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THE AXISYMMETRIC CRACK PROBLEM IN A SEMI-INFINITE NONHOMOGENEOUS MEDIUM

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ABSTRACT

In this study, the basic axisymmetric crack problem in a nonhomogeneous semi-infinite medium with continuously varying elastic properties is examined. The problem is encountered in studying the fracture mechanics of functionally graded materials which are mostly two-phase particulate composites with continuously varying volume fractions. The objective of this study is to determine the effect of the material nonhomogeneity parameters on the stress intensity factors in functionally graded materials containing an axisymmetric crack parallel to the surface.

Using Hankel integral transforms for the displacements in the axisymmetric crack problem, the mixed boundary conditions are analytically reduced to a system of dual integral equations, and then, by a systematic approach, to a system of singular integral equations. After converting the system of singular integral equations to a system of functional equations, such physically important quantities as stress intensity factors and crack opening displacements are obtained numerically by using certain approximate techniques.

Chapter 1

Introduction

1.1 Introduction

The various forms of composites and bonded materials have always been widely used in technological applications such as power generation, transportation, aerospace, and microelectronics. However, for the demands of future technologies, the use of homogeneous materials and standard composites is becoming more and more difficult so that a greater emphasis in current research is placed on material design; more specificaly, on developing new materials or material systems tailored for specific applications. Increasing concerns with mechanical failure initiating at the interfacial regions require a better understanding of the interaction between flaws that may exist in these regions and applied loads and the other enviromental factors. The conventional approach of studying the thermomechanics of such materials is based on the assumption that the composite medium is piecewise homogeneous and the flaws may be represented by plane cuts or cracks. On the other hand, in most bonded materials the interfacial region appears to have a structure which is generally different than that of the adjacent materials. In many cases, such as in plasma spray coating, sputtering, ion plating and in some diffusion bonded materials, the thermomechanical properties of the region are graded in the sense that the interfacial region is a nonhomogeneous continuum of finite thickness with very steep property gradients.

In the 1980's the concept of functionally graded materials (FGMs) was proposed in Japan to process thermal barrier coatings that may be used to shield the high temperature components of the space plane. FGMs for this application are composite materials with a gradual compositional variation from ceramic to metal from one surface to the other. These continuous changes result in property gradients which can be adjusted by controlling the composition. In this sense, material property grading is just another means to get optimal performance from the material. Generally, the objective of the optimal design is to provide such properties as sitiffness, strength, toughness, ductility, hardness and wear, corrosion and temperature resistance wherever needed in the structural component. In this respect the concept of FGM provides the engineer with a highly versatile tool. One of the important potential applications of FGMs is, for example, their use as an interfacial zone in bonding dissimilar materials. By eliminating the abrupt change in thermomechanical properties along the interface through property grading, it is possible not only to reduce or eliminate the stress concentrations but also to increase the bonding strength quite considerably. [1]-[9]

In this study the axisymetric crack problem for a nonhomogeneous elastic half space is considered. It is assumed that the external loads as well as geometry are axisymmetric. A brief review of the fracture problems in conventional composite materials may be found in [10]. Delale and Erdogan considered the crack problem for a nonhomogeneous plane [11] and the interface crack in a nonhomogeneous medium [12]. The axisymmetric crack problem for a nonhomogeneous infinite medium and two semi-infinite homogeneous half-spaces bonded through a nonhomogeneous interfacial zone were considered by Ozturk and Erdogan [13]-[14].

In this study, it is assumed that the shear modulus is a function of z approximated by

$$\mu(z) = \mu_0 \exp(\alpha z) .$$

This is a simple simulation of materials and interfacial zones with intentionally or naturally graded properties. With the application to fracture mechanics in mind, the main result given in this study are the stress intensity factors as a function of the nonhomogeneity parameter α and the dimensionless length parameter h/a for various loading conditions. Some sample results showing the crack opening displacements are also given.

1.2 The Organization

The statement of the problem and the description of the geometric and material parameters used in this study are given in Chapter 2 which includes formulation of the problem by using the governing differential equations and the boundary conditions. The solution of the differential equations and the derivation of the dual integral equations are given in Chapter 3. In Chapter 4 the numerical procedure used in this study is described. The results of this study which consist of crack opening displacements, normalized stress intensity factors and the effect of Poisson's ratio on stress intensity factors are given in Chapter 5. Finally, some analytical details, including the asymptotic examination of the kernels are given in the Appendices.

Chapter 2

Formulation of the Problem

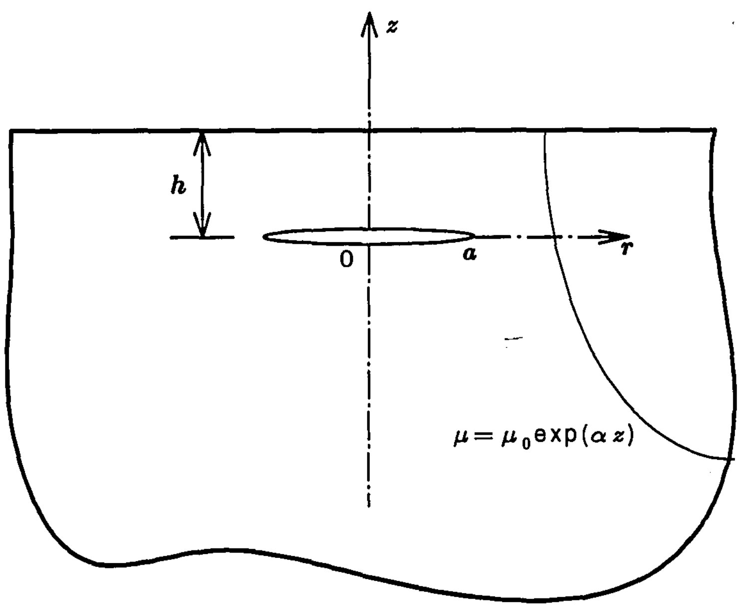

Figure 2.1: Crack geometry and notation

Consider the axisymmetric crack problem in a nonhomogeneous semi-infinite medium. It is assumed that the elastic moduli are functions of z only and are given by

$$\mu(z) = \mu_0 \exp(\alpha z), \qquad \lambda(z) = \lambda_0 \exp(\alpha z). \tag{2.1}$$

From the kinematic relations and the Hooke's law for the axisymmetric problem the nonzero stress components may be expressed as

$$\sigma_{rr} = (2\mu + \lambda)\frac{\partial u}{\partial r} + \lambda\left(\frac{u}{r} + \frac{\partial w}{\partial z}\right), \tag{2.2}$$

$$\sigma_{\theta\theta} = (2\mu + \lambda)\frac{u}{r} + \lambda\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right), \tag{2.3}$$

$$\sigma_{zz} = (2\mu + \lambda)\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right), \tag{2.4}$$

$$\sigma_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right). \tag{2.5}$$

where μ and λ are Lame's constants. In cyclindrical coordinates , the equilibrium equations are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + f_r = 0, \tag{2.6}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + f_r = 0, \tag{2.7}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + fz = 0. \tag{2.8}$$

In the absence of body forces , the equilibrium equations for the axisymmetric problem can be reduced to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \tag{2.9}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} = 0. \tag{2.10}$$

Substituting stresses which are found from equations (2.2)-(2.5) into the equilibrium equations (2.9) and (2.10), the following system of equations can be obtained

$$(\kappa + 1)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 w}{\partial r \partial z}\right) + (\kappa - 1)\alpha\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$$

$$+(\kappa-1)\left(\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 w}{\partial r\partial z}\right)=0, \tag{2.11}$$

$$(\kappa+1)\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}+\frac{\partial^2 w}{\partial z^2}\right)-(3-\kappa)\alpha\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)$$

$$(\kappa+1)\alpha\frac{\partial w}{\partial z}-(\kappa-1)\left(\frac{\partial^2 u}{\partial r\partial z}-\frac{\partial^2 w}{\partial r^2}\right)$$

$$-\frac{(\kappa-1)}{r}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\right)=0. \tag{2.12}$$

where $\kappa=3-4\nu$, $\lambda/\mu=2\nu/(1-2\nu)$, ν being the Poisson's ratio. Now , to solve the differential equations (2.11) and (2.12) we use the following Hankel transforms

$$F(z,\rho)=\int_0^\infty u(r,z)rJ_1(r\rho)\,d\rho, \tag{2.13}$$

$$G(z,\rho)=\int_0^\infty w(r,z)rJ_0(r\rho)\,d\rho. \tag{2.14}$$

The functions $u(r,z)$ and $w(r,z)$ are the r and z components of the displacement vector which are given by the following inverse transformations [15]:

$$u(r,z)=\int_0^\infty F(z,\rho)\rho J_1(r\rho)\,d\rho, \tag{2.15}$$

$$w(r,z)=\int_0^\infty G(z,\rho)\rho J_0(r\rho)\,d\rho. \tag{2.16}$$

where J_0 and J_1 are the Bessel functions of the first kind. Substituting (2.15) and (2.16) into (2.11) and (2.12) yields the following system of differential equations with constant coefficients.

$$(\kappa+1)\frac{d^2F}{dz^2}+\alpha(\kappa-1)\frac{dF}{dz}-(\kappa+1)\rho^2F-2\rho\frac{dG}{dz}-\alpha(\kappa-1)\rho G=0, \tag{2.17}$$

$$(\kappa+1)\frac{d^2G}{dz^2}+\alpha(\kappa+1)\frac{dG}{dz}-(\kappa-1)\rho^2G+2\rho\frac{dF}{dz}+\alpha(3-\kappa)\rho F=0. \tag{2.18}$$

where the following relationships have been used :

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)J_1(r\rho)=-\rho^2J_1(r\rho), \tag{2.19}$$

$$\left(\frac{d}{dr}+\frac{1}{r}\right)J_1(r\rho)=\rho J_0(r\rho), \tag{2.20}$$

$$\frac{d^2}{dr^2}J_0(r\rho)=\frac{\rho}{r}J_1(r\rho)-\rho^2J_0(r\rho). \tag{2.21}$$

Assuming a solution of the form

$$F(z,\rho)=A(\rho)e^{mz}, \tag{2.22}$$

$$G(z,\rho)=B(\rho)e^{mz}, \tag{2.23}$$

after substituting (2.22) and (2.23) into (2.17) and (2.18) , we obtain

$$F(z,\rho)=\sum_{k=1}^{4}A_k(\rho)e^{m_kz}, \tag{2.24}$$

$$G(z,\rho)=\sum_{k=1}^{4}B_k(\rho)e^{m_kz}, \tag{2.25}$$

where m_k, $(k=1,2,3,4)$ satisfies the following characteristic equation :

$$\left(m^2+\alpha m-\rho^2\right)^2+(\delta\alpha\rho)^2=0, \tag{2.26}$$

$$\delta=\frac{3-\kappa}{\kappa+1}.$$

The roots of the characteristic equation are given by

$$m_1=\frac{1}{2}\left(-\alpha+\sqrt{\alpha^2+4\rho^2+4i\alpha\delta\rho}\right), \tag{2.27}$$

$$m_2 = \frac{1}{2}\big(- \alpha - \sqrt{\alpha^2 + 4\rho^2 + 4i\alpha\delta\rho}\big), \tag{2.28}$$

$$m_3 = \frac{1}{2}\big(- \alpha + \sqrt{\alpha^2 + 4\rho^2 - 4i\alpha\delta\rho}\big), \tag{2.29}$$

$$m_4 = \frac{1}{2}\big(- \alpha - \sqrt{\alpha^2 + 4\rho^2 - 4i\alpha\delta\rho}\big). \tag{2.30}$$

From (2.27)-(2.30), it may be seen that

$$m_1 = \overline{m}_3 = \frac{1}{2}\big(- \alpha + \sqrt{\alpha^2 + 4\rho^2 + 4i\alpha\delta\rho}\big), \tag{2.31}$$

$$m_2 = \overline{m}_4 = \frac{1}{2}\big(- \alpha - \sqrt{\alpha^2 + 4\rho^2 + 4i\alpha\delta\rho}\big). \tag{2.32}$$

The arbitrary unknown functions $A_k(\rho)$ and $B_k(\rho)$ are not independent of each other. The relationship between them can be found by substituting (2.24) and (2.25) into (2.18) as follows:

$$B_k(\rho) = a_k(\rho)A_k(\rho), \qquad (k = 1, 2, 3, 4) \tag{2.33}$$

where

$$a_k(\rho) = -\frac{2m_k + \alpha(3-\kappa)}{2\rho + i\alpha(1+\kappa)\delta}, \qquad (k = 1, 2, 3, 4) \tag{2.34}$$

and

$$a_1 = \overline{a}_3 \quad , \quad a_2 = \overline{a}_4. \tag{2.35}$$

Using the relationship between $A_k(\rho)$ and $B_k(\rho)$, we find

$$F(z,\rho) = \sum_{k=1}^{4} A_k(\rho)e^{m_k z}, \tag{2.36}$$

$$G(z,\rho) = \sum_{k=1}^{4} a_k(\rho)A_k(\rho)e^{m_k z}. \tag{2.37}$$

By observing that $\Re(m_1, m_2) > 0$ and $\Re(m_2, m_4) < 0$, since both u and w vanish as $r^2 + z^2 \to \infty$, we must delete terms involving A_2 and A_4 for $z < 0$. Thus (2.36) and (2.37) reduce to

$$F_1(z,\rho) = \sum_{k=1}^{4} A_{1k}(\rho)e^{m_k z}, \qquad 0 \le z < h, \tag{2.38}$$

$$G_1(z,\rho) = \sum_{k=1}^{4} a_k(\rho)A_{1k}(\rho)e^{m_k z}, \qquad 0 \le z < h, \tag{2.39}$$

$$F_2(z,\rho) = A_{21}(\rho)e^{m_1 z} + A_{23}(\rho)e^{m_3 z}, \qquad -\infty < z \le 0, \tag{2.40}$$

$$G_2(z,\rho) = a_1(\rho)A_{21}(\rho)e^{m_1 z} + a_3(\rho)A_{23}(\rho)e^{m_3 z}, \qquad -\infty < z \le 0. \tag{2.41}$$

The coefficients A_{1k} and A_{2j}, $(k = 1, 2, 3, 4)$, $(j = 1, 3)$, can be obtained by using the following boundary and continuity conditions

$$\sigma_{zz}^{(1)}(r,h) = 0, \tag{2.42}$$

$$\sigma_{rz}^{(1)}(r,h) = 0, \tag{2.43}$$

$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0), \tag{2.44}$$

$$\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0), \tag{2.45}$$

$$w^{(1)}(r,0) - w^{(2)}(r,0) = 0, \qquad a < r < \infty, \tag{2.46}$$

$$u^{(1)}(r,0) - u^{(2)}(r,0) = 0, \qquad a < r < \infty, \tag{2.47}$$

$$\sigma_{zz}^{(2)}(r,0) = P_1(r), \qquad 0 \le r < a, \tag{2.48}$$

$$\sigma_{rz}^{(2)}(r,0) = P_2(r), \qquad 0 \le r < a. \tag{2.49}$$

From (2.42),(2.43),(2.4) and (2.5) it follows that

$$(\kappa+1)\frac{\partial G_1}{\partial z} + \rho(3-\kappa)F_1 = 0, \tag{2.50}$$

$$\frac{\partial F_1}{\partial z} - \rho G_1 = 0. \tag{2.51}$$

Similarly, from the equality of stresses at $z = 0$ and $|r| \geq 0$, it can be shown that

$$(\kappa+1)\frac{\partial}{\partial z}(G_1 - G_2) + \rho(3-\kappa)(F_1 - F_2) = 0, \tag{2.52}$$

$$\frac{\partial}{\partial z}(F_1 - F_2) - \rho(G_1 - G_2) = 0. \tag{2.53}$$

Now, by substituting from (2.38)-(2.41) into (2.46)-(2.49) we obtain

$$\sum_{k=1}^{4}(m_k a_k(\kappa+1) + \rho(3-\kappa))A_{1k}e^{m_k h} = 0, \tag{2.54}$$

$$\sum_{k=1}^{4}(m_k - \rho a_k)A_{1k}e^{m_k h} = 0, \tag{2.55}$$

$$\sum_{k=1}^{4}(m_k a_k(\kappa+1) + \rho(3-\kappa))A_{1k}$$
$$- (m_1 a_1(\kappa+1) + \rho(3-\kappa))A_{21} - (m_3 a_3(\kappa+1) + \rho(3-\kappa))A_{21} = 0, \tag{2.56}$$

$$\sum_{k=1}^{4}(m_k - \rho a_k)A_{1k} - (m_1 - \rho a_1)A_{21} - (m_3 - \rho a_3)A_{23} = 0. \tag{2.57}$$

The solution of this system of equations is,

$$A_{11} = \left(\lambda_1 E_1 e^{-2\xi h} + \overline{\lambda}_3 E_2 e^{-(\xi+\overline{\xi})h}\right)A_{21} + \left(\lambda_1 \overline{E}_2 e^{-2\xi h} + \overline{\lambda}_3 \overline{E}_1 e^{-(\xi+\overline{\xi})h}\right)A_{23}, \tag{2.58}$$

$$A_{12} = E_1 A_{21} + \overline{E}_2 A_{23}, \tag{2.59}$$

$$A_{13} = \left(\overline{\lambda}_1 E_2 e^{-2\overline{\xi} h} + \lambda_3 E_1 e^{-(\xi+\overline{\xi})h}\right) A_{21} \\ + \left(\overline{\lambda}_1 \overline{E}_1 e^{-2\overline{\xi} h} + \lambda_3 \overline{E}_2 e^{-(\xi+\overline{\xi})h}\right) A_{23}, \tag{2.60}$$

$$A_{14} = E_2 A_{21} + \overline{E}_1 A_{23}. \tag{2.61}$$

where the expression for A_{21}, A_{23}, $E_1, E_2, \lambda_1, \lambda_2$ and ξ are given in the Appendix A. The unknowns A_{21} and A_{23} may be determined from the mixed boundary conditions (2.46)-(2.49)

Chapter 3

The Integral Equations

3.1 Derivation of the Integral Equations

Referring to the previous chapter, the two remaining unknown functions $A_{21}(\rho)$ and $A_{23}(\rho)$ must be obtained by using the following mixed boundary conditions:

$$u^{(1)}(r, +0) = u^{(2)}(r, -0), \qquad (a < r < \infty), \tag{3.1}$$

$$w^{(1)}(r, +0) = w^{(2)}(r, -0), \qquad (a < r < \infty), \tag{3.2}$$

$$\sigma_{zz}^{(1)}(r, 0) = \sigma_{zz}^{(2)}(r, 0) = P_1(r), \qquad (0 \leq r < a), \tag{3.3}$$

$$\sigma_{rz}^{(1)}(r, 0) = \sigma_{rz}^{(2)}(r, 0) = P_2(r), \qquad (0 \leq r < a). \tag{3.4}$$

To reduce these conditions to a system of integral equations, we first define the following new unknown functions:

$$\phi_1(r) = \frac{\partial}{\partial r}(w(r, +0) - w(r, -0)), \qquad (0 \leq r < \infty), \tag{3.5}$$

$$\phi_2(r) = \frac{1}{r}\frac{\partial}{\partial r}(ru(r,+0) - ru(r,-0)), \qquad (0 \le r < \infty), \tag{3.6}$$

where

$$\phi_1(r) = \begin{cases} \phi_1(r) & 0 \le r < a\ , \\ 0 & a < r < \infty. \end{cases} \tag{3.7}$$

$$\phi_2(r) = \begin{cases} \phi_2(r) & 0 \le r < a\ , \\ 0 & a < r < \infty. \end{cases} \tag{3.8}$$

After substituting the equations (2.38)-(2.41) into (3.7) and (3.8), ϕ_1 and ϕ_2 may be expressed as

$$\phi_1(r) = \frac{\partial}{\partial r}\left(\int_0^\infty G_1(0,\rho)\rho J_0(r\rho)\,d\rho - \int_0^\infty G_2(0,\rho)\rho J_0(r\rho)\,d\rho\right), \tag{3.9}$$

$$\phi_2(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(\int_0^\infty F_1(0,\rho)\rho r J_0(r\rho)\,d\rho - \int_0^\infty F_2(0,\rho)\rho r J_0(r\rho)\,d\rho\right). \tag{3.10}$$

By using the following properties of Bessel functions [16]

$$\frac{\partial}{\partial r}J_0(r\rho) = -\rho J_1(r\rho), \tag{3.11}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\Big[rJ_1(r\rho)\Big] = \rho J_0(r\rho). \tag{3.12}$$

it can be shown that

$$\phi_1(r) = \int_0^\infty \Big\{G_2(0,\rho) - G_1(0,\rho)\Big\}\rho^2 J_1(r\rho)\,d\rho, \tag{3.13}$$

$$\phi_2(r) = \int_0^\infty \Big\{F_1(0,\rho) - F_2(0,\rho)\Big\}\rho^2 J_0(r\rho)\,d\rho. \tag{3.14}$$

Using inverse Hankel transformation, from (3.13) and (3.14) we find

$$G_2(0,\rho) - G_1(0,\rho) = \frac{1}{\rho}\int_0^\infty \phi_1(r) r J_1(r\rho)\,dr, \tag{3.15}$$

$$F_1(0,\rho) - F_2(0,\rho) = \frac{1}{\rho}\int_0^{\infty} \phi_2(r) r J_0(r\rho)\, dr. \tag{3.16}$$

Also by defining

$$\Phi_1(\rho) = \frac{1}{\rho}\int_0^{a} \phi_1(r) r J_1(r\rho)\, dr, \tag{3.17}$$

$$\Phi_2(\rho) = \frac{1}{\rho}\int_0^{a} \phi_2(r) r J_0(r\rho)\, dr. \tag{3.18}$$

from (3.7),(3.8),(3.15) and (3.16) it follows that

$$G_2(0,\rho) - G_1(0,\rho) = \Phi_1(\rho), \tag{3.19}$$

$$F_1(0,\rho) - F_2(0,\rho) = \Phi_2(\rho). \tag{3.20}$$

Now, by substituting from (2.38)-(2.41) into (3.19) and (3.20) it may be seen that

$$a_1 A_{21} + \overline{a}_1 A_{23} - a_1 A_{11} - a_2 A_{12} - \overline{a}_1 A_{13} - \overline{a}_2 A_{14} = \Phi_1(\rho), \tag{3.21}$$

$$A_{11} + A_{12} + A_{13} + A_{14} - A_{21} - A_{23} = \Phi_2(\rho). \tag{3.22}$$

Thus, from (2.54)-(2.57), (3.21) and (3.22), we obtain

$$b_1 A_{21} + \overline{b}_1 A_{23} = \Phi_1(\rho), \tag{3.23}$$

$$b_2 A_{21} + \overline{b}_2 A_{23} = \Phi_2(\rho), \tag{3.24}$$

or

$$A_{21} = \frac{\overline{b}_2 \Phi_1 - \overline{b}_1 \Phi_2}{\Delta_3}, \tag{3.25}$$

$$A_{23} = \frac{b_1 \Phi_2 - b_2 \Phi_1}{\Delta_3}. \tag{3.26}$$

where the functions b_1 and b_2 are defined in Appendix A. Now, by using the stress-displacement relations, (3.5) and (3.6) may be written as

$$\lim_{z\to -0}\left((\kappa+1)\frac{\partial w^2}{\partial z}+(3-\kappa)\left(\frac{\partial u^2}{\partial r}+\frac{u^2}{r}\right)\right)=\frac{(\kappa-1)}{\mu_0}P_1(r), \tag{3.27}$$

$$\lim_{z\to -0}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)=\frac{1}{\mu_0}P_2(r). \tag{3.28}$$

Then, from (2.15),(2.16),(3.27) and (3.28) it follows that

$$\lim_{z\to -0}\int_0^\infty \left((\kappa+1)\frac{\partial G_2}{\partial z}+(3-\kappa)\rho F_2\right)\rho J_0(r\rho)\,d\rho=\frac{(\kappa-1)}{\mu_0}P_1(r), \tag{3.29}$$

$$\lim_{z\to -0}\int_0^\infty \left(\frac{\partial F_2}{\partial z}-\rho G_2\right)\rho J_1(r\rho)\,d\rho=\frac{1}{\mu_0}P_2(r). \tag{3.30}$$

Now, by substituting from (2.40) and (2.41) into equations (3.29) and (3.30), after some manipulations we obtain

$$\int_0^\infty (d_{11}(\rho)\Phi_1+d_{12}(\rho)\Phi_2)\rho J_0(r\rho)\,d\rho=\frac{(\kappa-1)}{\mu_0}P_1(r), \tag{3.31}$$

$$\int_0^\infty (d_{21}(\rho)\Phi_1+d_{22}(\rho)\Phi_2)\rho J_1(r\rho)\,d\rho=\frac{1}{\mu_0}P_2(r), \tag{3.32}$$

where the function d_{ij} are given in Appendix A. Referring to (3.17) and (3.18), from (3.31) and (3.32) we finally obtain the following integral equations for the unknown functions ϕ_1 and ϕ_2 as follows :

$$\int_0^a (K_{11}(s,r)\phi_1(s)+K_{12}(s,r)\phi_2(s))s\,ds=\frac{(\kappa-1)}{\mu_0}P_1(r), \qquad (0\le r<a), \tag{3.33}$$

$$\int_0^a (K_{21}(s,r)\phi_1(s)+K_{22}(s,r)\phi_2(s))s\,ds=\frac{1}{\mu_0}P_2(r), \qquad (0\le r<a), \tag{3.34}$$

$$K_{11}(s,r) = \int_0^\infty d_{11}(\rho) J_0(r\rho) J_1(s\rho)\, d\rho\,, \tag{3.35}$$

$$K_{12}(s,r) = \int_0^\infty d_{12}(\rho) J_0(r\rho) J_0(s\rho)\, d\rho\,, \tag{3.36}$$

$$K_{21}(s,r) = \int_0^\infty d_{21}(\rho) J_1(r\rho) J_1(s\rho)\, d\rho\,, \tag{3.37}$$

$$K_{22}(s,r) = \int_0^\infty d_{22}(\rho) J_1(r\rho) J_0(s\rho)\, d\rho\,. \tag{3.38}$$

To make the asymptotic expansions somewhat more convenient we define $d_{ij} = \rho d'_{ij}$, $(i, j = 1, 2)$. The kernels of the integral equations may then be written as

$$K_{11}(s,r) = \int_0^\infty d'_{11}(\rho) J_0(r\rho) J_1(s\rho)\rho\, d\rho\,, \tag{3.39}$$

$$K_{12}(s,r) = \int_0^\infty d'_{12}(\rho) J_0(r\rho) J_0(s\rho)\rho\, d\rho\,, \tag{3.40}$$

$$K_{21}(s,r) = \int_0^\infty d'_{21}(\rho) J_1(r\rho) J_1(s\rho)\rho\, d\rho\,, \tag{3.41}$$

$$K_{22}(s,r) = \int_0^\infty d'_{22}(\rho) J_1(r\rho) J_0(s\rho)\rho\, d\rho\,. \tag{3.42}$$

where d'_{ij}, $(i, j = 1, 2)$, are defined in Appendix B. Now, by using the asymptotic results given in Appendix B the kernels of the integral equations (3.33) and (3.34) may be expressed as

$$K_{11}(s,r) =$$

$$d'^{\infty}_{11}\left(\int_0^\infty \left(\frac{d'_{11}(\rho)}{d'^{\infty}_{11}} - 1\right) J_0(r\rho) J_1(s\rho)\rho\, d\rho + \int_0^\infty J_0(r\rho) J_1(s\rho)\rho\, d\rho\right), \tag{3.43}$$

$$K_{12}(s,r) = d'^{\infty}_{11}\int_0^\infty \frac{d'_{12}(\rho)}{d'^{\infty}_{11}} J_0(r\rho)J_0(s\rho)\rho\, d\rho\,, \tag{3.44}$$

$$K_{21}(s,r) = d'^{\infty}_{22}\int_0^\infty \frac{d'_{21}(\rho)}{d'^{\infty}_{22}} J_1(r\rho)J_1(s\rho)\rho\, d\rho, \tag{3.45}$$

$$K_{22}(s,r) =$$

$$d'^{\infty}_{22}\left(\int_0^\infty J_1(r\rho)J_1(s\rho)\rho\, d\rho + \int_0^\infty \left(\frac{d'_{22}(\rho)}{d'^{\infty}_{22}} - 1\right) J_1(r\rho)J_0(s\rho)\rho\, d\rho\right), \tag{3.46}$$

$$d'^{\infty}_{11} = 2\frac{(\kappa-1)}{(\kappa+1)}, \tag{3.47}$$

$$d'^{\infty}_{22} = -2\frac{1}{(\kappa+1)}. \tag{3.48}$$

Also, defining the functions D_{ij}, $(i,j = 1,2)$ by (see Appendix B)

$$D_{11}(\rho) = \frac{d'_{11}(\rho)}{d'^{\infty}_{11}} - 1\,, \tag{3.49}$$

$$D_{12}(\rho) = \frac{d'_{12}(\rho)}{d'^{\infty}_{11}}\,, \tag{3.50}$$

$$D_{21}(\rho) = \frac{d'_{21}(\rho)}{d'^{\infty}_{22}}\,, \tag{3.51}$$

$$D_{22}(\rho) = \frac{d'_{22}(\rho)}{d'^{\infty}_{22}} - 1\,, \tag{3.52}$$

equations (3.33) and (3.34) become

$$\int_0^a \left(\int_0^\infty D_{11}(\rho)J_0(r\rho)J_1(s\rho)\rho\, d\rho\right) s\phi_1(s)\, ds$$

$$+\int_0^a\left(\int_0^\infty D_{12}(\rho)J_0(r\rho)J_0(s\rho)\rho\,d\rho\right)s\phi_2(s)\,ds$$

$$+\int_0^a\left(\int_0^\infty J_0(r\rho)J_1(s\rho)\rho\,d\rho\right)s\phi_1(s)\,ds=\frac{(\kappa+1)}{2\mu_0}P_1(r), \qquad (3.53)$$

$$\int_0^a\left(\int_0^\infty D_{21}(\rho)J_1(r\rho)J_1(s\rho)\rho\,d\rho\right)s\phi_1(s)\,ds$$

$$+\int_0^a\left(\int_0^\infty D_{22}(\rho)J_1(r\rho)J_0(s\rho)\rho\,d\rho\right)s\phi_2(s)\,ds$$

$$+\int_0^a\left(\int_0^\infty J_1(r\rho)J_0(s\rho)\rho\,d\rho\right)s\phi_2(s)\,ds=-\frac{(\kappa+1)}{2\mu_0}P_2(r). \qquad (3.54)$$

By examining the singular behaviour of the kernels and by separating the leading terms, the integral equations (3.53) and (3.54) may now be expressed as (see Appendix C)

$$\frac{1}{\pi}\int_0^a\left(\frac{1}{s-r}+\frac{1}{s+r}\right)\phi_1(s)\,ds+\frac{1}{\pi}\int_0^a\sum_{j=1}^{2}k_{1j}(s,r)\phi_j(s)\,ds$$

$$=\frac{(\kappa+1)}{2\mu_0}P_1(r), \qquad (3.55)$$

$$\frac{1}{\pi}\int_0^a\left(\frac{1}{s-r}-\frac{1}{s+r}\right)\phi_2(s)\,ds+\frac{1}{\pi}\int_0^a\sum_{j=1}^{2}k_{2j}(s,r)\phi_j(s)\,ds$$

$$=-\frac{(\kappa+1)}{2\mu_0}P_2(r), \qquad (3.56)$$

where

$$k_{11}(s,r)=\frac{M_2(s,r)-1}{s-r}+\frac{M_2(s,r)-1}{s+r}+\pi s\int_0^\infty D_{11}(\rho)J_0(r\rho)J_1(s\rho)\rho\,d\rho, \qquad (3.57)$$

$$k_{12}(s,r)=\pi s\int_0^\infty D_{12}(\rho)J_0(r\rho)J_0(s\rho)\rho\,d\rho, \qquad (3.58)$$

$$k_{21}(s,r) = \pi s \int_0^\infty D_{21}(\rho) J_1(r\rho) J_1(s\rho) \rho \, d\rho, \tag{3.59}$$

$$k_{22}(s,r) = \frac{M_4(s,r)-1}{s-r} - \frac{M_4(s,r)-1}{s+r} + \pi s \int_0^\infty D_{22}(\rho) J_1(r\rho) J_0(s\rho) \rho \, d\rho; \tag{3.60}$$

and $M_2(s,r)$ and $M_4(s,r)$ are defined in Appendix C. Note that the dominant kernels of the system of integral equations (3.55) and (3.56) are of the generalized Cauchy type [17]. The domain of integration can be extended from $(0,a)$ to $(-a,a)$ by using the following symmetry properties of functions $\phi_1(s)$ and $\phi_2(s)$

$$\phi_1(s) = -\phi_1(-s), \tag{3.61}$$

$$\phi_2(s) = \phi_2(-s). \tag{3.62}$$

Thus, by observing that

$$\frac{1}{\pi} \int_0^a \left(\frac{1}{s-r} + \frac{1}{s+r} \right) \phi_1(s) \, ds = \frac{1}{\pi} \int_{-a}^a \frac{\phi_1(s)}{s-r} \, ds. \tag{3.63}$$

$$\frac{1}{\pi} \int_0^a \left(\frac{1}{s-r} - \frac{1}{s+r} \right) \phi_2(s) \, ds = \frac{1}{\pi} \int_{-a}^a \frac{\phi_2(s)}{s-r} \, ds. \tag{3.64}$$

The integral equations (3.55) and (3.56) may be expressed as follows :

$$\frac{1}{\pi} \int_{-a}^a \frac{\phi_1(s)}{s-r} \, ds + \frac{1}{\pi} \int_0^a \sum_{j=1}^2 k_{1j}(s,r) \phi_j(s) \, ds = \frac{(\kappa+1)}{2\mu_0} P_1(r), \qquad 0 \le r < a \tag{3.65}$$

$$\frac{1}{\pi} \int_{-a}^a \frac{\phi_2(s)}{s-r} \, ds + \frac{1}{\pi} \int_0^a \sum_{j=1}^2 k_{2j}(s,r) \phi_j(s) \, ds = -\frac{(\kappa+1)}{2\mu_0} P_2(r), \quad 0 \le r < a \tag{3.66}$$

where the Fredholm kernels $k_{ij}(s,r)$, $(i,j=1,2)$, are given by (3.57)-(3.60). In solving equations such as (3.65) and (3.66), the accuracy is very higly dependent on the correct evaluation of the kernels k_{ij}, $(i,j=1,2)$. For this, it is necessary that the asymptotic behavior of k_{ij} for $s \to r$ be examined and the weak singularities, if any, be separated. Referring to Appendix C, the complete elliptic integral of the first kind has the behavior [18]

$$K(\eta) = \log \frac{4}{\sqrt{1-\eta^2}} \qquad \text{for } \eta \to 1. \tag{3.67}$$

Thus, as $s \to r$, it was shown in Appendix C that the kernels k_{ij} have logarithmic singularities which may be extracted as follows :

$$\frac{M_2(s,r)-1}{s-r} = -\frac{1}{2r}\log|s-r| - \frac{1}{r}(1-\log\sqrt{8r}) + m_{22}(s,r), \tag{3.68}$$

$$\frac{M_4(s,r)-1}{s-r} = \frac{1}{2r}\log|s-r| + \frac{1}{r}(2-\log\sqrt{8r}) + m_{22}(s,r), \tag{3.69}$$

$$k_{12}(s,r) = \pi s \int_0^\infty \left(D_{12}(\rho)\rho - \frac{\alpha}{2}\right) J_0(r\rho)J_0(s\rho)\,d\rho$$
$$+\pi s\frac{\alpha}{2}\int_0^\infty J_0(r\rho)J_0(s\rho)\,d\rho, \tag{3.70}$$

$$k_{21}(s,r) = \pi s \int_0^\infty \left(D_{21}(\rho)\rho - \frac{\alpha}{2}\right) J_1(r\rho)J_1(s\rho)\,d\rho$$
$$+\pi s\frac{\alpha}{2}\int_0^\infty J_1(r\rho)J_1(s\rho)\,d\rho, \tag{3.71}$$

where the integrals of Bessel functions are given in Appendix C which, by virtue of (3.67), are also seen to have logarithmic singularities and m_{22} and m_{44} are known functions which are bounded in the closed interval $0 \leq (s, r) \leq a$.

3.2 The Fundamental Function

In equations (3.65) and (3.66), ϕ_j and k_{ij}, $(i, j = 1, 2)$, are H-continuous functions [19]. Also second terms in equations (3.65) and (3.66) are bounded functions of r. Hence, the singular behavior of ϕ may be obtained by studying only the dominant part of (3.65) and (3.66), namely

$$\frac{1}{\pi}\int_{\mathcal{L}} \frac{\phi(s)}{s-r}\, ds = P(r), \qquad \mathcal{L} = (-a, a), \qquad r \in \mathcal{L}, \tag{3.72}$$

where $P(r)$ contains the input function P_j, $(j = 1, 2)$, and the terms coming from the part of the integral equations with Fredholm kernels. Let

$$\Phi(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\phi(s)}{s-z}\, ds. \tag{3.73}$$

The boundary values of the sectionally holomorphic function, $\Phi(z)$, are related by the following Plemelj formulas [19]:

$$\Phi^+(r) - \Phi^-(r) = \phi(r), \tag{3.74}$$

$$\Phi^+(r) + \Phi^-(r) = \frac{1}{\pi i}\int_{\mathcal{L}} \frac{\phi(s)}{s-r}\, ds. \tag{3.75}$$

From the equation (3.72)-(3.75) it may be seen that

$$\Phi^+(r) + \Phi^-(r) = -iP(r), \qquad \mathcal{L} = (-a, a), \qquad r \in \mathcal{L}. \tag{3.76}$$

The solution of the Riemann-Hilbert problem vanishing at infinity may easily be expressed as

$$\Phi(z) = -\frac{X(z)}{2\pi}\int_{\mathcal{L}}\frac{P(r)}{(s-r)X^{+}(s)}\,ds + C_n(z)X(z), \tag{3.77}$$

where $X(z)$ is the fundamental solution of the problem satisfying the following homogeneous boundary conditions

$$X^{+}(r) - X^{-}(r) = 0, \qquad (|r| > a), \tag{3.78}$$

$$X^{+}(r) + X^{-}(r) = 0, \qquad (|r| < a). \tag{3.79}$$

The solution for $X(z)$, is found to be

$$X(z) = (z-a)^{\alpha_1 + i\beta_1 + \lambda_1}(z+a)^{\alpha_2 + i\beta_2 + \lambda_2}, \tag{3.80}$$

$$\alpha_1 + i\beta_1 = \frac{\log e^{i\pi}}{2\pi i} = \frac{1}{2}, \qquad \alpha_2 + i\beta_2 = \frac{\log e^{-i\pi}}{2\pi i} = -\frac{1}{2}, \tag{3.81}$$

where $X(z)$ will be taken as the branch for which $z^{-\lambda_1-\lambda_2}, X(z)\to 1$ as $z\to\infty$. The index of the problem, κ, is given by [20]

$$\kappa = -\sum_{m=1}^{2}\lambda_m, \tag{3.82}$$

where λ's are integers. For this problem index $\kappa = 1$. As z tends to infinity,

$$X(z)\to 1, \tag{3.83}$$

meaning that $C_n(z)$ is a constant, C. On the other hand, from equations (3.74) and (3.77) the solution of the integral equation (3.72) is found to be

$$\phi(r) = \Phi^+(r) - \Phi^-(r) = 2CX^+(r) - \frac{X^+(r)}{\pi}\int_{\mathcal{L}} \frac{P(s)}{(s-r)X^+(s)}\,ds, \qquad r \in \mathcal{L}, \quad (3.84)$$

where from equations (3.80) and (3.81) for $\kappa = 1$, the fundamental function of the singular integral equation is seen to be

$$X(z) = \left(z^2 - a^2\right)^{-\frac{1}{2}}. \quad (3.85)$$

After the normalization of the problem the real function may be obtained by

$$w(s) = i(-1)^{\lambda_1} X^+(s) = (1-s)^{\frac{1}{2}+\lambda_1}(1+s)^{-\frac{1}{2}+\lambda_2}. \quad (3.86)$$

Since the function ϕ has integrable singularities at ends, then $\lambda_1 = -1, \lambda_2 = 0$ in equation (3.80). Hence, equation (3.82),(3.84) and (3.86) lead to

$$\kappa = 1, \qquad X(z) = (z^2 - 1)^{-1}, \qquad w(s) = (1 - s^2)^{-1}. \quad (3.87)$$

Chapter 4

Numerical Procedure

The integral equations (3.63) and (3.64) can be solved by using the properties of Chebyshev polynomials of the first and the second kind, T_n and U_n, respectively. In the previous chapter, the fundamental function $X(z)$ was found as

$$X(z) = (z^2 - a^2)^{-\frac{1}{2}}, \tag{4.1}$$

or

$$X(s) = i\sqrt{a^2 - s^2}. \tag{4.2}$$

To use Chebyshev polynomials for solving the integral equations (3.63) and (3.64), the interval of integral from $-a$ to a should be converted to from -1 to 1. Let

$$s = a\widehat{s}, \qquad r = a\widehat{r}, \qquad \text{and} \qquad a\rho = \widehat{\rho}. \tag{4.3}$$

Then, from (3.87)

$$w = \sqrt{1 - \widehat{s}^2} \tag{4.4}$$

becomes the corresponding weight function and the unknown functions $\phi_1(s)$ and $\phi_2(s)$ may be expressed as follows:

$$\phi_1(\widehat{s}) = \frac{F_1(\widehat{s})}{\sqrt{1-\widehat{s}^2}}, \qquad (-1 < \widehat{s} < 1), \tag{4.5}$$

$$\phi_2(\widehat{s}) = \frac{F_2(\widehat{s})}{\sqrt{1-\widehat{s}^2}}, \qquad (-1 < \widehat{s} < 1). \tag{4.6}$$

The bounded functions $F_1(\widehat{s})$ and $F_2(\widehat{s})$ may be expressed as

$$F_1(\widehat{s}) = \sum_{m=0}^{\infty} A_m T_m(\widehat{s}), \tag{4.7}$$

$$F_2(\widehat{s}) = \sum_{m=0}^{\infty} B_m T_m(\widehat{s}), \tag{4.8}$$

where A_m and B_m are unknown coefficients. By using the following conditions of compatibility

$$\int_{-1}^{1} \phi_1(\widehat{s})\, d\widehat{s} = 0, \tag{4.9}$$

$$\int_{-1}^{1} \widehat{s}\phi_2(\widehat{s})\, d\widehat{s} = 0, \tag{4.10}$$

and substituting, for example, the following representation of the density function $\phi_1(\widehat{s})$ in terms of Chebychev polynomials

$$\phi_1(\widehat{s}) = \sum_{m=0}^{N} A_m \frac{T_m(\widehat{s})}{\sqrt{1-\widehat{s}^2}}, \tag{4.11}$$

into (4.9), and noting that $T_0(\widehat{s}) = 1$, we obtain

$$\sum_{m=0}^{N} A_m \int_{-1}^{1} \frac{T_m(\widehat{s})\, T_0(\widehat{s})}{\sqrt{1-\widehat{s}^2}}\, d\widehat{s} = 0. \tag{4.12}$$

Since the Chebychev polynomial of the first kind are orthogonal with respect to the weight function $\frac{1}{\sqrt{1-\widehat{s}^2}}$, we conclude that

$$A_0 = 0. \tag{4.13}$$

Similarly, substituting

$$\phi_2(\widehat{s}) = \sum_{m=0}^{N} B_m \frac{T_m(\widehat{s})}{\sqrt{1-\widehat{s}^2}} \tag{4.14}$$

into (4.10), noting that $T_1(\widehat{s}) = \widehat{s}$, we obtain

$$\sum_{m=0}^{N} B_m \int_{-1}^{1} \frac{T_m(\widehat{s})\, T_1(\widehat{s})}{\sqrt{1-\widehat{s}^2}}\, d\widehat{s} = 0. \tag{4.15}$$

Then, because of the orthogonality we obtain

$$B_1 = 0. \tag{4.16}$$

By using symmetry considerations the bounded functions $F_1(\widehat{s})$ and $F_2(\widehat{s})$ may be expanded as

$$F_1(\widehat{s}) = \sum_{n=1}^{\infty} A_{2n-1} T_{2n-1}(\widehat{s}), \tag{4.17}$$

$$F_2(\widehat{s}) = \sum_{n=0}^{\infty} B_{2n} T_{2n}(\widehat{s}). \tag{4.18}$$

Now substituting (4.17) and (4.18) into (3.65) and (3.66), truncating the series at Nth term, the first part of the integral equations, which have the Cauchy type of singularity, becomes

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi_1(a\widehat{s})}{\widehat{s}-\widehat{r}}\,d\widehat{s} = \frac{1}{\pi}\sum_{n=1}^{N}A_{2n-1}\int_{-1}^{1}\frac{T_{2n-1}(\widehat{s})}{(\widehat{s}-\widehat{r})\sqrt{1-\widehat{s}^2}}\,d\widehat{s}, \tag{4.19}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi_2(a\widehat{s})}{\widehat{s}-\widehat{r}}\,d\widehat{s} = \frac{1}{\pi}\sum_{n=0}^{N}B_{2n}\int_{-1}^{1}\frac{T_{2n}(\widehat{s})}{(\widehat{s}-\widehat{r})\sqrt{1-\widehat{s}^2}}\,d\widehat{s}. \tag{4.20}$$

The integrals in (4.19) and (4.20) are given by [18]

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_j(\widehat{s})}{(\widehat{s}-\widehat{r})\sqrt{1-\widehat{s}^2}}\,d\widehat{s} = \begin{cases} 0, & j=0, \quad -1<\widehat{r}<1, \\ U_{j-1}(\widehat{r}), & j>0, \quad -1<\widehat{r}<1, \\ -\dfrac{\left(\widehat{r}-\dfrac{|\widehat{r}|}{\widehat{r}}\sqrt{\widehat{r}^2-1}\right)^j}{\dfrac{|\widehat{r}|}{\widehat{r}}\sqrt{\widehat{r}^2-1}}, & -1<\widehat{r}<1. \end{cases} \tag{4.21}$$

For the second part of, for example, the integral equation (3.63), it can be shown that

$$\frac{a}{\pi}\int_{0}^{1}\left(\widehat{k}_{11}(a\widehat{s},a\widehat{r})\phi_1(a\widehat{s})+\widehat{k}_{12}(a\widehat{s},a\widehat{r})\phi_2(a\widehat{s})\right)d\widehat{s}$$

$$=\frac{a}{\pi}\int_{0}^{1}\left(\widehat{k}_{11}(a\widehat{s},a\widehat{r})\frac{F_1(\widehat{s})}{\sqrt{1-\widehat{s}^2}}+\widehat{k}_{12}(a\widehat{s},a\widehat{r})\frac{F_2(\widehat{s})}{\sqrt{1-\widehat{s}^2}}\right)d\widehat{s}, \tag{4.22}$$

where

$$a\widehat{k}_{11}(a\widehat{s},a\widehat{r}) = \frac{M_2(a\widehat{s},a\widehat{r})-1}{\widehat{s}-\widehat{r}}+\frac{M_2(a\widehat{s},a\widehat{r})-1}{\widehat{s}+\widehat{r}}+$$

$$a^2\pi\widehat{s}\int_0^\infty D_{11}(\rho)J_0(a\widehat{r}\rho)J_1(a\widehat{s}\rho)\rho\,d\rho, \tag{4.23}$$

$$M_2(s,r) = M_2(a\widehat{s},a\widehat{r}). \tag{4.24}$$

Let

$$D_{11}\left(\frac{\widehat{\rho}}{a}\right) = \widehat{D}_{11}(\widehat{\rho}). \tag{4.25}$$

Then

$$a\widehat{k}_{11}(a\widehat{s},a\widehat{r}) = \frac{M_2(a\widehat{s},a\widehat{r})-1}{\widehat{s}-\widehat{r}} + \frac{M_2(a\widehat{s},a\widehat{r})-1}{\widehat{s}-\widehat{r}} +$$

$$\pi\widehat{s}\int_0^\infty \widehat{D}_{11}(\widehat{\rho})J_0(\widehat{r}\widehat{\rho})J_1(\widehat{s}\widehat{\rho})\widehat{\rho}\,d\widehat{\rho}\,. \tag{4.26}$$

Simlarly, if we let

$$D_{12}\left(\frac{\widehat{\rho}}{a}\right) = \widehat{D}_{12}(\widehat{\rho}), \tag{4.27}$$

it may be seen that

$$a\widehat{k}_{12}(a\widehat{s},a\widehat{r}) = \pi\widehat{s}\int_0^\infty \widehat{D}_{12}(\widehat{\rho})J_0(\widehat{r}\widehat{\rho})J_0(\widehat{s}\widehat{\rho})\widehat{\rho}\,d\widehat{\rho}\,. \tag{4.28}$$

Also, we observe that

$$\frac{a}{\pi}\int_0^1 \Big(\widehat{k}_{21}(a\widehat{s},a\widehat{r})\phi_1(a\widehat{s}) + \widehat{k}_{22}(a\widehat{s},a\widehat{r})\phi_2(a\widehat{s})\Big)\,d\widehat{s}$$

$$= \frac{1}{\pi}\int_0^1 \left(\widehat{k}_{21}(a\widehat{s},a\widehat{r})\frac{F_1(\widehat{s})}{\sqrt{1-\widehat{s}^2}} + \widehat{k}_{22}(a\widehat{s},a\widehat{r})\frac{F_2(\widehat{s})}{\sqrt{1-\widehat{s}^2}}\right)d\widehat{s}\,, \tag{4.29}$$

where

$$a\widehat{k}_{21}(a\widehat{s},a\widehat{r}) = a^2\pi\widehat{s}\int_0^\infty D_{21}(\rho)J_1(a\widehat{r}\rho)J_1(a\widehat{s}\rho)\rho\,d\rho. \tag{4.30}$$

$$a\widehat{k}_{22}(a\widehat{s}, a\widehat{r}) = \frac{M_4(a\widehat{s}, a\widehat{r}) - 1}{\widehat{s} - \widehat{r}} - \frac{M_4(a\widehat{s}, a\widehat{r}) - 1}{\widehat{s} + \widehat{r}} +$$

$$a^2 \pi \widehat{s} \int_0^\infty D_{22}(\rho) J_1(a\widehat{r}\rho) J_0(a\widehat{s}\rho) \rho \, d\rho, \tag{4.31}$$

$$M_4(s, r) = M_4(a\widehat{s}, a\widehat{r}). \tag{4.32}$$

Thus by letting

$$D_{21}\left(\frac{\widehat{\rho}}{a}\right) = \widehat{D}_{21}(\widehat{\rho}), \tag{4.33}$$

$$D_{22}\left(\frac{\widehat{\rho}}{a}\right) = \widehat{D}_{22}(\widehat{\rho}), \tag{4.34}$$

it may be seen that

$$a\widehat{k}_{21}(a\widehat{s}, a\widehat{r}) = \pi \widehat{s} \int_0^\infty \widehat{D}_{21}(\widehat{\rho}) J_1(\widehat{r}\widehat{\rho}) J_1(\widehat{s}\widehat{\rho}) \widehat{\rho} \, d\widehat{\rho}, \tag{4.35}$$

$$a\widehat{k}_{22}(a\widehat{s}, a\widehat{r}) = \frac{M_4(a\widehat{s}, a\widehat{r}) - 1}{\widehat{s} - \widehat{r}} - \frac{M_4(a\widehat{s}, a\widehat{r}) - 1}{\widehat{s} + \widehat{r}} +$$

$$\pi \widehat{s} \int_0^\infty \widehat{D}_{22}(\widehat{\rho}) J_1(\widehat{r}\widehat{\rho}) J_0(\widehat{s}\widehat{\rho}) \widehat{\rho} \, d\widehat{\rho}. \tag{4.36}$$

Referring to (3.66) and (3.67), the kernels $\widehat{k}_{ij}$, $(i, j = 1, 2)$, have the following asymptotic behavior :

$$\frac{M_2(a\widehat{s}, a\widehat{r}) - 1}{\widehat{s} - \widehat{r}} = -\frac{1}{2a\widehat{r}} \log|a\widehat{s} - a\widehat{r}| - \frac{1}{a\widehat{r}}(1 - \log\sqrt{8a\widehat{r}}) + m_{22}(a\widehat{s}, a\widehat{r}), \tag{4.37}$$

$$\frac{M_4(a\widehat{s}, a\widehat{r}) - 1}{\widehat{s} - \widehat{r}} = \frac{1}{2a\widehat{r}} \log|a\widehat{s} - a\widehat{r}| + \frac{1}{a\widehat{r}}(2 - \log\sqrt{8a\widehat{r}}) + m_{22}(a\widehat{s}, a\widehat{r}), \tag{4.38}$$

$$a\widehat{k}_{12}(a\widehat{s},a\widehat{r}) = \pi\widehat{s}\int_0^\infty \left(\widehat{D}_{12}(\widehat{\rho})\widehat{\rho} - \frac{\widehat{\alpha}}{2}\right) J_0(\widehat{r}\widehat{\rho})J_0(\widehat{s}\widehat{\rho})\,d\widehat{\rho}$$

$$+\pi\widehat{s}\frac{\widehat{\alpha}}{2}\int_0^\infty J_0(\widehat{r}\widehat{\rho})J_0(\widehat{s}\widehat{\rho})\,d\widehat{\rho}, \tag{4.39}$$

$$a\widehat{k}_{21}(a\widehat{s},a\widehat{r}) = \pi\widehat{s}\int_0^\infty \left(\widehat{D}_{21}(\widehat{\rho})\widehat{\rho} - \frac{\widehat{\alpha}}{2}\right) J_1(\widehat{r}\widehat{\rho})J_1(\widehat{s}\widehat{\rho})\,d\widehat{\rho}$$

$$+\pi\widehat{s}\frac{\widehat{\alpha}}{2}\int_0^\infty J_1(\widehat{r}\widehat{\rho})J_1(\widehat{s}\widehat{\rho})\,d\widehat{\rho}, \tag{4.40}$$

where

$$\widehat{\alpha} = a\alpha.$$

Now, by using the equation (4.17) and (4.18), the integral equations may be approximated by

$$\sum_{n=1}^{N} A_{2n-1}\,U_{2n-2} + \sum_{n=1}^{N} A_{2n-1}\,\frac{1}{\pi}\int_0^1 \widehat{k}_{11}(a\widehat{s},a\widehat{r})\frac{T_{2n-1}(\widehat{s})}{\sqrt{1-\widehat{s}^2}}\,d\widehat{s}$$

$$+\sum_{n=0}^{N} B_{2n}\,\frac{1}{\pi}\int_0^1 \widehat{k}_{12}(a\widehat{s},a\widehat{r})\frac{T_{2n}(\widehat{s})}{\sqrt{1-\widehat{s}^2}}\,d\widehat{s} = \frac{(\kappa+1)}{2\mu_0}P_1(a\widehat{r}), \tag{4.41}$$

$$\sum_{n=0}^{N} B_{2n}\,U_{2n-1} + \sum_{n=1}^{N} A_{2n-1}\,\frac{1}{\pi}\int_0^1 \widehat{k}_{21}(a\widehat{s},a\widehat{r})\frac{T_{2n-1}(\widehat{s})}{\sqrt{1-\widehat{s}^2}}\,d\widehat{s}$$

$$+\sum_{n=0}^{N} B_{2n}\frac{1}{\pi}\int_0^1 \widehat{k}_{22}(a\widehat{s},a\widehat{r})\frac{T_{2n}(\widehat{s})}{\sqrt{1-\widehat{s}^2}}\,d\widehat{s} = -\frac{(\kappa+1)}{2\mu_0}P_2(a\widehat{r}). \tag{4.42}$$

Because of the nature of the problem it is necessary to increase the density of the collocation points near the ends $r = \mp 1$. Thus, these points may be selected as follows [17]:

$$T_N(r_i) = 0, \quad r_i = \cos\left(\frac{(2i-1)\pi}{2N}\right), \qquad i = 1, 2, ..., N. \tag{4.43}$$

Using the definition of Chebyshev polynomial, $T_n(t)$

$$T_n(x) = \cos(n \arccos(x)), \tag{4.44}$$

and letting

$$\widehat{s} = \cos\theta, \qquad (0 < \theta < \pi), \tag{4.45}$$

equations (4.31) and (4.32) may be written as

$$\sum_{n=1}^{N} A_{2n-1} U_{2n-2}(r)$$

$$+\sum_{n=1}^{N} A_{2n-1} \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \widehat{k}_{11}(a\cos\theta, a\widehat{r}) \cos[(2n-1)\theta]\, d\theta$$

$$+\sum_{n=0}^{N} B_{2n} \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \widehat{k}_{12}(a\cos\theta, a\widehat{r}) \cos(2n\theta)\, d\theta = \frac{(\kappa+1)}{2\mu_0} P_1(a\widehat{r}), \tag{4.46}$$

$$\sum_{n=1}^{N} B_{2n} U_{2n-1}(r)$$

$$+\sum_{n=1}^{N} A_{2n-1}\,\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\widehat{k}_{21}(a\cos\theta, a\widehat{r})\cos[(2n-1)\theta]\,d\theta$$

$$+\sum_{n=0}^{N} B_{2n}\,\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\widehat{k}_{22}(a\cos\theta, a\widehat{r})\cos(2n\theta)\,d\theta = -\frac{(\kappa+1)}{2\mu_0}P_2(a\widehat{r}) \qquad (4.47)$$

Finally, after evaluating the integrals 0 to $\frac{\pi}{2}$ in (4.46) and (4.47), the problem reduces to a system of algebraic equations of the form

$$\sum_{n=1}^{N} A_{2n-1}a_{in}(\widehat{r}_i) + \sum_{n=0}^{N} B_{2n}b_{in}(\widehat{r}_i) = \widehat{P}_1(\widehat{r}_i), \qquad (i=1,2,.....,N), \qquad (4.48)$$

$$\sum_{n=1}^{N} A_{2n-1}c_{in}(\widehat{r}_i) + \sum_{n=0}^{N} B_{2n}d_{in}(\widehat{r}_i) = \widehat{P}_2(\widehat{r}_i), \qquad (i=1,2,.....,N), \qquad (4.49)$$

where $\widehat{r}_i, (i=1,2,....,N)$ are appropriate collocation points. In order to obtain a $2N$ by $2N$ system of linear algebraic equations , the coefficient B_0 should be defined in terms of B_{2i}, $(i=1,2,3,...,N)$. Thus, by using the following equality :

$$\sum_{n=0}^{N} B_{2n}\int_0^1 \frac{T_n(\widehat{s})\,T_1(\widehat{s})}{\sqrt{1-\widehat{s}^2}}\,d\widehat{s} = \sum_{n=0}^{N} B_{2n}\left(\frac{(-1)^n}{4n^2-1}\right), \qquad (4.50)$$

from (4.15) it follows that

$$B_0(-1) + \sum_{n=1}^{N} B_{2n}\frac{(-1)^n}{4n^2-1} = 0, \qquad (4.51)$$

and

$$B_0 = \sum_{n=1}^{N} B_{2n}\frac{(-1)^n}{4n^2-1}. \qquad (4.52)$$

Finally, we obtain the following system of linear algebraic equations such that

$$\sum_{n=1}^{N}\left[A_{2n-1}a_{in}(\widehat{r}_i)+B_{2n}\left(\frac{(-1)^n}{4n^2-1}b_{i0}+b_{in}(r_i)\right)\right]=\widehat{P}_1(\widehat{r}_i),\qquad (i=1,...,N),\quad (4.53)$$

$$\sum_{n=1}^{N}\left[A_{2n-1}c_{in}(\widehat{r}_i)+B_{2n}\left(\frac{(-1)^n}{4n^2-1}d_{i0}+d_{in}(r_i)\right)\right]=\widehat{P}_2(\widehat{r}_i),\qquad (i=1,...,N),\quad (4.54)$$

$$a_{in}(\widehat{r}_i)=U_{2n-2}(r_i)+\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\widehat{k}_{11}(a\cos\theta,a\widehat{r})\cos[(2n-1)\theta]\,d\theta,\qquad (4.55)$$

$$b_{in}(\widehat{r}_i)=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\widehat{k}_{12}(a\cos\theta,a\widehat{r})\cos(2n\theta)\,d\theta,\qquad (4.56)$$

$$c_{in}(\widehat{r}_i)=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\widehat{k}_{21}(a\cos\theta,a\widehat{r})\cos[(2n-1)\theta]\,d\theta,\qquad (4.57)$$

$$d_{in}(\widehat{r}_i)=U_{2n-1}(r_i)+\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\widehat{k}_{22}(a\cos\theta,a\widehat{r})\cos(2n\theta)\,d\theta.\qquad (4.58)$$

The solution of this system of algebraic equations would then give the coefficients $A_{2n\text{-}1}$ and B_{2n}.

From the derivation of integral equations (3.65) and (3.66), we observe that the right hand side of these integral equations represent $\sigma_{zz}(r,0)$ and $\sigma_{rz}(r,0)$ for $a<r<\infty$ as well as for $0\leq r<a$. Thus, defining the modes I and II stress intensits factors by [9]

$$k_1=\lim_{r\to a}\sqrt{2(r-a)}\sigma_{zz}(r,0),\qquad k_2=\lim_{r\to a}\sqrt{2(r-a)}\sigma_{rz}(r,0),\qquad (4.59)$$

and by using the properties of Chebychev polynomials and (3.65),(3.66) it can be shown that

$$k_1 = -\sqrt{2a}\sum_{1}^{\infty} A_{2n-1}, \qquad k_2 = -\sqrt{2a}\sum_{0}^{\infty} B_{2n}. \tag{4.60}$$

For a homogeneous infinite medium modes I and II crack problems are uncoupled and the stress intensity factors are given by

$$k_1 = -\frac{2}{\pi\sqrt{a}}\int_0^a \frac{rP_1(r)}{\sqrt{a^2-r^2}}dr, \tag{4.61}$$

$$k_2 = -\frac{2}{\pi\sqrt{a^3}}\int_0^a \frac{r^2P_2(r)}{\sqrt{a^2-r^2}}dr. \tag{4.62}$$

Chapter 5

The Results

The main results of this study are the stress intensity factors calculated for various loading conditions as functions of the nonhomogeneity constant α defined by $\mu(z) = \mu_0 \exp(\alpha z)$, and h/a which is the basic dimensionless length parameter in the problem. Table 5.1 shows the six different loading conditions used in the calculation.

Table 5.1: Loading conditions used and the corresponding stress intensity factors for the homogeneous infinite medium ($\alpha = 0$).

$P_1(r)$	$-p_0$	$-p_1\left(\frac{r}{a}\right)^2$	$-p_2\left(\frac{r}{a}\right)^4$	0	0	0
$P_2(r)$	0	0	0	$-q_0\left(\frac{r}{a}\right)$	$-q_1\left(\frac{r}{a}\right)^3$	$-q_2\left(\frac{r}{a}\right)^5$
k_1	$\frac{2}{\pi}p_0\sqrt{a}$	$\frac{4}{3\pi}p_1\sqrt{a}$	$\frac{16}{15\pi}p_2\sqrt{a}$	0	0	0
k_2	0	0	0	$\frac{4}{3\pi}q_0\sqrt{a}$	$\frac{16}{15\pi}q_1\sqrt{a}$	$\frac{32}{35\pi}q_2\sqrt{a}$

This table also shows the corresponding modes I and II stress intensity factors in a homogeneous medium containing a penny-shaped crack of radius a obtained from (4.61) and (4.62). Comparing the results given in Table 5.2 and Table 5.3 with the results of axisymetric crack problem in a nonhomogeneous infinite medium obtained by Ozturk and Erdogan [13], it may be seen that the stress intensity factors k_1 and k_2 obtained from the two solutions are almost the same for large values of the length parameter h/a.

For the problem under consideration the normalized stress intensity factors calculated for constant Poisson's ratio $(\nu = 0.3)$ and different h/a values such as $(h/a = 10., 2., 1., 0.75, 0.50, 0.25, 0.10)$ are shown in Tables 5.2 – 5.17. Note that the results given in these tables may be used to obtain the stress intensity factors for arbitrary crack surface tractions by superposition to the extent that the tractions may be approximated by a second degree polynomials in r.

After determining the coefficients A_{2n-1} and B_{2n} shown in (4.53) and (4.54), the crack opening displacements may be obtained from (3.5) and (3.6) as follows:

$$\phi_1(r) = \frac{\partial}{\partial r}\Big\{w(r,+0) - w(r,-0)\Big\} = \frac{\kappa+1}{2\mu_0}\sum_1^\infty A_{2n-1}\frac{T_{2n-1}(r/a)}{\sqrt{1-(r/a)^2}}, \tag{5.1}$$

$$\Big(w(s,+0) - w(s,-0)\Big)\Big|_{-a}^{r} = \frac{\kappa+1}{2\mu_0}\sum_1^\infty A_{2n-1}\int_{-a}^{r}\frac{T_{2n-1}(s/a)}{\sqrt{1-(s/a)^2}}ds, \tag{5.2}$$

$$w(r,+0) - w(r,-0) = \frac{\kappa+1}{2\mu_0}\sum_1^\infty A_{2n-1}\int_{-1}^{r/a}\frac{aT_{2n-1}(\hat{s})}{\sqrt{1-\hat{s}^2}}d\hat{s}, \tag{5.3}$$

by defining a new variable

$$\hat{s} = \cos\theta, \qquad \pi \leq \theta \leq \arccos(r/a), \tag{5.4}$$

right side of the equation (5.3) becomes

$$w(r,+0)-w(r,-0) = -\frac{a(\kappa+1)}{2\mu_0}\sum_1^{\infty} A_{2n-1}\int_{\pi}^{\arccos(r/a)} \cos(2n-1)\theta\, d\theta, \tag{5.5}$$

then

$$w(r,+0)-w(r,-0) = -\frac{a(\kappa+1)}{2\mu_0}\sum_1^{\infty} A_{2n-1}\frac{\sin\{(2n-1)\arccos(r/a)\}}{2n-1}. \tag{5.6}$$

Note that A_k has the dimension of stress. By using the relation

$$U_n(t) = \frac{\sin\{(n+1)\arccos t\}}{\sin(\arccos t)}, \tag{5.7}$$

it can be shown that z-component of the normalized crack opening displacement is,

$$W(r) = \frac{w(r,+0)-w(r,-0)}{w_0} = -\sqrt{1-(r/a)^2}\sum_1^{\infty}\frac{A_{2n-1}}{p_i}\frac{U_{2n-2}(r/a)}{2n-1}, \tag{5.8}$$

where

$$w_0 = \frac{(\kappa+1)}{2\mu_0} a p_i, \qquad (i=0,1,2). \tag{5.9}$$

Similarly, by using the equation (3.6) we find

$$\phi_2(r) = \frac{1}{r}\frac{\partial}{\partial r}\Big\{ru(r,+0)-ru(r,-0)\Big\} = \frac{(\kappa+1)}{2\mu_0}\sum_0^{\infty} B_{2n}\frac{T_n(r/a)}{\sqrt{1-(r/a)^2}}, \tag{5.10}$$

$$\Big(ru(s,+0)-ru(s,-0)\Big)\Big|_{-a}^{r} = \frac{(\kappa+1)}{2\mu_0}\sum_0^{\infty} B_{2n}\int_{-a}^{r}\frac{sT_n(s/a)}{\sqrt{1-(s/a)^2}}ds, \tag{5.11}$$

$$r\Big\{u(r,+0)-u(r,-0)\Big\} = \frac{a^2(\kappa+1)}{2\mu_0}B_0\int_{-1}^{r/a}\frac{\hat{s}}{\sqrt{1-\hat{s}^2}}d\hat{s}$$

$$+\frac{a^2(\kappa+1)}{2\mu_0}\sum_1^\infty B_{2n}\int_{-1}^{r/a}\frac{\widehat{s}\,T(\widehat{s})}{\sqrt{1-\widehat{s}^2}}d\widehat{s}. \tag{5.12}$$

By using the relation

$$2tT_n(t) = T_{n+1}(t) + T_{n-1}(t), \tag{5.13}$$

it may be shown that

$$r\{u(r,+0)-u(r,-0)\} = -\frac{a^2(\kappa+1)}{2\mu_0}B_0\sqrt{1-(r/a)^2}$$

$$-\frac{a^2(\kappa+1)}{4\mu_0}\sum_1^\infty B_{2n}\left\{\frac{\sin\{(2n+1)\arccos(r/a)\}}{2n+1}+\frac{\sin\{(2n-1)\arccos(r/a)\}}{2n-1}\right\}. \tag{5.16}$$

We again note that B_k has the dimension of stress. From (5.7) and (5.16) it then follows that

$$U(r) = \frac{u(r,+0)-u(r,-0)}{u_0} =$$

$$-\frac{\sqrt{1-(r/a)^2}}{2(r/a)}\left\{\frac{2B_0}{q_i}+\sum_1^\infty\frac{B_{2n}}{q_i}\left(\frac{U_{2n}(r/a)}{2n+1}+\frac{U_{2n-2}(r/a)}{2n-1}\right)\right\}, \tag{5.17}$$

where

$$u_0 = \frac{(1+\kappa)}{2\mu_0}aq_i, \qquad (i=0,1,2). \tag{5.18}$$

For different values of αa and h/a, z- and r- components of normalized crack opening displacements are given in Figures 5.11 – 5.30.

Table 5.2: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 10$.

$\sigma_{zz}(r,0) = P_1(r),\ \sigma_{rz}(r,0) = 0$						
$a\alpha$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_1\sqrt{a}}$	$\frac{k_2}{p_1\sqrt{a}}$	$\frac{k_1}{p_2\sqrt{a}}$	$\frac{k_2}{p_2\sqrt{a}}$
0.0	.6369	.0000	.4245	.0000	.3396	.0000
0.1	.6381	.0106	.4250	.0042	.3399	.0024
0.2	.6414	.0212	.4263	.0085	.3406	.0048
0.3	.6465	.0319	.4284	.0127	.3418	.0073
0.4	.6531	.0425	.4310	.0170	.3433	.0097
0.5	.6608	.0532	.4341	.0213	.3451	.0121
0.6	.6695	.0639	.4376	.0255	.3470	.0146
0.7	.6790	.0747	.4414	.0298	.3492	.0170
0.8	.6893	.0855	.4455	.0341	.3516	.0195
0.9	.7001	.0963	.4498	.0384	.3541	.0219
1.0	.7115	.1073	.4544	.0428	.3567	.0244
1.5	.7741	.1628	.4795	.0647	.3710	.0368
2.0	.8435	.2202	.5073	.0872	.3869	.0495
3.0	.9943	.3412	.5676	.1339	.4214	.0757
4.0	1.1561	.4712	.6320	.1833	.4581	.1031
5.0	1.3266	.6101	.6996	.2353	.4965	.1316

Table 5.3: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 10$.

$\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = P_2(r)$						
$a\alpha$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_1\sqrt{a}}$	$\frac{k_2}{q_1\sqrt{a}}$	$\frac{k_1}{q_2\sqrt{a}}$	$\frac{k_2}{q_2\sqrt{a}}$
0.0	.0000	.4244	.0000	.3395	.0000	.2910
0.1	.0000	.4245	.0000	.3396	.0000	.2910
0.2	.0000	.4246	.0000	.3396	.0000	.2911
0.3	.0000	.4249	.0000	.3398	.0000	.2912
0.4	.0000	.4252	.0000	.3400	.0000	.2913
0.5	.0000	.4256	.0000	.3402	.0000	.2915
0.6	.0000	.4262	.0000	.3405	.0000	.2917
0.7	.0000	.4268	.0000	.3409	.0000	.2919
0.8	.0000	.4275	.0000	.3413	.0000	.2922
0.9	.0000	.4282	.0000	.3417	.0000	.2925
1.0	.0000	.4290	.0000	.3422	.0000	.2928
1.5	.0000	.4341	.0000	.3451	.0000	.2947
2.0	.0000	.4403	.0000	.3487	.0000	.2971
3.0	.0000	.4550	.0000	.3571	.0000	.3028
4.0	.0000	.4712	.0000	.3666	.0000	.3092
5.0	.0000	.4881	.0000	.3765	.0000	.3159

Table 5.4: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 5$.

$\sigma_{zz}(r,0) = P_1(r),\ \sigma_{rz}(r,0) = 0$						
$a\alpha$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_1\sqrt{a}}$	$\frac{k_2}{p_1\sqrt{a}}$	$\frac{k_1}{p_2\sqrt{a}}$	$\frac{k_2}{p_2\sqrt{a}}$
0.0	.6392	−.0002	.4254	.0000	.3401	.0000
0.1	.6402	.0104	.4258	.0041	.3403	.0024
0.2	.6431	.0210	.4270	.0084	.3410	.0048
0.3	.6478	.0317	.4289	.0127	.3421	.0072
0.4	.6540	.0424	.4314	.0169	.3435	.0098
0.5	.6614	.0531	.4343	.0212	.3452	.0121
0.6	.6699	.0638	.4377	.0255	.3471	.0146
0.7	.6793	.0746	.4415	.0298	.3493	.0170
0.8	.6895	.0855	.4456	.0341	.3516	.0195
0.9	.7001	.0963	.4498	.0384	.3541	.0219
1.0	.7115	.1073	.4544	.0428	.3567	.0244
1.5	.7742	.1628	.4795	.0647	.3710	.0368
2.0	.8435	.2202	.5073	.0872	.3869	.0495
3.0	.9943	.3412	.5676	.1339	.4214	.0757
4.0	1.1561	.4712	.6320	.1833	.4581	.1031
5.0	1.3266	.6108	.6996	.2353	.4965	.1316

Table 5.5: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 5$.

$\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = P_2(r)$						
$a\alpha$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_1\sqrt{a}}$	$\frac{k_2}{q_1\sqrt{a}}$	$\frac{k_1}{q_2\sqrt{a}}$	$\frac{k_2}{q_2\sqrt{a}}$
0.0	−.0001	.4244	.0000	.3395	.0000	.2910
0.1	−.0001	.4245	.0000	.3396	.0000	.2911
0.2	−.0001	.4246	.0000	.3397	.0000	.2911
0.3	−.0001	.4249	.0000	.3398	.0000	.2912
0.4	−.0001	.4252	.0000	.3400	.0000	.2913
0.5	.0000	.4257	.0000	.3402	.0000	.2915
0.6	.0000	.4262	.0000	.3405	.0000	.2917
0.7	.0000	.4268	.0000	.3409	.0000	.2919
0.8	.0000	.4275	.0000	.3413	.0000	.2922
0.9	.0000	.4282	.0000	.3417	.0000	.2925
1.0	.0000	.4290	.0000	.3422	.0000	.2928
1.5	.0000	.4341	.0000	.3451	.0000	.2947
2.0	.0000	.4403	.0000	.3487	.0000	.2971
3.0	.0000	.4550	.0000	.3571	.0000	.3028
4.0	.0000	.4712	.0000	.3666	.0000	.3092
5.0	.0000	.4881	.0000	.3765	.0000	.3159

Table 5.6: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 2$.

$\sigma_{zz}(r,0) = P_1(r),\ \sigma_{rz}(r,0) = 0$						
$a\alpha$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_1\sqrt{a}}$	$\frac{k_2}{p_1\sqrt{a}}$	$\frac{k_1}{p_2\sqrt{a}}$	$\frac{k_2}{p_2\sqrt{a}}$
0.0	.6673	− .0070	.4364	− .0027	.3463	− .0015
0.1	.6679	.0037	.4367	.0016	.3464	.0009
0.2	.6698	.0145	.4374	.0059	.3469	.0034
0.3	.6730	.0253	.4387	.0102	.3476	.0059
0.4	.6773	.0363	.4405	.0146	.3486	.0084
0.5	.6828	.0473	.4427	.0190	.3499	.0109
0.6	.6893	.0584	.4453	.0234	.3514	.0134
0.7	.6967	.0695	.4483	.0278	.3531	.0159
0.8	.7049	.0807	.4516	.0323	.3550	.0185
0.9	.7139	.0920	.4552	.0368	.3571	.0210
1.0	.7236	.1033	.4591	.0412	.3593	.0235
1.5	.7803	.1604	.4818	.0638	.3723	.0363
2.0	.8465	.2189	.5084	.0867	.3875	.0493
3.0	.9949	.3409	.5678	.1338	.4215	.0756
4.0	1.1562	.4711	.6321	.1833	.4581	.1031
5.0	1.3266	.6101	.6996	.2353	.4965	.1316

Table 5.7: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 2$.

$\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = P_2(r)$						
$a\alpha$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_1\sqrt{a}}$	$\frac{k_2}{q_1\sqrt{a}}$	$\frac{k_1}{q_2\sqrt{a}}$	$\frac{k_2}{q_2\sqrt{a}}$
0.0	− .0025	.4253	− .0014	.3400	− .0009	.2913
0.1	− .0025	.4254	− .0014	.3401	− .0009	.2914
0.2	− .0025	.4255	− .0014	.3401	− .0009	.2914
0.3	− .0024	.4257	− .0013	.3403	− .0009	.2915
0.4	− .0023	.4260	− .0013	.3405	− .0009	.2916
0.5	− .0022	.4265	− .0012	.3407	− .0008	.2918
0.6	− .0020	.4269	− .0011	.3410	− .0008	.2920
0.7	− .0019	.4275	− .0011	.3413	− .0007	.2922
0.8	− .0017	.4281	− .0010	.3417	− .0007	.2924
0.9	− .0016	.4289	− .0009	.3421	− .0006	.2927
1.0	− .0015	.4296	− .0008	.3425	− .0005	.2930
1.5	− .0009	.4345	− .0005	.3453	− .0003	.2949
2.0	− .0005	.4406	− .0003	.3488	− .0002	.2972
3.0	− .0001	.4550	− .0001	.3572	.0000	.3028
4.0	.0000	.4712	.0000	.3666	.0000	.3092
5.0	.0000	.4881	.0000	.3765	.0000	.3159

Table 5.8: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 1$.

$\sigma_{zz}(r,0) = P_1(r),\ \sigma_{rz}(r,0) = 0$						
$a\alpha$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_1\sqrt{a}}$	$\frac{k_2}{p_1\sqrt{a}}$	$\frac{k_1}{p_2\sqrt{a}}$	$\frac{k_2}{p_2\sqrt{a}}$
0.0	.7781	−.0520	.4783	−.0186	.3694	−.0100
0.1	.7782	−.0410	.4783	−.0143	.3695	−.0075
0.2	.7790	−.0299	.4787	−.0099	.3697	−.0050
0.3	.7806	−.0187	.4794	−.0054	.3701	−.0025
0.4	.7829	−.0073	.4804	−.0009	.3707	−.0001
0.5	.7860	.0043	.4817	.0037	.3715	.0027
0.6	.7899	.6060	.4833	.0083	.3724	.0053
0.7	.7944	.0278	.4852	.0130	.3735	.0080
0.8	.7995	.0398	.4873	.0177	.3747	.0106
0.9	.8053	.0519	.4897	.0224	.3761	.0133
1.0	.8117	.0640	.4923	.0272	.3776	.0160
1.5	.8513	.1264	.5085	.0516	.3870	.0298
2.0	.9018	.1905	.5290	.0765	.3989	.0437
3.0	1.0262	.3230	.5794	.1274	.4278	.0722
4.0	1.1729	.4607	.6382	.1796	.4615	.1011
5.0	1.3351	.6044	.7027	.2333	.4982	.1306

Table 5.9: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 1$.

$\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = P_2(r)$						
$a\alpha$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_1\sqrt{a}}$	$\frac{k_2}{q_1\sqrt{a}}$	$\frac{k_1}{q_2\sqrt{a}}$	$\frac{k_2}{q_2\sqrt{a}}$
0.0	−.0153	.4342	−.0086	.3449	−.0057	.2945
0.1	−.0153	.4342	−.0086	.3449	−.0056	.2945
0.2	−.0152	.4344	−.0085	.3450	−.0056	.2945
0.3	−.0151	.4345	−.0085	.3451	−.0056	.2946
0.4	−.0149	.4348	−.0084	.3452	−.0055	.2947
0.5	−.0147	.4351	−.0082	.3454	−.0054	.2948
0.6	−.0144	.4355	−.0081	.3456	−.0053	.2950
0.7	−.0141	.4360	−.0079	.3459	−.0052	.2952
0.8	−.0138	.4365	−.0077	.3462	−.0051	.2954
0.9	−.0135	.4371	−.0076	.3465	−.0050	.2956
1.0	−.0131	.4377	−.0073	.3469	−.0048	.2959
1.5	−.0111	.4417	−.0062	.3492	−.0041	.2974
2.0	−.0091	.4467	−.0051	.3521	−.0033	.2994
3.0	−.0056	.4591	−.0031	.3594	−.0020	.3043
4.0	−.0032	.4737	−.0018	.3679	−.0012	.3101
5.0	−.0018	.4895	−.0010	.3772	−.0006	.3164

Table 5.10: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 0.75$.

	$\sigma_{zz}(r,0) = P_1(r),\ \sigma_{rz}(r,0) = 0$					
$a\alpha$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_1\sqrt{a}}$	$\frac{k_2}{p_1\sqrt{a}}$	$\frac{k_1}{p_2\sqrt{a}}$	$\frac{k_2}{p_2\sqrt{a}}$
0.0	.8763	−.1013	.5150	−.0352	.3896	−.0185
0.1	.8763	−.0901	.5150	−.0308	.3896	−.0160
0.2	.8766	−.0788	.5152	−.0264	.3897	−.0135
0.3	.8776	−.0672	.5156	−.0218	.3900	−.0109
0.4	.8791	−.0555	.5163	−.0172	.3904	−.0083
0.5	.8813	−.0436	.5173	−.0125	.3910	−.0057
0.6	.8841	−.0315	.5185	−.0078	.3917	−.0030
0.7	.8874	−.0193	.5199	−.0030	.3926	−.0003
0.8	.8913	−.0070	.5216	.0018	.3935	.0024
0.9	.8958	.0055	.5234	.0067	.3946	.0051
1.0	.9007	.0181	.5255	.0116	.3959	.0079
1.5	.9324	.0831	.5387	.0368	.4035	.0221
2.0	.9741	.1503	.5559	.0627	.4135	.0366
3.0	1.0813	.2902	.5996	.1161	.4388	.0663
4.0	1.2127	.4355	.6526	.1709	.4692	.0965
5.0	1.3626	.5861	.7125	.2270	.5035	.1273

Table 5.11: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 0.75$.

	$\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = P_2(r)$					
$a\alpha$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_1\sqrt{a}}$	$\frac{k_2}{q_1\sqrt{a}}$	$\frac{k_1}{q_2\sqrt{a}}$	$\frac{k_2}{q_2\sqrt{a}}$
0.0	−.0258	.4445	−.0146	.3504	−.0096	.2981
0.1	−.0258	.4445	−.0146	.3505	−.0096	.2981
0.2	−.0257	.4446	−.0145	.3505	−.0096	.2981
0.3	−.0256	.4448	−.0145	.3506	−.0096	.2982
0.4	−.0254	.4450	−.0144	.3507	−.0095	.2983
0.5	−.0252	.4453	−.0142	.3509	−.0094	.2984
0.6	−.0249	.4456	−.0141	.3511	−.0093	.2985
0.7	−.0246	.4460	−.0139	.3513	−.0092	.2988
0.8	−.0243	.4465	−.0137	.3516	−.0091	.2989
0.9	−.0239	.4470	−.0135	.3519	−.0089	.2991
1.0	−.0235	.4476	−.0133	.3522	−.0088	.2993
1.5	−.0213	.4511	−.0120	.3543	−.0079	.3007
2.0	−.0188	.4555	−.0105	.3569	−.0069	.3024
3.0	−.0139	.4665	−.0078	.3634	−.0051	.3068
4.0	−.0098	.4795	−.0055	.3711	−.0036	.3121
5.0	−.0067	.4938	−.0037	.3796	−.0024	.3179

Table 5.12: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 0.50$.

$\sigma_{zz}(r,0) = P_1(r),\ \sigma_{rz}(r,0) = 0$						
$a\alpha$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_1\sqrt{a}}$	$\frac{k_2}{p_1\sqrt{a}}$	$\frac{k_1}{p_2\sqrt{a}}$	$\frac{k_2}{p_2\sqrt{a}}$
0.0	1.1061	−.2297	.6005	−.0777	.4362	−.0400
0.1	1.1052	−.2179	.6003	−.0731	.4361	−.0374
0.2	1.1047	−.2060	.6002	−.0685	.4361	−.0348
0.3	1.1048	−.1938	.6004	−.0638	.4363	−.0321
0.4	1.1053	−.1815	.6007	−.0590	.4365	−.0295
0.5	1.1062	−.1689	.6012	−.0542	.4368	−.0268
0.6	1.1077	−.1562	.6019	−.0493	.4373	−.0240
0.7	1.1095	−.1434	.6028	−.0443	.4378	−.0212
0.8	1.1118	−.1303	.6038	−.0393	.4385	−.0184
0.9	1.1145	−.1171	.6051	−.0342	.4392	−.0156
1.0	1.1176	−.1038	.6064	−.0291	.4400	−.0127
1.5	1.1390	−.0350	.6156	−.0027	.4455	.0019
2.0	1.1689	.0369	.6282	.0246	.4529	.0171
3.0	1.2501	.1875	.6618	.0814	.4725	.0485
4.0	1.3551	.3452	.7046	.1403	.4972	.0808
5.0	1.4800	.5088	.7550	.2008	.5262	.1138

Table 5.13: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 0.50$.

$\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = P_2(r)$						
$a\alpha$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_1\sqrt{a}}$	$\frac{k_2}{q_1\sqrt{a}}$	$\frac{k_1}{q_2\sqrt{a}}$	$\frac{k_2}{q_2\sqrt{a}}$
0.0	−.0456	.4685	−.0264	.3637	−.0176	.3068
0.1	−.0456	.4686	−.0263	.3637	−.0176	.3067
0.2	−.0455	.4687	−.0263	.3638	−.0176	.3067
0.3	−.0454	.4688	−.0262	.3639	−.0176	.3068
0.4	−.0453	.4690	−.0261	.3640	−.0175	.3068
0.5	−.0451	.4692	−.0260	.3641	−.0174	.3069
0.6	−.0448	.4695	−.0259	.3643	−.0173	.3070
0.7	−.0445	.4698	−.0257	.3645	−.0172	.3072
0.8	−.0442	.4702	−.0255	.3647	−.0171	.3073
0.9	−.0439	.4706	−.0253	.3649	−.0170	.3075
1.0	−.0435	.4711	−.0251	.3652	−.0168	.3077
1.5	−.0413	.4740	−.0238	.3669	−.0159	.3089
2.0	−.0387	.4777	−.0223	.3691	−.0149	.3103
3.0	−.0330	.4869	−.0190	.3746	−.0126	.3141
4.0	−.0274	.4977	−.0157	.3810	−.0104	.3185
5.0	−.0223	.5096	−.0127	.3882	−.0084	.3235

Table 5.14: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 0.25$.

$\sigma_{zz}(r,0) = P_1(r),\ \sigma_{rz}(r,0) = 0$						
$a\alpha$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_1\sqrt{a}}$	$\frac{k_2}{p_1\sqrt{a}}$	$\frac{k_1}{p_2\sqrt{a}}$	$\frac{k_2}{p_2\sqrt{a}}$
0.0	1.9620	−.7704	.9160	−.2562	.6069	−.1289
0.1	1.9599	−.7568	.9154	−.2511	.6066	−.1261
0.2	1.9580	−.7430	.9149	−.2460	.6064	−.1232
0.3	1.9564	−.7290	.9145	−.2407	.6063	−.1204
0.4	1.9551	−.7149	.9142	−.2355	.6062	−.1175
0.5	1.9542	−.7006	.9140	−.2301	.6061	−.1145
0.6	1.9535	−.6862	.9139	−.2247	.6062	−.1116
0.7	1.9531	−.6716	.9140	−.2193	.6063	−.1086
0.8	1.9530	−.6568	.9141	−.2137	.6064	−.1056
0.9	1.9531	−.6418	.9144	−.2082	.6066	−.1025
1.0	1.9536	−.6267	.9147	−.2025	.6069	−.0994
1.5	1.9597	−.5490	.9181	−.1737	.6091	−.0836
2.0	1.9720	−.4679	.9238	−.1436	.6127	−.0673
3.0	2.0129	−.2973	.9416	−.0807	.6234	−.0331
4.0	2.0729	−.1173	.9667	−.0148	.6382	.0026
5.0	2.1500	.0705	.9983	.0536	.6566	.0394

Table 5.15: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 0.25$.

$\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = P_2(r)$						
$a\alpha$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_1\sqrt{a}}$	$\frac{k_2}{q_1\sqrt{a}}$	$\frac{k_1}{q_2\sqrt{a}}$	$\frac{k_2}{q_2\sqrt{a}}$
0.0	−.0925	.5411	−.0559	.4061	−.0386	.3351
0.1	−.0925	.5412	−.0559	.4061	−.0386	.3351
0.2	−.0925	.5412	−.0559	.4062	−.0386	.3351
0.3	−.0924	.5413	−.0558	.4062	−.0385	.3352
0.4	−.0923	.5414	−.0558	.4063	−.0385	.3352
0.5	−.0921	.5416	−.0557	.4064	−.0384	.3353
0.6	−.0920	.5418	−.0556	.4065	−.0383	.3354
0.7	−.0918	.5421	−.0555	.4067	−.0383	.3355
0.8	−.0915	.5423	−.0553	.4068	−.0382	.3356
0.9	−.0913	.5426	−.0552	.4070	−.0381	.3357
1.0	−.0910	.5430	−.0550	.4072	−.0380	.3353
1.5	−.0893	.5451	−.0540	.4084	−.0373	.3367
2.0	−.0872	.5478	−.0527	.4100	−.0364	.3378
3.0	−.0823	.5546	−.0497	.4140	−.0343	.3405
4.0	−.0768	.5626	−.0464	.4188	−.0320	.3438
5.0	−.0712	.5714	−.0430	.4241	−.0296	.3475

Table 5.16: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 0.10$.

$\sigma_{zz}(r,0) = P_1(r),\ \sigma_{rz}(r,0) = 0$						
$a\alpha$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_1\sqrt{a}}$	$\frac{k_2}{p_1\sqrt{a}}$	$\frac{k_1}{p_2\sqrt{a}}$	$\frac{k_2}{p_2\sqrt{a}}$
0.0	5.5317	−3.2759	2.1875	−1.0886	1.2795	−.5435
0.1	5.5271	−3.2579	2.1860	−1.0821	1.2788	−.5401
0.2	5.5228	−3.2397	2.1847	−1.0756	1.2782	−.5366
0.3	5.5186	−3.2215	2.1834	−1.0690	1.2776	−.5331
0.4	5.5146	−3.2030	2.1822	−1.0624	1.2770	−.5296
0.5	5.5107	−3.1845	2.1810	−1.0557	1.2765	−.5260
0.6	5.5071	−3.1658	2.1799	−1.0490	1.2760	−.5225
0.7	5.5036	−3.1469	2.1789	−1.0423	1.2755	−.5189
0.8	5.5003	−3.1280	2.1779	−1.0355	1.2751	−.5153
0.9	5.4972	−3.1088	2.1770	−1.0286	1.2747	−.5116
1.0	5.4942	−3.0896	2.1762	−1.0217	1.2743	−.5079
1.5	5.4820	−2.9914	2.1729	−.9866	1.2730	−.4893
2.0	5.474	−2.890	2.171	−.9504	1.273	−.470
3.0	5.469	−2.679	2.172	−.8752	1.274	−.430
4.0	5.476	−2.459	2.177	−.7967	1.277	−.389
5.0	5.497	−2.230	2.187	−.7155	1.284	−.346

Table 5.17: The variation of stress intensity factors with $a\alpha$ for various loading conditions shown in Table 5.1, for the value of $\nu = 0.3$, $h/a = 0.10$.

$\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = P_2(r),\ h/a = 0.10$						
$a\alpha$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_1\sqrt{a}}$	$\frac{k_2}{q_1\sqrt{a}}$	$\frac{k_1}{q_2\sqrt{a}}$	$\frac{k_2}{q_2\sqrt{a}}$
0.0	−.1856	.7173	−.1167	.5157	−.0832	.4122
0.1	−.1856	.7173	−.1167	.5157	−.0831	.4122
0.2	−.1856	.7174	−.1166	.5157	−.0831	.4122
0.3	−.1855	.7174	−.1166	.5157	−.0831	.4123
0.4	−.1855	.7175	−.1166	.5158	−.0831	.4123
0.5	−.1854	.7176	−.1165	.5159	−.0831	.4123
0.6	−.1853	.7178	−.1165	.5159	−.0830	.4124
0.7	−.1852	.7179	−.1164	.5160	−.0830	.4124
0.8	−.1850	.7181	−.1163	.5161	−.0829	.4125
0.9	−.1849	.7183	−.1162	.5162	−.0829	.4126
1.0	−.1847	.7185	−.1161	.5163	−.0828	.4127
1.5	−.1837	.7198	−.1155	.5171	−.0824	.4132
2.0	−.1824	.7216	−.1147	.5183	−.0818	.4139
3.0	−.1791	.7261	−.1128	.5207	−.0805	.4156
4.0	−.1752	.7314	−.1105	.5239	−.0789	.4178
5.0	−.1711	.7374	−.1079	.5274	−.0771	.4202

Table 5.18: The variation of stress intensity factors with ν for various loading conditions shown in Table 5.1, $h/a = 10.0$, $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$.

ν	$\alpha a = 0.1$		$\alpha a = 0.5$		$\alpha a = 1.0$		$\alpha a = 1.5$	
	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$
0.00	.6376	.0106	.6522	.0532	.6880	.1069	.7355	.1619
0.10	.6378	.0106	.6545	.0532	.6945	.1070	.7463	.1621
0.20	.6379	.0106	.6573	.0532	.7021	.1071	.7589	.1624
0.30	.6381	.0106	.6608	.0532	.7115	.1073	.7741	.1628
0.40	.6384	.0106	.6652	.0532	.7231	.1074	.7929	.1633
0.45	.6385	.0106	.6679	.0532	.7301	.1075	0.8041	.1635
	$\alpha a = 2.0$		$\alpha a = 3.0$		$\alpha a = 4.0$		$\alpha a = 5.0$	
ν	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$
0.00	.7908	.2183	.9173	.3369	1.0595	.4638	1.2138	.5995
0.10	.8055	.2188	.9390	.3381	1.0867	.4659	1.2455	.6025
0.20	.8229	.2194	.9643	.3395	1.1184	.4683	1.2824	.6060
0.30	.8435	.2202	.9943	.3412	1.1561	.4712	1.3266	.6101
0.40	.8689	.2211	1.0309	.3434	1.2020	.4747	1.3803	.6151
0.45	.8839	.2216	1.0526	.3446	1.2292	.4768	1.4122	.6180

Table 5.19: The variation of stress intensity factors with ν for various loading conditions shown in Table 5.1, $h/a = 10.0$, $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$.

ν	$\alpha a = 0.1$		$\alpha a = 0.5$		$\alpha a = 1.0$		$\alpha a = 1.5$	
	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$
0.00	.0000	.4244	.0000	.4253	.0000	.4278	.0000	.4316
0.10	.0000	.4245	.0000	.4254	.0000	.4281	.0000	.4323
0.20	.0000	.4245	.0000	.4255	.0000	.4285	.0000	.4331
0.30	.0000	.4245	.0000	.4256	.0000	.4290	.0000	.4341
0.40	.0000	.4245	.0000	.4258	.0000	.4297	.0000	.4353
0.45	.0000	.4245	.0000	.4260	0000	.4301	.0000	.4361
	$\alpha a = 2.0$		$\alpha a = 3.0$		$\alpha a = 4.0$		$\alpha a = 5.0$	
ν	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$
0.00	.0000	.4367	.0000	.4492	.0000	.4638	.0000	.4796
0.10	.0000	.4377	.0000	.4508	.0000	.4659	.0000	.4820
0.20	.0000	.4389	.0000	4527	.0000	4683	.0000	.4848
0.30	.0000	.4403	.0000	.4550	.0000	.4712	.0000	.4881
0.40	.0000	.4422	.0000	.4578	.0000	.4747	.0000	.4921
0.45	.0000	.4433	.0000	.4595	.0000	.4768	.0000	.4944

Table 5.20: The variation of stress intensity factors with ν for various loading conditions shown in Table 5.1, $h/a = 2.0$, $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$.

	$\alpha a = 0.1$		$\alpha a = 0.5$		$\alpha a = 1.0$		$\alpha a = 1.5$	
ν	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$
0.00	.6676	.0037	.6769	.0469	.7041	.1021	.7451	.1585
0.10	.6677	.0037	.6784	.0470	.7094	.1024	.7547	.1591
0.20	.6678	.0037	.6804	.0471	.7157	.1028	.7662	.1597
0.30	.6679	.0037	.6828	.0473	.7236	.1033	.7803	.1604
0.40	.6681	.0037	.6860	.0475	.7337	.1038	.7978	.1613
0.45	.6681	.0037	0.6880	.0476	0.7398	.1041	0.8084	.1618
	$\alpha a = 2.0$		$\alpha a = 3.0$		$\alpha a = 4.0$		$\alpha a = 5.0$	
ν	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$
0.00	.7962	.2162	.9188	.3361	1.0598	.4636	1.2138	.5994
0.10	.8101	.2170	.9402	.3375	1.0870	.4657	1.2455	.6024
0.20	.8266	.2179	.9652	.3391	1.1186	.4682	1.2825	.6059
0.30	.8465	.2189	.9949	.3409	1.1562	.4711	1.3266	.6101
0.40	.8710	.2201	1.0313	.3431	1.2021	.4747	1.3804	.6151
0.45	.8857	.2208	1.0529	.3445	1.2292	.4768	1.4122	.6180

Table 5.21: The variation of stress intensity factors with ν for various loading conditions shown in Table 5.1, $h/a = 2.0$, $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$.

	$\alpha a = 0.1$		$\alpha a = 0.5$		$\alpha a = 1.0$		$\alpha a = 1.5$	
ν	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$
0.00	−.0025	.4253	−.0023	.4261	−.0018	.4285	−.0012	.4322
0.10	−.0025	.4254	−.0023	.4262	−.0017	.4288	−.0011	.4328
0.20	−.0025	.4254	−.0022	.4263	−.0016	.4292	−.0010	.4336
0.30	−.0025	.4254	−.0022	.4265	−.0015	.4296	−.0009	.4345
0.40	−.0025	.4254	−.0021	.4266	−.0013	.4303	−.0007	.4357
0.45	−.0025	.4254	−.0020	.4267	−.0012	.4307	−.0006	.4364
	$\alpha a = 2.0$		$\alpha a = 3.0$		$\alpha a = 4.0$		$\alpha a = 5.0$	
ν	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$
0.00	−.0008	.4370	−.0003	.4493	−.0001	.4638	−.0000	.4796
0.10	−.0007	.4380	−.0002	.4509	−.0001	.4659	.0000	.4820
0.20	−.0006	.4391	−.0002	.4528	.0000	.4683	.0000	.4848
0.30	−.0005	.4406	−.0001	.4550	.0000	.4712	.0000	.4881
0.40	−.0004	.4423	−.0001	.4578	.0000	.4747	.0000	.4921
0.45	−.0003	.4434	−.0001	.4595	.0000	.4768	.0000	.4944

Table 5.22: The variation of stress intensity factors with ν for various loading conditions shown in Table 5.1, $h/a = 1.0$, $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$.

ν	$\alpha a = 0.1$		$\alpha a = 0.5$		$\alpha a = 1.0$		$\alpha a = 1.5$	
	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$
0.00	.7780	−.0411	.7825	.0035	.7989	.0614	.8264	.1215
0.10	.7781	−.0411	.7834	.0037	.8023	.0621	.8331	.1229
0.20	.7781	−.0410	.7846	.0040	.8065	.0630	.8412	.1244
0.30	.7782	−.0410	.7860	.0043	.8117	.0640	.8513	.1264
0.40	.7783	−.0410	.7880	.0047	.8184	.0654	.8642	.1287
0.45	.7783	−.0410	0.7892	.0050	0.8226	.0662	0.8721	.1301

ν	$\alpha a = 2.0$		$\alpha a = 3.0$		$\alpha a = 4.0$		$\alpha a = 5.0$	
	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$
0.00	.8634	.1836	.9616	.3130	1.0852	.4485	1.2281	.5903
0.10	.8739	.1856	.9795	.3159	1.1095	.4521	1.2579	.5944
0.20	.8864	.1878	1.0006	.3192	1.1383	.4561	1.2930	.5991
0.30	.9018	.1905	1.0262	.3230	1.1729	.4607	1.3351	.6044
0.40	.9211	.1937	1.0582	.3274	1.2157	.4661	1.3870	.6106
0.45	.9329	.1956	1.0773	.3299	1.2413	.4691	1.4179	.6142

Table 5.23: The variation of stress intensity factors with ν for various loading conditions shown in Table 5.1, $h/a = 1.0$, $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$.

ν	$\alpha a = 0.1$		$\alpha a = 0.5$		$\alpha a = 1.0$		$\alpha a = 1.5$	
	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$
0.00	−.0153	.4342	−.0149	.4349	−.0139	.4368	−.0124	.4399
0.10	−.0153	.4342	−.0149	.4349	−.0137	.4371	−.0121	.4404
0.20	−.0153	.4342	−.0148	.4350	−.0134	.4373	−.0116	.4410
0.30	−.0153	.4342	−.0147	.4351	−.0131	.4377	−.0111	.4417
0.40	−.0153	.4342	−.0146	.4353	−.0127	.4382	−.0105	.4426
0.45	−.0153	.4343	−.0145	.4353	−.0125	.4385	−.0101	.4432

ν	$\alpha a = 2.0$		$\alpha a = 3.0$		$\alpha a = 4.0$		$\alpha a = 5.0$	
	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$
0.00	−.0108	.4440	−.0075	.4545	−.0049	.4673	−.0029	.4818
0.10	−.0103	.4447	−.0069	.4557	−.0043	.4691	−.0025	.4839
0.20	−.0097	.4456	−.0063	.4573	−.0038	.4712	−.0021	.4865
0.30	−.0091	.4467	−.0056	.4591	−.0032	.4734	−.0018	.4895
0.40	−.0084	.4481	−.0049	.4615	−.0027	.4768	−.0014	.4932
0.45	−.0079	.4490	−.0045	.4629	−.0024	.4787	−.0012	.4954

Table 5.24: The variation of stress intensity factors with ν for various loading conditions shown in Table 5.1, $h/a = 0.50$, $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$.

ν	$\alpha a = 0.1$		$\alpha a = 0.5$		$\alpha a = 1.0$		$\alpha a = 1.5$	
	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$
0.00	1.1051	−.2180	1.1043	−.1698	1.1101	−.1070	1.1236	−.0414
0.10	1.1051	−.2180	1.1048	−.1696	1,1121	−.1062	1.1277	−.0396
0.20	1.1051	−.2180	1.1054	−.1693	1.1146	−.1051	1.1327	−.0375
0.30	1.1052	−.2179	1.1062	−.1689	1.1176	−.1038	1.1390	−.0350
0.40	1.1052	−.2179	1.1073	−.1684	1.1217	−.1021	1.1471	−.0317
0.45	1.1052	−.2179	1.1080	−.1681	1.1242	−.1010	1.1520	−.0297

ν	$\alpha a = 2.0$		$\alpha a = 3.0$		$\alpha a = 4.0$		$\alpha a = 5.0$	
	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{p_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$
0.00	1.1442	.0269	1.2053	.1709	1.2902	.3233	1.3963	.4830
0.10	1.1508	.0297	1.2175	.1755	1.3081	.3296	1.4194	.4904
0.20	1.1589	.0329	1.2322	.1810	1.3293	.3368	1.4467	.4989
0.30	1.1689	.0369	1.2501	.1875	1.3551	.3452	1.4800	.5088
0.40	1.1816	.0418	1.2727	.1954	1.3875	.3554	1.5215	.5205
0.45	1.1894	.0448	1.2865	.2001	1.4070	.3614	1.5465	.5273

Table 5.25: The variation of stress intensity factors with ν for various loading conditions shown in Table 5.1, $h/a = 0.50$, $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$.

ν	$\alpha a = 0.1$		$\alpha a = 0.5$		$\alpha a = 1.0$		$\alpha a = 1.5$	
	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{q_0\sqrt{a}}$
0.00	−.0456	.4686	−.0453	.4690	−.0443	.4704	−.0427	.4727
0.10	−.0456	.4886	−.0452	.4691	−.0441	.4706	−.0423	.4731
0.20	−.0456	.4886	−.0451	.4691	−.0438	.4708	−.0418	.4735
0.30	−.0456	.4686	−.0451	.4692	−.0435	.4711	−.0413	.4740
0.40	−.0456	.4686	−.0449	.4693	−.0431	.4715	−.0406	.4747
0.45	−.0456	.4686	−.0449	.4694	−.0429	.4717	−.0401	.4751

ν	$\alpha a = 2.0$		$\alpha a = 3.0$		$\alpha a = 4.0$		$\alpha a = 5.0$	
	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$	$\frac{k_1}{q_0\sqrt{a}}$	$\frac{k_2}{p_0\sqrt{a}}$
0.00	−.0407	.4757	−.0359	.4836	−.0308	.4933	−.0257	.5043
0.10	−.0401	.4763	−.0351	.4845	−.0298	.4945	−.0247	.5058
0.20	−.0395	.4769	−.0341	.4856	−.0287	.4959	−.0236	.5075
0.30	−.0387	.4777	−.0330	.4869	−.0274	.4977	−.0223	.5096
0.40	−.0377	.4787	−.0316	.4885	−.0259	.4999	−.0208	.5123
0.45	−.0371	.4793	−.0308	.4895	−.0250	.5012	−.0200	.5138

Table 5.26: The variation of stress intensity factors with ν for various loading conditions shown in Table 5.1, $h/a = 0.25$, $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$.

ν	$\alpha a = 0.1$ $\frac{k_1}{p_0\sqrt{a}}$	$\alpha a = 0.1$ $\frac{k_2}{p_0\sqrt{a}}$	$\alpha a = 0.5$ $\frac{k_1}{p_0\sqrt{a}}$	$\alpha a = 0.5$ $\frac{k_2}{p_0\sqrt{a}}$	$\alpha a = 1.0$ $\frac{k_1}{p_0\sqrt{a}}$	$\alpha a = 1.0$ $\frac{k_2}{p_0\sqrt{a}}$	$\alpha a = 1.5$ $\frac{k_1}{p_0\sqrt{a}}$	$\alpha a = 1.5$ $\frac{k_2}{p_0\sqrt{a}}$
0.00	1.9598	−.7568	1.9530	−.7014	1.9490	−.6297	1.9502	−.5553
0.10	1.9598	−.7568	1.9533	−.7012	1.9502	−.6289	1.9527	−.5536
0.20	1.9598	−.7568	1.9537	−.7010	1.9517	−.6280	1.9558	−.5515
0.30	1.9599	−.7568	1.9542	−.7006	1.9536	−.6267	1.9597	−.5490
0.40	1.9599	−.7567	1.9548	−.7002	1.9560	−.6251	1.9648	−.5457
0.45	1.9599	−.7567	1.9552	−.6999	1.9576	−.6241	1.9679	−.5437

ν	$\alpha a = 2.0$ $\frac{k_1}{p_0\sqrt{a}}$	$\alpha a = 2.0$ $\frac{k_2}{p_0\sqrt{a}}$	$\alpha a = 3.0$ $\frac{k_1}{p_0\sqrt{a}}$	$\alpha a = 3.0$ $\frac{k_2}{p_0\sqrt{a}}$	$\alpha a = 4.0$ $\frac{k_1}{p_0\sqrt{a}}$	$\alpha a = 4.0$ $\frac{k_2}{p_0\sqrt{a}}$	$\alpha a = 5.0$ $\frac{k_1}{p_0\sqrt{a}}$	$\alpha a = 5.0$ $\frac{k_2}{p_0\sqrt{a}}$
0.00	1.9564	−.4780	1.9832	−.3156	2.0285	−.1435	2.0911	.0375
0.10	1.9606	−.4753	1.9913	−.3106	2.0407	−.1362	2.1073	.0467
0.20	1.9657	−.4720	2.0010	−.3046	2.0552	−.1276	2.1265	.0576
0.30	1.9720	−.4679	2.0129	−.2973	2.0729	−.1173	2.1500	.0705
0.40	1.9802	−.4627	2.0280	−.2881	2.0952	−.1045	2.1793	.0864
0.45	1.9852	−.4596	2.0372	−.2827	2.1088	−.0969	2.1971	.0957

Table 5.27: The variation of stress intensity factors with ν for various loading conditions shown in Table 5.1, $h/a = 0.25$, $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$.

ν	$\alpha a = 0.1$ $\frac{k_1}{q_0\sqrt{a}}$	$\alpha a = 0.1$ $\frac{k_2}{q_0\sqrt{a}}$	$\alpha a = 0.5$ $\frac{k_1}{q_0\sqrt{a}}$	$\alpha a = 0.5$ $\frac{k_2}{q_0\sqrt{a}}$	$\alpha a = 1.0$ $\frac{k_1}{q_0\sqrt{a}}$	$\alpha a = 1.0$ $\frac{k_2}{q_0\sqrt{a}}$	$\alpha a = 1.5$ $\frac{k_1}{q_0\sqrt{a}}$	$\alpha a = 1.5$ $\frac{k_2}{q_0\sqrt{a}}$
0.00	−.0925	.5412	−.0923	.5415	−.0915	.5425	−.0903	.5441
0.10	−.0925	.5412	−.0922	.5415	−.0914	.5426	−.0901	5444
0.20	−.0925	.5412	−.0922	.5416	−.0912	.5428	−.0897	.5447
0.30	−.0925	.5412	−.0921	.5416	−.0910	.5430	−.0893	.5451
0.40	−.0925	.5412	−.0920	.5417	−.0907	.5432	−.0888	.5456
0.45	−.0925	.5412	−.0920	.5417	−.0906	.5434	−.0885	.5459

ν	$\alpha a = 2.0$ $\frac{k_1}{q_0\sqrt{a}}$	$\alpha a = 2.0$ $\frac{k_2}{q_0\sqrt{a}}$	$\alpha a = 3.0$ $\frac{k_1}{q_0\sqrt{a}}$	$\alpha a = 3.0$ $\frac{k_2}{q_0\sqrt{a}}$	$\alpha a = 4.0$ $\frac{k_1}{q_0\sqrt{a}}$	$\alpha a = 4.0$ $\frac{k_2}{q_0\sqrt{a}}$	$\alpha a = 5.0$ $\frac{k_1}{q_0\sqrt{a}}$	$\alpha a = 5.0$ $\frac{k_2}{q_0\sqrt{a}}$
0.00	−.0887	.5463	−.0847	.5520	−.0799	.5591	−.0746	.5673
0.10	−.0883	.5467	−.0840	.5527	−.0790	.5601	−.0736	.5684
0.20	−.0878	.5472	−.0832	.5535	−.0780	.5612	−.0725	.5698
0.30	−.0872	.5478	−.0823	.5546	−.0768	.5626	−.0712	.5714
0.40	−.0865	.5486	−.0811	.5558	−.0754	.5642	−.0696	.5733
0.45	−.0860	.5490	−.0804	.5566	−.0745	.5652	−.0687	.5745

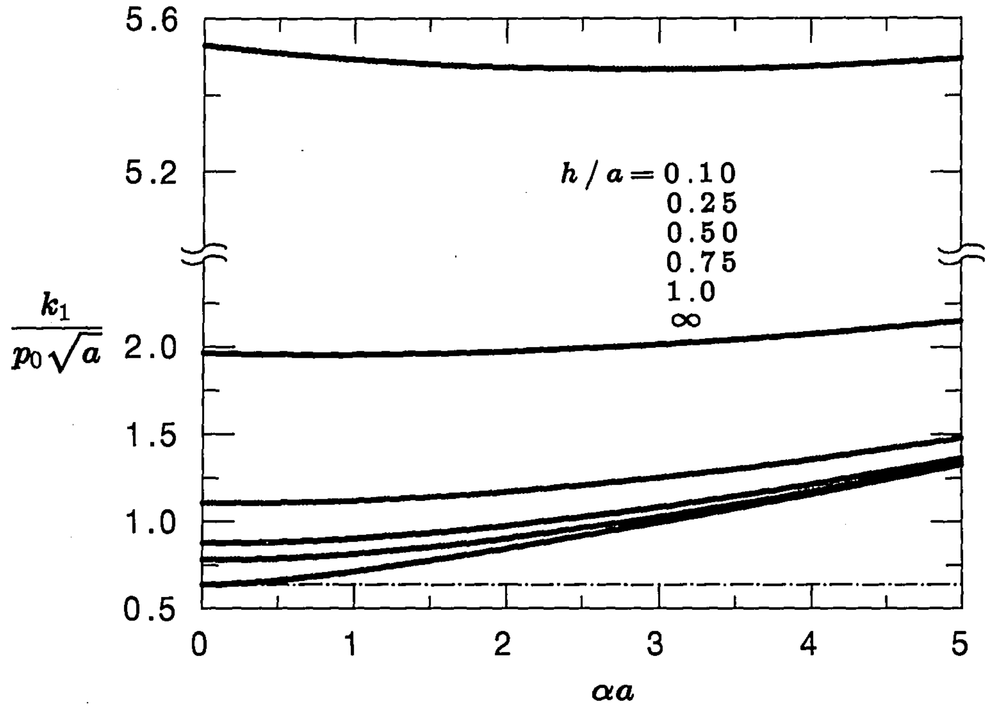

Figure 5.1: Normalized stress intensity factors for various h/a values when $\sigma_{zz}(r,0) = -p_0,\ \sigma_{rz}(r,0) = 0.$

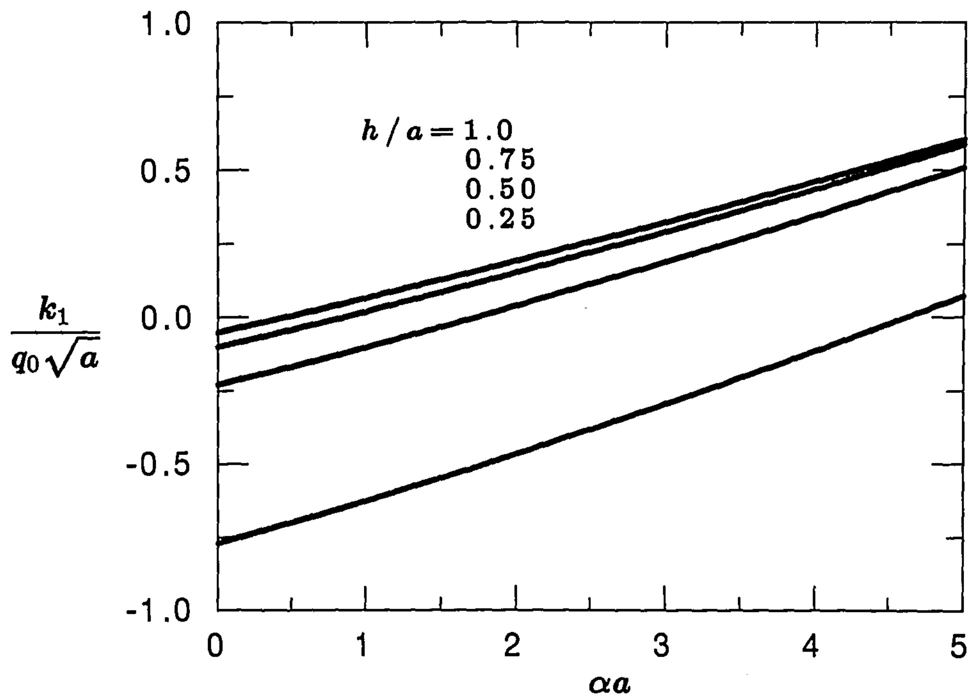

Figure 5.2: Normalized stress intensity factors for various h/a values when $\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = -q_0.$

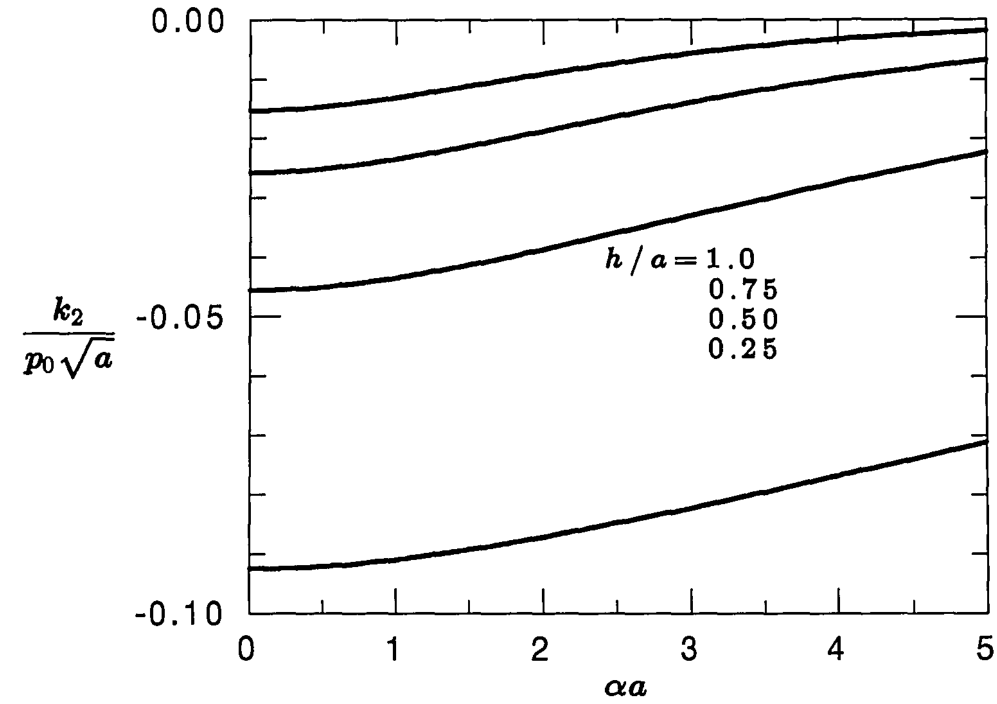

Figure 5.3: Normalized stress intensity factors for various h/a values when $\sigma_{zz}(r,0) = -p_0,\ \sigma_{rz}(r,0) = 0.$

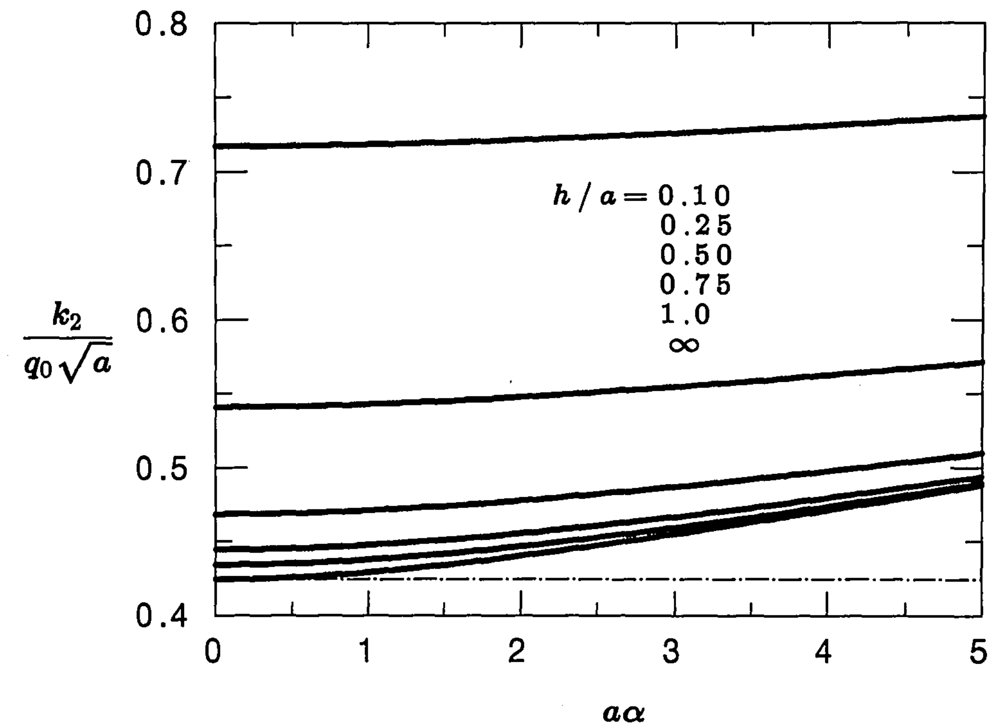

Figure 5.4: Normalized stress intensity factors for various h/a values when $\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = -q_0.$

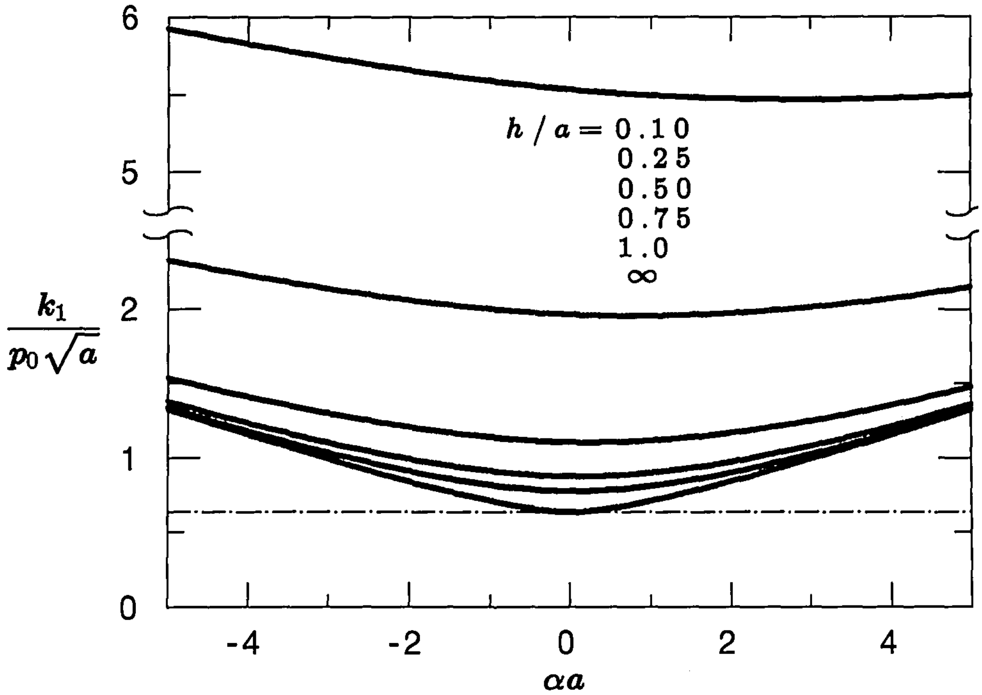

Figure 5.5: Normalized stress intensity factors for various h/a values when $\sigma_{zz}(r,0) = -p_0,\ \sigma_{rz}(r,0) = 0.$

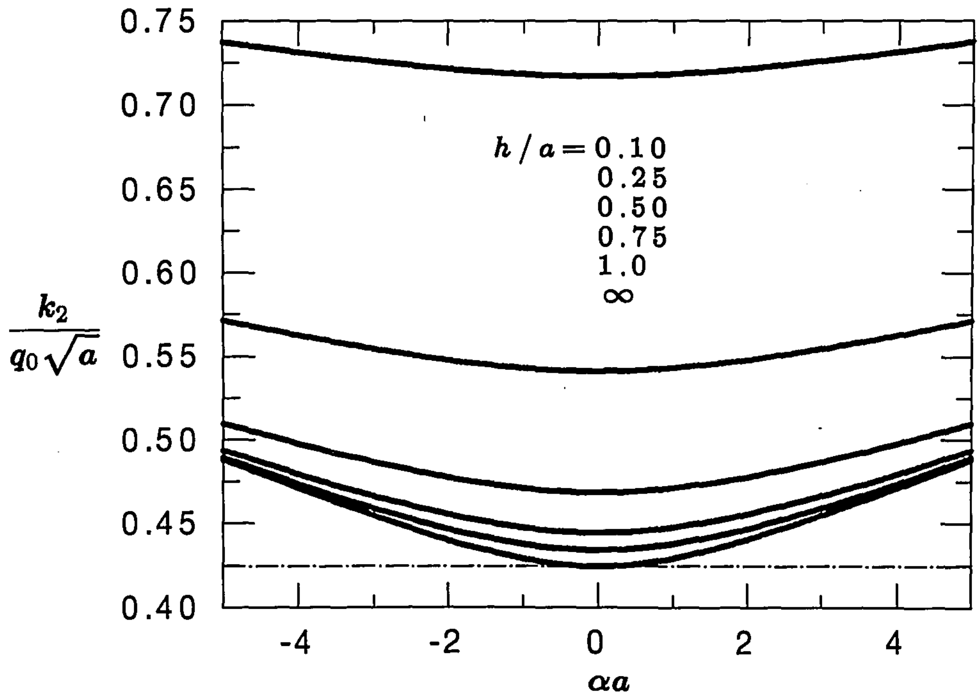

Figure 5.6: Normalized stress intensity factors for various h/a values when $\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = -q_0.$

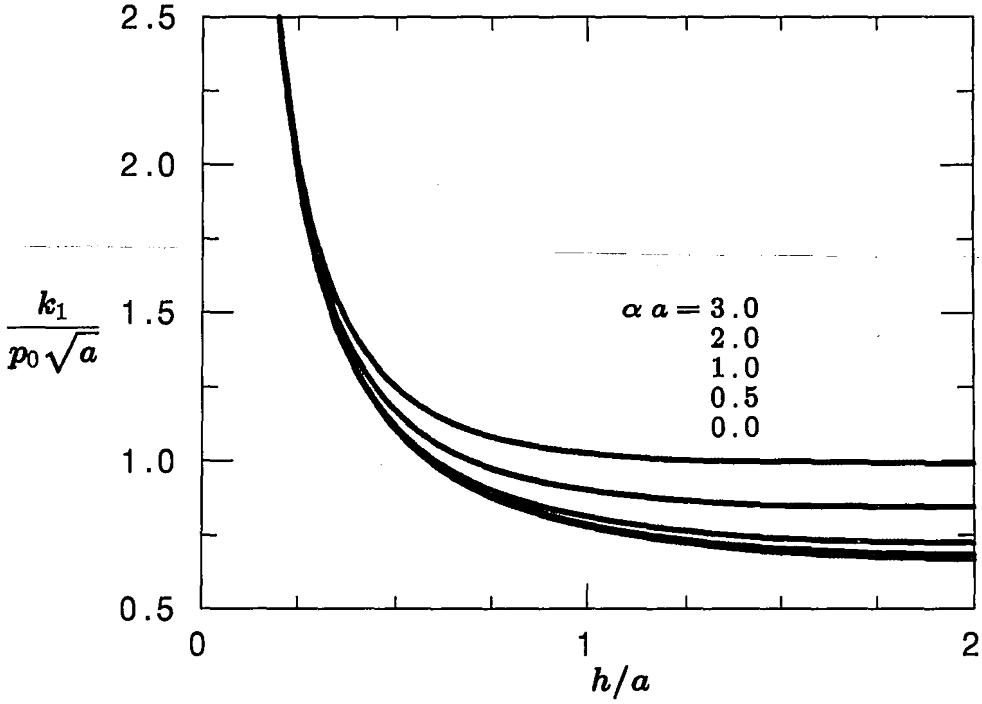

Figure 5.7: Normalized stress intensity factors for various αa values when $\sigma_{zz}(r,0) = -p_0,\ \sigma_{rz}(r,0) = 0.$

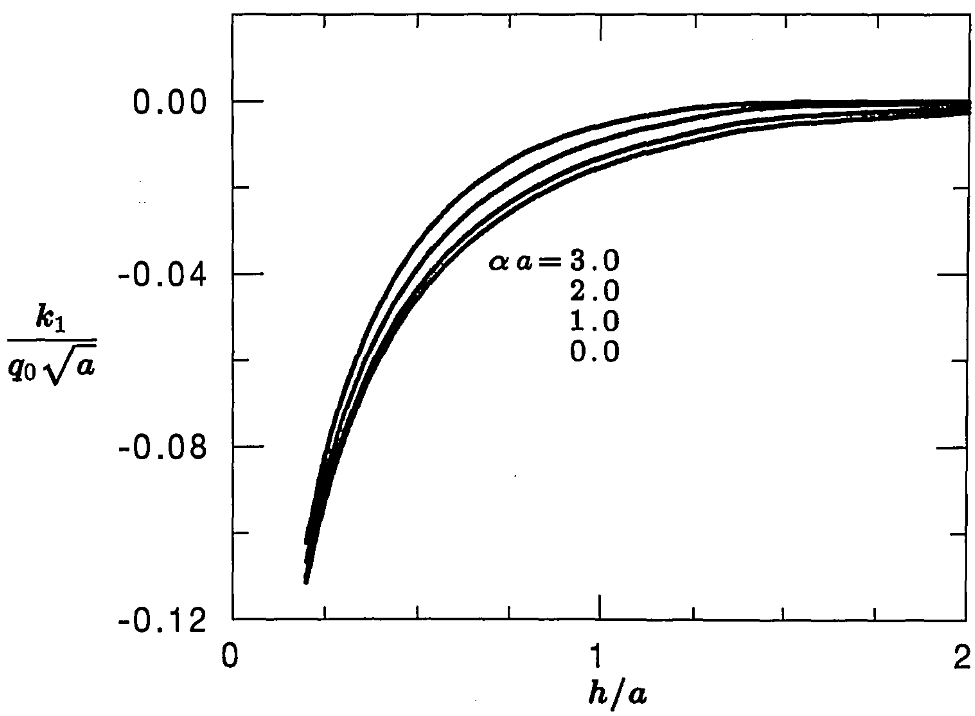

Figure 5.8: Normalized stress intensity factors for various αa values when $\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = -q_0.$

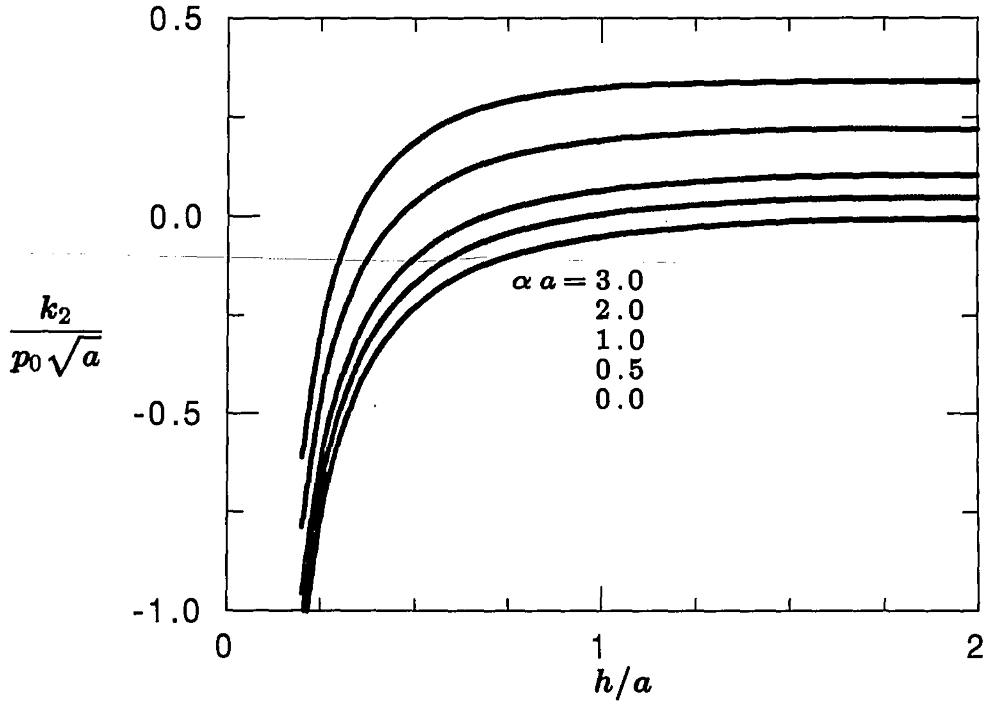

Figure 5.9: Normalized stress intensity factors for various αa values when $\sigma_{zz}(r,0) = -p_0,\ \sigma_{rz}(r,0) = 0.$

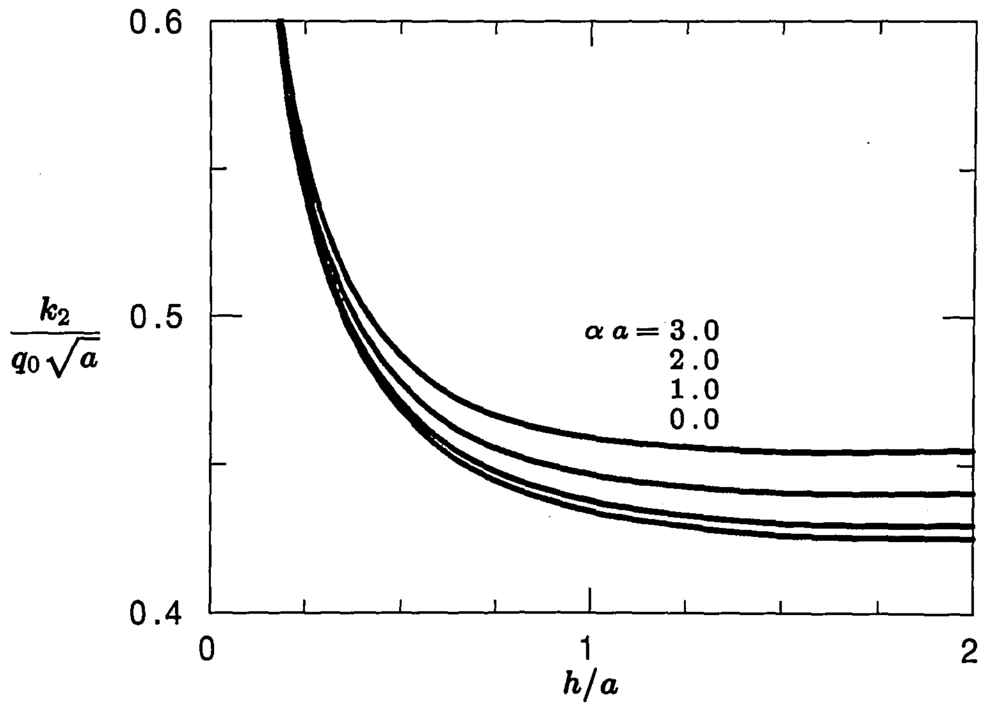

Figure 5.10: Normalized stress intensity factors for various αa values when $\sigma_{zz}(r,0) = 0,\ \sigma_{rz}(r,0) = -q_0.$

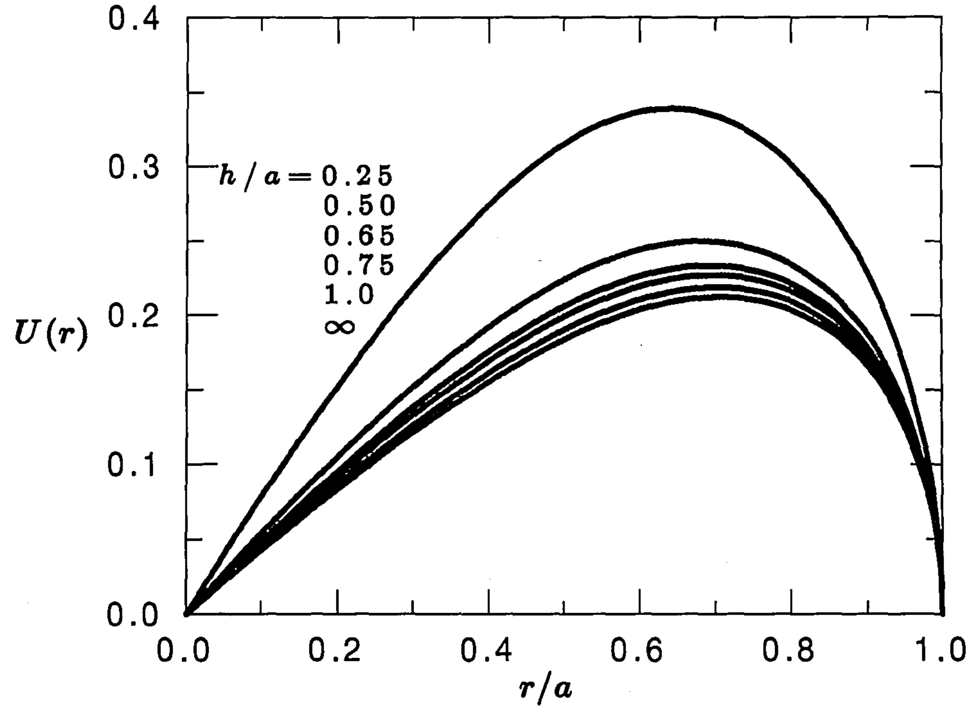

Figure 5.11: r- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$, and $\alpha a = 0$.

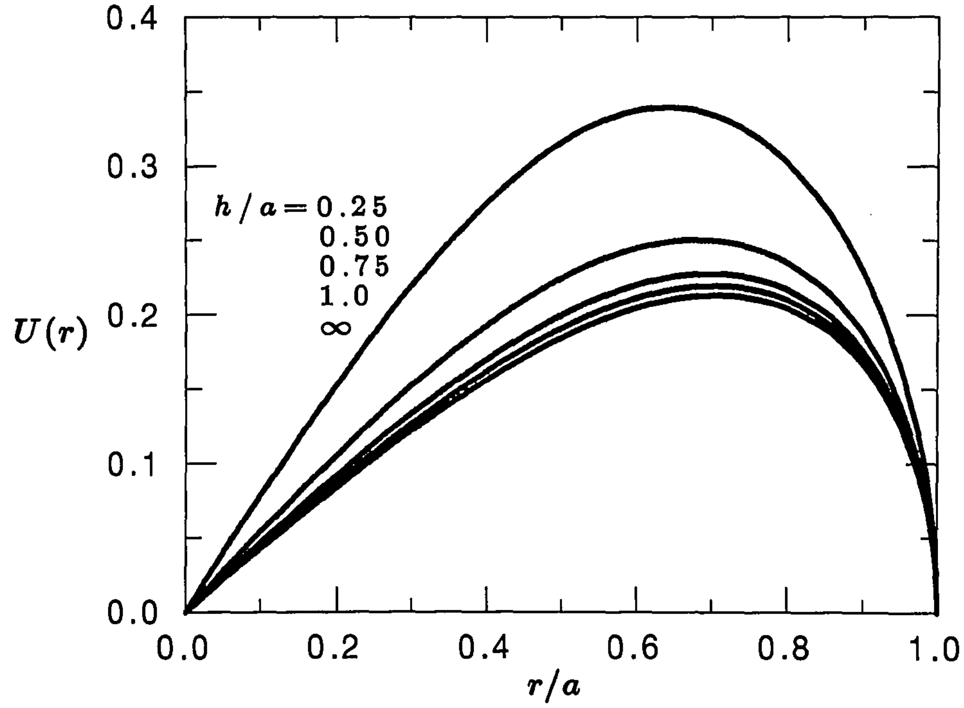

Figure 5.12: r- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$, and $\alpha a = 0.5$.

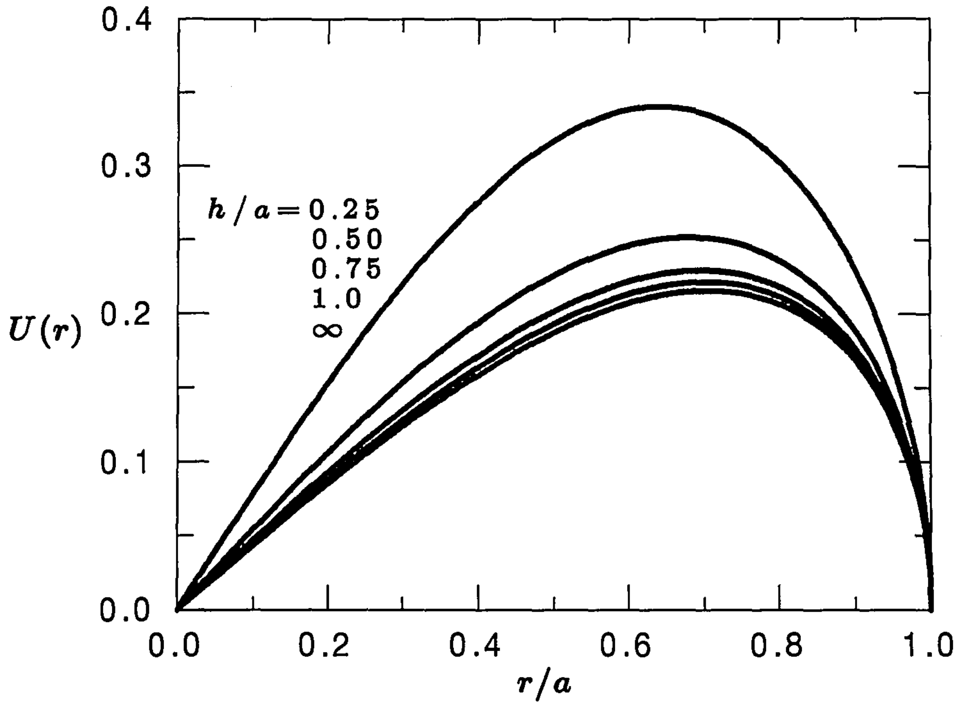

Figure 5.13: r- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$, and $\alpha a = 1.0$.

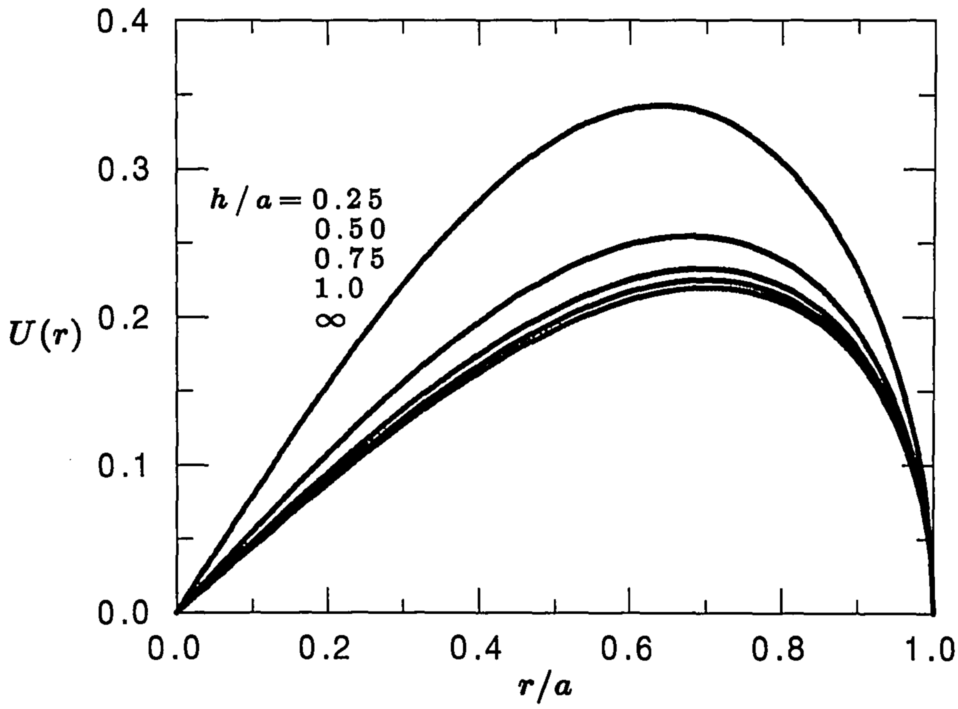

Figure 5.14: r- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$, and $\alpha a = 1.50$.

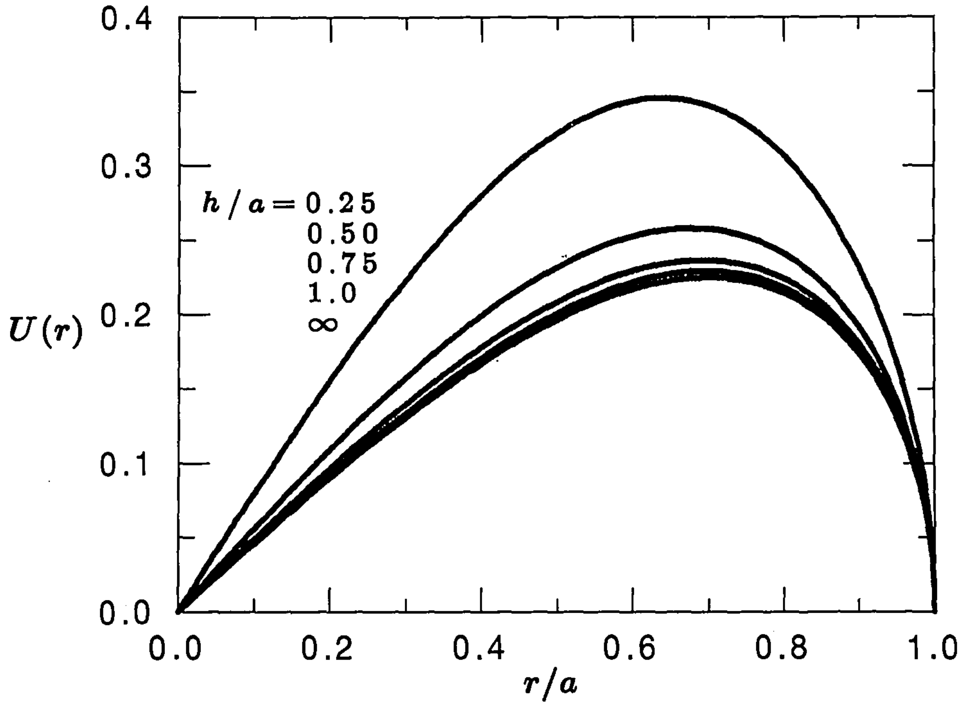

Figure 5.15: r- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$, and $\alpha a = 2.0$.

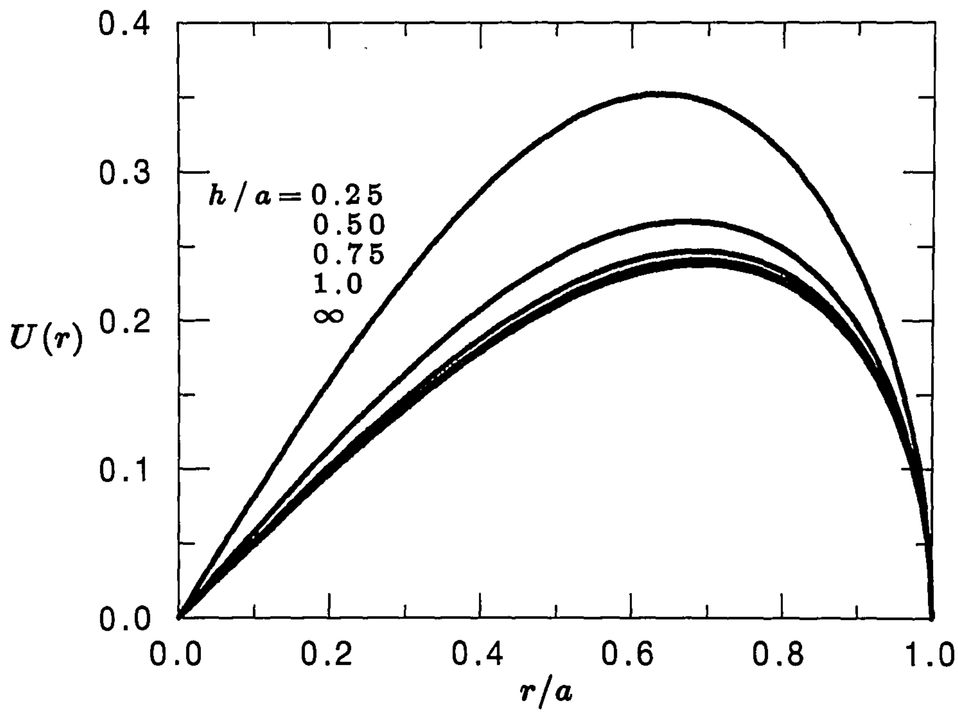

Figure 5.16: r- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$, and $\alpha a = 3.0$.

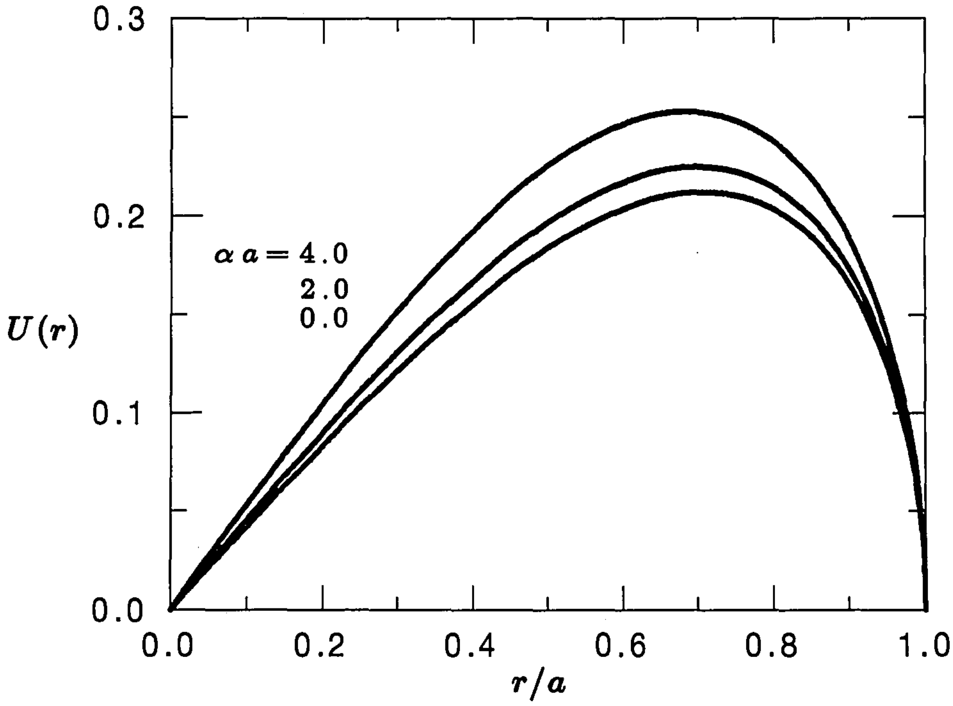

Figure 5.17: r- component of the normalized crack opening displacement for various αa values in case of the external loading $\sigma_{zz}(r, 0) = 0$, $\sigma_{rz}(r, 0) = -q_0$, and $h/a = 5.0$.

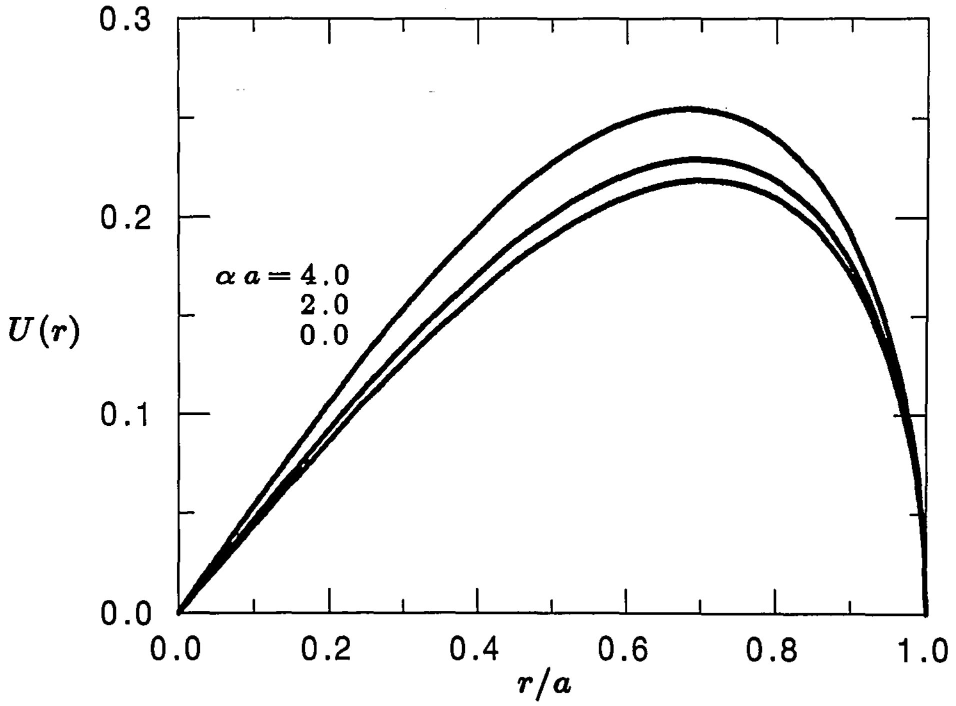

Figure 5.18: r- component of the normalized crack opening displacement for various αa values in case of the external loading $\sigma_{zz}(r, 0) = 0$, $\sigma_{rz}(r, 0) = -q_0$, and $h/a = 1.0$.

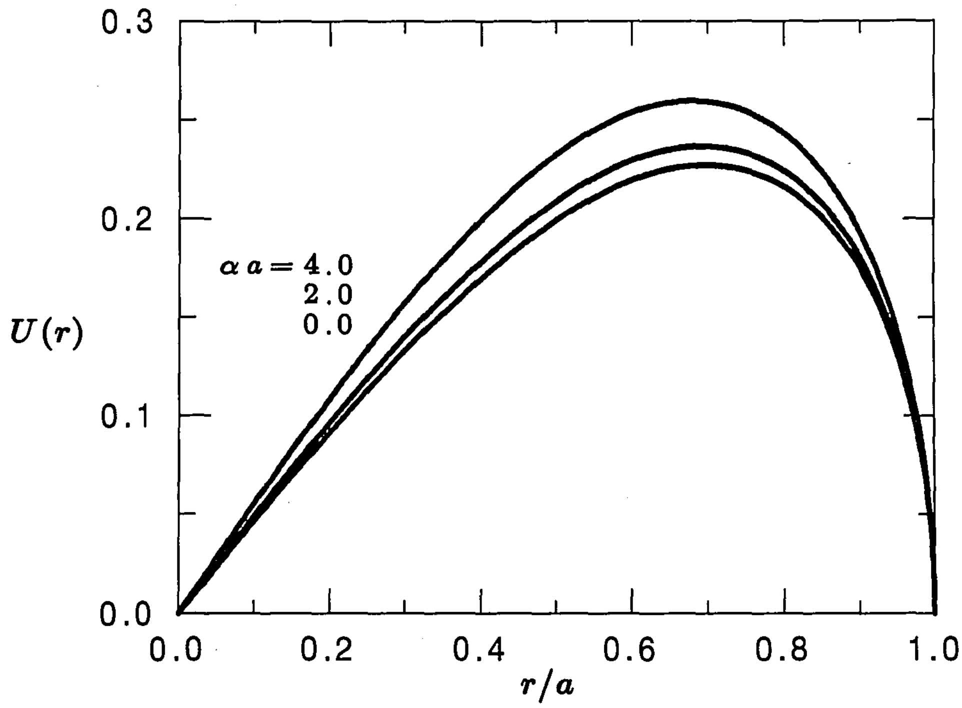

Figure 5.19: r- component of the normalized crack opening displacement for various αa values in case of the external loading $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$, and $h/a = 0.75$.

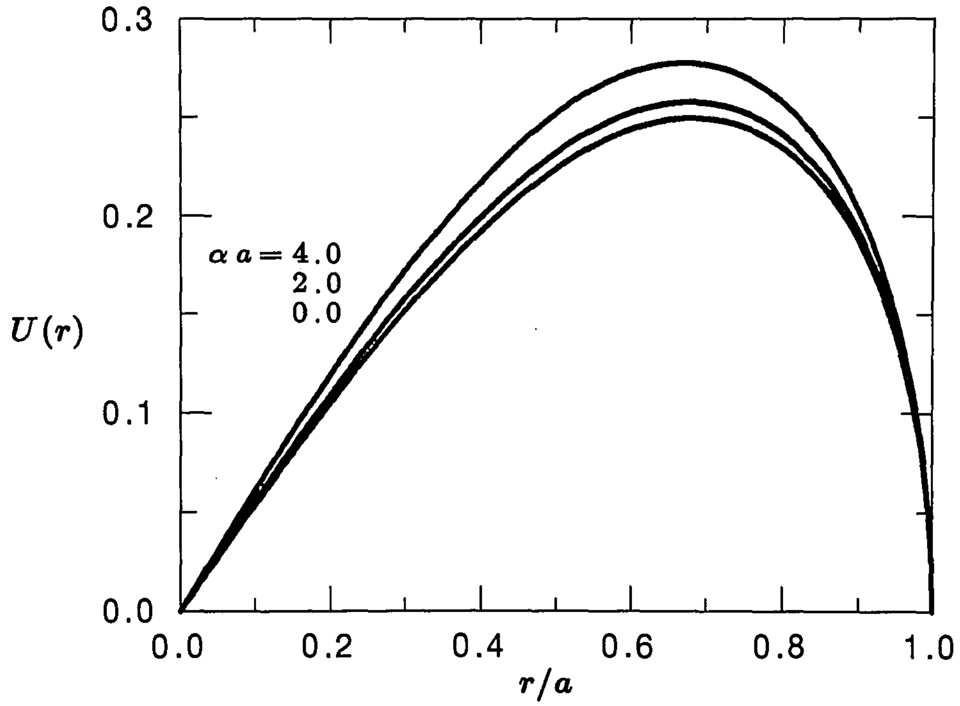

Figure 5.20: r- component of the normalized crack opening displacement for various αa values in case of the external loading $\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0$, and $h/a = 0.50$.

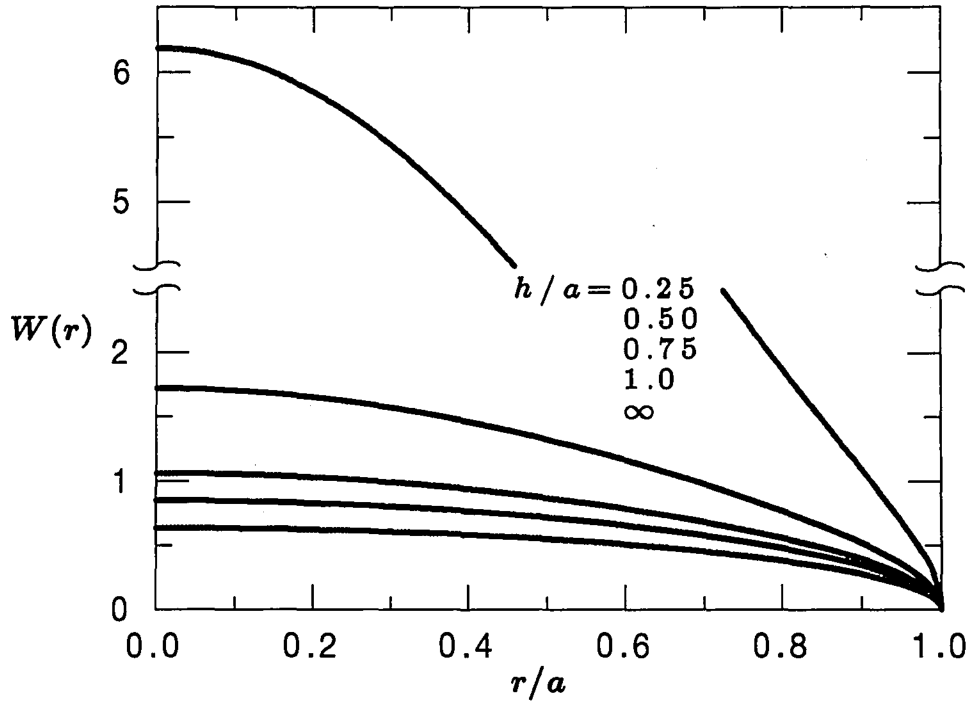

Figure 5.21: z- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$, and $\alpha a = 0$.

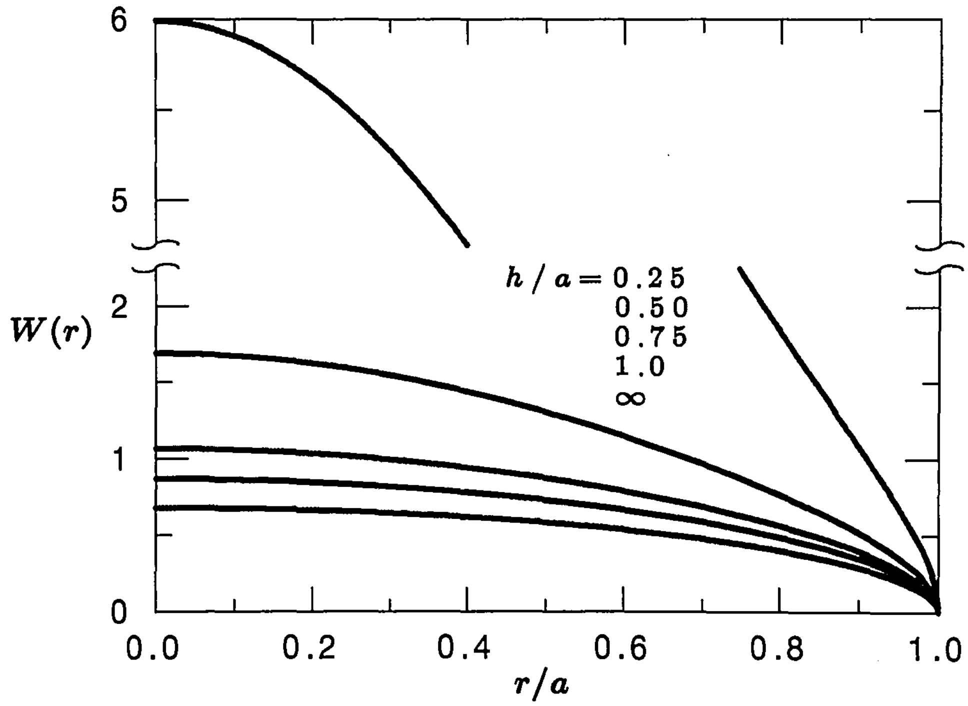

Figure 5.22: z- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$, and $\alpha a = 0.50$.

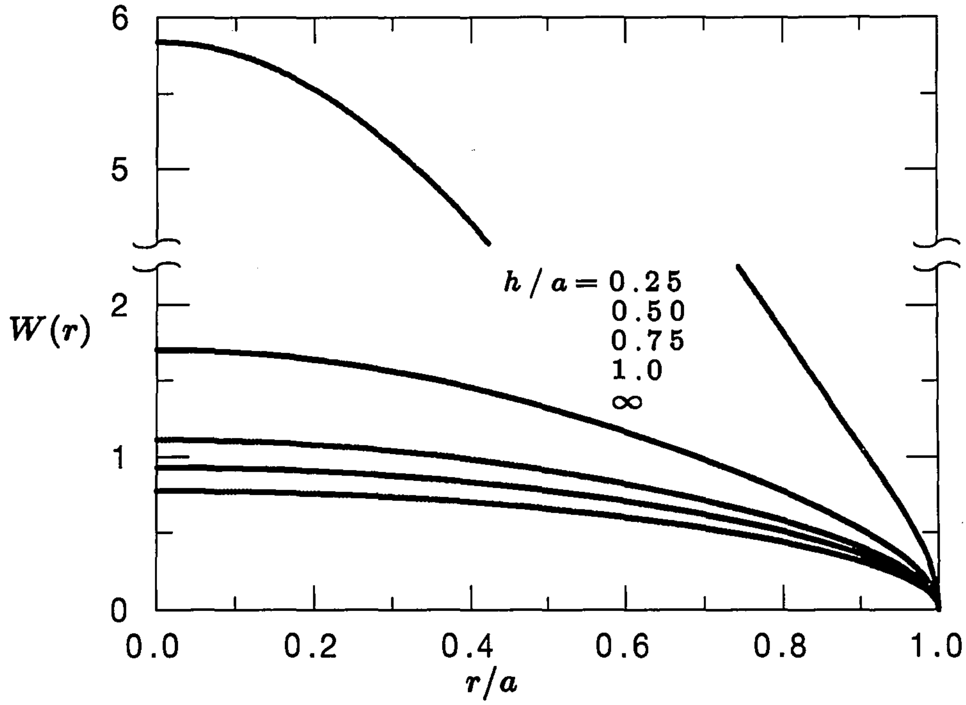

Figure 5.23: z- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$, and $\alpha a = 1.0$.

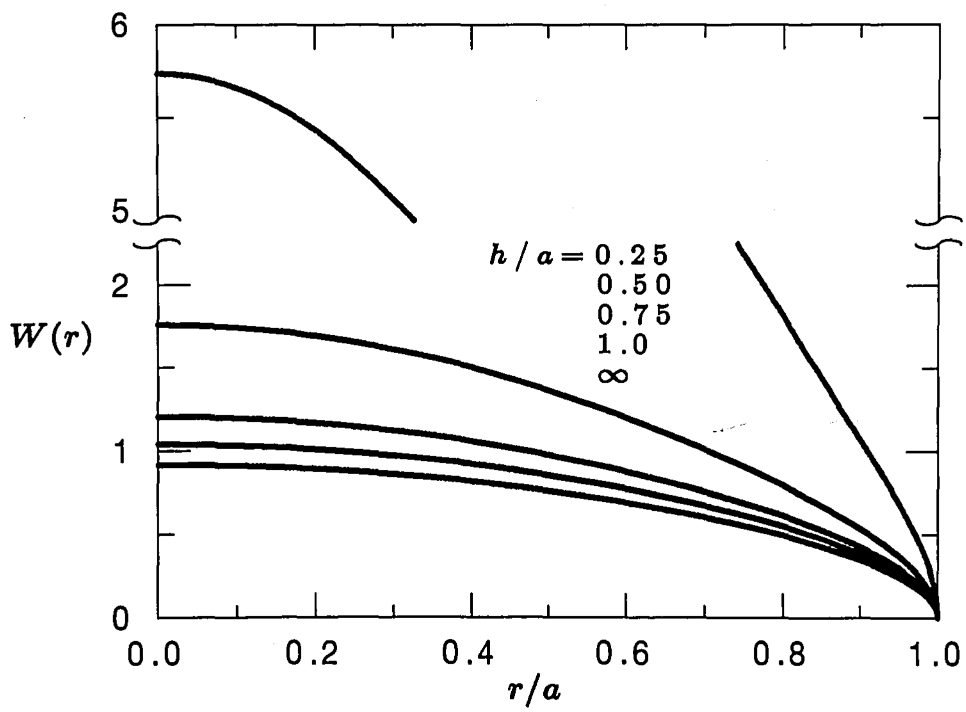

Figure 5.24: z- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$, and $\alpha a = 1.50$.

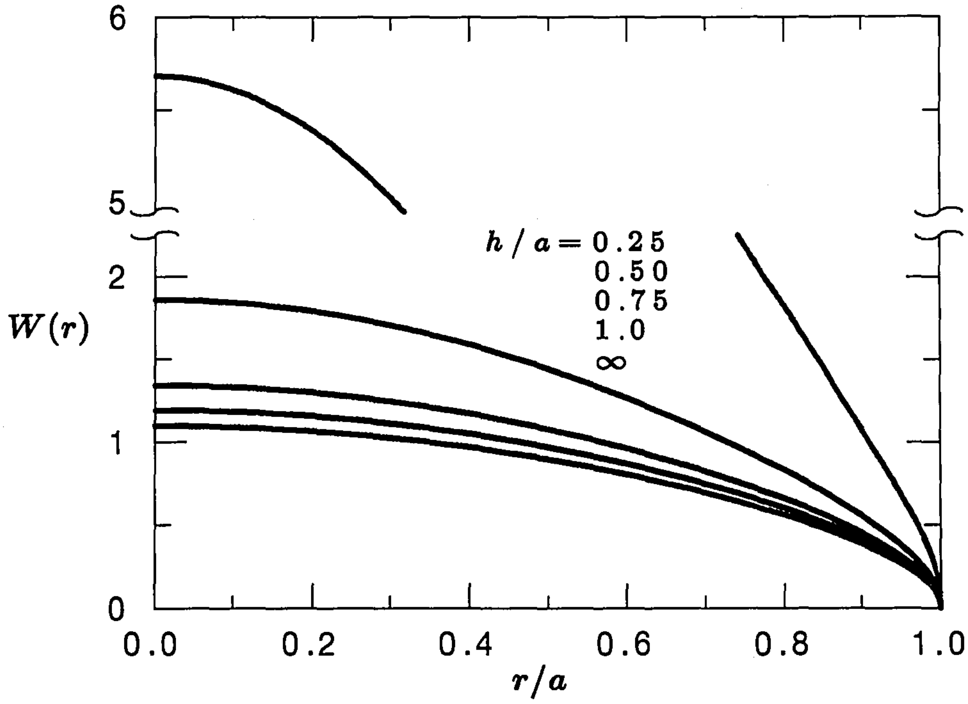

Figure 5.25: z- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$, and $\alpha a = 2.0$.

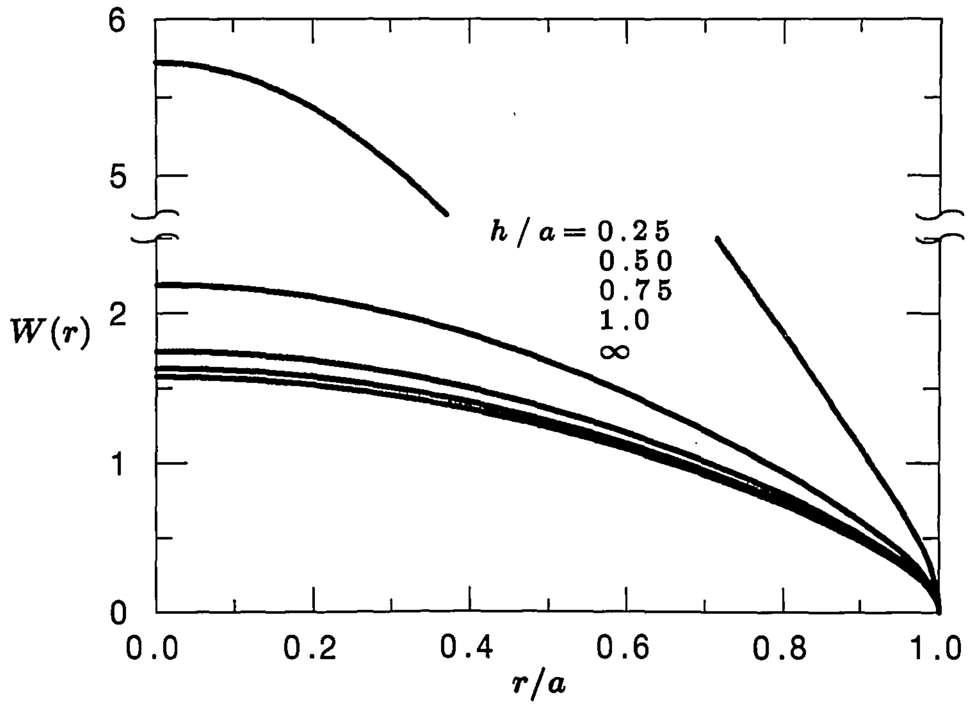

Figure 5.26: z- component of the normalized crack opening displacement for various h/a values in case of the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$, and $\alpha a = 3.0$.

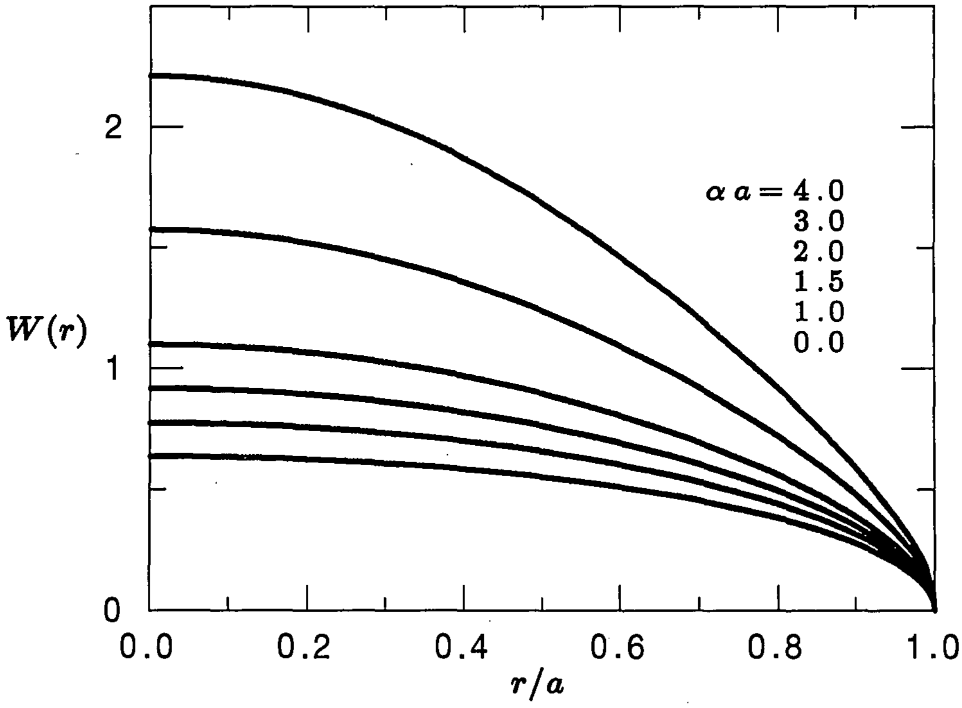

Figure 5.27: z- component of the normalized crack opening displacement for various αa values in case of the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$, and $h/a = 5.0$.

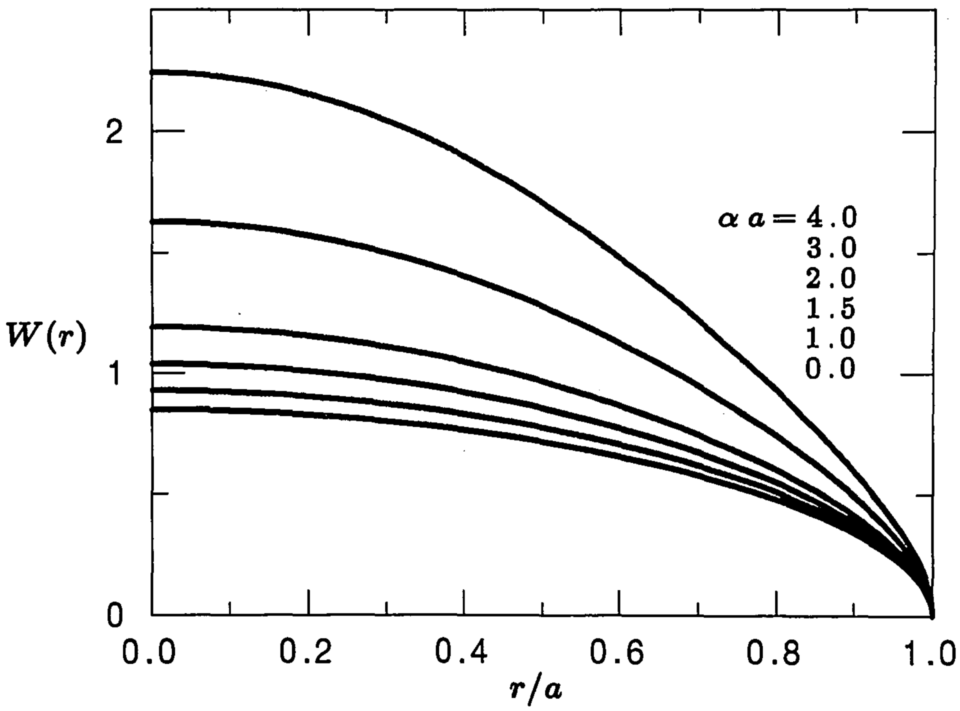

Figure 5.28: z- component of the normalized crack opening displacement for various αa values in case of the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$, and $h/a = 1.0$.

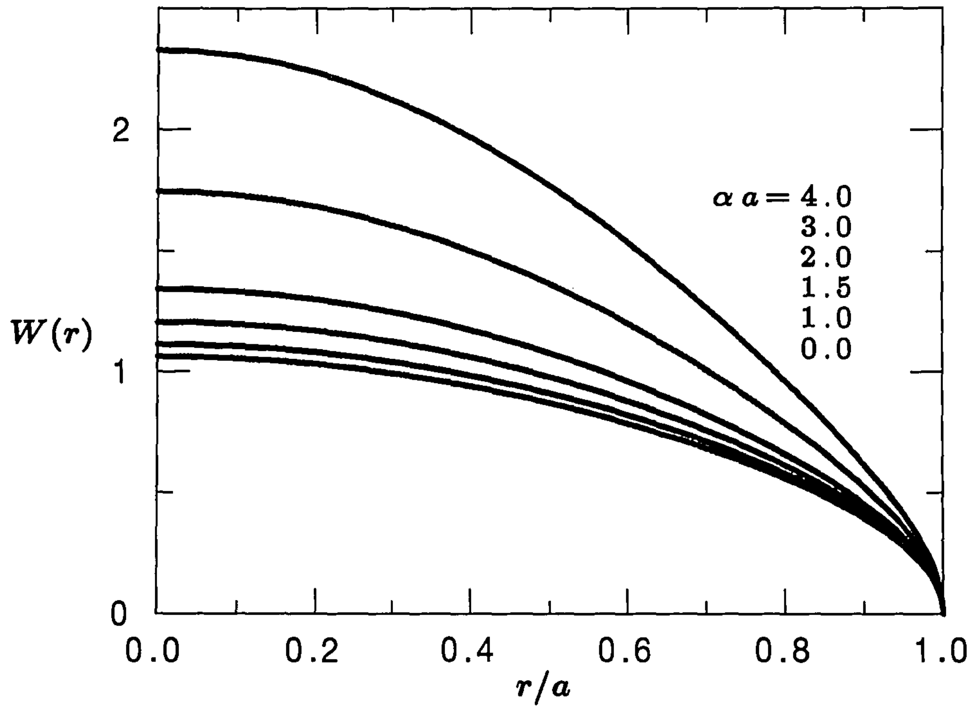

Figure 5.29: z- component of the normalized crack opening displacement for various αa values in case of the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$, and $h/a = 0.75$.

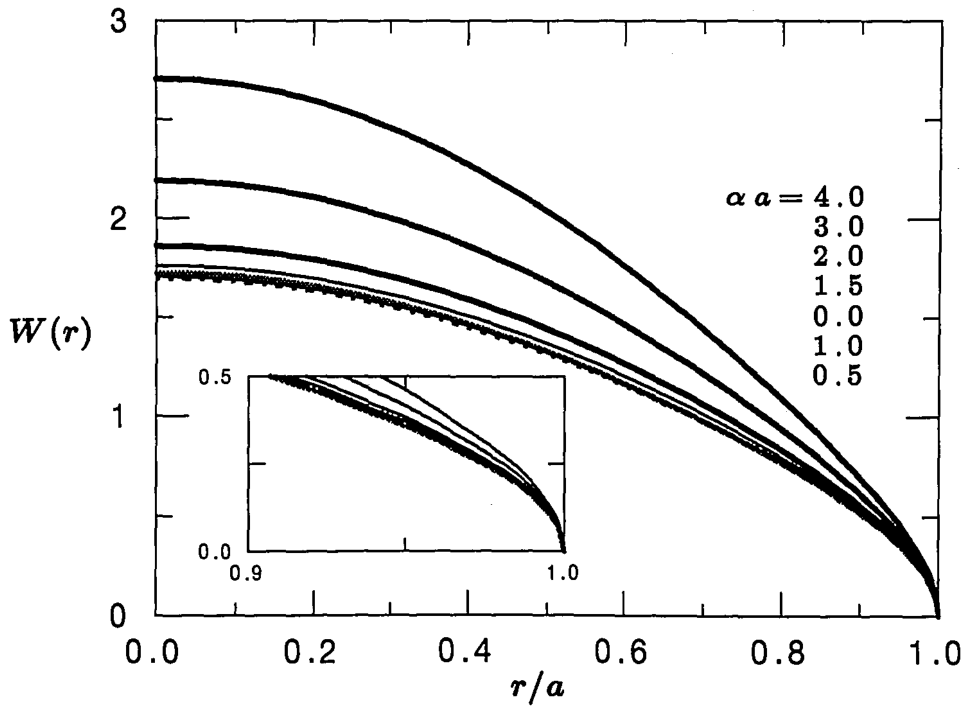

Figure 5.30: r- component of the normalized crack opening displacement for various αa values in case of the external loading $\sigma_{zz}(r,0) = -p_0$, $\sigma_{rz}(r,0) = 0$, and $h/a = 0.50$.

Conclusions and Future Works

In this study stress intensity factors and the crack opening displacements for different values of non-homogeneity parameter α and length parameter h have been investigated. Also studied is the effect of the Poisson's ratio ν on the stress intensity factors. The main conclusions may be summarized as follows :

(a) – For large h/a values, the calculated stress intensity factors agree with the results given in [13] within at least three digits.

(b) – When there was only normal loading $(\sigma_{zz}(r,0) = -p_0, \sigma_{rz}(r,0) = 0)$, it was observed that for large values of h/a, normalized stress intensity factor k_1 increases slowly as the non-homegeneity parameter α increases. However, for small values of h/a, (such as $h/a = 0.10$), the normalized stress intensity factor k_1 first decreases and then slowly increases with increasing α (Figure 5.1). Under the same loading k_2 increases with increasing α for all values of h/a. On the other hand for shear loading $(\sigma_{zz}(r,0) = 0$, $\sigma_{rz}(r,0) = -q_0)$, stress intensity factor k_1 increases for all values of h/a with increasing α, however, the values of k_1 are small. Similarly, k_2 increases for all values of h/a with increasing α, but the values of k_2 were small compared to k_1 under the normal loading.

Also it was observed that for negative α values k_1 under normal loading and k_2 under shear loading were almost symmetric for the large values of h/a.
Since the stress intensity factors do not depend on the magnitude of the shear modulus μ_0 for a crack in an infinite medium, this result is expected.

$(c) -$ It was observed that stress intensity factors k_1 and k_2 under respectively normal and shear loading tend to certain limiting values as h/a increases. On the other hand as expected, same stress intensity factors tend to infinity when h/a goes to zero. For large values of h/a the results agree with [13]. Also, for fixed values of α the stress intensity factors k_1 and k_2 under shear and normal loading, respectively, tend to certain limiting values which are, however, negligibly small.

$(d) -$ It was observed that under shear loading r-component of the normalized crack opening displacements $U(r)$ increases slowly when the length parameter h/a decreases. On the other hand, it is easy to see that under normal loading z-component of the normalized crack opening displacement $W(r)$ increases rather significantly with decreasing values of h/a. In both cases the results agree with [13] for large values of h/a.

$(e) -$ It was observed that stress intensity factors are relatively insensetive to variations in the Poisson's ratio for the small values of non-homogeneity parameter α and for all values af h/a. But for large α and small h/a the effect of Poisson's ratio may not be negligible. Some results are presented in Tables $5.18 - 5.27$ to give an idea about the influence of the variations in ν on the stress intensity factors. It may be seen that, generally, the influence of ν on the stress intensity factors is not very significant.

Among the possible continuation of this research one may mention the investigation of the axisymmetric interface crack problem in a FGM coating bonded to a homogeneous substrate and the spallation phenomenon resulting from the buckling instability. The in-plane compression that may cause buckling of the coating may be mechanical or thermal in nature.

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Appendix A

Expressions for various functions that appear in chapter 2 and chapter 3.

$$\delta = \sqrt{\frac{(3-\kappa)}{(\kappa+1)}}\,, \tag{A.1}$$

$$\xi = \frac{1}{2}\sqrt{\alpha^2 + 4\rho^2 + 4i\alpha\rho\delta}\,, \tag{A.2}$$

$$\overline{\xi} = \frac{1}{2}\sqrt{\alpha^2 + 4\rho^2 - 4i\alpha\rho\delta}\,, \tag{A.3}$$

$$m_1 = -\frac{1}{2}\alpha + \xi\,, \tag{A.4}$$

$$m_2 = -\frac{1}{2}\alpha - \xi\,, \tag{A.5}$$

$$\overline{m}_1 = -\frac{1}{2}\alpha + \overline{\xi}\,, \tag{A.6}$$

$$\overline{m}_2 = -\frac{1}{2}\alpha - \overline{\xi}\,, \tag{A.7}$$

$$a_1 = -\frac{2m_1 + \alpha(3-\kappa)}{2\rho + i\alpha\delta(\kappa+1)}\,, \tag{A.8}$$

$$a_2 = -\frac{2m_2 + \alpha(3-\kappa)}{2\rho + i\alpha\delta(\kappa+1)}\,, \tag{A.9}$$

$$\overline{a}_1 = -\frac{2\overline{m}_1 + \alpha(3-\kappa)}{2\rho + i\alpha\delta(\kappa+1)}, \tag{A.10}$$

$$\overline{a}_2 = -\frac{2\overline{m}_2 + \alpha(3-\kappa)}{2\rho + i\alpha\delta(\kappa+1)}, \tag{A.11}$$

$$n_1 = (3-\kappa)\rho + (\kappa+1)a_1 m_1, \tag{A.12}$$

$$n_2 = (3-\kappa)\rho + (\kappa+1)a_2 m_2, \tag{A.13}$$

$$\overline{n}_1 = (3-\kappa)\rho + (\kappa+1)\overline{a}_1\overline{m}_1, \tag{A.14}$$

$$\overline{n}_2 = (3-\kappa)\rho + (\kappa+1)\overline{a}_2\overline{m}_2, \tag{A.15}$$

$$v_1 = m_1 - a_1\rho, \tag{A.16}$$

$$v_2 = m_2 - a_2\rho, \tag{A.17}$$

$$\overline{v}_1 = \overline{m}_1 - \overline{a}_1\rho, \tag{A.18}$$

$$\overline{v}_2 = \overline{m}_2 - \overline{a}_2\rho, \tag{A.19}$$

$$\Delta_1 = n_1\overline{v}_1 - v_1\overline{n}_1, \qquad \Delta_1 = -\overline{\Delta}_1 \tag{A.20}$$

$$\lambda_1 = \frac{\overline{n}_1 v_2 - \overline{v}_1 n_2}{\Delta_1}, \tag{A.21}$$

$$\overline{\lambda}_1 = \frac{n_1\overline{v}_2 - v_1\overline{n}_2}{\overline{\Delta}_1}, \tag{A.22}$$

$$\lambda_3 = \frac{n_2 v_1 - v_2 n_1}{\Delta_1}, \tag{A.23}$$

$$\overline{\lambda}_3 = \frac{\overline{n}_2\overline{v}_1 - \overline{v}_2\overline{n}_1}{\overline{\Delta}_1}, \tag{A.24}$$

$$G_1 = n_2 + n_1\lambda_1 e^{-2\xi h} + \overline{n}_1\lambda_3 e^{-(\xi+\overline{\xi})h}, \tag{A.25}$$

$$G_2 = v_2 + v_1\lambda_1 e^{-2\xi h} + \overline{v}_1\lambda_3 e^{-(\xi+\overline{\xi})h}, \tag{A.26}$$

$$\overline{G}_1 = \overline{n}_2 + \overline{n}_1\overline{\lambda}_1 e^{-2\overline{\xi} h} + n_1\overline{\lambda}_3 e^{-(\xi+\overline{\xi})h}, \tag{A.27}$$

$$\overline{G}_2 = \overline{v}_2 + \overline{v}_1 \lambda_1 e^{-2\overline{\xi} h} + v_1 \overline{\lambda}_3 e^{-(\xi+\overline{\xi})h}, \tag{A.28}$$

$$\Delta_2 = G_1 \overline{G}_2 - \overline{G}_1 G_2, \qquad \Delta_2 = -\overline{\Delta}_2 \tag{A.29}$$

$$E_1 = \frac{n_1 \overline{G}_2 - v_1 \overline{G}_1}{\Delta_2}, \tag{A.30}$$

$$E_2 = \frac{-n_1 G_2 + v_1 G_1}{\Delta_2}, \tag{A.31}$$

$$\overline{E}_1 = \frac{\overline{n}_1 G_2 - \overline{v}_1 G_1}{\overline{\Delta}_2}, \tag{A.32}$$

$$\overline{E}_2 = \frac{-\overline{n}_1 \overline{G}_2 + \overline{v}_1 \overline{G}_1}{\overline{\Delta}_2}, \tag{A.33}$$

$$E_3 = \lambda_1 E_1 e^{-2\xi h} + \overline{\lambda}_3 E_2 e^{-(\xi+\overline{\xi})h}, \tag{A.34}$$

$$E_4 = \overline{\lambda}_1 E_2 e^{-2\overline{\xi} h} + \lambda_3 E_1 e^{-(\xi+\overline{\xi})h}, \tag{A.35}$$

$$\overline{E}_3 = \overline{\lambda}_1 \overline{E}_1 e^{-2\overline{\xi} h} + \lambda_3 \overline{E}_2 e^{-(\xi+\overline{\xi})h}, \tag{A.36}$$

$$\overline{E}_4 = \lambda_1 \overline{E}_2 e^{-2\xi h} + \overline{\lambda}_3 \overline{E}_1 e^{-(\xi+\overline{\xi})h}, \tag{A.37}$$

$$b_1 = a_1 - a_2 E_1 - \overline{a}_2 E_2 - a_1 E_3 - \overline{a}_1 E_4, \tag{A.38}$$

$$b_2 = E_1 + E_2 + E_3 + E_4 - 1, \tag{A.39}$$

$$\overline{b}_1 = \overline{a}_1 - \overline{a}_2 \overline{E}_1 - a_2 \overline{E}_2 - \overline{a}_1 \overline{E}_3 - a_1 \overline{E}_4, \tag{A.40}$$

$$\overline{b}_2 = \overline{E}_1 + \overline{E}_2 + \overline{E}_3 + \overline{E}_4 - 1, \tag{A.41}$$

$$\Delta_3 = b_1 \overline{b}_2 - \overline{b}_1 b_2, \qquad \Delta_3 = -\overline{\Delta}_3 \tag{A.42}$$

$$d_{11} = \frac{n_1 \overline{b}_2 - \overline{n}_1 b_2}{\Delta_3}, \tag{A.43}$$

$$d_{12} = \frac{-n_1 \overline{b}_1 + \overline{n}_1 b_1}{\Delta_3}, \tag{A.44}$$

$$d_{21} = \frac{v_1 \overline{b}_2 - \overline{v}_1 b_2}{\Delta_3}, \tag{A.45}$$

$$d_{22} = \frac{- v_1 \overline{b}_1 + \overline{v}_1 b_1}{\Delta_3}, \tag{A.46}$$

Appendix B

Asymptotic Analysis of Kernels

By defining a new variable

$$R = \frac{\alpha}{2\rho}, \tag{B.1}$$

from

$$m_1 = -\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 + 4\rho^2 + 4i\alpha\rho\delta} = -\frac{\alpha}{2} + \xi \tag{B.2}$$

$$m_1 = \rho M_{1,} \tag{B.3}$$

we find

$$M_1 = -R + \sqrt{1 + 2i\delta R + R^2}. \tag{B.4}$$

Similarly,

$$m_2 = \rho M_2, \qquad M_2 = -R - \sqrt{1 + 2i\delta R + R^2}, \tag{B.5}$$

$$m_1 = \rho \bar{M}_1, \qquad \bar{M}_1 = -R + \sqrt{1 - 2i\delta R + R^2}, \tag{B.6}$$

$$\bar{m}_2 = \rho \bar{M}_2, \qquad \bar{M}_2 = -R - \sqrt{1 - 2i\delta R + R^2}. \tag{B.7}$$

Defining

$$\eta = \sqrt{1 + 2i\delta R + R^2}, \tag{B.8}$$

$$\bar{\eta} = \sqrt{1 - 2i\delta R + R^2}, \tag{B.9}$$

we have

$$\xi = \rho\eta, \qquad \overline{\xi} = \rho\overline{\eta}. \tag{B.10}$$

By substituting the value of m_j in terms of M_j, $(j = 1, 2)$ into (2.34) we find

$$a_1 = -\frac{M_1 + (3-\kappa)R}{1 + i(\kappa+1)\delta R}, \tag{B.11}$$

$$a_2 = -\frac{M_2 + (3-\kappa)R}{1 + i(\kappa+1)\delta R}, \tag{B.12}$$

$$a_1 = \overline{a}_3 \qquad \text{and} \qquad a_2 = \overline{a}_4. \tag{B.13}$$

Expressing (A.12)-(A.19) as

$$n_1 = \rho N_1, \qquad N_1 = (3-\kappa) + (\kappa+1)a_1 M_1, \tag{B.14}$$

$$n_2 = \rho N_2, \qquad N_2 = (3-\kappa) + (\kappa+1)a_2 M_2, \tag{B.15}$$

$$\overline{n}_1 = \rho \overline{N}_1, \qquad \overline{N}_1 = (3-\kappa) + (\kappa+1)\overline{a}_1 \overline{M}_1, \tag{B.16}$$

$$\overline{n}_2 = \rho \overline{N}_2, \qquad \overline{N}_2 = (3-\kappa) + (\kappa+1)\overline{a}_2 \overline{M}_2, \tag{B.17}$$

$$v_1 = \rho V_1, \qquad V_1 = M_1 - a_1, \tag{B.18}$$

$$v_2 = \rho V_2, \qquad V_2 = M_2 - a_2, \tag{B.19}$$

$$\overline{v}_1 = \rho \overline{V}_1, \qquad \overline{V}_1 = \overline{M}_1 - \overline{a}_1, \tag{B.20}$$

$$\overline{v}_2 = \rho \overline{V}_2, \qquad \overline{V}_2 = \overline{M}_2 - \overline{a}_2. \tag{B.21}$$

(A.20) becomes

$$\Delta_1 = \rho^2(N_1\overline{V}_1 - \overline{N}_1 V_1), \qquad \Delta_1^{'} = N_1\overline{V}_1 - \overline{N}_1 V \tag{B.22}$$

$$\Delta_1^{'} = -\overline{\Delta}_1^{'}. \tag{B.23}$$

Define the coefficients A_{11} and A_{13} in terms of $M_i, a_i,\ N_i$ and $V_i,\ (i=1,2)$, as

$$A_{11} = \lambda_1 e^{-2\eta\rho h} A_{12} + \overline{\lambda}_3 e^{-(\eta+\overline{\eta})\rho h} A_{14}, \tag{B.25}$$

$$A_{13} = \lambda_3 e^{-(\eta+\overline{\eta})\rho h} A_{12} + \overline{\lambda}_1 e^{-2\overline{\eta}\rho h} A_{14}, \tag{B.26}$$

where

$$\lambda_1 = \frac{\overline{N}_1 V_2 - \overline{V}_1 N_2}{\Delta_1'}, \tag{B.27}$$

$$\lambda_3 = \frac{N_2 V_1 - V_2 N_1}{\Delta_1'}. \tag{B.28}$$

Referring to (A.25)-(A.27), we may write

$$G_1' = N_2 + N_1\lambda_1 e^{-2\eta\rho h} + \overline{N}_1 \lambda_3 e^{-(\eta+\overline{\eta})\rho h}, \tag{B.29}$$

$$\overline{G}_1' = \overline{N}_2 + \overline{N}_1 \overline{\lambda}_1 e^{-2\overline{\eta}\rho h} + N_1 \overline{\lambda}_3 e^{-(\eta+\overline{\eta})\rho h}, \tag{B.30}$$

$$G_2' = V_2 + V_1\lambda_1 e^{-2\eta\rho h} + \overline{V}_1 \lambda_3 e^{-(\eta+\overline{\eta})\rho h}, \tag{B.31}$$

$$\overline{G}_2' = \overline{V}_2 + \overline{V}_1 \overline{\lambda}_1 e^{-2\overline{\eta}\rho h} + V_1 \overline{\lambda}_3 e^{-(\eta+\overline{\eta})\rho h}, \tag{B.32}$$

and

$$\Delta_2 = \rho^2\left(G_1'\overline{G}_2' - \overline{G}_1' G_2'\right), \qquad \Delta_2' = G_1'\overline{G}_2' - \overline{G}_1' G_2', \tag{B.33}$$

$$\Delta_2' = -\overline{\Delta}_2'. \tag{B.34}$$

Then, the coefficients A_{12} and A_{14} may be expressed in terms of $M_i,\ a_i,\ N_i,\ V_i$ and G_i', $(i=1,2)$, as follows :

$$A_{12} = E_1 A_{21} + \overline{E}_2 A_{23}, \tag{B.35}$$

$$A_{14} = E_2 A_{21} + \overline{E}_1 A_{23}, \tag{B.36}$$

where

$$E_1 = \frac{\left(N_1 \overline{G}_2' - V_1 \overline{G}_1'\right)}{\Delta_2'}, \tag{B.37}$$

$$\overline{E}_1 = \frac{\left(\overline{N}_1 G_2' - \overline{V}_1 G_1'\right)}{\overline{\Delta}_2'}, \tag{B.38}$$

$$E_2 = -\frac{\left(N_1 G_2' - V_1 G_1'\right)}{\Delta_2'}, \tag{B.39}$$

$$\overline{E}_2 = -\frac{\left(\overline{N}_1 \overline{G}_2' - \overline{V}_1 \overline{G}_1'\right)}{\overline{\Delta}_2'}. \tag{B.40}$$

As a result, all four coefficients A_{1j},(j=1,2,3,4), can be expressed in terms of A_{21} and A_{22}, as

$$A_{11} = \left(\lambda_1 E_1 e^{-2\eta\rho h} + \overline{\lambda}_3 E_2 e^{-(\eta+\overline{\eta})\rho h}\right) A_{21} + \left(\lambda_1 \overline{E}_2 e^{-2\eta\rho h} + \overline{\lambda}_3 \overline{E}_1 e^{-(\eta+\overline{\eta})\rho h}\right) A_{23}, \tag{B.41}$$

$$A_{12} = E_1 A_{21} + \overline{E}_2 A_{23}, \tag{B.42}$$

$$A_{13} = \left(\overline{\lambda}_1 E_2 e^{-2\overline{\eta}\rho h} + \lambda_3 E_1 e^{-(\eta+\overline{\eta})\rho h}\right) A_{21} + \left(\overline{\lambda}_1 \overline{E}_1 e^{-2\overline{\eta}\rho h} + \lambda_3 \overline{E}_2 e^{-(\xi+\overline{\xi})h}\right) A_{23}, \tag{B.43}$$

$$A_{14} = E_2 A_{21} + \overline{E}_1 A_{23}, \tag{B.44}$$

where

$$E_3 = \lambda_1 E_1 e^{-2\eta\rho h} + \overline{\lambda}_3 E_2 e^{-(\eta+\overline{\eta})\rho h}, \tag{B.45}$$

$$\overline{E}_3 = \overline{\lambda}_1 \overline{E}_1 e^{-2\overline{\eta}\rho h} + \lambda_3 \overline{E}_2 e^{-(\eta+\overline{\eta})\rho h}, \tag{B.46}$$

$$E_4 = \overline{\lambda}_1 E_2 e^{-2\overline{\eta}\rho h} + \lambda_3 E_1 e^{-(\eta+\overline{\eta})\rho h}, \tag{B.47}$$

$$\overline{E}_4 = \lambda_1 \overline{E}_2 e^{-2\eta\rho h} + \overline{\lambda}_3 \overline{E}_1 e^{-(\eta+\overline{\eta})\rho h}, \tag{B.48}$$

$$b_1 = a_1 - a_1 E_3 - a_2 E_1 - \overline{a}_1 E_4 - \overline{a}_2 E_2, \tag{B.49}$$

$$\overline{b}_1 = \overline{a}_1 - \overline{a}_1 \overline{E}_3 - \overline{a}_2 \overline{E}_1 - a_1 \overline{E}_4 - a_2 \overline{E}_2, \tag{B.50}$$

$$b_2 = E_1 + E_2 + E_3 + E_4 - 1, \tag{B.51}$$

$$\overline{b}_2 = \overline{E}_1 + \overline{E}_2 + \overline{E}_3 + \overline{E}_4 - 1, \tag{B.52}$$

$$\Delta_3 = b_1 \overline{b}_2 - \overline{b}_1 b_2, \qquad \Delta_3 = -\overline{\Delta}_3, \tag{B.53}$$

$$d'_{11}(R) = \frac{1}{\Delta_3}((\kappa+1)(M_1 a_1 \overline{b}_2 - \overline{M}_1 \overline{a}_1 b_2) + (3-\kappa)(\overline{b}_2 - b_2)), \tag{B.54}$$

$$d'_{12}(R) = \frac{1}{\Delta_3}((\kappa+1)(\overline{M}_1 \overline{a}_1 b_1 - M_1 a_1 \overline{b}_1) + (3-\kappa)(b_1 - \overline{b}_1)), \tag{B.55}$$

$$d'_{21}(R) = \frac{1}{\Delta_3}((M_1 - a_1)\overline{b}_2 - (\overline{M}_1 - \overline{a}_1) b_2), \tag{B.56}$$

$$d'_{22}(R) = \frac{1}{\Delta_3}((\overline{M}_1 - \overline{a}_1) b_1 - (M_1 - a_1)\overline{b}_1). \tag{B.57}$$

When ρ goes to infinity, R will go to 0 and d'_{ij}, $(i, j = 1, 2)$, can be expressed as :

$$d'_{ij}(R) = \sum_{k=0}^{\infty} d'^{k}_{ij} R^k$$

where

$$d'^{0}_{11} = 2\frac{(\kappa-1)}{(\kappa+1)}, \tag{B.58}$$

$$d'^{2}_{11} = 2\frac{(\kappa-1)(\kappa-9)}{(\kappa+1)^2}, \tag{B.59}$$

$$d'^{4}_{11} = 2\frac{(\kappa-1)(\kappa^2-26\kappa+99)}{(\kappa+1)^3}, \tag{B.60}$$

$$d_{11}^{'6} = 2\frac{(\kappa-1)(\kappa^3-51\kappa^2+489\kappa-1175)}{(\kappa+1)^4}, \tag{B.61}$$

$$d_{11}^{'8} = 2\frac{(\kappa-1)(\kappa^4-84\kappa^3+1458\kappa^2-8268\kappa+14499)}{(\kappa+1)^5}, \tag{B.62}$$

$$d_{11}^{'10} = 2\frac{(\kappa-1)(\kappa^5-125\kappa^4+3390\kappa^3-33270\kappa^2+132735\kappa-183195)}{(\kappa+1)^6}, \tag{B.63}$$

$$d_{11}^{'1} = d_{11}^{'3} = d_{11}^{'5} = d_{11}^{'7} = d_{11}^{'9} = d_{11}^{'11} = 0. \tag{B.64}$$

$$d_{12}^{'1} = 2\frac{(\kappa-1)}{(\kappa+1)}, \tag{B.65}$$

$$d_{12}^{'3} = 2\frac{(\kappa-1)(\kappa-9)}{(\kappa+1)^2}, \tag{B.66}$$

$$d_{12}^{'5} = 2\frac{(\kappa-1)(\kappa^2-26\kappa+99)}{(\kappa+1)^3}, \tag{B.67}$$

$$d_{12}^{'7} = 2\frac{(\kappa-1)(\kappa^3-51\kappa^2+489\kappa-1175)}{(\kappa+1)^4}, \tag{B.68}$$

$$d_{12}^{'9} = 2\frac{(\kappa-1)(\kappa^4-84\kappa^3+1458\kappa^2-8268\kappa+14499)}{(\kappa+1)^5}, \tag{B.69}$$

$$d_{12}^{'11} = 2\frac{(\kappa-1)(\kappa^5-125\kappa^4+3390\kappa^3-33270\kappa^2+132735\kappa-183195)}{(\kappa+1)^6}, \tag{B.70}$$

$$d_{12}^{'0} = d_{12}^{'2} = d_{12}^{'4} = d_{12}^{'6} = d_{12}^{'8} = d_{12}^{'10} = 0. \tag{B.71}$$

$$d_{21}^{'1} = -2\frac{1}{(\kappa+1)}, \tag{B.72}$$

$$d_{21}^{'3} = -2\frac{(\kappa-9)}{(\kappa+1)^2}, \tag{B.73}$$

$$d_{21}^{'5} = -2\frac{(\kappa^2-26\kappa+99)}{(\kappa+1)^3}, \tag{B.74}$$

$$d_{21}^{'7} = -2\frac{(\kappa^3-51\kappa^2+489\kappa-1175)}{(\kappa+1)^4}, \tag{B.75}$$

$$d_{21}^{'9} = -2\frac{(\kappa^4-84\kappa^3+1458\kappa^2-8268\kappa+14499)}{(\kappa+1)^5}, \tag{B.76}$$

$$d_{21}^{'11} = -2\frac{(\kappa^5-125\kappa^4+3390\kappa^3-33270\kappa^2+132735\kappa-183195)}{(\kappa+1)^6}, \tag{B.77}$$

$$d_{21}^{'0} = d_{21}^{'2} = d_{21}^{'4} = d_{21}^{'6} = d_{21}^{'8} = d_{21}^{'10} = 0. \tag{B.78}$$

$$d_{22}^{'0} = -2\frac{1}{(\kappa+1)}, \tag{B.79}$$

$$d_{22}^{'2} = -2\frac{(\kappa-1)}{(\kappa+1)^2}, \tag{B.80}$$

$$d_{22}^{'4} = -2\frac{(\kappa^2-10\kappa+3)}{(\kappa+1)^3}, \tag{B.81}$$

$$d_{22}^{'6} = -2\frac{(\kappa^3-27\kappa^2+105\kappa+1)}{(\kappa+1)^4}, \tag{B.82}$$

$$d_{22}^{'8} = -2\frac{(\kappa^4-52\kappa^3+498\kappa^2-1132\kappa-253)}{(\kappa+1)^5}, \tag{B.83}$$

$$d_{22}^{'10} = -2\frac{(\kappa^5-85\kappa^4+1470\kappa^3-8070\kappa^2+12255\kappa+5085)}{(\kappa+1)^6}, \tag{B.84}$$

$$d_{22}^{'1} = d_{22}^{'3} = d_{22}^{'5} = d_{22}^{'7} = d_{22}^{'9} = d_{22}^{'11} = 0. \tag{B.85}$$

When ρ tends to infinity $d_{11}^{'}, d_{12}^{'}, d_{21}^{'}$ and $d_{22}^{'}$ behave as follows :

$$d_{11}^{'}(\rho) = 2\frac{(\kappa-1)}{(\kappa+1)} + 2\frac{(\kappa-1)(\kappa-9)}{(\kappa+1)^2}\left(\frac{\alpha}{2\rho}\right)^2 + O\left(\frac{1}{\rho^4}\right), \tag{B.86}$$

$$d_{12}^{'}(\rho) = 2\frac{(\kappa-1)}{(\kappa+1)}\left(\frac{\alpha}{2\rho}\right) + 2\frac{(\kappa-1)(\kappa-9)}{(\kappa+1)^2}\left(\frac{\alpha}{2\rho}\right)^3 + O\left(\frac{1}{\rho^5}\right), \tag{B.87}$$

$$d_{21}^{'}(\rho) = -2\frac{1}{(\kappa+1)}\left(\frac{\alpha}{2\rho}\right) - 2\frac{(\kappa-9)}{(\kappa+1)^2}\left(\frac{\alpha}{2\rho}\right)^3 + O\left(\frac{1}{\rho^5}\right), \tag{B.88}$$

$$d_{22}^{'}(\rho) = -2\frac{1}{(\kappa+1)} - 2\frac{(\kappa-9)}{(\kappa+1)^2}\left(\frac{\alpha}{2\rho}\right)^2 + O\left(\frac{1}{\rho^4}\right). \tag{B.89}$$

Also, dividing $d_{11}^{'}$ and $d_{12}^{'}$ by $d_{11}^{'0}$ and $d_{21}^{'}$ and $d_{22}^{'}$ by $d_{22}^{'0}$, it can be shown that (see equations (3.49)-(3.52)).

$$D_{11}(\rho) = \frac{(\kappa-9)}{(\kappa+1)}\left(\frac{\alpha}{2\rho}\right)^2 + \frac{(\kappa^2-26\kappa+99)}{(\kappa+1)^2}\left(\frac{\alpha}{2\rho}\right)^4 + O\left(\frac{1}{\rho^6}\right), \tag{B.90}$$

$$D_{12}(\rho) = \left(\frac{\alpha}{2\rho}\right) + \frac{(\kappa-9)}{(\kappa+1)}\left(\frac{\alpha}{2\rho}\right)^3 + \frac{(\kappa^2-26\kappa+99)}{(\kappa+1)^2}\left(\frac{\alpha}{2\rho}\right)^5 + O\left(\frac{1}{\rho 7}\right), \tag{B.91}$$

$$D_{21}(\rho) = \left(\frac{\alpha}{2\rho}\right) + \frac{(\kappa-9)}{(\kappa+1)}\left(\frac{\alpha}{2\rho}\right)^3 + \frac{(\kappa^2-26\kappa+99)}{(\kappa+1)^2}\left(\frac{\alpha}{2\rho}\right)^5 + O\left(\frac{1}{\rho^7}\right), \tag{B.92}$$

$$D_{22}(\rho) = \frac{(\kappa-9)}{(\kappa+1)}\left(\frac{\alpha}{2\rho}\right)^2 + \frac{(\kappa^2-10\kappa+3)}{(\kappa+1)^2}\left(\frac{\alpha}{2\rho}\right)^4 + O\left(\frac{1}{\rho^4}\right). \tag{B.93}$$

Appendix C

Examination of the Logarithmic Kernels

C.1 Elliptic Integrals

The expressions of the integrals giving the kernels contain complete elliptic integrals of the first and the second kind [21] which are defined by

$$K = K(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 sin^2\theta}}, \tag{C.1.1}$$

$$E = E(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 sin^2\theta}\, d\theta. \tag{C.1.2}$$

derivatives with respect to modulus k are given by

$$\frac{dK}{dk} = \frac{E - k'^2 K}{kk'^2}, \tag{C.1.3}$$

$$\frac{dE}{dk} = \frac{E - K}{k}, \tag{C.1.4}$$

where k' is the complementary modulus

$$k' = \sqrt{1 - k^2}. \tag{C.1.5}$$

When the modulus k tends to 1, the complete elliptic integrals have the following asymptotic properties:

$$K(k) = \log\left(\frac{4}{\sqrt{1-k^2}}\right), \quad k \to 1 \tag{C.1.6}$$

$$E(k) = 1, \qquad k \to 1. \tag{C.1.7}$$

By using the properties of the complete elliptic integrals, we now examine parts of the kernels that can be expressed in closed form.

C.2 $K_{11}^{\infty}(s,r) = \int_0^{\infty} J_0(r\rho) J_1(s\rho)\rho \, d\rho$

In equation (3.53) the third integral, namely $K_{11}^{\infty}(s,r)$, may be expressed in terms of the complete elliptic integrals as follows :

$$\int_0^{\infty} J_0(r\rho)J_1(s\rho)\rho\, d\rho = \frac{2}{\pi}\begin{cases} \dfrac{r}{s}\dfrac{1}{s^2-r^2}E\left(\dfrac{s}{r}\right) + \dfrac{1}{rs}K\left(\dfrac{s}{r}\right), & s < r, \\ \dfrac{1}{s^2-r^2}E\left(\dfrac{r}{s}\right), & s > r. \end{cases} \tag{C.2.1}$$

Rewriting (C.2.1) for $s < r$, we have

$$\frac{1}{s^2-r^2}\left(\frac{r}{s}E\left(\frac{s}{r}\right) + \frac{s^2-r^2}{rs}K\left(\frac{s}{r}\right)\right), \tag{C.2.2}$$

adding the expression,

$$\mp\frac{1}{2}\left(\frac{1}{s-r}+\frac{1}{s+r}\right), \tag{C.2.3}$$

and by using the identity

$$\frac{1}{s^2-r^2}=\frac{1}{2s}\left(\frac{1}{s-r}+\frac{1}{s+r}\right), \tag{C.2.4}$$

for $s<r$, the integral (C.2.1) becomes

$$\frac{1}{2s}\left(\frac{1}{s-r}+\frac{1}{s+r}\right)$$
$$+\frac{1}{2s}\left(\frac{\frac{r}{s}E\left(\frac{s}{r}\right)+\frac{s^2-r^2}{rs}K\left(\frac{s}{r}\right)-1}{s-r}+\frac{\frac{r}{s}E\left(\frac{s}{r}\right)+\frac{s^2-r^2}{rs}K\left(\frac{s}{r}\right)-1}{s+r}\right). \tag{C.2.5}$$

Similarly, for $s>r$ we find

$$\frac{1}{2s}\left(\frac{1}{s-r}+\frac{1}{s+r}\right)+\frac{1}{2s}\left(\frac{E\left(\frac{r}{s}\right)-1}{s-r}+\frac{E\left(\frac{r}{s}\right)-1}{s+r}\right),\ s>r, \tag{C.2.6}$$

or, by defining

$$M_2(s,r)=\begin{cases}\frac{r}{s}E\left(\frac{s}{r}\right)+\frac{s^2-r^2}{rs}K\left(\frac{s}{r}\right), & s<r,\\[2ex] E\left(\frac{r}{s}\right), & s>r,\end{cases} \tag{C.2.7}$$

the integral , $K_{11}^{\infty}(s,r)$, becomes,

$$sK_{11}^{\infty}(s,r)=\frac{1}{\pi}\left(\frac{1}{s-r}+\frac{1}{s+r}+\frac{M_2(s,r)-1}{s-r}+\frac{M_2(s,r)-1}{s+r}\right). \tag{C.2.8}$$

C.3 $K_{22}^{\infty}(s,r) = \int_0^{\infty} J_1(r\rho) J_0(s\rho) \rho \, d\rho$

Referring to (3.54), the third integral, namely $K_{22}^{\infty}(s,r)$, may be expressed as

$$\int_0^{\infty} J_1(r\rho) J_0(s\rho) \rho \, d\rho = \frac{2}{\pi} \begin{cases} \dfrac{s}{r} \dfrac{1}{(r^2 - s^2)} E\left(\dfrac{r}{s}\right) + \dfrac{1}{rs} K\left(\dfrac{r}{s}\right), & s > r, \\ \dfrac{1}{r^2 - s^2} E\left(\dfrac{s}{r}\right), & s < r, \end{cases} \quad \text{(C.3.1)}$$

Following the procedure described in (C.2) and defining

$$M_4(s,r) = \begin{cases} \dfrac{s^2}{r^2} E\left(\dfrac{r}{s}\right) + \dfrac{r^2 - s^2}{r^2} K\left(\dfrac{r}{s}\right), & s > r, \\ \dfrac{s}{r} E\left(\dfrac{s}{r}\right), & s > r, \end{cases} \quad \text{(C.3.2)}$$

it can be shown that

$$sK_{22}^{\infty}(s,r) = \frac{1}{\pi} \left(\frac{1}{s-r} - \frac{1}{s+r} + \frac{M_4(s,r) - 1}{s-r} - \frac{M_4(s,r) - 1}{s+r} \right). \quad \text{(C.3.3)}$$

C.4 $H_{11}(s,r) = \int_0^{\infty} J_0(r\rho) J_0(s\rho) \, d\rho$

By adding

$$\pm \pi s \frac{\alpha}{2} \int_0^{\infty} J_0(r\rho) J_0(s\rho) \, d\rho, \quad \text{(C.4.1)}$$

from (3.58), it can be shown that

$$k_{12}(s,r) = \pi s \int_0^{\infty} \left(D_{12}(\rho)\rho - \frac{\alpha}{2}\right) J_0(r\rho) J_0(s\rho)\, d\rho$$
$$+ \pi s \frac{\alpha}{2} \int_0^{\infty} J_0(r\rho) J_0(s\rho)\, d\rho, \qquad \text{(C.4.2)}$$

where

$$D_{12}(\rho)\rho - \frac{\alpha}{2} = \frac{(\kappa-9)}{(\kappa+1)}\left(\frac{\alpha}{2}\right)^3 \frac{1}{\rho^2} + \frac{(\kappa^2 - 26\kappa + 99)}{(\kappa+1)^2}\left(\frac{\alpha}{2}\right)^5 \frac{1}{\rho^4} + O\left(\frac{1}{\rho^6}\right). \qquad \text{(C.4.3)}$$

The second integral in (C.4.2) or $H_{11}(s,r)$, has a closed-form expression given by

$$\int_0^{\infty} J_0(r\rho) J_0(s\rho)\, d\rho = \frac{2}{\pi} \begin{cases} \dfrac{1}{r} K\left(\dfrac{s}{r}\right), & s < r, \\ \dfrac{1}{s} K\left(\dfrac{r}{s}\right), & s > r. \end{cases} \qquad \text{(C.4.4)}$$

Hence, referring to (C.1.6) it is seen that at $r = s$ $k_{12}(r,s)$ has a logarithmic singularity .

C.5 $H_{22}(s,r) = \int_0^{\infty} J_1(r\rho) J_1(s\rho) \rho\, d\rho$

Similarly, by adding and subtracting the integral

$$\pm \pi s \frac{\alpha}{2} \int_0^{\infty} J_1(r\rho) J_1(s\rho)\, d\rho, \qquad \text{(C.5.1)}$$

the kernel $k_{21}(r,s)$ may be expressed as

$$k_{21}(s,r) = \pi s \int_0^\infty \left(D_{21}(\rho)\rho - \frac{\alpha}{2}\right) J_1(r\rho) J_1(s\rho)\, d\rho$$

$$+ \pi s \frac{\alpha}{2} \int_0^\infty J_1(r\rho) J_1(s\rho)\, d\rho, \qquad \text{(C.5.2)}$$

where

$$D_{21}(\rho)\rho - \frac{\alpha}{2} = \frac{(\kappa-9)}{(\kappa+1)}\left(\frac{\alpha}{2}\right)^3 \frac{1}{\rho^2} + \frac{(\kappa^2 - 26\kappa + 99)}{(\kappa+1)^2}\left(\frac{\alpha}{2}\right)^5 \frac{1}{\rho^4} + O\left(\frac{1}{\rho^6}\right). \qquad \text{(C.5.3)}$$

The second integral in (C.5.2),namely $H_{22}(s,r)$, has a closed-form expression which is given by,

$$\int_0^\infty J_1(r\rho) J_1(s\rho)\, d\rho = \frac{2}{\pi} \begin{cases} \frac{1}{s}\left(K\left(\frac{s}{r}\right) - E\left(\frac{s}{r}\right)\right), & s < r, \\ \frac{1}{r}\left(K\left(\frac{r}{s}\right) - E\left(\frac{r}{s}\right)\right), & s > r. \end{cases} \qquad \text{(C.5.4)}$$

Also, referrring to (C.1.6) it is seen that at $r = s$ $k_{21}(r,s)$ has a logarithmic singularity .

Vita

Ali Sahin was born in Istanbul, Turkey on the 25th of June, 1968. After graduating from Haydarpasa Technical High School in 1986, he had woked in Northern Telecommunication as a quality control expert. In 1987, he had started his undergraduate study in Deparment of Engineering Mathematics of Yildiz University, Istanbul, Turkey. After receiving his B.S. degree, he had worked as a teaching assistant in the same University. In 1993, he had worked as a research assistant in Gebze Advanced Institute of Technology. He joined Lehigh University in 1994.

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