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The impact of transaction costs on active portfolio management, index tracking and mixed strategies

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by

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To my parents

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by

Chach Wanapat

Submitted to the Department of Industrial and Systems Engineering
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Master of Science

Abstract

In this thesis, we investigate the impact of transaction costs on portfolio management, with a focus on active management, index-tracking and mixed strategies where we alternate between active and passive portfolio optimization. We measure the quality of tracking using the concepts of Mean Absolute Deviation (MAD) and Mean Absolute Downside Deviation (MADD) and compare our results with active portfolio management. We first consider a simple static model with one stock and one bond, then extend the analysis to multi-period problems. Finally, we present extensive numerical results illustrating the promise of the approach.

Chapter 1

Introduction

1.1 Background

The field of modern financial theory was pioneered in 1959 by Markowitz [14], who proposed the mean-variance model to balance expected return with portfolio risk measured by its standard deviation. Nowadays, portfolio management strategies can be broadly classified in two categories:

1. In *active portfolio management*, the investor seeks to outperform the market, i.e., the benchmark index, by actively picking stocks based on his own estimates of future stock returns.

2. In *passive portfolio management*, the investor seeks to obtain the same returns as the benchmark index.

In [2], Beasley describes the strengths and weaknesses of both strategies as follows:

- Active management entails high fixed costs (associated with payments to the management team) and high transaction costs (due to the frequent trading involved in stock picking). The hope is that these costs will be offset by the returns obtained.

- Passive management has lower fixed costs and lower transaction costs, but presents the disadvantage that if the stock market falters, so inevitably will the return of the index fund, while a few individual stocks might still perform strongly.

As Clarke et. al. point out [6], the naive approach consisting in replicating the index by purchasing stocks in the exact same proportions, called full replication, is impractical due to the number of stocks involved and the need to repeatedly re-balance the portfolio to maintain the correct proportions, which would lead to enormous transaction costs. Hence, the investor choosing a passive management approach will attempt to track the performance of the index as well as possible with a much smaller number of stocks.

Several measures of tracking error between the returns on the benchmark and index-tracking portfolio have been investigated. For example, Clarke et. al. [6] consider the absolute deviation tracking error. Consiglio and Zenios [7] and Worzel, Vassiadou-Zeniou and Zenios [18] study the tracking of fixed-income securities problem. Fang and Wang [9] analyze a fuzzy model using a mean absolute downside deviation tracking error. Konno and Yamazaki [11] also use the mean absolute deviation tracking error, while Roll [16] focuses on the sum of the squared deviation tracking error and Wolter and Zimmerman [17] use linear deviation tracking error, to name just a few.

1.2 Thesis overview and contributions

The purpose of this thesis is to investigate a mixed strategy where the investor attempts to gain "the best of both worlds" by alternating between active and passive management, depending on which one is performing better. The presence of transaction costs introduces friction in the management model, i.e., the investor stays with a strategy longer than he would like because the gains from switching must at least cover the cost of the transaction. In this work, we seek to evaluate the potential of such an upside tracking approach.

The thesis is structured as follows. In Chapter 2, we describe the basic models of both active and passive management, derive closed-form solutions and derive a lower bound on the objective function when the distribution of the stock return is not known precisely. In Chapter 3 we first review ways to simulate the underlying stock returns through binomial lattice and scenario tree models, before extending the models of Chapter 2 to multi-period settings. Chapter 4 investigates the index-tracking model using the mean absolute deviation (MAD) and the mean absolute downside deviation (MADD) measures of tracking error. Chapter 5 summarizes our results and discusses future work.

Chapter 2

The Basic Models

In this chapter we introduce two basic models of portfolio optimization, which correspond to active and passive management strategies.

2.1 Active portfolio management

In this section, we present our results for active portfolio management over two time periods when there are one underlying risky asset and one riskless asset. Transaction costs were first studied by Davis and Norman [8], who proved optimality conditions on selling and buying times in a continuous-time infinite-horizon problem when the risky asset obeys a lognormal distribution. The purpose of our analysis is to give more insights in the structure of the optimal strategy. We will use the following notations:

Notations:

- W_0 *the wealth at the beginning of time period 0.*
- W_0^α *the wealth after a transaction cost incurred during time period 0.*
- W_1 *the wealth at the end of time period 1.*
- X_0 *the number of shares of the underlying stock in the portfolio during period 0.*
- X_1 *(decision variable) the number of shares in the portfolio during period 1, determined at the end of period 0.*

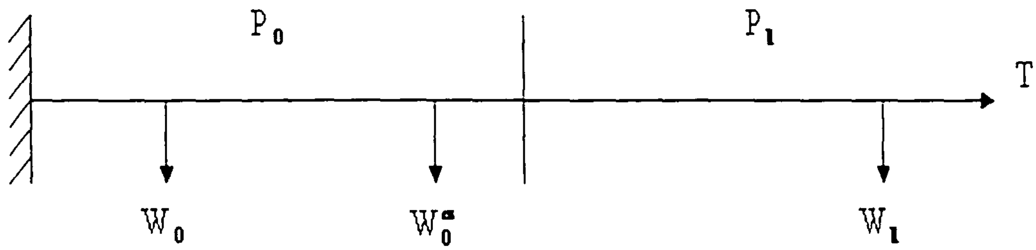

Figure 2-1: *Sequence of decisions in active portfolio management.*

- B_0' *the amount of cash (bond value) in the portfolio before a transaction cost is incurred during period 0.*
- B_0 *the amount of cash after the transaction cost is paid during period 0.*
- B_1' *the amount of cash in the portfolio during period 1.*
- p_0 *the known unit price of the underlying stock during period 0.*
- r_1 *the risky return of the underlying stock during period 1.*
- $\bar{r}_1$ *the mean return of the underlying stock during period 1.*
- r_f *the risk-less return of the bond, i.e., interest rate of cash.*
- α *the transaction cost factor, i.e., the percentage of the transaction amount that is paid.*

We assume that short-selling is not allowed, i.e., the financial institutions (banks, brokerage firms) do not allow the investors to borrow their stocks to buy other securities. In mathematical terms, this means that the number of shares, the investor's wealth and the dollar amount in cash cannot be negative numbers. Our other assumption concerns transaction costs. We assume that we pay a transaction cost proportional to the change, in absolute value, in the amount of money held in stock. The investor is obligated to pay this cost immediately after it incurred. Furthermore, we assume that we have been able to estimate the mean return for the underlying stock from the historical data. For simplicity, we do not constrain the number of stock shares to be an integer. Consequently, the problem can be reformulated as a linear programming problem, instead of a mixed-integer programming problem.

Lemma 2.1.1 *We have the following relations between the mathematical quantities considered:*

$$
\begin{aligned}
W_0 &= p_0 X_0 + B_0' \\
W_0^\alpha &= W_0 - \alpha p_0 |X_1 - X_0| \\
&= p_0 X_1 + B_0 \\
W_1 &= p_1 X_1 + B_1' \\
&= (1+r_1) p_0 X_1 + (1+r_f) B_0 \\
&= (1+r_1) p_0 X_1 + (1+r_f)(W_0 - p_0 X_1 - \alpha p_0 |X_1 - X_0|) \\
&= (1+r_f) W_0 + (r_1 - r_f) p_0 X_1 - \alpha (1+r_f) p_0 |X_1 - X_0| \\
\mathbf{E}_{r_1}[W_1] &= (1+r_f) W_0 + (\bar{r}_1 - r_f) p_0 X_1 - \alpha (1+r_f) p_0 |X_1 - X_0|
\end{aligned}
\tag{2.1}
$$

The portfolio model is then formulated as follows:

$$
\begin{array}{ll}
\max & \mathbf{E}_{r_1}[W_1] \\
\text{s.t.} & 0 \leq X_1 \leq X_0 + \left(\frac{1}{1+\alpha}\right)\left[\frac{W_0}{p_0} - X_0\right]
\end{array}
\tag{2.2}
$$

where we have used that $B_0 \geq 0$ with $B_0 = W_0 - \alpha\, p_0\, |X_1 - X_0| - p_0\, X_1$ and $p_0\, X_0 \leq W_0$ ($B_0' \geq 0$ by assumption) to obtain the upper bound on X_1.

2.1.1 Optimal solution for the active portfolio model

We analyze the optimal solution by considering three possible cases for the parameters of the system. The derivations are straightforward; hence, the results are stated without proof.

Theorem 2.1.2 *Optimal allocation in stocks and optimal wealth are given by the following relations.*

Case (i): $\bar{r}_1 < (1-\alpha)r_f - \alpha$

$$
\begin{aligned}
X_1^* &= 0 \\
W_1^* &= (1+r_f)(W_0 - p_0X_0 - \alpha p_0X_0) + (1+r_f)p_0X_0 \qquad (2.3)\\
&= (1+r_f)(B_0' - \alpha p_0X_0) + (1+r_f)p_0X_0
\end{aligned}
$$

Case (ii): $1-\alpha)r_f - \alpha \le \bar{r}_1 < (1+\alpha)r_f + \alpha$

$$
\begin{aligned}
X_1^* &= X_0 \\
W_1^* &= (1+r_f)(W_0 - p_0X_0) + (1+\bar{r}_1)p_0X_0 \qquad (2.4)\\
&= (1+r_f)B_0' + (1+\bar{r}_1)p_0X_0
\end{aligned}
$$

Case (iii): $(1+\alpha)r_f + \alpha \le \bar{r}_1$

$$
\begin{aligned}
X_1^* &= X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0] \\
W_1^* &= (\frac{1+\bar{r}_1}{1+\alpha})[\frac{W_0}{p_0} - p_0X_0] + (1+\bar{r}_1)p_0X_0 \qquad (2.5)\\
&= (\frac{1+\bar{r}_1}{1+\alpha})B_0' + (1+\bar{r}_1)p_0X_0
\end{aligned}
$$

2.1.2 Sensitivity to the transaction cost

Here we investigate the relation between the transaction cost factor, and the optimal number of shares in the underlying stock that should be held in the next period. We first consider the term:

$$X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0] \qquad (2.6)$$

which is the upper bound on X_1.

Of course, if α is relatively large, then $(\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0]$ becomes very small, so that $X_1 \approx X_0$. Intuitively, we are more likely not to change the portfolio allocation when the transaction cost is high. Moreover, we can see that the range($= 2\alpha(1+r_f)$) from value $(1-\alpha)r_f - \alpha$ to $(1+\alpha)r_f + \alpha$ (case (ii) where $X_1^* = X_0$) is farther spread when the transaction cost increases.

Most brokerage firms charge their clients with a transaction cost up to 0.60%(0.006).

Therefore, in our research we adopt a transaction cost of 0.30% (0.003). The change in a transaction cost factor will directly affect a limit number of shares in underlying stock as in Eq.(2.6). However, this effect is not too significant as long as the factor is still close to 0.30% as shown in Table 2.1.

Table 2.1: *The sensitivity between a transaction cost factor and a limit number of underlying stock*

Transaction Cost Factor		# Limit Stock
α	Increasing(%)	Decreasing(%)
0.003	0	0.000
0.0045	50	0.149
0.006	100	0.298
0.009	200	0.595
0.012	300	0.889
0.015	400	1.182

2.2 Passive portfolio management

In this section, we outline the approach associated with passive portfolio management. We consider a basic portfolio model in presence of transaction cost with a benchmark. For the simplicity, we construct this model based on one underlying risky asset, say, stock; one risk-less asset, say, bond or cash; one benchmark index, say, S&P500 index; for two consecutive periods. We use the same notations as in Section (2.1), and add one more notation for a benchmark index.

Notations:

- K_1 *the benchmark value in period 1.*

Using Lemma (2.1.1), we can formulate this model as follows:

$$\begin{aligned} \min \quad & \mathbf{E}_{\mathcal{F}_1} \max(0, K_1 - W_1) \\ \text{s.t.} \quad & 0 \le X_1 \le X_0 + \left(\frac{1}{1+\alpha}\right)\left[\frac{W_0}{p_0} - X_0\right] \end{aligned} \tag{2.7}$$

This objective function is based on aversion to regret. Indeed, the investor feels regret when it turns out that the benchmark beats his portfolio. Hence, we minimize the

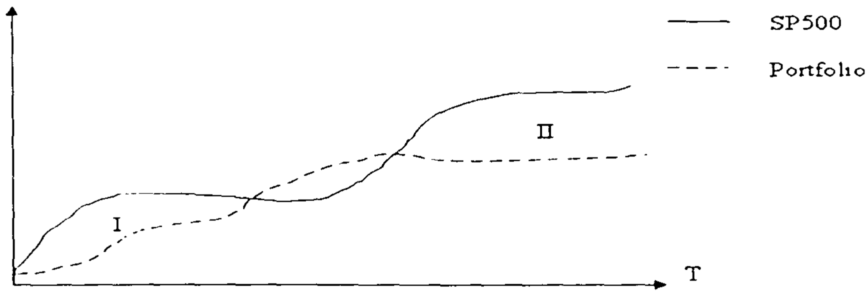

Figure 2-2: *The passive portfolio management*

deviation between the benchmark, and our portfolio only when our portfolio lies under the benchmark as shown in areas I, II in Fig.(2-2). Note that $\max(0, K_1 - W_1)$ is equal to the smallest number z that satisfies $z \geq (K_1 - W_1)$ and $z \geq 0$. For this reason, the optimization problem under consideration is equivalent to the linear programming problem:

$$
\begin{aligned}
\min \quad & \mathbf{E}_{\bar{r}_1}[Z] \\
\text{s.t.} \quad & Z \geq 0 \\
& Z \geq K_1 - W_1 \\
& 0 \leq X_1 \leq X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0]
\end{aligned}
\tag{2.8}
$$

2.2.1 A specific case

We first make the assumption that the return of the underlying stock and the benchmark can take two values each as follows:

$$
\begin{aligned}
r_1 &= \begin{cases} \bar{r}_1 + \sigma_1 & \text{prob}= \frac{1}{2}, \\ \bar{r}_1 - \sigma_1 & \text{prob}= \frac{1}{2}, \end{cases} & , \mathbf{E}_{\bar{r}_1}[r_1] = \bar{r}_1, \mathbf{V}[r_1]= \sigma_1^2 \\
K_1 &= \begin{cases} K^{+} & \text{prob}= \frac{1}{2}, \\ K^{-} & \text{prob}= \frac{1}{2}, \end{cases} & , 0 \leq K^{-} \leq K_1^{+}
\end{aligned}
\tag{2.9}
$$

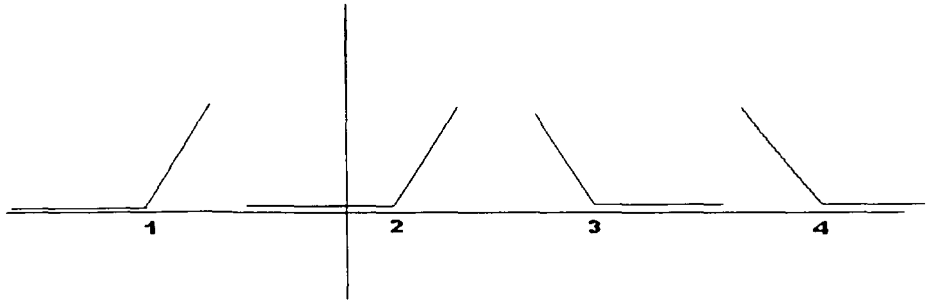

Figure 2-3: *The breaking points of max(0, .) function*

The portfolio selection can be formulated as the following model:

$$
\begin{aligned}
\min \quad & \frac{1}{4}\max(0, K_1^{+} - (1+r_f)W_0 - (\bar{r}_1 + \sigma_1 - r_f)p_0X_1 + \alpha(1+r_f)p_0|X_1 - X_0|) \\
& + \frac{1}{4}\max(0, K_1^{+} - (1+r_f)W_0 - (\bar{r}_1 - \sigma_1 - r_f)p_0X_1 + \alpha(1+r_f)p_0|X_1 - X_0|) \\
& + \frac{1}{4}\max(0, K_1^{-} - (1+r_f)W_0 - (\bar{r}_1 + \sigma_1 - r_f)p_0X_1 + \alpha(1+r_f)p_0|X_1 - X_0|) \\
& + \frac{1}{4}\max(0, K_1^{-} - (1+r_f)W_0 - (\bar{r}_1 - \sigma_1 - r_f)p_0X_1 + \alpha(1+r_f)p_0|X_1 - X_0|)
\end{aligned} \tag{2.10}
$$

$$\text{s.t.} \quad 0 \le X_1 \le X_0 + (\tfrac{1}{1+\alpha})[\tfrac{W_0}{p_0} - X_0]$$

We now analyze the objective function, and in particular study the breaking points and the slope of the linear pieces involved. For example, all values after breaking points for pieces (1),(2) are greater than zero, and a slopes of function are positive. On the other hand, all values before breaking points (3),(4) are greater than zero, and slopes of function are negative as illustrated in Fig.(2-3). Our objective function has four max(0, ·) pieces; therefore, we have four breaking points as described above.

To analyze these breaking points, we consider the following two cases:

(1) $X_0 \le X_1 \le X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0]$

(2) $0 \le X_1 \le X_0$.

Case I: $X_0 \leq X_1 \leq X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0]$

There are four breaking points that make a slope of objective function change.

$$X_1^1 = \frac{K_1^+ - (1+r_f)(W_0 + \alpha p_0 X_0)}{(\bar{r}_1 - (1+\alpha)r_f - \alpha + \sigma_1)p_0}, \quad X_1^2 = \frac{K_1^+ - (1+r_f)(W_0 + \alpha p_0 X_0)}{(\bar{r}_1 - (1+\alpha)r_f - \alpha - \sigma_1)p_0}$$
$$X_1^3 = \frac{K_1^- - (1+r_f)(W_0 + \alpha p_0 X_0)}{(\bar{r}_1 - (1+\alpha)r_f - \alpha + \sigma_1)p_0}, \quad X_1^4 = \frac{K_1^- - (1+r_f)(W_0 + \alpha p_0 X_0)}{(\bar{r}_1 - (1+\alpha)r_f - \alpha - \sigma_1)p_0} \tag{2.11}$$

Case II: $0 \leq X_1 \leq X_0$

There are four breaking points that make a slope of objective function change.

$$X_1^1 = \frac{K_1^+ - (1+r_f)(W_0 - \alpha p_0 X_0)}{(\bar{r}_1 - (1+\alpha)r_f + \alpha + \sigma_1)p_0}, \quad X_1^2 = \frac{K_1^+ - (1+r_f)(W_0 - \alpha p_0 X_0)}{(\bar{r}_1 - (1+\alpha)r_f + \alpha - \sigma_1)p_0}$$
$$X_1^3 = \frac{K_1^- - (1+r_f)(W_0 - \alpha p_0 X_0)}{(\bar{r}_1 - (1+\alpha)r_f + \alpha + \sigma_1)p_0}, \quad X_1^4 = \frac{K_1^- - (1+r_f)(W_0 - \alpha p_0 X_0)}{(\bar{r}_1 - (1+\alpha)r_f + \alpha - \sigma_1)p_0} \tag{2.12}$$

All numerators are nonnegative since $K_1^- \geq (1+r_f)(W_0 + \alpha p_0 X_0)$; hence, the worst-case regret is incurred when the lowest value of the benchmark is still greater than a total wealth from investing a predecessor wealth plus the highest transaction cost in risk-less asset. Figure(2-4) shows the various regions, which are allotted by the values of known mean of risky return in both cases. Therefore, we now have the optimal solutions for the situations that all slopes in objective function are greater or less than zero.However, there are some situations in both cases, when some slopes are greater than zero, and some are less than zero as illustrated in ambiguous area A, and B.

The optimal solutions:

1. $\bar{r}_1 < (1+\alpha)r_f - \alpha - \sigma_1$

$$\begin{aligned} X_1^* &= 0 \\ W_1^* &= \frac{1}{2}\max(0, K_1^+ - (1+r_f)W_0 + \alpha(1+r_f)p_0X_0) \\ &+ \frac{1}{2}\max(0, K_1^- - (1+r_f)W_0 + \alpha(1+r_f)p_0X_0) \end{aligned} \tag{2.13}$$

2. $(1+\alpha)r_f - \alpha - \sigma_1 \leq \bar{r}_1 \leq (1+\alpha)r_f + \alpha + \sigma_1$

First, consider two main sub-conditions(A,B), then we provide the optimal

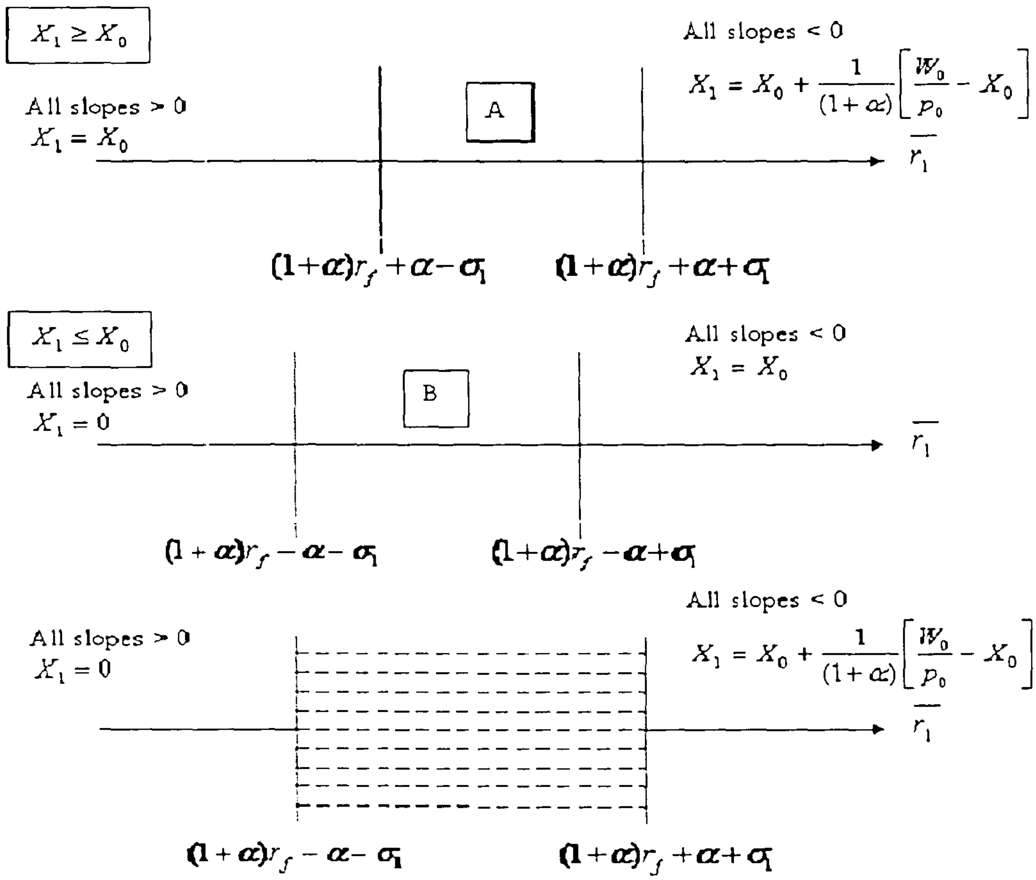

Figure 2-4: *The optimal solution's regions*

solutions for the grey area in Fig.(2-4):

(a) $X_0 \le X_1 \le X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0]$, and

$(1+\alpha)r_f + \alpha - \sigma_1 \le \bar{r}_1 \le (1+\alpha)r_f + \alpha + \sigma_1$

The optimal solutions:

i. $\bar{r}_1 - (1+\alpha)r_f - \alpha < 0$

$$X_1^* = X_0 \tag{2.14}$$

ii. $0 \le \bar{r}_1 - (1+\alpha)r_f - \alpha \le \frac{\sigma_1}{3}$

$$X_1^* = \begin{cases} X_0 & \text{if } X_1^3 < X_0, \\ X_1^3 & \text{if } X_0 \le X_1^3 < X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0], \\ \frac{W_0}{p_0} & \text{if } X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0] \le X_1^3. \end{cases} \tag{2.15}$$

iii. $\frac{\sigma_1}{3} < \bar{r}_1 - (1+\alpha)r_f - \alpha$

$$X_1^* = \begin{cases} X_0 & \text{if } X_1^1 < X_0, \\ X_1^1 & \text{if } X_0 \le X_1^1 < X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0], \\ \frac{W_0}{p_0} & \text{if } X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0] \le X_1^1. \end{cases} \tag{2.16}$$

(b) $0 \le X_1 < X_0$, and $(1+\alpha)r_f - \alpha - \sigma_1 \le \bar{r}_1 \le (1+\alpha)r_f - \alpha + \sigma_1$

The optimal solutions:

i. $\bar{r}_1 - (1+\alpha)r_f + \alpha < 0$

$$X_1^* = 0 \tag{2.17}$$

ii. $0 \le \bar{r}_1 - (1+\alpha)r_f + \alpha \le \frac{\sigma_1}{3}$

$$X_1^* = \begin{cases} X_1^3 & \text{if } 0 \le X_1^3 < X_0, \\ X_0 & \text{if } X_0 \le X_1^3. \end{cases} \tag{2.18}$$

iii. $\frac{\sigma_1}{3} < \overline{r}_1 - (1+\alpha)r_f + \alpha$

$$X_1^* = \begin{cases} X_1^1 & \text{if } 0 \leq X_1^1 < X_0, \\ X_0 & \text{if } X_0 \leq X_1^1. \end{cases} \tag{2.19}$$

Then,we can exploit the optimal solutions from the conditions above to be analyzed in the following condition.

- $(1+\alpha)r_f - \alpha + \sigma_1 > (1+\alpha)r_f + \alpha - \sigma_1$
  - $(1+\alpha)r_f - \alpha - \sigma_1 \leq \overline{r}_1 < (1+\alpha)r_f + \alpha - \sigma_1$

    Optimal $sol^n \Rightarrow$ compare X_0,and 2(b).
  - $(1+\alpha)r_f + \alpha - \sigma_1 \leq \overline{r}_1 < (1+\alpha)r_f - \alpha + \sigma_1$

    Optimal $sol^n \Rightarrow$ compare 2(a),and 2(b).
  - $(1+\alpha)r_f - \alpha + \sigma_1 \leq \overline{r}_1 < (1+\alpha)r_f + \alpha + \sigma_1$

    Optimal $sol^n \Rightarrow$ compare X_0,and 2(a).
- $(1+\alpha)r_f + \alpha - \sigma_1 > (1+\alpha)r_f - \alpha + \sigma_1$
  - $(1+\alpha)r_f - \alpha - \sigma_1 \leq \overline{r}_1 < (1+\alpha)r_f - \alpha + \sigma_1$

    Optimal $sol^n \Rightarrow$ compare X_0.and 2(b).
  - $(1+\alpha)r_f - \alpha + \sigma_1 \leq \overline{r}_1 < (1+\alpha)r_f + \alpha - \sigma_1$

    Optimal sol^n is at X_0.
  - $(1+\alpha)r_f + \alpha - \sigma_1 \leq \overline{r}_1 < (1+\alpha)r_f + \alpha + \sigma_1$

    Optimal $sol^n \Rightarrow$compare X_0.and 2(a).

3. $(1+\alpha)r_f + \alpha + \sigma_1 \leq \overline{r}_1$

$$\begin{aligned} X_1^* &= X_0 + (\frac{1}{1+\alpha})[\frac{W_0}{p_0} - X_0] \\ W_1^* &= \frac{1}{4}\max(0, K_1^+ - (\frac{\overline{r}_1 + \sigma + 1}{1+\alpha})(W_0 + \alpha p_0 X_0)) \\ &+ \frac{1}{4}\max(0, K_1^+ - (\frac{\overline{r}_1 - \sigma + 1}{1+\alpha})(W_0 + \alpha p_0 X_0)) \\ &+ \frac{1}{4}\max(0, K_1^- - (\frac{\overline{r}_1 + \sigma + 1}{1+\alpha})(W_0 + \alpha p_0 X_0)) \\ &+ \frac{1}{4}\max(0, K_1^- - (\frac{\overline{r}_1 - \sigma + 1}{1+\alpha})(W_0 + \alpha p_0 X_0)) \end{aligned} \tag{2.20}$$

2.2.2 General case

In this section, we formulate the model more generally. We use a closed-form solution to form a distribution-free objective function based on the approach proposed in ([4],[12]).

$$\max_{X\sim(\mu,\sigma^2)^+} \quad \mathbf{E}[\max(0, X-k)] = \begin{cases} \frac{1}{2}[\mu - k + \sqrt{\sigma^2 + (\mu-k)^2}] & \text{if } k \geq \frac{\mu^2+\sigma^2}{2\mu} \\ \mu - k + k(\frac{\sigma^2}{\mu^2+\sigma^2}) & \text{if } k < \frac{\mu^2+\sigma^2}{2\mu} \end{cases} \tag{2.21}$$

The solution gives the optimal upper bound on the price of a European call option with strike k, on a stock whose price at maturity has a known mean μ and variance σ^2. However, our formulation is a minimizing problem; moreover, this closed-form will provide a lower bound on optimal solution.In order to formulate our formulation, we need new definitions for a known mean and variance as follows:

Definition 2.2.1

$$\begin{aligned} \overline{W}_1 &= (1+r_f)W_0 + (\bar{r}_1 - r_f)p_0X_1 - \alpha(1+r_f)p_0|X_1 - X_0| \\ \mathbf{V}[W_1] &= p_0^2X_1^2\mathbf{V}[r_1] \\ &= \sigma_{w1}^2 \end{aligned} \tag{2.22}$$

Then, we can exploit from a closed-form above to use in our formulation.

$$\begin{aligned} &\min_{0\leq X_1\leq X_0+(\frac{1}{1+\alpha})[\frac{W_0}{p_0}-X_0]} \quad \mathbf{E}[\max(0, K_1 - W_1)] \\ &= [K_1 - \overline{W}_1 + \mathbf{E}\max(0, W_1 - K_1)] \\ &= K_1 - \overline{W}_1 + \begin{cases} \frac{1}{2}[\overline{W}_1 - K_1 + \sqrt{\sigma_{w1}^2 + (\overline{W}_1 - K_1)^2}] & \text{if } K_1 \geq \frac{\overline{W}_1^2+\sigma_{w1}^2}{2\overline{W}_1} \\ \overline{W}_1 - K_1 + K_1(\frac{\sigma_{w1}^2}{\overline{W}_1^2+\sigma_{w1}^2}) & \text{if } K_1 < \frac{\overline{W}_1^2+\sigma_{w1}^2}{2\overline{W}_1} \end{cases} \end{aligned} \tag{2.23}$$

The optimal solutions:

This objective function, established by a closed-form solution, is a convex differentiable function as seen in Fig.(2-5). Therefore, we differentiate the objective and set the slope to zero to find the global minimum of this function. We distinguish

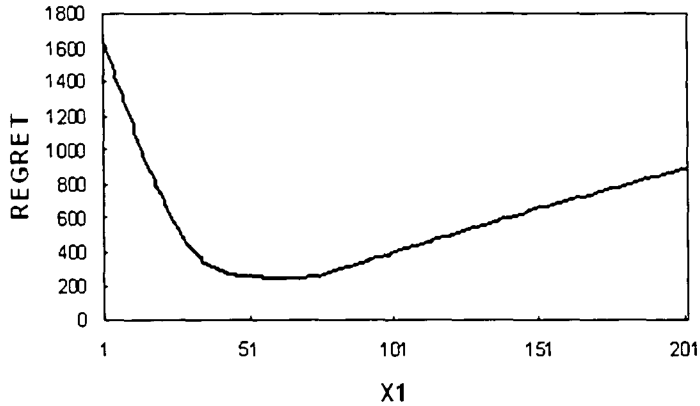

Figure 2-5: *A characteristic of an objective function using a closed-form solution*

between the cases where X_1 is greater than or smaller than X_0, respectively.

$$X_1^* = \begin{cases} -\dfrac{2(\bar{r}_1-(1+\alpha)r_f-\alpha)((1+r_f)(W_0+\alpha p_0 X_0)-K_1)}{(\mathbf{V}[r_1]+(\bar{r}_1-(1+\alpha)r_f-\alpha)^2)p_0} \\ \\ -\dfrac{2(\bar{r}_1-(1-\alpha)r_f+\alpha)((1+r_f)(W_0-\alpha p_0 X_0)-K_1)}{(\mathbf{V}[r_1]+(\bar{r}_1-(1-\alpha)r_f+\alpha)^2)p_0} \end{cases} \tag{2.24}$$

Then we compare these two solutions with X_0 to preserve the bounds. For example, both are less than X_0, then the optimal lower bound is at an extremity of the interval. The results above in Table 2.2 are computed when we assume weekly values of

Table 2.2: *An optimal lower bound on a regret objective function at* $X_0 = 100$

Benchmark	1st Sol.	2nd Sol.	Regret	Lower Bound on X_1
2001	-15.97	28.61	1.00	28.61
2003	-7.99	57.19	2.00	57.19
2005	-0.01	85.76	3.00	85.76
2035	119.64	514.33	29.40	119.64
2045	159.53	657.19	39.20	159.53
2055	199.41	800.04	49.00	199.41

$\bar{r}_1 = 0.005, \sigma_1 = 0.007, r_f = 0.001, \alpha = 0.003$. Moreover, an initial wealth W_0 begins with \$2000 where we have an initial underlying stock $X_0 = 100$ with price $p_0 = \$10$.

Chapter 3

The Multi-Period Models

In this chapter we first review methods to simulate the price of the underlying stock. Then we extend the formulation developed in Chapter 2 to the case of several time periods, and investigate how efficiently our formulation can track a benchmark in a passive portfolio management setting.

3.1 Simulating the stock prices

3.1.1 The Binomial lattice model

In this section we review the binomial lattice model as a way to simulate stock prices conveniently and realistically. If the price is known at the beginning of a period, the price at the beginning of the next period can only take two possible values. Usually, these two possibilities are defined as multiples of the price at the previous period, i.e., multiplied by u for an upward direction, and by d for a downward direction. Both u and d are positive with $u > 1$, and $d < 1$. Hence, if the price at the beginning of period is S, it will be either uS or dS at the next period. The probabilities of these outcomes are pr and 1-pr, respectively, for some given probability pr, $0 < pr < 1$. For example, if the current price is S, there is a probability pr that the price at the next time period will be uS, and a probability 1-pr that it will be dS.

The general form of such a lattice is shown in Fig.(3-1). The stock price can be visualized as moving from node to node in a rightward direction. At each time period, the probability of an upward, resp. downward, movement from any node is pr, resp. 1-pr. The set of the possible trajectories of the stock price has a lattice form since an up movement followed by a down movement yields the same final value for the stock price as a down followed by an up: both produce ud times the price.

The binomial model may appear overly simple because it allows only two possible values at the next period. But if the period length is small, many values are possible after several short steps.

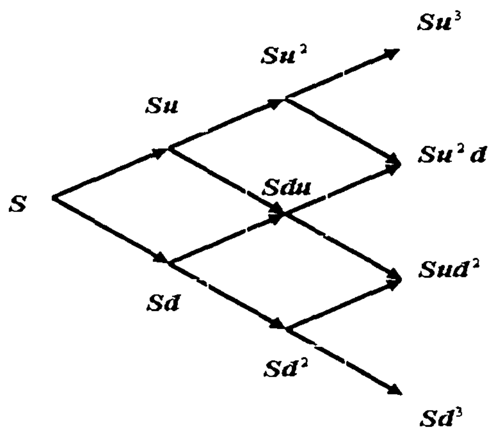

Figure 3-1: *The binomial lattice stock model*

To specify the model completely, we must now select numerical values for u and d and the probability pr. We note that the price can never become negative (the value at the next time period being uS or dS, with $u > 0$, $d > 0$). It is therefore possible to consider the logarithm of the stock price. Consequently, we define ν as the expected interval growth rate as follows:

$$\nu = E[ln(\frac{S_T}{S_0})]$$

where S_0 is the initial underlying stock price, and S_T is the price at the end of a specific interval.

Similarly, we define σ as the interval standard deviation as follows:

$$\sigma^2 = VAR[ln(\frac{S_T}{S_0})]$$

If a period length of Δt is chosen, which is small enough compared to the entire interval, it is well known that the parameters of the binomial lattice can be selected as follows:

$$
\begin{aligned}
pr &= \frac{1}{2} + \frac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t} \\
u &= e^{\sigma\sqrt{\Delta t}} \\
d &= e^{-\sigma\sqrt{\Delta t}}
\end{aligned}
\tag{3.1}
$$

With this choice, the binomial model will closely match the value of ν and σ; therefore, the expected growth rate of $\ln S$ in the binomial lattice will be nearly ν, and the variance of that rate will be nearly σ^2 . The closeness of the match improves if Δt is made smaller($\Delta t \to 0$). For further details, see [1],[13].

3.1.2 The scenario tree model

In decision-making under uncertainty, we cannot completely observe the uncertainty (stock returns) when we make our decisions (an amount of stock in each period X_t): we can only observe the returns that have already taken place. Stochastic programming incorporates the fact that our decisions are non-anticipative of future outcomes. According to two possible values of the risky returns of stock (generated by the prices from binomial model), we assume that, over the N decision periods, 2^N possible scenarios may occur. These scenarios are represented by a symmetric tree. To build a tree of scenarios that allows the decision-maker to use past outcomes, i.e., for which there is only one way to reach any node, we transform the binomial lattice model into a scenario tree as shown in Figure 3-2. The advantage of the scenario tree over the binomial lattice model is that each scenario in scenarios tree corresponds to a traceable sample path (see Figure 3-3).

Figure 3-3 also allows us to illustrate non-anticipativity. For instance, if we consider both A and B in Fig.3-3, we note that they use the same path between node 0 and node 1. Consequently, all parameters and decision variables made for those nodes in time periods 0 and 1 must be identical. Similarly, if we assume three decision periods as shown in Fig.(3-3), the decision variables at time periods 0 and 1 will be identical for the four scenarios in the upper part of the tree.

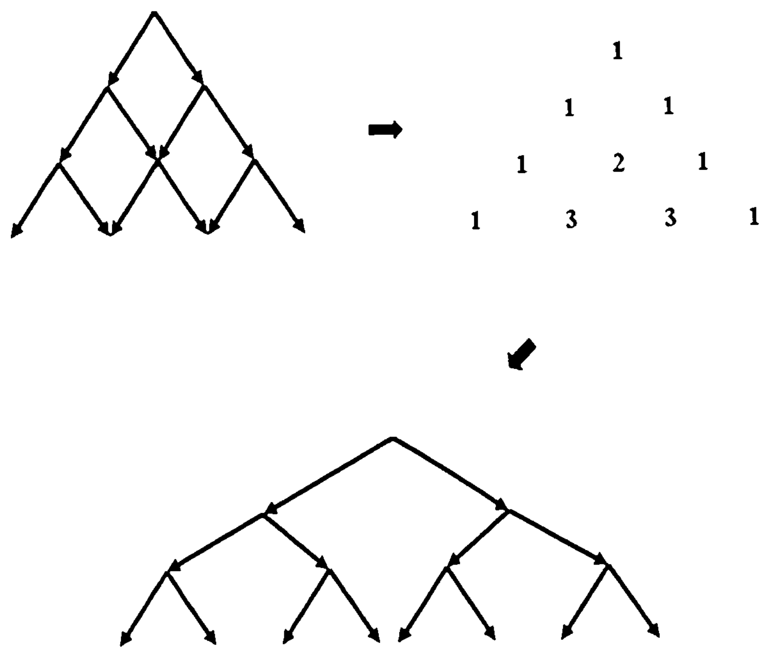

Figure 3-2: *The transformation between binomial lattice model and tree of scenarios for three periods*

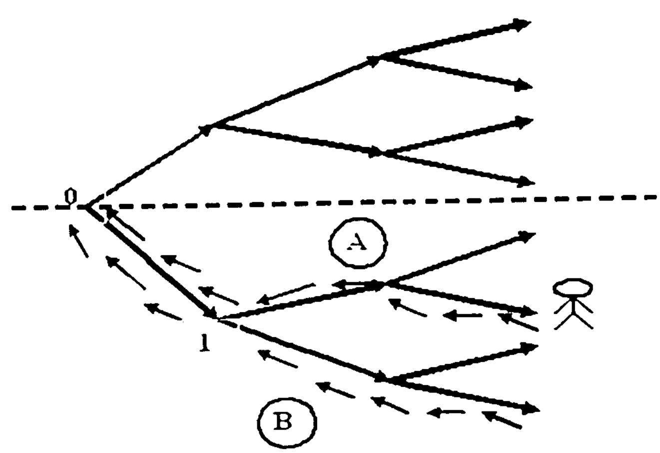

Figure 3-3: *The traceable sample paths*

3.2 Multi-Period Formulations

We now extend the formulations developed in Chapter 2 to the multi-period setting. We will consider models with one underlying risky asset (stock), one risk-less asset (bond or cash), one benchmark index (e.g.,S&P500 index). We do not allow short sales. We use the following notations:

Notations:

- W_t^s *the wealth at the beginning of period t in scenario s.*
- $W_t^{a^s}$ *the wealth after a transaction cost incurred during period t in scenario s.*
- W_{t+1}^s *the wealth at the end of period t+1 in scenario s.*
- W_0 *the initial wealth.*
- X_t^s *the number of shares of stock invested during period t in scenario s.*
- X_{t+1}^s *the number of shares of underlying stock invested during period t+1, and decided at the end of period t in scenario s.*
- X_0 *the initial of shares of underlying stock.*
- $B_t'^s$ *the amount of cash or bond before a transaction cost incurred during period t in scenario s.*
- B_t^s *the amount of cash or bond after a transaction cost paid during period t in scenario s.*
- $B_{t+1}'^s$ *the amount of cash or bond in the portfolio during period t+1 in scenario s.*
- p_t^s *the unit price of underlying stock during period t in scenario s.*
- p_{t+1}^s *the unit price of underlying stock during period t+1 in scenario s.*
- r_{t+1}^s *the return of the underlying stock during period t in scenario s.*
- r_f *the riskless return of bond or interest rate of cash.*
- α *the transaction cost factor.*

- K_t *the benchmark during period t.*
- Z_t^s *the decision variable modeling regret during period t in scenario s.*
- Ω^s *the probability of scenario s*

Lemma 3.2.1 *We have the following relations:*

$$
\begin{aligned}
W_t^s &= p_t^s X_t^s + B_t^{\prime s} \\
W_t^{\alpha^s} &= W_t^s - \alpha p_t^s |X_{t+1}^s - X_t^s| \\
&= p_t^s X_{t+1}^s + B_t^s \\
W_{t+1}^s &= p_{t+1}^s X_{t+1}^s + B_{t+1}^{\prime s} \qquad\qquad ,\forall t, s \\
&= (1 + r_{t+1}^s) p_t^s X_{t+1}^s + (1 + r_f) B_t^s \\
&= (1 + r_{t+1}^s) p_t^s X_{t+1}^s + (1 + r_f)(W_t^s - p_t^s X_{t+1}^s - \alpha p_t^s |X_{t+1}^s - X_t^s|) \\
&= (1 + r_f) W_t^s + (r_{t+1}^s - r_f) p_t^s X_{t+1}^s - \alpha(1 + r_f) p_t^s |X_{t+1}^s - X_t^s|
\end{aligned}
\tag{3.2}
$$

The planning horizon has T time periods, which gives us $S = 2^T$ possible scenarios. Scenarios and nodes on the graph are defined in decreasing order on the graph. Hence, if $s = S$, then a sample path leading to this scenario observes only downward movements in the stock price. With the scenarios defined here, we assign probabilities to each scenario as follows:

$$\Omega^s = (1 - pr)^N pr^{1-N} \tag{3.3}$$

where we define pr, resp. 1-pr, as the up, resp. down, probability. N is the number of downward movements on the sample path.

Due to the non-anticipativity of decisions, X_t^s will be identical for all scenarios s that cannot be distinguished from each other (as they have the same historical realizations) at time t. For example, with $T = 3, S = 2^3 = 8$:

$$
\begin{aligned}
t = 0 &\longrightarrow X_0^1 = X_0^2 = X_0^3 = X_0^4 = X_0^5 = X_0^6 = X_0^7 = X_0^8 \\
t = 1 &\longrightarrow X_1^1 = X_1^2 = X_1^3 = X_1^4; X_1^5 = X_1^6 = X_1^7 = X_1^8 \\
t = 2 &\longrightarrow X_2^1 = X_2^2; X_2^3 = X_2^4; X_2^5 = X_2^6; X_2^7 = X_2^8 \\
t = 3 &\longrightarrow X_3^1; X_3^2; X_3^3; X_3^4; X_3^5; X_3^6; X_3^7; X_3^8
\end{aligned}
\tag{3.4}
$$

For more details how to formulate a stochastic model, see [5].

3.2.1 The multi-period active management model

The active portfolio management problem maximizes final expected wealth. It is straightforward to formulate this problem as:

$$
\begin{aligned}
\max \quad & \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{S}\Omega^s W_t^s \\
\text{s.t.} \quad & W_t^s - p_t^s X_{t+1}^s - \alpha p_t^s |X_{t+1}^s - X_t^s| \geq 0,\ t = 0,\ldots T-1, \forall s \\
& W_{t+1}^s = (1+r_f)W_t^s + (r_{t+1}^s - r_f)p_t^s X_{t+1}^s - \alpha(1+r_f)p_t^s|X_{t+1}^s - X_t^s|,\ t = 0,\ldots,T-1, \forall s \\
& t = 0 \quad \longmapsto \quad W_0^1 = W_0^2 = .. = .. = W_0^{S-1} = W_0^S = W_0 \\
& t = 0 \quad \longmapsto \quad X_0^1 = X_0^2 = .. = .. = X_0^{S-1} = X_0^S = X_0 \\
& t = 1 \quad \longmapsto \quad X_1^1 = .. = X_1^{\frac{S}{2}} ; X_1^{\frac{S}{2}+1} = .. = X_1^S \\
& \vdots \\
& t = T-1 \quad \longmapsto \quad X_{T-1}^1 = X_{T-1}^2 ; .. ; X_{T-1}^{S-1} = X_{T-1}^S \\
& X_t^s \geq 0, \forall t, s
\end{aligned}
\tag{3.5}
$$

Note that Problem (3.5) can be rewritten as a linear programming problem by transforming the constraint:

$$W_t^s - p_t^s X_{t+1}^s \geq \alpha p_t^s |X_{t+1}^s - X_t^s| \tag{3.6}$$

for all t and s into a set of two equations:

$$W_t^s - p_t^s X_{t+1}^s \geq \alpha p_t^s (X_t^s - X_{t+1}^s) \tag{3.7}$$

and

$$W_t^s - p_t^s X_{t+1}^s \geq \alpha p_t^s (X_{t+1}^s - X_t^s). \tag{3.8}$$

3.2.2 The multi-period passive management model

The goal here is to minimize the average final regret. The passive management model can be formally formulated as follows:

$$
\begin{aligned}
\min\quad & \tfrac{1}{T}\textstyle\sum_{t=1}^{T}\sum_{s=1}^{S}\Omega^{s}Z_t^{s} \\
s.t.\quad & W_t^{s}-p_t^{s}X_{t+1}^{s}-\alpha p_t^{s}|X_{t+1}^{s}-X_t^{s}|\geq 0,\ \forall t,s \\
& W_{t+1}^{s}=(1+r_f)W_t^{s}+(r_{t+1}^{s}-r_f)p_t^{s}X_{t+1}^{s}-\alpha(1+r_f)p_t^{s}|X_{t+1}^{s}-X_t^{s}|\ \forall t,\ s \\
& t=0\longmapsto W_0^{1}=W_0^{2}=..=..=W_0^{S-1}=W_0^{S}=W_0 \\
& t=0\longmapsto X_0^{1}=X_0^{2}=..=..=X_0^{S-1}=X_0^{S}=X_0 \\
& t=1\longmapsto X_1^{1}=..=X_1^{\frac{S}{2}};X_1^{\frac{S}{2}+1}=..=X_1^{S} \\
& \vdots \\
& t=T-1\longmapsto X_{T-1}^{1}=X_{T-1}^{2};..;X_{T-1}^{S-1}=X_{T-1}^{S} \\
& X_t^{s}\geq 0,\ \forall t,s \\
& Z_t^{s}\geq 0,\ \forall t\geq 1,\forall s \\
& Z_t^{s}\geq K_t-W_t^{s},\ \forall t\geq 1,\forall s
\end{aligned}
\tag{3.9}
$$

Again, Problem (3.9) can be rewritten as a linear programming problem.

3.3 The numerical experiments

In section 3.2, we have shown how to extend the models over several periods. To examine the results of our formulations in practical situations, then, we turn to computational examples, where we use historical stock data publicly available at *http://finance.yahoo.com* and *http://www.indices.standardandpoors.com*. This data set consists of weekly stock price observations over a period of about two years of a major stock index, say, the S&P500 and its top five components. Therefore, we collect historical data of these stocks from January, 2004 to December, 2005. In terms of weekly observations, the data set covers 105 periods. The details of the top five S&P500's components are provided in Table 3.1. With this data in hand, we can now investigate the performances of both proposed active and passive models. We consider the following numerical values: initial wealth $W_0 = 20000$, annual risk-less

Table 3.1: The top five components of S&P500 index

Company	Index Weight	Initial Price($)
Exxon Mobil Corp.	3.32	38
General Electric	3.1	29
Citigroup Inc.	2.12	34
Bank of America Corp.	1.96	36
Microsoft Corp.	1.73	24

return $r_f = 0.06$, and a transaction cost factor $\alpha = 0.003$. Each stock has a distinct probability distribution. We compute for each the two-year expected return or expected growth rate and standard deviation corresponding to this data set, respectively, by using the binomial lattice model as shown in Table 3.2. We then

Table 3.2: The probabilistic data of stocks

Stock Data	2 Years Expected Return(%)	2 Years Std Dev(%)
S&P500	23.46	7.17
Exxon	36.79	27.95
GE	16.90	18.54
Citi	39.65	31.75
Bank of America	23.55	17.86
Microsoft	7.51	23.70

generate a scenario tree for each set of five underlying stock data over eight periods (weeks). We include S&P500 later for the experiments about passive portfolio management. This procedure yields $2^8 = 256$ scenarios in the tree. Several numerical experiments are performed as described in the following pages.

3.3.1 The numerical results from the active management model

First series of experiments

In the first set of experiments, we compare the results of the active portfolio optimized over 8 weeks (256 scenarios) using Formulation (3.5). Initially, the portfolio starts with all portfolio weight in underlying stock. We provide the following results: (1) the optimal final wealth (W_8) (2) the optimal stock weight (3) expected growth rate over 8 weeks as shown in Table 3.3. These problems are solved by using the XPress software and Excel on a PC. It appears that we would maximize our expected return over eight

Table 3.3: The results of the multi-period active model

Stock Data	Wealth($)	Risky Security Weight(%)	Expected Growth Rate(%)
Exxon	21375	54.26	6.65
GE	20833	52.92	4.08
Citi	21573	54.03	7.57
Bank of America	20834	54.26	4.08
Microsoft	21033	50.99	5.04

weeks by investing into Citigroup Inc. However, to protect against risk, we should not invest solely in that company's stock: instead, our results indicate our portfolio should hold about fifty percent of risky asset.

Second series of experiments

In the second series of experiments we investigate how efficiently those results can be exploited on the real data (Jan, 2004 to Dec, 2005). We make our investment according to the optimal weight and keep it constant over 105 periods (2 years) using the relations in Lemma (3.2.1). For example, we use the add-in Solver in Excel to adjust the weight of stock investment in our portfolio to be constant each week. We now see the drawback of using the ideal optimal active management strategy with a fixed allocation rule when the data comes to be realized. The initial portfolio starts with weight according to the optimal strategy. Each weekly risky return is realized on this real weekly data. We provide the following results: (1) two years expected growth rate (2) two year standard deviation.

Table 3.4: The results of optimal strategy based on the real data

Stock Data	Expected Growth Rate(%)	Std Dev(%)
Exxon	24.36	15.17
GE	14.11	9.83
Citi	25.55	17.13
Bank of America	17.82	9.70
Microsoft	8.93	12.02

Third series of experiments

In the third series of experiments we look at the effects on the optimal wealth and expected growth rate of varying the parameters. Thus, we vary (1) a weight or percent of risky security invested in the initial portfolio wealth W_0 (2) a transaction cost factor (3) a risk-less return. We only vary one parameter at a time. These procedures are performed with the binomial lattice and scenarios tree over eight weeks as in the first series of experiments. The effects of changing the initial stock weight are shown in

Table 3.5: *The effects of varying the initial stock weight*

Initial Stock Weight in portfolio(%)	Wealth($)	Expected Growth Rate(%)
(Exxon)		
0	21356	6.56
25	21362	6.59
50	21366	6.61
75	21370	6.63
100	21375	6.65
(GE)		
0	20819	4.01
25	20822	4.03
50	20825	4.04
75	20829	4.06
100	20831	4.07
(Citi)		
0	21554	7.48
25	21558	7.50
50	21563	7.52
75	21567	7.54
100	21572	7.57
(Bank of America)		
0	20817	4.00
25	20822	4.03
50	20826	4.05
75	20831	4.07
100	20835	4.09
(Microsoft)		
0	21027	5.01
25	21029	5.02
50	21030	5.02
75	21031	5.03
100	21032	5.03

Table 3.5. We note that changes in both the optimal wealth and expected growth are not significant. We note from Table 3.6 that the optimal wealth and the expected growth rate tend to decrease significantly when the transaction cost factor varies, but.

Table 3.6: The effects of a change in the transaction cost factor

Transaction Cost Factor(α)	Wealth($)	Expected Growth Rate(%)
(Exxon)		
0.0000	21510	7.28
0.0030	21375	6.65
0.0045	21309	6.34
0.0060	21242	6.03
0.0075	21176	5.71
(GE)		
0.0000	20966	4.72
0.0030	20831	4.07
0.0045	20764	3.75
0.0060	20698	3.43
0.0075	20631	3.11
(Citi)		
0.0000	21709	8.20
0.0030	21572	7.57
0.0045	21505	7.25
0.0060	21437	6.94
0.0075	21369	6.62
(Bank of America)		
0.0000	20966	4.72
0.0030	20835	4.09
0.0045	20770	3.78
0.0060	20706	3.47
0.0075	20641	3.16
(Microsoft)		
0.0000	21174	5.70
0.0030	21032	5.03
0.0045	20962	4.70
0.0060	20892	4.36
0.0075	20823	4.03

as indicated in Table 3.7, the optimal stock weight does not change much.

Finally, Table 3.8 shows the impact of a change in the risk-less return. The optimal wealth and expected growth rate slightly change in the same direction as the risk-less return. However, the range of change is narrow; therefore, the effects are quite small.

Table 3.7: The optimal stock(Exxon) weight in portfolio when a transaction cost factor changing

Transaction cost factor(α)	0.0000	0.0030	0.0045	0.0060	0.0075
(Exxon)Optimal Stock Weight(%)	54.27	54.26	54.26	54.27	54.25

Table 3.8: The effects of changing the risk-less return

Annual risk-less return(r_f)	Wealth($)	Expected Growth Rate(%)
(Exxon)		
0.01	21335	6.46
0.06	21375	6.65
0.09	21400	6.76
(GE)		
0.01	20790	3.88
0.06	20831	4.07
0.09	20856	4.19
(Citi)		
0.01	21531	7.28
0.06	21572	7.57
0.09	21597	7.68
(Bank of America)		
0.01	20797	3.90
0.06	20835	4.09
0.09	20859	4.20
(Microsoft)		
0.01	20988	4.82
0.06	21032	5.03
0.09	21059	5.16

3.3.2 The numerical results from the passive management model

First series of experiments

In this section, we use the multi-period passive management model to compute the numerical results as follows. We use the S&P500 index as benchmark. First of all, we generate the S&P500's returns for eight weeks using its probabilistic data as shown in Table 3.2. Furthermore, we assume that we invest all portfolio in this index for eight weeks as shown in Table 3.9. Unfortunately, we cannot hold all of the stocks that make up the S&P500 index and so perfectly reproduce it (full replication). For this reason, the top five stocks in S&P500 are chosen to track a benchmark. To introduce the numerical experiments, we first assume the initial wealth $W_0 = 20000$, annual

Table 3.9: The S&P500 index portfolio for 8 weeks(W_0 = \$20000)

Week	Wealth($)	Return
1	20121	0.0060
2	20286	0.0082
3	20337	0.0025
4	20393	0.0028
5	20162	-0.0113
6	20161	0.0000
7	20448	0.0142
8	20473	0.0012

risk-less return $r_f = 0.06$, and a transaction cost factor $\alpha = 0.003$.

In the first set of experiments, we compare the results of the passive portfolio optimized over 8 weeks(256 scenarios) using our formulation(3.9). The initial portfolio starts with all portfolio weight in risk-less security. These results are given (1) the optimal regret (2) the optimal stock weight as shown in Table 3.10. These optimized problems are solved by using X-press software and Excel on a PC. In the

Table 3.10: The results of multi-period passive model

Stock Data	Regret($)	Risky Security Weight(%)
Exxon	12.49	9.48
GE	18.62	15.88
Citi	12.84	8.04
Bank of America	17.31	16.49
Microsoft	18.51	11.74

conclusions, the regret is small for each stock models compared to the total wealth(0.06% – 0.09% of the wealth). The investor should hold the underlying stock about 10%-20% of his portfolio weight. As in the multi-period active model, the investor should hold the underlying stock for about 50% of his portfolio weight. Figure (3-4) shows the total wealth of holding underlying stock (Exxon) with a benchmark index(S&P500). Our passive portfolio curve try to keep the line above its benchmark. Therefore, the regret will be minimized. The regret-averse investor may use this type of model.

Second series of experiments

In the second series of experiments, we have extended the results to one-hundred and

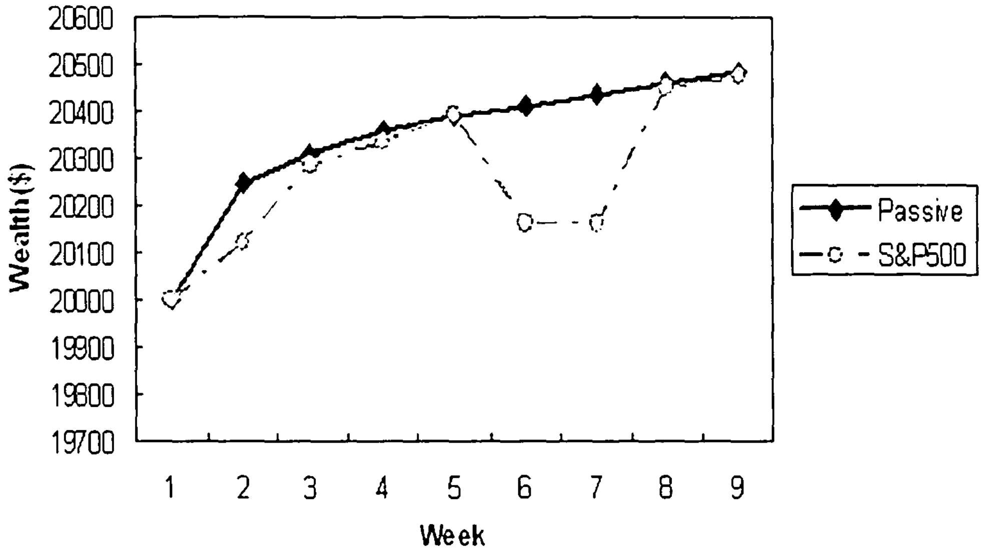

Figure 3-4: *The total wealth of Exxon vs S&P500 benchmark over 8 weeks*

five weeks using the real data. As in the active model experiment, we keep the constant optimal stock weight in our portfolio. In this experiment, we choose the Exxon stock as an underlying stock, since its optimal regret is the lowest of all stocks. The initial portfolio starts with weight according to the optimal strategy. Each weekly risky return is realized on this real weekly data.

The optimal regret = \$47.61

Number of weeks that passive portfolio under S&P500 curve = 16 weeks

Number of weeks that passive portfolio above S&P500 curve = 89 weeks

The percentage of time periods that passive portfolio under S&P500 curve = 15.24%

The percentage of time periods that passive portfolio above S&P500 curve = 84.76%

The results have explicitly shown that our multi-period passive model can track(or replicate) the benchmark efficiently. Moreover, the model try to adjust its curve to be above the benchmark, and can track a trend line of the benchmark as well.

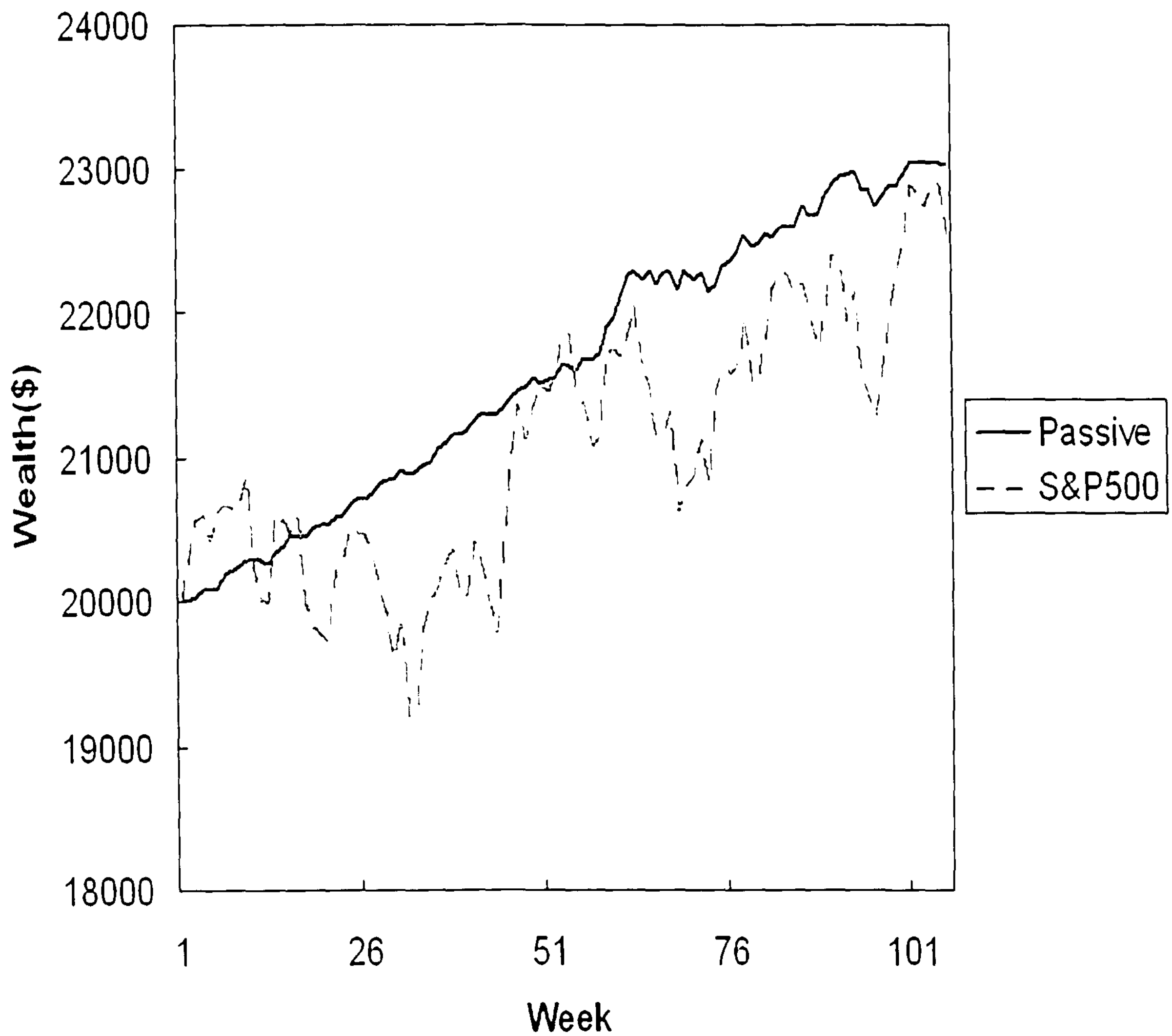

Figure 3-5: *The total wealth of Exxon vs S&P500 benchmark over 105 weeks*

Chapter 4

The index-tracking models

In this chapter we consider the problem of constructing the index-tracking portfolio out of a large universe of stocks. This universe can be an index such as S&P500. In order to choose a portfolio that closely tracks the return on a benchmark, we must first decide how to measure the performance of a tracking portfolio. This is a kind of passive portfolio management. We formulate the mathematical model, and then turn our attention to analyze the numerical results.

4.1 The tracking error models

We first quantify the degree of which the return on the index-tracking portfolio differs from the return on a benchmark, i.e., a measure of tracking error. Tracking error is measured using historical data, because the future return paths of the index-tracking portfolio and its benchmark are unknown. Some common measures of tracking error are given in ([2],[7],[9],[10],[15], [17],[18]).

The investor allocates his/her wealth among n underlying risky securities which are components of the benchmark index. This allocation is done at the beginning of investment, and the proportion of the wealth invested in each stock is constant throughout all the periods. Moreover, there is no short-selling allowed. We consider two mathematical formulations of tracking error: (1) the mean absolute deviations (2) the mean absolute downside deviations as in ([7],[9],[11],[17],[18]). We introduce the notations as follows:

Notations:

- *I_t the observed return of the benchmark at period t.*
- *r_t^i the observed return of underlying stock i at period t.*
- *x_i the proportion of the wealth invested in stock i.*

The tracking error based on the mean absolute deviation can be expressed as:

$$TE_{MAD} = \frac{1}{T}\sum_{t=1}^{T} \left|\sum_{i=1}^{n} r_t^i x_i - I_t\right|. \tag{4.1}$$

The tracking error based on the mean absolute downside deviations can be expressed as:

$$TE_{MADD} = \frac{1}{T}\sum_{t=1}^{T} \left|\min(0, \sum_{i=1}^{n} r_t^i x_i - I_t)\right|. \tag{4.2}$$

In the first approach (MAD model), we consider the deviations between the total return of underlying stock portfolio and the return on the benchmark. On the other hand, in the second approach (MADD model), we consider the deviations between the total return of underlying stock portfolio and the return on the benchmark when they move in the downside direction (the benchmark return is greater than the total return).

As mentioned in Chapter 1, a portfolio that tracks the benchmark perfectly can be obtained by the strategy of full replication. Trading a tracking portfolio with such a large number of different positions leads to high transaction costs. As a result, it is desirable to form a good tracking portfolio with as few stocks as possible.

4.1.1 The mean absolute deviations model

To formulate the model, suppose that we want to track the index with k out of n stocks. Therefore, we can formulate this problem of minimizing tracking error while restricting the number of assets. The mean absolute deviations model can be formally formulated as follows:

$$
\begin{aligned}
\min \quad & \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n} r_t^i x_i - I_t\right| \\
s.t. \quad & \sum_{i=1}^{n} \Psi(x_i) = k, \\
& \sum_{i=1}^{n} x_i = 1, \\
& x_i \geq 0 \qquad\qquad ,\forall i
\end{aligned}
\tag{4.3}
$$

where

$$
\Psi(x_i) = \begin{cases} 0 & \text{if } x_i = 0 \\ 1 & \text{otherwise} \end{cases}
$$

This can be reformulated as a mixed-integer model using binary variables for the Ψ function.

4.1.2 The mean absolute downside deviations model

Similarly, the mean absolute downside deviations model can be formulated as follows:

$$
\begin{aligned}
\min \quad & \frac{1}{T}\sum_{t=1}^{T}\left|\min(0, \sum_{i=1}^{n} r_t^i x_i - I_t)\right| \\
s.t. \quad & \sum_{i=1}^{n} \Psi(x_i) = k, \\
& \sum_{i=1}^{n} x_i = 1, \\
& x_i \geq 0 \qquad\qquad ,\forall i
\end{aligned}
\tag{4.4}
$$

where

$$
\Psi(x_i) = \begin{cases} 0 & \text{if } x_i = 0 \\ 1 & \text{otherwise} \end{cases}
$$

[9] notes that this problem can be reformulated by using the relation:

$$|\min(0, a)| = \frac{1}{2}|a| - \frac{1}{2}a \tag{4.5}$$

for any real number a. Hence, we can introduce the additional constraints to Eq.(4.4) by using the auxiliary variables b_t^+, b_t^-. The new model can be formulated as follows:

$$\begin{aligned}
\min\quad & \textstyle\sum_{t=1}^{T} \frac{2b_t^-}{T} \\
\text{s.t.}\quad & \textstyle\sum_{i=1}^{n} \Psi(x_i) = k, \\
& \textstyle\sum_{i=1}^{n} x_i = 1, \\
& b_t^+ - b_t^- = \frac{\sum_{i=1}^{n} r_t^i x_i - I_t}{2}, \forall t \\
& b_t^+, b_t^- \geq 0, \forall t \\
& x_i \geq 0, \ \forall i,
\end{aligned} \tag{4.6}$$

where

$$\Psi(x_i) = \begin{cases} 0 & \text{if } x_i = 0 \\ 1 & \text{otherwise} \end{cases}$$

4.2 The current methods

Suppose that we want to construct an index-tracking portfolio using 40 stocks out of 500 component stocks of S&P500. One may take the 40 largest stocks in this index and minimize the tracking error using these stocks. However, this is not necessarily the best (optimal) solution. Another approach would be to solve this problem by choosing 40 stocks out of 500 stocks to minimize the tracking error. It is very difficult to solve this problem. Because of the time involved, enumerate all the possibilities to find the combination of stocks that minimizes the tracking error is not achievable when we have to choose some stocks out of a large universe. The enumeration-based approach is only practical for very small or very large k or for very small n, since the number of ways to choose k stocks out of n stocks is:

$$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!} \tag{4.7}$$

For example, there are 10^{59} ways to form a 40 stock subset out of a given 500 stocks. Nowadays, we can take several steps to find the optimal solutions. Suppose that we want to construct an index-tracking portfolio using 10 stocks out of 500 component stocks of S&P500. First, we choose the 60 largest stocks out of S&P500 and minimize the tracking error using these stocks. Second, choose 40 stocks out of 60 stocks that give the minimal tracking error, and minimize the tracking error using these stocks. Third, choose 30 stocks out of 40 stocks that give the minimal tracking error, and minimize the tracking error using these stocks. Then, perform this procedure until we achieve to the desired 10 stock portfolio.

4.3 The numerical experiments

In this section, we perform two series of numerical experiments to illustrate our index-tracking models using minimized tracking error approach. The main goal is to track or replicate the benchmark index using a small desired portfolio. We make these computations using the underlying stock price data made publicly available at *http://finance.yahoo.com* and *http://www.indices.standardandpoors.com* as in Chapter 3. All computations are performed on a Window-PC using X-press solver and Excel.

First series of experiments

In the first series of experiments we formulate the mean absolute deviations and the mean absolute downside deviations portfolios. We have to decide the fixed proportion of our wealth on each stock that optimizes the tracking error. Thus, we choose the 60 largest stocks out of S&P500 index to be the stock universe of our problem. Then, we relax the limit number of stocks in the portfolio constraint, and optimize the models using these stock data over 50 weeks from January, 04 to November, 04. Hence, the results have shown the optimal number of stocks held in portfolio, and the optimal tracking error as in Table 4.1. We formulate the models repeatedly, and limit the number of the stocks available as the 50,40,30,20,10,5,3,2,1 largest S&P500 stocks. In MADD model, we can track S&P500 index efficiently using the 50 weeks historical return data of the 40 largest stocks in S&P500, since the tracking error are nearly zero

Table 4.1: The MAD and MADD index-tracking portfolios using the desired largest stock universe over 50 weeks

#Stock Universe	MAD		MADD	
	#Stock in Portfolio	TE	#Stock in Portfolio	TE
60	40	0.000426	13	0.000000
50	33	0.000676	24	0.000000
40	27	0.000931	21	0.000000
30	22	0.001572	21	0.000203
20	18	0.002563	12	0.001754
10	10	0.003605	9	0.002375
5	5	0.006024	5	0.003663
3	3	0.008623	3	0.005773
2	2	0.009124	2	0.006245
1	1	0.013112	1	0.009768

and the optimal number of stocks held in portfolio is 21 stocks. In both MAD and MADD model, if we reduce the size of the stock universe to be chosen to form our index-tracking portfolio, the tracking error will increase.

Now, we achieve the optimal weight or proportion of our wealth on each stock, which is optimized based on the fact that we have known all information about the returns on stocks for 50 weeks. To extend these results to the future use, we assume the future returns of S&P500 index and its components for 55 weeks using the data from December, 04 to December, 05. Thus, we measure how efficiently the current optimal stock weight can be used to form the new index-tracking model as in Table 4.2.

Figures (4-1) and (4-2) show the index-tracking portfolios using the 30 largest stocks and their benchmark over 55 weeks. Comparing the characteristics of both MAD and MADD models is explicitly shown that our MADD index-tracking model has higher deviation errors from its benchmark when the index-tracking returns are greater than the benchmark. However, this has lower deviation errors from its benchmark in Fig.(4-2) when the index-tracking returns are less than the benchmark.

Second series of experiments

In the second series of experiments we formulate the index-tracking model using the mean absolute deviations as a tracking error, and adopt the current method in section 4.2 to solve the problem. The current procedure is to sort the stock weights in the

Table 4.2: The MAD and MADD index-tracking portfolios using the desired largest stock universe over 55 weeks

#Stock Universe	MAD tracking error	MADD tracking error
60	0.003122	0.001268
50	0.002820	0.001268
40	0.004457	0.001337
30	0.004704	0.001807
20	0.004340	0.002167
10	0.005451	0.002186
5	0.006428	0.002887
3	0.009355	0.004288
2	0.008479	0.005693
1	0.022601	0.010285

Table 4.3: The MAD index-tracking portfolio using sorting approach over 50 weeks

#Stock Universe	#Stock in Portfolio		Tracking Error	
	MAD	MAD(SORTED)	MAD	MAD(SORTED)
60	40	40	0.000426	0.000426
50	33	40	0.000676	0.000426
40	27	40	0.000931	0.000426
30	22	29	0.001572	0.000532
20	18	20	0.002563	0.000816
10	10	10	0.003605	0.002361
5	5	5	0.006024	0.004628
3	3	3	0.008623	0.006712
2	2	2	0.009124	0.009054
1	1	1	0.013112	0.012142

portfolio in descending order. Afterwards, we choose a smaller stock universe that will be available to construct the new index-tracking portfolio, and optimized the model to find the next optimal stock weight. To achieve the desired portfolio size, we repeat the procedure until the result is satisfied, but we reduce the stock universe in each step.

The tracking errors in sorting approach are smaller than the former approach in the first series of experiments as shown in Table 4.3. Furthermore, we analyze the optimal stock weight results with the next 55 weeks as in Table 4.4 and Fig.(4-3). In conclusion, we obtain more efficient performance to track the benchmark using sorting the stock weight approach.

Table 4.4: The MAD index-tracking portfolio using sorting approach over 55 weeks

#Stock Universe	Tracking Error
60	0.003122
50	0.003122
40	0.003122
30	0.003253
20	0.003589
10	0.005295
5	0.007232
3	0.008876
2	0.009914
1	0.016554

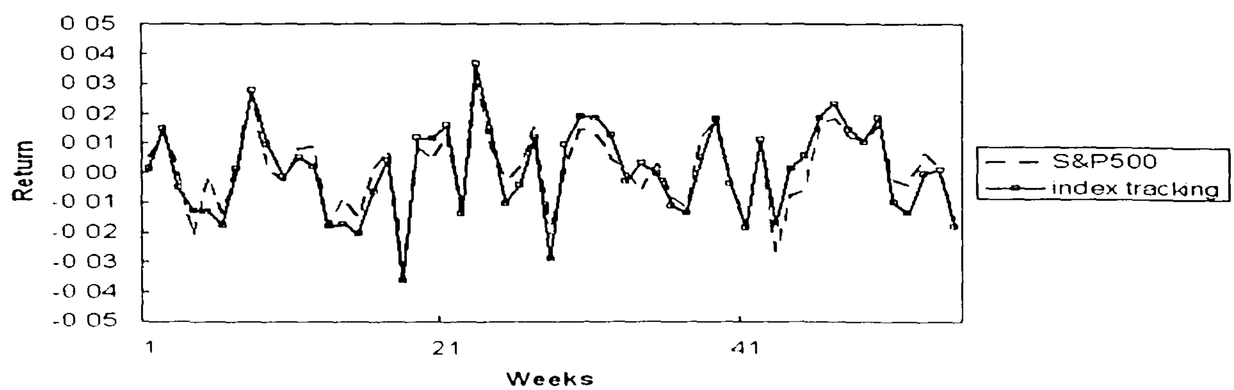

Figure 4-1: *The returns on S&P500 and MAD index-tacking portfolio using the 30 largest stocks over 55 weeks*

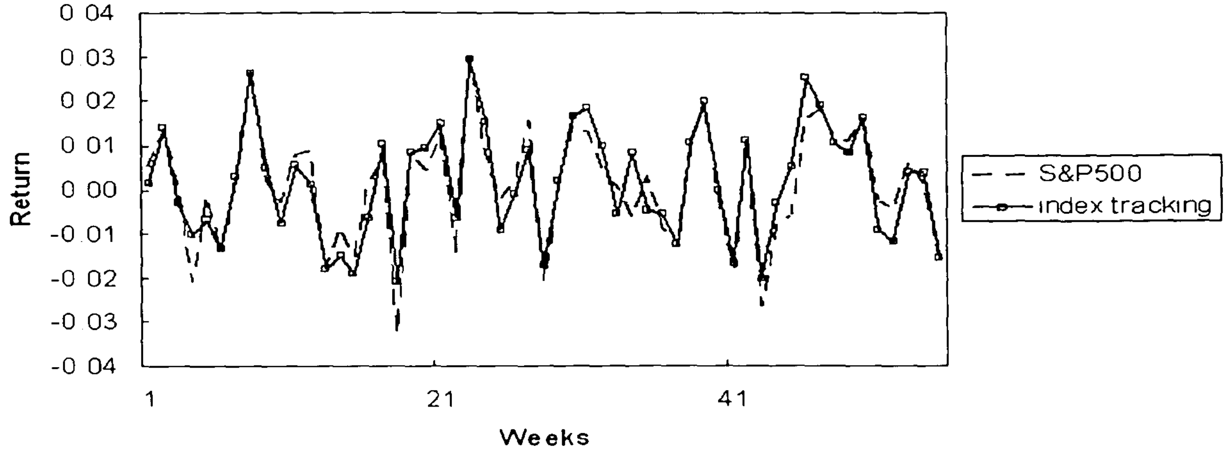

Figure 4-2: *The returns on S&P500 and MADD index-tacking portfolio using the 30 largest stocks over 55 weeks*

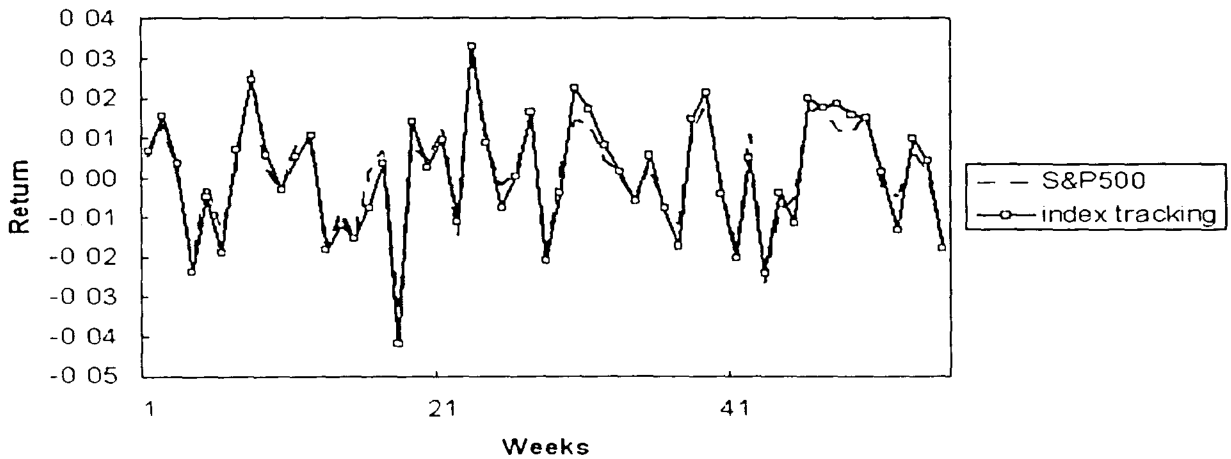

Figure 4-3: *The returns on S&P500 and MAD index-tacking portfolio using the 30 sorted stocks over 55 weeks*

Chapter 5

Conclusions

We have studied active, passive and mixed portfolio management models for two-stage and multi-period models, using mean average deviation and mean average downside deviation measures. Future work includes testing the models on larger problem sizes, with more risky assets allowed in the portfolio.

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Short Biography of the Candidate

Chach Wanapat was born on April 11th, 1982 in Khon Kaen, Thailand. He had a happy childhood, growing up with the love and support of his parents. After graduating from Triamudom High School, Bangkok, Thailand in 1999, he recieved a Bachelor of Engineering degree in electrical from Chulalongkorn University, Bangkok, Thailand in 2003. In August 2004, he began his M.S. study in Industrial and Systems engineering at Lehigh University where he worked under Professor Aurélie C. Thiele's research umbrella. His research interests include quantitative financial problems as well as stochastic problems.

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