95% or 99% generally) and deduce the relevant VaR, which is the difference between S_i and the α-th lowest terminal stock price. S_i is the stock price on the ith day.

The advantage of Monte Carlo simulation is that the Monte Carlo simulation approach can easily be adjusted to economic forecasts (Munniksma [20]).

The disadvantages of Monte Carlo simulation are: (i) it is computationally intensive, (ii) the manager must input specific theoretical distributions to generate samples from.

1.1.2 Shortcomings of VaR

Although VaR is widely used by financial institutions, it has three undesirable properties (Fabozzi [11]). First, it is not subadditive, so the risk as measured by the VaR of a portfolio of two funds may be higher than the sum of the risks of the two individual portfolios. This goes against the intuitive property that diversification should decrease risk. Second, when VaR is calculated from generated scenarios, it is a nonsmooth and nonconvex function of the decision variables, i.e., the portfolio allocation. Third, VaR does not take the magnitude of the losses beyond the VaR value into account. VaR tells us, for instance, that our weekly losses will not exceed a certain value 95% of the time, but we do not know how severe they will be if we do find ourselves in that 5% of adverse scenarios. In addition, since VaR highly depends on historical returns and/or the Gaussian assumption, there exists a significant possibility of prediction errors that will affect the quality of VaR estimation.

1.2 Goodness of Fit Test

The goodness of fit of a statistical model describes how well it fits a set of observations. Measures of goodness of fit quantify the discrepancy between observed values and the values expected under a model. In determining whether a given distribution is suited to a given data set, two tests are usually used: Pearson's Chi-squared test and Kolmogorov-Smirnov test (KS test).

1.2.1 Pearson's Chi-squared test

Pearson's Chi-squared test tests the null hypothesis that the frequency distribution observed in a sample is consistent with a theoretical distribution. The test statistic is (Greenwood and Nikulin [12]):

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}, \tag{1.4}$$

where

χ^2=Pearson's cumulative test statistic, which asymptotically approaches a chi-squared distribution,

O_i=an observed frequency,

E_i=an expected (theoretical) frequency, asserted by the null hypothesis,

n=the number of cells in the table.

According to this theory, the statistic χ^2 approaches a chi-square distribution. Hence, we can calculate the corresponding p value for the statistic. Given a significance level (e.g. 0.05), if the p value is less than the significance level, we reject the null hypothesis and conclude that the observations are not from the assumed theoretical distribution under this significance level, and vice versa.

1.2.2 Kolmogorov-Smirnov test

The Kolmogorov-Smirnov statistic quantifies the distance between the empirical distribution function of the sample and the cumulative distribution function of the reference distribution, or between the empirical distribution functions of two samples. The null hypothesis is that the samples are drawn from the same distribution (in the two-sample case) or that the sample is drawn from the reference distribution (in the one-sample case).

The empirical distribution function F_n for n i.i.d observations X_i is defined as:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{X_i} \leq x, \tag{1.5}$$

where I_{X_i} is the indicator function, equal to 1 if $X_i \leq x$ and equal to 0 otherwise.

The one-sample KS statistic for a given cumulative distribution function $F(x)$ is:

$$D_n = \sup_x |F_n(x) - F(x)|, \tag{1.6}$$

where $\sup_x$ is the supremum. If F is continuous and n is large enough, then under the null hypothesis the statistic $\sqrt{n}D_n$ converges to the Kolmogorov distribution, which does not depend on F (Kolmogorov [14]).

Therefore, we can find the corresponding p value according to $\sqrt{n}D_n$ in the Kolmogorov distribution. Hence, by comparing the p value with the given significance level, we can decide whether to reject the null hypothesis or not.

KS test for two samples

The Kolmogorov-Smirnov test may also be used to test whether two underlying one-dimensional probability distributions differ. The Kolmogorov-Smirnov test for two samples is very similar to the KS test above. Suppose that a first sample $X_1, \ldots, X_m$ of size m has distribution with CDF $F(x)$ and the second sample $Y_1, \ldots, Y_m$ of size n has distribution with CDF $G(x)$ and we want to test:

$$H_0: \; F = G \;\; vs. \;\; H_1 : F \neq G. \tag{1.7}$$

If $F_m(x)$ and $G_n(x)$ are the corresponding empirical CDFs then we have the following statistic:

$$D_{mm} = (\frac{mn}{m+n})^{1/2} \sup_x |F_m(x) - G_n(x)|. \tag{1.8}$$

This statistic also approaches the Kolmogorov distribution. Hence, we can check whether the two data samples come from the same distribution.

1.3 Basel II

The use of VaR in financial risk management has been heavily promoted by bank regulators (Jorion [13]). The landmark Basel Capital Accord of 1988 provided the first step toward strengthened risk management. The so-called Basel Accord sets minimum capital requirements that must be met by commercial banks to guard against credit risk. It is named after the city where the Bank for International Settlements (BIS) is located, namely Basel, Switzerland. Basel II, initially published in June 2004, is the successor to Basel I. It was intended to create an international standard for banking regulators to control how much capital banks need to put aside in order to guard against financial and operational risks. The BIS gives recommendations to banks and other financial institutions on how to manage capital (Munniksma [20]).

Basel II uses a “three pillars” concept(See Figure 1.1), where the three pillars are: (1) minimum capital requirements (addressing risk), (2) supervisory review, and (3) market discipline.

1.3.1 Types of risks in Basel II

As we can see from Figure 1.1, three types of risks are covered by the minimum capital requirement: Credit Risk, Market Risk, and Operational Risk.

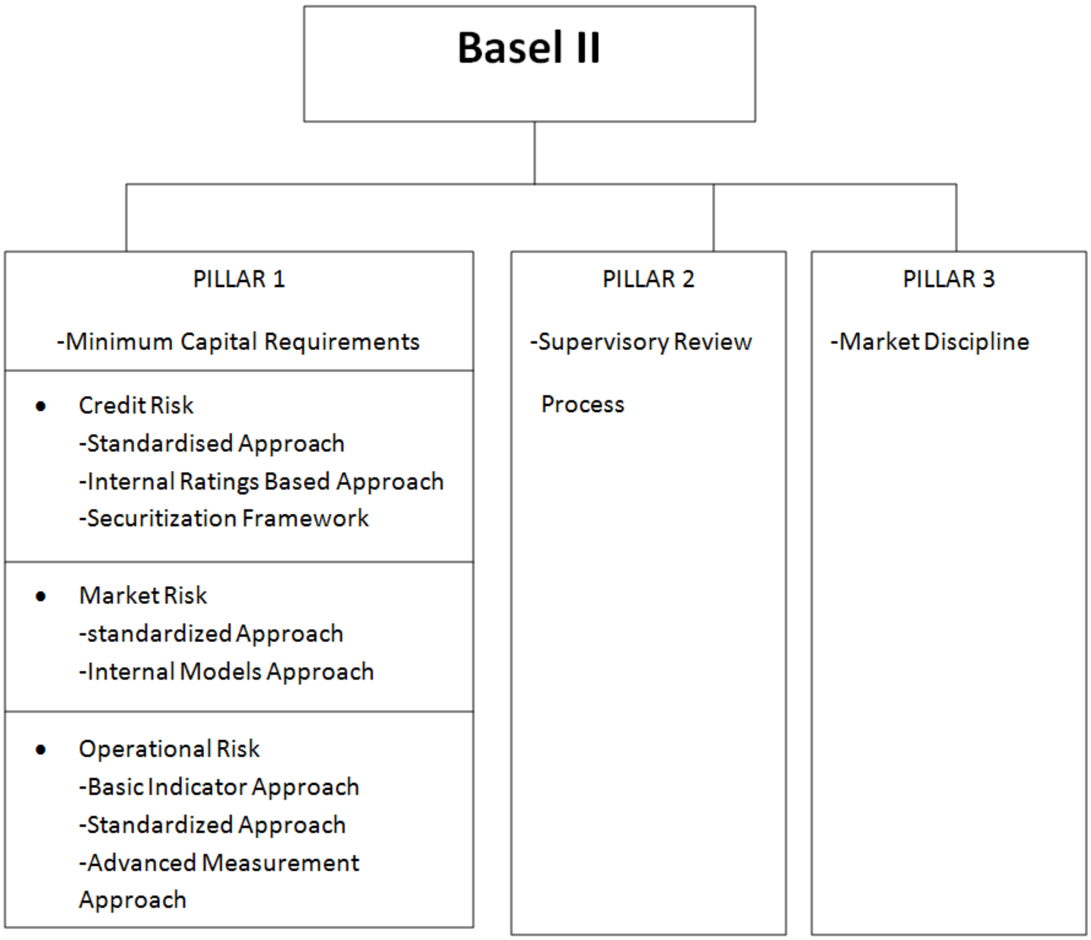


Figure 1.1: Structure of Basel II

Credit Risk

Credit risk is an investor's risk of loss arising from a borrower who does not make payments as promised (Basel II [7]). It is also called default risk and counterparty risk. According to Basel II, three methods can be used for managing credit risk: the Standardized Approach, the Foundation Internal Rating Based Approach, and the Advanced Rating Based Approach.

Operational Risk

In Basel II (Basel II [7]), operational risk is the risk of loss resulting from inadequate or failed internal processes, people and systems, or from external events. Since operational risk is not used to generate profit, the approach to managing operational risk differs from that applied to other types of risk. Three methods have been mentioned in Basel II: the Basic Indicator Approach, the Standardized Approach, and the Advanced Measurement Approach.

Market Risk

Market Risk refers to the risk that the value of a portfolio, either an investment portfolio or a trading portfolio, will decrease due to the change in value of the market risk factors. The four standard market risk factors are stock prices, interest rates, foreign exchange rates, and commodity prices (Basel II [7]). The two methods used to measure market risk in Basel II are: the Standardized Approach and the Internal Models Approach.

The focus of our thesis lies in the measurement of market risk, which will be discussed in detail in the following.

1.3.2 Market Risk

As mentioned above, market risk refers to the risk resulting from movements in market prices (changes in interest rates, foreign exchange rates, and equity and commodity prices). Market risk is often propagated by other forms of financial risk such as credit and market-liquidity risks (Hassan [15]). Under Basel II, banks are encouraged to develop sound and well informed strategies to manage market risk and are required to communicate their daily market risk estimates to the relevant authorities at the beginning of each trading day. In measuring their market risks, banks can choose between two methods. One is the standardized approach and the other one is internal model-based approach. For market risk, the preferred approach is the internal model-based approach. Under Basel II (Basel

II [7]), however, the internal model-base approach should be subject to seven sets of conditions, namely:

- Certain general criteria concerning the adequacy of the risk management system,
- Qualitative standards for internal oversight of the use of models, notably by management,
- Guidelines for specifying an appropriate set of market risk factors(i.e. the market rates and prices that affect the value of banks' positions),
- Quantitative standards setting out the use of common minimum statistical parameters for measuring risk,
- Guidelines for stress testing,
- Validation procedures for external oversight of the use of models,
- Rules for banks which use a mixture of models and the standardized approach.

Although banks have flexibility in devising their models, they must abide to the following rules (Basel II [7]):

- "Value-at-risk" must be computed on a daily basis,
- In calculating VaR, a 99th percentile one-tailed confidence interval is to be used,
- In calculating VaR, an instantaneous price shock equivalent to a 10-day movement in prices is to be used,
- The historical observation period is a minimum length of one year,
- Banks should update their data sets no less frequently than once every month.

In addition, Basel II regulates the functions for calculating capital requirement. Each bank must meet, on a daily basis, a capital requirement expressed as the higher of (i) its previous day's Value-at-Risk number measured according to the parameters specified above (VAR_{t-1}) and (ii) an average of the daily Value-at-Risk measures on each of the preceding sixty business days (VAR_{avg}), multiplied by a multiplication factor (m_c), which is at least 3. The model is then expressed as:

$$DCC = \max\left\{VAR_{t-1},\, (m_c + k) \cdot VAR_{avg}\right\}, \tag{1.9}$$

where DCC is the daily capital requirement.

Basel II [8] additionally requires that a bank must calculate a 'stressed value-at-risk' measure ($sVAR$) that captures a hypothetical period of stress on the relevant factors. Then according to Basel II, the capital requirement should be calculated according to the following new formula:

$$DCC = \max\left\{VAR_{t-1},\, (m_c + k) \cdot VAR_{avg}\right\} + \max\left\{sVAR_{t-1},\, (m_s + k) \cdot sVAR_{avg}\right\}.$$

The purpose of stressed VAR is to better take into account extreme or tail risks.

1.3.3 Backtesting Framework

As we have seen in Eq. (1.9), there is a factor named k. Under Basel II, banks will be required to add to the multiplication factor a "plus", k, related the ex-post performance of the model. This creates an incentive to develop models with good predictive qualities. k will range from 0 to 1 based on the outcome of backtesting. Since backtesting plays an important role when we use the internal model-based approach, in what follows we will discuss backtesting in detail.

Backtesting consists of a periodic comparison of the bank's daily VaR measure with the subsequent daily profit or loss ("trading outcome"). According to the number of VaR violations (violation means that the loss is larger than the relative VaR), banks can evaluate the accuracy of their capital requirement model and then make daily adjustment for k.

In reality, many factors influence the profit and losses, such as price movement, intra-day trading, portfolio composition shifts, and fee income, complicating the issue of back-testing. According to Basel II, the fee income and the trading gains or losses resulting from changes in the composition of the portfolio should not be included in the definition of the trading outcome because they do not relate to the risk inherent in the static portfolio that was assumed in computing VaR (Basel II [7]). Furthermore, where open positions remain at the end of the trading day, intra-day trading will tend to increase the volatility of trading outcomes, and may result in VaR figures underestimating the true risk of the portfolio.

On the other hand, the Value-at-Risk approach to risk measurement is generally based on analyzing the possible change in the value of the static portfolio due to price and rate movements over the assumed holding period. Therefore, it is unreasonable to compare the Value-at-Risk measure against actual trading outcomes directly. In order to overcome the comparison problem in our model, we need to set some conditions and assumptions in terms of Basel II:

- The backtesting described in our model involves the use of VaR with 99% confidence level, one tail, previously 250 observations, and a one-day holding period (although the value-at-risk in the capital requirement formula mentioned above uses ten-day holding period);

- Performance of backtesting is based on the hypothetical changes in portfolio value that would occur were end-of-day positions to remain unchanged;

- The fee incomes have been separated from the trading profit and losses.

1.3.4 Description of the Backtesting approach

The idea behind backtesting is that we want to test if the capital requirement calculated by the internal model-based approach has a true coverage level of 99% (Basel II [7]). For example, over 200 trading days, a 99% daily risk measure should cover, on average, 198 of the 200 trading outcomes, leaving two exceptions. If there are too many violations, the model we used may be inaccurate and we need to adjust k to get the 99% coverage level.

When doing backtesting, we will face two types of statistical errors: (i) false negative, i.e., the possibility that an accurate risk model would be classified as inaccurate on the basis of its backtesting result, and (ii) false positive, i.e., the possibility that an inaccurate model would not be classified that way based on its backtesting result. Hence, three violation zones have been defined in Basel II [7] and their boundaries chosen in order to balance the two types of error (see Table 1.1).

As we can see in the figure above, the green zone gives a penalty of zero, which means that four exceptions or less (out of 250 data points) will be quite likely to indicate a truly 99% coverage level. The red zone gives the biggest penalty of one, which means that it

Zone	Number of exceptions	Increase In scaling factor	Cumulative probability
Green Zone	0	0.00	8.11%
	1	0.00	28.58%
	2	0.00	54.32%
	3	0.00	75.81%
	4	0.00	89.22%
Yellow Zone	5	0.40	95.88%
	6	0.50	98.63%
	7	0.65	99.60%
	8	0.75	99.89%
	9	0.85	99.97%
Red Zone	10 or more	1.00	99.99%

Table 1.1: Three penalty zones (Basel II [7])

is extremely unlikely that an accurate model would independently generate ten or more exceptions from a sample of 250 trading outcomes. In addition to assigning a penalty, if a bank's model falls in the red zone, the supervisor should also begin investigating the reasons for the bad result. In the yellow zone, it is difficult to judge if the model is accurate (but generated outlier points) or inaccurate. In order to return the model to a 99% coverage level, the yellow zone uses some specific values for each number of value-at-risk violations. For example, five violations in a sample of 250 implies only 98% coverage. If the trading outcomes are Normally distributed, the ratio of 99th percentile to 98th percentile is approximately 1.14. Then the product of 1.14 and multiplication factor 3 will be 3.42, which is approximately equal to 3 plus k of 0.4.

Therefore, the backtesting model can be expressed as:

$$k = \begin{cases} 0 & \text{if } V \leq 4 \\ 0.4 + 0.1(V-5) & \text{if } 5 \leq V \leq 6 \\ 0.65 + 0.1(V-7) & \text{if } 7 \leq V \leq 9 \\ 1 & \text{if } V \geq 10, \end{cases}$$

where V means the number of violations. k must be evaluated and updated every day.

In conclusion, by incorporating backtesting with the internal model-based approach, we obtain the following steps to calculate the daily capital requirement of market risk:

- Calculating k according to the previous day's backtesting result, which is implemented by comparing the previous 250 days' one-day holding period Value-at-Risk against the correspondingly 250 trading outcomes beginning from yesterday backward,
- Respectively calculating (i) the previous day's ten-day holding period value-at-risk measured according to the parameters specified above and (ii) an average of the daily ten-day holding period value-at-risk measures on each of the preceding sixty business days, multiplied by a multiplication factor of (3+k),
- Getting today's capital requirement by using the higher of (i) and (ii),
- Repeating the three steps above for the following days.

Chapter 2

Testing of distributions used for VaR under Basel II

The main purpose of Basel II is to provide a risk management standard for banks and other financial institutions. For market risk, Basel II currently uses VaR as the risk measure. Although VaR is widely used by banks for market risk management, it has some undesirable weaknesses. One of the biggest problems for the VaR model is the assumption for the underlying distribution. After the recent financial crisis, more and more risk managers became aware of this issue. It is true that the Basel Committee is trying to compensate for the shortcomings of the distribution assumption by adding the stress-VaR to the original model. However, since the Basel Committee does not specify the stress period, and in fact requires that banks consider multiple stress periods, the measurement is still open to interpretation. Moreover, if banks implemented the requirement literally, they might be forced to run VaR models continuously to find the appropriate window of market stress, which would be computationally burdensome (Pengelly [23]). For these reasons, risk managers still need to find more efficient models. Our thesis focuses on improving the distribution used for the VaR model. In this chapter we will analyze the weaknesses of the distributions currently in use by checking the goodness of fit test and implementing backtesting under Basel II. Since Monte Carlo simulation has become the industry standard to generate samples, it will be used in our thesis for calculating VaR.

2.1 Goodness of Fit test for Benchmark distributions

The distribution fit directly influences the quality of VaR model: if the actual returns do not follow the assumed distribution, the VaR model will exhibit poor performance. In general, risk managers usually assume a Normal distribution for market returns. However, when compared to a Normal distribution, historical data has shown a significant degree of 'fat tail risk' in the returns of the US stock market. Throughout financial history, there have been a number of extreme, and often severe, events that cannot be predicted based on prior events. While Nassim Taleb famously referred to this as the Black Swan theory, it is more widely regarded as "Fat-tail Risk" (Cook Pine Capital [9]). The reader is referred to Cook Pine Capital [9] and Taleb [25] for more details about fat-tail risk. In the following we will take the stock of Exxon Mobil Corporation (XOM) as an example to show that the Normal distribution indeed ignores the fat tail of historical returns. Matlab has been used as our programming software. Our approach can be easily extended to the historical data of any stock or portfolio.

When testing the goodness of fit for a distribution, we need to first select a reasonable observation period. The overall historical close prices and returns for XOM are shown in Figures 2.1 and 2.2:

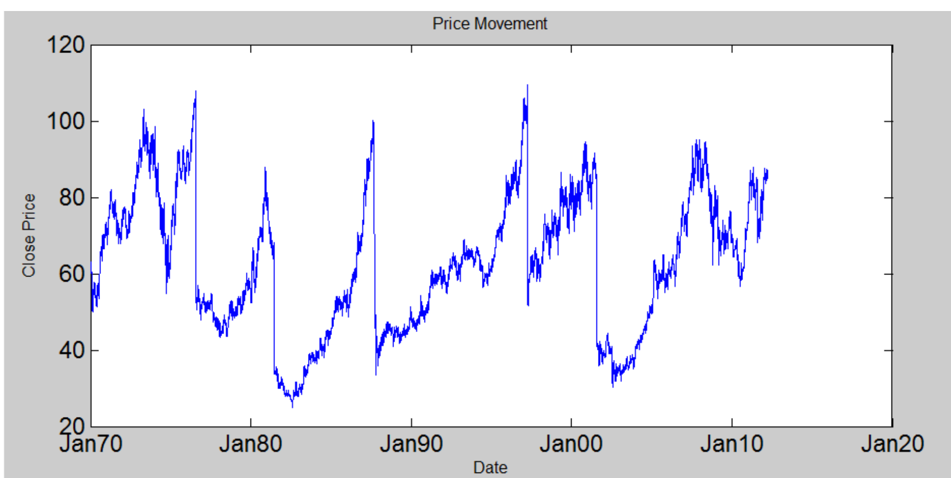


Figure 2.1: Historical Price Movement of XOM

As we can see from Figure 2.1, historical prices exhibit periodical movements while

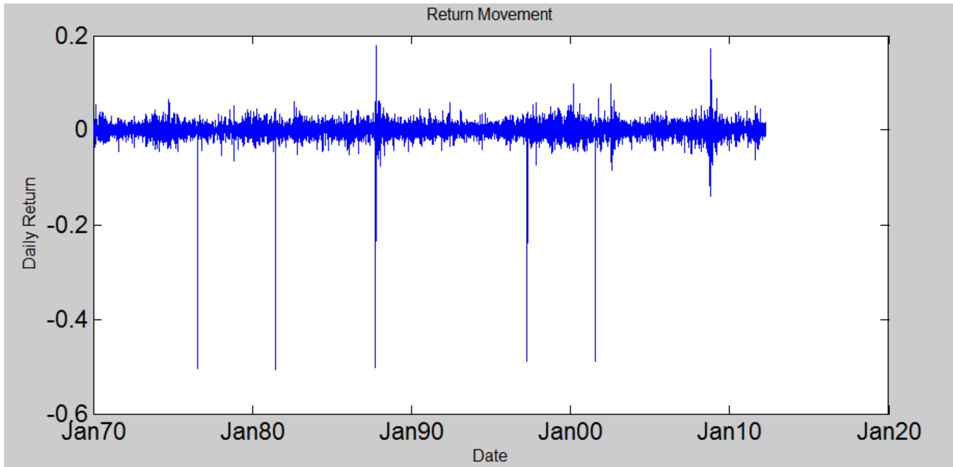


Figure 2.2: Historical Return Movement of XOM

the length of the cycle period changes every time. This movement has also been shown in Figure 2.2. The returns always go up and down around zero and then extreme changes happen. However, we should point out that the most recent extreme negative return is much smaller than what has happened in history. Hence, our selected observation period should reflect both the cycle movement and the recent market changes. In this case, we have chosen 1700 historical observations beginning from July 8th, 2005 to April 5th, 2012. Figure 2.3 shows the 1700 daily historical returns of XOM with a Normal distribution fit. The parameters used, which were identified by Matlab as those providing optimal fit, are μ=3.7009e-004 and σ=0.0180.

We can see that the Normal distribution ignores the extreme points and does not fit the fat tail very well, which means that it understates the tails of the actual distribution. In addition, the distribution of historical returns seems more 'peaked' than that of the Normal one. As a result, the VaR model using Normal distribution cannot protect banks from fat-tail risks. Alternatively, some financial institutions use a fat-tail distribution for Monte Carlo simulation: in practice, some risk managers prefer to use the Cauchy distribution which is a fat-tail distribution. According to Mandelbrot [18], the Cauchy distribution fits the tails of stock returns much better. The construction of its cumulative distribution function (CDF) is not overly difficult as it relies on two simple parameters: the median and the difference between the 75th and 25th percentile divided by 2 (called Gamma). For more details about Cauchy distribution, see Weisstein [26]. Figure 2.4

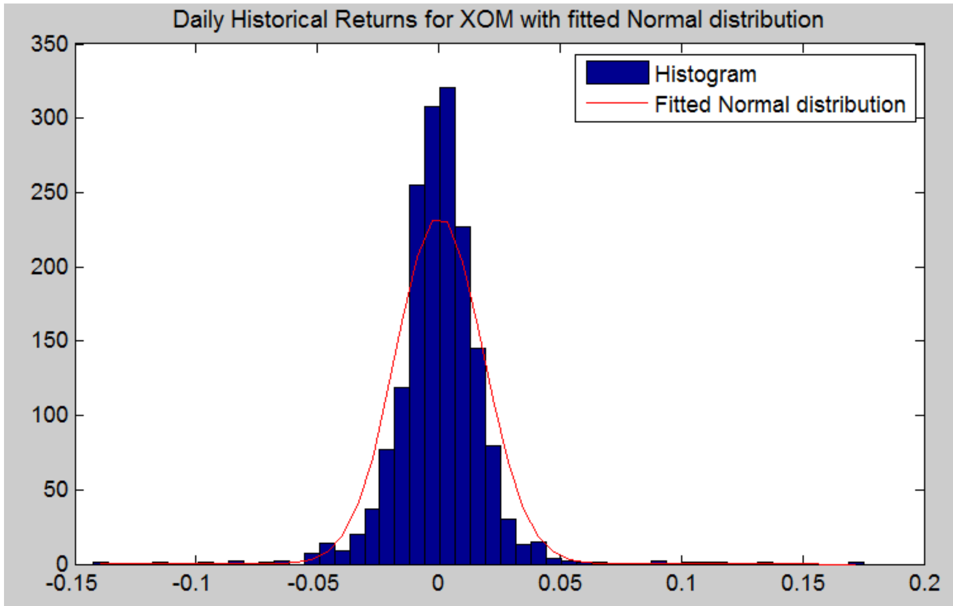


Figure 2.3: Daily Historical Returns for XOM with fitted Normal distribution

shows the 1700 historical returns fitted to a Cauchy distribution. The parameters used are location=6.5175e-004 and scale=0.0081.

We can see that the Cauchy distribution has fewer observations centered around the mean. Those are redistributed in the tails. Hence, the Cauchy distribution has included the extreme points. However, it seems that the tails of the Cauchy distribution are much longer and fatter than that of the actual returns, resulting in overstating the tails of the actual distribution. The fitting issues for both can also be seen in the following histogram counts table (Table 2.1). As shown, the Normal distribution fit is not good as it misses the 10 worst returns and 8 best returns. Likewise, the Cauchy distribution fit is also poor as its tails are too fat.

In order to verify the fitting performance for the Normal and Cauchy distributions, we need to perform a goodness of fit test. In our thesis, we will use both Pearson's Chi-squared test and Kolmogorov-Smirnov test (KS-test), then we will confirm the result using the Probability Plot (PP-plot). In the goodness of fit test, we have the Null hypothesis that the historical observations are from the specified theoretical distribution. Table 2.2 shows the result of the goodness of fit test.

The table shows that all of the p-values are zero, meaning that all the results are

Daily Returns Bins	Actual	Normal	Cauchy
-0.139525255	1	0	3
-0.127068042	1	0	1
-0.11461083	0	0	4
-0.102153617	1	0	6
-0.089696404	2	0	7
-0.077239191	2	0	8
-0.064781978	3	3	8
-0.052324765	22	19	17
-0.039867552	42	82	50
-0.027410339	143	192	128
-0.014953126	469	437	391
-0.002495913	624	458	684
0.0099613	283	344	191
0.022418512	72	130	60
0.034875725	22	29	35
0.047332938	5	6	18
0.059790151	1	0	10
0.072247364	0	0	11
0.084704577	2	0	9
0.09716179	2	0	6
0.109619003	1	0	5
0.122076216	1	0	3
0.134533429	0	0	0
0.146990642	0	0	0
0.159447854	0	0	0
0.171905067	1	0	0
	1700	1700	1700

Table 2.1: Histogram counts for Actual, Normal, and Cauchy distributions

	Statistic	P-value
Chi-squared test for Normal	166.1890	8.1754e-037
Chi-squared test for Cauchy	146.7310	1.9714e-028
KS two sample test for Normal	0.0904	1.7305e-009
KS two sample test for Cauchy	0.0655	3.4798e-005

Table 2.2: Goodness of fit test

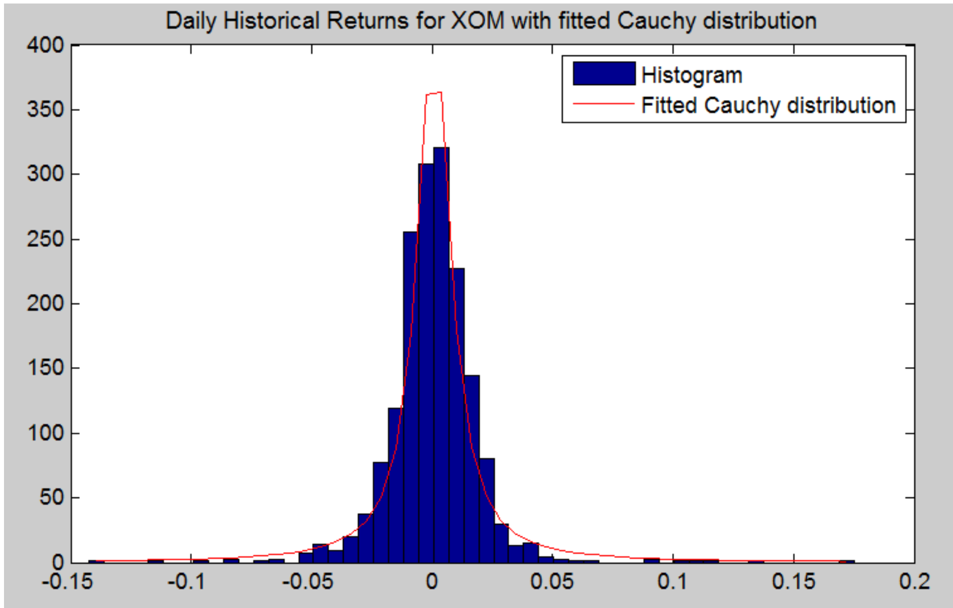


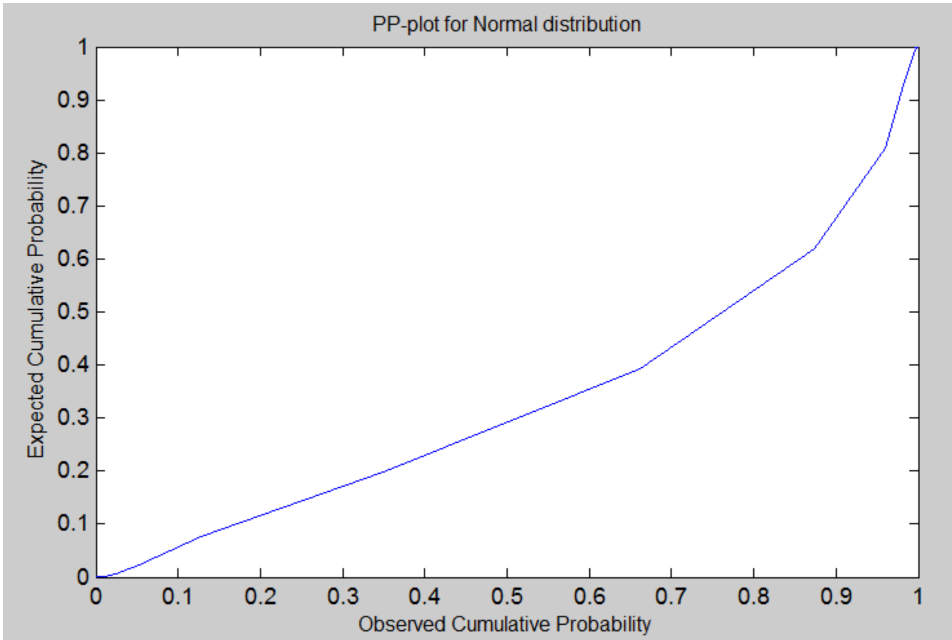
Figure 2.4: Daily Historical Returns for XOM with fitted Cauchy distribution

significant and hence we reject the Null hypothesis. The PP-plot in Figure 2.5 also confirms the result: we can see that neither of the plots are in a straight line, indicating that the distributions fits are poor.

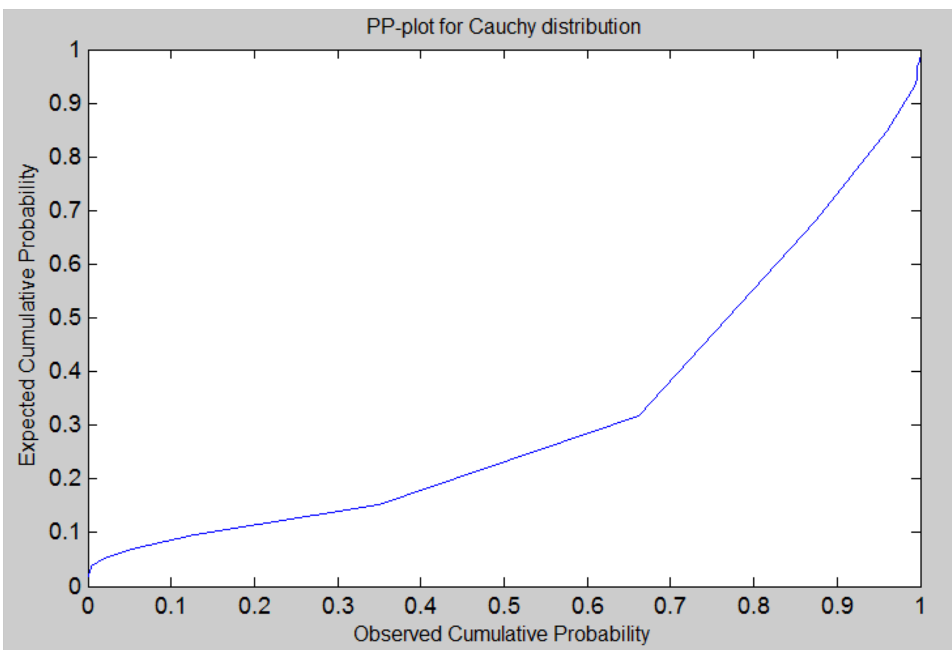
2.2 Performance of VaR model

In the analysis above we have tested the goodness of fit performance for both the Normal and Cauchy distributions. When choosing distributions for the VaR model, it is also important to evaluate the predictive quality or accuracy of the model using the distributions. If the VaR estimates are conservative, too much cash will be set aside and the portfolio profit will be very low. On the other hand, if the VaR estimates are subject to a lot of violations, there must exist serious problems in the VaR model. Under Basel II, the penalty zones (see Table 1.1) have been used to evaluate the quality of the VaR models. Hence, in the following we will do the backtesting under Basel II and then, according to the penalty zones, evaluate the model performance when either a Normal or Cauchy distribution has been applied.

Under Basel II, the VaR model used in backtesting should be based on a one-day



(a) PP-plot for Normal distribution



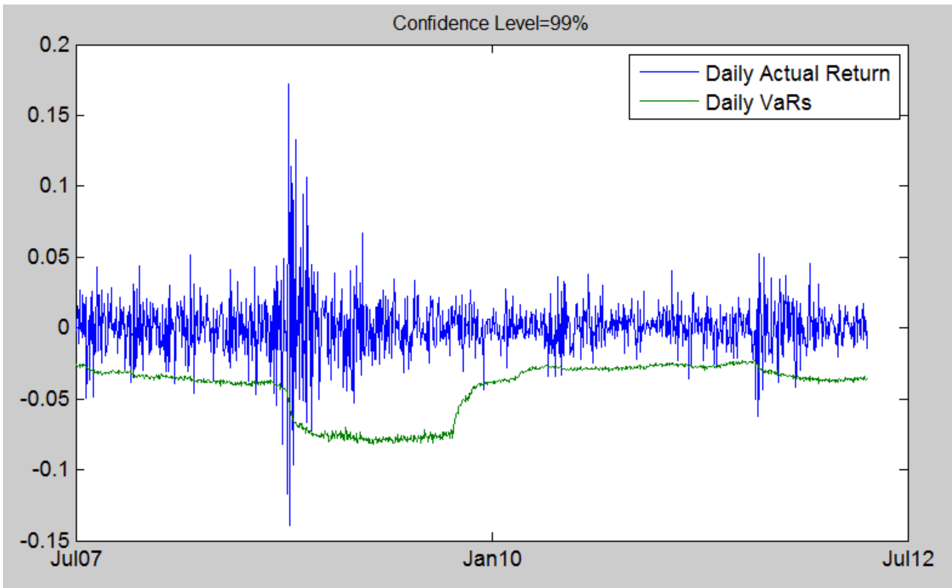
(b) PP-plot for Cauchy distribution

Figure 2.5: PP-plot

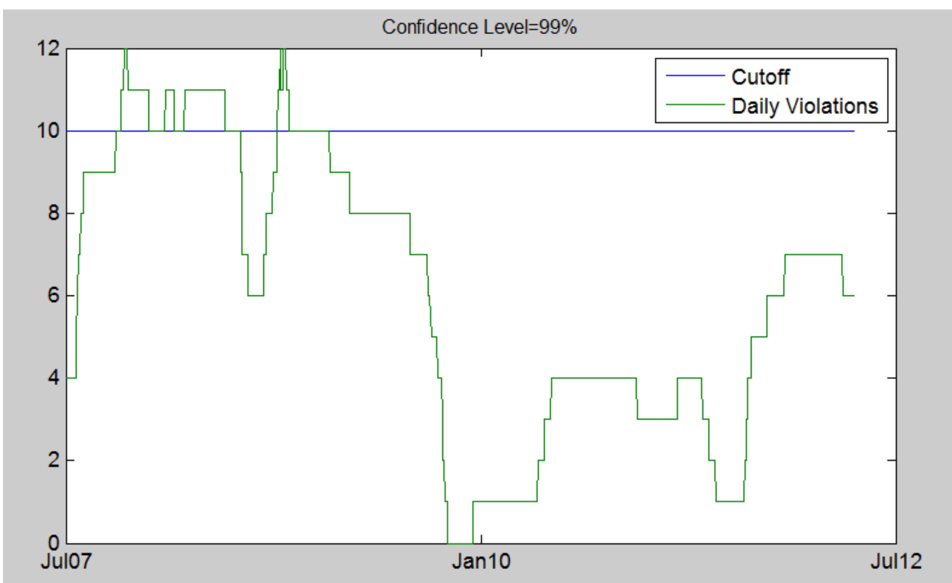
holding period. Hence, in the Monte Carlo simulation, we fit the previous 250 daily returns rather than ten-day returns used in calculating the Daily Capital Charge. In our study, simulation rounds are set to 5000. As pointed out by Fabozzi [11], simulations inevitably generate sampling variability, or variations in summary statistics due to the limited number of replications. More replications lead to more precise estimates but take longer to estimate. He points out that 1000 replications make the histogram representing the distribution of the ending price smooth and eventually should converge to the continuous distribution in the right panel. Here, 5000 simulation rounds is acceptable and time efficient.

The backtesting results for both the Normal and Cauchy distributions are displayed in Figures 2.6 and 2.7.

In theory, a good VaR model not only produces the 'correct' amount of violations but also violations that are evenly spread over time (Nieppola [21]). However, as we can see from Figure 2.6(a), the Normal VaR model shows a clustering of violations, indicating that the model does not accurately capture the changes in market volatility and correlations. In addition, Figure 2.6(b) indicates that the VaR model with Normal distribution leads to serious violations: there are too many days with daily violations greater than 9. On the other hand, the Cauchy VaR model is too conservative since the line of daily VaR estimate is much lower than the line of daily actual returns. A VaR model that is overly conservative is inaccurate and useless (Nieppola [21]). Many reasons could explain a conservative model. One of the most important ones is the selection of the confidence level. In the Cauchy distribution, we use 99% confidence level as regulated by Basel II. This confidence level may not be reasonable in conjunction with a Cauchy distribution model since the tail of that distribution is much fatter and longer, resulting in a very small VaR value at the 1% level. In order to better analyze the VaR model, we change the 99% confidence level for both the Cauchy and Normal VaR models to 95%. Figures 2.8 and 2.9 display the result. As we can see from the two graphs, the revised Cauchy VaR model performs much better. However, since all the daily violations fall into the green zone, the VaR estimates are still conservative. For the Normal VaR model, the violations become much more serious. Hence, we still need to find good substitutions for the Normal and Cauchy distributions.

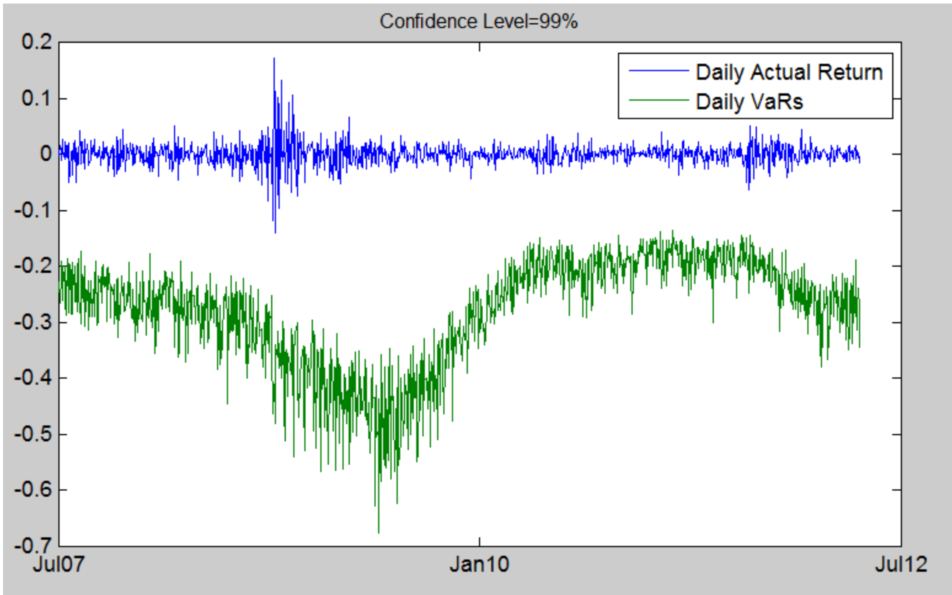


(a) VaR with Normal Distribution

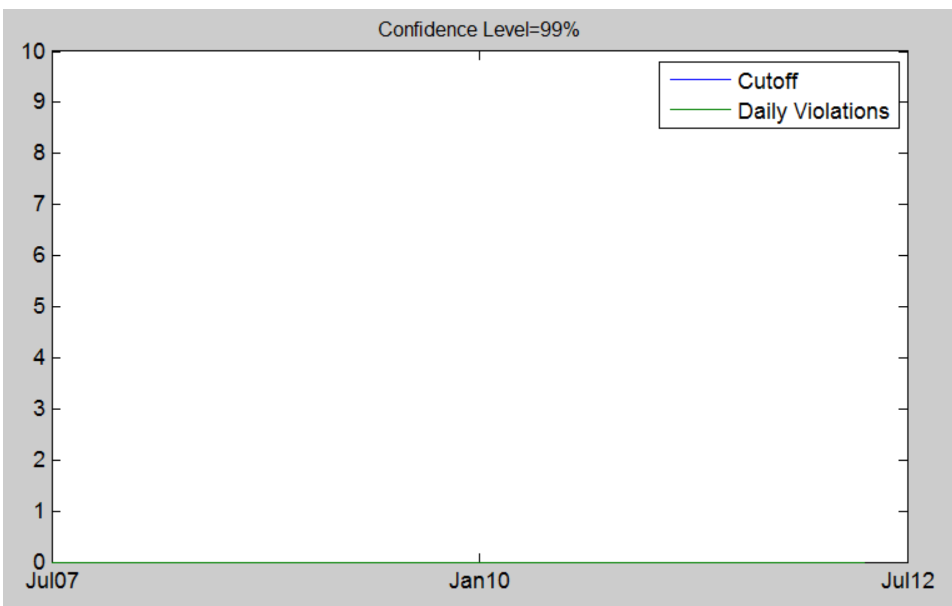


(b) Violations with Normal Distribution

Figure 2.6: Backtesting result for Normal VaR model

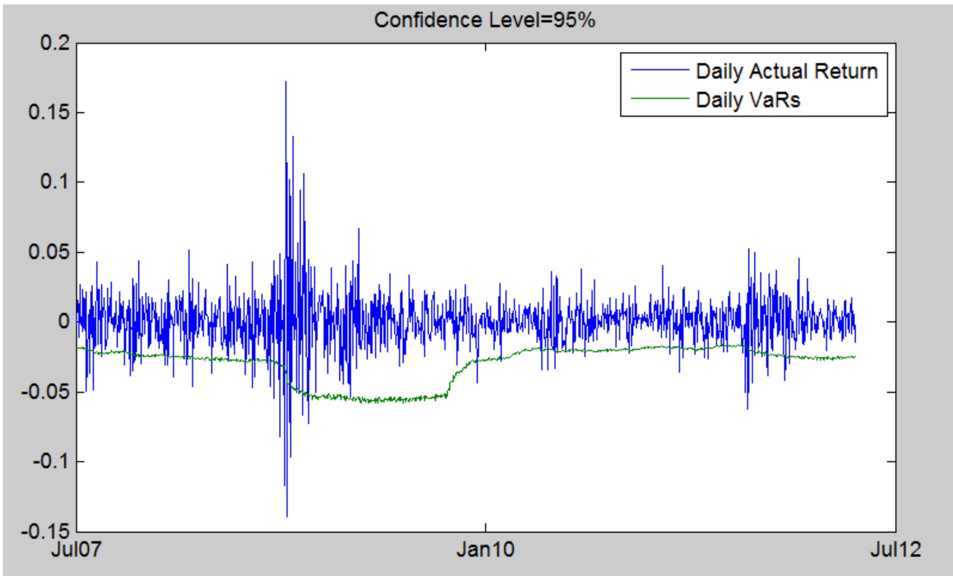


(a) VaR with Cauchy Distribution

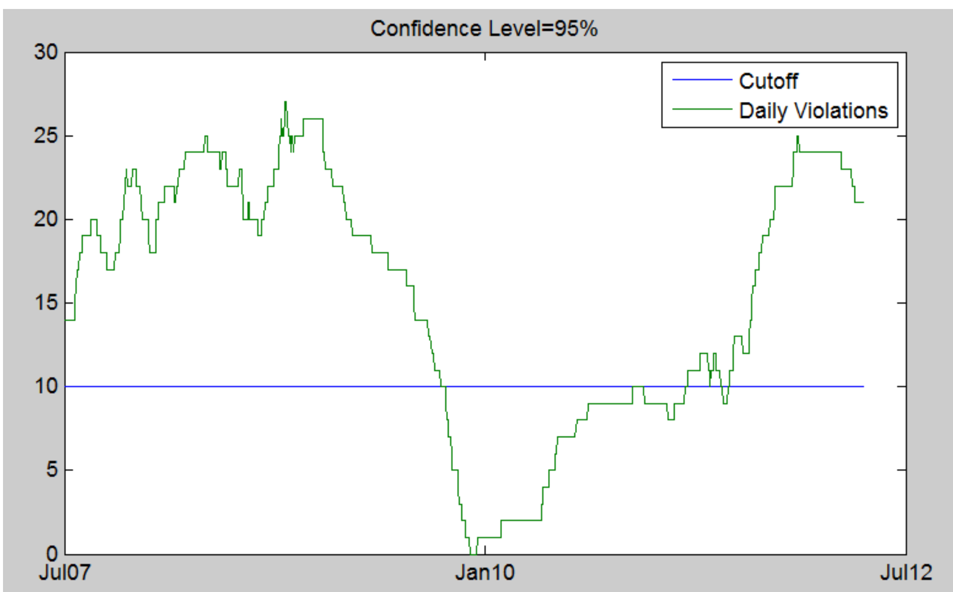


(b) Violations with Cauchy Distribution

Figure 2.7: Backtesting result for Cauchy VaR model

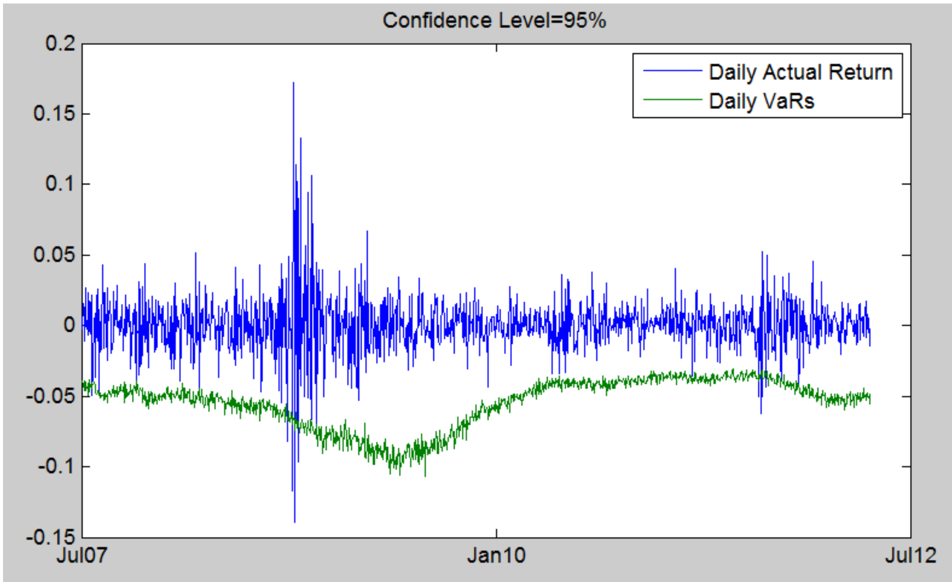


(a) VaR with Normal Distribution

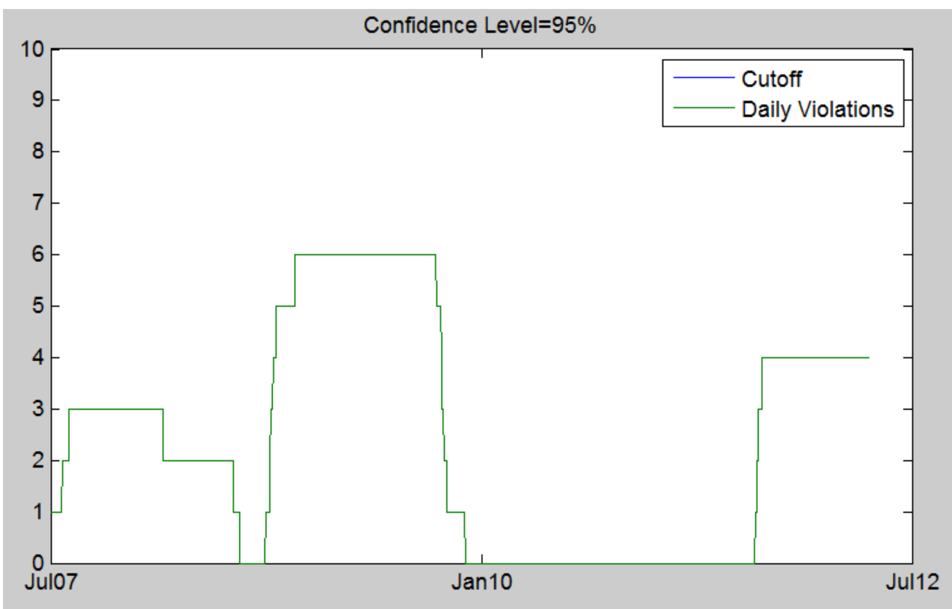


(b) Violations with Normal Distribution

Figure 2.8: Backtesting result for Normal VaR model with 95% confidence level



(a) VaR with Cauchy Distribution



(b) Violations with Cauchy Distribution

Figure 2.9: Backtesting result for Cauchy VaR model with 95% confidence level

2.3 Conclusions

Based on the analysis above, we draw the following conclusions:

- The distributions of actual historical returns present fat tails. The Normal distribution fit tends to ignore the tail while Cauchy distribution fit overstates it. As a result, both Normal and Cauchy distributions have been rejected by the goodness of fit test.

- When implementing backtesting under Basel II, the Normal VaR model suffers from a large number of violations while the Cauchy VaR model yields too conservative VaR estimates. In other words, neither of them provides good-quality VaR predictors.

- When using a 95% confidence level instead, the Cauchy VaR model performs much better. However, the VaR estimates are still conservative. We need to find some better distributions for the VaR model.

Chapter 3

Distribution Design and Implementation

The analysis in the previous chapter shows that the Normal distribution has too many violations and the Cauchy distribution is too conservative. This is mainly because the Normal (respectively, Cauchy) distribution always underestimates (respectively, overestimates) the fat tails. Hence, our idea is to create a new distribution by mixing the two distributions. We expect that the Cauchy-Normal mixture distribution will show balanced performance and, as a result, improve the quality of VaR prediction.

3.1 Model design

Before designing the distribution, we need to first analyze the historical observations. Figure 3.1 shows the scatter plot for the recent 1700 XOM stock returns. We have split the total scatter plot into several small periods of plot. For each period distribution of the returns, the shapes of the tail are different from others. We select the June 2006-December 2007 and January 2008-July 2009 periods for comparison (see Figure 3.2).

In the second half of 2008, we can see that the distribution is more 'peaked' and has much longer tails. On the other hand, in some normal (non-crisis) periods such as the year of 2007, the distributions do not contain extreme points and the shape of the plot seems more 'Normal'. Therefore, we can assume that the population of returns in each period is a mixture of Cauchy and Normal distributions while the weight for each component

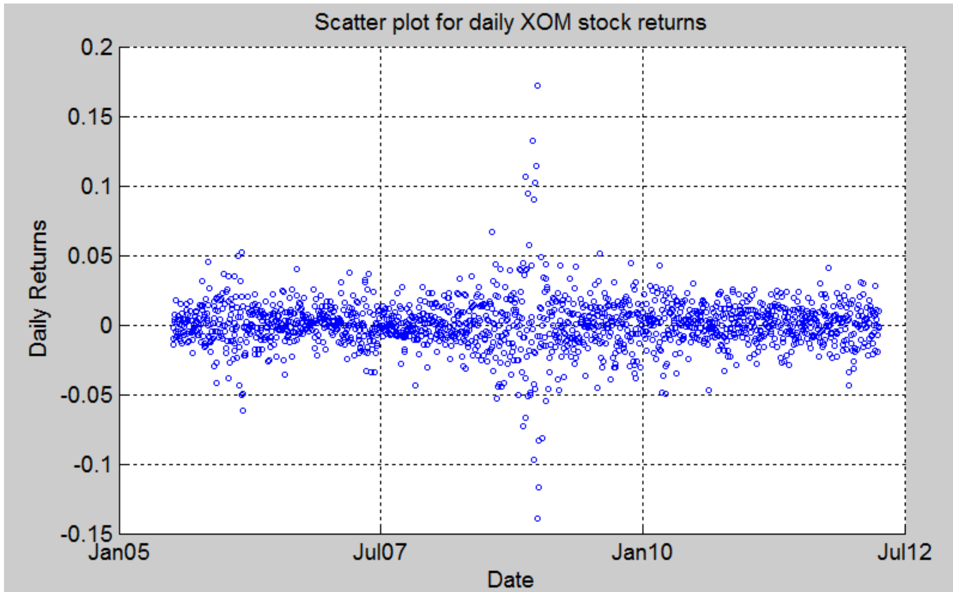


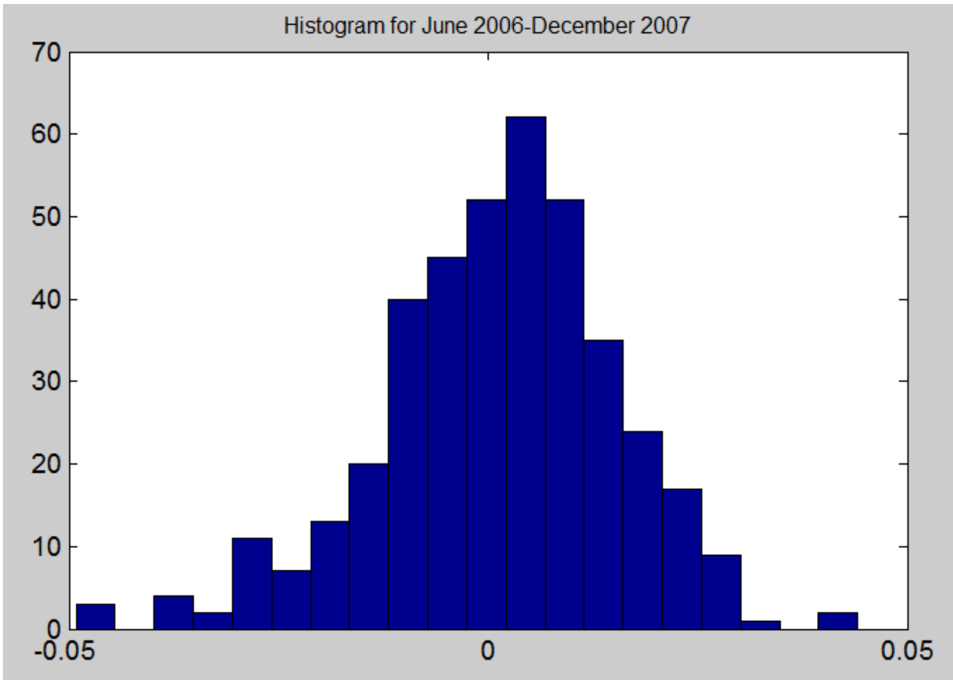
Figure 3.1: Scatter Plot for 1700 Daily XOM Stock Returns

changes with the observation period. For example, for each of the two selected periods, the population of returns consists of Cauchy and Normal sub-populations. However, the returns in Figure 3.2(a) are more likely Cauchy distributed while the returns in Figure 3.2(b) are more likely Normal distributed. In other words, we can say that, for the period considered in Figure 3.2(a), there are more returns that are from a Cauchy distribution than from a Normal distribution, and vice versa. According to this analysis, we can assign a probability or weight to each distribution to create a Cauchy-Normal mixture distribution, and then we update the weight every day to calculate daily VaR. The density function (PDF) is given by:

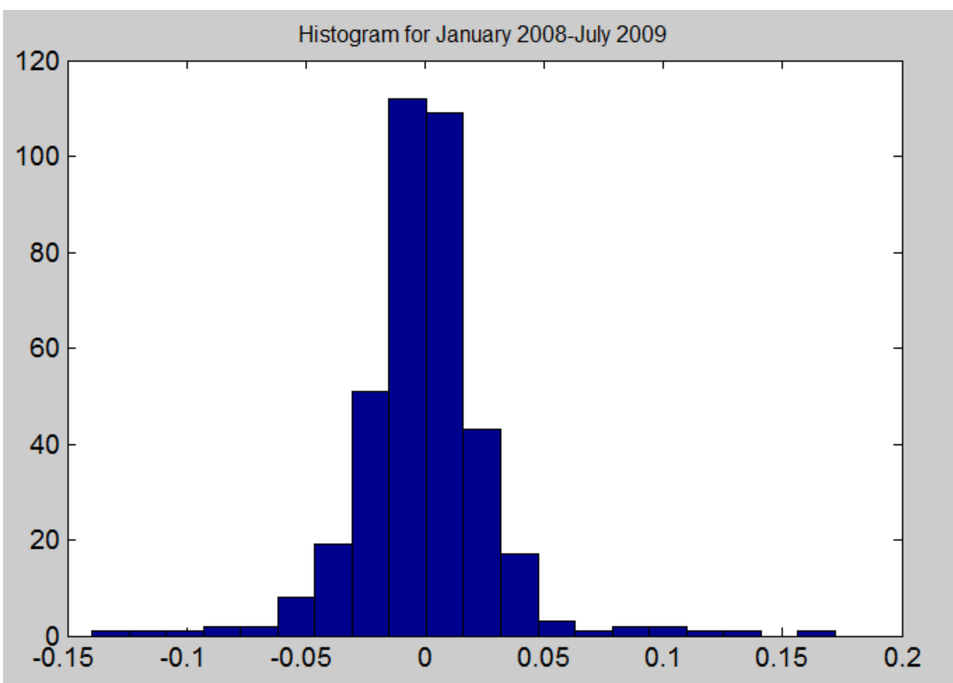
$$f_m(X;\Theta) = \alpha \cdot f_c(X;x_0,\gamma) + (1-\alpha)\cdot f_n(X;\mu,\sigma), \tag{3.1}$$

where the parameters are $\Theta = (\alpha, x_0, \gamma, \mu, \sigma)$. f_c is Cauchy density function parameterized by x_0 and γ, and f_n is Normal density function parameterized by μ and σ. Hence, we assume that we have Cauchy and Normal densities mixed together with mixing coefficient α. The log-likelihood expression for this density from the data X is given by:

$$\log(\mathrm{L}(\Theta;X)) = \log\prod_{i=1}^{N} f_m(x_i;\Theta) = \sum_{i=1}^{N}\log(\alpha\cdot f_c(x_i;x_0,\gamma) + (1-\alpha)\cdot(f_n(x_i;\mu,\sigma))). \tag{3.2}$$



(a) June 2006-December 2007



(b) January 2008-July 2009

Figure 3.2: Histogram Comparison for Two Periods

Next we fit the mixture models to the data using the maximum likelihood method (MLE). Finite mixture models with a fixed number of components are usually estimated with the expectation-maximization (EM) algorithm within a maximum likelihood framework (Dempster [10]). However, the EM algorithm is mostly used in mixture models within the same distribution family (e.g. Gaussian family). Using the EM algorithm, Swami [24] obtained the parameters for the estimation of the Cauchy-Gaussian mixture model (CGM). However, the complexity of Swami's approach is somewhat high owing to the iterative estimation for the triple parameters (α, σ, γ) (Li [16]). More tractable approximations and less computationally burdensome models still need to be developed. Furthermore, from the programming aspect, currently there is no EM algorithm package for Cauchy-Normal mixture models, and naive implementation of the EM algorithm can lead to computationally inefficient results (Cadez [4]). Therefore, we will not use the EM algorithm for the Cauchy-Normal mixture model.

On the other hand, if we set good initial parameter values and set reasonable iterations when implementing MLE using Matlab, we can see that the fitted parameters will converge, which means that the result is reliable. Hence, in the following we will use the MLE instead of EM algorithm for the distribution fit. The MLE expression is given by:

$$\Theta = \arg\max_{\Theta} \log(\mathrm{L}(\Theta; X)). \tag{3.3}$$

Therefore, after the MLE procedure, we can get the density function for the Cauchy-Normal mixture distribution. We also need to know the cumulative probability function (CDF), which is the integral of the density function:

$$\begin{aligned} F_m(x;\Theta) &= \int_{-\infty}^{x} f_m(x;\Theta) d_x \\ &= \alpha \cdot \int_{-\infty}^{x} f_c(x; x_0, \gamma) d_x + (1-\alpha) \cdot \int_{-\infty}^{x} f_n(x; \mu, \sigma) d_x \\ &= \alpha * F_c(x; x_0, \gamma) + (1-\alpha) * F_n(x; \mu, \sigma), \end{aligned} \tag{3.4}$$

We can see that the new CDF is just the mixture of the Cauchy and Normal CDF.

3.2 Goodness of Fit test

To evaluate how well the mixture model fits returns, we also use the recent 1700 historical observations as an example. In the following we will first analyze the histogram, then do the goodness of fit test, and finally use the PP-plot to verify the test result.

3.2.1 Histogram Analysis

By fitting the mixture distribution to the 1700 observations, we get the converged parameters $\Theta = (0.2574\ 0.0011\ 0.0060\ 0.0003\ 0.0146)$. Hence, the density function is:

$$f_m(X;\Theta) = 0.2574 \cdot f_c(X;0.0011, 0.006) + 0.7426 \cdot f_n(X;0.0003, 0.0146). \qquad (3.5)$$

Using the density function, we plot the historical fit (Figure 3.3). We can see that the center of the fitted plot nearly has the same peak as the histogram, and the tails of the historical observations have been covered well. The tails of the distribution fit are just as fat as that of the historical distribution. The fitting performance can also be seen from the histogram counts (Table 3.1).

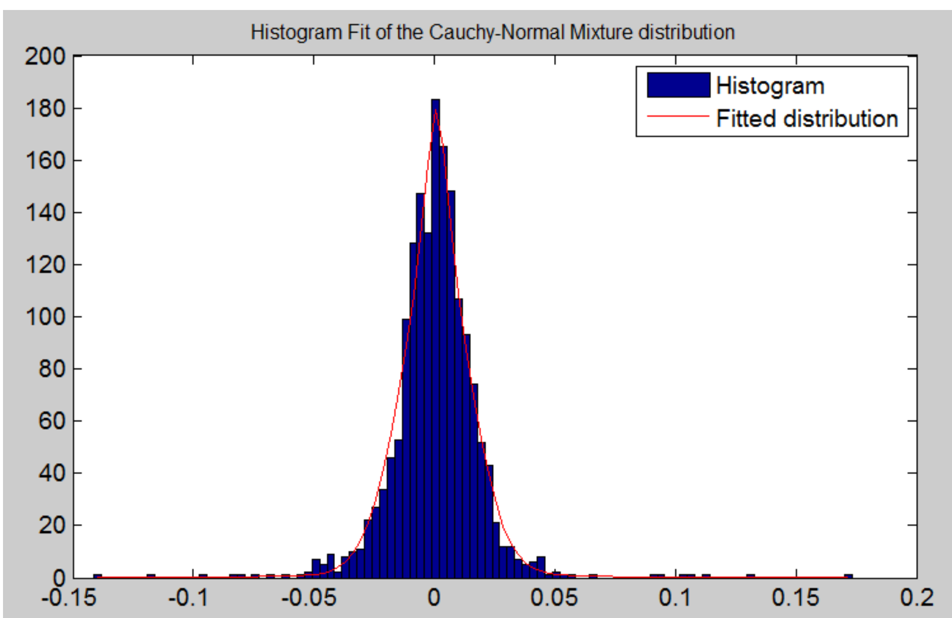


Figure 3.3: Histogram Fit for 1700 Daily XOM Stock Returns

Daily Returns Bins	Actual	Normal	Cauchy	Mixture
-0.139525255	1	0	7	1
-0.127068042	1	0	3	2
-0.11461083	0	0	5	0
-0.102153617	1	0	5	2
-0.089696404	2	0	6	1
-0.077239191	2	1	12	2
-0.064781978	3	1	20	1
-0.052324765	22	17	29	10
-0.039867552	42	85	50	36
-0.027410339	143	244	114	190
-0.014953126	469	419	363	444
-0.002495913	624	426	675	648
0.0099613	283	333	178	254
0.022418512	72	133	78	74
0.034875725	22	37	42	17
0.047332938	5	4	22	5
0.059790151	1	0	12	2
0.072247364	0	0	7	2
0.084704577	2	0	6	0
0.09716179	2	0	8	1
0.109619003	1	0	4	0
0.122076216	1	0	4	0
0.134533429	0	0	1	0
0.146990642	0	0	1	0
0.159447854	0	0	2	0
0.171905067	1	0	0	0
	1700	1700	1700	1700

Table 3.1: Histogram counts for Actual, Normal, Cauchy, and Mixture distributions

Compared with Normal and Cauchy distribution, the mixture model reflects much more accurately the tails of historical distribution: there is only two returns missed in the left tail and only four missed in the right tail. Hence, the mixture model indeed has improved the quality of fit.

3.2.2 Goodness of fit test

In Chapter Two, we have used both the Chi-squared and KS tests to check the goodness of fit. However, due to the complexity of the mixture CDF, it is hard to mathematically obtain the expression of the mixture quantile function, which is required by the Chi-squared test to set the equal-frequency bins. Hence, hereby we only use the KS two-sample test. For the 1700 XOM historical returns, the test result is shown in Table 3.2:

	Statistic	P-value
KS two sample test	0.0329	0.3100

Table 3.2: Goodness of fit test for Mixture distribution

We can see that the p value is much larger than 0.05, which means that we should accept the Null hypothesis that the historical observations are from the mixture distribution. The fitting performance is confirmed by the PP-plot due to the straight line:

Under Basel II, every day banks should use the previous 250 historical observations to recalculate VaR. Hence, testing the goodness of fit for each daily mixture distribution fit is necessary. Since there exists two types of errors (see Chapter One) when we test the quality of models, we cannot expect that all the daily fitted distributions will pass the test. Naturally, it is rarely the case that we observe the exact amount of exceptions suggested by the significance level. Each daily testing result either produces a rejection or not. This sequence of 'successes and failures' is known as Bernoulli trial (Jorion [13]). The number of rejections x follows a binomial probability distribution:

$$f(x) = \binom{T}{x} p^x (1-p)^{T-x}. \tag{3.6}$$

As the number of tests increases, the binomial distribution can be approximated by a Normal distribution:

$$z = \frac{x - pT}{\sqrt{p(1-p)T}} \approx N(0,1), \tag{3.7}$$

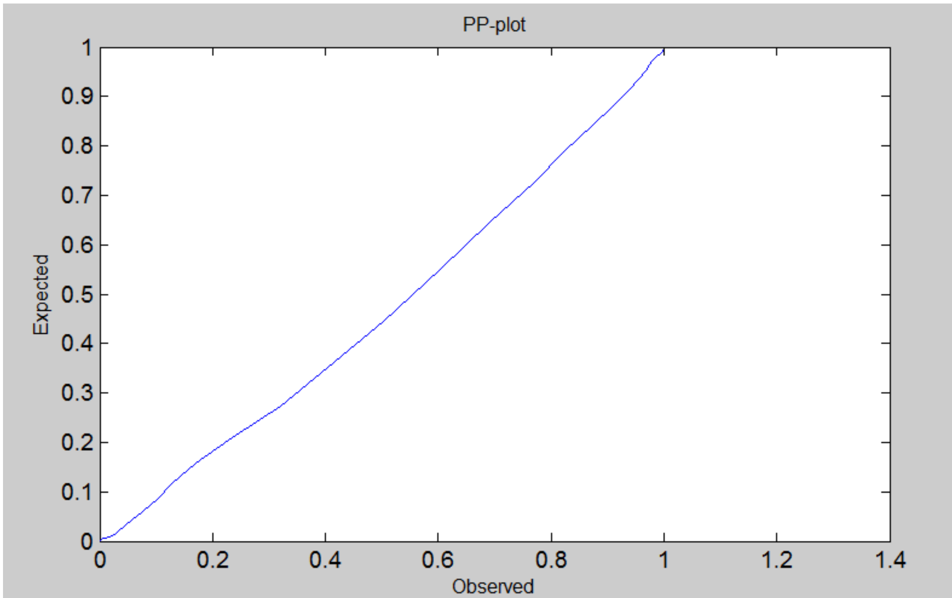


Figure 3.4: PP-plot for Mixture distribution

where pT is the expected number of rejections and $p(1-p)T$ the variance of rejections (Jorion [13]). Hence, since there is totally 1700-250=1450 daily distribution fits and since the significance level we used is 0.05, the expected number of rejections in our example would be 1450*0.05=72.5, and the variance of exceptions is 0.05*0.95*1450=68.875. Therefore, in light of the theory of Bernoulli trials we can evaluate the quality of the mixture model. Figure 3.5 shows the result for KS two-sample tests:

We can see that, for the KS two-sample test, there are approximately only five rejection days, which is much less than the expected number. Hence, we can make a conclusion that, under Basel II, the Cauchy-Normal mixture model fits the historical returns very well.

3.3 Backtesting Performance under Basel II

The fact that a distribution fits well the available data doesn't mean that it must have a good performance regarding VaR prediction. This makes sense since our model is only based on historical data and then we use the fitted distribution to predict the future. Therefore, it is important to evaluate the predictive quality of the selected distribution by

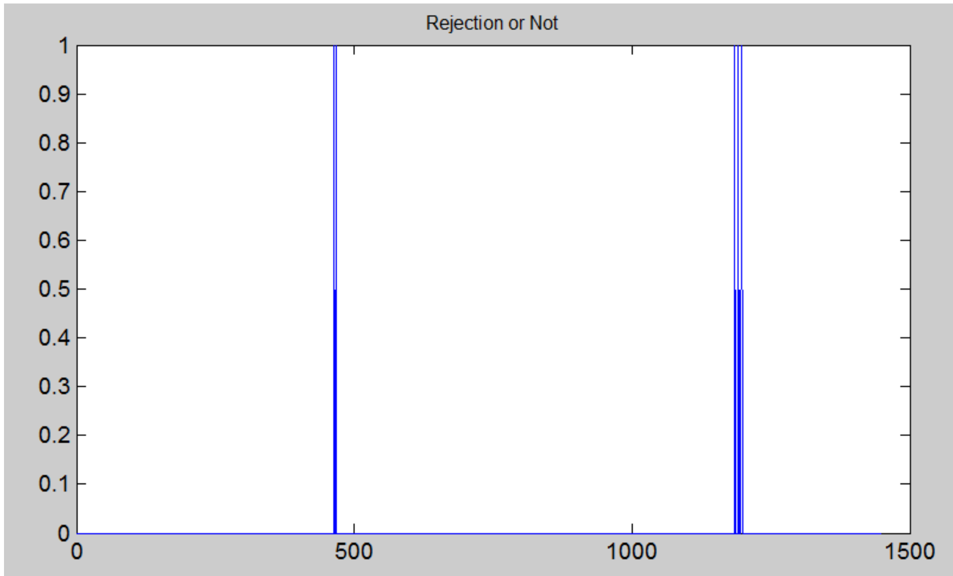


Figure 3.5: Reject or not (1=reject)

doing backtesting.

Before doing backtesting for the mixture model, there is an issue of sample size selection that we need to discuss. When we implement the MLE method in Matlab, sometimes there are 'warnings' as to the accuracy of the result. If we increase the sample size for estimation, the 'warnings' will disappear. Naturally, the MLE method requires a large sample size to ensure the accuracy of estimates, which means that 250 previous observations may not be enough for MLE estimation. However, for our mixture model, if the sample size is too large, we will not capture the change of the stock returns in time. Hence, it should be a tradeoff between the accuracy of the estimates and the reflection of market risks. In addition, we can make use of the goodness of fit test to check the quality of our estimate. If the distribution fit can pass the goodness of fit all the time, we can conclude that the distribution fit by using MLE is reliable. As shown in Section 3.2.2, the mixture model using 250 previous returns has good fit properties. Hence, we can still use the previous 250 observations for the mixture distribution fit and VaR calculations.

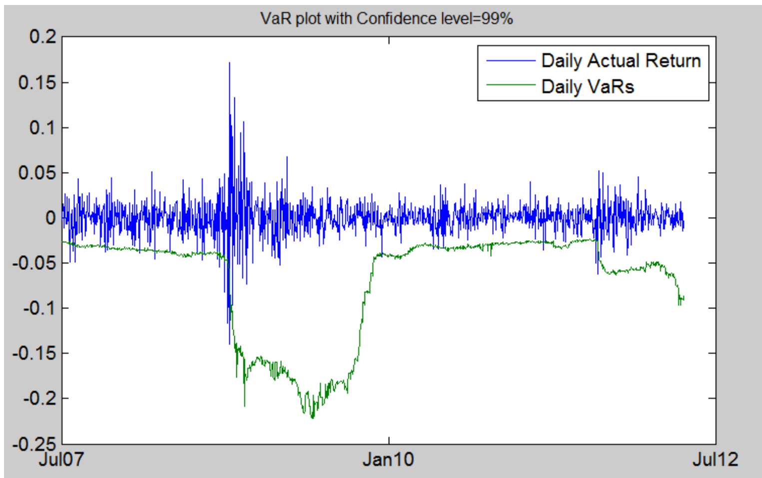
Another issue is about sample generation. When using Monte-Carlo simulation to calculate VaR, we need to first generate returns from our mixture distribution. In general, returns are generated by using the quantile function, which is just the inverse function of the CDF. However, as mentioned before, it is extremely hard to transform the mixture CDF into the quantile function. Forturenately, due to the nature of our mixture distribution, hereby we can use a more tractable method to generate random returns.

As we have explained for the mixture distribution, it can be considered that some of the historical returns originate from a Normal distribution and others from the Cauchy distribution, and the amount for each component is decided by the weight or probability parameter α. Hence, we can design a Bernoulli process. For example, if α is equal to 0.3, the weight for Cauchy will be 0.3 and for Normal 0.7. Next, we generate a random number from the Uniform(0,1) generator. If the number is less than or equal to 0.3, we generate a return from the Cauchy distribution; otherwise we generate it from the Normal distribution. Therefore, we can repeat the Bernoulli trial 5000 times to create 5000 sample returns, a sample size that is enough for calculating VaR. Because of the nature of our mixture distribution, the generating method can be a good substitution for the quantile function.

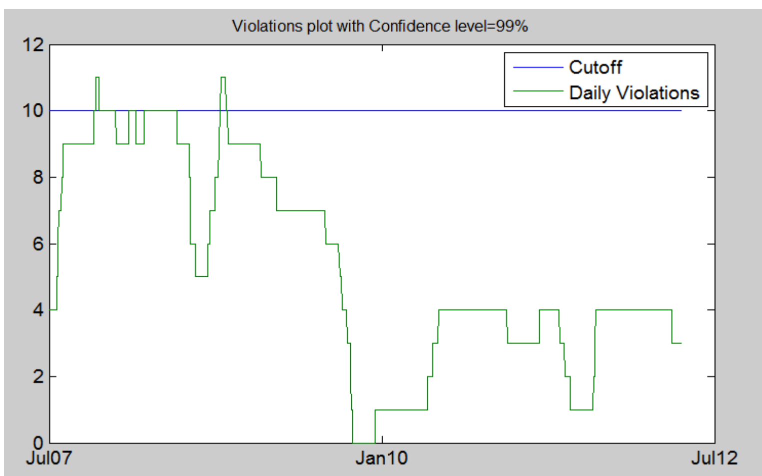
After generating the samples, we can calculate VaRs and do backtesting. As for the goodness of fit test, we also should not expect that all the VaRs are predicted well and there are no violations. Table 3.3 shows the violations distribution under the 99% VaR coverage level.

The table provides the exact probabilities of obtaining a certain number of violations from a sample of 250 independent observations assuming that the level of coverage is truly 99%. According to the table above combined with the two types of errors, Basel II creates the penalty zones (see Table 1.1). As a good VaR model, the predictions should neither be too conservative or suffer too many violations, and hence it should fall into the yellow zone rather than the green or red zones. During a crisis period, even a good VaR model may suffer serious violations. Hence, we should analyze violations in crisis periods separately. Figure 3.6(a) and Figure 3.6(b), respectively, show the daily VaR plot and the daily violation plot for the recent 1700-500=1200 XOM historical returns.

As we can see from Figure 3.6(b), before the crisis period (approximately the second half of 2008), most of the daily violations numbers are between 5 and 9, which is in the range of the yellow zone. In the period of crisis, there are several days for which the violations number is greater than or equal to 10, meaning a serious violation. However, compared with the Normal VaR model, the Cauchy-Normal mixture VaR model does not suffer from a big cluster of violations, and the serious violations only happen for several



(a) Daily VaRs with Mixture Distribution



(b) Daily Violations with Mixture Distribution

Figure 3.6: Backtesting Result for Cauchy-Normal Mixture distribution

Violations(out of 250)	exact	type 1
0	8.1%	100.0%
1	20.1%	91.9%
2	25.7%	71.4%
3	21.5%	45.7%
4	13.4%	24.2%
5	6.7%	10.8%
6	2.7%	4.1%
7	1.0%	1.4%
8	0.3%	0.40%
9	0.1%	0.1%
10	0.0%	0.0%
11	0.0%	0.0%
12	0.0%	0.0%
13	0.0%	0.0%
14	0.0%	0.0%
15	0.0%	0.0%

Table 3.3: Bernoulli Trial for 99% confidence level (Basel II [7])

days. Hence, it also exhibits better performance during the crisis period. After the crisis, the daily violations number again gradually falls into the yellow zone. In summary, the Cauchy-Normal mixture VaR model has a good prediction quality. The good performance should be due to the flexibility of the mixture distributions. When the number of extreme observations increases and the crisis occurs, the weight of the Cauchy distribution will be heavier and hence the VaR estimate will move downward quickly with the serious decrease in daily returns. As a result, the serious violations will be avoided. The movement of the weight α is shown in Figure 3.7.

We can see that the weight of Cauchy distribution is correspondingly updated with the change of extreme returns. In practice, we also tried the 99.5% VaR coverage level for the mixture model and found that it also has a better prediction quality. Figure 3.8 shows the result.

In conclusion, the Cauchy-Normal mixture distribution can greatly improve the quality of VaR prediction and can avoid too many serious violations during the crisis.

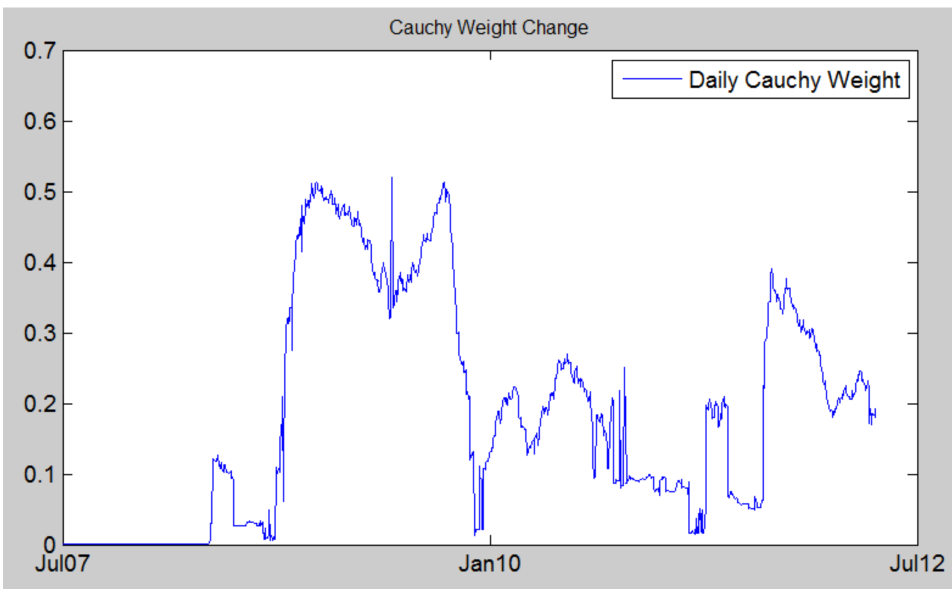
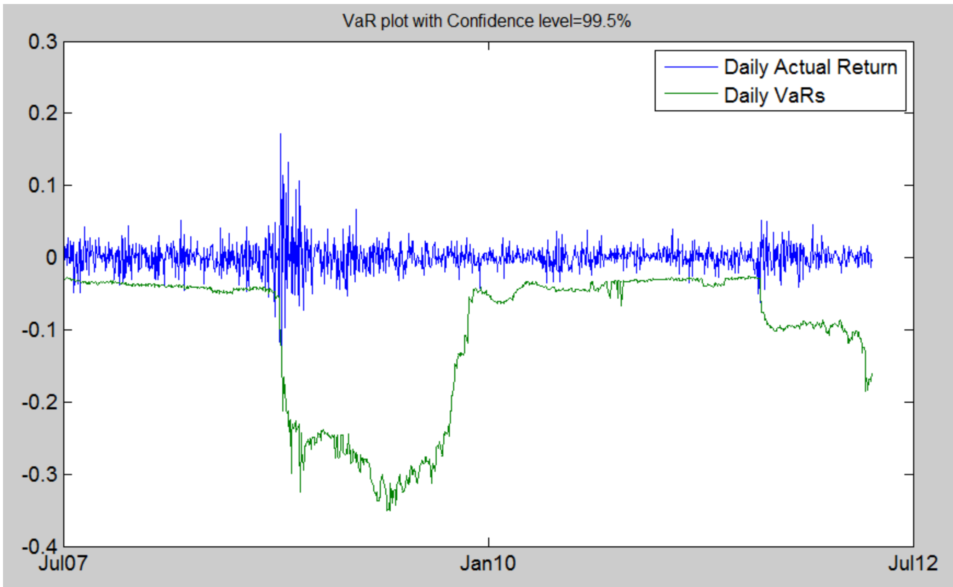
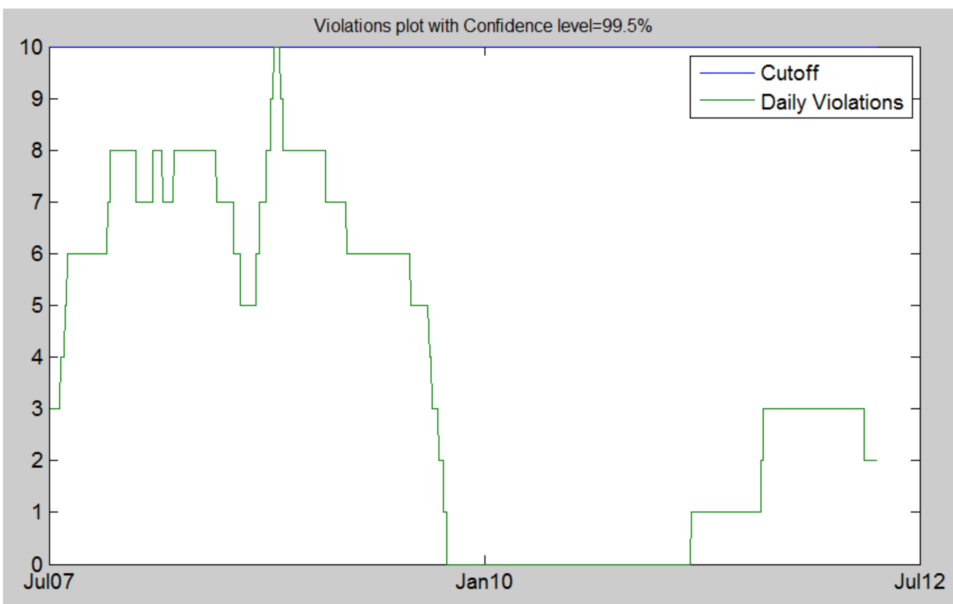


Figure 3.7: α Movement



(a) Daily VaRs with Mixture Distribution



(b) Daily Violations with Mixture Distribution

Figure 3.8: Backtesting Result with Confidence Level of 99.5%

Chapter 4

Conclusions

The analysis presented in this thesis allows us to draw the following conclusions.

1. The Cauchy and Normal distributions do not fit the fat tails of stock returns very well. The Cauchy distribution tends to overestimate the tails while the Normal distribution always underestimates it.

2. The backtesting results for the Cauchy and Normal distributions suggest that they both yield VaR predictions of poor quality. At the 99% confidence level, the Cauchy VaR model is too conservative while the Normal VaR model leads to too many violations.

3. The Cauchy-Normal mixture model improves the goodness of fit performance; it fits the tails of returns very well and also passes the goodness of fit test.

4. The prediction quality of the mixture model for VaR is good since (i) in normal market periods, most of the daily violations numbers fall into the yellow zone as defined by Basel II and (ii) the mixture VaR model can avoid the clustering of serious violations.

5. The mixture model exhibits great flexibility; in particular, the weight α for each distribution is updated according to changes in the market conditions.

To further improve the model, we recommend the following as directions for future work:

- Although the weight parameter α is updated every day, the movement of the VaR estimate is backward-looking in nature due to the estimation process using historical data, so the movement of VaR always lags behind the movement of actual returns. Hence, it would be interesting to develop an algorithm to update the weight parameter that also incorporates forward-looking analysis and the manager's beliefs regarding future movements.

- It would also be interesting to use Conditional Value-at-Risk (CVaR) instead of VaR, that is, the expected value of the losses conditional on being in a worst-case situation on a given time horizon, to better capture the scope of adverse events.

- A limitation of our current model is that it only builds upon a mixture of two components: one Gaussian distribution and one Cauchy distribution. Incorporating a larger number of Cauchy components might help generate a better fit.

- Because the shape of the mixture density function is nonlinear, MLE may return a local optimum rather than a global one. Therefore, the quality of the model may still be improved.

In summary, the contribution of this thesis is that we propose a new distribution that exhibits good fitness and good VaR prediction quality for stock return data. While additional improvements are possible, this represents a significant improvement on the pure Cauchy and Normal distribution models that are currently used by financial institutions. Our approach is easy to implement and captures the trade-off between over- and under-conservatism in financial risk management.

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Appendix A

Matlab code

A.1 PDF and CDF functions

```
%Density function
function Density=MixtureFunction(x,a,mu,sigma,location,scale)
Density=a*cauchypdf(x,location,scale)+(1-a)*normpdf(x,mu,sigma);
end

%Cumulative Probability function
function CDF=MixCDF(x,w,mubar,sigmabar,locationbar,scalebar)
CDF=w*cauchycdf(x,locationbar,scalebar)+(1-w)*normcdf(x,mubar,scalebar);
end
```

A.2 Histogram Fit and Goodness of Fit test

```
mydata= xlsread('XOM.xlsx','XOM','J1701:J2');
%1700 XOM stock returns beginning from July 8, 2005 to April 5,2012.

[mlepars, res]= cauchyfit(mydata);
Nlocation=mlepars(1);
Nscale=mlepars(2);
[muhat,sigmahat] = normfit(mydata);
```

```
Nmu=muhat;
Nsigma=sigmahat;
Default=[0.5 Nmu Nsigma Nlocation Nscale];
%Initial Value for mixture PDF

lb = [0 -Inf -Inf -Inf 0]; % lower constraint
ub = [1 Inf Inf Inf Inf]; % upper constraint

x=mydata;
mixpdf = @(x,a,mu,sigma,location,scale) MixtureFunction(x,a,mu,
sigma,location,scale) ;
options = statset('MaxIter',50000, 'MaxFunEvals',10000);
[phat,pci] = mle(x, 'pdf', mixpdf, 'start',Default, 'lowerbound',
lb,'upperbound',ub,'options',options) %MLE method


w=phat(1);
mubar=phat(2);
sigmabar=phat(3);
locationbar=phat(4);
scalebar=phat(5);

randn('state',3) %set the seeds (state) to have
rand ('state',3) %the constancy of result
G=[]; n=5000;
for i=1:n  %Bernoulli trial for generating returns
ra=rand(1,1);
if ra < phat(1)
add = cauchyrnd(phat(4),phat(5),1);
else
add = normrnd(phat(2),phat(3),1);
end
G = [G add];
```

```
end
[h1,p1,ks2stat]=kstest2(mydata,G)  %ks two sample test

%setting bins
MAX = max(mydata);
MIN = min(mydata);
interval=100;
STEP = (MAX - MIN) / interval;
area=STEP*numel(mydata);
bin=MIN:STEP:MAX;


PDF = area*MixtureFunction(bin,w,mubar,sigmabar,locationbar,scalebar);
figure(1) %Histogram Fit
hist(mydata,MIN:STEP:MAX)
hold on
plot(MIN:STEP:MAX, PDF,'r')
legend('Histogram','Fitted distribution')
title('Histogram Fit of the Cauchy-Normal Mixture distribution')
set(gca,'FontSize',12)
hold off


figure(2) %PP-plot for the mixture distribution
Counts=histc(mydata,bin);
ObservedDensity=Counts/1700;
ObservedProb=zeros(length(bin),1);
ExpectedProb=zeros(length(bin),1);
ExpectedDensity=PDF/1700;
for j=1:length(bin)
    ObservedProb(j)=sum(ObservedDensity(1:j));
    ExpectedProb(j)=sum(ExpectedDensity(1:j));
end
```

```
plot(ObservedProb,ExpectedProb);
xlabel('Observed')
ylabel('Expected')
title('PP-plot')
set(gca,'FontSize',12)


%Comparison of histogram counts between Actual, Normal, Cauchy, and Mixture
MAX = max(mydata);
MIN = min(mydata);
interval=25;
STEP = (MAX - MIN) / interval;
area=STEP*numel(mydata);
bin=MIN:STEP:MAX;

ObservCount=histc(x,bin);
ExpectCount=histc(G,bin);
[mlepars, res]=cauchyfit(mydata);
a=mlepars(1);
b=mlepars(2);
CR=cauchyrnd(a,b,45);
TransC=reshape(CR,[],1);
CauchyGenerator=TransC(1:1700);
CauchyCount=histc(CauchyGenerator,bin);

[muhat, sigmahat]=normfit(mydata);
a=muhat;
b=sigmahat;
NR=normrnd(a,b,45);
TransN=reshape(NR,[],1);
NormGenerator=TransN(1:1700);
NormalCount=histc(NormGenerator,bin);
```

```
Compare=zeros(interval+1,5);
Compare(1:interval+1,1)=bin;
Compare(1:interval+1,2)=ObservCount;
Compare(1:interval+1,3)=NormalCount;
Compare(1:interval+1,4)=CauchyCount;
Compare(1:interval+1,5)=ExpectCount;
Compare;

xlswrite('XOM.xlsx',Compare, 'Sheet1','K3:O28'); %output to the Excel file
```

A.3 Backtesting

```
[num,txt,raw]=xlsread('XOM.xlsx','XOM','A1201:A2');
%the data of trading days beginning from July 5,2007 to April 5, 2012.
mydata= xlsread('XOM.xlsx','XOM','J1701:J2');
%XOM returns beginning from July 8,2005 to April 5, 2012.
mydata1=xlsread('XOM.xlsx','XOM','J1451:J2');
%XOM returns beginning from August 17,2005 to April 5, 2012.

OneDayR=flipud(mydata1);
DailyR=flipud(mydata);
n=length(OneDayR);
m=n-250;
w=0.3; %weight of Cauchy distribution in the mixture model
VaR=zeros(n,1);
W=zeros(n,1);
H=zeros(n,1);

for j=1:n
```

```
[mlepars, res]= cauchyfit(DailyR(j:j+249));
Nlocation=mlepars(1);
Nscale=mlepars(2);
[muhat,sigmahat] = normfit(DailyR(j:j+249));
Nmu=muhat;
Nsigma=sigmahat;

Default=[w Nmu Nsigma Nlocation Nscale]; %Initial Value for mixture PDF




lb = [0 -Inf -Inf -Inf 0]; % lower constraint
ub = [1 Inf Inf Inf Inf]; % upper constraint

x=DailyR(j:249+j);
mixpdf = @(x,a,mu,sigma,location,scale) MixtureFunction(x,a,mu,sigma,location,scale) ;

options = statset('MaxIter',50000, 'MaxFunEvals',1000);
[phat,pci] = mle(x, 'pdf', mixpdf, 'start',Default,
'lowerbound',lb,'upperbound',ub,'options',options) ;%MLE

w=phat(1);
W(j)=phat(1);

rand ('state',3); %the constancy of result
G=[]; g=5000;
for i=1:g  %generate sample returns
ra=rand(1,1);
if ra < phat(1)
add = cauchyrnd(phat(4),phat(5),1);
else
add = normrnd(phat(2),phat(3),1);
end
```

```
G = [G add];
end
[h,p,ks2stat]=kstest2(DailyR(j:j+249),G); %KS two sample test
VaR(j)=prctile(G,0.5); %calculating VaR
H(j)=h;
end
VaR;
RealR=OneDayR(1:n);

Violations=zeros(m,1);
for p=1:m  %calculating violations
Compare=VaR(p:249+p)-RealR(p:249+p);
Violations(p)=sum(Compare>0);
end
%NumOfSerious=sum(Violations>9);
Violations;

Date=flipud(txt);
x=datenum(Date);

figure(1) %VaR plot
plot(x,RealR(251:n),x,VaR(251:n))
datetick('x','mmmyy')
title('VaR plot with Confidence level=99%')
legend('Daily Actual Return','Daily VaRs')
set(gca,'FontSize',12)

figure(2) % Violations plot
plot(x,10*ones(m,1),x,Violations)
datetick('x','mmmyy')
title('Violations plot with Confidence level=99%')
legend('Cutoff','Daily Violations')
set(gca,'FontSize',12)
```

```
figure(3) %the movement of the weight of Cauchy
plot(x,W(251:n))
datetick('x','mmmyy')
title('Cauchy Weight Change')
legend('Daily Cauchy Weight')
set(gca,'FontSize',12)

figure(4) %plot of daily goodness of fit rejections of KS two sample test.
'0' means accepted; '1', otherwise.
plot(H)
title('Rejection or Not')
set(gca,'FontSize',12)
```

Appendix B

VITA

Jingbo Li was born in 1986. He graduated in 2006 from Tangshan No. 1 High School in Tangshan, China. He received a Bachelor of Economics in Financial Risk Management in May 2010 from the Central University of Finance & Economics(CUFE) in Beijing, China. He was awarded a scholarship from CUFE for the 2006-2007 and 2007-2008 academic years. In 2007-2008, he was the president of Student Union in School of Finance, CUFE. Currently he plans to graduate in May 2012 from Lehigh University in Bethlehem, PA with a Master of Science in Management Science and Engineering.