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On multiperiod stock portfolio selection with taxes and transaction cost

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On Multiperiod Stock Portfolio Selection with Taxes and Transaction Costs

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Chapter 1: Introduction

The portfolio selection problem is a well-known financial and mathematical problem where an investor uses quantitative measures of risk and expected return to create a portfolio. The simplest form of the problem is to construct a collection of securities that cannot be outperformed by any other collection in the market, under the same conditions. An investor must first choose a relationship between risk and return, known as a utility function, before a solution can be found. With that in mind, the real problem is to strategically select securities based on the utility function into proportions that minimize risk for a given expected return. The efficient portfolio is the optimal solution and the efficient frontier is the set of all optimal portfolios for all possible expected returns.

Markowitz [7] pioneered the portfolio selection problem in the early 1950's. He quantified the need to diversify a portfolio of risky assets. Markowitz described risk as the variance of returns about an expected return (this was later modified to minimizing variance below expected return, or semivariance). Based on his theory, the covariance of the returns on the securities selected must be at a minimum in order to increase the certainty of attaining an expected return. His model for selecting optimal portfolios consisted of the objective minimizing (semi) variance (as a function of covariance) of returns for any given expected return subject to several constraints. This concept of mean-variance optimization was a breakthrough in portfolio selection and is widely recognized even today.

Soon after the seminal work of Markowitz, Sharpe [9] derived an equilibrium model that described expected return and risk of securities as a function of the expected return and risk of the market. This Capital Asset Pricing Model (CAPM) has a major influence on the behavior of the portfolio selection model and is incorporated into the portfolio selection model.

Based on the nonlinear objective function that describes an investor's utility, solutions to the portfolio selection model are determined through quadratic programming. Quadratic programming problems with a large number of variables can be difficult to solve efficiently. Sharpe [10] [11] (and Stone [12]) later developed a linear programming approximation method that efficiently solves the portfolio selection problem for many variables.

By incorporating the previous methods together, an efficient portfolio can be created for a single period. Unfortunately, the selected portfolio may not remain efficient as the market changes over time. Efficient portfolios that dominate the initial model are sure to arise when the market changes.

For multiperiod investing, an investor must adhere to new objectives, constraints, and penalties[1] to maintain favorable returns over time. Penalties include transaction costs for buying and selling assets, and taxes on capital gains earned from selling an asset. Modeling a multiperiod portfolio that considers these penalties is a complex subject related to the original portfolio selection problem. To the author's knowledge, little effort has been made on this subject.

The purpose of this paper is to introduce an approach to maximize the returns from a portfolio in multiple periods that include taxes and transaction costs. Since most models ignore such penalties, the approach will enhance the practicality of the portfolio selection model. The concept behind the approach is simple: after an efficient portfolio is presented

[1] By penalties, we are referring to the combination of transaction costs from buying and selling shares and the taxes from capital gains, where applicable.

in a subsequent period, the objective would be to adjust the securities that benefit more from the transaction that the actual cost of the transaction. The portfolio would be allowed to go off course from the efficient set for a short period, and then converge back to efficiency after a specified time interval. The result will be a less efficient portfolio that will outperform the efficient portfolio when subject to multiperiod constraints. The portfolio may be considered inefficient under terms of single period portfolio selection, but it will be efficient to all other portfolios that consider the effect of transaction penalties. We find a solution with the use of a mixed integer knapsack program.

This paper is outlined as follows. Next we review the literature relevant to generating a single period portfolio. In chapter 2, the investment environment is described and modeled while a delineation of alternative strategies are discussed and compared to a three period investment horizon. Chapter 3 will detail the solution approach to the multiperiod model. In chapter 4, models of the alternative strategies over a 35 period investment horizon are compared to the stock market. The paper is finally summarized and further work is discussed in chapters 5 and 6. Chapter 7 contains the appendix.

Literature Review

1.1 Markowitz: Portfolio Selection

Markowitz developed a program that, based on a given risk measure and investor utility function, would select an efficient portfolio. With N decision variables X_i and X_j as the proportions of the total wealth allocated to assets i and j ($j \neq i$), the solution is to apportion wealth into a pool of assets in a manner that minimizes the variance of returns for a given expected return. The program is as follows:

min.

$$V_p = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j \sigma_{ij} \qquad (1)$$

s.t.

$$E_p = \sum_{i=1}^{N} X_i \mu_i \qquad (2)$$

$$\sum_{i=1}^{N} X_i = 1 \qquad (3)$$

$$X_i \geq 0 \qquad (4)$$

The objective is to minimize the variance (1), stated as a function of the covariance, for a given expected return (2). The expected return of the portfolio is the sum of returns of the individual assets

$$\mu_i = E(R_i) \qquad (5).$$

The objective function is subject to the constraints that the wealth of the investment be exhausted in the portfolio (4) and that all proportions of the investment be nonnegative (5), disallowing short sales. Based on the objective function, the solution is found with the use of a quadratic program. The optimal solution offers no lower risk for every given return and no higher return for a given risk. Other portfolios may be attainable, but the efficient portfolios created through the program have a dominant relationship between mean-variance of returns. The returns on the assets are simply the proportions of the wealth multiplied by the change in price of the asset $r_i = X_i R_i$, and the return on the portfolio is the sum of the returns of the assets $r_p = \sum_{i=1}^{N} X_i R_i$.

The assumptions behind the theory of the model include:

- Risk is defined by the variance of the returns of the securities in the portfolio.
- Decisions on portfolio are based on risk and expected return only.
- Risk Aversion: Investors must be averse to risk.

  For a given expected return, an investor prefers the *least* amount of risk. An investor is expected to want a less risky portfolio. This way, two investors devoting the same amount of wealth into the same market would have different portfolios only if their measures of risk were different.

  "Variance is an undesirable thing."-Markowitz
- No short sales are allowed.
- Portfolio includes the entire wealth of assets.
- Taxes and transactions costs are omitted.

The portfolio generated from this program has an expected return close to the expected return for every individual security, at a lower degree of risk. Computing all optimal portfolio combinations based on all possible expected rates of return generates the efficient frontier (see figure 1.1.1). Each investor is expected to hold one portfolio on the efficient frontier based on their utility function.

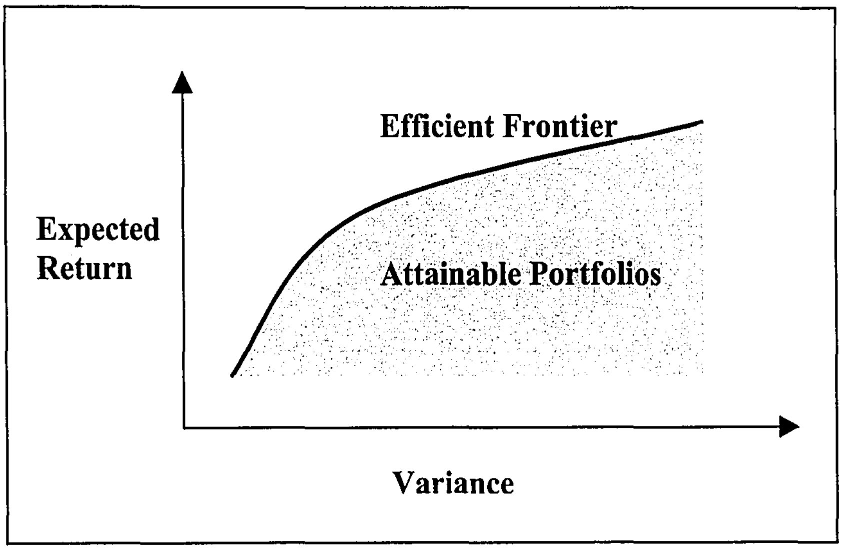


Figure 1.1.1
The surface of the graph represents the efficient frontier. The shaded region represents the set of feasible portfolios.

Before Markowitz's work on portfolio selection, investing in areas such as the stock market was by no means an exact science. Diversification of assets was the rule at the time, but the reason for diversification was unknown. Markowitz revealed that in a portfolio of risky assets, risk is dependent not only on the returns of individual assets but the covariance of the returns of assets as well. Therefore to minimize risk, one must minimize the covariance of assets. Diversification thus becomes most effective when the types of assets are varied and have low correlations of returns.

The reason why diversifying assets is effective is because mathematically increasing a pool of random variables will theoretically decrease the variance of random variables. By exploiting the covariance of returns, as opposed to simply diversifying by investing in multiple securities, the best possible portfolio can be selected. Markowitz stated, "Not only does the hypothesis imply diversification, it implies the "right kind" of diversification for the "right reason." The adequacy of diversification is not thought by investors to depend solely on the number of securities held. Similarly in trying to make variance small is not enough to invest in many securities. It is necessary to avoid investing in securities with high covariances among themselves. We should diversify across industries because firms in different industries, especially industries with different economic characteristics, have lower covariances than firms within an industry."[7]

1.2 Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk

Based on Markowitz's portfolio selection model, diversification for the purposes of minimizing the covariance between returns of assets will minimize the risk incurred on a portfolio. One would assume through this method that risk could be virtually eliminated through efficient diversification. In the real world, however, risk cannot be eliminated so easily because of other factors that influence the returns of a security. Sharpe developed a model in 1964 to better describe this behavior of a security with respect to the market. His method reinforces Marowitz's model in terms of the relationship between risk and expected return, but posits that the returns on the market (not just the assets alone) has a role in the performance of a portfolio. Sharpe's theory was that the total risk (and thus, return) of an asset does not determine the price, and the remaining risk of an asset, known as the systematic risk, cannot be completely diversified away. He proposed that the risk on each asset in the market (or a portfolio) could be related to the return of a common index. The valuation model developed by Sharpe describing the relationship between the market and securities is known as the Capital Asset Pricing Model.

The assumptions of this model include:

- Partial shares are allowed in the investment.
- Transactions are instantaneous and all orders are fulfilled.

- The riskless rate is the same for all investors.

Other Assumptions Similar to the Markowitz Model:

- No arbitrage or short sales: arbitrage is the near simultaneous purchase and sale of a security for the purpose of making a profit from the asset in another market. A short sale is the sale of a security sale that was borrowed and paid for after the price has fallen. If short sales and arbitrage were accepted, most optimization programs would exploit these opportunities with little regard for the main agenda of the portfolio.
- Risk Aversion.
- Mean Variance decisions.
- No taxes or transaction costs.

Sharpe proposes that by separating returns of assets into two basic components, a portfolio can be created with a superior risk and expected return profile than that of Markowitz's original efficient frontier. The components of the returns are either independent of the market, or based on the market.

Assets that are considered free of risk make up the independent component (and by definition can be diversified away). The changes in the risk free asset's returns are uncorrelated with the changes in the general market.

The theory behind the Capital Asset Pricing Model (CAPM) is that investors are rewarded by the market for taking on risk. The level of risk taken on by the investor is dependent on how sensitive their investment is to changes in the market (system). This systematic risk is quantified by the parameter beta, β. The total returns on an asset, R_i, as a function of the riskless asset and beta are:

$R_i = \alpha_i + \beta_i R_m + \varepsilon_i$ (1)

where α_i is the riskless (non-market) asset, R_m is the return on the market and ε_i is the non-market residual return. β is determined by linearly regressing the returns of a security or portfolio verses the returns on the market for a period of observations ($t = 1 \ldots N$). The slope of the regression line is the value of beta (see figure 1.2.1).

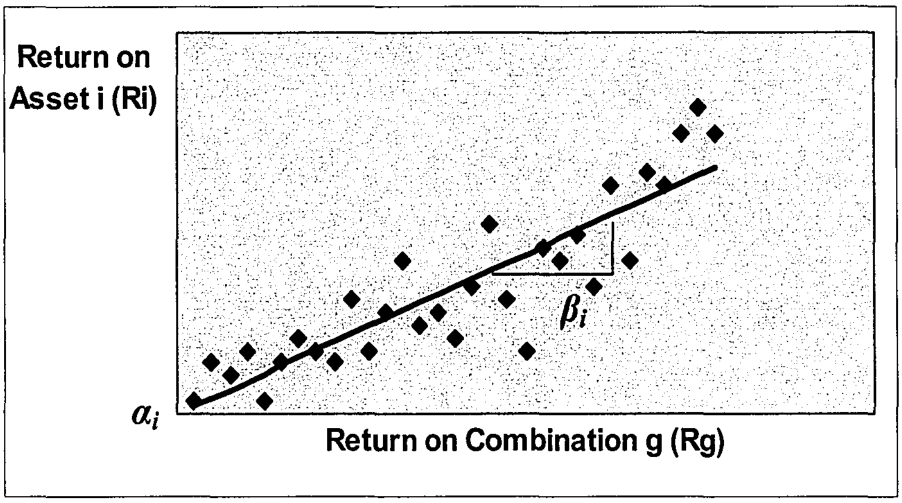


Figure 1.2.1
Beta is found through linear regression. Alpha is equal to the regression line as it intercepts the y-axis.

The equation for beta,

$$\beta_i = \frac{\sum_{t=1}^{N}\left[\left(R_{it} - \overline{R}_{it}\right)\left(R_{mt} - \overline{R}_{mt}\right)\right]}{\sum_{t=1}^{N}\left(R_{mt} - \overline{R}_{mt}\right)^2} \qquad (2)$$

is the covariance of returns between the market and the security divided by the variance of returns on the market. α is a constant (variance = 0) found by setting the trend component of the regression line equal to zero. ε is the set of residual values, with a mean equal to zero, that keep α constant.

CAPM is an equilibrium model that describes the pricing of assets, as well as derivatives. The expected return of an asset (or derivative) equals the riskless return plus a measure of the assets non-diversifiable risk, beta, times the market-wide risk premium (excess expected return of the market portfolio over the riskless return).

$$E(R_i) = \alpha_i + \beta_i \overline{R}_m \qquad (3)$$

where $\overline{R}_m$ is the average return on the market.

The variance of a security based on CAPM is equal to

$$V(R_i) = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2 \qquad (4)$$

where σ_m^2 is the variance of the market return, and $\sigma_{\varepsilon_i}^2$ is the variance of the non-market residual return.

Using Sharpe's Capital Asset Pricing Model to describe returns and their derivatives will improve the risk function of Markowitz's Portfolio Selection problem. The optimization method, however, is still difficult to solve for large variables when the risk measure (objective of the program) is a quadratic function such as (semi)variance.

1.3 Sharpe: A Linear Programming Approximation for the General Portfolio Analysis Problem

In this paper, Sharpe reformulates Markowitz's quadratic program as a linear program using an approximation of the quadratic objective function. The general portfolio selection problem is solved with the use of a quadratic program (QP) since the objective, minimum variance, is a squared function defined by the covariance of returns. As the number of decision variables (n) increases, the number of values (equations) in the covariance matrix increases quadratically ($n(n-1)/2$). As of today, few computational packages exist that can derive an efficient solution for a QP with a large number of decision variables (n > 20). By expressing the variance as a piecewise linear function[2] approximate solutions can be found for large numbers of decision variables. As a result of this linear approximation, the number of equations needed to solve the program is reduced significantly and the solution is guaranteed for convex functions.

Example 1.3.1

Markowitz's Portfolio Selection problem will be approximated as an example to further explain this process. The objective is, of course, to minimize variance. Portfolio variance $V_p = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_{ij}$ is defined by the product sum of each pair of asset proportions multiplied by the covariance of their returns. Before this nonlinear function can be approximated, it must be expressed as a separable set of variables.

To separate, the function must first be restated as a sum of one squared term (and not the product of two), according to Sharpe [10] and Hillier and Lieberman [4].

Start by defining a new set of variables, Y_i, such that

$$Y_i = \sum_{j=i}^{N} \sigma'_{ij} x_j \qquad (1)$$

where σ'_{ij} is the transformed set of entries along and above the main diagonal of the original covariance matrix (Sharpe provides an algorithm for transforming the covariance matrix, the algorithm is presented in the appendix 7.1). Y_i is a linear function of the original variables and the variance of this function can be expressed as a sum of squared terms $V_p = \sum_{i=1}^{N} D_i Y_i^2$ (2)

and

[2] Piecewise linear function- a continuous function with linear segments so that its graph comprises of a polygon.

$$V_i = D_i Y_i^2 \qquad (3)$$

where

$$D_i = \left(\frac{1}{\sigma_{ii}}\right) \qquad (4)$$

D_i is the inverse of each asset's covariance unto itself (or inverse of the each asset's variance). V_i is asset i's contribution to the portfolio's variance (one stock in a portfolio). The relationship between V_i and Y_i is shown below

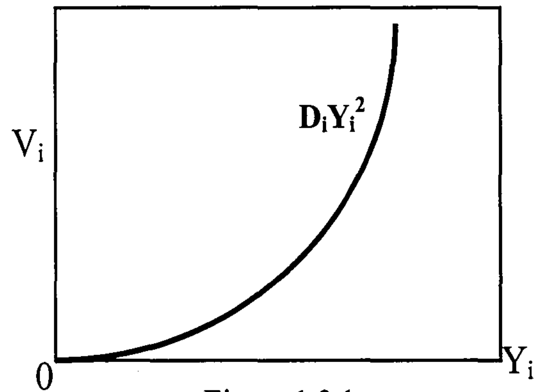


Figure 1.3.1
True variance as a function of single variable Y_i

Now that variance is expressed as a separable (quadratic) function of some set of decision variables, this function can now be separated and approximated. Start by choosing a number of breakpoints for the piecewise linear function. These breakpoints determine the ranges of linear segments and where the slope of the segments change. Let m_j be the number of breakpoints for the approximating function. Let $b_{j1}, b_{j1} + b_{j2}, \ldots, \sum_{i=1}^{m_j} b_{jl}$ be the values of Y_i where the breakpoints occur. The slopes at the breakpoints are found by taking the change in V_i (ΔV_i) over the distance ($b_{jl+1} - b_{jl}$, $l<m$). Y_i then decomposes into auxiliary variables (Y_{il}) that use the breakpoints as bounds. The auxiliary variables can be defined as

$$Y_{il} \equiv \begin{cases} 0, & \text{if } Y_i \le \sum_{l=1}^{k-1} b_{jl} \\ Y_i - \sum_{l=1}^{k-1} b_{jl}, & \text{if } \sum_{l=1}^{k-1} b_{jl} \le Y_i \le \sum_{l=1}^{k} b_{jl} \\ \sum_{l=1}^{k} b_{jl} - \sum_{l=1}^{k-1} b_{jl} & \text{if } Y_i \ge \sum_{l=1}^{k} b_{jl} \end{cases} \quad (5)$$

The segments that separate Y_i into parts (Y_{il}) are restricted to the range of the segments and sum back to Y_i. D_i is also broken into parts (D_{il}) that represent the slope of the line segments for respective values of Y_{il}.

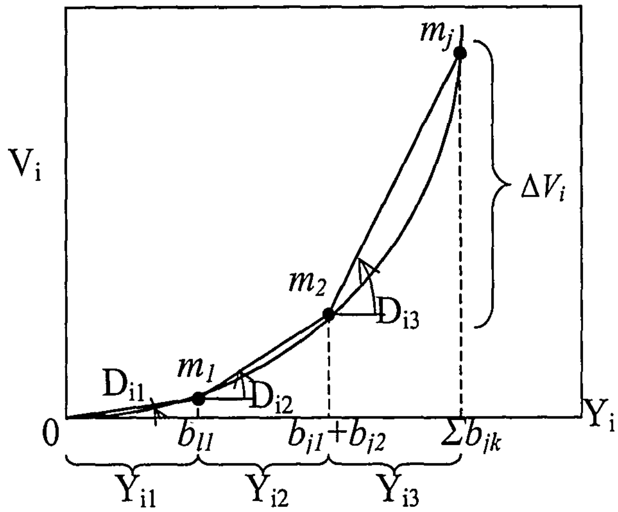


Figure 1.3.2
Piecewise linear approximation of variance

As a result, the piecewise linear function approximating variance $V_i = D_i Y_i^2$, is now a sum of the linear terms

$$V_i^a = D_{i1} Y_{i1} + D_{i2} Y_{i2} + D_{i3} Y_{i3} \quad (6).$$

The auxiliary variables substitute for the squared term and the linear sum approximates the original variance of each asset.

Once the approximation is completed, the objective function for the portfolio of N assets is now to minimize the sum of linear approximations

$$V_p^a = \left(\sum_{i=1}^{N} D_{i1} Y_{i1} + \sum_{i=1}^{N} D_{i2} Y_{i2} + \ldots + \sum_{i=1}^{N} D_{il} Y_{il} \right) \quad (7).$$

The number of breakpoints determines the approximation error and number of equations in the objective function. The two measures of error that are considered are the maximum possible error and the expected error. Expected error of each piece, or average area between the approximation segment and the original function, is found by integrating the difference between the two functions between each $b_{jl+1} - b_{jl}$, $l<m$. The maximum possible error for each approximation segment is the greatest available deviation from the original function. Figure [] gives a visual depiction of the two forms of error.

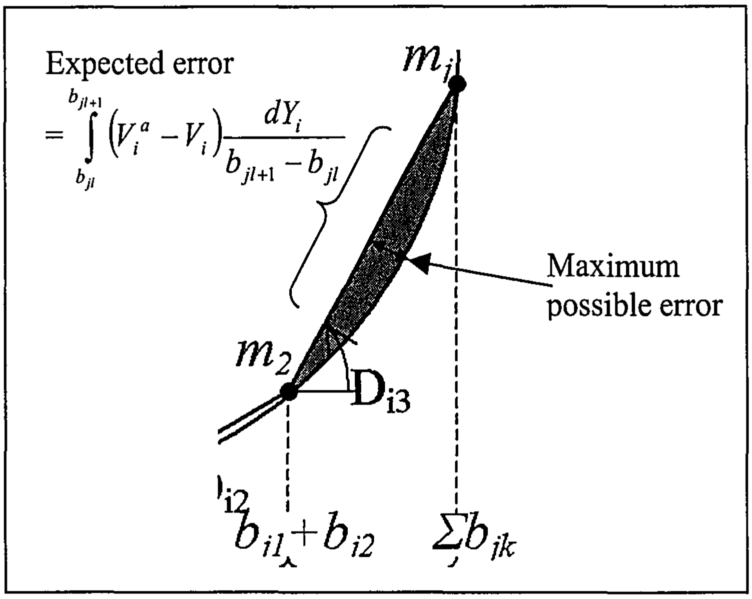


Figure 1.3.3
Two types of approximation error

There are several ways of minimizing approximation error. Setting the approximation segments to equal lengths, or making the breakpoints over Y_i equal, can initially minimize expected error. A greater number of breakpoints will decrease the maximum approximation error. The tradeoff between minimizing approximation error is that the number of equations increases as the number of breakpoints increases, thus increasing the complexity of the model. The creator of the model must determine a limiting error value that is satisfactory for the model; and thus the number of line segments used to approximate the original function is inversely proportional to this limiting error. Sharpe then applied this approximation method to his single index model. This model will be described in detail later.

In Markowitz's model, the risk function, accepted as variance, is expressed as a nonlinear function of multiple decision variables $V_p = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_{ij}$. The same risk function, using CAPM variance, is expressed as:

$$V_p = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon_i}^2 \qquad (8),$$

where σ^2_m is the variance of the market return and $\sigma_{\varepsilon_i}^2$ is the variance of the non-market residual return ε_i.

Chapter 2: Multiperiod Environment

The portfolio selection methods described thus far will create an efficient single period portfolio. The efficiency obtained from using these methods, however, will not last forever. In one month, one week, or even less, the market can change enough to render the selected portfolio inefficient. Rebalancing the portfolio periodically is necessary for maintaining efficiency. Unfortunately, in a real world investment, adjustments made to a portfolio are accompanied by mandatory penalty charges that reduce returns and complicate the time dependent model. Additional strategies that are independent of the utility function are necessary to properly maintain a portfolio in a multiperiod environment. In the following sections of this chapter we discuss several possible strategies for approaching the multiperiod investment, describe any inefficiencies that these strategies may face, and provide examples for each strategy.

2.1 Delineation of Strategy Alternatives for Multiperiod Investment

We present several strategies that an investor could follow to maintain a portfolio over an investment horizon:

1. **Continue to use the initial optimal portfolio and track its returns.**

   The advantage with holding the initial portfolio proportions throughout the investment is that there are no penalty costs, as no transactions take place. The problem with this method is that the static portfolio may no longer be efficient over time and eventually lose its edge when the market changes.

2. **Rebalance the portfolio periodically.**

   After a specified period, a new portfolio model is generated and the proper adjustments are implemented into the old model. This method will ensure that the portfolio is efficient throughout the investment horizon. Two major problems arise when executing this method.

1. A new model may suggest too many costly changes be made to maintain efficiency. The cost of the transactions could offset the benefits from making the changes to the portfolio.
2. Changes may be recommended too soon: A model may recommend changes in consecutive periods that, in a sense, contradict the previous period's recommendation. Had the model been left alone, the same outcome would have resulted, but in the absence of costly transactions.

3. **Combination of the first two strategies on an asset-by-asset basis in order to maximize returns and minimize penalties with the expectation of becoming efficient at a later time.**

This method combines each of the previous alternatives. If the investor has the option of choosing which stocks to adjust and which stocks to hold, unnecessary penalties can be avoided and decisions can be aggregated. This method can be incorporated through programming techniques and heuristics.

We show examples of these alternative strategies by creating stock portfolios from a pool of 27 of the 30 most popular stocks available on the market. The results of each alternative are compared to one another and compared to the S & P 500. The example investment horizon ran from March 1 to May 31, 2000. We used the portfolio selection methods from chapter 1 to create the portfolios with return characteristics based on CAPM. Beta, alpha, and the non-market residual were created for each stock by regressing the previous month's prices to the prices of the market (see appendix 7.2 for a sample of regression and other statistical values). The utility function minimized variance with an expected return of 2%. The upper bound on the proportion of allocation to each stock was 10%. The model was approximated using Sharpe's piecewise linear approximation method, with the number of linear segments set to four. The description of the formulation of the approximation method is available in appendix 7.3. Strategy 3 requires additional formulation that will be described later. Microsoft Excel's solver add-in was the software used to find our solutions.

Example 2.1

Strategy 1: Holding Portfolio for the Duration of the Investment

Table 2.1.1 lists the results of the model with respect to the first strategy. Table 2.1.2 lists the returns on the respective stocks over the investment horizon. The monthly returns for the portfolio were –3.91%, -1.36% and –3.18% for March, April, and May, respectively. This portfolio lost 8.2% over the three-month horizon. The S & P 500 gained 1.43% over the same horizon, with returns of 8.7%, -3.6% and –3.2% over the respective periods. The problem with the investment was that it began to lose interest toward the end of the horizon. We suspect the decrease in interest to be a result of changes to the efficient market portfolio that were not incorporated into the portfolio.

Stoclk	Mar 00
Wal-Mart	0.025
PM	0.025
Microsoft	0.025
Merck	0.05
McDonalds	0.05
J&J	0.025
Paper	0.025
Intel	0.05
Kodak	0.075
Disney	0.05
Coca-Cola	0.025
Caterpillar	0.05
Proctor & Gamble	0
SBC	0.025
Home Depot	0.05
United Technologies	0.025
Boeing	0.025
3M	0.05
AT&T	0.025
GE	0.05
GM	0.05
Exxon M	0.025
JP Morgan	0.05
American X	0.05
American A	0.05
Dupont	0.025
IBM	0.025

Table 2.1.1
Portfolio of stocks created for a three-month investment based on strategy 1

Stock	Mar-00	Apr-00	May-00
Wal-Mart	-0.09	-0.01	0.011
PM	-0.04	0.11	0.005
Microsoft	-0.23	-0.15	0.239
Merck	0.04	0.11	0.058
McDonalds	-0.01	-0.08	-0.075
J&J	0.14	0.09	0.162
Paper	-0.14	-0.05	-0.123
Intel	-0.03	-0.02	0.031
Kodak	-0.02	0.07	-0.012
Disney	0.04	0	-0.07
Coca-Cola	-0.04	0.15	0.082
Caterpillar	-0.05	-0.01	-0.147
P & G	0	0.11	-0.135
SBC	-0.02	-0.02	0
Home Depot	-0.17	-0.13	0.002
UTX	0.01	-0.02	-0.046
Boeing	0.07	0	0.039
3M	-0.06	-0.01	-0.022
AT&T	-0.21	-0.29	-0.391
GE	-0.02	-0.01	0.012
GM	0.1	-0.23	-0.168
Exxon M	-0.03	0.07	-0.054
JP Morgan	-0.22	0.01	-0.104
American X	-0.04	0.07	-0.032
American A	-0.1	-0.08	0.005
Dupont	-0.18	0.05	-0.111
IBM	-0.09	-0.04	0.034

Table 2.1.2
Stock returns over three-month investment horizon.

Example 2.2

Strategy 2: Adjusting Portfolio to the Efficient Set Every Period

Table 2.2.1 lists the stock proportions if the model followed the second strategy. The portfolio selection model was run three times over the investment horizon and adjusted to the efficient set at the end of each period. The stock proportions per period of this multiperiod portfolio are listed below.

Month	Mar 00	Apr 00	May 00
Wal-Mart	0.025	0	0.1
PM	0.025	0	0
Microsoft	0.025	0	0
Merck	0.05	0	0.1
McDonalds	0.05	0.1	0.1
J&J	0.025	0	0.1
Paper	0.025	0.1	0
Intel	0.05	0	0
Kodak	0.075	0.1	0
Disney	0.05	0.1	0.1
Coca-Cola	0.025	0.1	0.1
Caterpillar	0.05	0	0
Proctor & Gamble	0	0	0.1
SBC	0.025	0	0
Home Depot	0.05	0	0.1
United Technologies	0.025	0	0
Boeing	0.025	0.1	0
3M	0.05	0.1	0
AT&T	0.025	0	0
GE	0.05	0	0
GM	0.05	0	0
Exxon M	0.025	0.1	0.1
JP Morgan	0.05	0	0
American X	0.05	0	0
American A	0.05	0	0
Dupont	0.025	0.1	0
IBM	0.025	0.1	0.1

Table 2.2.1
Multiperiod portfolio based on strategy 2.

Theoretically, this portfolio would have lost 1.77% (with returns of –4.61%, 2.83%, and .14% for March, April and May, respectively) based on the returns from table 2.1.2. As promising as this strategy may seem, the penalty costs associated with altering the portfolio were left out. We describe the penalty costs in full before they are incorporated

Transaction Costs

Transaction costs are the costs of buying and selling shares of stock. It is attributed mostly to a combination of the commissions and fees that a brokerage firm charges for executing a trade of shares. The fixed cost can be either a dollar value, such as $20 a trade, or a percentage of the total transaction. Since the model used in this paper is not based on dollar figures (it is based on the interest accrued on the investment), there will be charge of 1% per transaction.

Capital Gains Tax

The difference between the price of an asset when purchased and sale price when terminated is the capital gain of the asset. U.S. law currently states that capital gains on assets held for less than one year are taxed at a rate of 39.6%, while capital gains on assets held for more than one year are taxed at a rate of 20%. These taxes are payable when shares are partially or entirely sold and if the gains are favorable to the investor (there is a common strategy that investors practice to save on capital gains taxes in appendix 7.4). If an asset fails to make a gain, no capital gains taxes are due from the asset.

If the results of the model from the second strategy were implemented in an investment without regard for the penalties, the charges on transactions and capital gains would consume the interest (if any) and possibly the investment.

Table 2.2.2 tallies the total number of transactions that would take place if we invested using the second strategy.

Stock	Transaction at period end Mar 00	Apr 00
Wal-Mart	2	1
PM	2	0
Microsoft	1	2
Merck	0	1
McDonalds	1	0
J&J	1	2
Paper	0	2
Intel	1	0
Kodak	2	1
Disney	1	1
Coca-Cola	1	2
Caterpillar	0	1
Proctor & Gamble	0	2
SBC	0	0
Home Depot	2	1
United Technologies	1	2
Boeing	0	0
3M	0	1
AT&T	0	0
GE	0	2
GM	0	1
Exxon M	0	2
JP Morgan	0	0
American X	0	0
American A	2	1
Dupont	2	0
IBM	0	2
Total Transactions	19	27

Table 2.2.2
Transactions associated with trading each stock in strategy 2. Buying stocks require a transaction cost, and selling stocks require both a transaction cost and a tax on capital gains, where applicable.

The investment consisted of 16 buys and 15 sales, including 6 instances (out of a possible 15) where a gain in capital was made. If a transaction cost of 1% of the value of the transaction were applied in addition to taxes on capital gains, the portfolio would have actually lost 3.19% (with returns of –6.29, 1.13, .014 for March, April, and May, respectively). In only three months, the penalty charges as a result of strategy 2 accumulated to 1.42% of the wealth of the investment and significantly reduced the value of the portfolio. A portfolio containing more securities over a longer investment horizon could potentially suffer a worse fate if it followed portfolio strategy 2.

Example 2.3

Strategy 3: Combination of Previous Strategies

Strategy 3 uses the best attributes from the previous two strategies by making the adjustment available (Ex. 2.2), while leaving the option to hold (Ex. 2.1). During the second period of the 3-period investment, the investor's strategy is to logically decide which stocks to alter and which to hold in order to maximize returns. This strategy is best described and modeled as an acyclic staged network.

Multiperiod Network

Figure 2.3.1 below depicts a network of all possible decisions for one stock (Wal-Mart) over several periods (Jan 2000 through June 2000) in a portfolio, had the model run repeatedly as in alternative 2. The nodes represent proportions of wealth allocated to the stock and are connected by arcs that represent the decisions that must be made to attain the proportions at the period's end (normally, arcs pointing up, down, or horizontal indicate buy, sell, or hold, respectively). Basically, a new model is created at the period's end ($t+1$) and is compared to proportions of the previous period (t). Each period after the creation of the original portfolio ($1 < t < m$, m = the investment horizon), the investor has the option of either adjusting the stock to the proportion of the next period's efficient set, or holding the stock at the present proportion. There will be a penalty cost associated with each arc that is not horizontal. s_p represents the set of stock i's proportions in each path from period 1 to m. Path p_t represent the set of choices made from period 1 to period m for each stock. Let the path $*p_t$ be the path that represents only efficiently derived proportions of the stock each period. Also let $*s_p$ be the set of proportions that belong to path $*p_t$. Path $*p_t$ is represented by dashed lines in the figure.

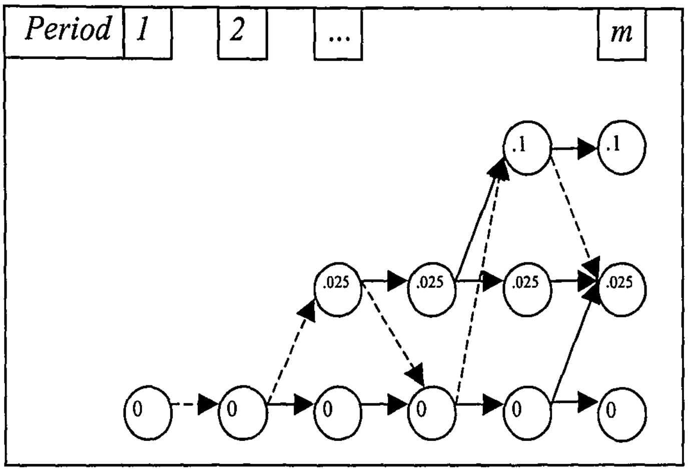


Figure 2.3.1
Network of security proportions over g periods. The Network represents one security in a portfolio. Arcs directed up, down, or straight represent buys, sells and holds, respectively. The efficient set of proportions is represented by the path of dashed lines.

Based on the network in figure 2.3.1, there exist 5 paths that contain less efficient sets of choices *but* also contain the efficient proportion at end period m[5]. More importantly, the 5 paths accumulated less penalty charges while they "traveled" the path. These less than efficient paths will outperform $*p_t$ unless the efficient path produces returns greater than the margin of its excess penalty charges. The significance of these less efficient paths is that we can use these paths to gain higher returns.

We also assign a period g ($1 < g < m$) to the network. At period g all proportions converge back to the efficient set. The purpose of the convergence period is to give the network a destination. With a destination, a state space can be determined, and the network can be traversed computationally. Since the interval that the investment

---

5 $*s_{tp} = (0,0,.025,0,0.1,.025)$ is the efficient set

$$s_{p1} = \begin{cases} (0,0,0,0,0,0.025) \\ (0,0,0,0,0.1,.025) \\ (0,0,0.025,0,0.1,0.025) \\ (0,0,0.025,0.025,0.1,0.025) \\ (0,0,0.025,0.025,0.025,0.025) \end{cases}$$ all carry less penalty and possibly higher returns.

converges to efficiency every g periods, and the investment horizon is m periods, there will exist $T = \left\lceil \frac{m}{g} \right\rceil$ periods of convergence. The state space is all possible paths that can reach the efficient proportion from origin to the period of convergence, as shown in figure 2.3.2.

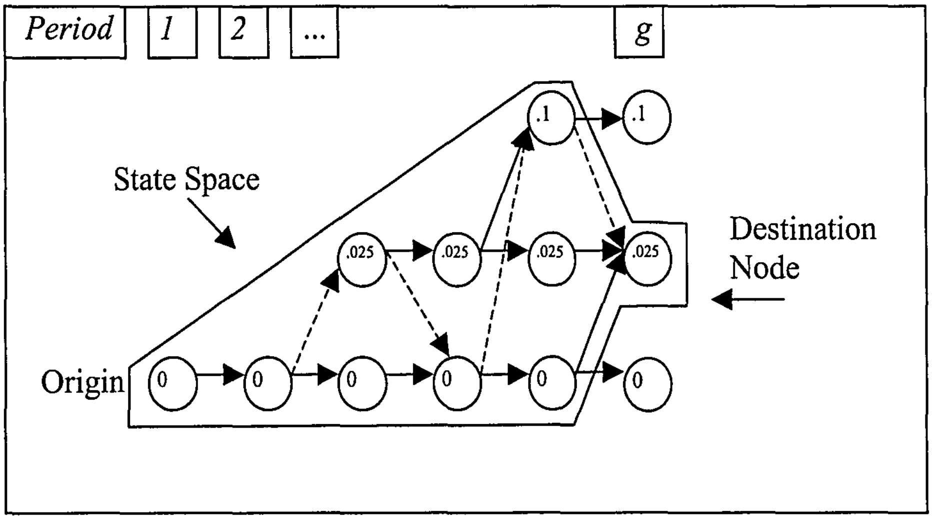


Figure 2.3.2
State Space created by destinations made from groups of g periods in investment horizon m

Decisions will be made in period t ($1 < t < g$, $g \subseteq m$, m = the investment horizon).

Choosing the Best Combination

There are several ways to exploit this strategy now that we know the portfolio is represented by a staged network. The three techniques that we will look at are methods used to optimally traverse a network. The first example is a routing heuristic developed by Clark and Wright [2]. The second method (the method used in the paper) is a mixed integer knapsack program [1][8].

Clark-Wright Savings Heuristic

The Clark-Wright savings heuristic for vehicle routing (see appendix) works by combining the routes that a fleet of vehicles were scheduled to travel in an effort to reduce total travel. From an origin ($s0$) a set of destinations ($1,\ldots,i, j$) must be visited. The lengths are known from the origin to each destination ($d_{s0\text{-}s1},\ldots, d_{s0\text{-}si}, d_{s0\text{-}sj}$) and between each destination ($d_{si\text{-}sj}$). Combining a pair of destinations in a route will save a total distance of the difference of returning to the origin from both destinations ($d_{s0\text{-}s1}$, $d_{s0\text{-}s2}$) and the distance from one destination to the other ($d_{s1\text{-}s2}$) (see figure 2.3.2). The savings from all pairs of destinations are then computed, ranked, and selected such that the total tour length is minimized.

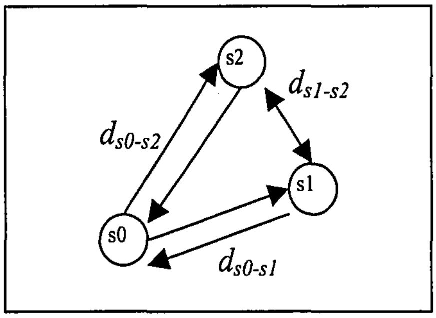
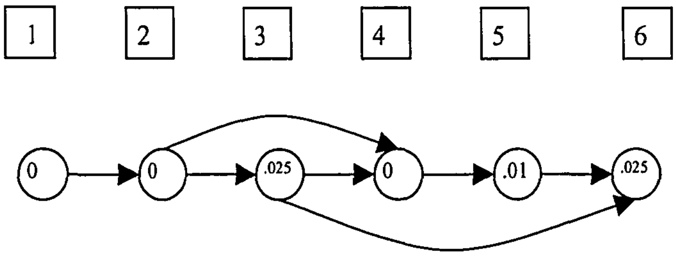


Figure 2.3.3
Clark-Wright Savings Heuristic.
Path (s0, s1, s2, s0) will save
$d_{s0\text{-}s2} + d_{s0\text{-}s1} - d_{s1\text{-}s2}$ in total travel versus
traveling to s1 and s2 separately

This theory can be applied, in part, to the multiperiod portfolio optimization problem. By adopting this strategy of ranking and savings in our multiperiod portfolio, we can choose proportions based on the least penalties. In our model, the arcs represent the cost to "travel" from one proportion to another. We can consider a path to be the set of decisions made from an initial proportion to a destination proportion at some period in the future.



Decision	Savings
Hold 2 4	2 Transactions + Possible Capital Gain
Hold 3 6	3 Transactions + 2 Possible Capital Gains

Figure 2.3.4
Model of $*p_{ts}$. Aggregating decisions for the future will save on penalty charges

Since the costs of the penalties can be determined before the transactions, and thus the savings from avoiding penalizing decisions, we can choose proportions based on the most savings. If the investors could see several periods ahead, they would want to reach that proportion as early as possible and hold throughout to save the most money from penalties throughout the interval. Even if the investors could see as little as two periods ahead, they would still choose that period's efficient set immediately. Next period's decision can be made at the present period in order to save on transaction costs or possible taxes on capital gains. Also, aggregating multiple transactions in a single period instead of making transactions in multiple periods will save additional transaction costs. We can see that minimizing penalties is no different than holding stocks at the previous period's proportions, as in example 2.2.1.

The returns on the investment, however, are just as dependent on the interest rate as they are on the savings from penalties. Therefore a savings heuristic alone may not be sufficient to maximize the returns. Also, in the environment where the savings heuristic is most effective, all the destinations are known. In the investment, the final destination is not known from the origin and thus no savings can be calculated.

Mixed Integer Knapsack

If we assign 0-1 logical decision variables to each node, with the objective of maximizing the returns (penalties included) on the investment, the network can be solved using a mixed integer program. The decisions can be made for pairs of periods, or the set of periods that end at the convergent period (although solving for more than two periods at a time will be more complicated). We want as much of the wealth as possible to be allocated into the investment (ΣX_i - $\varepsilon = 1$), making the problem a knapsack.

Solving Example 2.3 with MIP

We maximize returns of the three-period example, using the approach of alternative strategy 3 (combination of adjustments and holds). To correspond with parameter g, the third period must converge to the efficient set.

The problem is formulated using a mixed integer program and solved with excel solver. Table 2.3.1 lists a subset of the results.

A			B		C			D			E			F		
Returns					Efficient Set			Decisions			Proportions			Returns less Penalty		
Mar-00	Apr-00	May-00			Mar-00	Apr-00	May-00	Mar-00	Apr-00	May-00	Mar-00	Apr-00	May-00	Mar-00	Apr-00	May-00
0.1530			Mar-00	Wal-Mart	0.025			1	0	0	0.025			0.0038		
	-0.094		Apr-00			0			1	0		0			0	
		-0.005	May-00				0.1			1			0.1			-0.001
0.0151			Mar-00	PM	0.025			1	1	0	0.025			0.0003		
	-0.041		Apr-00			0			0	0		0.025			-0.0012	
		0.108	May-00				0			1			0			0.0027
0.1700			Mar-00	Microsoft	0.025			1	0	0	0.025			0.0042		
	-0.2324		Apr-00			0			1	0		0			-0.0019	
		-0.148	May-00				0			1			0			0

Table 2.3.1
Three-period investment solved using binary decisions on stocks to invest based on transaction costs and taxes on capital gains

Section C of table 2.3.1 represents the proportions of each period's efficient set through the course of the investment. The 0-1 decision variables were made in section D of the table. The decisions for the first and last periods are a mere formality since the efficient set and recommended adjustment must be chosen in those periods respectively. During the second period, a decision is made on whether to continue with the previous proportion or make the recommended adjustment for April. The adjustments are based on the stock returns for March and are penalized transaction costs and taxes on capital gains. If the decision is made to hold the previous proportion in April, 1 will be placed in the March row of the April column (meaning that March's proportion was chosen for April). If the decision is made to adjust to the April proportion, 1 will be placed in the April row of the column. Each column in section D sums to 27 since 1 decision must be made on every stock in the portfolio. Section E represents the proportions based on the decisions made in each period. Each column in section E must sum to one.

The formulation of this problem is reserved to the next chapter.

This portfolio had returns of -3.91%, 3.47% and .032% for March, April and May, respectively. The portfolio lost .8% over the 3 period investment horizon, outperforming the other strategies. The full results from this example are displayed in appendix 7.6.

Chapter 3: Solving a Thirty-Five Period Stock Portfolio

Three portfolios were created from the same pool of possible stocks from the previous examples. The portfolios followed the three strategies discussed earlier and are compared to the S & P 500. The same utility function (minimum variance at expected return of 2%) is used throughout the investment horizon[3]. The efficient frontier is updated monthly for just under 3 years using the data from the closing prices of the stocks and the market from January 3, 2000, to November 15, 2002. For strategy 3, the network will converge to the efficient set every 6 periods (g = 6, T = 6). The staged network is created for each stock using the previous algorithm.

The formulations for the first two strategies are in the appendix (7.3). The formulation for the mixed integer program follows:

Variables

X_{it} -Proportion of wealth invested in stock i in period t

Y_{it} -Decision to invest in stock i in period t

Parameters

R_{it} -Return on stock i in period t

$C_{it,t+1}$ -Transaction cost for buying or selling shares = $0.01 * |X_{it+1} - X_{it}|$

$G_{it,t+1}$ -Capital Gains Tax = $\max\{0,.396 R_{it}\} * \max\{0, X_{it_{buy}} - X_{it_{sell}}\}$

Maximize

$$\sum_{i=1}^{N} \sum_{t}^{t+1} \{(R_{it} Y_{it} X_{it}) - Y_{it}(C_{it,t+1} + G_{it_{buy},t_{sell}})\} \qquad (1)$$

---

[3] Normally, an investor's utility function will change over the course of the investment. This would not increase the complexity of the network or model since the objective can be easily adjusted to fit the investor's utility each period

Subject to

$$\sum_{i=1}^{N}\sum_{t}^{t+1} X_{it} = 1 \qquad (2)$$

$$\sum_{t}^{t+1} Y_{it} X_{it} = 1 \quad \text{for every } i \text{ in } N \qquad (3)$$

$$\sum_{i=1}^{N}\sum_{t}^{t+1} Y_{it} X_{it} = 27 \qquad (4)$$

$$X, Y \geq 0 \qquad (5)$$

$$Y_{it} \in \{0,1\} \qquad (6)$$

Because of constraint (2), the problem is a mixed-integer knapsack. Realistically, we allow the wealth proportions to sum to 1 - ε, where ε is a nonnegative value that makes a solution easier to find (equality constraints are not recommended for knapsack problems). ε would be allocated to a risk free investment such as a treasury bond. The results from the portfolios are listed in the next chapter.

Chapter 4: Computational Results

Each solution for the mixed integer knapsack problem from strategy 3 took approximately one minute to solve. Excel solver uses branch and bound to solve MIP problems and is most successful when solving for a small variables (around 20). The 0-1 knapsack is *NP*-complete, but can be solved in pseudo-polynomial time with dynamic programming. Solutions for strategies 1 and 2 were linear and found quickly. No multiperiod decisions were made for strategy 1 and strategy 2 was simply the incorporation of transaction costs and capital gains taxes.

An unforeseen advantage of the piecewise linear approximation was the simplicity of the range of transactions. Since there were few proportions available (0, 0.025, 0.05, 0.075, 0.1), few adjustments could be made.

All portfolios, including the market lost wealth over the course of the investment. Strategy 1 lost less than the other portfolios, but Strategy 3 had the smallest variance of returns. The strategy to hold throughout the investment had the best results, but the strategy involving the MIP also performed strong. The final results are listed in table 4.1.

	Combination Portfolio	Efficient Portfolio	Holding Portfolio	Market
Final Wealth	0.705133159	0.5339466	0.81709347	0.61827613
Mean	-0.009262404	-0.0173005	-0.0043969	-0.0116813
Variance	0.001955345	0.00199875	0.0030979	0.00231155

Table 4.1
Final Results of Multiperiod Portfolios using 3 Alternative Strategies. Portfolio returns compared to the S & P 500

Figures 4.1 tracks the returns of the portfolios versus the market over the investment.

Figure 4.2 tracks the proportional wealth of the portfolios versus the market.

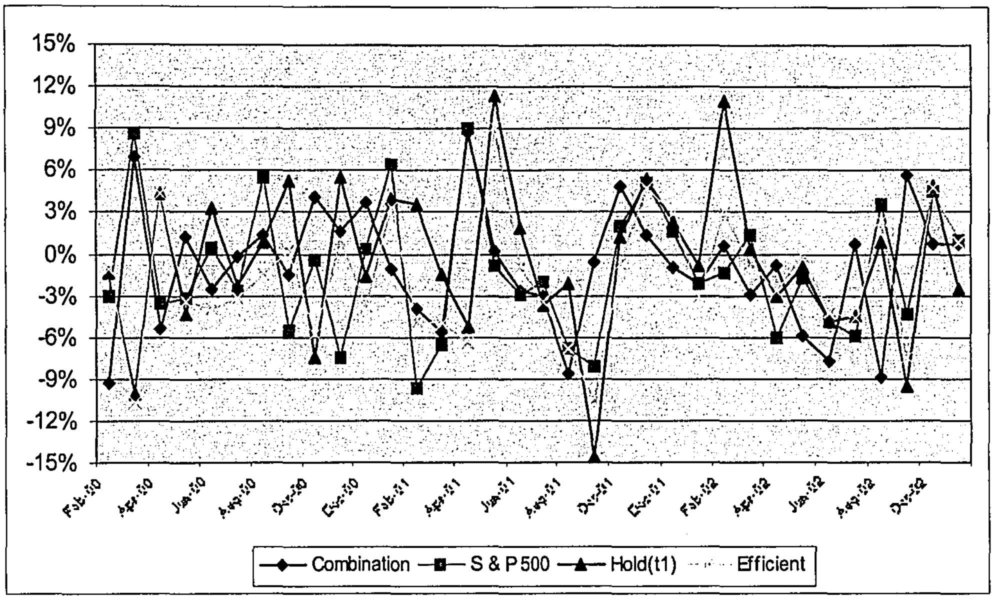


Figure 4.1
Monthly Returns over the Investment

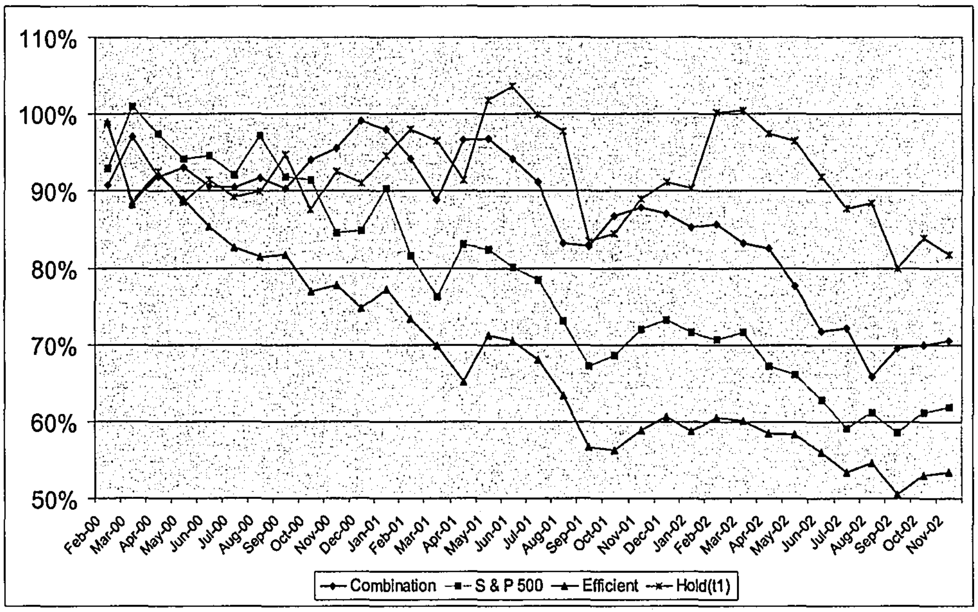


Figure 4.2
Portfolio Growth over

By the end of the first year of the investment, the strategy using the MIP outperformed all other strategies significantly. The wealth of the portfolio using the combination strategy (strategy 3) was greater than the theoretically efficient portfolio (strategy 2) and the holding portfolio (strategy 1) by 32.61% and 8.95%, respectively.

By the end of the second year, both strategies 1 and 3 significantly outperformed the market, while the static efficient portfolio continued to plummet in value. With a total loss of 8.91%, the portfolio following strategy 1 out performed strategy 3 by 4.28%. Both strategy 1 and 3 were some 20% above the market at the time.

The final values from the four portfolios revealed that the MIP strategy outperformed both the market portfolio and the constant rebalancing efficient portfolio. One particular result, the overall performance of the holding portfolio, was unexpected. The returns on the initial period of the holding portfolio example were non-complementary at the time and gave the best possible portfolio without satisfying every constraint. Nonetheless, the portfolio following strategy 1 reached the investment horizon with a value 20.79% greater than strategy 3 and 32.15% greater than the market. The efficient portfolio performed worse than both the market and the other strategies throughout the investment. The results from the program reveal that in order to maximize the return, transactions should be reserved for significant changes in the portfolio. Negligible changes in the model (such as buying and selling a small amount of shares) from period to period are not necessary; these changes are actually detrimental to the investment. It is not necessary to achieve the theoretically efficient portfolio every period. When maintaining efficiency, the penalty costs that are incurred consumed the interest of the portfolio.

Corrections to the Model

There were several issues that affected the integrity of the model. The first issue was with capital gains. Capital gains were not accurately calculated in the model. The taxes were subtracted from gains made in consecutive periods and not from the gains made from when the stocks were purchased. Also, taxes on capital gains are only assessed if gains were recorded for the year, not from consecutive periods. According to our model, situations can exist where stocks that lose money over the course of a year incur a tax simply because it sold after a gaining period.

The first mistake was a result of incorrectly implementing the formulation of capital gains taxes into the model. The formulation for capital gains tax is correct:

$G_{it,t+1} = \max\{0, .396 R_{it}\} * \max\{0, X_{it_{buy}} - X_{it_{sell}}\}$, (3.1)

but the formulation was not accurately applied to the model. Correcting the first mistake is simply a matter of calculating the appropriate gains made from the period when the stocks were purchase to the period the stocks are recommended for sale. This does require more nodes in the acyclic network, but should be manageable.

The second mistake would be difficult to correct. Since decisions are made over consecutive periods, an investor would not know if a stock will make a gain by the end of the year. A conservative approach would be to leave the model as it were and refund taxes if no gains were made. This decision could also change throughout the year since, as more information on the stock's performance would be available.

Another issue was with the convergent period g. This parameter was introduced to the model so that a solution could be found with a dynamic program, which requires a finite horizon. The parameter was also added so the model would return to an efficient state (at least) periodically. If the convergent period were not added to the model, the portfolio using strategy 3 would have performed slightly better.

The main problem with our initial model was in our constraint matrix. The initial constraint to sum proportions of wealth to one (Eq. 4.2) was invalid. The total wealth would be less than one if any penalty costs took place in the investment. It would have been wiser to incorporate the effect of transaction costs and taxes paid to both the wealth of the investment and the objective function, and not the objective function alone.

Most every facet of a multiperiod investment is influenced by transaction costs. If an investor expects to make the same returns from a portfolio that lost wealth paying transaction costs than before, (s)he must accept a higher amount of risk to do so. The investor's wealth becomes dependent on the transaction costs pulled from the investment. Unfortunately, this methodology was not incorporated in the model.

Chapter 5: Conclusion

This paper assessed several strategies to maximize the returns of a portfolio subject to capital gains taxes and transactions, over an investment horizon. An investor initially provides an investment objective that describes a measure of risk for an expected return. A model is developed that chooses securities whose performance follows the investor's utility function. Markowitz and Sharpe's methods are chosen for the single period investment. Over time, changes in the market will undoubtedly call for adjustments to a portfolio in order to maintain efficiency. The adjustments come in the form of buying and selling proportions of the portfolio. Any change in a portfolio is accompanied by mandatory penalty charges. The penalties are substantial enough that actual returns under perform theoretical returns as portions of the investment can be consumed altogether. The investor needs a strategy for the length of the investment, but not for the state of the portfolio for every moment of the investment. Some stocks may be more volatile, providing incentive to adjust frequently, while other stocks may perform better if left alone. Therefore, adjusting portions of the portfolio appears to be more beneficial than adjusting the entire portfolio constantly.

We presented a method to increase the practicality and realism of a portfolio selection model. The multiperiod objective is to create paths for each security that combine to create a portfolio with the highest after tax returns. Through this method, we find less efficient portfolios that make better returns that the theoretically efficient portfolio. The market is uncertain and the future can never be predicted with absolute certainty. We use advanced techniques with present data to make decisions on the future with promising results.

Chapter 6: Future Research

There are many implementations in portfolio selection that were left out of this model. Hopefully, in the future, these topics can be integrated into a more realistic model.

The inclusion of actual dollar figures would have been an important amendment to make the model more realistic. A wide array of applications can be implemented into the model such as stock shares and cash transaction costs if actual dollar values were involved.

The model could have also been developed using better software than Microsoft Excel. Checking our model's results to varying parameters such as expected return, convergence interval g, or transaction costs, with the use of better software, would provide insightful information on the performance of our multiperiod models. The expected return would have been a most useful parameter to vary in order to observe the efficient frontiers of the stocks over the investment horizon.

The risk measure that was used in the model is outdated. The two reasons why variance was the accepted risk measure in this paper was (1) out of respect for Markowitz and Sharpe, and (2) variance is easy to approximate. The use of a better risk measure (semivariance, mean absolute deviation [5]) in the future is expected.

There are also advanced utility criterion such as mean-variance-skeweness that can be employed to minimize the uncertainty of returns, developed by Stone [12].

Mansini and Speranza [6] present heuristic methods in portfolio that include the creation of integer shares of stocks, making the portfolio much more realistic.

Borrowing, lending and adding money to the investment are other useful transactions that take place in a portfolio, but were left out of our model.

Chapter 7: Appendix

7.1 Algorithm for Transforming Covariance Matrix

```
FOR P = 1 TO N
    LET D(P) = 1/C(P,P)
    FOR I = P + 1 TO N
        FOR J = I TO N
            LET C(I,J) = (C(P,P)*C(I,J) – C(P,I)*C(P,J))/C(P,P)
        J = J + 1
    I = I + 1
P = P + 1
```

The algorithm creates 2 outputs

1. Diagonalized two variable covariance matrix
2. New set of nonnegative values D_i

7.2 Sample Beta, Alpha Table

t	p	**Market** S&P % Change	**WAL-MART** Ending	Price Close	**R(i)** Stock % chng(y)	**= Alpha +** alpha	**Beta*Rm +** Beta*Rm	**residual** Residual	calculated Beta	R(i)*R(m)	Market Var(M)	Market std(M)	Residual Var®	Residual std®
1	1		3-Jan-00	66.81		-0.728			1.2658		2.647	1.627	5.223	2.285
2		-3.83	4-Jan-00	64.31	-3.74198	-0.728	-4.8519	1.8389		14.342				
3		0.192	5-Jan-00	63	-2.03702	-0.728	0.2432	-1.5517		-0.396				
4		0.0955	6-Jan-00	63.69	1.09521	-0.728	0.1209	1.7027		0.1047				
5		2.70	7-Jan-00	68.5	7.5526	-0.728	3.4275	4.8529		20.452				
6		1.118	10-Jan-00	67.25	-1.82485	-0.728	1.4155	-2.5124		-2.046				
7		-1.306	11-Jan-00	66.25	-1.48698	-0.728	-1.6524	0.8944		1.949				
8		-0.4381	12-Jan-00	65.06	-1.79624	-0.728	-0.5552	-0.5126		0.788				
9		1.2163	13-Jan-00	65.12	0.09226	-0.728	1.5391	-0.7197		0.113				
10		1.0671	14-Jan-00	64.5	-0.95205	-0.728	1.3509	-1.5749		-1.010				
11		-0.6835	18-Jan-00	65.56	1.64349	-0.728	-0.8641	3.2361		-1.129				
12		0.0527	19-Jan-00	64.06	-2.28795	-0.728	0.0667	-1.6259		-0.110				
13		-0.7098	20-Jan-00	63.38	-1.06158	-0.728	-0.8975	0.5649		0.757				
14		-0.2916	21-Jan-00	62.44	-1.48317	-0.728	-0.3682	-0.3867		0.434				
15		-2.7634	24-Jan-00	59.38	-4.90077	-0.728	-3.4964	-0.6753		13.542				
16		0.6061	25-Jan-00	61.13	2.94712	-0.728	0.7676	2.9083		1.787				
17		-0.4216	26-Jan-00	61.94	1.3255	-0.728	-0.5338	2.5862		-0.550				
18		-0.3934	27-Jan-00	59.13	-4.53664	-0.728	-0.4985	-3.3105		1.786				
19		-2.7453	28-Jan-00	55.13	-6.76476	-0.728	-3.4741	-2.5626		18.576				
20		2.5211	31-Jan-00	54.75	-0.68929	-0.728	3.1906	-3.1517		-1.730				

7.3 Piecewise Linear Approximation for the Capital Asset Pricing Model

Recall the return function of a security based on CAPM

$R_i = \alpha_i + \beta_i R_m + \varepsilon_i$, where α_i is the riskless asset, β_i is the asset's sensitivity to the market, R_m is the return on the market, ε_i is the residual return with mean = 0. Through linear regression α and β can be found

Variables based on the single index model

In a portfolio with asset proportions X_i, the return for the asset becomes:

$X_i R_i = X_i(\alpha_i + \beta_i R_m + \varepsilon_i)$

The return for the portfolio of N stocks

$R_p = \sum_{i=1}^{N} X_i(\alpha_i + \beta_i + \varepsilon_i)$

Since the riskless asset (α) is constant and the average residual (ε) is zero, the expected return of a stock based on the single index model is:

$E_i = X_i\left(\alpha_i + \beta_i \overline{R}_m\right)$ where $\overline{R}_m$ is the average return on the market.

The expected return of the portfolio based on the CAPM

$$E_p = \sum_{i=1}^{N} X_i\left(\alpha_i + \beta_i \overline{R}_m\right)$$

The actual variance of a stock is equal to

$$V_i = x_i^2 \beta_i^2 \sigma_m^2 + x_i^2 \sigma_{\varepsilon_i}^2$$

where σ_m^2 is the variance of the market return, and $\sigma_{\varepsilon_i}^2$ is the variance of the non-market residual return. The actual variance of the portfolio

$$V_p = \sigma_m^2 \sum_{i=1}^{N} x_i^2 \beta_i^2 + \sum_{i=1}^{N} x_i^2 \sigma_{\varepsilon_i}^2$$

The piecewise linear approximation of variance for the single index model is simple since the actual variance is already a sum of squared terms. Therefore Yi = Ri = $\alpha_i + \beta_i R_m + \varepsilon_i$. The upperbound on the investment for each stock is 10% or 0.10. Four breakpoints were chosen ($m = 4$), and the breakpoints are equally separated.

$$Y_{il} \equiv \begin{cases} 0 \le b_{i1} \le 0.025 \\ 0.025 \le b_{i1_i} \le 0.05 \\ 0.05 \le b_{i1} \le 0.075 \\ 0.075 \le b_{i1} \le 0.1 \end{cases}$$

Therefore the range (b_{jl}) of the auxiliary variables (Y_{il}) are:

$$Y_{il} \equiv \begin{cases} 0 \le Y_{i1} \le b_{i1} \\ 0 \le Y_{i2} \le b_{i2} - b_{i1} \\ 0 \le Y_{i3} \le b_{i3} - b_{i2} \\ 0 \le Y_{i4} \le b_{i4} - b_{i3} \end{cases}$$

$$Y_{il} \equiv \begin{cases} Y_{i1}, & \text{if } 0 \le Y_i \le 0.025 \\ Y_{i2}, & \text{if } 0.025 \le Y_i \le 0.05 \\ Y_{i3}, & \text{if } 0.05 \le Y_i \le 0.075 \\ Y_{i4}, & \text{if } 0.075 \le Y_i \le 0.1 \end{cases}$$

The auxiliary variables sum to Y_i:

$Y_i = Y_{i1} + Y_{i2} + Y_{i3} + Y_{i4}$

Recall that D_i is the inverse of each asset's actual variance. The values for the slopes D_{il} depend on the contribution V_i and the auxiliary variable Y_{il}.

$D_{il} = \Delta V_i / (b_{jl+1} - b_{jl}), \quad l<m$

The approximating functions are obtained by plugging the auxiliary variables and slopes into the single index model functions. Therefore:

$$V_i^a = (Y_{i1} + Y_{i2} + Y_{i3} + Y_{i4})\left(\beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2\right) \qquad (6)$$

is the variance approximation of each stock and

$$V_p^a = \left(\sum_{i=1}^N D_{i1}Y_{i1} + \sum_{i=1}^N D_{i2}Y_{i2} + \sum_{i=1}^N D_{i3}Y_{i3} + \sum_{i=1}^N D_{i4}Y_{i4}\right)\left(\beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2\right) \qquad (7)$$

is the variance approximation of the portfolio.

Thus for the piecewise linear approximation model, with four chosen breakpoints and utility function of minimum variance for an expected return of 2%, the objective function is as follows. For every

$$E_p = \sum_{i=1}^N Y_i\left(\alpha_i + \beta_i R_m\right) = 0.02$$

Minimize

$$\left(\sum_{i=1}^N D_{i1}Y_{i1} + \sum_{i=1}^N D_{i2}Y_{i2} + \sum_{i=1}^N D_{i3}Y_{i3} + \sum_{i=1}^N D_{i4}Y_{i4}\right)\left(\beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2\right)$$

Subject to

$$\sum_{i=1}^{N} \left(Y_{i1} + Y_{i2} + Y_{i3} + Y_{i4}\right) = 1$$

$$\sum_{l=1}^{4} Y_{il} \leq 0.1 \text{ for all in } N$$

$$Y_{il} \geq 0$$

7.4 Common Practice to Save on Capital Gains Tax

Suppose an investor buys 50 shares of a stock at $100 a share, and buys another 50 shares of the same stock, at a later period, at $120 a share. Say the stock gains another $15 per share and allures the investor to sell 50 shares. The investor would sell shares from the second investment and suffer the least loss in capital gains (thus the least loss on capital gains tax). This method is common practice among investors and a description of the savings is shown below.

Bought	Sold	Gain	Less Tax	Loss from Taxes
50 shares at $100	50 shares at $135	($135 –$100)* 50 = $1,750	$1,750 * .396	$693
50 shares at $120	50 shares at $135	($135 - $120)*50 = $750	$ 750 * .396	$297
			Savings in taxes	$396

Modeling this practice on a spreadsheet can be difficult because an investor would have to computationally assess all prices of shares purchased before deciding to sell.

7.5 Sample of MIP

				Decision						
R(i)		Jan-00	Feb-00	Jan-00	Feb-00	sum	prop	returns		
-0.16936447	Wal-Mart	0	0	0	1	1	0	0		
-0.03857142	PM	0.1	0.1	0	1	1	0.1	-0.0038571		
-0.13182434	Microsoft	0	0	0	1	1	0	0		
-0.19859525	Merck	0	0	0	1	1	0	0		
-0.13293984	McDonalds	0	0	0	1	1	0	0		
-0.14787525	J&J	0	0.1	1	0	1	0	0		
-0.22961808	Paper	0	0	0	1	1	0	0		
0.11442191	Intel	0	0	0	1	1	0	0		
-0.05423039	Kodak	0.1	0.1	0	1	1	0.1	-0.005423		
-0.05568116	Disney	0.1	0.1	0	1	1	0.1	-0.0055681		
-0.16354558	Coca-Cola	0.1	0.1	0	1	1	0.1	-0.0163546		
-0.18816385	Caterpillar	0.1	0	0	1	1	0	-0.001		
-0.11674752	P & G	0	0.1	0	1	1	0.1	-0.0126748		
-0.10924797	SBC	0	0	0	1	1	0	0		
-0.07246614	Home Depot	0	0	0	1	1	0	0		
-0.04773911	UTX	0	0	0	1	1	0	0		
-0.15281754	Boeing	0.1	0.1	0	1	1	0.1	-0.0152818		
-0.06308226	3M	0	0	0	1	1	0	0		
-0.05944186	AT&T	0	0	0	1	1	0	0		
-0.02675899	GE	0	0	0	1	1	0	0		
-0.10217443	GM	0	0.1	0	1	1	0.1	-0.0112174		
-0.09462729	Exxon M	0.1	0	1	0	1	0.1	-0.0094627		
-0.04924848	JP Morgan	0	0	0	1	1	0	0		
-0.20609571	American X	0	0	1	0	1	0	0		
-0.04707603	American A	0.1	0.1	0	1	1	0.1	-0.0047076	sum y	27
-0.15116494	Dupont	0.1	0	0	1	1	0	-0.001	total p	1
-0.06590599	IBM	0.1	0.1	0	1	1	0.1	-0.0065906	total r	-0.0931377

7.6 Clarke Wright Savings Heuristic

The algorithm is concerned with minimizing a total distance traveled in a vehicle routing network. The objective is to find a set of tours for each vehicle to minimize sum of tour lengths. This is found by aggregating routes to eliminate costly and needless trips. The algorithm works by selecting the routes with the highest savings. The optimal solution consists of the path with the largest total savings. s0 is the origin (depot) with destinations s(1,...i,j). Route s(1) is implied as traveling from origin to destination 1 (at a destination

$d_{s0\text{-}s1}$) and back to origin. Route s(ij) means to travel from origin to destination i then to destination j, and back to origin.

The heuristic works as follows:

Step 1 Compute the savings s(ij) for all pairs of customers

Step 2 Sort the savings into descending order

Step 3 Select the highest savings and determine if it is feasible. If so, construct a new route by joining them. If not, discard this possibility and choose the next savings.

Step 4 Continue with Step 3 as long as the savings are positive.

7.7 Example 2.3 Results

Returns					Recommendations			Decisions			Proportions			Returns less Penalty		
Mar-00	Apr-00	May-00			Mar-00	Apr-00	May-00	Mar-00	Apr-00	May-00	Mar-00	Apr-00	May-00	Mar-00	Apr-00	May-00
-0.095			Mar-00	Wal-Mart	0.025			1	0	0	0.025			-0.002		
	-0.005		Apr-00			0			1	0		0			0.000	
		0.011	May-00				0.1			1			0.1			0.000
-0.041			Mar-00	PM	0.025			1	1	0	0.025			-0.001		
	0.109		Apr-00			0			0	0		0.025			0.003	
		0.005	May-00				0			1			0			0.000
-0.232			Mar-00	Microsoft	0.025			1	0	0	0.025			-0.006		
	-0.148		Apr-00			0			1	0		0			0.000	
		0.239	May-00				0			1			0			0.000
0.041			Mar-00	Merck	0.05			1	1	0	0.05			0.002		
	0.109		Apr-00			0			0	0		0.05			0.005	
		0.058	May-00				0.1			1			0.1			0.005
-0.011			Mar-00	McDonalds	0.05			1	0	0	0.05			-0.001		
	-0.079		Apr-00			0.1			1	0		0.1			-0.008	
		-0.075	May-00				0.1			1			0.1			-0.008
0.142			Mar-00	J&J	0.025			1	1	0	0.025			0.004		
	0.086		Apr-00			0			0	0		0.025			0.002	
		0.162	May-00				0.1			1			0.1			0.015
-0.142			Mar-00	Paper	0.025			1	1	0	0.025			-0.004		
	-0.053		Apr-00			0.1			0	0		0.025			-0.001	
		-0.123	May-00				0			1			0			0.000
-0.029			Mar-00	Intel	0.05			1	0	0	0.05			-0.001		
	-0.019		Apr-00			0			1	0		0.05			-0.001	
		0.031	May-00				0			1			0			-0.001
-0.023			Mar-00	Kodak	0.075			1	0	0	0.075			-0.002		
	0.074		Apr-00			0.1			1	0		0.1			0.007	
		-0.012	May-00				0			1			0			-0.001
0.043			Mar-00	Disney	0.05			1	0	0	0.05			0.002		
	0.005		Apr-00			0.1			1	0		0.1			0.000	
		-0.070	May-00				0.1			1			0.1			-0.007
-0.042			Mar-00	Coca-Cola	0.025			1	1	0	0.025			-0.001		
	0.154		Apr-00			0.1			0	0		0.025			0.004	
		0.082	May-00				0.1			1			0.1			0.007
-0.051			Mar-00	Caterpillar	0.05			1	1	0	0.05			-0.003		
	-0.015		Apr-00			0			0	0		0.05			-0.001	
		-0.147	May-00				0			1			0			-0.001
0.004			Mar-00	P&G	0			1	0	0	0			0.000		

	0.114		Apr-00			0			1	0		0			0.000	
		-0.135	May-00				0.1			1			0.1			-0.015
-0.016			Mar-00	SBC	0.025			1	1	0	0.025			0.000		
	-0.021		Apr-00			0			0	0		0.025			-0.001	
		0.000	May-00				0			1			0			0.000
-0.169			Mar-00	Home Depot	0.05			1	0	0	0.05			-0.008		
	-0.127		Apr-00			0			1	0		0			-0.001	
		0.002	May-00				0.1			1			0.1			-0.001
0.005			Mar-00	UTX	0.025			1	0	0	0.025			0.000		
	-0.022		Apr-00			0			1	0		0.025			-0.001	
		-0.046	May-00				0			1			0			0.000
0.067			Mar-00	Boeing	0.025			1	0	0	0.025			0.002		
	0.004		Apr-00			0.1			1	0		0.1			0.000	
		0.039	May-00				0			1			0			-0.003
-0.057			Mar-00	3M	0.05			1	1	0	0.05			-0.003		
	-0.007		Apr-00			0.1			0	0		0.05			0.000	
		-0.022	May-00				0			1			0			-0.001
-0.209			Mar-00	AT&T	0.025			1	0	0	0.025			-0.005		
	-0.287		Apr-00			0			1	0		0			0.000	
		-0.391	May-00				0			1			0			0.000
-0.023			Mar-00	GE	0.05			1	1	0	0.05			-0.001		
	-0.009		Apr-00			0			0	0		0.05			0.000	
		0.012	May-00				0			1			0			-0.001
0.104			Mar-00	GM	0.05			1	1	0	0.05			0.005		
	-0.234		Apr-00			0			0	0		0.05			-0.012	
		-0.168	May-00				0			1			0			-0.001
-0.035			Mar-00	Exxon M	0.025			1	0	0	0.025			-0.001		
	0.068		Apr-00			0.1			1	0		0.1			0.006	
		-0.054	May-00				0.1			1			0.1			-0.005
-0.216			Mar-00	JP Morgan	0.05			1		0	0.05			-0.011		
	0.013		Apr-00			0			1	0		0			-0.001	
		-0.104	May-00				0			1			0			0.000
-0.037			Mar-00	American X	0.05			1	0	0	0.05			-0.002		
	0.070		Apr-00			0			1	0		0			-0.002	
		-0.032	May-00				0			1			0			0.000
-0.096			Mar-00	American A	0.05			1	0	0	0.05			-0.005		
	-0.075		Apr-00			0			1	0		0			-0.001	
		0.005	May-00				0			1			0			0.000
-0.176			Mar-00	Dupont	0.025			1	1	0	0.025			-0.004		
	0.046		Apr-00			0.1			0	0		0.025			0.001	
		-0.111	May-00				0			1			0			0.000
-0.086			Mar-00	IBM	0.025			1	1	0	0.025			-0.002		
	-0.042		Apr-00			0.1			0	0		0.025			-0.001	
		0.034	May-00				0.1			1			0.1			0.003
				Totals	1	1	1	27	27	27	1	1	1	-0.048	-0.002	-0.013
														Final	Return	0.965

7.8 Results from 35-Period Investment

Period	Strategy 1 Holding Portfolio Returns	Accum. Interest	Strategy 2 Efficient Portfolio Returns	Accum. Interest	Strategy 3 Combination Portfolio Returns	Accum. Interest	S&P500 Market Returns	Accum. Interest
Jan-00	-	-	-	-	-	-	-0.04175	0.95824
Feb-00	-0.01370	0.98630	-0.01084	0.98916	-0.09314	0.90686	-0.03041	0.92910
Mar-00	-0.10118	0.88650	-0.10750	0.88282	0.06985	0.97021	0.08657	1.00953
Apr-00	0.04395	0.92546	0.04337	0.92111	-0.05338	0.91842	-0.03555	0.97364
May-00	-0.04383	0.88490	-0.03402	0.88977	0.01267	0.93006	-0.03245	0.94204
Jun-00	0.03269	0.91383	-0.04024	0.85397	-0.02597	0.90590	0.00400	0.94581
Jul-00	-0.02301	0.89281	-0.03163	0.82696	-0.00187	0.90421	-0.02634	0.92089
Aug-00	0.00780	0.89977	-0.01492	0.81462	0.01360	0.91651	0.05534	0.97185
Sep-00	0.05264	0.94714	0.00245	0.81662	-0.01461	0.90312	-0.05541	0.91801
Oct-00	-0.07504	0.87606	-0.05720	0.76991	0.04046	0.93966	-0.00476	0.91364
Nov-00	0.05568	0.92484	0.00933	0.77709	0.01648	0.95515	-0.07477	0.84532
Dec-00	-0.01655	0.90954	-0.03842	0.74724	0.03748	0.99094	0.00384	0.84857
Jan-01	0.03994	0.94587	0.03314	0.77200	-0.01077	0.98027	0.06448	0.90328
Feb-01	0.03541	0.97936	-0.04875	0.73437	-0.03984	0.94121	-0.09722	0.81546
Mar-01	-0.01537	0.96431	-0.05015	0.69754	-0.05657	0.88797	-0.06518	0.76232
Apr-01	-0.05212	0.91404	-0.06492	0.65226	0.08773	0.96587	0.09040	0.83123
May-01	0.11330	1.01760	0.09182	0.71215	0.00140	0.96722	-0.00839	0.82426
Jun-01	0.01841	1.03634	-0.01006	0.70499	-0.02716	0.94095	-0.02879	0.80053
Jul-01	-0.03674	0.99826	-0.03494	0.68036	-0.03078	0.91199	-0.02061	0.78403
Aug-01	-0.02159	0.97671	-0.06770	0.63430	-0.08685	0.83278	-0.06773	0.73093
Sep-01	-0.14569	0.83441	-0.10615	0.56697	-0.00628	0.82756	-0.08120	0.67158
Oct-01	0.01182	0.84428	-0.00720	0.56289	0.04826	0.86749	0.02044	0.68531
Nov-01	0.05433	0.89015	0.04677	0.58922	0.01333	0.87906	0.05106	0.72030
Dec-01	0.02326	0.91085	0.02900	0.60631	-0.00972	0.87052	0.01609	0.73189
Jan-02	-0.00898	0.90266	-0.02996	0.58814	-0.02112	0.85213	-0.02119	0.71638
Feb-02	0.10918	1.00122	0.02971	0.60561	0.00553	0.85684	-0.01379	0.70650
Mar-02	0.00378	1.00500	-0.00753	0.60105	-0.02895	0.83203	0.01379	0.71624
Apr-02	-0.03064	0.97421	-0.02623	0.58529	-0.00832	0.82511	-0.06072	0.67275
May-02	-0.00947	0.96498	-0.00222	0.58399	-0.05862	0.77674	-0.01778	0.66079
Jun-02	-0.04858	0.91810	-0.04188	0.55953	-0.07735	0.71667	-0.04887	0.62850
Jul-02	-0.04500	0.87678	-0.04609	0.53375	0.00747	0.72202	-0.05888	0.59149
Aug-02	0.00797	0.88377	0.02377	0.54643	-0.08873	0.65795	0.03551	0.61249
Sep-02	-0.09509	0.79973	-0.07466	0.50563	0.05643	0.69508	-0.04311	0.58609
Oct-02	0.04840	0.83844	0.04728	0.52954	0.00735	0.70019	0.04464	0.61225
Nov-02	-0.02546	0.81709	0.00832	0.53395	0.00706	0.70513	0.00985	0.61828
Mean	-0.00440	0.92180	-0.01730	0.68752	-0.00926	0.86629	-0.01168	0.78239
variance	0.00310	0.00355	0.00200	0.01729	0.00196	0.00864	0.00231	0.01627

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Vita

Nicholas Idorenyin Umoh, II was born on April 3, 1975 in Dayton, Ohio to Dr. Frances and Linus Umoh. He grew up and attended high school in nearby Trotwood before moving to Columbus to attend the Ohio State University. He received his Bachelor of Science degree in Industrial & Systems Engineering from the Ohio State University in January 2001. He also received a Bachelor of Arts degree in the Humanities from the Ohio State University in June 2001. After graduation, he plans to pursue a career in teaching or pursue a Doctorate.

END OF TITLE