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Disturbance Propagation in Interconnected Linear Dynamical Networks

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Disturbance Propagation in Interconnected Linear Dynamical Networks

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To my parents and my wife, I love you and I dedicate this thesis to you.

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Abstract

We consider performance analysis of interconnected linear dynamical networks subject to external stochastic disturbances. For stable linear networks, we define scalar performance measures by considering weighted $\mathcal{H}_2$–norms of the underlying systems, which are defined from the disturbance input to a desired output. It is shown that the performance measure of a general stable linear network can be tightly bounded from above and below using some spectral functions of the state matrix of the network. This result is applied to a class of cyclic linear networks and shown that the performance measure of such networks scales quadratically with the network size. Next, we focus on first– and second–order linear consensus networks and introduce the notion of Laplacian energy for such networks, which in fact measures the expected steady-state dispersion of the state of the entire network. We develop a graph-theoretic framework in order to relate graph characteristics to the Laplacian energy of the network and show that how the Laplacian energy scales asymptotically with the network size. We quantify several inherent fundamental limits on Laplacian energy in terms of graph diameter, node degrees, and the number of spanning trees, and several other graph specifications. Particularly we characterize several versions of fundamental tradeoffs between Laplacian energy and sparsity measures of a linear consensus network, showing that more sparse networks have higher levels of Laplacian energies. At the end, we show that several existing performance measures in real–world applications, such as total power loss in synchronous power networks and flock energy of a group of autonomous vehicles in a formation, are indeed special forms of Laplacian energies.

Chapter 1

Introduction

The issue of fundamental limits and their tradeoffs in large-scale interconnected dynamical systems design lies at the very core of theory of distributed feedback control systems as it reveals what is achievable and conversely what is not achievable by distributed feedback control laws. Improving global performance as well as robustness to external disturbances in dynamical networks are crucial for sustainability, from engineering infrastructures to living cells; examples include a group of autonomous agents such as UAVs in a formation, distributed emergency response systems, interconnected transportation networks, energy and power networks, metabolic pathways and even social networks [1–9]. One of the fundamental design problems in dynamical networks is to develop a mathematical framework to study and characterize intrinsic fundamental limits and their tradeoffs in networks of interconnected systems. This enables us to devise underpinning principles to design robust-by-design dynamical networks that are less fragile to external disturbances.

The focus of this thesis is on revealing foundational role of underlying graph of dynamical networks in emergence of severe theoretical hard limits on the global performance and robustness. The structure of the underlying graph of a dynamical network depends on the coupling structure among the subsystems which are usually imposed by physical laws or global objectives. We consider the class of linear time-invariant networks in closed-loop operation, i.e., the linear dynamical network is stabilized by a linear state feedback control law. The topology of an information structure in a spatially distributed feedback system determines the communication requirements in the controller array, i.e., each subsystem should communicate with which of the neighboring subsystems to exchange state information with regard to global objectives. As a

result, the controller architecture usually imposes a sparsity-constraint on the structure of underlying graph of the closed-loop dynamical network.

The impacts of such fundamental limits usually appear as fundamental tradeoffs between various measures of performance and robustness in the presence of external disturbances, subsystem addition or deletion and various modeling uncertainties. In this thesis, we are particularly interested in linear networks driven by a stochastic disturbances. We propose a comprehensive approach based on ideas from graph theory to quantify limits of performance and robustness due to the structure of the underlying graph of linear dynamical networks.

There have been several recent works on the performance and robustness analysis of first- and second-order linear consensus networks; only to name a few, we refer to [1,5,7,10–14] and references in there. The reference papers [1, 10–12, 15] study performance of a class of linear consensus networks under influence of some external stochastic disturbances. The common approach of the above-mentioned papers is to adopt the $\mathcal{H}_2$–norm of the system (from the disturbance input to the performance output of the system) as a scalar performance measure. The basic assumption in these papers is that the state matrix of the system is normal. Based on this assumption, the $\mathcal{H}_2$–norm of the system can be exactly calculated as a function of the eigenvalues of the state matrix of the system [1]. When state matrix of the system is the Laplacian matrix of the underlying graph of the system, this scalar measure is proportional to the total effective resistance of the system. The concept of effective resistance has been used in several disciplines and applications. In the context of electric circuit analysis, the effective resistance of an edge is the resistance measured between endpoints of that edge. In the context of random walks and Markov chains on networks, the effective resistance of an edge can be interpreted as the commute time between the endpoints of that edge. Another interesting version of the notion of effective resistance appears in the context of graph scarification, where the goal is to approximate a given graph by a sparse graph. In this setting, the effective resistance is defined as probability of appearing an edge in a random spanning tree of the graph (see [16] and references in there). In [17], the authors demonstrate a physical interpretation of the effective resistance in least-squares estimations as well as motion control problems.

The $\mathcal{H}_2$–norm of a system can be interpreted as a macroscopic performance measure, that captures the notion of coherence in dynamical networks. In [1], the asymptotic scaling of upper bounds on this scalar performance measure is investigated in terms of the network size for linear networks with d-dimensional

discrete torus interconnection topologies. In [12], the authors consider the $\mathcal{H}_2$ performance measure for a class of first-order consensus networks with exogenous inputs in the form of process and sensor noises. The performance measure used in [12] is different from those scalar measures considered in [1, 3, 5, 10]. The proposed analysis method in [12] applies the edge agreement protocol by considering a minimal realization of the edge interpretation system. Another related work is reported in [15], where the authors use the 2-norm coefficient of ergodicity to find upper bounds on the $\mathcal{H}_2$ performance measure.

In this thesis, we propose a graph–theoretic approach to analyze global performance of linear consensus networks using trace operator. We introduce the notions of first-order and second-order *Laplacian energies* for linear consensus networks, which are indeed weighted $\mathcal{H}_2$–norms of the system. This new performance measure depends linearly or quadratically on the pseudo–inverse of the Laplacian matrix of the underlying graph of the network. This formulation has several advantages. First, the Laplacian energy of a linear consensus network is well-defined and the marginally stable mode of the system is not observable through this performance measure. Second, the trace representation of the Laplacian energy enables us to reveal the foundational role of the topology of the underlying graph of the network in quantifying tight lower and upper bounds for the Laplacian energy in terms of various characteristics of the underlying graph.

In Chapter 3, we consider general closed-loop linear dynamical networks and calculate new tight lower bound for the $\mathcal{H}_2$–norm of the system in terms of the eigenvalues of the closed-loop matrix. This main result enables us to quantify inherent fundamental limits on the $\mathcal{H}_2$–norm of several interesting dynamical networks, in particular linear networks with nonnormal matrices. In Section 3.2, we apply our main result to analyze global performance of the class of cyclic dynamical networks. This class of networks usually arises in modeling biological networks such as Glycolysis pathway [3, 4, 18]. We show that the $\mathcal{H}_2$–norm of a cyclic dynamical network scales quadratically with the size of the system. In Chapter 4, we define the notion of first-order Laplacian energy for first-order consensus linear networks. We characterize inherent fundamental limits on the best achievable Laplacian energy. Several lower and upper bound for Laplacian energy have been obtained in terms of graph diameter, node degrees, and the number of spanning trees. Specifically, we identify an uncertainty principle like inequality in order to show interplay between Laplacian energy and sparsity measures of the underlying graph of the network. It is shown that Laplacian energy times a sparsity measure of the network is lower bounded by a constant that scales

with the size of the network. This implies that networks with more sparse topologies incur higher levels of Laplacian energies. In Chapter 5, we introduce the notion of second-order Laplacian energy for two classes of second-order linear consensus networks. In Section 5.1, it is shown that several existing performance measures in real-world applications are special forms of the second-order Laplacian energy. In particular, we show that in synchronous power networks the concept of Laplacian energy can be interpreted as the total resistive power loss, and the flock energy of controlled vehicles in a formation is a second-order Laplacian energy [19]. We characterize a fundamental limit in the form of an inequality that explains interplay between the second-order Laplacian energy of a group of autonomous vehicles in a formation and a sparsity measure of the formation graph.

Chapter 2

Mathematical Preliminaries

The set of all nonnegative real numbers is denoted by $\mathbb{R}_+$. The $n \times 1$ vector of all ones is denoted by $\mathbf{1}_n = [1, 1, \ldots, 1]^T$, the $n \times n$ identity matrix by I_n, and the $n \times n$ matrix of all ones by $J_n = \mathbf{1}_n \mathbf{1}_n^T$. Throughout this thesis, it is assumed that all graphs are finite, simple, undirected and connected. A weighted graph $\mathcal{G}$ is represented by a treble $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), w^{(\mathcal{G})})$, where $V(\mathcal{G})$ is the set of nodes, $E(\mathcal{G}) \subset \big\{\{i,j\}\big|\, i,j \in V(\mathcal{G}),\ i \neq j\big\}$ is the set of edges, and $w^{(\mathcal{G})} : E(\mathcal{G}) \to \mathbb{R}_+$ is the weight function. For each node $i \in V(\mathcal{G})$, the degree of i is defined by

$$d_i \triangleq \sum_{e=\{i,j\} \in E(\mathcal{G})} w^{(\mathcal{G})}(e).$$

For a given graph, we assume that the degree sequence of the graph is indexed in ascending order $d_1 \leq d_2 \leq \cdots \leq d_n$. The adjacency matrix $A = [a_{ij}]$ of graph $\mathcal{G}$ is defined by setting $a_{ij} = w^{(\mathcal{G})}(e)$ if $e = \{i,j\} \in E(\mathcal{G})$, otherwise $a_{ij} = 0$. The Laplacian matrix of $\mathcal{G}$ is defined by $L_{\mathcal{G}} \triangleq \Delta - A$, where $\Delta = \mathbf{diag}(d_1, \ldots, d_n)$ is a diagonal matrix. The eigenvalues of $L_{\mathcal{G}}$ are indexed in ascending order $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\lambda_1 = 0$. The eigenvalue decomposition of the Laplacian matrix is given by $L_{\mathcal{G}} = \mathbf{U}\Lambda\mathbf{U}^T$ where $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$ and $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n]$ is the corresponding orthonormal matrix of eigenvectors.

The class of all connected graphs with n nodes is denoted by $\mathbb{G}_n$. A dumbbell graph $\mathcal{D}(n; n_1, n_2)$ in $\mathbb{G}_n$ is a graph consisting of two node-disjoint stars $\mathcal{S}_{n_1}$ and $\mathcal{S}_{n_2}$ and a path $\mathcal{P}_{n-n_1-n_2+2}$ joining them having only its end-nodes in common with the center of the two stars (See Fig. 4.4(b) and [20]). A tree

is a connected graph on n nodes and with exactly $n-1$ edges. For comparison purposes throughout the thesis, we consider the standard graphs in Table 2.1 in several occasions. Everyone of these graphs has its own comparable characteristics. For instance, among all graphs in $\mathbb{G}_n$ a complete graph has the maximum number of edges and a star graph has the maximum number of nodes of degree one. A path graph is a tree with minimum number of nodes of degree one. We refer to reference [21] for more details and discussions. An edge is called a *cut-edge* whose deletion increases the number of connected components (see Fig. 4.3).

Definition 2.0.1. *For a given Laplacian matrix $L_{\mathcal{G}}$, the $L_{\mathcal{G}}$–semi–norm of a vector $x \in \mathbb{R}^n$ is defined by*

$$\|x\|_{L_{\mathcal{G}}}^2 \triangleq x^T L_{\mathcal{G}} x = \sum_{e=\{i,j\}\in E(\mathcal{G})} w(e)\big(x_i - x_j\big)^2, \tag{2.1}$$

where $w(e)$ is the weight of edge $e = \{i,j\} \in E(\mathcal{G})$,

Definition 2.0.2. *The Moore-Penrose pseudo-inverse of $L_{\mathcal{G}}$ is denoted by $L_{\mathcal{G}}^{\dagger} = [l_{ji}^{\dagger}]$ which is a square, symmetric, doubly-centered and positive semidefinite matrix.*

Definition 2.0.3. *For a given Laplacian matrix $L_{\mathcal{G}}$, the corresponding resistance matrix $R_{\mathcal{G}} = [r_{ij}]$ is defined using the Moore-Penrose pseudo-inverse of $L_{\mathcal{G}}$ by setting $r_{ij} = l_{ii}^{\dagger} + l_{jj}^{\dagger} - l_{ji}^{\dagger} - l_{ij}^{\dagger}$, where r_{ij} is called the effective resistance between nodes i and j.*

Definition 2.0.4. *For a given Laplacian matrix $L_{\mathcal{G}}$, the total effective resistance $\mathbf{r}_{total}$ is defined as the sum of the effective resistances between all distinct pairs of nodes, i.e.,*

$$\mathbf{r}_{total} = \frac{1}{2}\mathbf{1}_n^T R_{\mathcal{G}} \mathbf{1}_n = \frac{1}{2}\sum_{i,j=1}^{n} r_{ij}. \tag{2.2}$$

Theorem 2.0.5. *For a given $n \times n$ Laplacian matrix $L_{\mathcal{G}}$, the following equalities hold*

$$\mathbf{r}_{total} = n\sum_{i=2}^{n}\frac{1}{\lambda_i}, \tag{2.3}$$

$$\sum_{e=\{i,j\}\in E(\mathcal{G})} r_{ij}w(e) = \frac{1}{2}\mathbf{Tr}(L_{\mathcal{G}}R_{\mathcal{G}}) = n-1, \tag{2.4}$$

where r_{ij} and $w(e)$ are the effective resistance and the weight of edge $e = \{i,j\} \in E(\mathcal{G})$, respectively.

Proof. We refer to [22, Lemma 2] for a proof. □

Graph Families in $\mathbb{G}_n$	Notation
Complete graph	$\mathcal{K}_n$
Star graph	$\mathcal{S}_n$
Cycle graph	$\mathcal{C}_n$
Path graph	$\mathcal{P}_n$
Dumbbell graph	$\mathcal{D}(n; n_1, n_2)$
Complete bipartite graph of size (n_1, n_2)	$\mathcal{K}_{n_1,n_2}$

Table 2.1: For comparison purposes throughout the thesis, we consider the standard graphs in this table in several occasions.

In the rest of this section, we review some concepts from majorization theory. The following definition is from [23].

Definition 2.0.6. *For every $x \in \mathbb{R}^n_+$, let us define $x^\downarrow$ to be a vector whose elements are a permuted version of elements of x in descending order. We say that x majorizes y, which is denoted by $x \trianglerighteq y$, if and only if $\mathbf{1}^T x = \mathbf{1}^T y$ and*

$$\sum_{i=1}^{k} x_i^\downarrow \geq \sum_{i=1}^{k} y_i^\downarrow, \tag{2.5}$$

for all $k = 1, \ldots, n-1$.

We should emphasize that majorization is not a partial ordering. This is because from relations $x \trianglerighteq y$ and $y \trianglerighteq x$ one can only conclude that the entries of these two vectors are equal, but not necessarily in the same order. Therefore, relations $x \trianglerighteq y$ and $y \trianglerighteq x$ do not imply $x = y$. The following theorem is from [24] which suggests equivalent methods to verify majorization.

Theorem 2.0.7. *For every $x, y \in \mathbb{R}^n_+$, the following statements are equivalent:*

(i) $x \trianglerighteq y$*;*

(ii) *For all scalar convex functions f, $F(x) \geq F(y)$ where $F(x) = \sum_{i=1}^n f(x_i)$; and*

(iii) $y = Dx$ *for some doubly stochastic matrix D.*

Definition 2.0.8. *The real-valued function $F : \mathbb{R}^n_+ \to \mathbb{R}$ is called Schur–convex if $F(x) \geq F(y)$ for every two vectors x and y with property $x \trianglerighteq y$. Similarly, a function F is Schur–concave if $-F$ is Schur–convex.*

Chapter 3

Performance Measures for General Linear Dynamical Networks

The steady-state variance of outputs of linear systems driven by external stochastic disturbances can be regarded as a measure of performance. We consider a linear time-invariant network

$$\dot{x} = Ax + \xi, \tag{3.1}$$

$$y = Cx, \tag{3.2}$$

with $x(0) = 0$, where $x \in \mathbb{R}^n$ is the state and $y \in \mathbb{R}^m$ the output of the system. The input signal $\xi \in R^n$ is a white noise process with zero mean and identity covariance, i.e.,

$$\mathbf{E}\left[\xi(t)\xi(\tau)^T\right] = I_n\delta(t-\tau),$$

where $\delta(.)$ is the delta function. It is assumed that the state matrix A is Hurwitz.

Definition 3.0.9. *The $\mathcal{H}_2$–norm of linear system* (3.1)-(3.2) *from ξ to y is defined as the square root of the following quantity*

$$\mathbf{H}_Q(A) \triangleq \lim_{t\to\infty} \mathbf{E}[y(t)^T y(t)] = \lim_{t\to\infty} \mathbf{E}[\|x(t)\|_Q^2]. \tag{3.3}$$

where $Q = C^T C$.

For unstable linear systems, the outputs of the system have finite steady state variance as along as the

unstable modes of the system are not observable from the output of the system (cf. [1]). The value of $\mathbf{H}_Q(A)$ for (3.1)-(3.2) can be quantified as

$$\mathbf{H}_Q(A) \;=\; \mathbf{Tr}(P_c Q), \tag{3.4}$$

where P_c is the controllability Grammian of linear system (3.1)-(3.2) which is the unique solution of

$$AP_c + P_c A^T + I_n = 0. \tag{3.5}$$

One can also calculate the $\mathbf{H}_Q(A)$ using the observability Grammian P_o,

$$\mathbf{H}_Q(A) \;=\; \mathbf{Tr}(P_o), \tag{3.6}$$

where P_o is the unique solution of the Lyapunov equation

$$P_o A + A^T P_o + Q = 0. \tag{3.7}$$

3.1 The Main Result

We show that the $\mathcal{H}_2$–norm of a general linear system (3.1)-(3.2) from external disturbance input to the output of the system can be bounded from above and below using some real-valued functions of the eigenvalues of the state matrix A. The following result was originally reported in [3, 5].

Theorem 3.1.1. *Suppose that in linear system* (3.1)-(3.2) *the disturbance input is a white stochastic process xi with zero mean and identity covariance, the state matrix A is Hurwitz, and $C = I_n$. Then, we have*

$$-\sum_{i=1}^{n} \frac{1}{2\mathbf{Re}\{\lambda_i(A)\}} \;\leq\; \mathbf{H}_Q(A) \;\leq\; -\sum_{i=1}^{n} \frac{1}{2\lambda_i(A_s)}, \tag{3.8}$$

where $A_s = \frac{A^T + A}{2}$ is the systematic part of matrix A.

Proof. We refer the reader to the Appendix for a proof. □

The following corollaries explore several special cases and show that how the performance measure $\mathbf{H}_Q(A)$ depends on the general properties of A and the size of the network.

Corollary 3.1.2. *Suppose that the assumptions of Theorem 3.1.1 hold. Furthermore, if we assume that matrix A is normal, i.e., $A^TA = AA^T$, then* (3.8) *reduces to*

$$\mathbf{H}_Q(A) \;=\; -\sum_{i=1}^{n} \frac{1}{2\mathbf{Re}\{\lambda_i(A)\}} \;=\; -\sum_{i=1}^{n} \frac{1}{2\lambda_i(A_s)}. \tag{3.9}$$

Proof. According to the Schur decomposition for normal matrices, there exists a unitary $V \in \mathbb{C}^{n\times n}$, such that $A = V\Gamma V^H$ where $\Gamma = \mathbf{diag}\{\lambda_1, \ldots, \lambda_n\}$ and V^H denotes the conjugate transpose of matrix V. Using this we have

$$\begin{aligned} A_s &= \frac{A + A^H}{2} = V\left(\frac{\Gamma + \Gamma^H}{2}\right)V^H \\ &= V\mathbf{diag}\big(\mathbf{Re}\{\lambda_1\}, \ldots, \mathbf{Re}\{\lambda_n\}\big)V^H. \end{aligned} \tag{3.10}$$

This implies that $\lambda_i(A_s) = \mathbf{Re}\{\lambda_i\}$ for all $i = 1, \ldots, n$. Thus, the lower and upper bounds in (3.8) coincide. □

Corollary 3.1.3. *Suppose that the assumptions of Theorem 3.1.1 hold. Then, it follows that*

$$\frac{n^{1.5}}{2\sqrt{\mathbf{Tr}(AA_s)}} \;\leq\; \mathbf{H}_Q(A). \tag{3.11}$$

Proof. From the definition of trace operator, we have

$$\mathbf{Tr}(A^2) = \sum_{i=1}^{n} \mathbf{Re}\{\lambda_i(A)\}^2 - \sum_{i=1}^{n} \mathbf{Im}\{\lambda_i(A)\}^2. \tag{3.12}$$

According to the definition of the Frobenius norm, we have

$$\begin{aligned} \mathbf{Tr}(AA^T) &= \|A\|_F^2 \\ &\geq \sum_{i=1}^{n} |\lambda_i(A)|^2 \\ &= \sum_{i=1}^{n} \mathbf{Re}\{\lambda_i(A)\}^2 + \sum_{i=1}^{n} \mathbf{Im}\{\lambda_i(A)\}^2. \end{aligned} \tag{3.13}$$

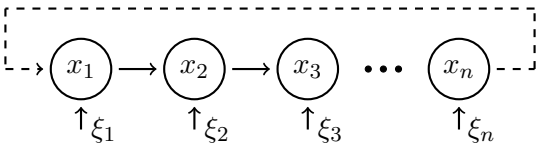


Figure 3.1: Schematic diagram of negative feedback noisy cyclic system. The dashed link indicates a negative (inhibitory) feedback signal.

Therefore based on (3.12) and (3.13), it follows that

$$\sum_{i=1}^{n} \mathbf{Re}\{\lambda_i(A)\}^2 \leq \frac{\|A\|_F^2 + \mathbf{Tr}(A^2)}{2} = \mathbf{Tr}(AA_s). \tag{3.14}$$

By applying the root-mean square and harmonic mean inequalities and (3.14), one can conclude inequality (3.11). □

A more conservative lower bound can be obtained by considering the following inequality

$$\mathbf{Tr}(AA_s) \leq n^2 \max_{i,j} |a_{ij}|^2, \tag{3.15}$$

which leads to the following inequality

$$\frac{n^{0.5}}{2\max_{i,j}|a_{ij}|} \leq \mathbf{H}_Q(A). \tag{3.16}$$

3.2 Example of a Linear Network with Nonnormal Matrix

In this part, we apply our main result to a nontrivial example. We consider the class of linear dynamical networks with cyclic interconnection topologies. An example of a cyclic network is an autocatalytic pathway in biology with ring topology which consists of a sequence of biochemical reactions where the system's product (output) is necessary to power and catalyze its own function [3, 4, 25]. We consider a cyclic linear dynamical network consists of a group of linear-time invariant systems $\mathcal{S}_i$ with state-space representations

$$\dot{x}_i = -a_i x_i + u_i + \xi_i, \tag{3.17}$$

$$v_i = c_i x_i, \tag{3.18}$$

for $i = 1, \ldots, n$, where a_i, c_i are strictly positive numbers. The scalar quantities u_i, v_i, x_i are the input, output and state variables of subsystem $\mathcal{S}_i$, respectively. By considering series interconnection of subsystems $\mathcal{S}_i$ for $i = 1, 2, \cdots, n$ and applying the output of subsystem i as the input of subsystem $i+1$ (see Fig. 3.1), we obtain the dynamics of the cyclic network as follows

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 - v_n + \xi_1, \\ \dot{x}_2 &= -a_2 x_2 + v_1 + \xi_2, \\ &\vdots \\ \dot{x}_n &= -a_n x_n + v_{n-1} + \xi_n, \end{aligned} \tag{3.19}$$

where ξ_i for $i = 1, 2, \cdots, n$ are independent white stochastic processes with identical statistics. The resulting dynamical system can be represented in the following compact form

$$\dot{x} = Ax + \xi, \tag{3.20}$$

$$y = Cx, \tag{3.21}$$

where

$$A = \begin{bmatrix} -a_1 & 0 & \ldots & 0 & -c_n \\ c_1 & -a_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -a_{n-1} & 0 \\ 0 & 0 & \ldots & c_{n-1} & -a_n \end{bmatrix} \tag{3.22}$$

and $C = I$ and $\xi \in \mathbb{R}^n$ is a zero-mean white stochastic process with identity covariance. Our goal is to investigate robustness properties of the cyclic linear dynamical network (3.19) driven by external white stochastic disturbances.

Theorem 3.2.1. *For the cyclic linear dynamical network (3.19) driven by a zero-mean white stochastic*

process $\xi \in \mathbb{R}^n$ with identity covariance, we define

$$\mathfrak{a} \triangleq \sqrt[n]{a_1 a_2 \cdots a_n}, \tag{3.23}$$

$$\mathfrak{c} \triangleq \sqrt[n]{c_1 c_2 \cdots c_n}. \tag{3.24}$$

If $\gamma > \cos(\frac{\pi}{n})$ where $\gamma = \frac{\mathfrak{a}}{\mathfrak{c}}$, then the cyclic linear dynamical network is stable. Moreover, if we assume that $\mathfrak{a} = a_1 = \cdots = a_n$, then

$$\mathbf{H}_Q(A) \geq -\sum_{i=1}^{n} \frac{1}{2\mathbf{Re}\{\lambda_i(A)\}} = \begin{cases} \frac{n \tan \frac{\beta}{2}}{2\mathfrak{c} \sin \frac{\beta}{n}} & , \quad \gamma < 1 \\ \frac{n^2}{4\mathfrak{c}} & , \quad \gamma = 1 \\ \frac{n \tanh \frac{\beta}{2}}{2\mathfrak{c} \sinh \frac{\beta}{n}} & , \quad \gamma > 1 \end{cases} \tag{3.25}$$

where

$$\beta \triangleq \begin{cases} \arccos(\gamma) n & , \quad \gamma \leq 1 \\ \operatorname{arccosh}(\gamma) n & , \quad \gamma > 1 \end{cases}. \tag{3.26}$$

Proof. The stability condition $\gamma > \cos(\frac{\pi}{n})$ implies that A is Hurwitz. Therefore, the $\mathcal{H}_2$–norm squared is finite and given by $\mathbf{Tr}(P)$ (see [10, 26] for more details), where P is the unique positive definite solution of the Lyapunov equation

$$AP + PA^T = -I_n. \tag{3.27}$$

When $\mathfrak{a} = a_1 = a_2 = \cdots = a_n$, it is straightforward to verify that the characteristic equation of A is given by

$$(\lambda + \mathfrak{a})^n + c_1 c_2 \cdots c_n = 0.$$

Therefore, the eigenvalues of the matrix are

$$\lambda_k = -\mathfrak{a} + \mathfrak{c}\, e^{i(\frac{\pi}{n} + \frac{2\pi k}{n})},$$

for $k = 0, 1, \cdots, n-1$. By substituting these eigenvalues into the lower bound of (3.8), we get

$$\begin{aligned} -\sum_{i=1}^{n} \frac{1}{2\mathbf{Re}\{\lambda_i(A)\}} &= \sum_{k=0}^{n-1} \frac{1}{2\mathbf{Re}\left\{-\mathfrak{a} + \mathfrak{c}\, e^{i(\frac{\pi}{n}+\frac{2\pi k}{n})}\right\}} \\ &= \sum_{k=0}^{n-1} \frac{1}{2\mathfrak{c}\left(\gamma - \cos(\frac{\pi}{n} + \frac{2\pi k}{n})\right)}. \end{aligned} \tag{3.28}$$

First, let us assume that $\gamma < 1$ and substitute $\gamma = \cos(\frac{\beta}{n})$ in (3.28). It follows that

$$\begin{aligned} -\sum_{i=1}^{n} \frac{1}{2\mathbf{Re}\{\lambda_i(A)\}} &= \frac{1}{2\mathfrak{c}} \sum_{k=0}^{n-1} \frac{1}{\cos(\frac{\beta}{n}) - \cos(\frac{\pi}{n} + \frac{2\pi k}{n})} \\ &= \frac{1}{4\mathfrak{c}} \sum_{k=0}^{n-1} \csc(\tfrac{(2k+1)\pi}{2n} + \tfrac{\beta}{2n}) \csc(\tfrac{(2k+1)\pi}{2n} - \tfrac{\beta}{2n}) \\ &= \frac{n \tan\frac{\beta}{2}}{2\mathfrak{c} \sin\frac{\beta}{n}}, \end{aligned}$$

where the Birkhoff Ergodic theorem is used to conclude the last equation. Similar steps can be taken when $\gamma \geq 1$. In each case by substituting γ from (3.26) in (3.28), one can obtain the desired result in the right hand side of (3.25). □

The classical secant criterion reported in [27] and [28] for cyclic linear dynamical network (3.19) provides a stability condition when all a_i for $i = 1, \ldots, n$ are identical and implies that the unperturbed system with $\xi = 0$ in (3.19) is stable if and only if $\gamma > \cos(\frac{\pi}{n})$. For a fixed parameter β, the stability condition of the cyclic network is not affected when the number of intermediate subsystems changes. However, the result of Theorem 3.2.1 asserts that the lower bound of the performance measure $\mathbf{H}_Q(A)$ increases when the size of network increases. We show that the lower bound of the performance measure $\mathbf{H}_Q(A)$ is in

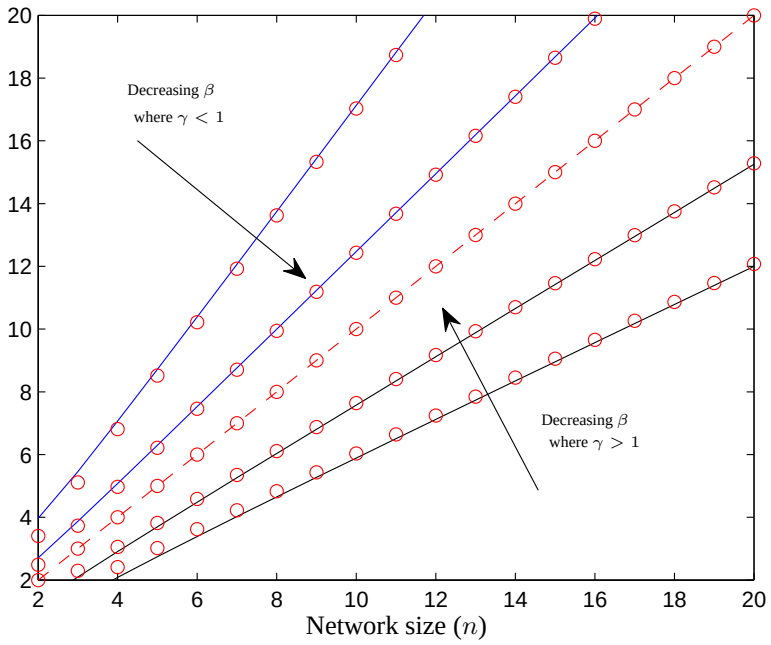


Figure 3.2: The lower bound in (3.19), which is depicted by small red circles (∘), is compared asymptotically to its approximation in (3.29). It can be observed that (3.29) tightly approximates the lower bound in (3.19).

order of $\mathcal{O}(n^2)$ when parameter β is fixed. More explicitly, we obtain the following approximation

$$-\sum_{i=1}^{n}\frac{1}{2\mathbf{Re}\{\lambda_i(A)\}}\approx\begin{cases}\frac{\tan\frac{\beta}{2}}{2\mathfrak{c}\beta}n^2 & , \quad \gamma<1\\ \frac{1}{4\mathfrak{c}}n^2 & , \quad \gamma=1\\ \frac{\tanh\frac{\beta}{2}}{2\mathfrak{c}\beta}n^2 & , \quad \gamma>1\end{cases}. \tag{3.29}$$

From this result, we conclude that the lower bound on $\mathcal{H}_2$–norm of the network scales with $\mathcal{O}(n)$. Figure 3.2 depicts such linear relationship.

Corollary 3.2.2. *Suppose that the following condition holds for the cyclic linear dynamical network* (3.19)

$$\frac{\mathfrak{a}}{\mathfrak{c}}>\cos\left(\frac{\pi}{n}\right), \tag{3.30}$$

where $\mathfrak{a} \triangleq a_1 = \ldots = a_n$, $\mathfrak{c} \triangleq c_1 = \ldots = c_n$, *and the output of the system is defined by*

$$y \;=\; Cx \;=\; \left[\begin{array}{cccc} 0 & \ldots & 0 & 1 \end{array}\right] x.$$

Then, the steady-state output dispersion is bounded from above by

$$\mathbf{H}_Q(A) \triangleq \lim_{t\to\infty} \mathbf{E}[y(t)^2] \;\leq\; \frac{1}{2(\mathfrak{a} - \mathfrak{c}\cos(\frac{\pi}{n}))}.$$

Proof. The steady-state output dispersion is given by

$$\mathbf{H}_Q(A) = \mathbf{Tr}(CPC^T),$$

where P is the unique solution of the Lyapunov equation (3.5). According to Theorem 3.2.1, our assumption (3.30) implies that all the eigenvalues of A have strictly negative real parts. Therefore, the unique solution of (3.5) can be written in the following closed form

$$P = \int_0^\infty e^{A^T t} e^{At} dt. \tag{3.31}$$

The state matrix defined by (3.22) is normal, i.e., $A^T A = AA^T$. According to the spectral theorem, there exists a unitary matrix $V \in \mathbb{C}^{n\times n}$ such that $A = V\Lambda V^{\mathrm{H}}$ where $\Lambda = \mathbf{diag}(\lambda_1, \cdots, \lambda_n)$. We now consider the integrand of (3.31)

$$\begin{aligned} P &= \int_0^\infty e^{A^{\mathrm{H}} t} e^{At} dt \\ &= \int_0^\infty V e^{\Lambda^{\mathrm{H}} t} e^{\Lambda t} V^{\mathrm{H}} dt \\ &= V\mathbf{diag}\left(\frac{1}{2\mathbf{Re}\{\lambda_1\}}, \cdots, \frac{1}{2\mathbf{Re}\{\lambda_n\}}\right) V^{\mathrm{H}} \end{aligned} \tag{3.32}$$

Since $\|C\|_2 = \|C^T\|_2 = 1$, it follows that

$$\begin{aligned} \mathbf{Tr}(CPC^T) &\leq \max_i \lambda_i(P) \\ &= \max_i \frac{1}{2\mathbf{Re}\{\lambda_i\}} \end{aligned}$$

$$= \frac{1}{2(\mathfrak{a} - \mathfrak{c}\cos(\frac{\pi}{n}))}. \tag{3.33}$$

□

In the following chapters, we apply the main result of this section to linear consensus algorithms in large-scale dynamical networks.

Chapter 4

First-Order Linear Consensus Networks

We consider linear networks with first-order consensus dynamics over a weighted connected graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), w^{(\mathcal{G})})$ with n nodes and m edges. For this class of networks, each node (i.e., subsystem) corresponds to a scalar state variable. Therefore, the state of the entire network can be represented by $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ where x_i for $i = 1, \dots, n$ is the state variable of i'th node. We assume that the dynamics of this class of dynamical networks is given by the following continuous-time first-order linear consensus dynamics

$$\dot{x} = -L_{\mathcal{G}} x + \xi, \tag{4.1}$$

where $L_{\mathcal{G}}$ is the Laplacian matrix of the underlying graph $\mathcal{G}$ and $\xi \in \mathbb{R}^n$ is an external stochastic white noise with zero-mean and identity covariance. The output of the network is defined using the incidence matrix of the output graph $\mathcal{Q} = (V(\mathcal{Q}), E(\mathcal{Q}), w^{(\mathcal{Q})})$ as follows

$$y = C_{\mathcal{Q}} x, \tag{4.2}$$

where

$$L_{\mathcal{Q}} = C_{\mathcal{Q}}^T C_{\mathcal{Q}}, \tag{4.3}$$

is the Laplacian of graph $\mathcal{Q}$. We should emphasize that, in general, the output matrix $C_{\mathcal{Q}}$ may not be an incidence matrix. The only requirement for $C_{\mathcal{Q}}$ is to satisfy (4.3). For example, let us consider the

following output matrix

$$C_{\mathcal{Q}} = I_n - \frac{1}{n} J_n.$$

It is straightforward to verify that

$$L_{\mathcal{Q}} = C_{\mathcal{Q}}^T C_{\mathcal{Q}} = I_n - \frac{1}{n} J_n$$

is the Laplacian matrix of a complete graph. Thus, the corresponding output graph is $\mathcal{Q} = \mathcal{K}_n$ with weight function $w^{(\mathcal{Q})}(e) = \frac{1}{n}$ for all $e \in E(\mathcal{Q})$.

Definition 4.0.3. *The first-order Laplacian energy of the linear consensus network (4.1)-(4.2) is defined as the steady state variance of the output signal, i.e.,*

$$\begin{aligned} \mathbf{H}_{\mathcal{Q}}^{(1)}(L_{\mathcal{G}}) &= \lim_{t\to\infty} \mathbf{E}\left[y(t)^T y(t)\right] \\ &= \lim_{t\to\infty} \mathbf{E}\left[\|x(t)\|_{L_{\mathcal{Q}}}^2\right] \\ &= \lim_{t\to\infty} \mathbf{E}\left[\sum_{e=\{i,j\}\in E(\mathcal{Q})} w^{(\mathcal{Q})}(e)\big(x_i(t) - x_j(t)\big)^2\right], \end{aligned}$$

where $w^{(\mathcal{Q})}(e)$ is the weight of edge $e = \{i, j\}$ in the output graph $\mathcal{Q}$.

We recall that the Laplacian matrix $L_{\mathcal{G}}$ has a simple zero eigenvalue with eigenvector $\mathbf{1}_n = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$. This implies that the linear consensus network (4.1)-(4.2) is marginally stable. Since $L_{\mathcal{Q}}\mathbf{1}_n = 0$, the marginally stable mode of the system does not affect the Laplacian energy of the linear consensus network.

Theorem 4.0.4. *For the linear consensus network (4.1)-(4.2), the first-order Laplacian energy is given by*

$$\mathbf{H}_{\mathcal{Q}}^{(1)}(L_{\mathcal{G}}) = \frac{1}{2}\mathbf{Tr}(L_{\mathcal{Q}} L_{\mathcal{G}}^{\dagger}). \tag{4.4}$$

where $L_{\mathcal{G}}^{\dagger}$ is the Moore–Penrose pseudoinverse of the underlying graph of the network.

Proof. According to (3.6), we need to calculate the unique solution of the Lyapunov equation

$$L_{\mathcal{G}} P + P L_{\mathcal{G}} = L_{\mathcal{Q}}. \tag{4.5}$$

By multiplying each side of (4.5) by $L_{\mathcal{G}}^{\dagger}$, we get

$$L_{\mathcal{G}}^{\dagger}L_{\mathcal{G}}P + L_{\mathcal{G}}^{\dagger}PL_{\mathcal{G}} = L_{\mathcal{G}}^{\dagger}L_{\mathcal{Q}}. \tag{4.6}$$

One can verify that $L_{\mathcal{G}}^{\dagger}L_{\mathcal{G}} = I_n - \frac{1}{n}J_n$. By applying trace operate to the left hand side of (4.6), it follows that

$$\mathbf{Tr}\big((I_n - \frac{1}{n}J_n)P\big) + \mathbf{Tr}(PL_{\mathcal{G}}L_{\mathcal{G}}^{\dagger}) = 2\mathbf{Tr}((I_n - \frac{1}{n}J_n)P).$$

Since $L_{\mathcal{Q}}$ and $L_{\mathcal{G}}$ are both symmetric matrices with zero row and column sums, we have

$$\mathbf{H}_{\mathcal{Q}}^{(1)}(L_{\mathcal{G}}) = \mathbf{Tr}\,(P) = \frac{1}{2}\mathbf{Tr}(L_{\mathcal{Q}}L_{\mathcal{G}}^{\dagger}).$$

□

When output graph $\mathcal{Q}$ is a complete graph with weight function $w^{(\mathcal{Q})}(e) = \frac{1}{n}$ for all $e \in E(\mathcal{Q})$, then $L_{\mathcal{Q}} = I_n - \frac{1}{n}J_n$. The first-order Laplacian energy of the corresponding network is given by

$$\mathbf{H}_{\mathcal{K}_n}^{(1)}(L_{\mathcal{G}}) = \sum_{i=2}^{n} \frac{1}{2\lambda_i}. \tag{4.7}$$

Therefore, the first-order Laplacian energy of the first-order linear consensus network (4.1)-(4.2) reduces to the concept of first-order network coherence and the expected dispersion of the state of the system in steady state [1, 10]. It turns out that the total effective resistance of (4.1)-(4.2) depends on the spectrum of the Laplacian matrix that is given by

$$\mathbf{r}_{\text{total}} = n\sum_{i=2}^{n} \frac{1}{\lambda_i}. \tag{4.8}$$

We refer to [1] for more details. Therefore, the first-order Laplacian energy for linear consensus network (4.1)-(4.2) is

$$\mathbf{H}_{\mathcal{K}_n}^{(1)}(L_{\mathcal{G}}) = \frac{\mathbf{r}_{\text{total}}}{2n}. \tag{4.9}$$

In the next section, we derive several combinatorial and graph-theoretical lower and upper bounds on

the first-order Laplacian energy (4.4).

4.1 Graph-Dependent Scaling Laws for the First-Order Laplacian Energy

We consider a class of first-order linear consensus networks (4.1)-(4.2) that are defined over a simple connected unweighted graph. In order to apply our results from Subsection 3.1, we will treat an unweighted graph as a weighted graph with constant weight values equal to 1 for all edges, i.e., $w^{(\mathcal{G})}(e) = 1$ for all $e \in E(\mathcal{G})$. It is also assumed that the output graph is a complete graph with $w^{(\mathcal{Q})}(e) = \frac{1}{n}$ for all $e \in E(\mathcal{Q})$. For simplicity of our notation, we adopt the simple notation $\mathbf{H}^{(1)}(L_{\mathcal{G}})$ instead of $\mathbf{H}^{(1)}_{\mathcal{K}_n}(L_{\mathcal{G}})$ whenever the output graph is a complete graph. In the following subsections, we consider several scenarios and reveal the foundational role of the underlying graph of the network on how the first-order Laplacian energy of a linear consensus network depends on various characteristics of the underlying graph.

4.1.1 General Lower and Upper Bounds

The result of the following theorem relates the first-order Laplacian energy to the diameter of the underlying graph of the network. The diameter of a graph is one of the key features of a graph and defined as the largest distance between every two nodes in a graph. The diameter of a simple connected graph $\mathcal{G}$ is denoted by $\mathbf{diam}(\mathcal{G})$.

Theorem 4.1.1. *For the linear consensus network (4.1)-(4.2), the first-order Laplacian network is bounded by*

$$\mathfrak{L}_{L_{\mathcal{G}}} \leq \mathbf{H}^{(1)}(L_{\mathcal{G}}) \leq \mathfrak{U}_{L_{\mathcal{G}}}, \tag{4.10}$$

where

$$\mathfrak{L}_{L_{\mathcal{G}}} = \frac{(n-1)^{1.5}}{2\sqrt{s_1 + s_2}} \tag{4.11}$$

$$\mathfrak{U}_{L_{\mathcal{G}}} = \frac{n-1}{2n}\left[1 + \left(\binom{n}{2} - m\right)\mathbf{diam}(\mathcal{G})\right] \tag{4.12}$$

where m is the number of edges, d_i the degree of node i, and $s_\alpha \triangleq \sum_{i=1}^{n} d_i^\alpha$ for $\alpha = 1, 2$.

Proof. For the lower bound, we apply the result of Theorem 3.1.3 and the fact that $\|L_{\mathcal{G}}\|_F = \sqrt{s_1 + s_2}$ to obtain

$$\frac{(n-1)^{1.5}}{2\sqrt{s_1+s_2}} \quad \leq \quad \mathbf{H}^{(1)}(L_{\mathcal{G}}). \tag{4.13}$$

For the upper bound, according to [29, Th. 1] it follows that

$$\frac{n}{1 + \left(\binom{n}{2} - m\right) \mathbf{diam}(\mathcal{G})} \leq \lambda_2. \tag{4.14}$$

From (4.7) and the fact that λ_2 is the smallest nonzero eigenvalue of $L_{\mathcal{G}}$, we get

$$\mathbf{H}^{(1)}(L_{\mathcal{G}}) = \sum_{i=2}^{n} \frac{1}{2\lambda_i} \leq \frac{n-1}{2\lambda_2}, \tag{4.15}$$

By combining inequalities (4.14) and (4.15), we get the desired upper bound

$$\mathbf{H}^{(1)}(L_{\mathcal{G}}) \leq \frac{n-1}{2n}\left[1 + \left(\binom{n}{2} - m\right) \mathbf{diam}(\mathcal{G})\right].$$

□

For a complete graph $\mathcal{G} = \mathcal{K}_n$, both lower and upper bounds in Theorem 4.1.1 coincide and we have

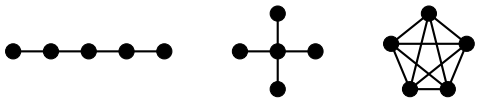
$$\mathbf{H}^{(1)}(L_{\mathcal{K}_n}) = \frac{n-1}{2n}.$$

Proposition 1. *For the linear consensus network (4.1)-(4.2) defined over a graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}), w^{(\mathcal{G})})$, the corresponding first-order Laplacian energy is bounded from below by*

$$\frac{(n-1)^2}{2\mathbf{Tr}(L_{\mathcal{G}})} \quad \leq \quad \mathbf{H}^{(1)}(L_{\mathcal{G}}). \tag{4.16}$$

Proof. It can be shown that $\mathbf{H}^{(1)}(L_{\mathcal{G}})$ is a Schur–convex function respect to $(\lambda_2, \ldots, \lambda_n)^T \in \mathbb{R}^{n-1}_{++}$ where λ_i for $i = 2, \ldots, n$ are eigenvalues of $L_{\mathcal{G}}$. On the other hand, we have

$$\frac{\mathbf{Tr}(L)}{n-1}\mathbf{1}_n^T \trianglelefteq (\lambda_2, \ldots, \lambda_n)^T.$$



(a) (b) (c)

Figure 4.1: According to Theorems 4.1.3 and 4.1.10, we can consider the following extreme cases: the first-order Laplacian energy is (a) maximal for $\mathcal{P}_5$ among all connected graphs with five nodes as well as among all graphs with tree structures with five nodes, (b) minimal for $\mathcal{S}_5$ among all graphs with tree structures with five nodes, and (c) minimal for $\mathcal{K}_5$ among all graphs with five nodes.

Therefore, according to the definition of Schur–convex functions, we can conclude inequality (4.16). □

The next theorem shows that the Laplacian energy of a graph is always less than or equal to the Laplacian energy of its connected spanning subgraphs.

Theorem 4.1.2. *Suppose that connected graph $\mathcal{G}$ is the underlying graph of the linear consensus network (4.1)-(4.2) with n nodes. If $\mathcal{P}$ is a connected spanning subgraph of $\mathcal{G}$, then*

$$\mathbf{H}^{(1)}(L_{\mathcal{G}}) \leq \mathbf{H}^{(1)}(L_{\mathcal{P}}), \tag{4.17}$$

and the equality holds if and only if $\mathcal{G} = \mathcal{P}$.

Proof. For every $x \in \mathbb{R}^n$, we have

$$\begin{aligned} x^T L_{\mathcal{G}} x &= \sum_{e=\{i,j\}\in E(\mathcal{G})} w(e)\,(x_i - x_j)^2 \\ &\geq \sum_{e=\{i,j\}\in E(\mathcal{P})} w(e)\,(x_i - x_j)^2 \\ &= x^T L_{\mathcal{P}} x. \end{aligned} \tag{4.18}$$

This inequality implies that

$$L_{\mathcal{P}} \leq L_{\mathcal{G}}, \tag{4.19}$$

and equivalently, we have

$$L_{\mathcal{G}}^{\dagger} \leq L_{\mathcal{P}}^{\dagger}. \tag{4.20}$$

From the linearity property of the trace operator and the fact that $L_{\mathcal{P}}^{\dagger} - L_{\mathcal{G}}^{\dagger}$ is a positive semi-definite matrix, we get

$$\begin{aligned} \frac{1}{2}\mathbf{Tr}(L_{\mathcal{P}}^{\dagger} - L_{\mathcal{G}}^{\dagger}) &= \frac{1}{2}\mathbf{Tr}(L_{\mathcal{P}}^{\dagger}) - \frac{1}{2}\mathbf{Tr}(L_{\mathcal{G}}^{\dagger}) \\ &= \mathbf{H}^{(1)}(L_{\mathcal{P}}) - \mathbf{H}^{(1)}(L_{\mathcal{G}}) \\ &\geq 0. \end{aligned}$$

□

In the following theorem, we characterize the maximal and minimal values of the first-order Laplacian energy over all graphs with n nodes.

Theorem 4.1.3. *The maximal and minimal values of the first-order Laplacian energy for the class of linear consensus networks (4.1)-(4.2) are given by*

$$\frac{(n-1)}{2n} \leq \mathbf{H}^{(1)}(L_{\mathcal{G}}) \leq \frac{n^2-1}{12}. \tag{4.21}$$

Furthermore, the lower bound is achieved if and only if $\mathcal{G} = \mathcal{K}_n$, and the upper bound is reached if and only if $\mathcal{G} = \mathcal{P}_n$.

Proof. According to Theorem 4.1.2, the lower bound in (4.21) can be achieved for a complete graph. The reason is that every connected graph with n nodes is a spanning subgraph of $\mathcal{K}_n$. On the other hand, $\mathbf{H}^{(1)}$ reaches its maximal value when the underlying graph is a tree. We refer to Theorem 4.1.10 in Subsection 4.1.4 for more details and a proof. □

The result of Theorem 4.1.3 is applied to different graphs with five nodes and the result is compared in Figure 4.1.

4.1.2 Tradeoffs Between Sparsity and the Laplacian Energy

In this subsection, we show that a fundamental limit emerges between sparsity of the underlying graph of the network and its first-order Laplacian energy. First, we consider the total number of nonzero elements of a Laplacian matrix $L_{\mathcal{G}} = [l_{ij}]$ as a sparsity measure for the underlying graph. In fact, this measure is

equal to the total number of edges in the underlying graph and denoted by

$$\|L_{\mathcal{G}}\|_0 \;=\; \frac{1}{2}\sum_{i\neq j}^{n} |l_{ij}|^0. \tag{4.22}$$

Corollary 4.1.4. *For the linear consensus network* (4.1)-(4.2)*, there is a fundamental tradeoff between the first-order Laplacian energy and the sparsity measure* (4.22) *that is characterized in the multiplicative form by the following inequality*

$$\mathbf{H}^{(1)}(L_{\mathcal{G}})\,\|L_{\mathcal{G}}\|_0 \;\geq\; \frac{(n-1)^2}{4}, \tag{4.23}$$

and in the additive form by

$$\left(\frac{2\mathbf{H}^{(1)}(L_{\mathcal{G}})-1}{\mathbf{diam}(\mathcal{G})}\right) \;+\; \|L_{\mathcal{G}}\|_0 \;\leq\; \frac{n(n-1)}{2}. \tag{4.24}$$

Proof. The proof is a direct consequence of Proposition 1 and Theorem 4.1.1. □

Let us consider the class of graphs with identical number of nodes and compare several scenarios. The inequality (4.23) asserts that the minimum achievable levels of first-order Laplacian energy for sparse networks is higher. For all networks with identical diameters, inequality (4.24) implies that graphs with more edges have smaller levels of first-order Laplacian energies. Among all networks with identical number of edges, the ones with larger diameters have higher levels of first-order Laplacian energies.

Corollary 4.1.5. *Let us consider the class of all linear consensus networks* (4.1)-(4.2) *with identical first-order Laplacian energies, the sparsity measure* (4.22) *can be bounded by*

$$\frac{(n-1)^2}{4\mathbf{H}^{(1)}(L_{\mathcal{G}})} \;\leq\; \|L_{\mathcal{G}}\|_0 \;\leq\; \frac{n(n-1)}{2} - \frac{2\mathbf{H}^{(1)}(L_{\mathcal{G}})-1}{\mathbf{diam}(\mathcal{G})}. \tag{4.25}$$

Proof. The proof is a direct consequence of Proposition 1 and Theorem 4.1.1. □

This result provides us with a criterion to determine what the minimum and maximum number of required edges are for a linear consensus network with a priori given level of first-order Laplacian energy.

We can also consider the following sparsity measure for linear consensus networks

$$\|L_{\mathcal{G}}\|_{\mathcal{S}_{0,1}} \triangleq \max\Big\{ \max_{1\leq i\leq n} \|L_{\mathcal{G}}(i,.)\|_{\ell^0}, \max_{1\leq j\leq n} \|L_{\mathcal{G}}(.,j)\|_{\ell^0} \Big\}$$

where $L_{\mathcal{G}}(i,.)$ represents the i'th row and $L_{\mathcal{G}}(.,j)$ the j'th column of matrix $L_{\mathcal{G}}$. The value of ℓ^0–measure $\|\,.\,\|_{\ell^0}$ returns the total number of nonzero elements in a vector. In fact, the value of the $\mathcal{S}_{0,1}$–measure of a sparse matrix is the maximum number of nonzero elements among all rows and columns of that matrix. We refer to [30] for more details and discussions on this sparsity measure. The $\mathcal{S}_{0,1}$–measure of incidence matrix of an unweighted graph is equal to the maximum node degree in that graph. The result of the following theorem provides us with a mean to quantity tradeoffs between the Laplacian energy and this sparsity measure.

Theorem 4.1.6. *For the class of linear consensus networks* (4.1)-(4.2) *with* $n \geq 3$ *nodes, we have*

$$2\mathbf{H}^{(1)}(L_{\mathcal{G}}) \geq (d_1 + d_2 - 1)^{-1} + \sum_{i=2}^{n-1} d_i^{-1} + (d_n + 1)^{-1},$$

where $d_1 \leq d_2 \leq \ldots \leq d_n$ *are the node degrees in ascending order. Moreover, the equality holds if and only if* $\mathcal{G} = \mathcal{S}_n$ *or* $\mathcal{G} = \mathcal{K}_3$.

Proof. For a given convex function $f : \mathbb{R}_+ \to \mathbb{R}$, let us define $F(x) = \sum_{i=1}^n f(x_i)$ where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}_+^n$. According to [23, Sec. 3.C], $F(x)$ is a Schur–convex function. Therefor, it follows that

$$\mathbf{H}^{(1)}(L_{\mathcal{G}}) = \sum_{i=2}^{n} \frac{1}{2\lambda_i}$$

is a Schur–convex function. This is because function $f(\lambda_i) = \frac{1}{2\lambda_i}$ is a convex function from $\mathbb{R}_+$ to $\mathbb{R}$. Moreover, the following relationship holds

$$(\lambda_2, \ldots, \lambda_n) \trianglerighteq (d_1 + d_2 - 1, d_2, \ldots, d_{n-1}, d_n + 1)$$

according to [31, Lemma 2]. From this relationship and the definition of Schur–convex function, we get

$$2\mathbf{H}^{(1)}(L_{\mathcal{G}}) \;\; = \;\; f(\lambda_2, \ldots, \lambda_n)$$

$$\begin{aligned} &\geq \quad \frac{1}{d_1+d_2-1}+\frac{1}{d_2}+\ldots+\frac{1}{d_n+1} \\ &= \quad (d_1+d_2-1)^{-1}+\sum_{i=2}^{n-1} d_i^{-1}+(d_n+1)^{-1}. \end{aligned}$$

The interested reader is referred to [31] for more details and similar arguments. □

Corollary 4.1.7. *For the class of linear consensus networks* (4.1)-(4.2) *with* $n \geq 3$ *nodes, we have*

$$\mathbf{H}^{(1)}(L_{\mathcal{G}})\, S_{\mathcal{G}} \;\geq\; \frac{n-1}{2}, \tag{4.26}$$

where

$$S_{\mathcal{G}} = \max\Big\{\|L_{\mathcal{G}}\|_{\mathcal{S}_{0,1}}+1,\; 2\|L_{\mathcal{G}}\|_{\mathcal{S}_{0,1}}-1\Big\} \tag{4.27}$$

is a measure of sparsity.

Proof. The proof is a direct consequence of Theorem 4.1.6 and the definition of $S_{\mathcal{G}}$. □

For the class of first-order linear consensus networks with identical number of nodes, the result of this corollary asserts that by improving local connectivity in a network the minimum achievable level of first-order Laplacian energy decreases.

Remark 4.1.8. *The value of the* $\mathcal{S}_{0,1}$ *sparsity measure reveals some valuable information about sparsity as well as the spatial locality features of a given sparse matrix, while* (4.22) *does not. Moreover,* (4.22) *does not exhibit any interesting algebraic property and cannot be used in infinite-dimensional settings.*

4.1.3 Role of the Characteristic Polynomial of the Laplacian.

The first-order Laplacian energy of the linear consensus network (4.1)-(4.2) depends on the coefficients of the characteristic polynomial of the Laplacian matrix of the underlying graph of the network, which is represented by

$$\Phi_{L_{\mathcal{G}}}(\lambda) = \sum_{k=0}^{n} (-1)^{n-k}\, c_k(L_{\mathcal{G}})\, \lambda^k. \tag{4.28}$$

From (4.7) and Vieta's formulas for (4.28), it follows that

$$\mathbf{H}^{(1)}(L_{\mathcal{G}}) = \frac{c_2(L_{\mathcal{G}})}{2c_1(L_{\mathcal{G}})}. \tag{4.29}$$

The total number of spanning trees of graph $\mathcal{G}$ can be characterized by

$$\mathfrak{T}(\mathcal{G}) = \frac{1}{n}\lambda_2 \cdots \lambda_n = \frac{1}{n}c_1(L_{\mathcal{G}}). \tag{4.30}$$

This result can be deducted from the following formula that establishes a relationship between the coefficients of the characteristic polynomial (4.28) and the structure of graph $\mathcal{G}$

$$c_k(L_{\mathcal{G}}) = \sum_{F \in \mathcal{F}_k(\mathcal{G})} \gamma(F), \tag{4.31}$$

in which F stands for a spanning forest, $\mathcal{F}_k(\mathcal{G})$ the set of all spanning forests of $\mathcal{G}$ with exactly k components, and $\gamma(F)$ the product of the number of nodes of the components of F [32, 33]. Therefore, from (4.29) and (4.30) one can conclude that

$$\mathbf{H}^{(1)}(\mathcal{G}) = \frac{c_2(L_{\mathcal{G}})}{2n\mathfrak{T}(\mathcal{G})}.$$

It is worth mentioning that there are methods to compute the coefficients of (4.28) in an iterative manner. For instance, the following recursive formulae is proposed by Fadeev (see [34] for more details)

$$c_{n-k} = \frac{1}{k}\mathbf{Tr}(\mathcal{L}_{\mathcal{G}}^{(k)}), \tag{4.32}$$

where

$$\mathcal{L}_{\mathcal{G}}^{(k)} = L_{\mathcal{G}}^{k} - (-1)^{k-1}c_{n-1}L_{\mathcal{G}}^{k-1} - \cdots - (-1)^{1}c_{n-k+1}L_{\mathcal{G}}.$$

The next theorem shows that a lower bound in terms of the total number of spanning trees can be obtained for the first-order Laplacian energy of a linear consensus network.

Theorem 4.1.9. *For the linear consensus network (4.1)-*(4.2)*, the first-order Laplacian energy is bounded*

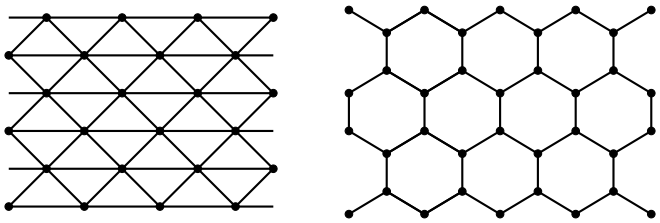


Figure 4.2: Two symmetric lattices of order 3: (a) Triangular lattice (b) Honeycomb lattice which is the planar dual of triangular lattice.

from below by

$$\frac{n-1}{2\sqrt[n-1]{n\mathfrak{T}(\mathcal{G})}} \leq \mathbf{H}^{(1)}(L_{\mathcal{G}}). \tag{4.33}$$

Proof. By applying the inequality of arithmetic and geometric means to (4.7) and using equation (4.30), we get

$$\begin{aligned} \frac{\mathbf{H}^{(1)}(L_{\mathcal{G}})}{n-1} &= \frac{\sum_{i=2}^{n} \frac{1}{2\lambda_i}}{n-1} \\ &\geq \sqrt[n-1]{\prod_{i=2}^{n} \frac{1}{2\lambda_i}} \\ &= \frac{1}{2\sqrt[n-1]{n\mathfrak{T}(\mathcal{G})}}. \end{aligned} \tag{4.34}$$

□

For a complete graph $\mathcal{G} = \mathcal{K}_n$, it can be shown that $\mathfrak{T}(\mathcal{G}) = n^{n-2}$. Therefore, complete graphs achieve the lower bound in (4.33). The result of this theorem implies that if the number of spanning trees increases, the minimum achievable levels of first-order Laplacian energy decreases accordingly. The result of Theorem 4.1.9 can be applied to graphs with regular lattice topologies and show that the first-order Laplacian energy scales asymptotically with network size. Let us consider the number of spanning trees in a finite subgraph of a lattice. It can be shown that $\mathfrak{T}(\mathcal{G})$ grows exponentially with the number of nodes. We refer to [35] for detailed discussions and proofs.

4.1.4 Graphs with Tree Structure

In this subsection, we assume that the underlying graph of the linear consensus network (4.1)-(4.2) is a tree graph that is denoted by $\mathcal{T}$. One of the invariant characteristics of a graph is its Wiener number which

is denoted by $\mathbf{W}(\mathcal{T})$ [33] and is equal to the sum of distances between all pairs of nodes of $\mathcal{T}$. It is well known that the second coefficient of the Laplacian characteristic polynomial of a tree coincides with the Wiener number, i.e.,

$$c_2(L_{\mathcal{T}}) = \mathbf{W}(\mathcal{T}).$$

According to this fact and (4.29), it follows that

$$\mathbf{H}^{(1)}(L_{\mathcal{T}}) = \frac{c_2(L_{\mathcal{T}})}{2n} = \frac{\mathbf{W}(\mathcal{T})}{2n}. \tag{4.35}$$

We apply this result in order to characterize trees that have minimal and maximal first-order Laplacian energies among all trees with n nodes.

Theorem 4.1.10. *For the class of linear consensus networks* (4.1)-(4.2) *with underlying tree graphs with at least five nodes, the first-order Laplacian energy is bounded by*

$$\frac{(n-1)^2}{2n} \;\leq\; \mathbf{H}^{(1)}(L_{\mathcal{T}}) \;\leq\; \frac{n^2-1}{12}, \tag{4.36}$$

Moreover, the lower bound is achieved if and only if $\mathcal{T} = \mathcal{S}_n$, and the upper bound is achieved if and only if $\mathcal{T} = \mathcal{P}_n$.

Proof. According to reference [36], if $\mathcal{T}$ is a tree with n nodes that is neither $\mathcal{P}_n$ nor $\mathcal{S}_n$, then

$$\mathbf{W}(\mathcal{S}_n) \;<\; \mathbf{W}(\mathcal{T}) \;<\; \mathbf{W}(\mathcal{P}_n). \tag{4.37}$$

Furthermore, it is shown that (see [36] for more details)

$$\mathbf{W}(\mathcal{P}_n) = \binom{n+1}{3}, \;\text{ and }\; \mathbf{W}(\mathcal{S}_n) = (n-1)^2. \tag{4.38}$$

From (4.37) and (4.35), we have

$$\frac{(n-1)^2}{2n} \;<\; \mathbf{H}^{(1)}(L_{\mathcal{T}}) \;<\; \frac{n^2-1}{12},$$

On the other hand, it follows from (4.38) and (4.35) that

$$\mathbf{H}^{(1)}(\mathcal{P}_n) = \frac{n^2-1}{12}, \text{ and } \mathbf{H}^{(1)}(\mathcal{S}_n) = \frac{(n-1)^2}{2n}. \tag{4.39}$$

Therefore, the lower bound in (4.36) is achieved if and only if $\mathcal{T} = \mathcal{S}_n$, and the upper bound is achieved if and only if $\mathcal{T} = \mathcal{P}_n$. □

For comparison purposes, the result of Theorem 4.1.10 is applied to three different graphs with five nodes and the result is explained in Figure 4.1.

Remark 4.1.11. *We should note that there is a connection between our results in this subsection and those of [12]. However, our results are more general. In [12], the authors consider the $\mathcal{H}_2$–norm of the system as a performance measure for first-order consensus networks driven by white stochastic process. This class of systems are marginally stable as the Laplacian matrix of the underlying graph of the network has a simple zero eigenvalue. It turns out that due to the existence of this marginally stable mode, the $\mathcal{H}_2$–norm of the consensus network is unbounded. In [12], this analysis is performed using the edge agreement protocol by considering a minimal realization of the edge interpretation system. The result of [12] shows that all spanning trees have identical $\mathcal{H}_2$–norm. More specifically, their results imply that graphs with path and star topologies have identical $\mathcal{H}_2$–norm. On the other hand, the result of Theorem 4.1.10 show that tree graphs with larger λ_2 have lower levels of first-order Laplacian energy.*

4.1.5 Graphs with Cut Edges

An edge is called a cut edge of the graph $\mathcal{G}$ if removing that edge from $\mathcal{G}$ results in more components than $\mathcal{G}$.

Theorem 4.1.12. *Suppose that the underlying graph of the linear consensus network* (4.1)-(4.2) *has exactly k cut edges. Then the Laplacian energy is bounded from below by*

$$\mathbf{H}^{(1)}(L_{\mathcal{G}}) \;\geq\; \frac{k+1}{2} - \frac{n+k}{2n(n-k)}.$$

The equality holds if and only if $\mathcal{G} = \mathcal{S}_n(\mathcal{K}_{n-k}; \mathcal{K}_1, \cdots, \mathcal{K}_1)$, i.e., $\mathcal{G}$ is a star graph that is formed by replacing the center of the star with a clique $\mathcal{K}_{n-k}$.

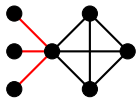


Figure 4.3: $\mathcal{S}_4(\mathcal{K}_4;\mathcal{K}_1,\mathcal{K}_1,\mathcal{K}_1)$ has the minimal $\mathbf{H}^{(1)}$ index among all connected graphs of order 7 with exactly 3 cut edges (red edges).

Proof. In equation (4.40), we show that the first-order Laplacian energy for linear consensus network (4.1)-(4.2) is

$$\mathbf{H}^{(1)}_{\mathcal{K}_n}(L_{\mathcal{G}}) = \frac{\mathbf{r}_{\text{total}}}{2n}. \tag{4.40}$$

In reference [37], it is shown that the $\mathbf{r}_{\text{total}}$ can be bounded from below as

$$\mathbf{r}_{\text{total}} \geq n(k+1) + 1 - \frac{2n}{n-k}, \tag{4.41}$$

for all connected graphs with n nodes and k cut edges. The lower bound can be achieved if and only if $\mathcal{G} = \mathcal{S}_n(\mathcal{K}_{n-k};\mathcal{K}_1,\cdots,\mathcal{K}_1)$. □

If the underlying graph of the linear consensus network (4.1)-(4.2) is a tree graph, then $k = n-1$. In this case, the result of Theorem 4.1.12 reduces to that of Theorem 4.1.10, which gives explicit lower bounds for the first-order Laplacian energy among all trees with n nodes. On the other hand, complete graphs has no cut edge, $k = 0$. In this case, the result of Theorem 4.1.12 reduces to that of Theorem 4.1.3, which provides explicit lower bounds on the first-order Laplacian energy among all connected graphs with n nodes.

4.1.6 Bipartite Graphs

We characterize the minimal and maximal achievable levels of the first-order Laplacian energy among all linear network consensus networks with bipartite graphs topologies. For instance, Figure 4.4 shows graphs with minimal and maximal Laplacian energies among all $(2,7)$–bipartite graphs.

Theorem 4.1.13. *Suppose that the underlying graph of the linear consensus network* (4.1)-(4.2) *is a bi-*

partite graph with n nodes. Then, the Laplacian energy is bounded by

$$\frac{(n-1)\big(n^2-2n\lfloor\frac{n}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor^2\big)}{2n\lfloor\frac{n}{2}\rfloor\big(n-\lfloor\frac{n}{2}\rfloor\big)} \;\leq\; \mathbf{H}^{(1)}(L_{\mathcal{G}}) \;\leq\; \frac{n^2-1}{12}.$$

Furthermore, the lower bound is achieved if and only if $\mathcal{G}=\mathcal{K}_{\lfloor\frac{n}{2}\rfloor,n-\lfloor\frac{n}{2}\rfloor}$, and the upper bound is achieved if and only if $\mathcal{G}=\mathcal{P}_n$, where $\lfloor.\rfloor$ is the floor operator.

Proof. According to Theorem 4.1.3, a path graph $\mathcal{P}_n$ has the maximal level of first-order Laplacian energy among all graphs with n nodes. Moreover, $\mathcal{P}_n$ is in fact a bipartite graph. Therefore, we get

$$\mathbf{H}^{(1)}(L_{\mathcal{G}}) \;\leq\; \frac{n^2-1}{12}.$$

The best achievable lower bound can be obtained from (4.40) and the result of [20, Th. 3.1]. □

Theorem 4.1.14. *For the linear consensus network* (4.1)-(4.2) *with a (n_1,n_2)–bipartite underlying graph where $n_1\leq n_2$, the first-order Laplacian energy is bounded by*

$$m_{L_{\mathcal{G}}} \;\leq\; \mathbf{H}^{(1)}(L_{\mathcal{G}}) \;\leq\; M_{L_{\mathcal{G}}} \tag{4.42}$$

where

$$m_{L_{\mathcal{G}}}=\frac{(n_1+n_2-1)(n_1^2+n_2^2)-n_1n_2}{2n_1n_2(n_1+n_2)} \tag{4.43}$$

and

$$M_{L_{\mathcal{G}}}=\begin{cases}\frac{-3+n_1+3n_1^2-n_1^3-6n_1n_2+6n_1^2n_2+3n_2^2+3n_1n_2^2}{12(n_1+n_2)} & \text{if} \quad n_2\overset{2}{\equiv}1+n_1\\[2ex] \frac{-2n_1+3n_1^2-n_1^3-6n_1n_2+6n_1^2n_2+3n_2^2+3n_1n_2^2}{12(n_1+n_2)} & \text{if} \quad n_2\overset{2}{\equiv}n_1\end{cases}$$

in which $\overset{2}{\equiv}$ is the modulo operation with divisor 2. The lower bound is achieved if and only if $\mathcal{G}=\mathcal{K}_{n_1,n_2}$, and the upper bound is achieved if and only if

$$\mathcal{G}=\mathcal{D}\left(n_1+n_2,\left\lfloor\frac{n_2-n_1+1}{2}\right\rfloor+1,\left\lfloor\frac{n_2-n_1+1}{2}\right\rfloor+1\right).$$

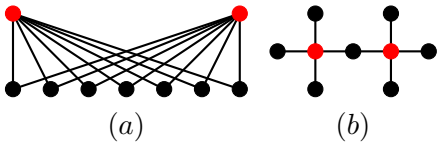


Figure 4.4: As a consequence of Theorem 4.1.14, (a) $\mathcal{K}_{2,7}$ has the least first-order Laplacian energy, and (b) $\mathcal{D}(9,4,4)$ has the highest level of first-order Laplacian energy among all linear consensus networks with $(2,7)$-bipartite graphs.

Proof. The proof is based on using equality (4.40) and the result of [20, Th. 2.6]. □

Chapter 5

Second-Order Linear Consensus Networks

In this Chapter, we turn our attention to the class of second-order linear consensus networks. We consider the following class of controlled linear time-invariant networks

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ F & G \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix} \xi, \tag{5.1}$$

$$y = C \begin{bmatrix} x \\ v \end{bmatrix} \tag{5.2}$$

where

$$C = \begin{bmatrix} C_{\mathcal{Q}_x} & 0 \\ 0 & C_{\mathcal{Q}_v} \end{bmatrix}, \tag{5.3}$$

and F and G are some stabilizing static linear feedback matrices. It is assumed that $\xi \in \mathbb{R}^{2n}$ is a zero-mean white noise process with identity covariance [1]. We associate two output graphs in order to define the second-order Laplacian energies. The output matrix $C_{\mathcal{Q}_x}$ is the incidence matrix of the position output graph $\mathcal{Q}_x$ with Laplacian matrix

$$L_{\mathcal{Q}_x} = C_{\mathcal{Q}_x}^T C_{\mathcal{Q}_x}, \tag{5.4}$$

and the output matrix $C_{\mathcal{Q}_v}$ is the incidence matrix of the velocity output graph $\mathcal{Q}_v$ with Laplacian matrix

$$L_{\mathcal{Q}_v} = C_{\mathcal{Q}_v}^T C_{\mathcal{Q}_v}. \tag{5.5}$$

Definition 5.0.15. *The second-order Laplacian energy of the linear dynamical network (5.1)-(5.2) is defined as the steady state variance of the output signal, i.e.,*

$$\begin{aligned}\mathbf{H}_Q^{(2)}(A) &= \lim_{t\to\infty} \mathbf{E}\left[y(t)^T y(t)\right] \\ &= \lim_{t\to\infty} \mathbf{E}\left[\|x(t)\|^2_{L_{\mathcal{Q}_x}}\right] + \lim_{t\to\infty} \mathbf{E}\left[\|v(t)\|^2_{L_{\mathcal{Q}_v}}\right].\end{aligned}$$

where $Q = C^T C$ and

$$A \;=\; \begin{bmatrix} 0 & I_n \\ F & G \end{bmatrix}. \tag{5.6}$$

Depending on how the output matrix C in (5.2) is defined, we can define more specific Laplacian energies. The second-order Laplacian energy of the linear dynamical network (5.1)-(5.2) with respect to the position output graph $\mathcal{Q}_x$ is defined by

$$\begin{aligned}\mathbf{H}_{x,\mathcal{Q}_x}^{(2)}(A) &= \lim_{t\to\infty} \mathbf{E}\left[\|x(t)\|^2_{L_{\mathcal{Q}_x}}\right] \\ &= \lim_{t\to\infty} \mathbf{E}\left[\sum_{e=\{i,j\}\in E(\mathcal{Q}_x)} w^{(\mathcal{Q}_x)}(e)\big(x_i(t) - x_j(t)\big)^2\right],\end{aligned}$$

where $w^{(\mathcal{Q}_x)}(e)$ is the weight of edge $e = \{i, j\}$ in the position output graph $\mathcal{Q}_x$. This case corresponds to $C_{\mathcal{Q}_v} = 0$. Similarly, the second-order Laplacian energy of the linear dynamical network (5.1)-(5.2) with respect to the velocity output graph $\mathcal{Q}_v$ is defined by

$$\begin{aligned}\mathbf{H}_{v,\mathcal{Q}_v}^{(2)}(A) &= \lim_{t\to\infty} \mathbf{E}\left[\|v(t)\|^2_{L_{\mathcal{Q}_v}}\right] \\ &= \lim_{t\to\infty} \mathbf{E}\left[\sum_{e=\{i,j\}\in E(\mathcal{Q}_v)} w^{(\mathcal{Q}_v)}(e)\big(v_i(t) - v_j(t)\big)^2\right],\end{aligned}$$

where $w^{(\mathcal{Q}_v)}(e)$ is the weight of edge $e = \{i, j\}$ in the position output graph $\mathcal{Q}_v$. This case corresponds to $C_{\mathcal{Q}_x} = 0$.

From the above definitions, the second-order Laplacian energy of (5.1)-(5.2) can be expressed as

$$\mathbf{H}_Q^{(2)}(A) \;=\; \mathbf{H}_{x,\mathcal{Q}_x}^{(2)}(A) \;+\; \mathbf{H}_{v,\mathcal{Q}_v}^{(2)}(A). \tag{5.7}$$

The specific structure of the state feedback matrices F and G depend on the types of sensor measurements available to form the feedback loop. We refer to [1] for more discussion and details on this. Therefore, we consider two specific class of second-order linear consensus networks. Suppose that $L_{\mathcal{G}}$ is the underlying graph of the consensus network.

In the first case, we assume that $F = -L_{\mathcal{G}}$ and $G = -\beta I_n$ for some design parameter $\beta > 0$. In this scenario, the dynamics of the second-order linear consensus network is governed by

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -L_{\mathcal{G}} & -\beta I_n \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix} \xi. \tag{5.8}$$

For this case, the state matrix A given by (5.6) only depends on the underlying Laplacian matrix $L_{\mathcal{G}}$ and parameter β. Therefore, we simplify our notation by replacing $\mathbf{H}^{(2)}_{\mathcal{Q}_x}(A)$ and $\mathbf{H}^{(2)}_{\mathcal{Q}_v}(A)$ by $\mathbf{H}^{(2)}_{\mathcal{Q}_x}(L_{\mathcal{G}})$ and $\mathbf{H}^{(2)}_{\mathcal{Q}_v}(L_{\mathcal{G}})$, respectively. Thus, the second-order Laplacian energies are given by

$$\mathbf{H}^{(2)}_{x,\mathcal{Q}_x}(L_{\mathcal{G}}) \ = \ \frac{1}{2\beta}\mathbf{Tr}(L_{\mathcal{Q}_x}L^{\dagger}_{\mathcal{G}}) \tag{5.9}$$

and

$$\mathbf{H}^{(2)}_{v,\mathcal{Q}_v}(L_{\mathcal{G}}) \ = \ \frac{1}{2\beta}\mathbf{Tr}(L_{\mathcal{Q}_v}). \tag{5.10}$$

In the second case, we consider linear dynamical networks (5.1)-(5.2) for which $F = -L_{\mathcal{G}}$ and $G = -\beta L_{\mathcal{G}}$ for some design parameter $\beta > 0$. The dynamics of the second-order linear consensus network is given by

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -L_{\mathcal{G}} & -\beta L_{\mathcal{G}} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix} \xi. \tag{5.11}$$

For this case, the second-order Laplacian energies are given by

$$\mathbf{H}^{(2)}_{x,\mathcal{Q}_x}(L_{\mathcal{G}}) \ = \ \frac{1}{2\beta}\mathbf{Tr}\big(L_{\mathcal{Q}_x}(L^{\dagger}_{\mathcal{G}})^2\big), \tag{5.12}$$

and

$$\mathbf{H}^{(2)}_{v,\mathcal{Q}_v}(L_{\mathcal{G}}) \ = \ \frac{1}{2\beta}\mathbf{Tr}(L_{\mathcal{Q}_v}L^{\dagger}_{\mathcal{G}}). \tag{5.13}$$

In the following chapter, we consider two important class of dynamical networks and show how one can compute their second-order Laplacian energies. Then, we obtain several scaling laws on how the Laplacian energies depend on the structure of the underlying graph and scale with the size of the network.

5.1 The Second-Order Laplacian Energy of Some Real-World Dynamical Networks

In this section, we evaluate the second-order Laplacian energy for an interconnected power networks and a controlled group of vehicles in a formation.

5.1.1 Total Power Loss in Synchronous Power Networks

We consider an interconnected network of synchronous generators with underlying graph $\mathcal{G}$ that consists of n buses (nodes) and m transmission lines (edges). A synchronous generator G_i is associated to each node i for $i = 1, \ldots, n$ with inertia constant M_i, damping constant β_i, voltage magnitude V_i. It is assumed that a reduced order model of synchronous generator G_i can be expressed using only two state variables: rotor angle θ_i and angular velocity ω_i. Moreover, we assume that all damping constants are identical, i.e., $\beta = \beta_1 = \ldots = \beta_n$. For each edge $e \in E(\mathcal{G})$, we denote the admittance over e by

$$y_e = g_e - \mathbf{j} b_e, \tag{5.14}$$

where g_e and b_e are the conductance and susceptance of the corresponding transmission line, respectively, and $\mathbf{j} = \sqrt{-1}$. For each edge e, the ratio of its conductance to its susceptance is denoted by

$$\alpha_e = \frac{g_e}{b_e}. \tag{5.15}$$

We define two graphs based on equation (5.14): conductance and susceptance graphs. The conductance graph is denoted and defined by $\mathcal{G}_g = (V(\mathcal{G}), E(\mathcal{G}), w^{(\mathcal{G}_g)})$ where $w^{(\mathcal{G}_g)}(e) = g_e$ for all $e \in E(\mathcal{G})$. Similarly, the susceptance graph is denoted and defined by $\mathcal{G}_b = (V(\mathcal{G}), E(\mathcal{G}), w^{(\mathcal{G}_b)})$ where $w^{(\mathcal{G}_g)}(e) = b_e$ for all $e \in E(\mathcal{G})$. In fact, the conductance and susceptance graphs are two identical copies of $\mathcal{G}$ but with different weight functions.

The governing nonlinear rotor dynamics of the interconnected network of synchronous generators (also known as swing equations) can be linearized around the zero equilibrium operating point of the network in order to obtain

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L_{\mathcal{G}_b} & -\beta I \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \xi, \tag{5.16}$$

where $\theta = \begin{bmatrix} \theta_1 & \dots & \theta_n \end{bmatrix}^T$ and $\omega = \begin{bmatrix} \omega_1 & \dots & \omega_n \end{bmatrix}^T$ are the state vectors of the entire network and ξ is a zero-mean white noise process with identity covariance that models external disturbances [38, 39].

The resistive power loss over each edge $e = \{i, j\}$ can be expressed as the following quantity

$$P_e = g_e |V_i - V_j|^2, \tag{5.17}$$

where g_e is the the conductance of edge e. Therefore, the total resistive power loss in the power network is given by

$$P_{\text{loss}} = \sum_{e=\{i,j\} \in E(\mathcal{G})} P_e. \tag{5.18}$$

If we consider the swing equations of the power network around its equilibrium point, we may apply the small angle approximation and replace the coupling terms $\sin(\theta_i - \theta_j)$ by $\theta_i - \theta_j$ to obtain the following relationship

$$\tilde{P}_{\text{loss}} = \sum_{e=\{i,j\} \in E(\mathcal{G})} g_e |\theta_i - \theta_j|^2. \tag{5.19}$$

According to our definitions in Section 5, the total resistive power loss $\tilde{P}_{\text{loss}}$ given by (5.19) is equal to the second-order Laplacian energy of the linearized swing equations (5.16) with respect to the angle output graph $\mathcal{Q}_\theta = \mathcal{G}_g$, where $\mathcal{G}_b$ is the corresponding conductance graph. Thus, we have

$$\mathbf{H}^{(2)}_{\theta, \mathcal{G}_g}(L_{\mathcal{G}_b}) \;=\; \tilde{P}_{\text{loss}}. \tag{5.20}$$

Theorem 5.1.1. *The second-order Laplacian energy (5.20) of the linearized swing equations (5.16) with*

respect to the angle output graph $\mathcal{G}_g$ is given by

$$\mathbf{H}^{(2)}_{\theta,\mathcal{G}_g}(L_{\mathcal{G}_b}) \;=\; \frac{\bar{\alpha}}{2\beta}(n-1), \tag{5.21}$$

and

$$\bar{\alpha} = \frac{\sum_{e\in E(\mathcal{G})}\nu_e\alpha_e}{\sum_{e\in E(\mathcal{G})}\nu_e} = \frac{\sum_{e\in E(\mathcal{G})}\nu_e\alpha_e}{n-1}. \tag{5.22}$$

in which $\nu_e = r_e b_e$ and r_e and b_e are the line resistance and the susceptance of edge e, respectively. Furthermore, the total resistive power loss is bounded by

$$\frac{\alpha_{\min}}{2\beta}(n-1) \;\le\; \mathbf{H}^{(2)}_{\theta,\mathcal{G}_g}(L_{\mathcal{G}_b}) \;\le\; \frac{\alpha_{\max}}{2\beta}(n-1), \tag{5.23}$$

where

$$\alpha_{\min} = \min_{e\in E(\mathcal{G})}\alpha_e, \quad \alpha_{\max} = \max_{e\in E(\mathcal{G})}\alpha_e. \tag{5.24}$$

Proof. From (3.3) and (3.4), we have

$$\mathbf{H}^{(2)}_{\theta,\mathcal{G}_g}(L_{\mathcal{G}_b}) = \frac{1}{\beta}\mathbf{Tr}(P_2), \tag{5.25}$$

where P_2 is the solution of the following Lyapunov equation

$$L_{\mathcal{G}_b}P_2 + P_2 L_{\mathcal{G}_b} = L_{\mathcal{G}_g}.$$

The trace of P_2 can be written as

$$\begin{aligned}
\mathbf{Tr}(P_2) &= \int_0^\infty \mathbf{Tr}(e^{-L_{\mathcal{G}_b}t}L_{\mathcal{G}_g}e^{-L_{\mathcal{G}_b}t})dt \\
&= \mathbf{Tr}\left(\int_0^\infty e^{-2L_{\mathcal{G}_b}t}dt\, L_{\mathcal{G}_g}\right) \\
&= \frac{1}{2}\mathbf{Tr}(L^\dagger_{\mathcal{G}_b}L_{\mathcal{G}_g}),
\end{aligned} \tag{5.26}$$

where $L^\dagger_{\mathcal{G}_b}$ is the Moore-Penrose generalized inverse of the Laplacian matrix $L_{\mathcal{G}_b}$. According to reference

[32], we have

$$L^{\dagger}_{\mathcal{G}_b} = -\frac{1}{2}\left(R_{\mathcal{G}_b} - \frac{1}{n}(R_{\mathcal{G}_b}J_n + J_nR_{\mathcal{G}_b}) + \frac{1}{n^2}J_nR_{\mathcal{G}_b}J_n\right)$$

where $R_{\mathcal{G}_b}$ is the resistance matrix of the Laplacian matrix $L_{\mathcal{G}_b}$. For a given Laplaican matrix $L_{\mathcal{G}_g}$, it is straightforward to verify that $L_{\mathcal{G}_g}J_n = J_nL_{\mathcal{G}_g} = 0$. Therefore, we get

$$\begin{aligned}
&\mathbf{Tr}(L^{\dagger}_{\mathcal{G}_b}L_{\mathcal{G}_g}) \\
&= -\frac{1}{2}\mathbf{Tr}\left((R_{\mathcal{G}_b} - \frac{1}{n}(R_{\mathcal{G}_b}J_n + J_nR_{\mathcal{G}_b}) + \frac{1}{n^2}J_nR_{\mathcal{G}_b}J_n)L_{\mathcal{G}_g}\right) \\
&= -\frac{1}{2}\mathbf{Tr}\left(R_{\mathcal{G}_b}L_{\mathcal{G}_g}\right) = \sum_{e\in E(\mathcal{G})} r_eb_e\frac{g_e}{b_e} \\
&= \sum_{e\in E(\mathcal{G})} \nu_e\alpha_e, \qquad (5.27)
\end{aligned}$$

where $\nu_e = r_eb_e$. From the result of Theorem 2.0.5, we have that $\sum_{e\in E(\mathcal{G})}\nu_e = n-1$. Using this, we can define the weighted mean of the edge parameters α_e for all $e \in E(\mathcal{G})$ as follows

$$\bar{\alpha} = \frac{\sum_{e\in E(\mathcal{G})}\nu_e\alpha_e}{\sum_{e\in E(\mathcal{G})}\nu_e} = \frac{\sum_{e\in E(\mathcal{G})}\nu_e\alpha_e}{n-1}. \qquad (5.28)$$

From (5.28), (5.27) and (5.25), we conclude that the desired result (5.21). □

According to (5.21), the total resistive power loss depends on the specific structure of the underlying graph of the power network through $\bar{\alpha}$. However, the inequality (5.23) shows that the lower and upper bounds of the total resistive power loss does not depend on the specific topology of the underlying graph of the network. For the special case when $\alpha_1 = \cdots = \alpha_m$, the result of Theorem 5.1.1 reduces to the results reported in reference [38]. Under the assumption that all α_e are identical, the process of calculating the total resistive power loss benefits greatly from the symmetric structure of normal matrices [1].

Definition 5.1.2. *We say that graph $\mathcal{G}$ is an edge-transitive graph if there is an automorphism of $\mathcal{G}$ that maps e_1 to e_2 for all edges $e_1, e_2 \in E(\mathcal{G})$.*

Intuitively Speaking, in an edge-transitive graph all edges have identical local environments, such that an edge can not be distinguished from other edges based on its neighboring nodes and edges. Examples of edge-transitive graphs include biregular, star, cycle and complete graphs [40].

Theorem 5.1.3. *Suppose that the underlying graph of the linearized power network (5.16) is edge-transitive and the internal conductances of all edges are identical. Then, the total resistive power loss is given by*

$$\mathbf{H}^{(2)}_{\theta,\mathcal{G}_g}(L_{\mathcal{G}_b}) \;=\; \frac{\sum_{e\in E(\mathcal{G})}\alpha_e}{2\beta m}(n-1). \tag{5.29}$$

Proof. Similar to the proof of Theorem 5.1.1 we have

$$\mathbf{H}^{(2)}_{\theta,\mathcal{G}_g}(L_{\mathcal{G}_b}) = \frac{1}{\beta}\mathbf{Tr}(P_2) = \frac{1}{2\beta}\mathbf{Tr}(L^{\dagger}_{\mathcal{G}_b}L_{\mathcal{G}_g}). \tag{5.30}$$

Based on (5.27) we obtain

$$\mathbf{Tr}(L^{\dagger}_{\mathcal{G}_b}L_{\mathcal{G}_g}) = \sum_{e\in E(\mathcal{G})}\nu_e\alpha_e. \tag{5.31}$$

Since the underlying graph is edge-transitive and $\sum_{e\in E(\mathcal{G})}\nu_e = n-1$, it follows that $\nu_e = \frac{n-1}{m}$. This completes the proof. □

Theorem 5.1.4. *Suppose that the underlying graph of the linearized power network (5.16) is a tree. Then, the total resistive power loss is given by*

$$\mathbf{H}^{(2)}_{\theta,\mathcal{G}_g}(L_{\mathcal{G}_b}) \;=\; \frac{\sum_{e\in E(\mathcal{G})}\alpha_e}{2\beta m}(n-1). \tag{5.32}$$

Proof. Similar to the proof of Theorem 5.1.1, we have

$$\mathbf{H}^{(2)}_{\theta,\mathcal{G}_g}(L_{\mathcal{G}_b}) = \frac{1}{\beta}\mathbf{Tr}(P_2) = \frac{1}{2\beta}\mathbf{Tr}(L^{\dagger}_{\mathcal{G}_b}L_{\mathcal{G}_g}). \tag{5.33}$$

From (5.27), we get

$$\mathbf{Tr}(L^{\dagger}_{\mathcal{G}_b}L_{\mathcal{G}_g}) = \sum_{e\in E(\mathcal{G})}\nu_e\alpha_e. \tag{5.34}$$

Since the underlying graph is a tree graph and $\sum_{e\in E(\mathcal{G})}\nu_e = n-1$, it follows that $\nu_e = 1$. □

5.1.2 Flock Energy of Controlled Vehicles in a Formation

We consider the formation control problem for n vehicles. It is assumed that each vehicle can be modeled using two state variables: position and velocity. The global objective is for each vehicle to travel at a constant given velocity while maintaining a fixed pre-specified distance from its neighboring vehicles [11]. We assume that the dynamics of the vehicles in the group formation is given by the following second-order linear consensus network model

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -L_{\mathcal{G}} & -L_{\mathcal{G}} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix} \xi, \tag{5.35}$$

where $L_{\mathcal{G}}$ is the Laplacian matrix of the underlying graph of the group formation and $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $v = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}^T$ are the position and velocity state vectors of the entire network.

The $L_{\mathcal{Q}_v}$–semi–norm of the velocity vector is given by

$$\|v\|^2_{L_{\mathcal{Q}_v}} = \sum_{e=\{i,j\}\in E(\mathcal{Q}_v)} w^{(\mathcal{Q}_v)}(e)(v_i - v_j)^2, \tag{5.36}$$

where $w^{(\mathcal{Q}_v)}(e)$ is the weight of each edge $e \in E(\mathcal{Q}_v)$. It is interesting to note that this quantity coincides with the energy of flock (cf. [19]). According to (5.13), the second-order Laplacian energy with respect to the velocity output graph $\mathcal{Q}_v = \mathcal{K}_n$ with weight function $w^{(\mathcal{Q}_v)}(e) = \frac{1}{n}$ for all $e \in E(\mathcal{Q}_v)$ is given by

$$\mathbf{H}^{(2)}_{v,\mathcal{K}_n}(L_{\mathcal{G}}) = \lim_{t\to\infty} \mathbf{E}\left[\|v(t)\|^2_{L_{\mathcal{K}_n}}\right] = \frac{1}{2}\mathbf{Tr}(L^{\dagger}_{\mathcal{G}}) = \sum_{i=2}^{n} \frac{1}{2\lambda_i}.$$

The above interpretation implies that the results of Section 4.1 also hold for the second-order consensus network (5.35) with the second-order Laplacian energy $\mathbf{H}^{(2)}_{v,\mathcal{K}_n}(L_{\mathcal{G}})$.

The second-order Laplacian energy of the linear dynamical network (5.1)-(5.2) with respect to the position output graph $\mathcal{Q}_x = \mathcal{K}_n$ is given by

$$\mathbf{H}^{(2)}_{x,\mathcal{K}_n}(L_{\mathcal{G}}) = \sum_{i=2}^{n} \frac{1}{2\lambda_i^2}, \tag{5.37}$$

which coincides with the second-order network coherence (cf. [11]).

Theorem 5.1.5. *Suppose that the underlying graph of the second-order consensus network* (5.35) *is connected and denoted by $\mathcal{G}$. The second-order Laplacian energy* (5.37) *satisfies the following inequality*

$$\frac{(2m)^4}{2(s_2+s_1)^3} \leq \mathbf{H}^{(2)}_{x,\mathcal{K}_n}(L_{\mathcal{G}}), \tag{5.38}$$

where m is the number of edges, d_i the degree of node i, and $s_\alpha \triangleq \sum_{i=1}^n d_i^\alpha$ for $\alpha = 1, 2$.

Proof. From the Hölder's inequality, it follows that

$$\begin{aligned}
\sum_{i=2}^n \lambda_i &= \sum_{i=2}^n \left(\frac{1}{\lambda_i^{\frac{1}{2}}}\right)\left(\lambda_i^{\frac{3}{2}}\right) \\
&\leq \left(\sum_{i=2}^n \left(\frac{1}{\lambda_i^{\frac{1}{2}}}\right)^4\right)^{\frac{1}{4}} \left(\sum_{i=2}^n \left(\lambda_i^{\frac{3}{2}}\right)^{\frac{4}{3}}\right)^{\frac{3}{4}} \\
&= \left(\sum_{i=2}^n \frac{1}{\lambda_i^2}\right)^{\frac{1}{4}} \left(\sum_{i=2}^n \lambda_i^2\right)^{\frac{3}{4}}.
\end{aligned} \tag{5.39}$$

The inequality (5.39) can be rewritten in the following form

$$\frac{\sum_{i=2}^n \lambda_i}{\left(\sum_{i=2}^n \lambda_i^2\right)^{\frac{3}{4}}} \leq \left(\sum_{i=2}^n \frac{1}{\lambda_i^2}\right)^{\frac{1}{4}}. \tag{5.40}$$

By combining (5.37) and (5.40) and using the facts that $\sum_{i=2}^n \lambda_i = 2m$ and $\|L_{\mathcal{G}}\|_F^2 = \sum_{i=2}^n \lambda_i^2 = s_1 + s_2$, we have

$$\frac{2m}{(s_2+s_1)^{\frac{3}{4}}} \leq \left(2\mathbf{H}^{(2)}_{x,\mathcal{K}_n}(L_{\mathcal{G}})\right)^{\frac{1}{4}}. \tag{5.41}$$

Thus, one can conclude that (5.38) holds. □

Theorem 5.1.6. *Suppose that the underlying graph of the second-order linear consensus networks* (5.35) *is connected and denoted by $\mathcal{G}$. If $\mathcal{P}$ is a connected spanning subgraph of $\mathcal{G}$, then*

$$\mathbf{H}^{(2)}_{x,\mathcal{K}_n}(L_{\mathcal{G}}) \leq \mathbf{H}^{(2)}_{x,\mathcal{K}_n}(L_{\mathcal{P}}), \tag{5.42}$$

$$\mathbf{H}^{(2)}_{v,\mathcal{K}_n}(L_{\mathcal{G}}) \leq \mathbf{H}^{(2)}_{v,\mathcal{K}_n}(L_{\mathcal{P}}), \tag{5.43}$$

and the equalities hold if and only if $\mathcal{G} = \mathcal{P}$. This result also holds for linear consensus network (5.11).

Proof. From our assumptions, we have

$$L_{\mathcal{P}} \leq L_{\mathcal{G}}. \tag{5.44}$$

From the definition, one can verify that

$$(L_{\mathcal{G}}^{\dagger})^2 \leq (L_{\mathcal{P}}^{\dagger})^2. \tag{5.45}$$

By using the fact that the trace of a positive semi-definite matrix is always nonnegative, we get

$$\begin{aligned} \frac{1}{2}\mathbf{Tr}\left((L_{\mathcal{P}}^{\dagger})^2 - (L_{\mathcal{G}}^{\dagger})^2\right) &= \frac{1}{2}\mathbf{Tr}\left((L_{\mathcal{P}}^{\dagger})^2\right) - \frac{1}{2}\mathbf{Tr}\left((L_{\mathcal{G}}^{\dagger})^2\right) \\ &= \mathbf{H}_{x,\mathcal{K}_n}^{(2)}(L_{\mathcal{P}}) - \mathbf{H}_{x,\mathcal{K}_n}^{(2)}(L_{\mathcal{G}}). \\ &\geq 0. \end{aligned}$$

From linearity property of the trace operator, one can conclude that inequality (5.42) holds. □

Theorem 5.1.7. *Suppose that the underlying graph of the second-order consensus network* (5.35) *is connected with at least three nodes and denoted by $\mathcal{G}$. Then, the second-order Laplacian energy* (5.37) *is bounded by*

$$2\mathbf{H}_{x,\mathcal{K}_n}^{(2)}(L_{\mathcal{G}}) \geq (d_1 + d_2 - 1)^{-2} + \sum_{i=2}^{n-1} d_i^{-2} + (d_n + 1)^{-2}.$$

Moreover, the equality holds if and only if $\mathcal{G} = \mathcal{S}_n$ or $\mathcal{G} = \mathcal{K}_3$.

Proof. Let us define a composite function $F(x) = \sum_{i=1}^{n} f(x_i)$, where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}_+^n$ and $f : \mathbb{R}_+ \to \mathbb{R}$ is a convex function. According to reference [23, Sec. 3.C], $F(x)$ is a Schur–convex function. Since $f(\lambda_i) = (2\lambda_i)^{-2}$ is a convex function from $\mathbb{R}_+$ to $\mathbb{R}$, we can conclude that

$$\mathbf{H}_{x,\mathcal{K}_n}^{(2)}(L_{\mathcal{G}}) = \sum_{i=2}^{n} \frac{1}{2\lambda_i^2}$$

is a Schur–convex function. According to the result of [31, Lemma 2], it follows that

$$(\lambda_2, \ldots, \lambda_n) \trianglerighteq (d_1 + d_2 - 1, d_2, \ldots, d_{n-1}, d_n + 1).$$

Using this relationship and the definition of a Schur–convex function, we get

$$\begin{aligned} 2\mathbf{H}^{(2)}_{x,\mathcal{K}_n}(L_{\mathcal{G}}) &= \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_n} \\ &\geq \frac{1}{(d_1 + d_2 - 1)^2} + \frac{1}{d_2^2} + \ldots + \frac{1}{(d_n + 1)^2} \\ &= (d_1 + d_2 - 1)^{-2} + \sum_{i=2}^{n-1} d_i^{-2} + (d_n + 1)^{-2}. \end{aligned}$$

The interested reader is referred to reference [31] for some related discussions and results □

Corollary 5.1.8. *For the class of second-order linear consensus networks* (5.35) *with connected underlying graphs and at least three nodes, the second-order Laplacian energy* (5.37) *satisfies*

$$\mathbf{H}^{(2)}_{x,\mathcal{K}_n}(L_{\mathcal{G}})\, S_{\mathcal{G}}^2 \;\geq\; \frac{n-1}{2}. \tag{5.46}$$

where the sparsity measure $S_{\mathcal{G}}$ is defined by (4.27).

Proof. The proof is a direct application of Theorem 5.1.7 and the definition of $S_{\mathcal{G}}$. □

Chapter 6

Conclusions and Future Directions

We exploit structural properties of the underlying graph of linear dynamical networks in order to characterize their inherently existing fundamental limits on performance with respect to stochastic disturbances. Several performance measures are defined based on weighted $\mathcal{H}_2$–norms of the network. We develop a graph-theoretic framework in order to relate underlying graph characteristics to the Laplacian energy of the network. It is shown that these performance measures depend on various characteristics of the underlying graph of the network such as graph diameter, node degrees, and the number of spanning trees, and several other graph specifications. Specifically we show how these measures scale asymptotically with the network size. More importantly, we establish a connection between sparsity and performance measures of linear dynamical networks, and prove several uncertainty principle like inequalities.

In this thesis, we study the first- and second-order laplacian energies as performance and robustness measures, future research includes extensions to the case of more general class of performance measures that have been used in control theory. Moreover, the generalization of this work to the case of time-varying underlying graph seems more useful for analyzing real-world dynamical networks.

Chapter 7

Appendix

PROOF OF THEOREM 3.1.1

In this appendix, we consider a more general case than the statement of Theorem 3.1.1. Suppose that A is a Hurwitz matrix and its corresponding Lyapunov equation is given by

$$A^T P + PA + Q = 0, \tag{7.1}$$

where Q is a positive semidefinite matrix. For simplicity of our notations, we represent the eigenvalues $\lambda_i(A)$ and $\lambda_i(Q)$ by α_i and β_i, respectively. Furthermore, it is assumed that $\alpha_1 \geq \ldots \geq \alpha_n$ and $\beta_1 \geq \ldots \geq \beta_n$.

Lemma 7.0.9. *The trace of the positive semidefinite solution of the Lyapunov equation (7.1) is bounded from below by*

$$\mathbf{Tr}(P) \;\geq\; -\sum_{i=1}^{n} \frac{\beta_n}{2\mathbf{Re}\{\alpha_i\}}. \tag{7.2}$$

Proof. Every symmetric matrix Q can be decomposed as $Q = UDU^T$ where $UU^T = U^T U = I$ and $D = \mathbf{diag}[\beta_{\mathbf{1}}, \cdots \beta_{\mathbf{n}}]$. Using this fact, we can rewrite (7.1) in the following form

$$\bar{A}^T \bar{P} + \bar{P}\bar{A} + D = 0, \tag{7.3}$$

where $\bar{A} = U^T AU$ and $\bar{P} = U^T PU$. Since A is a Hurwitz matrix, all eigenvalues of $\bar{A}$ have strictly

negative real parts. Therefore, the unique solution of (7.3) can be expressed in the following closed form

$$\bar{P} = \int_0^\infty e^{\bar{A}^T t} D e^{\bar{A} t} dt. \tag{7.4}$$

According to Schur decomposition, there exists a unitary matrix $V \in \mathbb{C}^{n\times n}$ such that $\bar{A} = V(\Gamma + N)V^{\mathrm{H}}$ where $\Gamma = \mathbf{diag}(\alpha_1, \cdots, \alpha_n)$, N is strictly upper triangular, and V^{H} is the conjugate transpose of V. Next, let us consider the integrand of (7.4)

$$\begin{aligned}
\mathbf{Tr}(e^{\bar{A}^T t} D e^{\bar{A} t}) &= \mathbf{Tr}(e^{\bar{A}^{\mathrm{H}} t} D e^{\bar{A} t}) \\
&= \mathbf{Tr}(e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t} V^{\mathrm{H}} D V e^{(\Gamma+N)t} V^{\mathrm{H}} V) \\
&= \mathbf{Tr}(V^{\mathrm{H}} D V e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t} e^{(\Gamma+N)t}) \\
&= \mathbf{Tr}(D V e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t} e^{(\Gamma+N)t} V^{\mathrm{H}}) \\
&\geq \beta_n \mathbf{Tr}(V e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t} e^{(\Gamma+N)t} V^{\mathrm{H}}) \\
&= \beta_n \mathbf{Tr}(e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t} e^{(\Gamma+N)t}).
\end{aligned} \tag{7.5}$$

Furthermore, we have

$$e^{(\Gamma+N)t} = e^{\Gamma t} + M_t, \tag{7.6}$$

$$e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t} = e^{\Gamma^{\mathrm{H}} t} + M_t^{\mathrm{H}}, \tag{7.7}$$

where M_t is an upper-triangular Nilpotent matrix. From (7.6) and (7.7), we have

$$\begin{aligned}
\mathbf{Tr}(e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t} e^{(\Gamma+N)t}) &= \mathbf{Tr}(e^{\Gamma t} e^{\Gamma^{\mathrm{H}} t} + M_t M_t^{\mathrm{H}}) \\
&\geq \mathbf{Tr}(e^{(\Gamma^{\mathrm{H}}+\Gamma)t}).
\end{aligned} \tag{7.8}$$

From (7.5) and (7.8), it follows that

$$\begin{aligned}
\mathbf{Tr}(e^{\bar{A}^T t} D e^{\bar{A} t}) &\geq \beta_n \mathbf{Tr}(V e^{(\Gamma^{\mathrm{H}}+N^{\mathrm{H}})t} e^{(\Gamma+N)t} V^{\mathrm{H}}) \\
&\geq \beta_n \mathbf{Tr}(e^{(\Gamma^{\mathrm{H}}+\Gamma)t}) \\
&= \beta_n \mathbf{Tr}(e^{2\mathbf{Re}\{\Gamma\}t}).
\end{aligned} \tag{7.9}$$

Since $\mathbf{Re}\{\alpha_i\} \neq 0$ for all $i = 1, \ldots, n$, from (7.4) and (7.9) we have

$$\begin{aligned} \mathbf{Tr}(P) &= \mathbf{Tr}(\bar{P}) \\ &= \int_0^\infty \mathbf{Tr}(e^{\bar{A}^T t} D e^{\bar{A} t}) dt \\ &\geq -\sum_{i=1}^{n} \frac{\beta_n}{2\mathbf{Re}\{\alpha_i\}}. \end{aligned} \tag{7.10}$$

In the above inequality, we apply the fact that the trace and sum operators are linear and they can commute with the integral. □

Remark 7.0.10. *We should emphasize that if $Q = qI_{n\times n}$ for $q > 0$, the lower bound in Theorem 7.0.9 is tighter than the lower bounds reported in reference papers [41–44].*

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