2013

Degree Sequences of Edge-Colored Graphs in Specified Families and Related Problems

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by

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A Dissertation
Presented to the Graduate Committee
of Lehigh University
in Candidacy for the Degree of
Doctor of Philosophy
in
Mathematics

Lehigh University
September 2013

Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Date

Garth Isaak, Dissertation Director, Chair

Accepted Date

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Acknowledgements

First and foremost, I would like to thank my advisor Garth Isaak, a deeply honorable, patient, and clever man, without whom my upcoming life as a mathematician would not be possible. While many Lehigh professors have impacted my life, I wish specifically to give thanks to Wei-Min Huang for always finding the resources to support my professional development, and to Colton Magnant and Vincent Coll for providing me with extracurricular mathematical opportunities.

I am indescribably grateful to my parents for their unwavering support in all aspects of my life. They have sacrificed much for one simple goal, the happiness of their children, and I am who I am today because of them. (Thanks for the Starbucks gift cards, too, Mom and Dad!) I am also deeply thankful for my siblings and their spouses, who have provided me with a constant stream of love and encouragement and who have given me my favorite title, that is, 'aunt.' In specific, I must recognize my sister, whose greatest work of art is knitting our family together, a craft she most certainly learned from my loving mother. Finally, I would like to thank my nieces, nephews, and godchildren, whose hugs are therapy for the soul.

I have been granted three confidants in life who are my inspiration. I am thankful seventy times seven times over for Tom, the keeper of my memories and the caretaker of my soul. My life would be empty without Sarah, my life coach and my best friend, who fills my days with constant laughter and sends me flowers whenever there are tears. As for my nights, I would have no one to philosophize with at midnight if it were not for Spyro, who listens faithfully and never judges. The size of his heart humbles me.

I cannot end this list of thanks without thanking two angels in my life. I am deeply grateful to Mary Ann, whose thoughtfulness and love has no bounds. My greatest joy about staying in Bethlehem is remaining close to her. I would also like to thank Peachy for keeping her promise to let me always remain a part of her family and for reminding me how deep the inner strength of a woman can be.

Last, but certainly not least, I am thankful to the Lord for giving me all of the blessings that I have listed here as well as so very many more.

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Abstract

Movement has been made in recent times to generalize the study of degree sequences to k-edge-colored graphs and doing so requires the notion of a *degree vector*. The degree vector of a vertex v in a k-edge-colored graph is a $(k \times 1)$ column vector in which entry i indicates the number of edges of color i incident to v. Consider the following question which we refer to as the *k-Edge-Coloring Problem*. Given a set of column vectors $\mathcal{C}$ and a graph family $\mathcal{F}$, when does there exist some k-edge-colored graph in $\mathcal{F}$ whose set of degree vectors is $\mathcal{C}$? This question is NP-Complete in general but certain graph families yield tractable results. In this document, I present results on the k-Edge-Coloring Problem and the related Factor Problem for the following families of interest: unicyclic graphs , disjoint unions of paths (DUPs), disjoint union of cycles (DUCs), grids, and 2-trees. Specifically, in Chapter 1, I characterize the degree vector sequences of k-edge-colored unicylcic graphs, and in Chapter 2, I characterize degree sequences of factors of fixed DUPs, fixed DUCs, and fixed graphs with maximum degree at most 2. In Chapter 3, I characterize degree vector sequences of 2-edge-colored fixed DUPs and fixed DUCs, and in doing so, I show that one restricted case for each is NP-hard. Finally, I characterize degree sequences of grid factors in a subset of cases in Chapter 4 and degree sequences of partial 2-trees in Chapter 5.

Introduction

The concept of characterizing the degree sequences of graphs is natural amongst graph theorists, and at its core, the impetus of all our results stems from this simple concept. Movement has been made in recent times to generalize the study of degree sequences to edge-colored graphs, a notion pertinent to the field of Discrete Tomography. In an attempt to progress the study of degree sequences of edge-colored graphs, we introduce the notion of the *k-Edge-Coloring Problem* and its sibling, the *Factor Problem*. We now give the background required to define these problems and then we explain the connection between them. Finally, at the close of this introductory section, we discuss what contributions others have made to the k-Edge-Coloring Problem and we summarize our contributions as well.

We begin with a few simple definitions. A *graph* consists of a set of vertices and a set of edges where each edge represents a pair of vertices. We are concerned only with simple graphs, meaning we do not consider graphs with multiple edges or loops. The *size* of a graph is the number of edges and the *order* of a graph is the number of vertices. A *graph family* is a set of graphs. Typically, the graphs in a graph family share a common descriptive property. Given a subgraph H of G, the notation $G - H$ refers to all vertices and edges of G which do not appear in H.

The *degree* of a vertex in a graph is the number of edges incident to the vertex. In a graph G with order n, the *degree sequence* of a graph is a list of n non-negative integers which consists of the degrees of the vertices in G. A list of integers D is *realizable* if there is some graph whose degree sequence is D. The use of the word "sequence" in "degree sequence" is a bit of a misnomer since the order of the entries does not truly matter, and so some places in literature refer to a "degree sequence" as a "degree list." Nonetheless, we use the traditional term "degree sequence." Also, we adhere to convention of writing degree sequences in descending order. If the reader requires any further details about these definitions or others, we refer the reader to [15].

Asking whether or not a list of integers is realizable is a well-studied question. To generalize this question to graphs whose edges are colored with 1 of k colors, we now define a *k-edge-colored graph*, a *degree vector*, and a *degree vector sequence*.

Definition 0.0.1. *For a positive integer k, a **k-edge-colored graph** is a graph whose edges are colored (not necessarily properly) with colors from the set $\{1, 2, \ldots, k\}$. A k-edge-colored graph has a **k-coloring**.*

Definition 0.0.2. *The **degree vector** of vertex v in a k-edge-colored graph is the $(k \times 1)$ column vector where the entry in row i is the degree of color i at v. Given a k-edge-colored graph on n vertices, the **degree vector sequence** of the graph is the collection of n degree vectors of the graph.*

Definition 0.0.3. *A k-edge-colored graph G **realizes** a set of vectors $\mathcal{D}$ if the degree vector sequence of G is $\mathcal{D}$. In this case, we say that $\mathcal{D}$ is **realizable**, and that $\mathcal{D}$ **is realized by** G.*

Instead of saying that a k-edge-colored graph realizes $\mathcal{D}$, we sometimes say that a k-coloring realizes $\mathcal{D}$, and by this we mean that the k-edge-colored graph with the given k-coloring realizes $\mathcal{D}$. Given a k-edge-colored graph G, we define a *color i degree sequence* of G. Note that the color i degree sequence is precisely the list of entries in row i of the degree vector sequence of G.

Definition 0.0.4. *For $1 \leq i \leq k$, the **color** i **degree sequence** of a k-edge-colored graph is the degree sequence of the color i subgraph.*

Given a collection of $(k \times 1)$ column vectors $\mathcal{D}$, we wish to determine whether or not there exists some k-edge-colored graph from a specified family which realizes $\mathcal{D}$. This is precisely the k-Edge-Coloring Problem. When $k = 1$, this problem reduces to asking whether or not a list of integers is realizable by some graph in a certain family.

Problem 0.0.1. ***(The** k**-Edge-Coloring Problem)*** *Given a graph family* $\mathcal{F}$*, determine criteria characterizing when a collection* $\mathcal{D}$ *of* $(k \times 1)$ *column vectors is realized by some* k*-edge-colored graph in* $\mathcal{F}$*.*

We now explain the sibling of the k-Edge-Coloring Problem, that is, the *Factor Problem.* A *factor* of a graph G is a spanning subgraph of G. G is referred to as the *host graph* of the factor. Factors have been well-studied in the context of f-factors. Given a graph G and a function $f : V(G) \to \mathbb{Z}^+$, an f*-factor* of G is a factor in which vertex v has degree $f(v)$. Tutte's f-factor Theorem indicates when an arbitrary graph has an f-factor. As another example, a theorem of Ore indicates when a bipartite graph has an f-factor.

Instead of specifying a function f, we instead specify the desired degree sequence of a factor and this gives impetus for the following definition.

Definition 0.0.5. *Let* G *be a graph with max degree* Δ*. A* $[d_0, d_1, \ldots, d_\Delta]$***-factor*** *of* G *is a factor with* d_i *vertices of degree* i *for* $0 \leq i \leq \Delta$*.*

Figure 2 shows an example of a $[d_0, d_1, d_2]$-factor of a path. While an f-factor indicates the degree of a specified vertex in the factor, a $[d_0, d_1, \ldots, d_\Delta]$-factor does not. We wish to determine when $[d_0, d_1, \ldots, d_\Delta]$-factors are present in graphs within a specified family and we define this problem as the Factor Problem.

Problem 0.0.2. ***(The Factor Problem)*** *Let* $\mathcal{F}$ *be a graph family. Given non-negative integers* $d_0, d_1, \ldots, d_r$*, determine criteria describing when some graph from* $\mathcal{F}$ *has a* $[d_0, d_1, \ldots, d_r]$*-factor.*

We often consider the Factor Problem before the k-Edge-Coloring Problem. This is because the Factor Problem gives insight into the 2-Edge-Coloring Problem, which in turn gives insight into the k-Edge-Coloring Problem for general k. This insight stems from the natural relationship between a 2-coloring of a graph and a factor of a graph. After deleting all color 2 edges from a given 2-coloring of a graph G, the remaining subgraph can be viewed as a factor H of G. Conversely, given any factor H of G, we can replace all edges in H with color 1 edges and all edges in $G - H$

with color 2 edges in order to obtain a 2-coloring of G. Figure 1 exemplifies this idea.

(a) A 2-coloring of G (b) A Factor H of G

Figure 1: Obtaining factors from 2-colorings

There is also a relationship between the degree vector sequence $\mathcal{D}$ of a 2-coloring of a graph G and the degree sequence of the factor H obtained by deleting all color 2 edges from G. The list of initial entries of each vector in $\mathcal{D}$ is both the degree sequence of the factor H and the color 1 degree sequence of the 2-coloring of G. For example, the degree sequence of the factor H in Figure 1(b) is $< 3, 2, 2, 2, 2, 1 >$. This is precisely the color 1 degree sequence of the degree vector sequence of G in Figure 1(a), which is

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Notice that the column sums of these vectors yield $< 4, 4, 3, 3, 2, 2 >$ which is precisely the degree sequence of G.

Therefore, if a k-edge-colored graph realizes a vector sequence, then the graph has a factor whose degree sequence is the color 1 degree sequence of the given vector sequence. However, given a factor of a graph G with degree sequence consisting of d_i entries of the number i, not every set of vectors whose first row entries consist of d_i entries of the number i is realizable by a 2-coloring of G. We give an example of this now.

Consider the path P and the $[1, 2, 3]$-factor of P shown in Figure 2. The $[1, 2, 3]$-factor of P clearly has degree sequence $< 2, 2, 2, 1, 1, 0 >$. The column sums of the degree vectors of any 2-coloring of P must be the degree sequence of P and so must

(a) A Path P (b) A $[1, 2, 3]$-factor of P

Figure 2: Example of a $[d_0, d_1, d_2]$-factor of a path

be $< 2, 2, 2, 2, 1, 1 >$. Below are the only two vector sequences with non-negative integer entries whose first row entries are $< 2, 2, 2, 1, 1, 0 >$ and whose column sums are $< 2, 2, 2, 2, 1, 1 >$.

$$2, 2, 2, 1, 1, 0 \left\{ \begin{array}{l} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad X \\ \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \checkmark \end{array} \right.$$

We cannot color P with 2 colors so that it has a degree vector sequence which matches the first vector sequence in the list above. However, we can do so for the second vector sequence. In general, if a graph G has a $[d_0, d_1, \ldots, d_\Delta]$-factor and if we let $\mathcal{S}$ be the set of vector sequences whose first row consists of exactly d_i entries of the number i and whose column sums match the degree sequence of G, then some non-empty but possibly proper subset of the vector sequences in $\mathcal{S}$ are realizable as a 2-coloring of G.

A *regular* graph is a graph in which every vertex has the same degree. Note that if G is regular, then the previously discussed set $\mathcal{S}$ consists of exactly one vector sequence. So in order to show that a vector sequence $\mathcal{D}$ is realizable by a 2-coloring of a regular graph G, it suffices to find any factor of G whose degree sequence consists of the color 1 degree sequence of $\mathcal{D}$ and vice versa. In this sense, the 2-Edge-Coloring Problem and Factor Problem are equivalent for regular graphs. Claim 0.0.6 and Claim 0.0.7 formalize these arguments.

Claim 0.0.6. *Let G be a graph with maximum degree Δ. Let $\mathcal{S}$ be the set of (2×1) vector sequences where row j consists of d_i entries of the integer i and* *whose column sums is the degree sequence of G. Then G has $[d_0, d_1, \ldots, d_\Delta]$-factor if and only if some non-empty (but possibly proper) subset of the vector sequences in $\mathcal{S}$ are realizable as a 2-coloring of G. If G is regular, $\mathcal{S}$ consists of exactly 1 vector sequence.*

Proof. Without loss of generality, assume row j is row 1. If G has $[d_0, d_1, \ldots, d_\Delta]$-factor H, color the edges in H with color 1 and the edges in $G-H$ with color 2. The degree vector sequence of the resulting 2-edge-colored graph is in $\mathcal{S}$. If any vector sequence in $\mathcal{S}$ is realizable by a 2-coloring of G, then the color 1 subgraph of this 2-coloring of G is a $[d_0, d_1, \ldots, d_\Delta]$-factor. Finally, if G is a r-regular, then row 2 is unique and is r minus the value in row 1. Then $\mathcal{S}$ consists of 1 vector sequence. $\square$

Claim 0.0.7. *Let G be a graph with maximum degree Δ. Let $\mathcal{D}$ be a sequence of (2×1) column vectors with entries in $[0, \ldots, \Delta]$ where the columns sums of the vectors in $\mathcal{D}$ is the degree sequence of G. For a fixed row $j \in \{1, 2\}$, let d_i where $0 \leq i \leq \Delta$ be the number of vectors in $\mathcal{D}$ where row j contains the integer i. If there exists a 2-coloring of G with degree vector sequence $\mathcal{D}$, then G has a $[d_0, d_1, \ldots, d_\Delta]$-factor. The converse is also true if G is a regular graph.*

Proof. Let $\mathcal{S}$ be the set of (2×1) vector sequences where row 1 consists of d_i entries of the integer i and whose column sums is the degree sequence of G. Then $\mathcal{D} \in \mathcal{S}$. $\square$

0.1 Summary of Our Results and Others'

Dürr, Guiñez, and Matamala while working with topics in Discrete Tomography considered the following problem. Given a collection of $(k \times 1)$ column vectors $\mathcal{D}$, determine whether or not there exists a k-coloring of a complete bipartite graph which realizes $\mathcal{D}$. Dürr, et. al., recently showed that this problem is NP-Complete for $k \geq 3$ [4]. As a consequence of this result, in the same paper they show that determining whether or not there exists *some* edge-colored graph which realizes $\mathcal{D}$ is also NP-Complete for $k \geq 2$. Their results exclude $k = 1$ because when $k = 1$, the problem reduces to asking whether or not a list of integers is realizable. We

Family $\mathcal{F}$ and Value of k	**What is known**
Arbitrary Graphs $k \geq 2$	NP-Complete [4]
Arbitrary Graphs $k = 1$	Solutions Exist, Ch. 3 of [12]
Bipartite Graphs $k \geq 2$	NP-Complete [4]
Bipartite Graphs $k = 1$	Gale Ryser Theorem [12] (if partite sets are given)
Complete Graph of Appropriate Size $k \geq 3$	NP-Complete [4]
Complete Graph of Appropriate Size $k = 2$	Reduces to Arbitrary Graphs $k = 1$
Complete Bipartite Graph of Appropriate Size $k \geq 3$	NP-Complete [4]
Complete Bipartite Graph of Appropriate Size $k = 2$	Reduces to Bipartite Graphs $k = 1$
Forests	Solution exists for all k [5]
Graphs With Max Degree 3	Solution exists for all k [10]

Table 1: Known results for the k-Edge-Coloring Problem

previously mentioned that this question is well-studied and many theorems exist which yield answers to this question. Thus, these authors showed that the k-Edge-Coloring Problem is NP-Complete for a complete bipartite graph when $k \geq 3$ and for the graph family of arbitrary graphs when $k \geq 2$.

On the other hand, there are for which the k-Edge-Coloring Problem yields tractable results. For example, Isaak and Carroll found conditions describing when $\mathcal{D}$ is realized by a k-edge-colored forest for $k \geq 1$ [5]. Also, Alpert et al. provide a different proof of the forest result and also give conditions characterizing when some graph with max degree 3 can be colored with $k \geq 1$ colors so as to realize $\mathcal{D}$ [10]. Table 1 summarizes what is known about the k-Edge-Coloring Problem for these and additional graph families. Note that some families in this table are families that consist of a single graph and others are infinite families.

In an effort to find results for specified families as our predecessors have done, our main focus has been to solve the k-Edge-Coloring Problem or Factor Problem for a set of 'plausible' graph families. By 'plausible', we mean families whose degree sequences are characterized, as such families yield an immediate answer to the k-Edge-Coloring Problem when $k = 1$. Characterizations exist for degree sequences of the following families and so these families are all plausible: disjoint unions of paths and cycles, partial 2-trees [13], and unicyclic graphs [2], cacti graphs [14], Halin graphs [3], and edge-maximal outerplanar graphs [11]. We now describe the results we have obtained for disjoint unions of paths and cycles, partial 2-trees, and unicyclic graphs. We also describe how we have organized these results in the upcoming chapters. We leave the other plausible graph families as avenues for future work.

In Chapter 1, we generalize the forest results of Isaak and Carroll in [5]. We characterize when a sequence of vectors is the degree vector sequence of a k-edge-colored graph with at most one cycle for $k \geq 1$. Thus, we answer the k-Edge-Coloring Problem for what we refer to as *at-most-unicyclic graphs.*

In Chapter 2, we concentrate on the Factor Problem for graphs of max degree 2. Such graphs are a union of paths and cycles. It is straightforward to give conditions for when some disjoint union of paths (DUP) exists with a given $[d_0, d_1, d_2]$-factor. So after doing so, we then consider a deeper question, specifically, we characterize *which* DUPs have a given $[d_0, d_1, d_2]$-factor. In doing so, we answer the Factor Problem for a *fixed* DUP with specified path sizes. We then give similar results for a disjoint union of cycles (DUC). When considering factors of DUCs, we show that a case of the Factor Problem for a fixed DUC is NP-Complete and reduces to the Subset Sum Problem. Finally, we combine the results for DUPs and DUCs to answer the Factor Problem in general for fixed graphs of max degree 2.

In Chapter 3, we concentrate on the 2-Edge-Coloring Problem for graphs of max degree 2. Since a DUP is a forest, the forest results yield conditions for when a vector sequence $\mathcal{D}$ is a degree vector sequence of a 2-edge-colored DUP. Similar to Chapter 2, we thus consider a deeper question. We characterize *which* DUPs can be 2-colored so that the resulting degree vector suquence is $\mathcal{D}$. In doing so, we answer the 2-Edge-Coloring Problem for a *fixed* DUP and we again explain a case of this problem that is NP-Complete and which reduces to the Subset Sum Problem. As expected, in Chapter 3 we use the insight given by the Factor Problem for DUPs in Chapter 2. Finally, we dissect the details of these proofs to give an algorithm for how to color a fixed DUP with 2 colors so that its degree vector sequence is a specified one.

In Chapter 3, we also characterize the degree vector sequences of 2-Edge-Colored DUCs. Since a fixed DUC is a regular graph, by Claim 0.0.7, the 2-Edge-Coloring Problem answers the Factor Problem for a fixed DUC and vice versa. We have chosen to give the details of the proofs in Chapter 3 and to state the result in Chapter 2 as a corollary. When doing this, we make it very clear that we create no circular references. We remark that the terminology associated with colorings is more clear and more natural to use than that associated with factors. This explains why we chose to keep the proofs in Chapter 3 instead of Chapter 2.

Finally, we close Chapter 3 by exemplifying why it is difficult to answer the k-Edge-Coloring Problem for DUPs and DUCs when $k \geq 3$. We remark that in Chapter 3, we do not combine the separate results concerning 2-colorings of DUPs and DUCs in order to answer the 2-Edge-Coloring Problem in general for fixed graphs of max degree 2. Doing so requires a case-by-case analysis that is less interesting as other work we chose to consider.

In Chapter 4, we discuss the Factor Problem for grids. Since the maximum degree of a grid is 4, the factors we seek are $[d_0, d_1, d_2, d_3, d_4]$-factors. DUP results are helpful in this section because a grid is the cartesian product of paths. In fact, when $d_3 = d_4 = 0$, we use DUP results to list the few cases where the desired factor does not exist in a grid. When $d_3 + d_4 > 0$, there are 4 challenges that arise: (a) when $d_1 + d_2$ is 'too small', that is, when $d_1 + d_2 < 4$, (b) when d_4 is 'too large', (c) when d_1 or d_2 is 0, and (d) when $d_1 + d_3 < 4$. When $d_1 + d_2 < 4$, the list of cases when a grid has the desired factor is short and we identify them. When $d_1 + d_2 = 4$, the shape of the possible factors is very restrictive and we conjecture what structure such factors have. Because this case is so specific, we leave this conjecture for future work and we present results when $d_1 + d_2 \geq 5$.

In the case when $d_1 + d_2 \geq 5$ and $d_4 > 0$ our greatest challenge is difficulty (b) above. When d_4 is 'large,' a certain number of degree 1, 2, and 3 vertices must exist in the factor. This is related to the fact that no vertex on the border of a grid or in a factor of a grid can be degree 4. We introduce the variable $B(n, m, d_4)$ to capture the minimum number of degree 1,2, or 3 vertices that are necessary in any factor of a $n \times m$ grid. Thus, $d_1 + d_2 + d_3 \geq B(n, m, d_4)$ in any $[d_0, d_1, d_2, d_3, d_4]$-factor of an $n \times m$ grid. We discuss the challenges involved in giving a closed formula for $B(n, m, d_4)$. We then present a lower bound for $B(n, m, d_4)$. Specifically, we show that $B(n, m, d_4) \geq \max\{2n_4 + 2, 2m_4 + 2\}$ where n_4 and m_4 are the least number of rows and columns, respectively, which must contain a degree 4 vertex in any factor of G. Hence, when $d_1 + d_2 + d_3 < \max\{2n_4 + 2, 2m_4 + 2\}$, a $[d_0, d_1, d_2, d_3, d_4]$-factor of an $n \times m$ grid G is impossible. We then show that when $d_1 + d_2 + d_3 \geq \min\{2n_4 + 2m - 1, 2m_4 + 2n - 1\}$ and when a few fairly weak additional assumptions are also made, G has the desired factor. When $\max\{2n_4+2, 2m_4+2\} \leq d_1+d_2+d_3 < \min\{2n_4 + 2m - 1, 2m_4 + 2n - 1\}$, we know of cases when G does and does not have a $[d_0, d_1, d_2, d_3, d_4]$-factor, and this range of d_4 values is left for future work.

Finally, in the case when $d_1 + d_2 \geq 5$ and when $d_3 > 0$ but $d_4 = 0$ we give a list of pathological cases and we show that no grid has a factor on this list. We then conjecture that the desired factor exists *except* for this list of cases. See Table 4.1 for a summary of our results.

In Chapter 5, we characterize the degree sequence of *partial 2-trees*, that is, factors of 2-trees. Determining this characterization is equivalent to answering the Factor Problem for 2-trees and the k-Edge-Coloring Problem for partial 2-trees when $k = 1$. Recall that we listed 2-trees as plausible graph families of interest for the k-Edge-Coloring Problem because the degree sequences of 2-trees are known [13]. Note that the color i subgraph of a k-edge-colored 2-tree is a partial 2-tree by definition. Hence, a necessary condition for a k-edge-colored 2-tree to have a degree vector sequence $\mathcal{D}$ is that the entries in row i of the vectors in $\mathcal{D}$ is the degree sequence of a partial 2-tree. Thus, the characterization of partial 2-tree degree sequences is crucial to the k-Edge-Coloring Problem for 2-trees, and hence, we have concentrated our efforts on this characterization.

We remark that work has been done as early as the 1980s to characterize the *degree sets*, that is, the set of vertex degrees, of k-trees. See [7] and [6]. Work also has been done to characterize the degree sequences of k-trees, and only in 2008, has this characterization been completed for the case $k = 2$ [13]. To date, the characterization is not complete for any $k \geq 3$. Prior to our results, no work had been done to characterize degree sequences of partial k-trees for any $k \geq 2$.

Chapter 1

The k-Edge-Coloring Problem for Unicyclic Graphs

In Theorem 1.0.5, Caroll and Isaak characterize when a sequence with $(k \times 1)$ with non-negative integer entries is the degree vector sequence of some k-edge-colored forest. In this chapter, we generalize their results to graphs which have at most one cycle.

We make use of the following helpful facts about forests. Claims 1.0.1 and 1.0.2 are well-known facts about forests. We omit the proof for Claim 1.0.1.

Claim 1.0.1. *The integer sequence $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$ is realizable as a forest F if and only if this sequence has even sum at most $2m-2$, where m is the number of nonzero d_i. If the sequence has sum less than $2m-2$, F is disconnected. If the sequence has sum exactly $2m-2$, then F is a tree and thus connected.*

Claim 1.0.2. *Let G be a forest with degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$. For any $d_i > 0$ and $d_j > 0$, $i \neq j$, where d_i and d_j are not both 1, there exists a realization in which a vertex of degree d_i is adjacent to a vertex of degree d_j.*

Proof. Assume $d_i \geq d_j$. Then $d_i > 1$. Let v_i and v_j be vertices of degree d_i and d_j, respectively, in some realization G of D. If v_i and v_j are adjacent, then we are done, so assume they are not. If v_i and v_j are in distinct trees in the forest, then let

x, y be neighbors of v_i, v_j, respectively. Delete v_ix and v_jy and add the edges v_iv_j and xy. (This process is called *2-switching* v_ix and v_jy in later chapters.) Doing so preserves degrees and does not create any cycles and so the result is the desired realization. If v_i and v_j are in the same tree T, then since $d_i > 1$ we know that v_i has at least two neighbors, x and z. Consider the unique path P from v_j to v_i. Both x and z cannot be on P or T would contain a cycle, so without loss of generality, we may assume x is not on P. Let y be the neighbor of v_j on P. Delete the edges v_ix and v_jy. This cuts T into three components, one with v_i, a second with v_j, and a third with x. Add the edges v_iv_j and xy to connect these components, which does not introduce any cycles. This results in the desired realization. $\square$

Given a degree vector sequence $\mathcal{D}$ of an edge-colored graph G, the entries in row i correspond to the degree sequence of the subgraph of G of color i. So when we refer to a *subset of colors* of $\mathcal{D}$, we are actually referring to a subset of row indices of $\mathcal{D}$. We now define the terms *sum degree sequence* and *support* of a subset of a colors.

Definition 1.0.3. *Given a subset of colors $\mathcal{C}$ of a degree vector sequence of an edge-colored graph, the* ***sum degree sequence*** *of $\mathcal{C}$, denoted $D_{\mathcal{C}}$, is the sequence of column sums of the rows in $\mathcal{C}$.*

Definition 1.0.4. *Given a subset of colors $\mathcal{C}$ of a degree vector sequence of an edge-colored graph, the* ***support*** *of $\mathcal{C}$ is the set of non-zero elements in the sum degree sequence $D_{\mathcal{C}}$. We let $m_{\mathcal{C}}$ refer to the size of the support, that is, $m_{\mathcal{C}}$ is the number of non-zero elements in $D_{\mathcal{C}}$.*

The main result of this chapter is Theorem 1.0.12, which relies on the following result by Carrol and Isaak.

Theorem 1.0.5 ([5])**.** *Let $\mathcal{D}$ be a sequence of $(k \times 1)$ column vectors with non-negative entries and non-zero columns sums. Then $\mathcal{D}$ is the degree vector sequence of a k-edge-colored forest if and only if for every subset $\mathcal{C}$ of the colors of $\mathcal{D}$, that is, $\mathcal{C} \subseteq \{1, 2, \ldots, k\}$, the sum degree sequence $D_{\mathcal{C}}$ is realizable as a forest.*

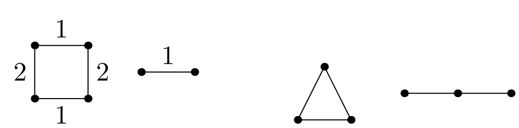
We introduce more definitions now.

Definition 1.0.6. *A* ***unicyclic*** *graph is a graph which is connected and has exactly one cycle. A* ***disconnected unicyclic*** *graph is a disconnected graph with exactly one cycle. A graph is* ***at-most-unicyclic*** *if the graph contains zero cycles or one cycle.*

Note that any subgraph of a unicyclic graph must be an at-most-unicyclic graph. Also, note that an at-most-unicyclic graph may or may not be connected and so is either a unicyclic graph, a disconnected unicyclic graph, or a forest. It is interesting to note that the degree sequences of disconnected unicyclic graphs are almost exactly those of forests, as shown by the claim below. Claim 1.0.7 shows a structural result as well, that is, that an integer sequence which is realizable as a disconnected unicylic graph can be realized by a such a graph where the unique cycle is a triangle on the vertices of largest degree. This structure is not guaranteed in a k-edge-colored disconnected unicylic graph. We demonstrate this now. Let $\mathcal{D}$ be the vector sequence below.

$$\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

Note that $\mathcal{D}$ is the degree vector sequence of the 2-edge-colored disconnected unicyclic graph G shown in Figure 1.1 (a). Figure 1.1(b) shows the unique disconnected unicyclic graph with a triangle whose degree sequence is the sum degree sequence of $\mathcal{D}$, which is $< 2, 2, 2, 2, 1, 1 >$. The reader can verify that there is no way to color the graph in Figure 1.1(b) so that its degree vector sequence is $\mathcal{D}$.



(a) **(b)**

Figure 1.1: Disconnected unicyclic graphs

Claim 1.0.7. *Given a sequence D of $n \geq 5$ positive integers, there exists a disconnected unicyclic graph with degree sequence D if and only if the sum of D is even and at most $2n - 2$ and at least 3 integers in D are greater than 1. Furthermore, if these conditions hold and d_1, d_2, d_3 are the three greatest integers in D, then there exists a disconnected unicylic graph with degree sequence D where the unique cycle is a triangle on vertices with degrees d_1, d_2, d_3.*

Proof. ($\Rightarrow$) Consider a disconnected unicyclic graph G with n vertices of positive degree and with degree sequence D. There must be a component with a cycle and a second component with at least one edge, and thus $n \geq 5$. Because G has a cycle, at least 3 elements in D are greater than 1. Add edges between components in G to obtain a (connected) unicyclic graph which has degree sum $2n$ by Theorem 1.0.8. Then the degree sum of the original graph G must have degree sequence with even sum less than $2n$.

($\Leftarrow$) *Proof 1:* Let d_1, d_2, d_3 be the largest three integers in D, all of which are greater than 1. Form the sequence D' with $n - 1$ integers by removing d_1 and d_2 from D and adding the positive integer $d_1 + d_2 - 2$. Note that $d_1 + d_2 - 2 \geq 2$. Then D' has even sum at most $2(n-1)-2$. By Claim 1.0.1 D' is realizable as a forest. Moreover, by Claim 1.0.2 D' is realizable as a forest G with a vertex v of degree $d_1 + d_2 - 2$ adjacent to a vertex w of degree d_3. Note that since $d_3 > 1$, w must have at least one neighbor $s \neq v$. Add a vertex u adjacent to v and w. Remove edges between v and $d_2 - 2$ of its neighbors that are not v or w. Add an edge between u and each of these $d_2 - 2$ neighbors. The resulting graph G' has a unique cycle on vertices v, u, w which have degrees $d_1 + 1, d_2, d_3 + 1$, respectively. Because $d_1 + 1 \geq 3$ by hypothesis, v has some neighbor t that is not on the triangle. Remove the edges vt and ws. Doing so disconnects the graph into at least 3 components since uvw is the unique cycle. Add the edge st. The resulting graph is a disconnected unicyclic graph where the unique cycle is a triangle on vertices v, u, w with degrees d_1, d_2, d_3, respectively.

Proof 2: Optionally, we can prove the claim by induction on n. The base case is $n = 5$. The only sequence which fits the conditions is $D =< 2, 2, 2, 1, 1 >$ which is realizable as a triangle on the three largest integers in D and a disjoint edge. Now assume the claim is true for a sequence D with $n-1 \geq 5$ positive integers. Consider a sequence D with $n \geq 6$ integers. The smallest element in D must be a 1 or otherwise D has sum at least $2n$. Obtain a new sequence D' with $n-1$ vertices by removing 1 from D and decreasing the largest degree $d_1 > 1$ by 1. The sum of D' is even and at most $2(n-1)-2$. If at least 3 elements in D' are not greater than 1, then D is the sequence $< 2, 2, 2, 1 \ldots, 1 >$. Realize this sequence with a triangle and a set of disjoint edges. Otherwise, at least 3 elements in D' are greater than 1. Then by induction, the claim holds and there exists a disconnected unicyclic graph G' with degree sequence D' where the unique cycle is a triangle on the vertices of largest degree in D'. Add a vertex adjacent to a vertex v of degree $d_1 - 1$ to obtain a unicyclic graph G with degree sequence D. If v is not on the triangle, then there are three degrees in D' larger than $d_1 - 1$ and so the four largest degrees in the original sequence D all have value d_1. Hence, the triangle in G is indeed a triangle on the vertices of largest degree in D and the claim holds.

□

Harary and Boesch in 1978 characterized the degree sequences of unicylic graphs by proving the two equivalent theorems below.

Theorem 1.0.8 ([8]). *Given a sequence D of n positive integers, there exists a unicyclic graph with degree sequence D if and only if the sum of D is $2n$ and D is graphic.*

Theorem 1.0.9 ([8]). *Given a sequence D of n positive integers, there exists a unicyclic graph with degree sequence D if and only if the sum of D is $2n$ and at least 3 elements in D are greater than 1.*

The following two claims follow almost immediately from Theorem 1.0.8 and Theorem 1.0.9.

Claim 1.0.10. *A sequence D of n positive integers is the degree sequence of an at-most-unicyclic graph if and only if D has even sum at most $2n$ and D is graphic when the sum is precisely $2n$.*

Proof. As noted before, an at-most-unicyclic graph is either a unicyclic graph, a disconnected unicyclic graph, or a forest. Any degree sequence of a disconnected unicyclic graph is the degree sequence of a forest per Claim 1.0.7. Thus, D is a degree sequence of a an at-most-unicyclic graph if and only if D is the degree sequence of a forest or unicyclic graph. By Claim 1.0.1 and Theorem 1.0.8, the result thus holds. □

Claim 1.0.11. *A sequence D of n positive integers is the degree sequence of an at-most-unicyclic graph if and only if D has even sum at most $2n$ and D contains at least three integers greater than 1 when the sum is precisely $2n$.*

Proof. As in Claim 1.0.10, D is a degree sequence of a an at-most-unicyclic graph if and only if D is the degree sequence of a forest or unicyclic graph. By Claim 1.0.1 and Theorem 1.0.9, the result thus holds. □

We present an example to demonstrate the necessity of condition 2 of Theorem 1.0.12. Consider the vector sequence below.

$$\begin{array}{l} \text{color 1} \\ \text{color 2} \\ \text{color 3} \end{array} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Color 1 is the degree sequence of a cycle and thus a unicyclic graph. Also, $D_{\{2,3\}}$, that is, the sum degree sequence of the colors 2 and 3, is $< 2, 2, 2 >$, which is again the degree sequence of cycle and thus a unicyclic graph. Thus, in any 3-edge-colored graph G that realizes this sequence of vectors, the color 1 subgraph must have a cycle C_1. Also, the color 2 and color 3 subgraph must have a cycle C_2. Since the colors of C_1 and C_2 are distinct, the cycles are distinct. Thus, G must have two cycles and cannot be unicyclic. In general, if we consider all subsets $\mathcal{C}$ of colors where the sum degree sequence of $\mathcal{C}$ is unicyclic realizable, the intersection of these subsets must be non-empty. Condition 2 of Theorem 1.0.12 ensures this.

Theorem 1.0.12. *Let $\mathcal{D}$ be a sequence of n $(k \times 1)$ column vectors with non-negative entries and non-zero columns sums. Then $\mathcal{D}$ is the degree vector sequence of a k-edge-colored at-most-unicyclic graph on n vertices if and only if the following two conditions hold.*

1. *Given any subset of colors $\mathcal{C}$, $D_{\mathcal{C}}$ has even sum at most $2n$ and $D_{\mathcal{C}}$ is graphic when the sum is precisely $2n$.*

2. *(Intersection Property) Let $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_s$ be the list of non-empty subsets of colors where $D_{\mathcal{C}_i}$ has sum $2m_{\mathcal{C}_i}$ for each i, $1 \leq i \leq s$. Then the intersection of all $\mathcal{C}_i$ is non-empty.*

Proof. $(\Rightarrow)$ Consider G, an edge-colored at-most-unicyclic graph with degree vector sequence $\mathcal{D}$. All subgraphs of G have 0 or 1 cycles, and so given any subset of colors $\mathcal{C}$, the subgraph consisting of the colors in $\mathcal{C}$ is an at-most-unicyclic graph. Then condition (1) holds by Claim 1.0.10. Now, assume G has exactly one cycle C. Let $\mathcal{C}$ be the colors that appear on C. If for some subset of colors $\mathcal{C}'$, $D_{\mathcal{C}'}$ has sum $2m_{\mathcal{C}'}$, then the subgraph of G whose edges have colors in $\mathcal{C}'$ must contain the unique cycle C. Thus, $\mathcal{C}$ must be contained in $\mathcal{C}'$. As a result, if $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_s$ is the list of non-empty subsets of colors where $D_{\mathcal{C}_i}$ has sum $2m_{\mathcal{C}_i}$ for each i, then the intersection of all $\mathcal{C}_i$ contains $\mathcal{C}$ and is thus non-empty.

$(\Leftarrow)$ If every subset of colors $\mathcal{C}$ has row entries which sum to at most $2m_{\mathcal{C}} - 2$, then the desired result follows from Theorem 1.0.5. Otherwise, there exists a non-empty subset of colors $\mathcal{S}$ with sum degree sequence $D_{\mathcal{S}}$ whose entries sum to exactly $2m_{\mathcal{S}}$. Consider all non-empty subsets of colors $\mathcal{C}$ where $D_{\mathcal{C}}$ has sum $2m_{\mathcal{C}}$. By the intersection property, there is a set of colors $\mathcal{I} \neq \emptyset$ which is a subset of any such $\mathcal{C}$. Hence, $\mathcal{I} \subseteq \mathcal{S}$. Choose any color $i \in \mathcal{I}$. We change our vector sequence $\mathcal{D}$ in the following manner so that the color i degree sequence has 2 more degree 1 vertices. Add 2 more column vectors to $\mathcal{D}$ where the entries in row i are 1 and all other entries are 0. Call this new vector sequence $\mathcal{D}'$

We now argue that all subsets of colors in $\mathcal{D}'$ are realizable as a forest. For any subset of colors $\mathcal{C}$ which excludes i, the support of $\mathcal{C}$ has not changed and so the vector sequence entries corresponding to $\mathcal{C}$ in $\mathcal{D}'$ still sum to at most $2m_{\mathcal{C}}-2$. Thus, the sum degree sequence of $\mathcal{C}$ is still realizable as a forest in $\mathcal{D}'$. Now consider any subset of colors $\mathcal{C}$ where $i \in \mathcal{C}$. Let $D'_{\mathcal{C}}$ be the sum degree sequence of $\mathcal{C}$ in the new vector sequence $\mathcal{D}'$. The number of non-zero entries in $D'_{\mathcal{C}}$ is $m_{\mathcal{C}}+2$. Because the sum of $D_{\mathcal{C}}$ is at most $2m_{\mathcal{C}}$, the sum of $D'_{\mathcal{C}}$ is at most $2m_{\mathcal{C}}+2 = 2(m_{\mathcal{C}}+2)-2$ and thus $D'_{\mathcal{C}}$ is realizable as a forest. In specific, note that $D'_{\mathcal{S}}$ has sum $2(m_{S}+2)-2$ and is thus realizable as a forest and any forest realization of $D'_{\mathcal{S}}$ is connected per Claim 1.0.1.

Since all subsets of colors are forest-realizable, Theorem 1.0.5 implies that there exists a k-edge-colored forest G with degree vector sequence $\mathcal{D}'$. Moreover, G has $n+2$ vertices and degree sum at most $2n+2 = 2(n+2)-2$. Let v and w be the vertices in G whose degree vectors correspond to the column vectors added to $\mathcal{D}$ to obtain $\mathcal{D}'$. Recall that any forest realization of $D'_{\mathcal{S}}$ is connected. Thus, all edges of colors from S appear in the same component U in G. Furthermore, $i \in \mathcal{I} \subseteq \mathcal{S}$ and so v and w are in this component. We argue that v and w cannot be adjacent. Otherwise, $G - \{vw\}$ is an edge-colored forest in which the subset of colors $\mathcal{S}$ with degree sum sequence $D_{\mathcal{S}}$ has sum $2m_{\mathcal{S}}$ and so is not realizable as a forest. This contradicts Theorem 1.0.5.

Since v and w are not adjacent, there exists edges va and wb of color i in G where $a \neq w$ and $b \neq v$ but possibly $a = b$. Remove v and w and add the edge ab, which may be a loop or a duplicate edge. Call the resulting graph G'.

Case 1: $a \neq b$ **-** If a and b are not adjacent in G, then the addition of the edge ab creates a unique cycle in U and thus G. Otherwise, the edge ab, which we assume has color j, exists in G and so G' is a forest with a duplicate edge between a and b. Note that possibly $j = i$. We now argue that there must be an edge of color i or j disjoint from a and b. Otherwise, the subgraph of G induced by edges of color i or j consists of a pair of duplicate edges between a and b with pendants incident to a and b. Then $D_{\{i,j\}}$ has too big of a sum to be a graphic degree sequence, thus violating condition 1.

Thus, there must be an edge xy of color i or j disjoint from a and b. We wish to show xy is in U. If xy has color i then xy must be in U since U contains all edges with a color from $\mathcal{S}$. Assume for a moment that xy has color $j \neq i$ and xy is in a component other than U and we show a contradiction. In this case, we can delete xy and the edge of color i between a and b and then add edges ay and bx of color i. Doing so joins two components but creates no cycles. Hence, the resulting graph is a forest which realizes $\mathcal{D}$. Then the subgraph consisting of edges from $\mathcal{S}$ has no cycle, which contradicts that $D_{\mathcal{S}}$ has sum $2m_{\mathcal{S}}$ and so is not a forest by Claim 1.0.1.

As a result, xy is in U, and without loss of generality, assume xy is of color i. Because $xy \in U$, there exists a path P between a and x. If b is on this path, switch the roles of a and b. If y is on this path, switch the roles of x and y. Thus, we may assume b and y are not on this path. Hence, ay and bx cannot be edges of any color in G' as the presence of either edge forces a cycle in G. Delete xy and the edge of color i between a and b and then add edges ay and bx of color i. Doing so creates a unique cycle which can be traversed by following ab, bx, and then the path P.

Case 2: $a = b$ **-** Then the vertex a has a loop of color i in G'. U is thus a tree with a loop in G'. If no edges of color i are disjoint from a then the degree sequence of color i is not graphic, thus violating condition 1. Let xy be an edge of color i disjoint from a. The vertex a cannot be adjacent to both x and y as this implies G has a cycle besides the loop, a contradiction. If a is adjacent to neither, then add edges ax and ay of color i and delete xy as well as the loop at vertex a. Since U was a tree with a loop, doing so creates a unique cycle. Otherwise, a is adjacent to exactly one of x or y, say x. Still add edges ax and ay of color i and delete xy as well as the loop at vertex a. The resulting graph is a forest with a a duplicate edge between a and x. Use the same technique as in the proof of Case 1 to replace this duplicate edge with a unique cycle.

$\square$

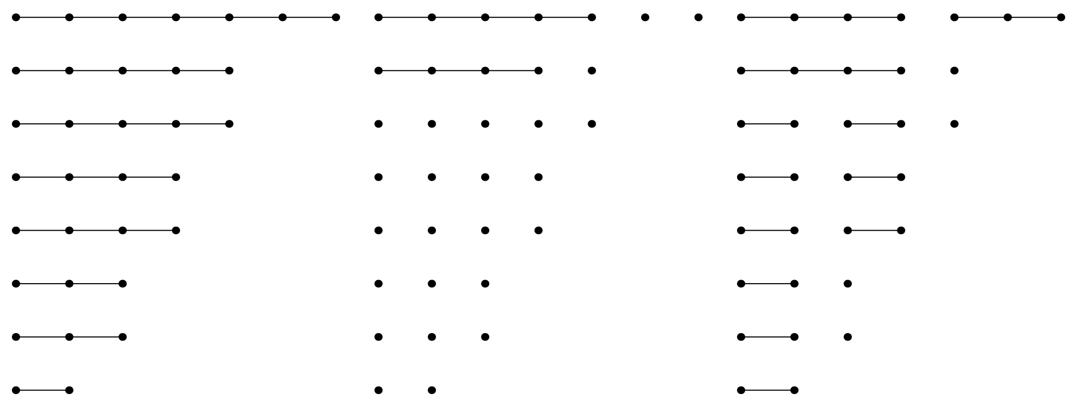
Chapter 2

The Factor Problem for Graphs of Max Degree 2

In any graph of max degree 2, each component is a path or a cycle. Hence, a graph of max degree 2 is a disjoint union of paths (DUP) together with a disjoint union of cycles (DUC). First we discuss factors of DUPs, then factors of DUCs, and finally, factors of graphs with max degree 2. Recall per Definition 0.0.5 that a $[d_0, d_1, d_2]$-factor of any graph with max degree 2 is a factor with d_i vertices of degree i for $0 \leq i \leq 2$.

2.1 Factors in a Disjoint Union of Paths (DUP)

The first graph family we consider is a disjoint union of paths (DUP) where each path has at least 2 vertices. Note that any factor of a DUP is in turn a DUP. We let p correspond to the number of paths and we let $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ be the non-decreasing list of the orders of the paths. Since a factor is spanning, if a DUP G has a $[d_0, d_1, d_2]$-factor, then $|V(G)| = d_0 + d_1 + d_2 = \sum_{i=1}^{p} C_i$. Consider an example of a DUP G with path orders $2, 3, 3, 4, 4, 5, 5, 7$. Here $|V(G)| = 33$ and $p = 8$. As an illustration, Figure 2.1 shows a $[24, 4, 5]$-factor and a $[4, 24, 5]$-factor of G.



(a) A DUP G **(b)** A $[24, 4, 5]$-factor of G **(c)** A $[4, 24, 5]$-factor of G

Figure 2.1: Factors in DUPs

Notice the $[24, 4, 5]$-factor in Figure 2.1 has $\frac{d_1}{2} = 2$ paths. Since each path requires two endpoints, the number of paths in a factor is exactly half the number of degree 1 vertices, $\frac{d_1}{2}$. Hence, d_1 must be even if a DUP has a $[d_0, d_1, d_2]$-factor. Since degree 2 vertices are internal to a path, if a factor has any degree 2 vertices, the factor must also have endpoints as well. Thus, if a DUP has a $[d_0, d_1, d_2]$-factor and $d_2 > 0$, then $d_1 > 0$ as well. Claim 2.1.1 shows that the two conditions we just described characterize when there exists *some* DUP with a $[d_0, d_1, d_2]$-factor. Moreover, if such a DUP exists, then more specifically, Claim 2.1.1 shows that there exists a *single path* with such a $[d_0, d_1, d_2]$-factor.

Claim 2.1.1. *Let d_0, d_1, d_2 be nonnegative integers. If $d_1 = 0$ and $d_2 > 0$, then no path or DUP has a $[d_0, d_1, d_2]$-factor. Otherwise, there exists a path with a $[d_0, d_1, d_2]$-factor if and only if d_1 is even. Furthermore, if $d_1 > 0$ and a path P with a $[d_0, d_1, d_2]$-factor exists, there is a realization of the factor in which an endpoint of P is a degree 1 vertex in the factor.*

Proof. The forward direction is immediate. Consider the backwards direction. If $d_1 = 0$, then $d_2 = 0$, and we obtain a $[d_0, 0, 0]$-factor by removing all edges from a path on d_0 vertices. Otherwise, $d_1 > 0$ and we let G be the DUP consisting of a path on d_2+2 vertices plus $\frac{d_1-2}{2}$ additional single-edge paths and d_0 isolated vertices.

Then G is a factor of a path P on $d_0 + d_1 + d_2$ vertices. Also, if we line up the paths in G from left to right so that the leftmost path is non-trivial, ie, is not an isolated vertex, then we obtain a factor of P in which the left endpoint is a degree 1 vertex in the factor G. $\square$

Since the question of when there exists *some* DUP with a $[d_0, d_1, d_2]$-factor is straightforward, we concentrate now on a more interesting question. We determine *which* DUPs have a $[d_0, d_1, d_2]$-factor. To do so, we must answer the Factor Problem for a DUP of specified path orders, which we do in Theorem 2.1.3. Before proving Theorem 2.1.3, we now present several enlightening examples that demonstrate necessary conditions for a $[d_0, d_1, d_2]$-factor to exist within a DUP.

Consider the DUP G in Figure 2.1. We argue that G does not have a $[14, 4, 15]$-factor. Such a factor would consist of 2 paths whose internal vertices sum to 15 and which are subpaths of paths in G. However, even the longest two paths in G only have a total of 8 internal vertices. Thus, the longest two paths in the factor require more internal vertices than are available in the longest two paths in G, thus making it impossible for G to have such a factor. This example demonstrates d_2 can be at most the number of internal vertices within the largest $\frac{d_1}{2}$ paths, i.e.,

$$d_2 \leq \sum_{i=p-\frac{d_1}{2}+1}^{p} (C_i - 2), \tag{2.1.1}$$

or equivalently,

$$d_1 + d_2 \leq \sum_{i=p-\frac{d_1}{2}+1}^{p} C_i. \tag{2.1.2}$$

Notice that the sums in the above conditions are only meaningful when $p \geq \frac{d_1}{2}$. Intuitively, this makes sense. If d_1 is small, we need $\frac{d_1}{2}$ of the p paths of G to be long enough to "fit" all d_2 internal vertices of degree 2 and this restricts the size of d_2. However, if d_1 is large, specifically, if $\frac{d_1}{2} \geq p$, then we do not have this restriction because we can use the internal vertices of any of the p paths to "fit" the d_2 internal vertices of degree 2. Furthermore, we point out that if $p = \frac{d_1}{2}$ then the above conditions immediately hold since $d_0 + d_1 + d_2 = \sum_{i=1}^{p} C_i \implies d_1 + d_2 = \sum_{i=p-\frac{d_1}{2}+1}^{p} (C_i - 2)$.

In our example, $33 = d_0 + d_1 + d_2$ and $d_1 = 4 \implies 29 = d_0 + d_2$, and so $d_2 \leq 10$ if and only if $d_0 \geq 19$. This suggests that we can restate the condition 2.1.1 as a lower bound on d_0 instead of an upper bound on d_2. We see that d_2 is no bigger than the number of internal vertices in the largest $\frac{d_1}{2}$ paths of G if and only if d_0 is at least the number of vertices in the smallest $p - \frac{d_1}{2}$ paths, and we formally show this now.

Claim 2.1.2. *Let* $d_0 + d_1 + d_2 = \sum_{i=1}^{p} C_i$ *and* $\frac{d_1}{2} \leq p$. *Then* $d_1 + d_2 \leq \sum_{i=p-\frac{d_1}{2}+1}^{p} C_i$ *if and only if* $\sum_{i=1}^{p-\frac{d_1}{2}} C_i \leq d_0$.

Proof. This follows directly from the fact that $d_0 + d_1 + d_2 = \sum_{i=1}^{p} C_i = \sum_{i=1}^{p-\frac{d_1}{2}} C_i + \sum_{i=p-\frac{d_1}{2}+1}^{p} C_i$. □

Now consider whether or not G from Figure 2.1(a) has a $[1, 30, 2]$-factor. Here $\frac{d_1}{2} > p$ so condition (2.1.1) is irrelevant. Since degree 1 vertices in a DUP must occur in pairs as endpoints of paths, the vertices in an odd path of G cannot each be degree 1 in any factor of G. Thus, each odd path of G must have at least one vertex of degree zero or two in any factor of G. Since G has 5 paths of odd order, this means that $d_0 + d_2$ must be at least 5, thus making it impossible for G to have a $[1, 30, 2]$-factor. Letting $o_p(G)$ refer to the number paths with odd order in G, this demonstrates that another necessary condition is

$$o_p(G) \leq d_0 + d_2. \tag{2.1.3}$$

The above explanation sheds light on why inequalities (2.1.1) and (2.1.3) appear as conditions in the following theorem.

Theorem 2.1.3. *Let G be a disjoint union of paths with orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ where p and all C_i are positive integers. Let $o_p(G)$ refer to the number of C_i which are odd. Given non-negative integers d_0, d_1, d_2 where $\sum_{i=1}^{p} C_i = d_0 + d_1 + d_2$, G has a $[d_0, d_1, d_2]$-factor if and only if the following conditions hold:*

1. d_1 *is even*

2. *If* $p > \frac{d_1}{2}$ *then* $d_2 \leq \displaystyle\sum_{i=p-\frac{d_1}{2}+1}^{p} (C_i - 2)$ *(The RHS sum is 0 when* $d_1 = 0$.*)*

3. $o_p(G) \leq d_0 + d_2$

Proof. ($\Rightarrow$) Let G' be a $[d_0, d_1, d_2]$-factor of G. The sum of any graphical degree sequence is even. The degree sum for G' is $0d_0 + 1d_1 + 2d_2$ which is even if and only if d_1 is even. Hence, condition (1) is clear.

A vertex is an endpoint to a path if and only if it has degree 1. Since G' has d_1 endpoints and each path requires two endpoints, G' has exactly $\frac{d_1}{2}$ paths. Also, a vertex is internal to a path if and only if the vertex has degree 2. Then the $\frac{d_1}{2}$ paths in G' have a total of d_2 internal vertices. If $d_1 = 0$, then G' has no paths and thus no internal vertices either and so it must be true that $d_2 = 0$. Thus, condition (2) holds when $d_1 = 0$. Now assume $0 < \frac{d_1}{2} < p$. Since the paths in G' are subpaths of paths in G, the $\frac{d_1}{2}$ paths of G' can be no longer than the longest $\frac{d_1}{2}$ paths in G. Hence, d_2 can be at most the number of internal vertices in the longest $\frac{d_1}{2}$ paths in G. Noting that a path of order $C_i \geq 2$ has $C_i - 2$ internal vertices, we see that condition (2) must be true, that is,

$$d_2 \leq \sum_{i=p-\frac{d_1}{2}+1}^{p} (C_i - 2).$$

Now assume that a path P of G corresponds to l subpaths in G'. Then $2l$ vertices of P are endpoints to paths in G'. So if P has odd order, then at least one vertex of P has degree zero or two in G'. Since there are $o_p(G)$ paths of odd order in G, we see that $o_p(G) \leq d_0 + d_2$. Thus, condition (3) holds.

($\Leftarrow$) We note that if $d_1 = 0$, then condition (2) implies that $d_2 = 0$ and so $d_0 = \sum_{i=1}^{p} C_i$, in which case the desired factor is a graph of d_0 isolated vertices.

Our proof is by induction on p. If $p = 1$, then G consists of one path with $C_1 = d_0 + d_1 + d_2$ vertices. If $d_1 > 0$, then the following is a $[d_0, d_1, d_2]$-factor of G: a path on $d_2 + 2$ vertices, $\frac{d_1 - 2}{2}$ paths on 2 vertices, and d_0 isolated vertices.

We now assume that for $p-1 \geq 1$ if $\sum_{i=1}^{p} C_i = d_0 + d_1 + d_2$ and if conditions (1)-(3) hold, then a DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ has a $[d_0, d_1, d_2]$-factor. We show the claim to be true for G, a DUP with $p \geq 2$ paths.

Let P be a path in G with order C_p. We use the notation $G - P$ to denote the DUP G with the path P removed. Our strategy will be to determine d_i' where $0 \leq d_i' \leq d_i$ for $i = 0, 1, 2$ so that $C_p = d_0' + d_1' + d_2'$ and thus $\sum_{i=1}^{p-1} C_i = (d_0 - d_0') + (d_1 - d_1') + (d_2 - d_2')$. It will be clear that conditions (1)-(3) hold for the chosen d_i' values. We verify that conditions (1)-(3) hold for the $d_i - d_i'$ values, and then we use the inductive hypothesis to find a $[d_0', d_1', d_2']$-factor of P and a $[d_0 - d_0', d_1 - d_1', d_2 - d_2']$-factor of $G - P$. The union of these two factors yields a $[d_0, d_1, d_2]$-factor of G. We do not show the details of applying induction to obtain a $[d_0', d_1', d_2']$-factor of P as this is just an instance of the base case.

If $p > \frac{d_1}{2}$, we choose the d_i' values so as to find a $[d_0', d_1', d_2']$-factor of P that consists of one long subpath with as many degree 2 vertices as possible. To accomplish this, we let $d_1' = 2, d_2' = \min\{d_2, C_p - 2\}, d_0' = C_p - d_1' - d_2'$. Then $d_0' + d_2' = C_p - 2$. We check now that our conditions hold for a $[d_0 - d_0', d_1 - d_1', d_2 - d_2']$-factor of $G - P$. Condition (1) holds since both d_i and d_i' are even. Since the inequality in condition (2) holds for $[d_0, d_1, d_2]$ and G, we see that for $[d_0 - d_0', d_1 - d_1', d_2 - d_2']$ and $G - P$, the RHS of the inequality decreases by $C_p - 2$ and the LHS either decreases by $C_p - 2$ or becomes 0. Thus, condition (2) holds. To show condition (3), we must show the number of path orders C_i, $1 \leq i \leq p - 1$, which are odd is at most $(d_0 - d_0') + (d_2 - d_2') = d_0 + d_2 - (C_p - 2)$. This follows since $p > \frac{d_1}{2}$ implies that $d_0 + d_2$ is more than the total number of internal vertices in G. Hence, $d_0 + d_2 - (C_p - 2)$ is more than the total number of internal vertices in $G - P$. Since each odd path in $G - P$ has an internal vertex, we see then that $d_0 + d_2 - (C_p - 2)$ is then an overcount for the number of path orders which are odd. Hence, condition (3) holds for $G - P$. Then by induction, $G - P$ has a $[d_0 - d_0', d_1 - d_1', d_2 - d_2']$-factor as desired. As previously discussed, the union of this factor plus a $[d_0', d_1', d_2']$-factor of P yields a $[d_0, d_1, d_2]$-factor of G.

Now assume $p \leq \frac{d_1}{2}$. It is helpful to let $s = d_1 - 2(p - 1)$. Then s is positive since $p \leq \frac{d_1}{2}$. This quantity s represents how many degree 1 vertices would be left

Case	Choice of d_i	$[d'_0, d'_1, d'_2]$-factor of P
Case A: $s \geq C_p$ and C_p is even	Let $d'_0 = d'_2 = 0$ and $d'_1 = C_p$	$\frac{C_p}{2}$ disjoint edges
Case B: $s \geq C_p$ and C_p is odd	Let $d'_1 = C_p - 1$ and let d'_0 or d'_2 be 1 and the other 0, depending on which is positive.	$\frac{C_p-1}{2}$ disjoint edges and one isolated vertex OR $\frac{C_p-3}{2}$ disjoint edges and one path on 3 vertices
Case C: $s < C_p$	Let $d'_1 = s$ and $d'_2 + d'_1 = C_p - s$	A path on $d'_2 + 2$ vertices, $\frac{d'_1-2}{2}$ disjoint edges, and d'_2 isolated vertices

Table 2.1: Inductive cases of Theorem 2.1.3 when $p \leq \frac{d_1}{2}$

over if some realization of the factor were to have exactly one non-trivial subpath in each of the $p-1$ smallest paths of G. We will choose d'_1 so that we lower d_1 no more than s. This forces that $p - 1 \leq \frac{d_1 - d'_1}{2}$, which is desirable because than condition (2) is irrelevant for $G - P$. Table 2.1 shows how to choose d'_i. We include the last column in order to give insight into this choice.

In all cases within Table 2.1, condition (1) holds for $[d_0 - d'_0, d_1 - d'_1, d_2 - d'_2]$ and $G - P$. We already noted that condition (2) is irrelevant by choice of d'_i. In case A, the inequality of condition (3) matches that for $[d_0, d_1, d_2]$ and G and so holds. As for case B, each side of the inequality decreases by 1 and thus holds. In case C, the quantity $(d_0 - d'_0) + (d_2 - d'_2) = d_0 + d_2 - (C_p - s)$ is more than the total number of internal vertices in $G - P$. Since each odd path in $G - P$ has at least one internal vertex, we see then that $d_0 + d_2 - (C_p - s)$ is an overcount for the number of path orders C_i, $1 \leq i \leq p - 1$, which are odd. Hence, condition (3) holds. Then by induction $G - P$ has a $[d_0 - d'_0, d_1 - d'_1, d_2 - d'_2]$-factor. Again, the union of this factor plus a $[d'_0, d'_1, d'_2]$-factor of P yields a $[d_0, d_1, d_2]$-factor of G.

□

2.2 Factors in a Disjoint Union of Cycles (DUC)

We now consider a disjoint union of cycles (DUC) with cycle sizes at least 3. We let m correspond to the number of cycles and we let $3 \leq C_1^\circ \leq C_2^\circ \leq \cdots \leq C_m^\circ$ be the ordered list of cycle sizes. Since a factor is spanning, if a DUC G has a $[d_0, d_1, d_2]$-factor, then $|V(G)| = d_0 + d_1 + d_2 = \sum_{i=1}^{m} C_i^\circ$. Figure 2.2 illustrates a disjoint union of cycles (DUC) that has a $[3, 8, 1]$-factor and a $[8, 0, 4]$-factor. As with DUPs, degree 1 vertices in any factor are endpoints to paths and so there are $\frac{d_1}{2}$ non-trivial paths in any $[d_0, d_1, d_2]$-factor of a DUC. Hence, when $d_1 = 0$, a $[d_0, d_1, d_2]$-factor consists of original cycles from G with isolated vertices. Figure 2.2(c) exemplifies this.

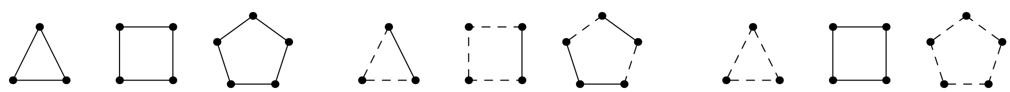

(a) A DUC G **(b)** A $[3, 8, 1]$-factor of G **(c)** An $[8, 0, 4]$-factor of G

Figure 2.2: Factors in DUCs

Claim 2.2.1 determines when there exists some DUC with a $[d_0, d_1, d_2]$-factor.

Claim 2.2.1. *Let d_0, d_1, d_2 be nonnegative integers whose sum is at least three. If $[d_0, d_1, d_2]$ are either one of the pathological cases below, then no DUC has a $[d_0, d_1, d_2]$-factor.*

1. $[d_0, 0, d_2]$, $d_0 = 1$ *or* $d_0 = 2$
2. $[d_0, 0, d_2]$, $d_2 = 1$ *or* $d_2 = 2$

Otherwise, there exists a DUC with a $[d_0, d_1, d_2]$-factor if and only if d_1 is even.

Proof. ($\Rightarrow$) If a DUC has a $[d_0, d_1, d_2]$-factor , then the factor has degree sum $0d_0 + d_1 + 2d_2$, which is even if and only if d_1 is even. We now consider the pathological cases. Consider a $[d_0, d_1, d_2]$-factor H of a DUC G where $d_1 = 0$. We can imagine removing edges from G to obtain H. Since $d_1 = 0$, H has no degree 1 vertices and thus no non-trivial paths. Thus, for each cycle in G, we must remove all edges or no edges to obtain the factor H. If we remove no edges from some cycle in G, then

$d_0 \geq 3$. Otherwise, $d_0 = 0$. If we remove all edges from some cycle in G, then $d_2 \geq 3$. Otherwise, $d_2 = 0$. Thus, each pathological case in the claim is not the factor of any DUC.

($\Leftarrow$) If $d_1 = 0$, then let G be a DUC with a cycle on d_0 vertices and a cycle on d_2 vertices. Remove all edges from the cycle with d_0 vertices to obtain a [d_0,0,d_2]-factor of G. Otherwise if $d_1 > 0$, it follows from Claim 2.1.1 that some path P has a $[d_0, d_1, d_2]$-factor H. Add an edge between the endpoints of P to obtain a cycle C and note that H is a $[d_0, d_1, d_2]$-factor of C. $\square$

Since it is straightforward to determine when there exists *some* DUC with a $[d_0, d_1, d_2]$-factor, we ask a more interesting question, that is, *which* DUCs have a $[d_0, d_1, d_2]$-factor? Theorem 2.2.2 answers this question when $d_1 > 0$, and in doing so, answers the Factor Problem for a DUC of specified sizes when $d_1 > 0$. As with DUPs, since degree 1 vertices must occur in pairs in the factor of a cycle, we see that d_1 must be even and that each odd cycle of a DUC must have at least one vertex of degree 0 or 2 in the factor. Therefore, as with inequality (2.1.3) for DUPs, it must be true that

$$o_c(G) \leq d_0 + d_2. \tag{2.2.1}$$

Furthermore, a DUC has a pathological case. Consider a DUC G with three cycles of sizes $C_1^\circ = C_2^\circ = C_3^\circ = 5$. We explain why G has no [9,2,4]-factor now. Assume such a factor does exist. Since $d_1 = 2$, the factor must have exactly one non-trivial path P. Since $C_i^\circ = 5$, P is a path on at most 5 vertices. Thus, P (and so each C_i) is just small enough where P cannot contain all of the $d_2 = 4$ degree 2 vertices in the factor plus the $d_1 = 2$ endpoints. This implies that the vertices of some other cycle C from G must all be degree 2 in the factor and so G must contain all of its edges in the factor. However, because $d_2 = C_m^\circ - 1 = 4$, d_2 is just small enough that this is impossible. In general, this situation occurs when $d_1 = 2$, all cycles have the same size, and d_2 is one less than a multiple of the cycle size, which forces that d_0 is also one less than a multiple of the cycle size. So this situation occurs when $[d_0, d_1, d_2] = [rC_m^\circ - 1, 2, (m-r)C_m^\circ - 1]$ for some integer $r \in (0, m)$, or equivalently, when $[d_0, d_1, d_2] = [(m-r)C^\circ_m - 1, 2, rC^\circ_m - 1]$.

Theorem 2.2.2. *Let G be a DUC with cycle sizes $3 \leq C^\circ_1 \leq C^\circ_2 \leq \cdots \leq C^\circ_m$. Let $o_c(G)$ be the number of C°_i which are odd. Let d_0, d_1, d_2 be non-negative integers which sum to $|V(G)|$ where $d_1 > 0$. If $C^\circ_1 = C^\circ_2 = \cdots = C^\circ_m$ and if $[d_0, d_1, d_2] = [rC^\circ_m - 1, 2, (m-r)C^\circ_m - 1]$ for some integer $r \in (0, m)$, then G does not have a $[d_0, d_1, d_2]$-factor. Otherwise, G has a $[d_0, d_1, d_2]$-factor if and only if d_1 is even and $o_c(G) \leq d_0 + d_2$.*

Theorem 3.3.4 answers the 2-Edge-Coloring Problem for DUCs, and we wait until after Theorem 3.3.4 to give the proof of Theorem 2.2.2. The reason is as follows. A DUC is a regular graph. Per Claim 0.0.7, the 2-Edge-Coloring Problem and Factor Problem are equivalent for regular graphs, meaning, an answer to one leads to an answer to the other. Thus, Theorem 2.2.2 naturally follows from Theorem 3.3.4. Care has been taken to prevent any circular arguments.

2.2.1 NP-Completeness of [d_0,0,d_2]-factors of DUCs

It is very important to notice that Theorem 2.2.2 requires that $d_1 > 0$. When $d_1 = 0$, we can show that an answer to the decision problem 'Does a DUC with specified path orders have a $[d_0, 0, d_2]$-factor?' yields an answer to the Subset Sum Problem which is a well-known NP-Complete Problem that is solvable in pseudo-polynomial time [9].

Problem 2.2.1 ([9])**.** *The **Subset Sum Problem** asks the following question: Given a finite set $\mathcal{A}$ of positive integers and a positive integer s, does there exist a subset $\mathcal{A}' \subseteq \mathcal{A}$ such that the sum of the integers in $\mathcal{A}'$ is exactly s?*

The reduction between the Subset Sum Problem and the Factor Problem for DUCs when $d_1 = 0$ follows immediately from Theorem 2.2.3.

Theorem 2.2.3. *Let G be a DUC with cycle sizes $3 \leq C^\circ_1 \leq C^\circ_2 \leq \cdots \leq C^\circ_m$ and let d_0, d_2 be non-negative integers which sum to $|V(G)|$. Then G has a $[d_0, 0, d_2]$-factor if and only if some subset of cycle sizes sum to d_0, or equivalently, some subset of cycle sizes sum to d_2.*

Proof. If a $[d_0, 0, d_2]$-factor exists, since there are no degree 1 vertices, the factor can be obtained by removing edges from a set of cycles in G whose sizes sum to d_0 and by leaving all edges in a set of cycles whose sizes sum to d_2. Also, if if some subset $\mathcal{S}$ of cycle sizes sum to d_0, then remove all edges from the cycles in $\mathcal{S}$ and leave all edges in the cycles that are not in $\mathcal{S}$ to obtain the desired $[d_0, 0, d_2]$-factor. $\square$

Subset Sum is known to be pseudo-polynomial or weakly NP-Complete, meaning that the algorithmic complexity of the problem depends greatly on the encoding of the problem [9]. For our purposes, the pseudo-polynomial complexity of Subset Sum translates to the following. In asking whether a DUC with specified sizes has a $[d_0, 0, d_2]$-factor, the input is the set of d_i values. The question is NP-Complete with this encoding. However, if we change the encoding and we specify the desired degree sequence of the factor, i.e., if our input is a list of size $d_0 + d_1 + d_2$ consisting of d_i values of degree i for $i = 1, 2, 3$, then the encoding becomes unary. The algorithm implied by Theorem 2.2.3 is polynomial with the unary encoding.

In summary, even though Theorem 2.2.3 characterizes when a $[d_0, 0, d_2]$-factor is possible, we do not expect that an implementation of this characterization can be done efficiently. This contrasts Theorem 2.2.2 which yields an efficient algorithm to answer our question when $d_1 > 0$.

Finally, the reader may wonder why finding $[d_0, 0, d_2]$-factors of DUCs leads to a complexity issue whereas finding $[d_0,0,d_2]$-factors of DUPs does not. Note that if $d_1 = 0$ in a factor of a DUP, then the factor has no endpoints to non-trivial paths and thus has no internal vertices either. In other words, if $d_1 = 0$, all vertices in the factor of the DUP are forced to be isolated vertices. The hypotheses of Theorem 2.1.3 ensure this.

2.3 Factors of Graphs with Max Degree 2

Claim 2.3.1 describes when there exists *some* graph with max degree 2 which contains a $[d_0, d_1, d_2]$-factor.

Claim 2.3.1. *Given non-negative integers d_0, d_1, d_2, there exists a graph of max degree 2 with a $[d_0, d_1, d_2]$-factor if and only if the sequence D consisting of d_i entries of the integer i for $i = 1, 2, 3$ is graphical.*

Proof. If a graph has a $[d_0, d_1, d_2]$-factor , then D is the degree sequence of the factor and is thus graphical. If D is graphical, then D is clearly realizable as a graph with max degree 2 and is thus a factor of itself. □

We now concentrate on determining when a *fixed* graph of max degree 2 has a $[d_0, d_1, d_2]$-factor. Theorem 2.3.3 is the main result of this section and describes necessary and sufficient conditions for when a *fixed* graph of max degree 2 has a $[d_0, d_1, d_2]$-factor. We prove an auxiliary claim before proving Theorem 2.3.3.

Claim 2.3.2. *Consider a list of s integers t_1 through t_s, each of which is at least 2. Let d_0, d_1, d_2 be non-negative integers where $\sum_{i=1}^{s} t_i = d_0 + d_1 + d_2$. If $d_1 \leq 2s$, then the number of t_i which are odd is at most $d_0 + d_2$.*

Proof. Note that $d_0 + d_1 + d_2 = \sum_{i=1}^{s} t_i$ implies $d_0 + d_1 + d_2 - 2s = \sum_{i=1}^{p} (t_i - 2)$. Since $d_1 \leq 2s$, this implies $d_0 + d_2 \geq \sum_{i=1}^{s} (t_i - 2)$. Since $t_i \geq 2$, any odd t_i is at least 3 and so contributes at least one to $t_i - 2$. Hence, $\sum_{i=1}^{p} (t_i - 2)$ is an overcount for the number of t_i which are odd. Then the number of t_i which are odd is at most $\sum_{i=1}^{s} (t_i - 2) \leq d_0 + d_2$. □

The pathological cases and hypotheses of Theorem 2.3.3 are similar to those of Theorem 2.1.3 and Theorem 2.2.2. This is sensible since a graph with max degree 2 is simply a union of a DUP and DUC. Note that we assume that $d_1 > 0$ in Theorem 2.3.3. Given a graph G with max degree 2, if $d_1 = 0$, then all path vertices in G must be degree 0 in any $[d_0, d_1, d_2]$-factor of G. Thus, G has a $[d_0, d_1, d_2]$-factor if and only if the cycles of G have a $[d_0 - \sum_{i=1}^{p} C_i, 0, d_2]$-factor. See Section 2.2.1 for an explanation of why determining whether or not a DUC has such a factor is a complex question.

Theorem 2.3.3. *Let G be a graph with m cycles and p paths where $m + p > 0$. Let the cycle orders be $3 \leq C_1^\circ \leq C_2^\circ \leq \cdots \leq C_m^\circ$ and the path orders be $2 \leq C_1 \leq C_2 \leq$*

$\cdots \leq C_p$. Let $o_c(G)$ be the number of cycles with odd order in G. Let $o_p(G)$ be the number of paths with odd order in G. Let d_0, d_1, d_2 be non-negative integers which sum to $\sum_{i=1}^m C_i^\circ + \sum_{i=1}^p C_i$. In the following pathological cases, G does not have a $[d_0, d_1, d_2]$-factor.

(a) $m > 0, p = 0, d_1 = 2, d_2 = rC_m^\circ - 1$ *for integer* r *in* $(0, m), C_1^\circ = \cdots = C_m^\circ$

(b) $m > 0, p > 0, d_1 = 2, d_2 = rC_m^\circ - 1$ *for integer* r *in* $(0, m), C_p \leq C_1^\circ = \cdots = C_m^\circ$

(c) $m > 0, p > 0, d_1 = 2, d_2 = (\sum_{i=1}^m C_i^\circ) - 1, C_p \leq C_1^\circ$

(d) $m > 0, p > 0, d_1 \geq 2, d_2 = (\sum_{i=1}^m C_i^\circ) - 1, C_1 = \cdots = C_p = 2$

Assume $d_1 > 0$. Then with the exception of the above pathological cases, G has a $[d_0, d_1, d_2]$-factor if and only if the following conditions hold.

1. *d_1 is even*

2. *$o_c(G) + o_p(G) \leq d_0 + d_2$*

3. *If $p > \frac{d_1}{2}$, then $d_2 \leq \sum_{i=1}^m C_i^\circ + \sum_{i=p-\frac{d_1}{2}+1}^p (C_i - 2)$*
   (We let $\sum_{i=1}^m C_i^\circ = 0$ when $m = 0$.)

Proof. ($\Rightarrow$) Case (a) follows from Theorem 2.2.2. For case (b), assume such a $[d_0, d_1, d_2]$-factor H of G exists. Let G° refer to the cycles of G and G^- refer to the paths of G. Then $H = H^\circ \cup H^-$ where H° is a factor of G° and H^- is a factor of G^-. Since $C_p \leq C_1^\circ = \cdots = C_m^\circ$, we can add isolated vertices to H^- so that H^- is a factor of p cycles of size C_m°. After doing so, $H = H^\circ \cup H^-$ still has $rC_m^\circ - 1$ degree 2 vertices and two degree 1 vertices but now has $(p+m-r)C_m^\circ - 1$ degree 0 vertices. Let $s = p + m - r$. Then $H = H^\circ \cup H^-$ is a $[sC_m^\circ - 1, 2, (m-s)C_m^\circ - 1]$-factor of a DUC whose cycles all have size C_m°. This contradicts Theorem 2.2.2.

For case (c), assume $m > 0, p > 0, d_1 = 2$ and $C_p \leq C_1^\circ$ in G. Consider a $[d_0, d_1, d_2]$-factor H which satisfies these hypotheses. We argue that H cannot have $d_2 = (\sum_{i=1}^m C_i^\circ) - 1$ degree 2 vertices. Since $\frac{d_1}{2} = 1$, there is exactly one non-trivial path in H. If two vertices of some cycle of G are degree 1 in H, then because there is exactly one non-trivial path in H, the path vertices of G must all be degree 0 in H. Hence, the number of degree 2 vertices in H in this case is at most $(\sum_{i=1}^{m-1} C_i^\circ)-2$. If the vertices of some cycle of G are all degree 0 in H, then all degree 2 vertices of H are within the other $m-1$ cycles of G and additionally at most one path of G. Hence, $d_2 \leq (\sum_{i=1}^{m-1} C_i^\circ) + C_p - 2$. Since $C_p \leq C_1^\circ \leq C_m^\circ$, this implies $d_2 \leq (\sum_{i=1}^{m} C_i^\circ) - 2$. Otherwise, no cycle vertices in G are degree 1 in H and no cycle in G has vertices which are all degree 0 in H. This implies that all cycle vertices in G are degree 2 in H and so $d_2 \geq \sum_{i=1}^{m} C_i^\circ$. Therefore, no factor can contain exactly $d_2 = (\sum_{i=1}^{m} C_i^\circ)-1$ degree 2 vertices.

Now consider case (d). Because $C_p = 2$, any vertex which is degree 2 in the factor is a vertex on a cycle in G. Thus, if such a factor exists, the cycles of G have a $[0, 1, (\sum_{i=1}^{m} C_i^\circ) - 1]$-factor or $[1, 0, (\sum_{i=1}^{m} C_i^\circ) - 1]$-factor. Both such factors are impossible by Claim 2.2.1.

($\Leftarrow$) Assume none of the pathological cases hold. If $m = 0$, the claim follows from Theorem 2.1.3. If $p = 0$, the claim follows from Theorem 2.2.2. Now assume that $m > 0$ and $p > 0$. We break the proof into the following cases.

Case I: $d_1 = 2 + \sum_{i=1}^{p} C_i$, $d_2 = rC_m^\circ - 1$ for r in $(0, m)$, $C_1^\circ = \cdots = C_m^\circ$, all C_i even

Case II: $d_1 \geq 2p + 2$ and Case I does not hold.

Case III: $2 \leq d_1 \leq 2p$ and $d_2 \geq \sum_{i=1}^{m} C_i^\circ$

Case IV: $2 < d_1 \leq 2p$ and $d_2 = (\sum_{i=1}^{m} C_i^\circ) - 1$

Case V: $d_1 = 2$ and $d_2 = (\sum_{i=1}^{m} C_i^\circ) - 1$

Case VI: $d_1 = 2$ and $d_2 \leq (\sum_{i=1}^{m} C_i^\circ) - 2$

Subcase (a): $d_1 = 2, d_2 = rC_m - 1$ where $0 < r < m$ and $C_1^\circ = \cdots = C_m^\circ$

Subcase (b): $d_1 = 2$ and Subcase (a) does not hold.

Case VII: $2 < d_1 \leq 2p$ and $d_2 \leq (\sum_{i=1}^{m} C_i^\circ) - 2$

We employ the following strategy in each of the above cases. We define non-negative integers d_0', d_1', d_2' and $\hat{d}_0, \hat{d}_1, \hat{d}_2$ where the sum of the primed and hatted variables is $d_0 + d_1 + d_2$. We then use Theorem 2.1.3 to show that the $p > 0$ paths have a $[d_0', d_1', d_2']$-factor and Theorem 2.2.2 to show the $m > 0$ cycles have a $[\hat{d}_0, \hat{d}_1, \hat{d}_2]$-factor.

Case I: $d_1 = 2 + \sum_{i=1}^{p} C_i$, $d_2 = rC_m^\circ - 1$ for r in $(0, m)$, $C_1^\circ = \cdots = C_m^\circ$, all C_i even

The hypothesis that $d_0 + d_1 + d_2 = \sum_{i=1}^{m} C_i^\circ + \sum_{i=1}^{p} C_i$ and the assumptions $d_1 = 2 + \sum_{i=1}^{p} C_i$ and $d_2 = rC_m^\circ - 1$ imply that $d_0 \geq 2$. Also, if $m = 1$, then there is no r in $(0, m)$ for which $d_2 = rC_m^\circ - 1$ is non-negative. Hence, $m \geq 2$.

First assume $C_p = 2$. Then all paths in G consist of a single edge, and $d_1 = 2 + \sum_{i=1}^{p} C_i = 2p + 2$. Let $[d_0', d_1', d_2'] = [2, 2p - 2, 0]$. Let $[\hat{d}_0, \hat{d}_1, \hat{d}_2] = [d_0 - 2, 4, d_2]$. The primed and hatted variables are non-negative since $d_0 \geq 2$ and $p \geq 1$. Remove a single edge from the paths in G to obtain a $[d_0', d_1', d_2']$-factor of the paths. The cycles have a $[\hat{d}_0, \hat{d}_1, \hat{d}_2]$-factor by Theorem 2.2.2. These two factors combine to yield a $[d_0, d_1, d_2]$-factor of G.

Now assume $C_p \geq 4$. Since $d_1 = 2 + \sum_{i=1}^{p} C_i$, we see that $d_1 \geq 6$. Since $r > 0$ and $C_m \geq 3$, we see that $d_2 = rC_m^\circ - 1 \geq 2$. Let $[\hat{d}_0, \hat{d}_1, \hat{d}_2] = [(m - r)C_m^\circ - 1, 4, rC_m - 3]$. Then $d_0' + d_2' = mC_m^\circ - 4 \geq 3m - 4$. As previously noted, $m \geq 2$, and thus, $d_0' + d_2' \geq 3m - 4 = m + (2m - 4) \geq m$. Thus, $d_0' + d_2'$ is at least the number of cycles, m, and so is at least the number of odd cycles, $o_c(G)$. Then by Theorem 2.2.2, the m cycles of G have a $[\hat{d}_0, \hat{d}_1, \hat{d}_2]$-factor. By choice of the hatted values, the primed values are forced to be $[d_0', d_1', d_2'] = [0, d_1 - 4, 2]$. Then $d_1' \geq 2$, d_1' is even, all C_i are even, and $d_1' \geq 2p$. By Theorem 2.1.3, the p paths of G have a $[d_0', d_1', d_2']$-factor. These two factors combine to yield a $[d_0, d_1, d_2]$-factor of G.

Case II: $d_1 \geq 2p + 2$ and Case I does not hold.

Let $\sigma_i = 1$ if the path order C_i is odd and let $\sigma_i = 0$ otherwise. Let $d_1' = \min\{\sum_{i=1}^{p}(C_i - \sigma_i), d_1 - 2\}$. The hypotheses imply that we can find d_0' and d_2' such that $0 \leq d_0' \leq d_0$ and $0 \leq d_2' \leq d_2$ so that $d_0' + d_2' = \sum_{i=1}^{p} C_i - d_1'$. Then

$d_1' \geq 2$ and is even. The σ_i values ensure that $o_p(G)$ is at most $d_0' + d_2'$. Recall $d_1 - 2 \geq 2p$ by assumption. Also, $\sum_{i=1}^{p} C_i - \sigma_i \geq 2p$ since $C_i - \sigma_i \geq 2$. Then by the definition of d_1', $d_1' \geq 2p$. As a result, all conditions of Theorem 2.1.3 hold and so the p paths have a $[d_0', d_1', d_2']$-factor.

Let $[\hat{d}_0, \hat{d}_1, \hat{d}_2] = [d_0 - d_0', d_1 - d_1', d_2 - d_2']$ which sum to $\sum_{i=1}^{m} C_i^\circ$. The choice of primed variables imply that $\hat{d}_1$ is even and that $\hat{d}_1 \geq 2$. If $d_1' = \sum_{i=1}^{p}(C_i - \sigma_i)$, then the hypothesis that $o_c(G) + o_p(G) \leq d_0 + d_2$ together with the σ_i values imply that $o_c(G) \leq \hat{d}_0 + \hat{d}_2$. On the other hand, if $d_1' = d_1 - 2$, then $\hat{d}_1 = 2$ and Claim 2.3.2 yields that $o_c(G) \leq \hat{d}_0 + \hat{d}_2$. So if it is not true that $\hat{d}_1 = 2$, $\hat{d}_2 = rC_m^\circ - 1$, $C_1^\circ = \cdots = C_m^\circ$, and $\hat{d}_0 = (m - r)C_m^\circ - 1$ for some $r \in (0, m)$, then by Theorem 2.2.2, the m cycles have a $[\hat{d}_0, \hat{d}_1, \hat{d}_2]$-factor.

Otherwise, $\hat{d}_1 = 2$, $\hat{d}_2 = rC_m^\circ - 1$ and $d_0' = (m - r)C_m^\circ - 1$ for some $r \in (0, m)$ and $C_1^\circ = \cdots = C_m^\circ$. Hence, $\hat{d}_0 \geq 1$ and $\hat{d}_2 \geq 1$. Assume for a moment that $d_0' + d_2' > 0$. Then either d_0' or d_2' is non-zero. If $d_2' > 0$, decrease d_2' by one, increase d_0' by one, and to balance, decrease $\hat{d}_0$ by one and increase $\hat{d}_2$ by 1. Perform a similar procedure if $d_2' = 0$ and $d_0' > 0$. Since $\hat{d}_2$ no longer equals $rC_m^\circ - 1$, Theorem 2.2.2 yields that the m cycles have a $[\hat{d}_0, \hat{d}_1, \hat{d}_2]$-factor. Furthermore, the changes to the primed variables do not affect that the p paths have a $[d_0', d_1', d_2']$-factor. Now assume $d_0' + d_2' = 0$. Since we earlier argued that $o_p(G) \leq d_0' + d_2'$, we see that all paths have even order. Also, by choice of d_0' and d_2', we see $d_0' + d_2' = 0 = \sum_{i=1}^{p} C_i - d_1'$ and so $d_1' = \sum_{i=1}^{p} C_i$. Since $\hat{d}_1 = 2 = d_1 - d_1'$, then $d_1 = 2 + d_1' = 2 + \sum_{i=1}^{p} C_i$. Thus, we are in Case I, a contradiction.

Case III: $2 \leq d_1 \leq 2p$ and $d_2 \geq \sum_{i=1}^{m} C_i^\circ$

Since $d_2 \geq \sum_{i=1}^{m} C_i^\circ$, we can make all vertices of the m cycles degree 2 vertices in the factor by not removing any edges from the cycles. We now must argue that the paths have a $[d_0', d_1', d_2']$-factor where $[d_0', d_1', d_2'] = [d_0, d_1, d_2 - \sum_{i=1}^{m} C_i^\circ]$. Because $d_1' = d_1 \leq 2p$, Claim 2.3.2 yields that $o_p(G) \leq d_0' + d_2'$. Also, if $\frac{d_1'}{2} < p$, then by hypothesis, $d_2 \leq \sum_{i=1}^{m} C_i^\circ + \sum_{i=p-\frac{d_1}{2}+1}^{p}(C_i - 2)$ and so

$d_2' \le \sum_{i=p-\frac{d_1}{2}+1}^{p}(C_i - 2)$. Thus, the hypotheses of Theorem 2.1.3 hold and so the p paths have a $[d_0', d_1', d_2']$-factor.

Case IV: $2 < d_1 \le 2p$ and $d_2 = (\sum_{i=1}^{m} C_i^\circ) - 1$

Remove one edge from any cycle. This yields a $[0, 2, (\sum_{i=1}^{m} C_i^\circ) - 2]$-factor of the cycles. We now argue that for $[d_0', d_1', d_2'] = [d_0, d_1 - 2, 1]$ there is a $[d_0', d_1', d_2']$-factor of the paths. Since $d_1 \ge 4$, we know that by hypothesis, pathological case (d) does not hold and so $C_p \ge 3$. Hence $d_2' = 1 \le C_p - 2$ and so $d_2' \le \sum_{i=p-\frac{d_1'}{2}+1}^{p}(C_i - 2)$. Claim 2.3.2 yields that $o_p(G) \le d_0' + d_2'$ since $d_1' \le 2p$. Thus, by Theorem 2.1.3, the p paths have a $[d_0', d_1', d_2']$-factor.

Case V: $d_1 = 2$ and $d_2 = (\sum_{i=1}^{m} C_i^\circ) - 1$

Because pathological case (c) does not hold, $C_p > C_1^\circ$. Also, since $d_1 = 2$ and $d_2 = (\sum_{i=1}^{m} C_i^\circ) - 1$ by assumption, we see that $d_0 = (\sum_{i=1}^{p} C_i) - 1$. Thus, $d_0 \ge C_p - 1 \ge C_1^\circ$. Remove all edges from a cycle of size C_1° in the factor and leave all edges in the rest of the cycles. This yields a $[C_1^\circ, 0, \sum_{i=2}^{m} C_i^\circ]$-factor of the cycles. We argue that for $[d_0', d_1', d_2'] = [d_0 - C_1^\circ, 2, d_2 - \sum_{i=2}^{m} C_i^\circ]$, there is a $[d_0', d_1', d_2']$-factor of the paths. Claim 2.3.2 yields that $o_p(G) \le d_0' + d_2'$ since $d_1' \le 2p$. Also, since $d_2 = (\sum_{i=1}^{m} C_i^\circ) - 1$ and since $C_p > C_1^\circ$, we see that $d_2' = d_2 - \sum_{i=2}^{m} C_i^\circ = C_1^\circ - 1 \le C_p - 2$. Then since $d_1' = 2$, it is true that $d_2' \le \sum_{i=p-\frac{d_1'}{2}+1}^{p}(C_i - 2)$. Then by Theorem 2.1.3, the p paths have a $[d_0', d_1', d_2']$-factor.

Case VI: $d_1 = 2, d_2 \le (\sum_{i=1}^{m} C_i^\circ) - 2$

Subcase (a): $d_1 = 2, d_2 = rC_m - 1$ where $0 < r < m$, $C_1^\circ = \cdots = C_m^\circ$

The conditions of this case and the hypothesis that $d_0 + d_1 + d_2 = \sum_{i=1}^{m} C_i^\circ + \sum_{i=1}^{p} C_i$ imply that $d_0 = (m-r)C_m^\circ - 1 + \sum_{i=1}^{p} C_i$. By hypothesis, pathological case (b) does not hold and so $C_p > C_1^\circ = C_m^\circ$. Thus, $d_0 = (m-r)C_m - 1 + \sum_{i=1}^{p} C_i > (m-r)C_m^\circ - 1 + C_m^\circ = (m-r+1)C_m^\circ - 1$ and so $d_0 \ge (m-r+1)C_m^\circ$. As a result, we can remove all edges from $m-r+1$ of the cycles and leave all edges in the other $r-1$ cycles. This yields a $[(m-r+1)C^\circ_m, 0, (r-1)C^\circ_m]$-factor of the cycles. Leave the first C°_m edges remain in a path of order $C_p > C^\circ_m$. Remove all other edges. This yields a $[C_p - C^\circ_m - 1 + \sum_{i=1}^{p-1} C_i, 2, C^\circ_m - 1]$-factor of the paths.

Subcase (b): $d_1 = 2$ and Subcase (a) does not hold.

Since $d_0 + d_1 + d_2 = \sum_{i=1}^m C^\circ_i + \sum_{i=1}^p C_i$ and this case assumes $d_1 + d_2 \leq \sum_{i=1}^m C^\circ_i$, we see that $d_0 \geq \sum_{i=1}^p C_i$. Remove all edges from the paths to yield a $[\sum_{i=1}^p C_i, 0, 0]$ factor of the paths. Let $[\hat{d}_0, \hat{d}_1, \hat{d}_2] = [d_0 - \sum_{i=1}^p C_i, 2, d_2]$. Claim 2.3.2 yields that $o_c(G) \leq \hat{d}_0 + \hat{d}_2$ since $\hat{d}_1 \leq 2m$. Since Subcase (a) does not hold, all hypotheses of Theorem 2.2.2 hold and so the desired $[\hat{d}_0, \hat{d}_1, \hat{d}_2]$-factor of the cycles exists.

Case VII: $2 < d_1 \leq 2p$ and $d_2 \leq (\sum_{i=1}^m C^\circ_i) - 2$

Use Case VI to obtain a $[d_0 + d_1 - 2, 2, d_2]$-factor H of the m cycles and p paths in G. This factor has exactly two degree 1 vertices and so at most one path from G can have degree 1 vertices in H. Thus, the vertices from at least $p-1$ paths from G are degree 0 in H and so are isolated vertices. Thus, there are at least $2p - 2 \geq d_1 - 2 > 0$ isolated vertices in H. Remove $d_1 - 2$ of these isolated vertices and add $\frac{d_1-2}{2}$ paths on 2 vertices to H. The result is a $[d_0, d_1, d_2]$-factor of G.

$\square$

Chapter 3

The k-Edge-Coloring Problem for Graphs with Max Degree 2

In this chapter, we first explore when there exists *some* k-edge-colored DUP, DUC, or in general, graph of max degree at most 2 with a given degree vector sequence. These results extend from previously known results. We then concentrate on a determining when a *fixed* DUP, DUC, or graph with max degree at most 2 can be colored with $k = 2$ colors so as to realize a given degree vector sequence. Finally, we discuss why this same question proves so difficult when $k \geq 3$.

3.1 The k-Edge-Coloring Problem for DUPs and DUCs

In Theorem 1.0.5, Caroll and Isaak characterize when a sequence of $(k \times 1)$ column vectors with non-negative integer entries is the degree vector sequence of some k-edge-colored forest. Also, Alpert et al. provide a different proof of the same result in [10]. Since a DUP is a forest in which every vertex has degree one or two, the characterization of degree matrices of k-edge-colored DUPs follows as a corollary to Theorem 1.0.5. See Definition 1.0.3 and Definition 1.0.4 for the definitions of *sum degree sequence* and *support*.

Corollary 3.1.1. *Let $\mathcal{D}$ be a sequence of $(k \times 1)$ column vectors with non-negative integers. $\mathcal{D}$ is the degree vector sequence of a k-edge-colored DUP if and only if the following conditions hold.*

1. *The sum of the entries in row i of the vectors in $\mathcal{D}$ is even for all i where $1 \leq i \leq k$.*

2. *The sum of the entries in any column vector of $\mathcal{D}$ is at most 2.*

3. *For every subset of colors $\mathcal{I}$ of the colors of $\mathcal{D}$, that is, of $\{1, 2, \ldots, k\}$, the sum degree sequence $D_{\mathcal{I}}$ is not a sequence consisting of only 2's and possibly some 0's.*

Proof. ($\Rightarrow$) For $1 \leq i \leq k$, the sum of row i must be even because row i is the degree sequence of the subgraph of the k-edge-colored DUP induced by edges of color i. Also, the highest degree of any vertex in a DUP is 2 and so each column sum is at most 2 and so hypothesis (2) holds. For any subset of colors $\mathcal{I}$ in a k-edge-colored DUP, consider the subgraph H induced by edges with a color in $\mathcal{I}$. H is a subgraph of a forest and so is a forest. Thus, H cannot have degree sequence $2, \ldots, 2, 0, \ldots, 0$ and hypothesis (3) holds.

($\Leftarrow$) Let $\mathcal{I}$ be any subset of colors from $\{1, 2, \ldots k\}$. Let $D_{\mathcal{I}}$ be the sum degree sequence of $\mathcal{I}$. Because each column sum is at most 2, each entry in $D_{\mathcal{I}}$ is 0, 1, or 2, and also, every entry in the vectors in $\mathcal{D}$ is $0, 1$, or 2. Note that $D_{\mathcal{C}}$ has even sum since each row has even sum. Let $m_{\mathcal{I}}$ be the support of $D_{\mathcal{I}}$. If the sum of the entries in $D_{\mathcal{I}}$ is at least $2m_{\mathcal{I}}$ then because each entry in $D_{\mathcal{I}}$ is $0, 1$, or 2, the sum must be exactly $2m_{\mathcal{I}}$. Then $D_{\mathcal{I}}$ is the sequence $2, \ldots, 2, 0 \ldots, 0$, which contradicts hypothesis (3). Thus, the sum of $D_{\mathcal{I}}$ is even and at most $2m_{\mathcal{I}}$-2 and so $D_{\mathcal{I}}$ is realizable as a forest by Claim 1.0.1. It follows from Theorem 1.0.5 that $\mathcal{D}$ is the degree vector sequence of some k-edge-colored forest G. Since every vertex in G has degree at most 2, G is a k-edge-colored DUP. $\square$

In Theorem 3.1.2, Alpert, et al, characterize when a sequence of $(k \times 1)$ column vectors with non-negative integer entries is the degree vector sequence of some k-edge-colored graph with max degree at most 3. Hence, Theorem 3.1.2 answers the

k-Edge-Coloring Problem for graphs with max degree at most 2.

Theorem 3.1.2 ([10]). *Let $\mathcal{D}$ be a sequence of vectors with non-negative integers in which each of the column sums is at most 3. For a subset of colors $\mathcal{I}$ in $\{1, 2, \ldots, k\}$, let $D_{\mathcal{I}}$ be the sum degree sequence of $\mathcal{I}$. Then $\mathcal{D}$ is the degree vector sequence of a k-edge-colored graph with max degree 3 on n vertices if and only the sum degree sequence $D_{\mathcal{I}}$ is graphic for every $\mathcal{I} \subseteq \{1, 2, \ldots, k\}$.*

Since a DUC is clearly a graph with max degree at most 3, the characterization of degree matrices k-edge-colored DUCs is a corollary of Theorem 3.1.2. However, although a DUP is also a graph with max degree at most 3, the characterization of degree matrices of k-edge-colored DUPs is not an immediate corollary of Theorem 3.1.2. This is because not every realization of a degree sequence of a DUP is a DUP. For example, $1, 2, 2, 2, 1$ is realized by both a path with 5 vertices as well as by a single edge and a triangle. On other hand, *every* realization of the degree sequence of a DUC is in turn a DUC.

Corollary 3.1.3. *Let $\mathcal{D}$ be a sequence of $(k \times 1)$ column vectors with non-negative integers in which each of the column sums is 2. For a subset of colors $\mathcal{I}$ in $\{1, 2, \ldots, k\}$, let $D_{\mathcal{I}}$ be the sum degree sequence of colors in $\mathcal{I}$. Then $\mathcal{D}$ is the degree vector sequence of a k-edge-colored DUC if and only the sequence $D_{\mathcal{I}}$ is graphic for every $\mathcal{I} \subseteq \{1, 2, \ldots, k\}$.*

3.2 The 2-Edge-Coloring Problem for Fixed DUPs

Since determining when *some* DUP exists which can be colored with k colors so as to realize a given degree vector sequence, we now concentrate on determining *which* DUPs can be colored as such. In this section, we consider only when $k = 2$ because the case when $k \geq 3$ is less 'nice' and we discuss why in Section 3.5. In other words, we wish to know whether or not a *fixed* DUP has the desired coloring. We let p correspond to the number of paths in a DUP and we let $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ be the ordered list of path orders. Consider the 2-coloring of the path shown in Figure 3.1. The number above the edge indicates the color of the edge.

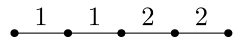


Figure 3.1: A 2-edge-colored path

As Figure 3.1 demonstrates, each vertex in a 2-coloring of a non-trivial path is either an endpoint or an internal vertex and is thus incident to exactly one color 1 edge, exactly one color 2 edge, an edge of each color, or two edges of the same color. Thus, there are five types of degree vectors which can be present in any 2-coloring of a nontrivial path and so in any 2-coloring of a DUP with path orders of at least 2. We now formally define these five types of vertices and vectors.

Definition 3.2.1. *We define type-a_1, type-a_2, type-x_{12}, type-z_1, and type-z_2 vertices and vectors as such:*

1. *A **type-**a_1 **vertex** is an endpoint of a 2-edge-colored path and is adjacent to exactly one color 1 edge ($\bullet\overset{1}{\text{——}}$). Its degree vector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which we define as a **type-**a_1 **vector**.*

2. *A **type-**a_2 **vertex** is an endpoint of a of a 2-edge-colored path and is adjacent to exactly one color 2 edge ($\bullet\overset{2}{\text{——}}$). Its degree vector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which we define as a **type-**a_2 **vector**.*

3. *A **type-**x_{12} **vertex** is internal to an 2-edge-colored path and is incident to an edge of each color ($\overset{1}{\text{——}}\bullet\overset{2}{\text{——}}$). Its degree vector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which we define as a **type-**x_{12} **vector**.*

4. *A **type-**z_1 **vertex** is internal to an 2-edge-colored path and is incident to exactly two color 1 edges ($\overset{1}{\text{——}}\bullet\overset{1}{\text{——}}$). Its degree vector is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ which we define as a **type-**z_1 **vector**.*

5. *A **type-**z_2 **vertex** is internal to an 2-edge-colored path and is incident to*

exactly two color 2 edges ($\overset{2}{\text{—}}\!\bullet\!\overset{2}{\text{—}}$)*. Its degree vector is* $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ *which we define as a* ***type-z_2 vector****.*

Because we define a type-x_{12} vertex to be incident to an edge of color 1 and an edge of color 2, it would be natural to say that a type-x_{11} vertex is adjacent to two edges of color 1 or a type-x_{22} vertex is adjacent to a two edges of color 2. However, in upcoming theorems, we wish to highlight the roles of these different types of vertices, and so we chose to use the terminology *type-z_1* instead of *type-x_{11}* and *type-z_2* instead of *type-x_{22}*.

It follows from Definition 3.2.1 that a sequence of (2×1) column vectors with $a_1, a_2, x_{12}, z_1, z_2$ vectors of type-a_1, type-a_2, type-x_{12}, type-z_1, and type-z_2 vectors, respectively, corresponds to the degree vector sequence of a 2-edge-colored DUP G with $a_1, a_2, x_{12}, z_1, z_2$ vertices of type-a_1, type-a_2, type-x_{12}, type-z_1, or type-z_2, respectively, and vice versa. We introduce more definitions now.

Definition 3.2.2. *If all edges of an edge-colored path P are color i, then P is* ***i-monochromatic****.*

Definition 3.2.3. *A DUP G is* ***$[a_1, a_2, x_{12}, z_1, z_2]$-colorable*** *if there exists a 2-coloring of the edges of G so that there are exactly $a_1, a_2, x_{12}, z_1, z_2$ vertices of type-a_1, type-a_2, type-x_{12}, type-z_1, and type-z_2, respectively. Such a 2-coloring is called an* ***$[a_1, a_2, x_{12}, z_1, z_2]$-coloring*** *of G.*

See Figure 3.3 for an example of a DUP which is $[2, 14, 2, 5, 10]$-colorable and $[11, 5, 11, 4, 2]$-colorable.

Definition 3.2.4. *A 2-edge-colored path P with an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring has the* ***form*** *$[a_1, a_2, x_{12}, z_1, z_2]$.*

Some examples of path forms are shown in 3.2.

Definition 3.2.5. *A* ***segment*** *i* ***subpath*** *in an edge-colored DUP is a maximal subpath whose edges are all color i.*

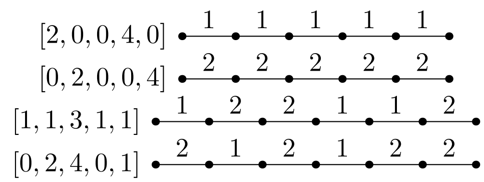


Figure 3.2: Different $[a_1, a_2, x_{12}, z_1, z_2]$ forms and $[a_1, a_2, x_{12}, z_1, z_2]$-colorings of paths

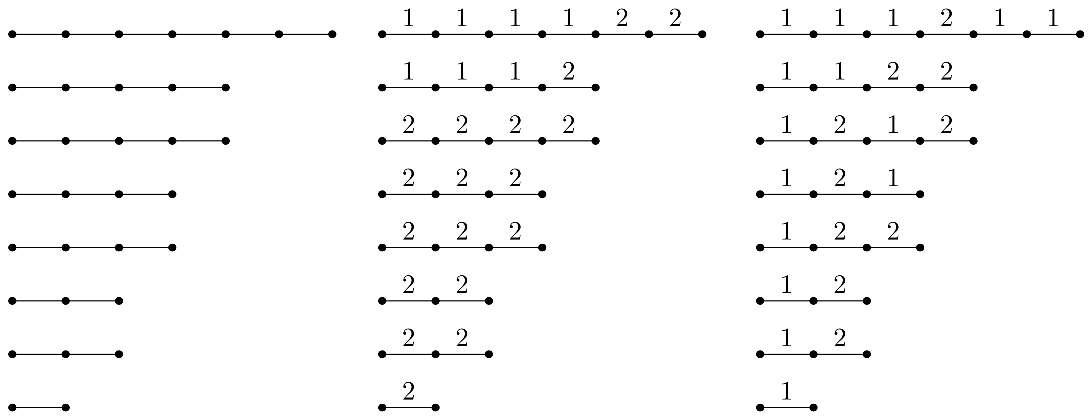
As an example of segment 1 and 2 subpaths, see Figure (3.3b) which has exactly two segment 1 subpaths, one in each of the top two paths, and eight segment 2 subpaths, one in each of the eight paths. Furthermore, Figure (3.3c) has an $[11, 5, 11, 4, 2]$-coloring with $\frac{a_1+x_{12}}{2} = \frac{11+11}{2} = 11$ segment 1 subpaths and $\frac{b+x}{2} = \frac{5+11}{2} = 8$ segment 2 subpaths. We now show that $\frac{a_1+x_{12}}{2}$ and $\frac{a_2+x_{12}}{2}$ are always the number of segment 1 and 2 subpaths, respectively, in any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G.

Claim 3.2.6. *In any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP G, there are $\frac{a_1+x_{12}}{2}$ segment 1 subpaths and $\frac{a_2+x_{12}}{2}$ segment 2 subpaths.*

Proof. The endpoints of a segment 1 subpath are type-a_1 ($\bullet\overset{1}{\text{———}}$) or type-$x_{12}$ ($\overset{1}{\text{——}}\bullet\overset{2}{\text{——}}$) vertices and so there are exactly $a_1 + x_{12}$ endpoints of segment 1 subpaths. Since each segment 1 subpath requires two endpoints, the number of endpoints, $a_1 + x_{12}$, must be twice the number of segment 1 subpaths. Similarly, the endpoints of the segment 2 subpaths are type-a_2 or type-x_{12} vertices and so $a_2 + x_{12}$ must be twice the number of segment 2 subpaths. □

We now present examples that demonstrate basic necessary conditions required for a DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ to be $[a_1, a_2, x_{12}, z_1, z_2]$-colorable. Let G be the DUP with $p = 8$ paths and path orders $2, 3, 3, 4, 4, 5, 5, 7$. See Figure 3.3.

Note that Figure (3.3) shows a $[2, 14, 2, 5, 10]$-coloring of G in which $\frac{a_1+a_2}{2} = \frac{2+14}{2} = 8 = p$ and a $[11, 5, 11, 4, 2]$-coloring in which $\frac{a_1+a_2}{2} = \frac{11+5}{2} = 8 = p$. In any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G, the endpoints are are either type-a_1 or type-a_2 and so there are a total of $a_1 + a_2$ endpoints. Since each path requires two endpoints,



(a) A DUP G **(b)** $[2, 14, 2, 5, 10]$-coloring **(c)** $[11, 5, 11, 4, 2]$-coloring

Figure 3.3: $[a_1, a_2, x_{12}, z_1, z_2]$-colorings in DUPs

the number of paths is precisely half the number of endpoints, that is,

$$p = \frac{a_1 + a_2}{2}. \tag{3.2.1}$$

In Figure 3.3, we also see that a_1, a_2, x_{12} have the same parity in both the $[2, 14, 2, 5, 10]$-coloring and the $[11, 5, 11, 4, 2]$-coloring. By Claim 3.2.6, we see that $a_i + x_{12}$ for $i = 1, 2$ must be twice the number of segment i subpaths in any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G. Thus, both $a_1 + x_{12}$ and $a_2 + x_{12}$ must be even, thus explaining why

$$a_1, a_2, x_{12} \text{ have the same parity in an } [a_1, a_2, x_{12}, z_1, z_2]\text{-coloring of a DUP.} \tag{3.2.2}$$

Finally, in Figure 3.3, we see that any type-z_1 vertex ($\overset{1}{\text{——}}\bullet\overset{1}{\text{——}}$) is internal to a segment 1 subpath which must end in a type-a_1 vertex ($\bullet\overset{1}{\text{——}}$) or a type-$x_{12}$ vertex ($\overset{1}{\text{——}}\bullet\overset{2}{\text{——}}$). Hence, if there are $z_1 > 0$ type-z_2 vertices in some $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP, then $a_1 + x_{12} > 0$ as well. The same is true for a segment 2 subpath and so

$$z_i > 0 \implies a_i + x_{12} > 0. \tag{3.2.3}$$

The basic necessary conditions we just exemplified in (3.2.1)-(3.2.3) are proven in Claim 3.2.8. In the proof of Claim 3.2.8 and others, we rely on results about

$[d_0, d_1, d_2]$-factors from Chapter 2. To convert a 2-coloring of a DUP to a $[d_0, d_1, d_2]$-factor, we can delete either color 1 or color 2 edges from the 2-coloring. For example, deleting the color 2 edges from the $[2, 14, 2, 5, 10]$-coloring in Figure (3.3b) yields the $[24, 4, 5]$-factor of G shown in Figure (2.1b). The degree 1 vertices in this factor are precisely the type-a_1 and type-x_{12} vertices from the $[2, 14, 2, 5, 10]$-coloring. Thus, there are $d_1 = a_1 + x_{12} = 4$ degree 1 vertices in the factor. The degree 0 vertices are precisely the type-a_2 and type-z_2 vertices and so $d_2 = a_2 + z_2 = 24$. Similarly, the degree 2 vertices are the type-z_1 vertices and so $d_2 = z_1 = 5$.

Claim 3.2.7. *The color 1 subgraph of an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP G is a $[d_0, d_1, d_2]$-factor of G where $[d_0, d_1, d_2] = [a_2 + z_2, a_1 + x_{12}, z_1]$. The color 2 subgraph is an $[a_1 + z_1, a_2 + x_{12}, z_2]$-factor of G.*

Proof. Deleting the color 2 edges from an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring G yields the color 1 subgraph H which is a $[d_0, d_1, d_2]$-factor of G. Then each degree 2 vertex in H is incident to two color 1 edges in the $[a_1, a_2, x_{12}, z_1, z_2]$-coloring and so is a type-z_1 vertex. Thus, $d_2 = z_1$. Each degree 1 vertex in H is incident to one color 1 edge in the 2-coloring and so is a type-a_1 or type-x_{12} vertex. Thus, $d_1 = a_1 + x_{12}$. Finally, each degree 0 vertex in H is incident to no color 1 edges and so is a type-a_2 or type-z_2 vertex. Thus, $d_2 = a_2 + z_2$. As a result, H is a $[a_2 + z_2, a_1 + x_{12}, z_1]$-factor. Similarly, deleting the color 1 edges from an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G yields the color 2 subgraph which is an $[a_1 + z_1, a_2 + x_{12}, z_2]$-factor of G. $\square$

Claim 3.2.8. *Let G be a DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$. Let $a_1, a_2, x_{12}, z_1, z_2$ be non-negative integers. If G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable, then the following are true:*

1. $\sum_{i=1}^{p} C_i = a_1 + a_2 + x_{12} + z_1 + z_2 = |V(G)|$
2. $p = \frac{a_1+a_2}{2}$
3. *If $z_1 > 0$, then $a_1 + x_{12} > 0$. If $z_2 > 0$, then $a_2 + x_{12} > 0$.*
4. *a_1, a_2, x_{12} have the same parity.*

Proof. Consider any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G. Such a coloring has exactly $a_1, a_2, x_{12}, z_1, z_2$ vertices of type-a_1, type-a_2, type-x_{12}, type-z_1, and type-z_2, respectively, and no other vertices. Thus, the number of vertices must be $a_1 + a_2 + x_{12} + z_1 + z_2$. Also, the total number of vertices in G is the sum of all path orders, namely, $\sum_{i=1}^{p} C_i$. This shows that $\sum_{i=1}^{p} C_i = a_1 + a_2 + x_{12} + z_1 + z_2 = |V(G)|$.

The endpoints to any path are either type-a_1 or type-a_2. Thus, there are $a_1 + a_2$ endpoints in G. Since each path has two endpoints and there are p paths, we see that $a_1 + a_2 = 2p \implies p = \frac{a_1+a_2}{2}$.

A type-z_1 vertex appears in a segment 1 subpath whose endpoints are either type-a_1 or type-x_{12} vertex. Therefore, if $z_1 > 0$ then $a_1 + x_{12} > 0$. A similar argument shows that if $z_2 > 0$, then $a_2 + x_{12} > 0$.

The color 1 subgraph of the $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G is a $[d_0, d_1, d_2]$-factor of G where $[d_0, d_1, d_2] = [a_2 + z_2, a_1 + x_{12}, z_1]$ by Claim 3.2.7. By Theorem 2.1.3, we thus know that $d_1 = a_1 + x_{12}$ is even. Similarly, the color 2 subgraph of the $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G is a $[a_1 + z_1, a_2 + x_{12}, z_2]$-factor of G by Claim 3.2.7. By Theorem 2.1.3, we thus know that $a_2 + x_{12}$ is even. Since $a_1 + x_{12}$ and $a_2 + x_{12}$ are both even, a_1, a_2, x_{12} must have the same parity.

□

Claim 3.2.9 proves that the equation $\sum_{i=1}^{p}(C_i - 2) = x_{12} + z_1 + z_2$ holds if the basic necessary assumptions from Claim 3.2.8 hold. In a DUP G with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$, since $C_i - 2$ is the number of internal vertices in a path of order C_i, it is sensible that $\sum_{i=1}^{p}(C_i - 2)$ equals the total number of internal vertices in G, that is, $x_{12} + z_1 + z_2$.

Claim 3.2.9. *Consider* $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ *where* C_i *are integers. Let* $\sum_{i=1}^{p} C_i = a_1 + a_2 + x_{12} + z_1 + z_2$ *for non-negative integers* $a_1, a_2, x_{12}, z_1, z_2$ *where* $p = \frac{a_1+a_2}{2}$. *Then* $\sum_{i=1}^{p}(C_i - 2) = x_{12} + z_1 + z_2$.

Proof.

$$\sum_{i=1}^{p}(C_i - 2) = \left(\sum_{i=1}^{p} C_i\right) - 2p = a_1 + a_2 + x_{12} + z_1 + z_2 - 2p = x_{12} + z_1 + z_2$$

□

We now show that the conditions from Claim 3.2.8 are sufficient for a *single* path to be $[a_1, a_2, x_{12}, z_1, z_2]$-colorable . We will see that this is not true in general for DUPs with more than one path.

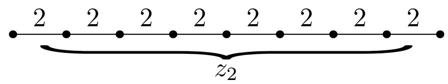
Theorem 3.2.10. *Let P be a path of order $C_1 \geq 2$. Let $a_1, a_2, x_{12}, z_1, z_2$ be non-negative integers. P is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable $\iff$ The following conditions hold:*

1. $C_1 = a_1 + a_2 + x_{12} + z_1 + z_2 = |V(P)|$
2. $a_1 + a_2 = 2$
3. *If $z_1 > 0$, then $a_1 + x_{12} > 0$. If $z_2 > 0$, then $a_2 + x_{12} > 0$.*
4. *a_1, a_2, x_{12} have the same parity.*

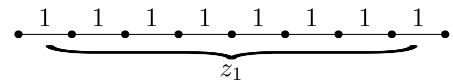
Proof. ($\Rightarrow$) Follows from Claim 3.2.8 when $p = 1$.

($\Leftarrow$) Assume first that $x_{12} = 0$. In this case, we show that the conditions imply that we can color P so that all edges have the same color. This makes sense because $x_{12} = 0$ implies that an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of P has no type-x_{12} vertices and so cannot switch colors and thus must be monochromatic. Since x_{12} is even, the parity condition tells us that a_1 and a_2 are even too. Since a_1 and a_2 sum to 2 by condition (2), we see that either $(a_1 = 0, a_2 = 2)$ or $(a_1 = 2, a_2 = 0)$. If $a_i = 0$ for $i = 1, 2$, then since $x_{12} = 0$ by assumption, condition (3) forces that $z_i = 0$. So if $a_1 = 0$ and $a_2 = 2$, then $z_2 = C_1 - 2$. We color all edges of P with color 2 so that P has the form $[0, 2, 0, 0, z_2]$ as shown in Figure 3.4(a). Similarly, if $a_1 = 2$ and $a_2 = 0$, the conditions imply that we can color all edges with color 1 so that P is 1-monochromatic and has form $[2, 0, 0, z_1, 0]$ as shown in Figure 3.4(b).

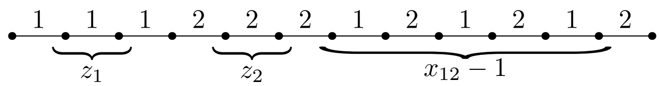
Now assume $x_{12} > 0$. Since a_1 and a_2 sum to 2, we see that $a_1 = a_2 = 1$ or one of a_1 or a_2 is 0 and the other is 2. If $a_1 = a_2 = 1$, then by the parity condition, x_{12} is odd too. Color the first edge with color 1. Continue coloring edges with color 1 until z_1 vertices are type-z_1. Color the next edge with color 2 thus making the



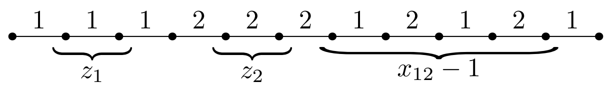
(a) A Path with Form $[0, 2, 0, 0, z_2]$



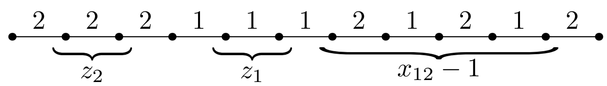
(b) A Path with Form $[2, 0, 0, z_1, 0]$



(c) A Path with Form $[1, 1, x_{12}, z_1, z_2]$ where x_{12} is odd



(d) A Path with Form $[2, 0, x_{12}, z_1, z_2]$ where x_{12} is even



(e) A Path with Form $[0, 2, x_{12}, z_1, z_2]$ where x_{12} is even

Figure 3.4: $[a_1, a_2, x_{12}, z_1, z_2]$-colorings of a path

next vertex a type-x_{12} vertex. Continue coloring the edges with color 2 until type-z_2 vertices are color 2. Alternate the colors of the remaining edges by coloring them as such: 1, 2, 1, 2, etc. This sequence has $x_{12} - 1$ terms which is an even number and so ends in a 2. The resulting coloring of P is a $[1, 1, x_{12}, z_1, z_2]$-coloring as shown in Figure 3.4(c).

If $a_1 = 2$ and $a_2 = 0$, then x_{12} is even too. Begin coloring the path as in the previous case. As before, alternate the colors of the final edges by coloring them as such: 1, 2, 1, 2, etc. Since this sequence has $x_{12} - 1$ terms which is now an odd number, this sequence instead ends in a 1, and the resulting coloring is a $[2, 0, x_{12}, z_1, z_2]$-coloring. See Figure 3.4(d). Finally, if $a_2 = 2$ and $a_1 = 0$, again x_{12} is even as well. Similar to the previous case, we can color the path with a $[0, 2, x_{12}, z_1, z_2]$-coloring, as shown in Figure 3.4(e).

The above arguments show that the only forms that a path can have are as follows: (i) $[0, 2, 0, 0, z_2]$, (ii) $[2, 0, 0, z_1, 0]$, (iii) $[0, 2, x_{12}, z_1, z_2]$ where $x_{12} > 0$ is even, (iv) $[2, 0, x_{12}, z_1, z_2]$ where $x_{12} > 0$ is even, and (v) $[1, 1, x_{12}, z_1, z_2]$ where $x_{12} > 0$ is odd.

□

The details of Theorem 3.2.10 yield the following corollary.

Corollary 3.2.11. *Let P be a path of order $C_1 \geq 2$ where $a_1, a_2, x_{12}, z_1, z_2$ are non-negative intgers and $a_1 + a_2 + x_{12} + z_1 + z_2 = C_1 = |V(P)|$. P is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable $\iff$ $[a_1, a_2, x_{12}, z_1, z_2]$ is one of the following:*

1. $[2, 0, 0, z_1, 0]$
2. $[0, 2, 0, 0, z_2]$
3. $[0, 2, x_{12}, z_1, z_2]$ *and* $x_{12} > 0$ *is even*
4. $[2, 0, x_{12}, z_1, z_2]$ *and* $x_{12} > 0$ *is even*
5. $[1, 1, x_{12}, z_1, z_2]$ *and* $x_{12} > 0$ *is odd*

Claim 3.2.12 highlights an important fact which is subtle in Corollary 3.2.11, that is, the parity of the number of type-x_{12} vertices in a 2-edge-colored path forces whether the first and last edges in the path have the same or different colors. Since a path switches colors at precisely the type-x_{12} vertices, an odd number of type-x_{12} vertices means the path switches colors an odd number of times, thus forcing the starting and ending colors to be opposite. See Figure 3.4(c) for an example. Similarly, each path with an even number of type-x_{12} vertices has the same color on its starting and ending edges, as in Figure 3.4(d)-(e).

Claim 3.2.12. *In any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a path P, x_{12} is odd $\iff$ P has exactly one type-a_1 endpoint and one type-a_2 endpoint. Also, x_{12} is even $\iff$ P has two type-a_1 endpoints or two type-a_2 endpoints.*

Proof. Corollary 3.2.11 lists all possible $[a_1, a_2, x_{12}, z_1, z_2]$-colorings of P. This corollary implies that x_{12} is odd $\iff$ P has an $[1, 1, x_{12}, z_1, z_2]$-coloring $\iff$ P has a exactly one type-a_1 and one type-a_2 endpoint. Equivalently, x_{12} is even $\iff$ P has two endpoints of the same type. □

When a DUP consists of more than just one path, the conditions from Claim 3.2.8 are not sufficient for the DUP to be $[a_1, a_2, x_{12}, z_1, z_2]$-colorable. We give examples of this now.

In some cases, an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP must have a certain number of i-monochromatic paths. Claim 3.2.13 gives us a lower bound for the number of i-monochromatic paths that must exist in any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP. Although the quantities in Claim 3.2.13 are only positive when $x_{12} < a_2$ and $x_{12} < a_1$, the statement is still true if these quantities are non-positive.

Claim 3.2.13. *In any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP, there are at least $\frac{a_2-x_{12}}{2}$ 2-monochromatic paths and $\frac{a_1-x_{12}}{2}$ 1-monochromatic paths.*

Proof. A 1-monochromatic path has no segment 2 subpaths. A 2-monochromatic path has no segment 1 subpaths. Claim 3.2.6 shows that there are $\frac{a_1+x_{12}}{2}$ segment 1 subpaths and $\frac{a_2+x_{12}}{2}$ segment 2 subpaths in any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G. Thus, there at most $\frac{a_1+x_{12}}{2}$ paths with a segment 1 subpath and $\frac{a_2+x_{12}}{2}$ paths with a segment 2 subpath. This implies that in a DUP with $p = \frac{a_1+a_2}{2}$ paths, there are at least $p - \frac{a_1+x_{12}}{2} = \frac{a_1+a_2}{2} - \frac{a_1+x_{12}}{2} = \frac{a_2-x_{12}}{2}$ paths without a segment 1 subpath and $p - \frac{a_2+x_{12}}{2} = \frac{a_1-x_{12}}{2}$ paths without a segment 2 subpath. □

Consider again G from Figure 3.3. We now show G is neither $[14, 2, 4, 5, 8]$-colorable nor $[2, 14, 4, 8, 5]$-colorable. For G to be $[14, 2, 4, 5, 8]$-colorable, there must be $z_1 = 5$ type-z_1 vertices in some coloring. By Claim 3.2.13, there must be at least $\frac{a_1-x_{12}}{2} = 5$ 1-monochromatic paths. The shortest 5 paths in G have a total of 6 internal vertices so if these paths were 1-monochromatic, they would require at least 6 type-z_1 vertices. Since $z_1 = 5$, we see that z_1 is too small to color even the smallest 5 paths 1-monochromatic. Thus, z_1 is too small to color any 5 paths 1-monochromatic. Hence, G is not $[14, 2, 4, 5, 8]$-colorable and this example demonstrates that z_1 must be at least as big as the number of internal vertices in the smallest $\frac{a_1-x_{12}}{2}$ paths. Thus, if G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable, then

$$x_{12} < a_1 \implies \sum_{i=1}^{\frac{a_1-x_{12}}{2}} (C_i - 2) \leq z_1. \tag{3.2.4}$$

A similar argument yields that if $x_{12} < a_2$, then since $\frac{a_2 - x_{12}}{2}$ paths must be 2-monochromatic, z_2 must be at least as big as the number of internal vertices in the shortest $\frac{a_2-x_{12}}{2}$ paths. Thus, if G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable, then

$$x_{12} < a_2 \implies \sum_{i=1}^{\frac{a_2-x_{12}}{2}} (C_i - 2) \leq z. \tag{3.2.5}$$

Then G cannot be $[2, 14, 4, 8, 5]$-colorable because such a coloring violates inequality (3.2.5). Additionally, switching colors 1 and 2 in a $[2, 14, 4, 8, 5]$-coloring would yield a $[14, 2, 4, 5, 8]$-coloring, thus contradicting that G is not $[14, 2, 4, 5, 8]$-colorable.

We now present a necessary a bound on the number of paths of order 3. There are three possible ways to color a path of order 3 with 2 colors. They are shown in Figure 3.5.

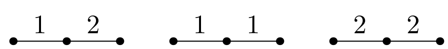


Figure 3.5: 2-colorings of paths of order 3

Consider again the DUP G in Figure 3.3. If G were $[0, 18, 14, 0, 1]$-colorable, then since $a_1 = 0$, we cannot color any order 3 path like the first or second colorings shown in Figure 3.5 both of which require a type-a_1 endpoint. Also, since $z_2 = 1$ only one path of order 3 in G can have a coloring like that of the third coloring shown in Figure 3.5 which requires a type-z_2 vertex. Thus, the given a_1 and z_2 values force that we can only successfully color at most one order 3 path in any $[0, 18, 14, 0, 1]$-coloring of any DUP. However, G has two order 3 paths and so G is not $[0, 18, 14, 0, 1]$-colorable. We have just illustrated that if a DUP is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable, then since the first and second colorings in Figure 3.5 require at least one type-a_1 endpoint and and the third coloring requires at least one type-z_2 vertex, the number of order 3 paths in G is at most $a_1 + z_2$. By symmetry, if a DUP is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable, then the number of order 3 paths in G is at most $a_2 + z_1$. Thus, G is not $[18, 0, 14, 1, 0]$-colorable. In general, it must be true that

$$\text{the number of order 3 paths is at most } \min\{a_1 + z_2, a_2 + z_1\}. \tag{3.2.6}$$

Moreover, we need a bound on all odd paths, not just order 3 paths. Any odd path must have at least one internal vertex. If all the internal vertices of an odd path are type-x_{12} then by Claim 3.2.12 the path requires a type-a_1 and a type-a_2 endpoint. Hence, any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP can have $\min\{a_1, a_2\}$ such odd paths. Any other odd path has a type-z_1 or type-z_2 internal vertex and there are at most $z_1 + z_2$ such odd paths. Hence, in any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP, it must be true that

$$\text{the number of odd paths is at most } \min\{a_1, a_2\} + z_1 + z_2. \tag{3.2.7}$$

For example, the DUP G from Figure 3.3 is not $[1, 15, 19, 1, 2]$-colorable. Since $\min\{a_1, a_2\} + z_1 + z_2 = 4$ but G has 5 odd paths, inequality (3.2.7) fails. By the same reasoning, G is not $[15, 1, 19, 2, 1]$-colorable.

The reader may ask why the order 3 paths are important enough that we specify a special bound for them but we do not specify a bound for any paths of larger order. The reason is that the colorings of a order 3 path are so specific that the the internal vertex actually defines the endpoints. As Figure 3.5 shows, a type-z_i vertex in an order 3 path forces two type-a_i endpoints, and a type-x_{12} vertex forces one type-a_2 and one type-a_1 endpoint. On the other hand, paths of larger order have flexibility between the endpoints and the internal vertex.

For example, consider the $[1, 9, 9, 7, 1]$-coloring of the DUP in Figure 3.6. Since $a_1 + z_2 = 2$, any DUP which is $[1, 9, 9, 7, 1]$-colorable can have at most 2 odd paths by inequality (3.2.6). Even though $z_1 = 7$ is relatively large, the number of order 3 paths must stay small because a type-z_1 vertex in a order 3 path requires additional type-a_1 vertices. On the other hand, paths of larger order can have a type-z_1 vertex without having type-a_1 endpoints as Figure 3.6 demonstrates.

The previous discussion exemplifies why inequalities (3.2.4)-(3.2.7) appear in Theorem 3.2.14. Later, in Theorem 3.2.22, we show that the hypotheses of Theorem 3.2.14 along with a few basic assumptions are sufficient.

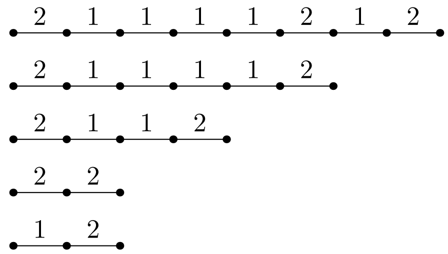


Figure 3.6: A $[1, 9, 9, 7, 1]$-coloring of a DUP with path orders $3, 3, 5, 7, 9$

Theorem 3.2.14. *Let G be a DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$. Let $a_1, a_2, x_{12}, z_1, z_2$ be non-negative integers. Let $o_p(G)$ refer to the number of path orders C_i in G which are odd. Let $t(G)$ refer to the number of path orders C_i in G which are 3. If G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable, then the following conditions hold:*

1. *If $x_{12} < a_1$, then $\sum_{i=1}^{\frac{a_1 - x_{12}}{2}} (C_i - 2) \leq z_1$.*
2. *If $x_{12} < a_2$ then $\sum_{i=1}^{\frac{a_2 - x_{12}}{2}} (C_i - 2) \leq z_2$.*
3. *$t(G) \leq \min\{a_1 + z_2, a_2 + z_1\}$*
4. *$o_p(G) \leq \min\{a_1, a_2\} + z_1 + z_2$*

Proof. By Claim 3.2.13, G has at least $\frac{a_1 - x_{12}}{2}$ 1-monochromatic paths. Since each 1-monochromatic path of order C_i has exactly $C_i - 2$ type-z_1 vertices and since the $\frac{a_1 - x_{12}}{2}$ 1-monochromatic paths can be no smaller than the first $\frac{a_1 - x_{12}}{2}$ path orders, we see that $\sum_{i=1}^{\frac{a_1 - x_{12}}{2}} (C_i - 2) \leq z_1$. A similar argument shows that $\sum_{i=1}^{\frac{a_2 - x_{12}}{2}} (C_i - 2) \leq z_2$.

An order 3 path has exactly one internal vertex. If any order 3 path has two type-a_2 endpoints, then its internal vertex is type-z_2 and so there are at most z_2 such order 3 paths in any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G. Otherwise, a order 3 path has at least one type-a_1 endpoint and there are at most a_1 such order 3 paths. Hence, there are at most $a_1 + z_2$ order 3 paths. A similar argument yields that there are at most $a_2 + z_1$ order 3 paths in any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G. Thus, $t(G) \leq \min\{a_1 + z_2, a_2 + z_1\}$.

Any odd path must have at least one internal vertex. If all the internal vertices of an odd path are type-x_{12} then by Claim (3.2.12) the path requires a type-a_1

and a type-a_2 endpoint. Hence, any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP can have $\min\{a_1, a_2\}$ such odd paths. Any other odd path contains a type-z_1 or type-z_2 internal vertex and there are at most $z_1 + z_2$ such odd paths. Hence, $o_p(G) \leq \min\{a_1, a_2\} + z_1 + z_2$.

□

In many cases, the bounds on the number of odd paths and paths of order 3 cannot fail. Claim 3.2.15 characterizes under what conditions these bounds must hold. This information simplifies upcoming proofs. Note that Claim 3.2.15 relies on Claim 3.2.16 which is proved in the upcoming Section 3.2.1. We did this so that all auxiliary claims are stated and proven within the same section. We remark that we took care to create no circular arguments are made.

Claim 3.2.15. *Let G be a DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$. Let $a_1, a_2, x_{12}, z_1, z_2$ be non-negative integers where $\sum_{i=1}^{p} C_i = a_1 + a_2 + x_{12} + z_1 + z_2 = |V(G)|$ and $p = \frac{a_1+a_2}{2}$. Let $o_p(G)$ refer to the number of path orders C_i in G which are odd. Let $t(G)$ refer to the number of path orders C_i in G which are 3.*

(a) *If $a_1 \leq a_2$ then $t(G) \leq a_2 + z_1$ and $o_p(G) \leq a_2 + z_1 + z_2$.*

(b) *If $a_2 \leq a_1$ then $t(G) \leq a_1 + z_2$ and $o_p(G) \leq a_1 + z_1 + z_2$.*

(c) *If $a_1 = a_2$ then $t(G) \leq \min\{a_1 + z_2, a_2 + z_1\}$ and $o_p(G) \leq \min\{a_1, a_2\} + z_1 + z_2$.*

(d) *Assume $x_{12} \leq \min\{a_1, a_2\}$. If $\sum_{i=1}^{\frac{a_1-x_{12}}{2}} (C_i - 2) \leq z_1$ when $x_{12} < a_1$ and if $\sum_{i=1}^{\frac{a_2-x_{12}}{2}} (C_i - 2) \leq z_2$ when $x_{12} < a_2$, then $t(G) \leq \min\{a_1 + z_2, a_2 + z_1\}$ and $o_p(G) \leq \min\{a_1, a_2\} + z_1 + z_2$.*

Proof. Observe that the number of odd paths and the number of order 3 paths can clearly be no bigger than p, ie, $t(G) \leq o_p(G) \leq p$. Also, Claim 3.2.16 yields the key observation that $p \leq \max\{a_1, a_2\}$. Thus, $a_1 \leq a_2 \implies \max\{a_1, a_2\} = a_2$ and thus $p \leq a_2$. Since a_2 is bigger than p, then $a_2 + z_1$ and $a_2 + z_1 + z_2$ are bigger than $t(G)$ and $o_p(G)$, respectively, as well. This proves statement (a). A similar argument proves statement (b).

If $a_1 = a_2$, then statements (a) and (b) both hold and this implies statement (c).

We now show statement (d) of the claim. If $a_1 = a_2$, then by statement (c), the claim follows, so we assume $a_1 < a_2$ or $a_2 < a_1$.

Assume first that $a_1 < a_2$. By hypothesis, $x_{12} \leq \min\{a_1, a_2\}$ and so we see $x_{12} \leq a_1 < a_2$. Since $a_1 < a_2$, statement (a) implies $t(G) \leq a_2 + z_1$ and $o_p(G) \leq a_2 + z_1 + z_2$. We now show $t(G) \leq a_1 + z_2$ and $o_p(G) \leq a_1 + z_1 + z_2$ to complete the claim.

Since $x_{12} \leq a_1 < a_2$, we see $a_2 > x_{12}$. For each $C_i = 3$, we see $C_i - 2 = 1$. Thus, each order 3 path in the shortest $\frac{a_2 - x_{12}}{2}$ paths contributes exactly one to the sum $\sum_{i=1}^{\frac{a_2-x_{12}}{2}}(C_i - 2)$. Furthermore, the largest $p - \frac{a_2-x_{12}}{2} = \frac{a_1+x_{12}}{2}$ paths can clearly have at most $\frac{a_1+x_{12}}{2}$ order 3 paths. Thus, it is true that $t(G) \leq \sum_{i=1}^{\frac{a_2-x_{12}}{2}}(C_i - 2) + \frac{a_1+x_{12}}{2}$. Since $\sum_{i=1}^{\frac{a_2-x_{12}}{2}}(C_i - 2) \leq z_2$ by assumption, the previous inequality implies that $t(G) \leq z_2 + \frac{a_1+x_{12}}{2}$. Recall $x_{12} \leq a_1$ and so $\frac{a_1+x_{12}}{2} \leq a_1$. Hence, $t(G) \leq z_2 + a_1$.

Now, since each odd C_i, is at least 3, each odd path contributes at least one to the sum $\sum_{i=1}^{p}(C_i - 2)$ and so this sum is an overcount for $o_p(G)$. Therefore, $o_p(G) \leq \sum_{i=1}^{p}(C_i - 2)$. By Claim 3.2.9, $\sum_{i=1}^{p}(C_i - 2) = x_{12} + z_1 + z_2$ which is at most $a_1 + z_1 + z_2$ since $x_{12} \leq a_1$. Thus, $o_p(G) \leq a_1 + z_1 + z_2$.

A similar argument yields the claim when $a_2 < a_1$.

□

3.2.1 Auxiliary Equations and Inequalities

In this section, we include claims which simplify the proofs in Section 3.2.2, where we show that the conditions of Theorem 3.2.14 are sufficient for determining when a DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable. Claim 3.2.16 proves immediate observations about p, a_1, a_2, x_{12} that result from basic assumptions.

Claim 3.2.16. *Let $a_1, a_2, x_{12}, z_1, z_2$ be non-negative integers where $p = \frac{a_1+a_2}{2}$ and a_1, a_2, x_{12} have the same parity. Then the following are all true.*

1. $p \leq \max\{a_1, a_2\} \leq 2p$

2. If $a_1 = a_2$, then $a_1 = a_2 = p$.

3. If $a_1 = p$ or $a_2 = p$, then $a_1 = a_2 = p$.

4. If $a_1 > x_{12}$ and $a_2 > x_{12}$, then $p > x_{12}$.

Proof. We prove each condition separately.

1. $p = \frac{a_1+a_2}{2} \leq \frac{2\max\{a_1,a_2\}}{2} = \max\{a_1, a_2\}$. Also, $2p = a_1 + a_2 = \max\{a_1, a_2\}$.

2. If $a_1 = a_2$, then $p = \frac{a_1+a_2}{2} = \frac{2a_2}{2} = a_1 = a_2$.

3. If $a_1 = p$, $2a_1 = 2p = a_1 + a_2 \implies 2a_1 = a_1 + a_2 \implies a_1 = a_2$. Similarly, $a_2 = p \implies a_1 = a_2 = p$.

4. $a_1 > x_{12}, a_2 > x_{12} \implies p = \frac{a_1+a_2}{2} > \frac{x_{12}+x_{12}}{2} = x_{12}$.

□

Claim 3.2.17 highlights a helpful parity condition that results from basic assumptions.

Claim 3.2.17. *Consider $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ where C_i are integers. Let $o_p(G)$ refer to the number of path orders C_i in G which are odd. Let $a_1, a_2, x_{12}, z_1, z_2$ be non-negative integers where $p = \frac{a_1+a_2}{2}$ and $\sum_{i=1}^{p} C_i = a_1 + a_2 + x_{12} + z_1 + z_2$. Assume a_1, a_2, x_{12} have the same parity. Then the quantities $o_p(G)$, $a_1 + z_1 + z_2$, and $a_2 + z_1 + z_2$ all have the same parity.*

Proof. The parity of the number of internal vertices in a DUP must equal the parity of the number of odd paths in the DUP, that is, $o_p(G)$. The number of internal vertices is $\sum_{i=1}^{p}(C_i - 2)$ which equals $x_{12} + z_1 + z_2$ by Claim 3.2.9. Thus, the quantities $o_p(G)$ and $x_{12} + z_1 + z_2$ must have the same parity. Since a_1, a_2, x_{12} have the same parity by hypothesis, we thus see that the quantities $o_p(G)$, $a_1 + z_1 + z_2$, and $a_2 + z_1 + z_2$ must have the same parity as well. □

Claim 3.2.18 proves that due to basic assumptions, if x_{12} and z_1 are "too small," then z_2 must be "large enough."

Claim 3.2.18. *Consider* $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ *where* C_i *are integers. Let* $a_1, a_2, x_{12}, z_1, z_2$ *be non-negative integers where* $p = \frac{a_1+a_2}{2}$ *and* $\sum_{i=1}^{p} C_i = a_1 + a_2 + x_{12} + z_1 + z_2$*. Assume* a_1, a_2, x_{12} *have the same parity and* $x_{12} < a_1$ *and* $x_{12} < a_2$*. If* $\sum_{i=2}^{\frac{a_1-x_{12}}{2}+1}(C_i - 2) > z_1$*, then* $\sum_{i=2}^{\frac{a_2-x_{12}}{2}+1}(C_i - 2) \leq z_2$*.*

Proof. Since $x_{12} < \min\{a_1, a_2\}$ by hypothesis, Claim 3.2.16 yields that $p > x_{12}$. Also, $p - x_{12} = \frac{a_1-x_{12}}{2} + \frac{a_2-x_{12}}{2}$. Since $x_{12} < a_1$, this implies $p - x_{12} > \frac{a_2-x_{12}}{2}$. Then we can partition the indexing interval $[1, p]$ into the following subintervals: $\left[1, \frac{a_2-x_{12}}{2}\right]$, $\left[\frac{a_2-x_{12}}{2} + 1, p - x_{12}\right]$, $[p - x_{12} + 1, p]$. Thus, we can write the sum of $C_1 - 2$ over the indices $i \in [1, p]$ as

$$\sum_{i=1}^{p}(C_i - 2) = \sum_{i=1}^{\frac{a_2-x_{12}}{2}}(C_i - 2) + \sum_{i=\frac{a_2-x_{12}}{2}+1}^{p-x_{12}}(C_i - 2) + \sum_{i=p-x_{12}+1}^{p}(C_i - 2). \tag{3.2.8}$$

Note that the sum over $[i \in \frac{a_2-x_{12}}{2} + 1, p - x]$ has $p - x_{12} - \frac{a_2-x_{12}}{2} = \frac{a_1-x_{12}}{2}$ terms and the sum over $i \in [p - x_{12} + 1, p]$ has x_{12} terms. We proceed by establishing two helpful facts.

Since the variables $C_i \geq 2$ are ordered by increasing order, if $C_{\frac{a_1-x_{12}}{2}+1} = 2$ then $C_i = 2$ for all $i \leq \frac{a_1-x_{12}}{2} + 1$. In this case, $\sum_{i=2}^{\frac{a_1-x_{12}}{2}+1}(C_i - 2) = 0 \leq z_1$, which contradicts the hypothesis of the claim. Thus, $C_{\frac{a_1-x_{12}}{2}+1} \geq 3$, which implies $C_i \geq 3$ for all $i \geq \frac{a_1-x_{12}}{2} + 1$. Recall $\frac{a_1-x_{12}}{2} < p - x_{12}$ and so $i = p - x_{12} + 1$ is a larger index than $i = \frac{a_1-x_{12}}{2} + 1$. Thus, $C_i \geq 3$ when $i \geq p - x_{12} + 1$. There are x_{12} terms in the sum $\sum_{i=p-x_{12}+1}^{p}(C_i - 2)$ and so $\sum_{i=p-x_{12}+1}^{p}(C_i - 2) - x_{12} = \sum_{i=p-x_{12}+1}^{p}(C_i - 3)$. Each term in this new sum is non-negative since $C_i \geq 3$ when $i \geq p - x_{12} + 1$. Hence, $\sum_{i=p-x_{12}+1}^{p}(C_i - 3)$ is at least as large as its largest term which is $C_p - 3$. Thus,

$$\sum_{i=p-x_{12}+1}^{p}(C_i - 2) - x_{12} \geq C_p - 3. \tag{3.2.9}$$

Recall the sum $\sum_{i=\frac{a_2-x_{12}}{2}+1}^{p-x_{12}}(C_i - 2)$ has $\frac{a_1-x_{12}}{2}$ terms. Since $x_{12} < a_2$, we see the initial index is $i = \frac{a_2-x_{12}}{2} + 1 \geq 2$. Thus, $\sum_{i=\frac{a_2-x_{12}}{2}+1}^{p-x}(C_i - 2)$ must be as large as the sum $\sum_{i=2}^{\frac{a_1-x_{12}}{2}+1}(C_i-2)$ which also has $\frac{a_1-x_{12}}{2}$ terms but which is a sum over lower indices. Hence, $\sum_{i=\frac{a_2-x_{12}}{2}+1}^{p-x_{12}}(C_i-2) \geq \sum_{i=2}^{\frac{a_1-x_{12}}{2}+1}(C_i-2) > z_1$. This implies that

$$\sum_{i=\frac{a_2-x_{12}}{2}+1}^{p-x_{12}}(C_i-2)-1 \geq z_1. \tag{3.2.10}$$

Assume now that $\sum_{i=2}^{\frac{a_2-x_{12}}{2}+1}(C_i-2) > z_2$ and we use these facts to show a contradiction.

$$\begin{aligned}
z_1+z_2 &= \sum_{i=1}^{p}(C_i-2)-x_{12} \text{ by Claim 3.2.9}\\
&= \sum_{i=1}^{\frac{a_2-x_{12}}{2}}(C_i-2)+\sum_{i=\frac{a_2-x_{12}}{2}+1}^{p-x_{12}}(C_i-2)+\sum_{i=p-x_{12}+1}^{p}(C_i-2)-x_{12} \text{ by (3.2.8)}\\
&\geq \sum_{i=1}^{\frac{a_2-x_{12}}{2}}(C_i-2)+\sum_{i=\frac{a_2-x_{12}}{2}+1}^{p-x_{12}}(C_i-2)+C_p-3 \text{ by (3.2.9)}\\
&= \sum_{i=1}^{\frac{a_2-x_{12}}{2}}(C_i-2)+\sum_{i=\frac{a_2-x_{12}}{2}+1}^{p-x_{12}}(C_i-2)+(C_p-2)-1\\
&\geq \sum_{i=1}^{\frac{a_2-x_{12}}{2}+1}(C_i-2)+\sum_{i=\frac{a_2-x_{12}}{2}+1}^{p-x_{12}}(C_i-2)-1 \text{ since } C_p \geq C_{\frac{a_2-x_{12}}{2}+1}\\
&\geq \sum_{i=1}^{\frac{a_2-x_{12}}{2}+1}(C_i-2)+z_1 \text{ by (3.2.10)}\\
&> z_2+z_1 \text{ by hypothesis}
\end{aligned}$$

Thus, we see $z_1+z_2 > z_1+z_2$, a contradiction. Hence, $\sum_{i=2}^{\frac{a_2-x_{12}}{2}+1}(C_i-2) \leq z_2$. ☐

Claim 3.2.19 simplifies induction in the proof of Theorem 3.2.22.

Claim 3.2.19. *Consider* $4 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ *where* C_i *are integers. Assume* $a_1, a_2, x_{12}, z_1, z_2$ *are non-negative integers where* $p = \frac{a_1+a_2}{2}$ *and* $\sum_{i=1}^{p} C_i = a_1 + a_2 + x_{12} + z_1 + z_2$. *Assume* a_1, a_2, x_{12} *have the same parity. Given* $j = 1, \ldots, p$, *let* P_j *be any path of order* C_j *in* G. *Let* $o_p(G - P_j)$ *be the number of odd paths in* $G - P_j$, *which is the number of* C_i *for* $i \neq j$ *which are odd. Then* $z_1 + z_2 \geq C_j - 4 + a_2 - x_{12} + a_1 + o_p(G - P_j)$. *Furthermore, if* $x_{12} < a_2$, *then* $z_1 + z_2 \geq C_j - 2 + o_p(G - P_j)$.

Proof. $p = \frac{a_1+a_2}{2} \implies 2p - a_1 = a_2 \implies$

$$2(p-1) - a_1 = a_2 - 2. \tag{3.2.11}$$

Consider the sum $\sum_{i \neq j}(C_i - 2)$. Since $C_i \geq 4$ for all i, we see $C_i - 2 \geq 2$ when C_i is even and $C_i - 2 \geq 3$ when C_i is odd. Thus, each of the $p - 1$ terms in the sum $\sum_{i \neq j}(C_i - 2)$ contributes at least 2 to the sum plus an additional 1 if C_i is odd. Hence,

$$\sum_{i \neq j}(C_i - 2) \geq 2(p-1) + o_p(G - P_j). \tag{3.2.12}$$

We now show the first of the desired inequalities.

$$\begin{aligned}
&z_1 + z_2 \\
&= \sum_{i=1}^{p}(C_i - 2) - x_{12} \text{ by Claim 3.2.9} \\
&= (C_j - 2) - x_{12} + \sum_{i \neq j}(C_i - 2) \\
&= (C_j - 4) + (a_2 - x_{12}) - (a_2 - 2) + \sum_{i \neq j}(C_i - 2) \\
&\geq (C_j - 4) + (a_2 - x_{12}) - (a_2 - 2) + 2(p-1) + o_p(G - P_j) \text{ by (3.2.11)} \\
&= (C_j - 4) + (a_2 - x_{12}) - (2(p-1) - a_1) + 2(p-1) + o_p(G - P_j) \text{ by (3.2.11)} \\
&= (C_j - 4) + (a_2 - x_{12}) + a_1 + o_p(G - P_j)
\end{aligned}$$

Thus, $z_1 + z_2 \geq (C_j - 4) + (a_2 - x_{12}) + a_1 + o_p(G - P_j)$. Since x_{12} and a_2 have the same parity, if $x_{12} < a_2$, then we see that $x_{12} \leq a_2 - 2$, or equivalently, $a_2 - x_{12} \geq 2$.

Then $z_1 + z_2 \geq (C_j - 4) + (a_2 - x_{12}) + a_1 + o_p(G - P_j) \geq C_j - 2 + a_1 + o_p(G - P_j)$. Since $a_1 \geq 0$, this implies $z_1 + z_2 \geq C_j - 2 + o_p(G - P_j)$, as desired.

□

3.2.2 Proofs of Sufficiency

The main result of this section is Theorem 3.2.22 which proves that the following four conditions are the key drivers for determining when a DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable. The necessity of these conditions are shown in Theorem 3.2.14.

1. If $x_{12} < a_1$, then $\sum_{i=1}^{\frac{a_1 - x_{12}}{2}} (C_i - 2) \leq z_1$.
2. If $x_{12} < a_2$ then $\sum_{i=1}^{\frac{a_2 - x_{12}}{2}} (C_i - 2) \leq z_2$.
3. If $a_1 < a_2$ and $a_1 < x_{12}$, then $t(G) \leq a_1 + z_2$ and $o_p(G) \leq a_1 + z_1 + z_2$.
4. If $a_2 < a_1$ and $a_2 < x_{12}$, then $t(G) \leq a_2 + z_1$ and $o_p(G) \leq a_2 + z_1 + z_2$.

We use Theorem 3.2.20 and Theorem 3.2.21 to prove two small cases of Theorem 3.2.22. Theorem 3.2.20 shows that the first two conditions are sufficient when all paths in a DUP have order 2 or 3. Theorem 3.2.21 shows that portions of the third and fourth conditions are sufficient when $z_1 = z_2 = 0$.

Theorem 3.2.20. *Let G be a DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p \leq 3$. Let $a_1, a_2, x_{12}, z_1, z_2$ be non-negative integers. Let $\sum_{i=1}^{p} C_i = a_1 + a_2 + x_{12} + z_1 + z_2 = |V(G)|$ and $p = \frac{a_1 + a_2}{2}$. Let a_1, a_2, x_{12} have the same parity. Then G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable $\iff$ The following conditions hold:*

1. *If $x_{12} < a_1$, then $\sum_{i=1}^{\frac{a_1 - x_{12}}{2}} (C_i - 2) \leq z_1$.*
2. *If $x_{12} < a_2$ then $\sum_{i=1}^{\frac{a_2 - x_{12}}{2}} (C_i - 2) \leq z_2$.*

Proof. ($\Rightarrow$) Follows from Theorem 3.2.14.

($\Leftarrow$) Since $2 \le C_i \le 3$, we see that $C_i - 2 = 1$ if $C_i = 3$ and $C_i - 2 = 0$ if $C_i = 2$. Thus, the sum $\sum_{i=1}^{p}(C_i - 2)$ counts the number of C_i which are 3 and so $t(G) = \sum_{i=1}^{p}(C_i - 2)$. Furthermore, Claim 3.2.9 yields $\sum_{i=1}^{p}(C_i - 2) = x_{12} + z_1 + z_2$. Thus, the number of order 3 paths in G is $x_{12} + z_1 + z_2$.

We will show that the hypotheses imply that $a_1 - x_{12} - 2z_1 \ge 0$ and $a_2 - x_{12} - 2z_2 \ge 0$. These inequalities imply that we can color the $x_{12} + z_1 + z_2$ order 3 paths and the remaining order 2 paths of G as such. (See Figure 3.7.) Color the $x_{12} + z_1 + z_2$ paths of order 3 so that x_{12} paths have the form $[1, 1, 1, 0, 0]$, z_1 paths have the form $[2, 0, 0, 1, 0]$, and z_2 paths have the form $[0, 2, 0, 0, 1]$. The other $p - (x_{12} + z_1 + z_2) = \frac{a_1 - x_{12} - 2z_1}{2} + \frac{a_2 - x_{12} - 2z_2}{2}$ paths are order 2. We color $\frac{a_1 - x_{12} - 2z_1}{2}$ of these order 2 paths with color 1 and the remaining $\frac{a_2 - x_{12} - 2z_2}{2}$ of the order 2 paths with color 2.

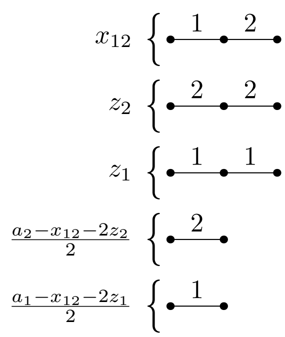


Figure 3.7: 2-coloring paths with order 2 or 3

We now show that $a_1 - x_{12} - 2z_1 \ge 0$ and $a_2 - x_{12} - 2z_2 \ge 0$. We may assume $a_1 \le a_2$ since we can switch colors 1 and 2 if $a_2 < a_1$. Recall that $t(G) = x_{12} + z_1 + z_2$. Since all paths are order 2 or 3, this implies $x_{12} \le p$. But $p \le \max\{a_1, a_2\} = a_2$ by Claim 3.2.16. Hence, $x_{12} \le a_2$. Then either $x_{12} = a_2$ or $x_{12} < a_2$. If $x_{12} = a_2$, then below we show $x_{12} + z_1 \le \frac{a_1 + x_{12}}{2}$, or equivalently, $a_1 - x_{12} - 2z_1 \ge 0$.

$$x_{12} + z_1 \le x_{12} + z_1 + z_2 = \sum_{i=1}^{p}(C_i - 2) \le p = \frac{a_1 + a_2}{2} = \frac{a_1 + x_{12}}{2}$$

If $x_{12} < a_2$, then the hypotheses yield $\sum_{i=1}^{\frac{a_2 - x_{12}}{2}}(C_i - 2) \le z_2$. Note $p - \frac{a_2 - x_{12}}{2} = \frac{a_1 + x_{12}}{2}$, and below we show $x_{12} + z_1 \le \frac{a_1 + x_{12}}{2}$.

$$\begin{aligned} x_{12} + z_1 &= \sum_{i=1}^{p}(C_i - 2) - z_2 \le \sum_{i=1}^{p}(C_i - 2) - \sum_{i=1}^{\frac{a_2 - x_{12}}{2}}(C_i - 2) \\ &= \sum_{\text{largest} \frac{a_1+x_{12}}{2} \text{paths}} (C_i - 2) \le \frac{a_1 + x_{12}}{2} \end{aligned}$$

Thus, $x_{12} + z_1 \le \frac{a_1 + x_{12}}{2}$ and so $a_1 - x_{12} - 2z_1 \ge 0$. This implies $x_{12} \le a_1$. By considering when $x_{12} = a_1$ and $x_{12} < a_1$, we can now apply the same argument to show that $a_2 - x_{12} - 2z_2 \ge 0$. ☐

Theorem 3.2.21. *Let G be a DUP with path orders $2 \le C_1 \le C_2 \le \cdots \le C_p$. Let a_1, a_2, x_{12} be non-negative integers. Let $\sum_{i=1}^{p} C_i = a_1 + a_2 + x_{12} = |V(G)|$ and $p = \frac{a_1+a_2}{2}$. Let a_1, a_2, x_{12} have the same parity. Then G is $[a_1, a_2, x_{12}, 0, 0]$-colorable if and only if the following conditions hold:*

1. *If $a_1 < a_2$ and $a_1 < x_{12}$, then $o_p(G) \le a_1$.*

2. *If $a_2 < a_1$ and $a_2 < x_{12}$, then $o_p(G) \le a_2$.*

Proof. ($\Rightarrow$) Follows from Theorem 3.2.14.
($\Leftarrow$) We assume $a_1 \le a_2$ since if $a_1 > a_2$ we could switch colors 1 and 2 to achieve $a_1 \le a_2$. By Claim 3.2.9, we see that $x_{12} = \sum_{i=1}^{p}(C_1 - 2)$ and so if G is $[a_1, a_2, x_{12}, 0, 0]$-colorable, all internal vertices must be type-x_{12}. Thus, each path must consist edges of alternating colors. We show how to color G. Since $x_{12} = \sum_{i=1}^{p}(C_1 - 2)$ and each odd path order contributes at least one to the this sum, we see that $x_{12} \ge o_p(G)$.

If $a_1 < a_2$ and $a_1 < x_{12}$, then condition (3) yields $o_p(G) \le a_1$. Otherwise, $a_1 = a_2$ or $x_{12} \le a_1$, in which case Claim 3.2.15 yields $o_p(G) \le a_1$. Since $o_p(G) \le a_1$ and $o_p(G) \le x_{12}$, a_1, a_2, x_{12} are large enough to color the $o_p(G)$ odd paths so that each has exactly one type-a_1 vertex, one type-a_2 vertex, and at least one type-x_{12} vertex. To do this, simply color the first edge of each odd path with color 1 and then alternate colors on the following edges, as shown in Figure 3.8. Because the path is odd, Claim 3.2.12 the second endpoint must be type-a_2 .

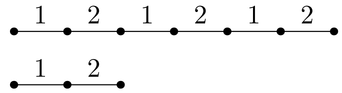


Figure 3.8: Paths of form $[1, 1, x_{12}, 0, 0]$

There are $p - o_p(G)$ even paths that we must color as well. By Claim 3.2.17, $o_p(G)$ and a_1 and a_2 have the same parity. Then since $o_p(G) \leq a_1 \leq a_2$, the quantities $\frac{a_1 - o_p(G)}{2}$ and $\frac{a_2 - o_p(G)}{2}$ are positive integers. Notice that $p - o_p(G) = \frac{a_1 + a_2}{2} - o_p(G) = \frac{a_1 - o_p(G)}{2} + \frac{a_2 - o_p(G)}{2}$. For each of the smallest $\frac{a_1 - o_p(G)}{2}$ even paths, color the first edge with color 1 and then alternate the colors. Again by Claim 3.2.12, this coloring forces each even path to have two type-a_1 endpoints. See Figure 3.9.

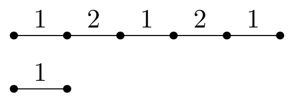


Figure 3.9: Paths of form $[2, 0, x_{12}, 0, 0]$

Finally, color each of the largest $\frac{a_2 - o_p(G)}{2}$ even paths starting with the color 2. Again by Claim 3.2.12, the path must have two type-a_2 endpoints. See Figure 3.10

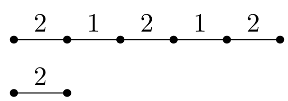


Figure 3.10: Paths of form $[0, 2, x_{12}, 0, 0]$

$\square$

We now prove Theorem 3.2.22, the main result of this section. We point out that the hypotheses of Theorem assume $x_{12} > 0$. If $x_{12} = 0$ in an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP, then all paths are monochromatic and this introduces a level of complexity explained in Section 3.2.3. The assumptions of Theorem 3.2.20 and Theorem 3.2.21 prevent this level of complexity and so these theorems did not include the hypothesis that x_{12} is positive.

Theorem 3.2.22. *Let G be a DUP with path orders $2 \le C_1 \le C_2 \le \cdots \le C_p$. Let $a_1, a_2, x_{12}, z_1, z_2$ be non-negative integers and let $x_{12} > 0$. Let a_1, a_2, x_{12} have the same parity. Let $\sum_{i=1}^{p} C_i = a_1 + a_2 + x_{12} + z_1 + z_2 = |V(G)|$ and $p = \frac{a_1+a_2}{2}$. Let $o_p(G)$ refer to the number of path orders C_i in G which are odd. Let $t(G)$ refer to the number of path orders C_i in G which are 3. Then G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable if and only if the following conditions hold:*

1. *If $x_{12} < a_1$, then $\sum_{i=1}^{\frac{a_1-x_{12}}{2}}(C_i - 2) \le z_1$.*
2. *If $x_{12} < a_2$ then $\sum_{i=1}^{\frac{a_2-x_{12}}{2}}(C_i - 2) \le z_2$.*
3. *$t(G) \le a_1 + z_2$ and $o_p(G) \le a_1 + z_1 + z_2$*
4. *$t(G) \le a_2 + z_1$ and $o_p(G) \le a_2 + z_1 + z_2$*

Proof. ($\Rightarrow$) Follows from Theorem 3.2.14.
($\Leftarrow$) We may assume $a_1 \le a_2$ since we can switch colors 1 and 2 if $a_2 < a_1$. By Claim 3.2.16 $p \le \max\{a_1, a_2\} = a_2$.

We proceed by induction on p. If $p = 1$, G consists of one path and Theorem 3.2.10 yields that G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable. We now assume that if G consists of $p-1$ paths where $p \ge 2$ and the conditions of the theorem hold, then G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable. Since $p \le a_2$ we note that

$$2 \le p \le a_2. \tag{3.2.13}$$

The cases we consider are below.

Case I: $x_{12} < a_1 \le b$

Case II: $a_1 \le x_{12}, a_1 \le b, C_1 = 2$

Case III: $a_1 \le x_{12}, a_1 \le b, C_1 = 3$

Case IV: $a_1 \le x_{12} < a_2, C_1 \ge 4$

Case: V $a_1 \le a_2 \le x_{12}, C_1 \ge 4$

In each case, we adhere to the same strategy, which we summarize now. We remove the shortest path P which has order C_1 from G and call the resulting DUP G'. We let $p' = p-1$ and we shift the indices on C_i so that $C_i' = C_i$ when $i < j$ and $C_i' = C_{i+1}$ for $i > j$. Then G' has path orders $2 \leq C_1' \leq C_2' \leq \cdots \leq C_{p'}'$. In each case, we choose a special $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}]$-coloring of P so that induction implies there exists a $[a_1 - \hat{a_1}, a_2 - \hat{a_2}, x_{12} - \hat{x_{12}}, z_1' = z_1 - \hat{z_1}$, and $z_2' = z_2 - \hat{z_2}]$-coloring of G'.

Thus, $\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}$ must be non-negative integers which sum to C_1. Also, $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}]$ must a form from Corollary 3.2.11 so that Corollary 3.2.11 yields that a path P of order C_1 is $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}]$-colorable. Per Corollary 3.2.11, our choices of $[\hat{a_1}, \hat{a_2}]$ are $[1,1]$, $[0,2]$, or $[2,0]$ and we choose $[\hat{a_1}, \hat{a_2}]$ so that $0 \leq \hat{a_1} \leq a_1$ and $0 \leq \hat{a_2} \leq a_2$. We also choose $\hat{x_{12}}$ so that the parity of $\hat{a_1}, \hat{a_2}$, and $\hat{x_{12}}$ match and so that $0 \leq \hat{x_{12}} < x_{12}$. This forces that $x - \hat{x_{12}} > 0$ which we need for the next step of our strategy. Finally, we choose $\hat{z_1}$ and $\hat{z_2}$ so that $0 \leq \hat{z_1} \leq z_1$ and $0 \leq \hat{z_2} \leq z_2$.

We use primed variables for the leftover amounts, ie, $a_1' = a_1 - \hat{a_1}, a_2' = a_2 - \hat{a_2}, x_{12}' = x - \hat{x_{12}}, z_1' = y - \hat{z_1}$, and $z_2' = z_2 - \hat{z_2}$. These primed variables will all be non-negative integers by choice of the hatted variables. Also, since we choose $\hat{x_{12}}$ so that $\hat{x_{12}} < x_{12}$, we see that $x_{12}' > 0$ which is required by the inductive hypothesis. We also see that $p' = \frac{a_1'+a_2'}{2}$ since $\frac{a_1'+a_2'}{2} = \frac{a_1+a_2}{2} - \frac{\hat{a_1}+\hat{a_2}}{2} = p - 1 = p'$. Furthermore, the parity of a_1, a_2, x_{12} match by hypothesis and we choose $\hat{a_1}, \hat{a_2}, \hat{x_{12}}$ so that their parity matches. These facts force that the parity of a_1', a_2', x_{12}' match as well. Finally, since $\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}$ sum to C_1, we see that

$$a_1'+a_2'+x_{12}'+z_1'+z_2' = a_1+a_2+x_{12}+z_1+z_2-C_1 = \left(\sum_{i=1}^{p} C_i\right) - C_1 = \sum_{i=1}^{p'} C_i' = |V(G')|.$$

In order to apply induction, we must show in each case that our choices for the hatted variables imply that the following conditions hold. We refer to these conditions as the *primed inequalities (PI)*. If we remove the primes, we refer to these conditions as the *original inequalities*.

1. If $x_{12}' < a_1'$, then $\sum_{i=1}^{\frac{a_1'-x_{12}'}{2}} (C_i' - 2) \leq z_1'$.

2. If $x'_{12} < a'_2$ then $\sum_{i=1}^{\frac{a'_2 - x'_{12}}{2}} (C'_i - 2) \leq z'_2$.

3. $t(G') \leq a'_1 + z'_2$ and $o_p(G') \leq a'_1 + z'_1 + z'_2$.

4. $t(G') \leq a'_2 + z'_1$ and $o_p(G') \leq a'_2 + z'_1 + z'_2$.

After proving that the primed inequalities hold, we then apply induction to obtain that G' is $[a'_1, a'_2, x'_{12}, z'_1, z'_2]$-colorable. Then any $[a'_1, a'_2, x'_{12}, z'_1, z'_2]$-coloring of G' and any $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}]$-coloring of P yield an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G, thus showing that G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable.

We apply our strategy now to each of the aforementioned cases.

Case I: $x_{12} < a_1 \leq a_2$

In this case, $x_{12} < a_1$ and $x_{12} < a_2$ and so hypotheses (1) and (2) of the theorem yield that $C_1 - 2 \leq z_1$ and $C_1 - 2 \leq z_2$. Also, $p > x_{12}$ by Claim 3.2.16 so any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G has at least $p - x_{12}$ monochromatic paths. Recall G' is the DUP G with a path P of order C_1 removed. Our goal is to color P so that it is i-monochromatic. Thus, we give P an $[2, 0, 0, C_1 - 2, 0]$-coloring or a $[0, 2, 0, 0, C_1 - 2]$-coloring.

Subcase A: $\sum_{i=2}^{\frac{a_1 - x_{12}}{2}+1} (C_i - 2) \leq z_1$

Let $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}] = [0, 2, 0, 0, C_1 - 2]$. Let $[a'_1, a'_2, x'_{12}, z'_1, z'_2] = [a_1, a_2 - 2, x, z_1, z_2 - (C_1 - 2)]$. Since $a_2 \geq 2$ by inequality (3.2.13) and since $C_1 - 2 \leq z_2$ by condition (2), the primed variables are nonnegative integers. Also $x'_{12} = x_{12} > 0$.

PI (1)	By the assumption of this subcase we see that $\sum_{i=1}^{\frac{a'_1 - x'_{12}}{2}} (C'_i - 2) = \sum_{i=2}^{\frac{a_1 - x_{12}}{2}+1} (C_i - 2) \leq z_1 = z'_1$
PI (2)	The primed inequality is obtained by decreasing both sides of the original inequality by $C_1 - 2$.
PI (3) & PI (4)	Since $x_{12} < a_1 \implies x'_{12} < a'_1$, these conditions follow from Claim 3.2.15(d).

By induction, G' is $[a_1, a_2-2, x_{12}, z_1, z_2-(C_1-2)]$-colorable. Any such coloring of G' together with a $[0, 2, 0, 0, C_1-2]$-coloring of P yields an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G.

Subcase B: $\sum_{i=2}^{\frac{a_1-x_{12}}{2}+1}(C_i-2) > z_1$

Let $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}] = [2, 0, 0, C_1-2, 0]$. Let $[a_1', a_2', x_{12}', z_1', z_2']$ be the values $[a_1-2, a_2, x_{12}, z_1-(C_1-2), z_2]$. Since $0 < x_{12} < a_1$, we know $a_1 \geq 2$. Also, $C_1 - 2 \leq z_1$ by original inequality (1). Then all primed variables are nonnegative integers and also $x_{12}' = x_{12} > 0$.

PI (1)	The primed inequality is obtained by decreasing both sides of the original inequality by C_1-2.
PI (2)	Since $\sum_{i=2}^{\frac{a_1-x_{12}}{2}+1}(C_i-2) > z_1$, Claim 3.2.18 shows us that $\sum_{i=1}^{\frac{a_2'-x_{12}'}{2}}(C_i'-2) = \sum_{i=2}^{\frac{a_2-x_{12}}{2}+1}(C_i-2) \leq z_2 = z_2'$
PI (3) & PI (4)	Since $x_{12} < \min\{a_1, a_2\} \implies x_{12}' \leq \min\{a_1', a_2'\}$, these inequalities follow from Claim 3.2.15(d).

By induction, G' is $[a_1-2, a_2, x_{12}, z_1-(C_1-2), z_2]$-colorable. Any such coloring of G' together with a $[2, 0, 0, C_1-2, 0]$-coloring of P yields an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G.

Case II: $a_1 \leq x_{12}, a_1 \leq a_2, C_1 = 2$

If all paths in G are either order 2 or 3 then it follows from Theorem 3.2.20 that G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable. Hence, we may assume that at least the largest path order C_p is greater than 3. Since $C_1 = 2$ and $C_p > 3$, at most $p-2$ paths have order 3. Thus,

$$t(G) \leq p-2. \tag{3.2.14}$$

G' is the DUP G with a path P of order $C_1 = 2$ removed. Then $o_p(G') = o_p(G)$ and $t(G') = t(G)$. We must color P so that P is a color 1 edge or a color 2 edge. Intuition from original inequality (3) indicates that some odd path may require a type-a_1 vertex so we choose to conserve type-a_1 vertices and to color P as a color 2 edge. Thus, let $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}] = [0, 2, 0, 0, 0]$. Let $[a'_1, a'_2, x'_{12}, z'_1, z'_2] = [a_1, a_2 - 2, x_{12}, z_1, z_2]$. Then since $a_2 \geq 2$ by (3.2.13), all primed variables are nonnegative integers and $x'_{12} = x_{12} > 0$.

PI (1)	Holds vacuously since $a_1 \leq x_{12} \implies a'_1 \leq x'_{12}$
PI (2)	Holds vacuously if $a'_2 \leq x'_{12}$. If $a'_2 < x'_{12}$, holds since $C_1 = 2$: $\sum_{i=1}^{\frac{a'_2 - x'_{12}}{2}} (C'_i - 2) = \sum_{i=2}^{\frac{a_2 - x_{12}}{2}} (C_i - 2) = \sum_{i=1}^{\frac{a_2 - x_{12}}{2}} (C_i - 2) =\leq z_2 = z'_2$
PI (3)	Holds. Both sides of the primed inequality match those of the original inequality since $o_p(G') = o_p(G)$ and $t(G') = t(G)$.

Primed inequality (4) follows from Claim 3.2.15(a) if $a'_1 \leq a'_2$. Otherwise, if $a'_2 < a'_1$, then $a_2 - 2 < a_1 \leq a_2$. Since a_1, a_2 have the same parity, we see that $a_1 = a_2$. Claim 3.2.16 thus yields $a_1 = a_2 = p$. By (3.2.14), $t(G) \leq p - 2$ and so the $t(G')$ inequality holds: $t(G') = t(G) \leq p - 2 = a_2 - 2 = a'_2 \leq a'_2 + z'_1$.

Furthermore, since $C_1 = 2$ is even, we know that $o_p(G) \leq p - 1 = a_2 - 1 \leq a_2 + z_1 + z_2 - 1$. By Claim 3.2.17, the parity of $o_p(G)$ and $a_2 + z_1 + z_2$ must match and so $o_p(G) \leq a_2 + z_1 + z_2 - 1 \implies o_p(G) \leq a_2 + z_1 + z_2 - 2$ and the $o_p(G')$ inequality holds: $o_p(G') = o_p(G) \leq a_2 + z_1 + z_2 - 2 = a'_2 + z'_1 + z'_2$.

Then primed inequality (4) holds. By induction, G' is $[a_1, a_2 - 2, x_{12}, z_1, z_2]$-colorable. Any such coloring of G' together with a $[0, 2, 0, 0, 0]$-coloring of P yields an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G.

Case III: $a_1 \leq x_{12}, a_1 \leq a_2, C_1 = 3$

We let G' be the DUP G with a path P of order $C_1 = 3$ removed. Then $o_p(G') = o_p(G) - 1$ and $t(G') = t(G) - 1$. Any 2-coloring of P has form $[1, 1, 1, 0, 0]$ or $[2, 0, 0, 1, 0]$ or $[0, 2, 0, 0, 1]$. To satisfy primed inequality (3), we may need one type-a_1, type-z_1, and type-z_2 vertex in each odd path if $o_p(G)$ is large. So we conserve these types of vertices and we color P with a $[1, 1, 1, 0, 0]$-coloring (which requires $a_1 > 0$) or a $[0, 2, 0, 0, 1]$-coloring (if $a_1 = 0$). However, if $x_{12} < a_2$, then some path must be 2-monochromatic in any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G so if $x_{12} < a_2$, then our goal is to color P via a $[0, 2, 0, 0, 1]$-coloring.

Subcase A: $a_1 = 0$ or $x_{12} < a_2$

Let $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}] = [0, 2, 0, 0, 1]$. Additionally, we let $[a'_1, a'_2, x'_{12}, z'_1, z'_2] = [a_1, a_2 - 2, x_{12}, z_1, z_2 - 1]$. We must show $a_2 - 2$ and $z_2 - 1$ are non-negative. By (3.2.13), $a_2 \geq 2$. If $x_{12} < a_2$, then original inequality (2) implies $z_2 \geq C_1 - 2 = 1$. Otherwise, $a_1 = 0$. Recall $t(G) \geq 1$ since $C_1 = 3$. Then since $a_1 = 0$, original inequality (3) implies $1 \leq t(G) \leq a_1 + z_2 = z_2$. Then all primed variables are nonnegative integers. Also, $x'_{12} = x_{12} > 0$ by hypothesis.

PI (1)	Holds vacuously since $a_1 \leq x_{12} \implies a'_1 \leq x'_{12}$
PI (2)	Since $C_1 - 2 = 1$, the primed inequality is obtained by decreasing each side of the original inequality by 1: $\sum_{i=1}^{\frac{a'_2 - x'_{12}}{2}} (C'_i - 2) = \sum_{i=1}^{\frac{a_2 - x_{12}}{2}} (C_i - 2) - 1 \leq z_2 - 1 = z'_2$
PI (3)	The primed inequality is obtained by decreasing each side of the original inequality by 1: $t(G') = t(G) - 1 \leq a_1 + z_2 - 1 = a'_1 + z'_2$ $o_p(G') = o_p(G) - 1 \leq a_1 + z_1 + z_2 - 1 = a'_1 + z'_1 + z'_2$
PI (4)	If $a_1 = 0$, then clearly $a'_1 \leq a'_2$. If $x_{12} < a_2$, then the case assumptions imply that $a_1 \leq x_{12} < a_2 \implies a'_1 \leq a'_2$. Hence, PI (4) follows from Claim 3.2.15(a).

By induction, G' is $[a_1, a_2 - 2, x_{12}, z_1, z_2 - 1]$-colorable. Any such coloring of G' together with a $[0, 2, 0, 0, 1]$-coloring of P yields an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G.

Subcase B: $a_1 > 0$ and $x_{12} \geq a_2$

Let $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}] = [1, 1, 1, 0, 0]$. Additionally, let $[a'_1, a'_2, x'_{12}, z'_1, z'_2] = [a_1 - 1, a_2 - 1, x_{12} - 1, z_1, z_2]$. We must show $a_1 - 1, a_2 - 1$, and $x_{12} - 1$ are all non-negative. By (3.2.13) $a_2 \geq 2$. Also, $a_1 > 0$ by assumption. We see $x_{12} \geq 2$ since $x_{12} \geq a_2 \geq 2$. Then the primed variables are all non-negative, and also, $x'_{12} = x_{12} - 1 > 0$.

PI (1) & PI (2)	Hold vacuously since $a_1 \leq a_2 \leq x_{12} \implies a_1' \leq a_2' \leq x_{12}'$.
PI (3)	Each side of the original inequality decreases by 1, thus yielding the primed inequalities: $t(G') = t(G) - 1 \leq a_1 + z_2 - 1 = a_1' + z_2'$ $o_p(G') = o_p(G) - 1 \leq a_1 + z_1 + z_2 - 1 = a_1' + z_1' + z_2'$
PI (4)	Holds for the same reasoning as PI (3).

By induction, G' is $[a_1 - 1, a_2 - 1, x_{12} - 1, z_1, z_2]$-colorable. Any such coloring of G' together with a $[1, 1, 1, 0, 0]$-coloring of P yields an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G.

Case IV: $a_1 \leq x_{12} < a_2, C_1 \geq 4$

G' is the DUP G with a path P of order $C_1 \geq 4$ removed. Since $a_2 > x_{12}$, some path must be 2-monochromatic in any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G and so our goal is to give P a $[0, 2, 0, 0, C_1 - 2]$-coloring.

Let $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}, \hat{z_1}, \hat{z_2}] = [0, 2, 0, 0, C_1 - 2]$. Let $[a_1', a_2', x_{12}', z_1', z_2'] = [a_1, a_2 - 2, x_{12}, z_1, z_2 - (C_1 - 2)]$. We must show $a_2 - 2$ and $z - (C_1 - 2)$ are non-negative. By (3.2.13), $a_2 \geq 2$. Also, since $x_{12} < a_2$, condition (2) implies $z_2 \geq C_1 - 2$. Then the primed variables are all nonnegative integers and $x_{12}' = x_{12} > 0$.

PI (1)	Holds vacuously since $a_1' \leq x_{12}'$.
PI (2)	The primed inequality is obtained by decreasing each side of the original inequality by $(C_1 - 2)$.
PI (4)	Since $a_1 < a_2 \implies a_1' \leq a_2'$, the primed inequality follows from Claim 3.2.15(a).

Claim 3.2.19 implies that $z_1 + z_2 \geq (C_1 - 2) + o_p(G')$. Thus, $a_1' + z_1' + z_2' \geq z_1 + z_2 - (C_1 - 2) \geq o_p(G')$. Also, $t(G') = t(G) = 0$ since $C_1 \geq 4$. Thus, primed inequality (3) holds.

By induction, G' is $[a_1, a_2-2, x_{12}, z_1, z_2-(C_1-2)]$-colorable. Any such coloring of G' together with a $[0, 2, 0, 0, C_1-2]$-coloring of P yields an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G.

Case V: $a_1 \le a_2 \le x_{12}, C_1 \ge 4$

If $a_2 > p$ then we know that some path must have two type-a_2 vertices in any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G. We lose this certainty when $a_2 \le p$ and we so proceed differently in each case.

We assume that $z_1 + z_2 > 0$ as otherwise, Theorem 3.2.21 yields that G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable.

Subcase A: $a_2 > p$

We let G' be the DUP G with a path P of order C_1 removed. Let $\sigma = 1$ if C_1 is odd and 0 if C_1 is even. Our goal is to color P so that P has as few type-z_1 and type-z_2 internal vertices as possible since this strategy helps us to ensure $o_p(G') \le a_1' + z_1' + z_2'$. We accomplish this by coloring P so that:

a. P has two type-a_2 endpoints and an even number of type-x_{12} vertices.

b. P has as many type-x_{12} vertices as possible. Specifically, we want P to have least two type-x_{12} vertices and up to $x_{12} - a_2$ more for a maximum of $x_{12} - a_2 + 2$ type-x_{12} vertices.

c. If P is odd ($\sigma = 1$), then P has at least 1 type-z_1 or type-z_2 vertex.

To satisfy criteria (a) and (c), P can have no more than $C_1 - 2 - \sigma$ type-x_{12} vertices. Let $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}] = [0, 2, \min\{x_{12}-a_2+2, C_1-2-\sigma\}]$. Then $\hat{a_1}, \hat{a_2}$, and $\hat{x_{12}}$ are even. Also, $\hat{x_{12}} > 0$ since $x_{12} \ge a_2$ implies $x_{12} - a_2 + 2 \ge 2$ and since $C_1 \ge 4$ implies $C_1 - 2 - \sigma \ge 2 - \sigma \ge 1$. Let $[a_1', a_2', x_{12}'] = [a_1, a_2 - 2, x_{12} - \hat{x_{12}}]$. Also, $\hat{x_{12}} < x_{12}$ since $a_2 > p \ge 2$ implies that $\hat{x_{12}} = \min\{x_{12} - a_2 + 2, C_1 - 2 - \sigma\} \le x_{12} - a_2 + 2 < x_{12} - p + 2 \le x_{12}$. Then $[a_1', a_2', x_{12}'] = [a_1, a_2 - 2, x_{12} - \hat{x_{12}}]$ are all nonnegative integers and $x_{12}' > 0$ since $\hat{x_{12}} < x_{12}$. With these choices of a_1', a_2', x_{12}', all primed inequalities but primed inequality (3) are immediate.

PI (1) & PI (2)	Holds vacuously since $x'_{12} \geq a'_1, x'_{12} \geq a'_2$
PI (4)	We see that is strictly greater than a_1 since $a_2 > p = \frac{a_1+a_2}{2} \geq \frac{a_1+a_1}{2} = a_1$. Thus, $a_1 < a_2 \implies a'_1 \leq a'_2$. The primed inequality then follows from Claim 3.2.15(a).

We show now how to choose $\hat{z_1}$ and $\hat{z_2}$. Then since $\hat{x_{12}} > 0$ is even, we know P is $[0, 2, \hat{x_{12}}, \hat{z_1}, \hat{z_2}]$-colorable by Corollary 3.2.11. Recall that $C_1 \geq 4$ so as long as z'_2 is non-negative, it is true that $t(G') = 0 \leq a'_1 + z'_2$. We need only show that $o_p(G') \leq a'_1 + z'_1 + z'_2$

Assume first that $C_1 - 2 - \sigma \leq x_{12} - a_2 + 2$. If C_1 is even, then our strategy is to color P with alternating colors. Thus, we let $\hat{z_1} = \hat{z_2} = \sigma = 0$, $z'_1 = z_1$, and $z'_2 = z_2$. Then $o_p(G') = o_p(G) \leq a_1 + z_1 + z_2 = a'_1 + z'_1 + z'_2$. Otherwise, C_1 is odd and so $\sigma = 1$ and we color P with alternating colors and an additional type-z_2 or type-z_1 vertex. Since we assumed that $z_1 + z_2 > 0$ prior to the start of this subcase, we know it is possible to either let $\hat{z_1} = 1$ and $\hat{z_2} = 0$ if z_1 is nonzero or to let $\hat{z_1} = 0$ and $\hat{z_2} = 1$ otherwise. Let $z'_1 = z_1 - \hat{z_1}$ and $z'_2 = z_2 - \hat{z_2}$. Hence, we see that primed inequality (3) hold when $C_1 - 2 - \sigma \leq x_{12} - a_2 + 2$.

Now assume $x_{12} - a_2 + 2 < C_1 - 2 - \sigma$, or equivalently, $C_1 - 4 - \sigma_1 + a_2 - x_{12} > 0$. We show it is possible to choose $\hat{z_1} \in [0, z_1]$ and $\hat{z_2} \in [0, z_2]$ so that $\hat{z_1} + \hat{z_2} = C_1 - (\hat{a_1} + \hat{a_2} + \hat{x_{12}}) = C_1 - 2 - (x_{12} - a_2 + 2) = C_1 - 4 + a_2 - x_{12}$. By Claim 3.2.19, $z_1 + z_2 \geq C_1 - 4 + a_2 - x_{12} + a_1 + o_p(G')$. We can drop a_1 and subtract σ from the right hand side to obtain $z_1 + z_2 \geq C_1 - 4 - \sigma + a_2 - x_{12} + o_p(G')$. We can thus choose non-negative $\hat{z_1}$ and $\hat{z_2}$ such that $\hat{z_1} + \hat{z_2} = C_1 - 4 - \sigma + a_2 - x_{12} > 0$ and $0 \leq \hat{z_1} \leq z_1$ and $0 \leq \hat{z_2} \leq z_1$. Then $z'_1 = z_1 - \hat{z_1}$ and $z'_2 = z_2 - \hat{z_2}$ satisfy $z'_1 + z'_2 = z_1 + z_2 - (C_1 - 4 - \sigma + a_2 - x_{12}) \geq o_p(G')$. Thus, primed inequality (3) holds when $x_{12} - a_2 + 2 < C_1 - 2 - \sigma$ as well.

By induction, G' is $[a_1, a_2 - 2, x - \hat{x_{12}}, z_1 - \hat{z_1}, z_2 - \hat{z_2}]$-colorable. Such a coloring of G' together with a $[0, 2, \hat{x_{12}}, \hat{z_1}, \hat{z_2}]$-coloring of P yields an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G.

Subcase B: $a_2 \leq p$

This subcase assumes $a_2 \leq p$ but Claim 3.2.16 yields $p \leq \min\{a_1, a_2\} = a_2$. Thus, $a_2 = p$, in which case, Claim 3.2.16 yields $a_1 = a_2 = p$. Then by inequality (3.2.13), $a_1 = a_2 = p \geq 2$.

We proceed similarly to the previous subcase. We let G' be the DUP G with a path P of order C_1 removed. Let $\tau = 1$ if C_1 is even and 0 if C_1 is odd. For this case, we wish to color P so that:

a. P has one type-a_1 and type-a_2 endpoint and an odd number of type-x_{12} vertices.

b. P has as many type-x_{12} vertices as possible. Specifically, P we want P to have least one type-x_{12} vertices and up to $x_{12} - a_2$ more for a maximum of $x_{12} - a_2 + 1$ type-x_{12} vertices.

c. If P is even ($\tau = 1$), then P has at least one type-z_1 or type-z_2 vertex.

To satisfy criteria (a) and (c), P can have no more than $C_1 - 2 - \tau$ type-x_{12} vertices. Let $[\hat{a_1}, \hat{a_2}, \hat{x_{12}}] = [1, 1, \min\{x_{12} - a_2 + 1, C_1 - 2 - \tau\}]$. Then $\hat{a_1}, \hat{a_2}$, and $\hat{x_{12}}$ are odd. We show now that $\hat{x_{12}} < x_{12}$.

Since $a_2 \geq 2$ by (3.2.13), we see that $\hat{x_{12}} = \min\{x_{12} - a_2 + 1, C_1 - 2 - \tau\} \leq x_{12} - a_2 + 1 \leq x_{12} - 1 < x_{12}$. Then $[a_1', a_2', x_{12}'] = [a_1 - 1, a_2 - 1, x_{12} - \hat{x_{12}}]$ are all nonnegative integers since $a_1 = a_2 \geq 2$, and also, $x_{12}' > 0$ since $\hat{x_{12}} < x_{12}$. We now show how to choose $\hat{z_1}$ and $\hat{z_2}$.

Assume $C_1 - 2 - \tau \leq x_{12} - a_2 + 1$. If C_1 is odd, then $\tau = 0$ and we set $\hat{z_1} = \hat{z_2} = 0$. If C_1 is even, then $\tau = 1$. Recall that prior to the subcases of Case V, we assumed that $z_1 + z_2 > 0$. Thus, it is possible to set $\hat{z_1} = 1$ and $\hat{z_2} = 0$ if z_1 is nonzero or to set $\hat{z_1} = 0$ and $\hat{z_2} = 1$ otherwise. As usual, let $z_1' = z_1 - \hat{z_1}$ and $z_2' = z_2 - \hat{z_2}$.

If $x_{12} - a_2 + 1 < C_1 - 2 - \tau$, or equivalently, if $C_1 - 4 - \tau + a_2 - x_{12} \geq 0$, then we show it is possible to choose $\hat{z_1} \in [0, z_1]$ and $\hat{z_2} \in [0, z_2]$ so that $\hat{z_1} + \hat{z_2} = C_1 - (\hat{a_1} + \hat{a_2} + \hat{x_{12}}) = C_1 - (x_{12} - a_2 + 3)$. As in the previous subcase, Claim 3.2.19 yields that $z_1 + z_2 \geq C_1 - 4 - \tau + a_2 - x_{12} \geq 0$. Hence, we can choose non-negative $\hat{z_1}$ and $\hat{z_2}$ such that $\hat{z_1} + \hat{z_2} = (C_1 - 4 - \tau) - (x_{12} - a_2)$. Again let $z_1' = z_1 - \hat{z_1}$ and $z_2' = z_2 - \hat{z_2}$. Then all conditions hold.

PI (1)&(2)	Hold vacuously since $a_1' \leq x_{12}', a_2' \leq x_{12}'$
PI (3)&(4)	Hold since $t(G') = 0$ and $o_p(G') \leq p - 1 = a_1' = a_2'$

By induction, G' is $[a_1 - 1, a_2 - 1, x_{12} - \hat{x_{12}}, z_1 - \hat{z_1}, z_2 - \hat{z_2}]$-colorable. Any such coloring of G' together with a $[1, 1, \hat{x_{12}}, \hat{z_1}, \hat{z_2}]$-coloring of P yields an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G.

□

3.2.3 NP-Completeness of $x_{12} = 0$

In this Section, we concentrate on $[a_1, a_2, x_{12}, z_1, z_2]$-colorings of DUPs when $x_{12} = 0$. Consider the following question.

Problem 3.2.1. *Let a_1, a_2, z_1, z_2 be non-negative integers where a_1 and a_2 are even and let $p = \frac{a_1 + a_2}{2}$. Given a DUP G with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ which sum to $a_1 + a_2 + z_1 + z_2$, is G $[a_1, a_2, 0, z_1, z_2]$-colorable?*

Let $n = |V(G)|$. The input is the set of integers $\{C_1, C_2, \cdots, C_p, a_1, a_2, z_1, z_2\}$. Each integer in this set is no bigger than n and so each requires at most $\log n$ bits. The size of the set is also no bigger than $p + 4 \approx p$ and so we assume the input size of the problem is $p \log n$. We call Problem 3.2.1 a *decision problem* because it elicits a 'yes' or 'no' response.

Consider any $[a_1, a_2, 0, z_1, z_2]$-coloring of a DUP. Since there are no type-x_{12} vertices, the edges of each path do not switch from color 1 to color 2 or vice versa and so each path is monochromatic. For example, let G be a DUP with path orders

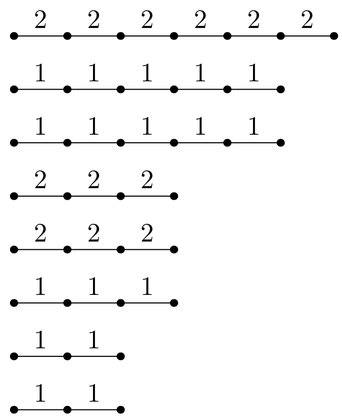


Figure 3.11: $[10, 6, 0, 12, 9]$-coloring of a DUP

$3, 3, 4, 4, 4, 6, 6, 7$. As shown in Figure 3.11, G is $[10, 6, 0, 12, 9]$-colorable and has only monochromatic paths.

We say that the *internal vertex count* of a path of order C_i is the number of internal vertices in a path, $C_i - 2$. In Figure 3.11 there are $\frac{a_1}{2} = \frac{10}{2} = 5$ paths which are 1-monochromatic and whose internal vertex counts sum to $z_1 = 8$. Also, there are $\frac{a_2}{2} = \frac{6}{2} = 3$ paths which are 2-monochromatic and whose internal vertex counts sum to $z_2 = 9$. As shown by Claim 3.2.23, determining whether or not a DUP is $[a_1, a_2, 0, z_1, z_2]$-colorable reduces to determining whether or not there exists $\frac{a_1}{2}$ path orders whose internal vertex count sums to z_1.

Claim 3.2.23. *Let a_1, a_2, z_1, z_2 be non-negative integers where a_1 and a_2 are even and let $p = \frac{a_1+a_2}{2}$. A DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ whose sum is $a_1 + a_2 + z_1 + z_2$ is $[a_1, a_2, 0, z_1, z_2]$-colorable if and only if there is a subset $\mathcal{A}$ of the internal vertex counts, i.e., of $\{C_i - 2\}_{i=1}^{i=p}$, where $|\mathcal{A}| = \frac{a_1}{2}$ and the sum of the elements in $\mathcal{A}$ is exactly z_1. [The other $\frac{a_2}{2}$ internal vertex counts are then forced to sum to exactly z_2.]*

Proof. Each 1-monochromatic path has exactly two type-a_1 vertices as endpoints and each internal vertex in a 1-monochromatic path is a type-z_1 vertex. Thus, we see that any $[a_1, a_2, 0, z_1, z_2]$-coloring of a DUP must have $\frac{a_1}{2}$ 1-monochromatic paths whose internal vertex counts sum to z_1. Similarly, any $[a_1, a_2, 0, z_1, z_2]$-coloring must have $\frac{a_2}{2}$ 2-monochromatic paths whose internal vertex counts sum to z_2. $\square$

Claim 3.2.23 gives us a polynomial time algorithm to check the correctness of a solution to Problem 3.2.1. Simply verify that the number of 1-monochromatic paths in a given 2-coloring of a DUP is $\frac{a_1}{2}$ and that these paths have internal vertex counts which sum to z_1.

Claim 3.2.23 is reminiscent of the NP-Complete Subset Sum Problem. See Problem 2.2.1 for details of the Subset Sum Problem. In Theorem 3.2.25, we show that Problem 3.2.1 reduces from the Subset Sum Problem, which thus shows that Problem 3.2.1 is NP-Complete. First we give an example.

Given the set $\mathcal{C} = \{1, 1, 2, 2, 2, 4, 4, 5\}$, we wish to know if some subset $\mathcal{A} \subset \mathcal{C}$ sums to $s = 12$. Notice that the entries of $\mathcal{C}$ sum to $t = 21$, so if $\mathcal{A}$ exists, the elements that are not in $\mathcal{A}$ sum to $t - s = 9$. We convert this question to an instance of Problem 3.2.1 by viewing $\mathcal{C}$ as a set of internal vertex counts in a DUP G. Then G has $p = |\mathcal{C}| = 8$ paths with orders $\{3, 3, 4, 4, 4, 6, 6, 7\}$. If there is some even $a_1 \in [0, 2p]$ such that G is $[a_1, 2p - a_1, 0, 12, 9]$-colorable, then by Claim 3.2.23, there is a subset $\mathcal{A}$ of $\mathcal{C}$ with $\frac{a_1}{2}$ elements whose sum is $s = 12$. If $a_1 = 10$ and $a_2 = 6$, then Figure 3.11 shows that G is indeed $[a_1, a_2, 0, 12, 9]$-colorable. We see that $\mathcal{A} = \{1, 1, 2, 4, 4\}$ is the set of internal vertex counts of the 1-monochromatic paths and $\mathcal{A}$ does indeed have sum 12. So by asking if there is some even $a_1 \in [0, 2p]$ such that G is $[a_1, 2p - a_1, 0, s, t - s]$-colorable, we can answer whether or not a subset $\mathcal{A}$ of $\mathcal{C}$ sums to s. The previous example helps explain Claim 3.2.24.

Claim 3.2.24. *Given a finite multiset of positive integers $\mathcal{C} = \{C_1, C_2, \cdots, C_p\}$ which sum to t and a positive integer s, there exists a subset $\mathcal{A}$ of $\mathcal{C}$ with sum s if and only if there exists even $a_1 \in [0, 2p]$ such that a DUP with path orders $\{C_1 + 2, C_2 + 2, \cdots, C_p + 2\}$ is $[a_1, a_2, 0, z_1, z_2]$-colorable, where $a_2 = 2p - a_1$, $z_1 = s$, and $z_2 = t - s$.*

Proof. ($\Rightarrow$) Let $\mathcal{A}$ be a subset of $\mathcal{C}$ with sum s. For each C_i in $\mathcal{A}$, create a path of order $C_i + 2$ and color its edges with color 1. For each $C_i \notin \mathcal{A}$, create a path of order $C_i + 2$ and color its edges with color 2. This process yields $|\mathcal{A}|$ 1-monochromatic paths whose internal vertices sum to s and $p - |\mathcal{A}|$ 2-monochromatic paths whose internal vertices sum to $t-s$. This is precisely an $[2|A|, 2p-2|A|, 0, s, t-s]$-coloring of a DUP with path orders $\{C_1+2, C_2+2, \cdots, C_p+2\}$.

($\Leftarrow$) Consider an $[a_1, a_2, 0, z_1, z_2]$-coloring of a DUP with path orders $\{C_1+2, C_2+2, \cdots, C_p+2\}$ where $a \in [0, 2p]$ and $a_2 = 2p - a_1$, $z_1 = s$, and $z_2 = t - s$. Notice that if a path of order $C_i + 2$ is 1-monochromatic, then it has C_i internal vertices. Also, the internal vertices of all 1-monochromatic paths are type-z_1 and so sum to $z_1 = s$. Let $\mathcal{A}$ be the multiset consisting of internal vertex counts of each 1-monochromatic paths. Then $\mathcal{A} \subset \mathcal{C}$ and the sum of the elements in $\mathcal{A}$ is $z_1 = s$.

□

Theorem 3.2.25 proves that Problem 3.2.1 is NP-Complete by showing that p iterations of Problem 3.2.1 yields a polynomial time solution to Subset Sum.

Theorem 3.2.25. *Problem 3.2.1 is NP-Complete.*

Proof. Assume there is an algorithm to solve Problem 3.2.1 which is polynomial in the size of its input, $p \log n$ where $n = |V(G)|$. Thus, we can solve Problem 3.2.1 in $O[f(p \log n)]$ time where $f(x)$ is a polynomial in x.

Consider an instance of the Subset Sum Problem. Let $\mathcal{C} = \{C_1, C_2, \cdots, C_p\}$ be a finite multiset of positive integers which sum to t. Let s be a positive integer. If $s > t$, the answer to the Subset Sum Problem is no. Otherwise, by Claim 3.2.24, there exists a subset $\mathcal{A}$ of $\mathcal{C}$ with sum s if and only if there exists even $a_1 \in [0, 2p]$ such that a DUP with p path orders $\{C_1+2, C_2+2, \cdots, C_p+2\}$ is $[a_1, a_2, 0, z_1, z_2]$-colorable, where $a_2 = 2p - a_1, z_1 = s$, and $z_2 = t - s$.

Since $a_1 \in [0, 2p]$ and a_1 is even, there are at most p even integers to iterate through to determine if a DUP G with path orders $\{C_1+2, C_2+2, \cdots, C_p+2\}$ is $[a_1, 2p-a_1, 0, s, t-s]$-colorable. Then each iteration is solvable in $O[f(p \log n)]$ time and so p iterations are solvable in $O[pf(p \log n)]$ time which is still polynomial in $p \log n$. Thus, the Subset Sum Problem is polynomial in $p \log n$ time, a contradiction. (To see the details of why Subset Sum is ploynomial with this input, see [9]). Thus, Problem 3.2.1 is NP-Complete, as shown by our reduction from Subset Sum. □

3.2.4 Algorithm for 2-coloring a fixed DUP G

We now know that there exists an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP G if the hypotheses of Theorem 3.2.22 hold. Furthermore, when these hypotheses hold, the inductive proof of Theorem 3.2.22 yields an algorithm for how to color a DUP G with an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring. It is easy to check the correctness of this algorithm by comparing its details of the proof of Theorem 3.2.22 .

ALGORITHM TO FIND AN $[a_1, a_2, x_{12}, z_1, z_2]$-COLORING OF A DUP
INPUT: a_1, a_2, x_{12}, z_1, z_2, $C[]$

- $a_1, a_2, x_{12}, z_1, z_2$ are non-negative integers.
- $C[]$ is a non-empty array of p integers of order at least 2 indexed 1 through p.
- $C[1], \ldots, C[p]$ are ordered smallest to highest.
- Hypotheses of Theorem 3.2.22 hold for a DUP with path orders $C[1], \ldots, C[p]$.

OUTPUT: An $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of a DUP with path orders $C[1], \ldots, C[p]$.
BEGIN ALGORITHM

Create an array $a[]$ where $a[1] = a_1$ and $a[2] = a_2$.

Create an array $z[]$ where $z[1] = z_1$ and $z[2] = z_2$.

Create an array $\hat{z}[]$ where $\hat{z}[1] = \hat{z}[2] = 0$.

Create integers $\hat{x_{12}}, m, M$.

Create a pointer P to a path.

Create a pointer G to an empty graph.

WHILE ($C[]$ is not empty)

Set $\hat{x_{12}} = \hat{z}[1] = \hat{z}[2] = 0$.

Set P =null.

IF $a[1] \leq a[2]$

Set $m = 1$ and $M = 2$.

ELSE

Set $m = 2$ and $M = 1$.

IF

* $\sum_{i=2}^{\frac{a_m - x_{12}}{2}+1} (C[i] - 2) \leq z[m], x_{12} < a[m]$ (Case IA) OR
* $a[m] \leq x_{12}, C[1] = 2, C[p] > 3$ (Case II) OR
* $a[m] \leq x_{12}, C[1] = 2, C[p] \leq 3, a[M] - x_{12} - 2z[M] > 0$ (Case II) OR
* $a[m] = 0, C[1] = 3$ (Case IIIA) OR
* $a[m] \leq x_{12} < a[M], C[1] \geq 3$ (Cases IIIA and IV)

THEN

Create a path of order $C[1]$ and set P to be this path.

Color all edges of P with color M.

Set $\hat{z}[M] = C_1 - 2$.

ELSE IF

* $\sum_{i=2}^{\frac{a_m - x_{12}}{2}+1} (C[i] - 2) > z[m], x_{12} < a_m$ (Case IB) OR
* $a[m] \leq x_{12}, C[1] = 2, C[p] \leq 3, a[M] - x_{12} - 2z[M] = 0$ (Case II)

THEN

Create a path of order $C[1]$ and set P to be this path.

Color all edges of P with color m.

Set $\hat{z}[m] = C_1 - 2$.

ELSE IF

* $a[m] \leq a[M] \leq x_{12}, C[1] = 3, a[m] > 0$ (Case IIIB) OR
* $a[m] \leq a[M] \leq x_{12}, C[1] \geq 4, z[1] = z[2] = 0$ (Case V)

THEN

Create a path of order $C[1]$ and set P to be this path.

Color the edges of P as such: $M, m, M, m, \ldots$.

Set $\hat{x_{12}} = C_1 - 2$.

ELSE:

{

IF $(p < a_M)$ (Case VA)

Let $\sigma = 0$ if $C[1]$ is even. Let $\sigma = 1$ otherwise.

Let $\hat{x_{12}} = \min\{x_{12} - a[M] + 2, C[1] - 2 - \sigma\}$.

ELSE (CASE VB)

Let $\sigma = 1$ if $C[1]$ is even. Let $\sigma = 0$ otherwise.

Let $\hat{x_{12}} = \min\{x_{12} - a[M] + 1, C[1] - 2 - \sigma\}$.

Create a path P' with $\hat{x_{12}} + 2$ vertices.

Color the edges of P' as such: $M, m, M, m, \ldots$.

WHILE (P' has less than $C[1]$ vertices and $\hat{z_1} < z[1]$)

Subdivide a color 1 edge of P'.

Increment $\hat{z_1}$ by 1.

WHILE (P' has less than $C[1]$ vertices and $\hat{z_2} < z[2]$)

Subdivide a color 2 edge of P'.

Increment $\hat{z_2}$ by 1.

Set $P = P'$.

}

IF the first and last edges of P are both colored m

THEN Set $a[m] = a[m] - 2$.

ELSE IF the first and last edges of P are both colored M

THEN Set $a[M] = a[M] - 2$.

ELSE:

Set $a[m] = a[m] - 1$ and set $a[M] = a[M] - 1$.

Delete $C[1]$ from $C[]$.

Set $G = G \cup P$.

Set $x_{12} = x_{12} - \hat{x_{12}}$.

Set $z[1] = z[1] - \hat{z_1}$.

Set $z[2] = z[2] - \hat{z_2}$.

Set $p = p - 1$.

END ALGORITHM

3.3 The 2-Edge-Coloring Problem for Fixed DUCs

We let m correspond to the number of cycles in a DUC. Additionally, we let $3 \leq C_1^\circ \leq C_2^\circ \leq \cdots \leq C_m^\circ$ be the ordered list of cycle sizes. Consider the 2-coloring of the DUC shown in Figure 3.12. There are only three types of degree vectors and vertices in a 2-coloring of a cycle, namely, type-x_{12}, type-z_1, and type-z_2 vertices as defined in Definition 3.2.1.

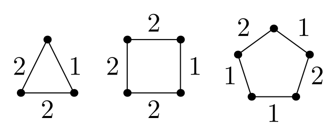


Figure 3.12: A DUC with 8 type-x_{12}, 1 type-z_1, and 3 type-z_2 vertices

Given non-negative integers x_{12}, z_1, z_2, Claim 3.3.1 gives necessary and sufficient conditions for a *single* cycle to be colored so that the cycle has x_{12}, z_1, z_2 vertices of type-x_{12}, type-z_1, and type-z_2, respectively.

Claim 3.3.1. *Let x_{12}, z_1, z_2 be non-negative integers where $C^\circ = x_{12} + z_1 + z_2$ for some integer $C^\circ \geq 3$. There exists a 2-coloring of a cycle with size C° with x_{12}, z_1, z_2 vertices of type-x_{12}, type-z_1, type-z_2, respectively, if and only if the following conditions hold.*

1. x_{12} *is even*

2. *If* $x_{12} = 0$ *then either* $z_1 = C$ *or* $z_2 = C$.

Proof. ($\Rightarrow$) If $x_{12} = 0$, then the 2-edge-colored cycle is monochromatic. Thus, $z_1 = C$ or $z_2 = C$. Moreover, since the colors of the edges must switch an even number of times in a 2-edge-colored cycle, the number of type-x_{12} vertices must be even. Hence, x_{12} is even.

($\Leftarrow$) If $x_{12} = 0$ and $z_1 = C^\circ$, then color all edges with color 1. If $x_{12} = 0$ and $z_2 = C^\circ$, then color all edges with color 2. Otherwise, $x_{12} > 0$ and so $x_{12} \geq 2$ since x_{12} is even. Then $x - 1 \geq 1$ and $x - 1$ is odd. Consider a path P with $C^\circ + 1$ vertices. By Corollary 3.2.11, we can color a path P so that it has the form $[1, 1, x - 1, z_1, z_2]$. Glue the endpoints of P together to obtain the desired 2-coloring of a cycle with C° vertices. $\square$

In Claim 3.3.2, we show that when $x_{12} = 2$, coloring a DUC yields a pathological case. Consider a DUC G with three cycles of sizes $C_1^\circ = C_2^\circ = C_3^\circ = 5$. We explain why we cannot color G so that G has $x_{12} = 2$ vertices of type-x_{12}, $z_1 = 4$ vertices of type-z_1, and $z_1 = 9$ vertices of type-z_2. Since $x_{12} = 2$ and type-x_{12} vertices appear in a 2-coloring of a cycle, exactly one cycle can have both color 1 and color 2 edges. Also, because $z_1 = C_m^\circ - 1 = 4$, z_1 is just small enough that no cycle can be 1-monochromatic. Therefore, some cycle must contain all of the type-z_1 vertices as well as the two type-x_{12} vertices. However, because $x_{12} + z_1 = 6$, the cycle size C_m° is just small enough where this is impossible. In general, this situation occurs when $x_{12} = 2$, all cycles have the same size, and $z_1 = rC_m^\circ - 1$ for some positive integer r, which forces that $z_2 = (m - r)C_m^\circ - 1$.

Claim 3.3.2. *Let* G *be a DUC with cycle sizes* $3 \leq C_1^\circ \leq C_2^\circ \leq \cdots \leq C_m^\circ$. *Let* x_{12}, z_1, z_2 *be non-negative integers where* $x_{12} = 2$ *and where* $\sum_{i=1}^{m} C_i^\circ = x_{12} + z_1 + z_2$. *There exists a 2-coloring of* G *with* x_{12}, z_1, z_2 *vertices of type-*x_{12}*, type-*z_1*, type-*z_2*, respectively, except when* $C_1^\circ = C_2^\circ = \cdots = C_m^\circ$ *and* $[z_1, z_2] = [rC_m^\circ - 1, (m - r)C_m^\circ - 1]$ *for some integer* $r > 0$.

Proof. We assume that $z_1 \leq z_2$ because otherwise, we can switch colors 1 and 2. We proceed by induction on the number of cycle sizes m. If $m = 1$, the claim holds by Claim 3.3.1. Assume for a moment that $m = 2$. If $z_2 < C_1^\circ$, then the assumption $z_1 \leq z_2$ and the hypothesis $C_1^\circ + C_2^\circ = 2 + z_1 + z_2$ force that $z_1 = z_2 = C_1^\circ - 1$ and $C_1^\circ = C_2^\circ$. Then $[z_1, z_2] = [rC_m^\circ - 1, (m-r)C_m^\circ - 1]$ for $r = 1$ and $m = 2$, a contradiction. Thus, it must be true that $z_2 \geq C_1^\circ$. Color the cycle of size C_1° with color 2. Claim 3.3.1 yields that we can color the second cycle with size C_2° with two vertices of type-x_{12}, z_1 vertices of type-z_1, and $z_2 - C_1^\circ$ vertices of type-z_2. Now, if $m \geq 3$, then below we argue that $2z_2 > 2C_{m-1}^\circ$ and so $z_2 > C_{m-1}^\circ$.

$$2z_2 \geq z_1 + z_2 = (\sum_{i=1}^{m} C_i^\circ) - 2 \geq C_m^\circ + C_{m-1}^\circ + C_1^\circ - 2 \geq 2C_{m-1}^\circ + C_1^\circ - 2 > 2C_{m-1}^\circ$$

Since $z_2 > C_{m-1}^\circ$, color a cycle of size C_{m-1}° with color 2. We wish to apply induction in order to color the remaining cycles so that these cycles have z_1 vertices of type-z_1, $z_2 - C_{m-1}^\circ$ vertices of type-z_1, and two vertices of type-x_{12}. If the inductive hypotheses fail, then $C_1^\circ = \cdots = C_{m-1}^\circ = C_m^\circ$ and $[z_1, z_2 - C_{m-1}^\circ] = [rC_m^\circ - 1, (m-1-r)C_m^\circ - 1]$ for some integer $r > 0$. This implies $[z_1, z_2] = [rC_m^\circ - 1, (m-r)C_m^\circ - 1]$, a contradiction. Hence, the hypotheses required for induction hold, and so we can color the remaining cycles as desired. □

Like with DUPs, if $x_12 = 0$, a solution to the 2-edge-coloring Problem for DUCs yields a solution to the NP-Complete Subset Sum Problem. The reduction between the Subset Sum Problem and the 2-edge-coloring Problem for DUCs when $x_{12} = 0$ follows immediately from Theorem 3.3.3.

Theorem 3.3.3. *Let G be a DUC with cycle sizes $3 \leq C_1^\circ \leq C_2^\circ \leq \cdots \leq C_m^\circ$ and let z_1, z_2 be non-negative integers which sum to $|V(G)|$. Then there exists a 2-coloring of G with z_1 vertices of type-z_1 and with z_2 vertices of type-z_2 if and only if some subset of $\{C_1^\circ, \ldots, C_m^\circ\}$ of G sum to z_1 and the rest sum to z_2.*

Proof. If such a coloring exists, all cycles are monochromatic. The cycles with color 1 have sizes which sum to z_1 and the cycles with color 2 have sizes which sum to z_2.

Also, if some subset $\mathcal{S}$ of $\{C_1^\circ, \ldots, C_m^\circ\}$ sum to z_1, color the corresponding cycles with color 1. Color the others with color 2 for the desired 2-coloring of G. $\square$

We now give necessary and sufficient conditions for when we can color a DUC G so that G has x_{12}, z_1, z_2 vertices of type-x_{12}, type-z_1, and type-z_2, respectively. As with DUCs, there is a bound on the number of odd cycles that G can have. Also, to prevent the aforementioned complexity issues, the hypotheses assume $x_{12} > 0$.

Theorem 3.3.4. *Let G be a DUC with cycle sizes $3 \leq C_1^\circ \leq C_2^\circ \leq \cdots \leq C_m^\circ$. Let $o_p(G)$ be the number of C_i° which are odd. Let x_{12}, z_1, z_2 be non-negative integers where $x_{12} > 0$ and where $\sum_{i=1}^m C_i^\circ = x_{12} + z_1 + z_2$. If $C_1^\circ = C_2^\circ = \cdots = C_m^\circ$ and $[x_{12}, z_1, z_2] = [2, rC_m^\circ - 1, (m-r)C_m^\circ - 1]$ for some integer $r \geq 0$, then there does not exist a 2-coloring of G with x_{12}, z_1, z_2 vertices of type-x_{12}, type-z_1, type-z_2, respectively. Otherwise, there exists such a 2-coloring of G if and only if x_{12} is even and $o_p(G) \leq z_1 + z_2$.*

Proof. ($\Rightarrow$) We need only $o_p(G) \leq z_1 + z_2$ since Claim 3.3.2 yields that the other conditions hold. If an odd cycle C° is monochromatic, then clearly C° contains a type-z_1 or type-z_2 vertex. Otherwise, C° switches between colors 1 and 2 at an even number of vertices and so contains an even number of type-x_{12} vertices. Thus, in order to be odd, C° must contain at least one type-z_i vertex. Hence, every odd cycle contains at least one type-z_1 or type-z_2 vertex and so $o_p(G) \leq z_1 + z_2$.

($\Leftarrow$) We proceed by induction on m. If $m = 1$, then Claim 3.3.1 yields the desired result. Now consider a DUC with $m \geq 2$ cycles. If $x_{12} = 2$, Claim 3.3.2 yields the desired result. We proceed differently for $2 < x_{12} \leq 2m$ and $x_{12} > 2m$. Our strategy in both cases is to choose a cycle C_i° and values x'_{12}, z'_1, z'_2 so that we can color a cycle of size C_i° with x'_{12}, z'_1, z'_2 vertices of type-x_{12}, type-z_1, type-z_2, respectively, and the other cycles with $x_{12} - x'_{12}, z_1 - z'_1, z_2 - z'_2$ vertices of type-x_{12}, type-z_1, type-z_2, respectively.

Assume $2 < x_{12} \leq 2m$. Let G' be the DUC with a cycle of size C_m removed. Let $x'_{12} = 2$. Since $x_{12} \leq 2m$, the hypothesis $\sum_{i=1}^m C_i = x_{12} + z_1 + z_2$ implies $\sum_{i=1}^m (C_i - 2) \leq z_1 + z_2$. Hence, for $i = 1, 2$, we can find z'_i such that $0 \leq z'_i \leq z_i$

and $C_m^\circ = 2 + z_1' + z_2'$. Since $C_m^\circ \geq 3$, at least one of z_1' or z_2', say z_2', is positive. If $C_1^\circ = \cdots = C_{m-1}^\circ$ and if $z - z_i' \equiv -1 \mod C_{m-1}^\circ$ for $i = 1, 2$, then the inductive hypotheses fail. In this case, decrease the positive value z_2' by 1 and add 1 to z_1'. With these changes, $z - z_i' \not\equiv -1 \mod C_{m-1}^\circ$ for $i = 1, 2$. In order to apply induction, we must show that the number of cycles that are odd in G' is at most $(z_1 - z_1') + (z_2 - z_2')$. It suffices to show that the number of cycles in G', that is, $m - 1$, is at most $(z_1 - z_1') + (z_2 - z_2')$. Since $\sum_{i=1}^m (C_i^\circ - 2) \leq z_1 + z_2$ and each C_i° is at least three, we see that $(C_m^\circ - 2) + (m - 1) \leq z_1 + z_2$. Since $C_m^\circ - 2 = z_1' + z_2'$, it then follows that $(m - 1) \leq (z_1 - z_1') + (z_2 - z_2')$. Then by induction, there exists a coloring of G' with $x_{12} - x_{12}', z - z_1', z - z_2'$ vertices of type-x_{12}, type-z_1, type-z_2, respectively. By Claim 3.3.1, we can color a cycle of size C_m° with x_{12}', z_1', z_2' vertices of type-x_{12}, type-z_1, type-z_2, respectively. These colorings yield the desired coloring of G.

Now assume $x_{12} > 2m$. We choose our primed variables so that $x_{12} - x_{12}' \geq 2(m - 1)$. In doing so, there are enough type-x_{12} vertices so that each cycle in G' can feasibly contain at least one pair of type-x_{12} vertices. This is desirable because cycles with at least one pair of type-x_{12} vertices contain both a color 1 and color 2 edge. Therefore, we can imagine subdividing a color i edge in order to insert additional type-z_i vertices as necessary.

We let $x_{12}' = \min\{C_j^\circ - 1, x_{12} - 2m + 2\}$ if there exists an odd cycle size C_j°. Otherwise, we let $x_{12}' = \min\{C_m^\circ, x_{12} - 2m + 2\}$. In either case, x_{12}' is even. Also, $x_{12}' \geq 2$ since $C_j^\circ \geq 3$ and $x_{12} > 2m$. Furthermore, since $x_{12}' \leq x_{12} - 2m + 2$, we see $x_{12} - x_{12}' \geq 2(m - 1)$, as desired. Note that if $x_{12} - x_{12}' = 2$, this inequality implies $m = 2$.

If some odd cycle size C_j° exists and $x_{12}' = C_j^\circ - 1$, then by hypothesis, $1 \leq o_p(G) \leq z_1 + z_2$. Thus, some z_i, say z_2, is positive. Let $z_2' = 1$. Then since $o_p(G) \leq z_1 + z_2$ in G, we see that the number of odd cycles in G' is at most $z_1 + z_2 - 1 = z_1 + (z_2 - z_2')$. The induction hypotheses thus hold except possibly when $x_{12} - x_{12}' = 2$. Recall that this implies $m = 2$, in which case, G' consists of one cycle and can be colored as desired by Claim 3.3.1. Otherwise, induction yields that there exists a coloring of G' with $x_{12} - x_{12}', z - z_1', z - z_2'$ vertices of type-x_{12}, type-z_1, type-z_2, respectively. Claim 3.3.1 also indicates that we can color a cycle of size C_j° with $x'_{12} = C_j^\circ - 1$ and $z'_2 = 1$ vertices of type-x_{12} and type-z_2, respectively. These colorings yield the desired coloring of G.

Now, if some odd cycle with size C_j° exists and $x'_{12} = x_{12} - 2m + 2$, then $x'_{12} < C_j^\circ$. The hypothesis $\sum_{i=1}^m C_i^\circ = x_{12} + z_1 + z_2$ implies $C_j^\circ - x'_{12} + \sum_{i \neq j}(C_i^\circ - 2) = z_1 + z_2$. Since $C_i^\circ - 2 \geq 1$, this implies $C_j^\circ - x'_{12} + (m-1) \leq z_1 + z_2$. Then we can choose z'_i so that $0 \leq z'_i \leq z_i$, $z'_1 + z'_2 = C_j^\circ - x'_{12}$, and $m - 1 \leq (z_1 - z'_1) + (z_2 - z'_2)$. Since G' has $m-1$ cycles, this implies that the number of odd cycles in G' is at most $(z_1 - z'_1) + (z_2 - z'_2)$. As in the previous case, the induction hypotheses hold except possibly when $x_{12} - x'_{12} = 2$, in which case $m = 2$, and G' thus consists of one cycle, so Claim 3.3.1 yields that G' can be colored as desired. Otherwise, induction yields that there exists a coloring of G' with $x_{12} - x'_{12}, z - z'_1, z - z'_2$ vertices of type-x_{12}, type-z_1, type-z_2, respectively. Claim 3.3.1 also indicates that we can color a cycle of size C_j° with $x'_{12} = x_{12} - 2m + 2, z'_1, z'_2$ vertices of type-x_{12}, type-z_1, type-z_2, respectively. These colorings yield the desired coloring of G.

Finally, if no cycle is odd, then $x'_{12} = \min\{C_m^\circ, x_{12} - 2m + 2\}$. Using logic similar to the previous cases, we can choose z'_i so that $0 \leq z'_i \leq z_i$ for $i = 1, 2$, and $z'_1 + z'_2 = C_m^\circ - x'_{12}$. The number of odd cycles in G' is 0 and so $o_p(G) \leq (z_1 - z'_1) + (z_2 - z'_2)$. If $x_{12} - x'_{12} = 2$, then $m = 2$ and we use Claim 3.3.1 to color G' as desired. Otherwise, if $x_{12} - x'_{12} > 2$, the induction hypotheses hold, and by induction, the desired coloring of G' exists.

□

In Theorem 2.2.2, we translate the details of Theorem 3.3.4 into terminology involving factors. Since DUCs are regular graphs, due to Claim 0.0.7, it is natural that an answer to the 2-Edge-Coloring Problem for DUCs leads to an answer to the Factor Problem for DUCs.

Theorem 2.2.2. *Let G be a DUC with cycle sizes $3 \leq C_1^\circ \leq C_2^\circ \leq \cdots \leq C_m^\circ$. Let $o_c(G)$ be the number of C_i° which are odd. Let d_0, d_1, d_2 be non-negative integers which sum to $|V(G)|$ where $d_1 > 0$. If $C_1^\circ = C_2^\circ = \cdots = C_m^\circ$ and if $[d_0, d_1, d_2] = [rC_m^\circ - 1, 2, (m-r)C_m^\circ - 1]$ for some integer $r \in (0, m)$, then G does not have a*

$[d_0, d_1, d_2]$-factor. Otherwise, G has a $[d_0, d_1, d_2]$-factor if and only if d_1 is even and $o_c(G) \leq d_0 + d_2$.

Proof. Label the vertices of G. We show a bijection between a 2-coloring of G with x_{12}, z_1, z_2 vertices of type-x_{12}, type-z_1, type-z_2, respectively, and a $[z_2, x_{12}, z_1]$-factor of G. Deleting the color 1 edges from a 2-coloring of G yields the color 2 subgraph H which is a $[d_0, d_1, d_2]$-factor of G. Then each degree 2 vertex in H is incident to two color 2 edges the 2-coloring of G and so is a type-z_2 vertex. Thus, $d_2 = z_2$. Each degree 1 vertex in H is incident to one color 1 edge and one color 2 edge in the 2-coloring and so is a type-x_{12} vertex. Thus, $d_1 = x_{12}$. Finally, each degree 0 vertex in H is incident to two color 1 edges and so is a type-z_1 vertex. Thus, $d_0 = z_1$. Hence, H is a $[z_1, x_{12}, z_2]$-factor of G. Similarly, given a $[d_0, d_1, d_2]$-factor of G, make each non-edge a color 1 edge and each edge a color 2 edge to obtain a 2-coloring of G with d_0, d_1, d_2 vertices of type-z_1, type-x_{12}, type-z_2, respectively.

This implies there exists a $[z_1, x_{12}, z_2]$-factor of G if and only if there exists a 2-coloring of G with x_{12}, z_1, z_2 vertices of type-x_{12}, type-z_1, type-z_2, respectively. By Theorem 3.3.4, if $C_1^\circ = C_2^\circ = \cdots = C_m^\circ$ and $[d_0, d_1, d_2] = [z_1, x_{12}, z_2] = [rC_m^\circ - 1, 2, (m-r)C_m^\circ - 1]$ for some integer $r > 0$, then there does not exist such a 2-coloring of G and thus no $[d_0, d_1, d_2]$-factor either. Otherwise, there exists such a 2-coloring, and thus a $[d_0, d_1, d_2]$-factor of G, if and only if $d_1 = x_{12}$ is even and $o_c(G) \leq z_1 + z_2 = d_0 + d_2$. □

3.4 Factors and 2-Colorings of DUPs

Let G be a DUP with specified path sizes. Let $\mathcal{S}$ be the set of (2×1) vector sequences whose column sums is the degree sequence of G and where the list of entries in row 1 of each vector consists of d_i entries of the integer i. Per Claim 0.0.6, G has a $[d_0, d_1, d_2]$-factor if and only if some non-empty (but possibly proper) subset $\mathcal{S}'$ of the vector sequences in $\mathcal{S}$ are realizable as a 2-coloring of G. It is interesting to note that we can use Theorem 3.2.22 to determine *which* vector sequences of $\mathcal{S}$ are in $\mathcal{S}'$ and are thus realizable as a 2-coloring of G. We illustrate this now. Using the terminology of the previous section, recall that any 2-coloring of a DUP is an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring with a degree vector sequence consisting of $a_1, a_2, x_{12}, z_1, z_2$ vectors of type-a_1, type-a_2, type-x_{12}, type-z_1, and type-z_2, respectively. Thus, we describe the the vector sequences in $\mathcal{S}'$ by listing their corresponding $[a_1, a_2, x_{12}, z_1, z_2]$ values.

Let G be a DUP with $p = 8$ paths with orders $\{3, 3, 3, 3, 3, 3, 5, 11\}$. The reader can verify that G has a $[d_0, d_1, d_2]$-factor where $d_0 = 16$, $d_1 = 16$, and $d_2 = 2$. For this example, the set $\mathcal{S}$ previously discussed is the set of (2×1) vector sequences where row 1 consists of 16 0's, 16 1's, and 2 2's and whose columns sums is the degree sequence of G. Also, $\mathcal{S}'$ is the set of the vector sequences in $\mathcal{S}$ which are the degree vector sequence of an $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G. Claim 0.0.6 tells us that $\mathcal{S}'$ is non-empty. Also, per Claim 3.2.7, the color 1 subgraph of any $[a_1, a_2, x_{12}, z_1, z_2]$-coloring of G with a degree vector sequence in $\mathcal{S}'$ is a $[d_0, d_1, d_2]$-factor of G where $[a_2 + z_2, a_1 + x_{12}, z_1] = [16, 16, 2]$. Additionally, if G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable, then per Claim 3.2.8, the number of paths in G is $p = 8 = \frac{a_1+a_2}{2}$ and so $a_1 + a_2 = 16$.

Table 3.1 lists all possible $[a_1, a_2, x_{12}, z_1, z_2]$ values for the sequences in the vector sequences in $\mathcal{S}'$. Since $a_1 + x_{12} = 16$, Table 3.1 lists all pairs of (a_1, x_{12}) values which sum to 16 by increasing a_1 from 0 to 16. The values of $(a_1, x_1 2)$ as well as the equations $a_1 + a_2 = 16$ and $a_2 + z_2 = 16$ then fix the values of a_2 and z_2 for each row. As mentioned before, $z_1 = 2$ for each row as well.

In Table 3.1, we bold those rows whose $[a_1, a_2, x_{12}, z_1, z_2]$ values satisfy the hypotheses of Theorem 3.2.22. Then these rows are precisely those where G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable and so the corresponding degree vector sequences are precisely the sequences in $\mathcal{S}'$, as desired.

Note that the bolded rows in Table 3.1 are a contiguous block of rows. This will be true in general for any $[d_0, d_1, d_2]$-factor H of G and the table corresponding to H. To see why this is the case, consider the hypothesis $\sum_{i=1}^{\frac{a_2-x_{12}}{2}} (C_i - 2) \leq z_2$. If this hypothesis holds for row i then it holds for all rows after i because z_2 increases as the rows increase. Now consider the hypothesis $t(G) \leq a_2 + z_1$. The quantity $a_2 + z_1$ decreases as the row numbers increase. Hence, if $t(G) \leq a_2 + z_1$ fails at row i, then it fails for all rows after i. All hypotheses of Theorem 3.2.22 behave in one of these manners, thus causing the contiguous block property. We remark that the only grey

a_1	a_2	x_{12}	z_1	z_2
0	16	16	2	0
1	15	15	2	1
2	14	14	2	2
3	13	13	2	3
4	**12**	**12**	**2**	**4**
5	**11**	**11**	**2**	**5**
6	**10**	**10**	**2**	**6**
7	**9**	**9**	**2**	**7**
8	**8**	**8**	**2**	**8**
9	**7**	**7**	**2**	**9**
10	**6**	**6**	**2**	**10**
11	5	5	2	11
12	4	4	2	12
13	3	3	2	13
14	2	2	2	14
15	1	1	2	15
16	0	0	2	16

Table 3.1: Colorings which yield a $[16, 16, 2]$-factor

area is that Theorem 3.2.22 requires $x_{12} > 0$. And so if a contiguous block is at the end of the table, to determine whether or not the final row where $x_{12} = 0$ should be included in the block reduces to the Subset Sum Problem by Claim 3.2.24.

3.5 k-Edge-Coloring Problem for a fixed DUP or DUC when $k \geq 3$

In order to generalize the notation of the previous section for $k \geq 3$, we give the following definitions.

Definition 3.5.1. *Given a k-edge-colored path P and colors i, j, we define a type-a_i, type-x_{ij}, and type-z_i vertices and vectors as such:*

1. *A* ***type-a_i vertex*** *is an endpoint of a P and is adjacent to exactly one color*

i edge ($\bullet\!\overset{i}{\text{——}}$). Its degree vector is is a column vector with 1 in row i and 0's elsewhere, which we define as a ***type-a_i vector****.*

2. *A* ***type-x_{ij} vertex****, where $i \neq j$, is internal to a P and is incident to an edge of color i and an edge of color j ($\overset{i}{\text{——}}\!\bullet\!\overset{j}{\text{——}}$). Its degree vector is a column vector with 1 in rows i and j and 0's elsewhere, which we define as a* ***type-x_{ij} vector****.*

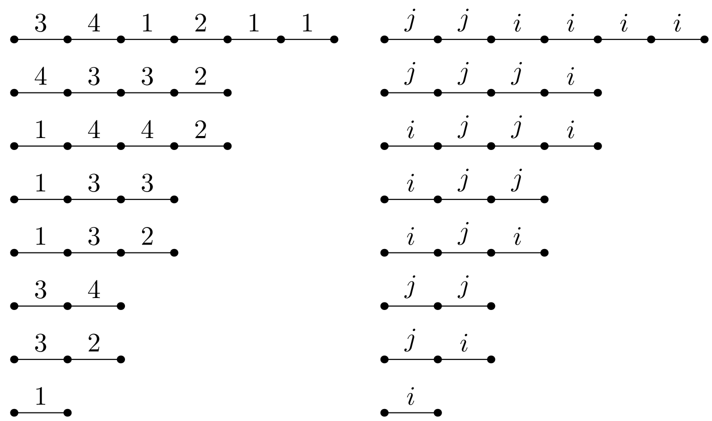
3. *A* ***type-z_i vertex*** *is internal to P and is incident to exactly two edges of color i edges ($\overset{i}{\text{——}}\!\bullet\!\overset{i}{\text{——}}$). Its degree vector is a column vector with 2 in row i and 0's elsewhere, which we define as a* ***type-z_i vector****.*

As before, we use the notation x_{ij} to count the number of column vectors of type-x_{ij} or the number of vertices of type-x_{ij}. Note that the subscript order is unimportant in x_{ij}, meaning, x_{ij} and x_{ji} refer to the same variable. To be consistent with the definition of a type-x_{ij} vertex when $i \neq j$, we always assume $x_{ii} = 0$. Additionally, we use the notation a_i to count the number of column vectors of type-a_i or the number of vertices of type-a_i, and we use z_i to count the number of column vectors of type-z_i or the number of vertices of type-z_i.

By combining colors, every k-coloring of a DUP yields a 2-coloring of a DUP. For example, in Figure 3.13(a) we show a 4-coloring of a DUP. We let the color i correspond to the colors 1 and 2 and we let color j correspond to the colors 3 and 4. This yields the 2-coloring shown in Figure 3.13(b). Thus, if a DUP G is $[a_1, a_2, x_{12}, z_1, z_2]$-colorable, then any partition of the colors into non-empty sets S_1 and S_2 must satisfy the $k = 2$ conditions given by Theorem 3.2.22. Claim 3.5.2 stems from this concept.

Claim 3.5.2. *Let G be a DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$. Let $k \geq 2$ be an integer. For $1 \leq i \leq k$, let a_i, x_{ij}, z_i be integers. Assume that subscript order does not matter and so x_{ij} and x_{ji} refer to the same variable. For any i, assume $x_{ii} = 0$. Let S_1 and S_2 be non-empty sets which partition the set $\{1, 2, \ldots, k\}$. Let $a_{S_1}, a_{S_2}, z_{S_1}, z_{S_2}, x_{S_1 S_2}$ be defined as follows.*

1. *Let $a_{S_1} = \sum_{i \in S_1} a_i$ and $a_{S_2} = \sum_{i \in S_2} a_i$.*



(a) **(b)**

Figure 3.13: Combining colors in DUPs when $k \geq 3$

2. *Let* $z_{S_1} = \sum_{i \in S_1} z_i + \sum_{i,j \in S_1} x_{ij}$ *and* $z_{S_2} = \sum_{i \in S_2} z_i + \sum_{i,j \in S_2} x_{ij}$.

3. *Let* $x_{S_1 S_2} = \sum_{i \in S_2, j \in S_2} x_{ij}$.

Let $o_p(G)$ *refer to the number of path orders* C_i *in* G *which are odd. Let* $t(G)$ *refer to the number of path orders* C_i *in* G *which are 3. If there exists a* k*-coloring of* G *with* a_i *vertices of type-*a_i, x_{ij} *vertices of type-*x_{ij}, *and* z_i *vertices of type-*z_i *for* $1 \leq i \leq k$, *then the following conditions hold.*

1. *For* $1 \leq i \leq k$, a_i, x_{ij}, z_i *are non-negative integers.*

2. *The total sum of* a_i, x_{ij}, z_i *for all* $1 \leq i \leq k$ *is the order of* G.

3. $z_i > 0 \implies a_i + \sum_j x_{ij} > 0$.

4. $p = \frac{\sum a_i}{2}$.

5. *For a fixed color* i, a_i *and* $\sum_j x_{ij}$ *have the same parity.*

6. $\sum_{l=1}^{\frac{a_{S_1} - x_{S_1 S_2}}{2}} (C_l - 2) \leq z_{S_1}$

7. $\sum_{l=1}^{\frac{a_{S_2} - x_{S_1 S_2}}{2}} (C_l - 2) \leq z_{S_2}$

8. $t(G) \leq a_{S_1} + z_{S_2}$ *and* $o_p(G) \leq a_{S_1} + z_{S_1} + z_{S_2}$

9. $t(G) \leq a_{S_2} + z_{S_1}$ *and* $o_p(G) \leq a_{S_2} + z_{S_1} + z_{S_2}$

Proof. Consider any such coloring of G. Re-color the edges so that all edges with a color in S_1 have color $\hat{1}$ and all edges with color in S_2 have color $\hat{2}$. All conditions then follow from Theorem 3.2.22. $\square$

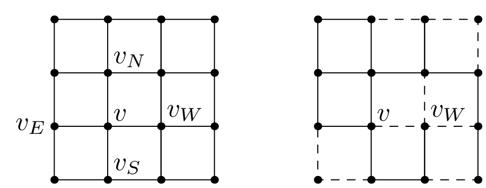
Recall that Theorem 3.2.25 proves that when $x_{12} = 0$, answering whether or not a DUP with path orders $2 \leq C_1 \leq C_2 \leq \cdots \leq C_p$ is $[a_1, a_2, 0, z_1, z_2]$-colorable is an NP-Complete Problem. Given a DUP G with specified path sizes and a sequence of a_i, z_i, x_{ij} values for $1 \leq i \leq k$ which satisfy the hypotheses of Claim 3.5.2, assume there exists a partition S_1, S_2 of the colors $\{1, 2, \ldots, k\}$ so that $x_{S_1 S_2} = 0$. Consider the question of whether or not there exists a k-coloring of G with a_i vertices of type-a_i, x_{ij} vertices of type-x_{ij}, and z_i vertices of type-z_i for $1 \leq i \leq k$. If we can answer this question, then we can answer whether or not G is $[a_{S_1}, a_{S_2}, 0, z_{S_1}, z_{S_2}]$-colorable, an NP-Complete Problem. Hence, if such a partition S_1, S_2 exists where $x_{S_1 S_2} = 0$, the k-Edge-Coloring Problem of a fixed DUP G is again NP-Complete.

One way to avoid this issue is to require that $x_{ij} > 0$ for all $i \neq j$. To date, even with this extra assumption, we do not know whether or not the conditions of Claim 3.5.2 are sufficient when $k \geq 3$. We have found no examples that show that the necessary conditions of Claim 3.5.2 are not sufficient. We conclude by giving a very basic reason to illustrate why $k = 2$ is a potentially much more handleable. If a path has a type-x_{12} vertex in the $k = 2$ case, then we know that the set of colors on the edges of the path is the set of all colors, which in this case is simply $\{1, 2\}$. So for example, we could subdivide edges of either color to increase the type-z_1 and type-z_2 vertices as needed. Thus, a helpful strategy in the $k = 2$ case is maximizing the number of paths with a type-x_{12} vertex. In the $k \geq 3$ case, knowing that a path has a type-x_{ij} vertex is not helpful if we want to increase the number of type-z_l vertices where $l \neq i$ or $l \neq j$.

Chapter 4

The Factor Problem for Grids

Formally, an $n \times m$ grid is the cartesian product of P_n and P_m, where P_n, P_m are paths with n, m vertices, respectively. If $n = 1$ or $m = 1$, then a grid is simply a path. Otherwise, a grid has a box shape. Our goal in this chapter is to characterize factors of a grids. Since the degree of a vertex in any grid is at most 4, the factors we seek are $[d_0, d_1, d_2, d_3, d_4]$-factors. Figure 4.1 illustrates an $n \times m$ grid that has a $[2, 3, 7, 3, 1]$-factor.



(a) A 4×4 Grid G **(b)** A $[2, 3, 7, 3, 1]$-factor of G

Figure 4.1: Factors in Grids

We formally define the border and interior of a grid now.

Definition 4.0.3. *Let G be an $n \times m$ grid where $n \geq 2$ and $m \geq 2$. The **border** of G is the the subgraph induced by all vertices of degree less than 4. The **interior** of G is the subgraph induced by all vertices of degree exactly 4. The corners of G are the degree 2 vertices.*

It is well-known that grids have Hamiltonian paths. In the very specific case when $d_3 = d_4 = 0$, if the desired $[d_0, d_1, d_2, 0, 0]$-factor exists, then clearly each vertex has degree 0, 1, or 2. In this case, the problem essentially reduces to finding a factor of a Hamiltonian path through the grid. Due to the DUP results, this is straightforward to do. When d_3+d_4 is positive, there are 4 main types of difficulties: (a) when d_1+d_2 is 'too small', (b) when d_4 is 'too large', (c) when d_1 or d_2 is 0, and (d) when $d_1 + d_3 < 4$.

We now explain what we mean by difficulty (a). It is obvious that each corner in a grid cannot be degree 3 or degree 4 in a resulting factor. As a result $d_0+d_1+d_2 \geq 4$ is a necessary condition in any $[d_0, d_1, d_2, d_3, d_4]$-factor. However, degree 0 vertices on the border in a factor tend to increase the required number of degree 1 and 2 vertices elsewhere in the factor. Therefore, although $d_0+d_1+d_2 \geq 4$ is necessary, one of the first facts we prove (Claim 4.4.1) shows that the more restrictive inequality $d_1 + d_2 \geq 4$ is necessary in most cases of interest, i.e., when $d_3 + d_4$ is positive. The case when $d_1 + d_2 = 4$ is very restrictive and reduces to a case-by-case analysis. We summarize what we conjecture about factors in the case where $d_1 + d_2 = 4$ but we concentrate on giving results when $d_1 + d_2 \geq 5$, specifically, when $d_2 \geq 5$. We will introduce the concept of an imperfect grid, which is in a sense a grid with 5 corners all of which are degree 2. The concept of an imperfect grid gives intuition for why the assumption that $d_2 \geq 5$ could be helpful.

Before discussing difficulty (b), recall that the *neighborhood* of a vertex v, denoted $N(v)$, is the set of vertices which are adjacent to v. Given a set of vertices S in a graph G, the neighborhood of S, denoted by $N(S)$, is the set of vertices in $G - S$ which are adjacent to a vertex in S. To understand difficulty (b), note two obvious facts. An $n \times m$ grid G does not have degree 4 vertices if $n \leq 2$ or $n \leq 2$. Hence, we always assume $n \geq 3$ and $m \geq 3$ when $d_4 > 0$. Also, note that no vertex on the border of a grid G is degree 4 in a factor of G. We generalize this second obvious fact now. Let H be a factor of a grid and let H_4 be the set of vertices of G which are degree 4 in H. If a vertex v is in the neighborhood of H_4 in G, then v must have positive degree in H. Furthermore, v must have degree 1, 2, or 3 in H since $v \notin H_4$. Thus, $d_1 + d_2 + d_3$ must be at least as big as the number of vertices in the neighborhood of H. We let $B(n,m,d_4)$ denote the minimum number of vertices in the neighborhood of H_4 for any factor H of an $n \times m$ grid G with $d_4 > 0$ vertices. Then it follows that $d_1 + d_2 + d_3 \geq B(n,m,d_4)$ in any $[d_0, d_1, d_2, d_3, d_4]$-factor of an $n \times m$ grid. In Claim 4.6.3, we show that a lower bound for $B(n,m,d_4)$ is $\max\{2n_4 + 2, 2m_4 + 2\}$ where n_4 and m_4 are the least number of rows and columns, respectively, which must contain a degree 4 vertex in any factor of G. Hence, when $d_4 > 0$, $d_1 + d_2 + d_3 \geq \max\{2n_4 + 2, 2m_4 + 2\}$ is a necessary condition for G to have a $[d_0, d_1, d_2, d_3, d_4]$-factor. We concentrate on the case when $d_4 > 0$ and $d_1 + d_2 + d_3 \geq \min\{2n_4 + 2m - 1, 2m_4 + 2n - 1\}$. When $\max\{2n_4 + 2, 2m_4 + 2\} \leq d_1 + d_2 + d_3 < \min\{2n_4 + 2m - 1, 2m_4 + 2n - 1\}$, we know of cases when G does and does not have a $[d_0, d_1, d_2, d_3, d_4]$-factor, and this range of d_4 values is left for future work.

Now consider difficulty (c). When d_1 or d_2 is 0, the placement of the edges in the factor tends to be very specific. Hence, the problem in this case often reduces to a case-by-case analysis determining the allowable placements. Thus, we often assume $d_1 > 0$. While discussing difficulty (a), we explained that we often assume $d_2 \geq 5$ and so this avoids issues that present themselves when $d_2 = 0$.

Finally, consider difficulty (d). Notice that each wall in a grid G is a path. Thus, any factor of a grid G yields a factor of the walls of G and thus in a sense yields a factor of a DUP with 4 paths. Recall that Theorem 2.1.3 shows that in any $[d_0, d_1, d_2]$-factor of a DUP, each odd path has a vertex which is degree 0 or degree 2 in a factor of the DUP. Similarly, there are cases where each odd wall requires a degree 1 or degree 3 vertex in the factor. Due to this fact, if all walls are odd the assumption that $d_1 + d_3 \geq 4$ is helpful. Since the walls of a grid could be even, $d_1 + d_3 \geq 4$ is certainly *not* a necessary condition. Nonetheless, the assumption that $d_1 + d_3 \geq 4$ is weak and we argue this now. We are able to show (Claim 4.2.1) that $d_1 + d_3$ is even in any $[d_0, d_1, d_2, d_3, d_4]$-factor of a grid. Thus, by assuming that $d_1 + d_3 \geq 4$ we are only excluding factors where $d_1 + d_3 = 0$ and $d_1 + d_3 = 2$. Due to difficulty (c), we already typically assume that $d_1 > 0$ in many cases, and so by parity, $d_1 > 0$ implies that $d_1 + d_3 \geq 2$. Thus, the assumption that $d_1 + d_3 \geq 4$ really only excludes one additional case, namely, the case when $d_1 + d_3 = 2$.

<table>
<tr><td rowspan="3">$d_1 + d_2 < 4$</td><td>$d_3 = d_4 = 0$</td><td colspan="2">All factors characterized (Thm 4.3.2)</td></tr>
<tr><td>$d_3 = 1, d_4 = 0$</td><td colspan="2">All factors characterized (Thm 4.4.4)</td></tr>
<tr><td>$d_3 \geq 2, d_4 > 0$</td><td colspan="2">Impossible (Clm 4.4.1)</td></tr>
<tr><td>$d_1 + d_2 = 4$</td><td colspan="3">We conjecture the structure of the factors in this case. (Conj 4.5.1)</td></tr>
<tr><td rowspan="7">$d_1 + d_2 \geq 5$</td><td rowspan="3">$d_4 > 0$, $d_1 > 0$
$d_2 \geq 5$,
$d_1 + d_3 \geq 4$</td><td>$B_I \leq d_1 + d_2 + d_3$</td><td>Possible (Thm 4.6.6)</td></tr>
<tr><td>$B_L \leq d_1 + d_2 + d_3 < B_I$</td><td>Open Question</td></tr>
<tr><td>$d_1 + d_2 + d_3 < B_L$</td><td>Impossible (Clm 4.6.3)</td></tr>
<tr><td>$d_4 > 0$, $d_1 = 0$</td><td colspan="2">Open Question</td></tr>
<tr><td>$d_4 > 0$, $d_2 < 5$</td><td colspan="2">Open Question</td></tr>
<tr><td>$d_4 > 0$, $d_1 + d_3 < 4$</td><td colspan="2">Open Question</td></tr>
<tr><td>$d_4 = 0$</td><td colspan="2">We identify a list of impossible factors which we conjecture is complete. (Conj 4.7.1)</td></tr>
</table>

Table 4.1: Summary of results for $n \geq 3$ and $m \geq 3$

Based on our previous discussion concerning difficulties (a)-(d), we summarize our results in Table 4.1 for the case when $n \geq 3$ and $m \geq 3$. When $n \leq 2$ or $m \leq 2$ we are able to characterize all $[d_0, d_1, d_2, d_3, d_4]$-factors . For ease of formatting, we use the variables B_L and B_I to refer to the true lower bound and our imposed lower bound, that is, $B_L = \max\{2n_4+2, 2m_4+2\}$ and $B_I = \min\{2n_4+2m-5, 2m_4+2n-5\}$. An entry of *Impossible* means that a grid does not have a $[d_0, d_1, d_2, d_3, d_4]$-factor in that case. An entry of *Possible* means that we have shown that a grid does have the desired factor in that case. An entry of *All factors characterized* means that we have proven which factors are and are not possible in that case. An entry of *Open Question* means that we have found examples of factors which are possible and others which are impossible in that case.

4.1 Definitions and Notation

In this section, we give basic definitions and notation used throughout the proofs of the upcoming sections.

It is sensible to use the directions *north (N), south (S), east (E),* and *west (W)* when describing structures of a grid in relation to another and the following definitions exhibit this.

Definition 4.1.1. *Let $v_{i,j}$ be the vertex in row i and column j of an $n \times m$ grid G where $1 \leq i \leq n$ and $1 \leq j \leq m$. Let H be a factor of G. If $i > 1$, then $v_{i-1,j}$ is the* ***N neighbor*** *of $v_{i,j}$ in G or in H. If $i < n$, then $v_{i+1,j}$ is the* ***S neighbor*** *of $v_{i,j}$ in G or in H. If $j > 1$, then $v_{i,j-1}$ is the* ***W neighbor*** *of $v_{i,j}$ in G or in H. Finally, if $j < m$, then $v_{i,j+1}$ is the* ***E neighbor*** *of $v_{i,j}$ in G or in H. We let v_N, v_S, v_E, and v_W refer to the N, S, E, and W neighbors, respectively, of a vertex v in G or in H.*

See Figure 4.1(a) which demonstrates v_N, v_S, v_E, and v_W, that is, the N, S, E, W neighbors, respectively, of a vertex v in the interior of a grid. Figure 4.1(b) shows a factor of a grid in which the vertex v is no longer adjacent to its W neighbor v_W. Although v and v_W are no longer adjacent in this factor H, we still refer to v_W as the W neighbor of v in H. Hence, we define N, S, E, and W neighbors by adjacencies in the original grid and not the factor.

Definition 4.1.2. *The* ***N wall*** *of an $n \times m$ grid G where $n \geq 2$ and $m \geq 2$ is the path between and including the NE and NW corners of G. The* ***S, E, and W walls*** *are defined similarly. A wall is* ***odd*** *if the wall is a path on an odd number of vertices. A wall is* ***even*** *otherwise.*

We define the concept of an imperfect grid now.

Definition 4.1.3. *Given an integers r, s where $1 \leq s \leq n-1$ and $1 \leq s \leq m-1$, an (n, m, r, s)**-imperfect grid** is an $n \times m$ grid with an $(n-r) \times s$ grid deleted from the SE corner.*

Typically, $s = 1$ for our needs and so we give the following definition.

Definition 4.1.4. *Given an integer r where $1 \leq r \leq n-1$, an (n, m, r)**-imperfect grid** is an $(n, m, r, 1)$-imperfect grid with an additional column m with exactly r vertices. In an imperfect grid, the* ***cutout vertex*** *is the vertex in row r and column $m-1$. When $r \geq 2$, the* ***fifth corner*** *of an imperfect grid is the vertex in row r and column m.*

See Figure 4.2 for an example of an imperfect grid where z is the cutout vertex and v is the fifth corner. Note that the N wall of an imperfect grid has m vertices, the S wall has $m-1$ vertices, and the W wall has n vertices. The E wall is not straight and we define the terms E1 wall and E2 wall of an imperfect grid for this reason.

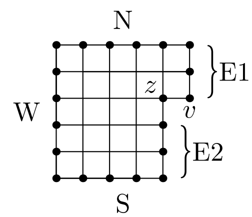


Figure 4.2: (n, m, r)-imperfect grid where $n = m = 6$ and $r = 3$

Definition 4.1.5. *For $2 \leq r \leq n-1$, the **E1 wall** of an (n, m, r)-imperfect grid is the path consisting of the r vertices in column m. The **E2 wall** of an (n, m, r)-imperfect grid is the path consisting of the final $n-r$ vertices in column $m-1$. To clarify, the endpoints of the E2 wall are the S neighbor of the cutout vertex and the final vertex in column $m-1$.*

Definition 4.1.6. *The **border** of an (n, m, r)-imperfect grid is the subgraph induced by all degree 1, 2, and 3 vertices as well as the cutout vertex.*

Claim 4.1.7. *The border of an $n \times m$ grid where $n \geq 2$ and $m \geq 2$ has $2n+2m-4$ vertices. The border of a (n, m, r)-imperfect grid G also has $2n+2m-4$ vertices. There are $2n+2m-5$ degree 2 and 3 vertices on the border of G, of which $2n+2m-10$ are degree 3. There is also 1 degree 4 vertex (the cutout vertex) on the border. G in total has $(n-2)(m-2)-(n-r-1)$ degree 4 vertices which includes the cutout vertex.*

Proof. The claim follows from simply counting the vertices on the border of a grid or imperfect grid. $\square$

4.2 Strategies

We now present observations and strategies that we rely on often throughout the upcoming sections. A simple observation is that if a grid has a $[d_0, d_1, d_2, d_3, d_4]$-factor, then the sequence consisting of d_i entries of the integer i is realizable. Claim 4.2.1 uses this fact to show that d_1 and d_3 have the same parity. This basic necessary condition repeatedly appears in our upcoming results.

Claim 4.2.1. *Let d_0, d_1, d_2, d_3, d_4 be nonnegative integers that sum to nm. Assume the sequence consisting of d_i entries of the integer i is realizable. Then d_1 and d_3 have the same parity. Also, the quantity $d_0 + d_2 + d_4$ has the same parity as nm.*

Proof. Because the sequence consisting of d_i entries of the integer i is realizable, the sum $0d_0 + d_1 + 2d_2 + 3d_3 + 4d_4$ is even. Then $d_1 + 3d_3$ and so $d_1 + d_3$ must be even as well. Thus, d_1 and d_3 have the same parity. Because the d_i values sum to nm and $d_1 + d_3$ is even, the quantity $d_0 + d_2 + d_4$ has the same parity as nm. □

We will see that results about factors of paths are very helpful when searching for factors of grids. For example, when $n = 1$, a $1 \times m$ grid is simply a path on m vertices and so results concerning factors of DUPs give an answer to the Factor Problem for a $1 \times m$ grid.

Claim 4.2.2. *Let G be an $n \times m$ grid where $n = 1$ or $m = 1$. Let d_0, d_1, d_2 be nonnegative integers that sum to nm where d_1 is even. G has a $[d_0, d_1, d_2]$-factor except when $d_1 = 0$ and $d_2 > 0$. Furthermore, if G has a $[d_0, d_1, d_2]$-factor and $d_1 > 0$, then there exists a factor in which an endpoint of G is a degree 1 vertex in the factor.*

Proof. Without loss of generality, assume $n = 1$. Let $d_3 = d_4 = 0$. Then d_1 is even per Claim 4.2.1. Because G is a $1 \times m$ grid, G is simply a path on m vertices. Then claim then follows from Claim 2.1.1. □

We now assume that $n > 1$ and $m > 1$ for the duration of this chapter. If $n = 2$ or $m = 2$, then an $n \times m$ grid has no degree 4 vertices and so $d_4 = 0$. If $n \geq 3$ and

$m \geq 3$, then because only the vertices interior to a grid can be degree 4 vertices in a factor, it must be true that $d_4 \leq (n-2)(m-2)$. Since no corner vertex in a factor of a grid can be degree 3 or degree 4, we also require that $d_3 + d_4 \leq nm - 4$. These facts are captured in Claim 4.2.3.

Claim 4.2.3. *Let $n \geq 2$ and $m \geq 2$ and assume an $n \times m$ grid has a $[d_0, d_1, d_2, d_3, d_4]$-factor. Then $d_3 + d_4 \leq nm - 4$. Also, if $n = 2$ or $m = 2$, then $d_4 = 0$. Otherwise, $d_4 \leq (n-2)(m-2)$.*

Claim 4.2.4 shows that a grid is a bipartite graph, and so any integer sequence which is not bipartite realizable is also not the degree sequence of a factor of a grid. Recall the well known fact that bipartite graphs are equivalently graphs with no odd cycles.

Claim 4.2.4. *An $n \times m$ grid G is a bipartite graph. Let X and Y be the partite sets of G. Then $|X| = |Y|$ if and only if nm is even. If nm is odd, the number of vertices in X and Y differ by 1. Finally, any factor of G is also a bipartite graph.*

Proof. Let v_{ij} be the vertex in row i and column j of G. Notice that for a fixed v_{ij}, if $i+j$ is even, then the neighbors of v_{ij} have row and column indices which have odd sum. Similarly, if $i+j$ is odd, then the row and column indices of the neighbors of v_{ij} have even sum. Hence, the grid has partite sets $X = \{v_{ij} | 1 \leq i \leq n, 1 \leq j \leq m, i+j$ is even $\}$ and $Y = \{v_{ij} | 1 \leq i \leq n, 1 \leq j \leq m, i+j$ is odd $\}$ and G is thus a bipartite graph. Furthermore, note that these partite sets have an equal number of vertices if and only if nm is even. If nm is odd, the number of vertices in the sets differ by 1. The final portion of the claim follows because any subgraph of a bipartite graph is a bipartite graph. $\square$

Corollary 4.2.5. *If the sequence consisting of d_i entries of the integer i is not bipartite realizable where $0 \leq i \leq 4$, then no grid has a $[d_0, d_1, d_2, d_3, d_4]$-factor.*

Corollary 4.2.6. *If every realization of the sequence consisting of d_i entries of the integer i where $0 \leq i \leq 4$ has an odd cycle, then no grid has a $[d_0, d_1, d_2, d_3, d_4]$-factor.*

We remind the reader of the definitions of a *Hamiltonian path* and *Hamiltonian cycle*.

Definition 4.2.7. *A* ***Hamiltonian path*** *in a graph G is a factor of G which is a single path on the vertices of G. A* ***Hamiltonian cycle*** *of G is a factor of G which is a single cycle on the vertices of G.*

It is well known that by traversing up and down the columns of an $n \times m$ grid G, we can find a Hamiltonian path. The same is clear for imperfect grids as well. If G has even order and has a Hamiltonian path then this yields a perfect matching of G.

Claim 4.2.8. *Every $n \times m$ grid or (n, m, r, s)-imperfect grid G has a Hamiltonian path P. If G is a grid, then endpoints of P are 2 corner vertices of G.*

Claim 4.2.9. *Every $n \times m$ grid or (n, m, r, s)-imperfect grid G with an even number of vertices has a perfect matching.*

Proof. Let P be a Hamiltonian path $v_1 v_2 \ldots v_n$ in G. Then n is even. The edges $v_i v_{i+1}$ where i is odd and $1 \leq i < n$ yields a perfect matching of P and so a perfect matching of G. $\square$

It is also well known that $n \times m$ grids with even order have Hamiltonian cycles when $n \geq 2$ and $m \geq 2$. Claim 4.2.11 gives the analog for imperfect grids. To avoid cumbersome notation which confuses a simple concept, the proof uses pictures to illustrate the desired Hamiltonian cycles.

Claim 4.2.10. *An $n \times m$ grid has a Hamiltonian cycle if and only if $n \geq 2$ and $m \geq 2$ and nm is even.*

Claim 4.2.11. *An (n, m, r, s)-imperfect grid G where $n \geq 3$ and $m \geq 3$ has a Hamiltonian cycle if and only if $r > 1$ and $m - s > 1$ and G has even order.*

Proof. ($\Rightarrow$)If $r = 1$ or $m - s = 1$, then G has a degree 1 vertex and so does not have a Hamiltonian cycle. Because G is bipartite, G cannot have an odd cycle. Hence, if G has a Hamiltonian cycle, then G must have even order.

(⇐) We can view G as an $n \times m$ grid with an $s \times (n-r)$ grid deleted from the SE corner. Thus, G has $nm - s(n-r) = nm - sn + sr$ vertices and since G has even order, this quantity must be even. If n is even, then either s or r must be even for $nm - sn + sr$ to be even. See Figure 4.3(a)-(b) for the Hamiltonian cycle through G in these cases. Figure 4.3(a) and Figure 4.3(b) require the hypotheses $m - s > 1$ and $r > 1$, respectively. Now assume m is even. Then s or $n - r$ must be even. By reflecting G across a line between the NW corner of G and the cutout vertex of G, G becomes an $(m, n, m-s, n-r)$-imperfect grid and the roles of n and m switch. Thus, finding an Hamiltonian cycle through G reduces to the previous case. Finally, if n and m are odd, then s and $n - r$ must be odd for $nm - s(n-r)$ to be even, or equivalently, s must be odd and r must be even. Then $m - s$ is even and Figure 4.3(c) demonstrates the desired Hamiltonian cycle.

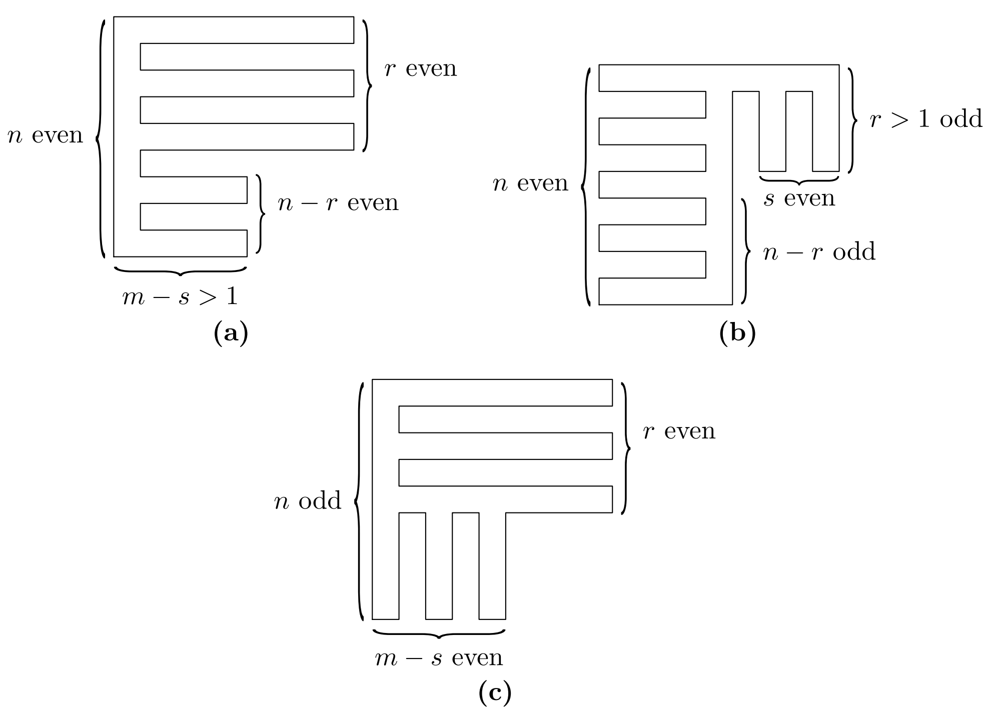


Figure 4.3: Hamiltonian cycles in imperfect grids

□

Corollary 4.2.12. *An (n, m, r)-imperfect grid G where $n \geq 3$ and $m \geq 3$ has a Hamiltonian cycle if and only if $r > 1$ and G has even order.*

Proof. Follows from letting $s = 1$ in Claim **??**. $\square$

Theorem 4.2.13. *Given integers $n \geq 2$ and $m \geq 2$, let G be an $n \times m$ grid. Let n', m', r' be positive integers where $n' \leq n, m' \leq m$, and $1 \leq r' < n$. Let d_0, d_1, d_2, d_3, d_4 and $d'_0, d'_1, d'_2, d'_3, d'_4$ be non-negative integers where $d'_i \leq d_i$ and where d_1 and d'_1 have the same parity. Let G' be an (n', m', r')-imperfect grid. If G' has a $[d'_0, d'_1, d'_2, d_3, d_4]$-factor H', then G has a $[d_0, d_1, d_2, d_3, d_4]$-factor if $d_1 - d'_1 > 0$ or if the fifth corner of G' is degree 1 in H'. Similarly, if an $n' \times m'$ grid G'' has a $[d'_0, d'_1, d'_2, d_3, d_4]$-factor H'', then G has a $[d_0, d_1, d_2, d_3, d_4]$-factor if $d_1 - d'_1 > 0$ or if any corner of G'' is degree 1 in H''.*

Proof. First assume an (n', m', r')-imperfect grid G' has a $[d'_0, d'_1, d'_2, d_3, d_4]$-factor H'. Imagine that G' corresponds to a subgraph in the NW corner of an $n' \times m$ grid $\hat{G}$. In other words, the NW corner of G' is the NW corner of $\hat{G}$. Let v be fifth corner of G'. Then v is the vertex in row r and column m of $\hat{G}$. Note that $\hat{G} - G'$ is an imperfect grid. There is a Hamiltonian path P through $\hat{G} - G'$ with endpoint v_S. Let b be the other endpoint of P and note that b is either the NE or SE corner of $\hat{G}$. If $n' = n$, then $G' \cup P$ is a factor of an G. Otherwise, $n' < n$. Subdivide the final edge ab of P $(n - n')m$ times so that $G' \cup P$ is a factor of G. See Figure 4.4 for clarification.

Note that $d_1 - d'_1$ is even since d_1 and d'_1 have the same parity. If $d_1 - d'_1 > 0$, then G' has a $[d'_0, d'_1, d'_2, d_3, d_4]$-factor H' by hypothesis and P has a $[d_0 - d'_0, d'_1 - d_1, d_2 - d'_2, 0, 0]$-factor by Claim 4.2.2. Hence, $H' \cup P'$ is a $[d_0, d_1, d_2, d_3, d_4]$-factor of $G' \cup P$ and thus of G.

Now assume that $d_1 = d'_1$. Assume the fifth vertex v of G' is degree 1 in H'. If $d_2 = d'_2$, H' with $d_0 - d'_0$ additional isolated vertices is the desired factor of G. If $d_2 - d'_2 = 1$, add the edge vv_S to H' and $d_0 - d'_0$ additional isolated vertices to obtain the desired factor of G. Otherwise, $d_2 - d'_2 \geq 2$. By Claim 4.2.2, P has a $[d_0 - d'_0, 2, d_2 - 2, 0, 0]$-factor where the endpoint v_S of P is degree 1 in P'. $H' \cup P'$

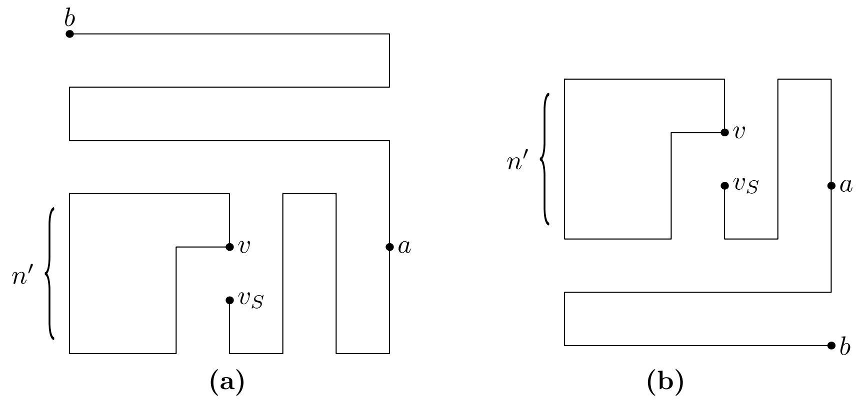


Figure 4.4: $G' \cup P$ is a factor of G

is a $[d_0, d_1 + 2, d_2 - 2, d_3, d_4]$-factor of $G' \cup P$. Add the edge vv_S to $H' \cup P'$ to yield a $[d_0, d_1, d_2, d_3, d_4]$-factor of $G' \cup P \cup vv_S$ and thus of G.

Follow a procedure similar as above to show that if an $n' \times m'$ grid G'' has a $[d'_0, d'_1, d'_2, d_3, d_4]$-factor H'', then G has a $[d_0, d_1, d_2, d_3, d_4]$-factor if $d_1 - d'_1 > 0$ or if any corner of G'' is degree 1 in H''. □

Our strategy in upcoming proofs is often to find factors H of an imperfect grid G' where G' is a subgraph within a grid G. We then manipulate H to obtain another factor of G. Furthermore, we often assume H has the property that any vertex which is degree 4 in G' is also degree 4 in H. Per Claim 4.1.7, if G' is an (n, m, r)-imperfect grid then G' has $(n-2)(m-2)-(n-r-1)$ degree 4 vertices. Hence, any factor H with this property is a $[d_0, d_1, d_2, d_3, (n-2)(m-2)-(n-r-1)]$ of G'. Claim 4.2.14 proves a few properties of such a factor H when $d_0 = 0$.

Claim 4.2.14. *Let G be an (n, m, r)-imperfect grid where $n \geq 3$ and $m \geq 3$ and $2 \leq r \leq n-1$. Let d_1, d_2, d_3 be non-negative integers where $d_2 \geq 5$. Let d_0, d_1, d_2, d_3 be non-negative integers. If G has a $[0, d_1, d_2, d_3, (n-2)(m-2)-(n-r-1)]$-factor, then the following must hold.*

1. *d_1, d_2, d_3 sum to $2n + 2m - 5$*

2. *d_1 and d_3 have the same parity*

3. $d_1 + d_3 \geq 2$

4. $d_1 + d_3 \geq 4$ *if* n *and* r *are both odd*

Proof. There are $2n - 2m - 5$ degree 2 and degree 3 vertices on the border of G per Claim 4.1.7. Thus, d_1, d_2, d_3 must sum to $2n + 2m - 5$. By Claim 4.2.1, d_1 and d_3 have the same parity.

We now show $d_1 + d_3 \geq 2$. Let G' be a DUP with 4 paths whose orders match the number of vertices on the N,S,W, and E1 walls. In any factor H of G, let P_N, P_S, P_{E1}, P_W be the subgraphs of H induced by the vertices on the N, S, E1, W walls of G, respectively. Then H yields a factor $H' = P_N \cup P_S \cup P_{E1} \cup P_W$ of the DUP G'. Per Theorem 2.1.3, the number of degree 0 and degree 2 vertices in H' is at least as large as the number of paths with odd order, or equivalently, the number of walls of the N,S,W, and E1 walls which are odd in G.

A degree 0 vertex v in H' corresponds to a degree 1 vertex in H since $d_0 = 0$. A degree 2 vertex v in H' corresponds to a degree 3 vertex in H. Hence, the number of degree 0 vertices and the number of degree 2 vertices in H' is at most $d_1 + d_3$. Thus, $d_1 + d_3$ is at least as big as the number of walls from the N,S,W, and E1 walls which are odd in G. Note that the N and S walls have m and $m - 1$ vertices, respectively, so one of these is always odd. Hence, $d_1 + d_3 \geq 1$ and so $d_1 + d_3 \geq 2$ by parity. If n and r are both odd, then the W wall and E1 are odd too. Then the W wall and E1 wall in addition to either the N or S wall are odd and so $d_1 + d_3 \geq 3$. Thus, $d_1 + d_3 \geq 4$ by parity. ☐

An imperfect grid has 5 degree 2 corners. For this reason, it is helpful to assume $d_2 \geq 5$ when examining factors of an imperfect grid. In fact, Theorem 4.2.16 proves that if $d_2 \geq 5$, then the conditions in Claim 4.2.14 are sufficient to find a $[0, d_1, d_2, d_3, (n-2)(m-2) - (n-r-1)]$-factor of an (n, m, r)-imperfect grid. Theorem 4.2.16 relies on the *full rung property*, which we define now.

Definition 4.2.15. *Let* $v_{i,j}$ *denote the vertex in row* i *and column* j *of an* $n \times m$ *grid* G. *Let* H *be a factor of* G. *The factor* H *has the* ***full rung property between rows*** i ***and*** $i+1$ *if the edge* $v_{i,j}v_{i+1,j}$ *exists in* H *for all* $j \in [1, m]$. *The factor* H

*has the **full rung property between columns** i **and** $i+1$ if the edge $v_{j,i}v_{j,i+1}$ exists in H for all $j \in [1, n]$.*

Essentially, a factor H of a grid G has the full rung property between rows i and $i+1$ if all edges between these rows in G exist in H as well. These edges look likes rungs of a ladder, thus explaining the name of this property. The full rung property between columns i and $i+1$ is defined similarly. See Figure 4.8 (a) for an example of a factor with the full rung property between rows 1 and 2 as well as between columns 1 and 2.

Theorem 4.2.16. *Let G be an (n, m, r)-imperfect grid where $n \geq 3$ and $m \geq 3$ and $2 \leq r \leq n-1$. Let d_1, d_2, d_3 be non-negative integers where $d_2 \geq 5$. G has a $[0, d_1, d_2, d_3, (n-2)(m-2)-(n-r-1)]$-factor if the following conditions hold.*

1. *d_1, d_2, d_3 sum to $2n+2m-5$*
2. *d_1 and d_3 have the same parity*
3. *$d_1 + d_3 \geq 2$*
4. *$d_1 + d_3 \geq 4$ when n and r are both odd.*

Furthermore, if $d_1 > 0$ and the above conditions hold, then except in the following cases, some factor H exists in which the fifth corner of G is degree 1 in H. However, except when $d_1 = 1$ and $r = 2$, we may assume a corner vertex other than the fifth corner of G is degree 1 in H.

Proof. We first make the following definitions.

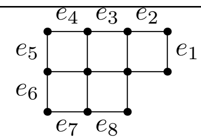
1. To *insert a column with 2 degree 3 endpoints* in a factor with the full rung property between columns i and $i+1$ means to do the following. Subdivide all edges between columns i and $i+1$. Let a_j be the vertex introduced into row j. Insert the edges $a_1a_2, a_2a_3, \ldots a_{n-1}a_n$. Then a_1 and a_3 are degree 3 in the resulting factor.

2. Let v_1, v_n be the northmost and southmost vertices in column i, respectively. Let w_1, w_n be the northmost and southmost vertices in column $i+1$, respectively. To *insert 2 columns with 4 degree 2 endpoints* between columns i and $i+1$ means to do the following. Subdivide all edges that exist between columns i and $i+1$ twice, introducing the vertices a_j and b_j and the edge a_jb_j into row j. Only the northmost and southmost edges maybe missing. If v_1w_1 is missing, add 2 vertices a_1, b_1 to row 1 and an edge between them. Else, delete a_1b_1. If v_nw_n is missing, add 2 vertices a_1, b_n to row 1 and an edge between them. Else, delete a_nb_n. Note that a_1, a_n, b_1, b_n are degree 2 in the resulting factor.

3. To *insert 1 column with 2 degree 1 endpoints* in a factor between columns i and $i+1$ where the northmost and southmost edges are missing, means to do the following. Subdivide all edges between columns i and $i+1$. Let a_j be the vertex introduced into row j. Also add the vertices a_1, a_2. Insert the edges $a_1a_2, a_2a_3, \ldots a_{n-1}a_n$.

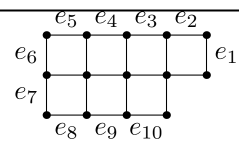
4. To *insert 1 column with a degree 1 and degree 3 endpoint* between columns i and $i+1$ where the northmost edge is missing but the southmost edge exists (or vice versa), means to do the following. Subdivide all edges between columns i and $i+1$, thus introducing a new vertex a_j in row j. Assume the northmost edge is missing. Add the vertex a_1 to row 1 between columns i and $i+1$. Insert the edges $a_1a_2, a_2a_3, \ldots a_{n-1}a_n$. Then a_1 is degree 1 and a_3 is degree 3 in the resulting factor.

We start by using induction on m to show the claim holds for $n = 3$. In this case, $r = 2$. Table 4.2 and Table 4.3 shows all factors which satisfy the hypotheses when $m = 3$ and $m = 4$. Notice that each factor has the full rung property between columns i and $i+1$ for some i. Also, if $d_3 \leq 1$, the factor does not have an edge between the northmost vertices in columns i and $i+1$ nor between the southmost vertices in columns i and $i+1$ for some $i \leq m-2$. Finally, if $d_1 > 0$, then except when $d_1 = 1$, the fifth corner of G is degree 1 in the given factors. Now consider an



d_2	d_1	d_3	edges to remove
5	0	4	
5	1	3	e_8
5	2	2	e_1, e_3
5	3	1	e_1, e_3, e_8
5	4	0	e_1, e_3, e_5, e_8

Table 4.2: Factors of a $(3, 4, 2)$-imperfect grid



d_2	d_1	d_3	edges to remove
7	0	4	e_4
7	1	3	e_4, e_{10}
7	2	2	e_1, e_4, e_9
7	3	1	e_1, e_4, e_6, e_9
7	4	0	e_1, e_3, e_4, e_6, e_9
5	0	6	
5	1	5	e_{10}
5	2	4	e_1, e_3
5	3	3	e_1, e_3, e_{10}
5	4	2	e_1, e_3, e_6, e_{10}
5	5	1	$e_1, e_3, e_4, e_6, e_{10}$
5	6	0	$e_1, e_3, e_4, e_6, e_9, e_{10}$

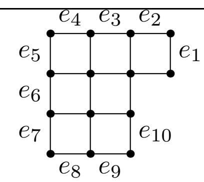
Table 4.3: Factors of a $(3, 5, 2)$-imperfect grid

(n, m, r)-imperfect grid G where $n = 3$ and $m \geq 5$.

If $d_2 \geq 9$, induction yields a $[0, d_1, d_2 - 4, d_3, d_4]$-factor H' of an $(n, m-2, r)$-imperfect grid G'. Insert 2 columns with 4 degree 2 endpoints between the columns 1 and 2 in H' for the desired factor. If $d_2 \leq 8$, because $d_1 + d_3$ is even and $d_1 + d_2 + d_3 = 2n + 2m - 5$ by hypothesis, we see that d_2 is odd and so $d_2 = 5$ or $d_2 = 7$. If $d_1 + d_3 = 4$, then $d_1+d_2+d_3 = 2n+2m-5$ implies that $m = 5$ and $d_2 = 7$, which is a base case. Thus, $d_1 + d_3 \geq 6$. If $d_3 \geq 2$, induction yields a $[0, d_1, d_2, d_3 - 2, d_4]$-factor H' of an $(n, m-1, r)$-imperfect grid G' and H' has the full rung property between columns i and $i+1$ for some i. Insert 1 column with 2 degree 3 endpoints between these columns for the desired factor. Otherwise, $d_3 \leq 1$ and so $d_1 \geq 5$. Induction yields a $[0, d_1 - 2, d_2, d_3, d_4]$-factor H' of an $(n, m-1, r)$-imperfect grid G', and for some $i \leq m-2$, H' does not have an edge between the northmost vertices in columns i and $i+1$ nor between the southmost vertices in columns i and $i+1$. Insert 1 column with 2 degree 1 endpoints between columns i and $i+1$ for the desired factor.

Hence, the claim holds for $n = 3$. We now show that the claim holds for $n = 4$. In this case $r = 2$ or $r = 3$. The reader can verify the claim holds for $m = 3$. Otherwise, Table 4.4 through Table 4.7 shows all factors when $m = 4$ and $m = 5$. Notice that each factor has the full rung property between columns i and $i+1$ for some i except when $d_3 = 0$ and $r = 2$, and in the case where $d_3 = 1$, the column with the full rung property has an edge in row 1 that is incident to a 2 vertex u. If $d_3 \leq 1$, the factor does not have an edge between the northmost vertices in columns i and $i+1$ nor between the southmost vertices in columns i and $i+1$ for some $i \leq m-2$, except again when $d_3 = 0$ and $r = 2$. If $d_3 = 0$ and $r = 2$, the factor has an edge between the northmost vertices in columns 1 and 2, one of which is degree 2 in the factor, and does not have an edge between the southmost vertices in columns 1 and 2. Finally, if $d_1 > 0$, then except when $d_1 = 1$, the fifth corner of G is degree 1 in the given factors.

If $d_2 \geq 9$, proceed as in the previous case. Otherwise, $d_2 = 5$ or $d_2 = 7$, as before. Since $d_1 + d_3 + 7 \geq d_1 + d_2 + d_3 = 2n + 2m - 5 \geq 8 + 10 - 5 = 10$, and so $d_1 + d_3 \geq 6$. If $d_3 \geq 2$ and it is not the case that $d_3 = 2$ and $r = 2$, induction yields a $[0, d_1, d_2, d_3 - 2, d_4]$-factor H' of an $(n, m-1, r)$-imperfect grid G' and H' has the full rung property between columns i and $i+1$ for some i. Insert 1 column with 2 degree 3 endpoints between these columns for the desired factor. If it is the case that $d_3 = 2$ and $r = 2$, induction yields a $[0, d_1 - 1, d_2, d_3 - 1, d_4]$-factor H' of an $(n, m-1, r)$-imperfect grid G' and H' has a column with the full rung property where the edge in row 1 is incident to a degree 2 vertex u. Insert 1 column with 2



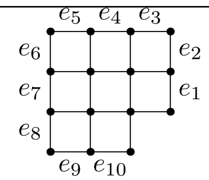
d_2	d_1	d_3	edges to remove
7	0	4	e_3
7	1	3	e_3, e_9
7	2	2	e_1, e_3, e_6
7	3	1	e_1, e_3, e_6, e_9
7	4	0	$e_1, e_3, e_6, e_8, e_{10}$
5	0	6	
5	1	5	e_9
5	2	4	e_1, e_6
5	3	3	e_1, e_6, e_9
5	4	2	e_1, e_3, e_5, e_6
5	5	1	e_1, e_3, e_5, e_6, e_9
5	6	0	$e_1, e_3, e_5, e_6, e_8, e_{10}$

Table 4.4: Factors of a $(4, 4, 2)$-imperfect grid

degree 3 vertices. Let v be the newly added vertex in row 1. v is a degree 3 vertex adjacent to u. Delete the edge uv for the desired factor.

Otherwise, $d_3 \leq 1$. Induction yields a $[0, d_1 - 2, d_2, d_3, d_4]$-factor H' of an $(n, m - 1, r)$-imperfect grid G'. If it is not the case that $d_3 = 0$ and $r = 2$, then H' does not have an edge between the northmost vertices in columns $m - 1$ and $m - 2$ nor between the southmost vertices in columns $m - 1$ and $m - 2$. Insert 1 column with 2 degree 1 endpoints between columns $m - 1$ and $m - 2$ for the desired factor. If it is the case that $d_3 = 0$ adn $r = 2$, then the factor has an edge between the northmost vertices in columns 1 and 2, and one of these, say w, is degree 2 in the factor, and the factor also does not have an edge between the southmost vertices in columns 1 and 2. Insert a degree 3 vertex t and a degree 1 vertex in this column. Then tv is an edge between a degree 3 and a degree 2 vertex. Delete this edge for the desired factor.

To finish the proof, proceed by induction on n and let $n = 3$ and $n = 4$ be the



d_2	d_1	d_3	edges to remove
7	0	4	e_7
7	1	3	e_1, e_7
7	2	2	e_1, e_7, e_{10}
7	3	1	e_1, e_4, e_8, e_{10}
7	4	0	$e_1, e_4, e_6, e_8, e_{10}$
5	0	6	
5	1	5	e_1
5	2	4	e_1, e_3
5	3	3	e_1, e_3, e_4
5	4	2	e_1, e_3, e_4, e_6
5	5	1	$e_1, e_3, e_4, e_6, e_{10}$
5	6	0	$e_1, e_3, e_4, e_6, e_7, e_{10}$

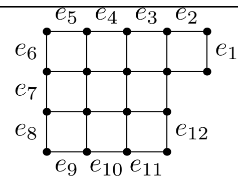
Table 4.5: Factors of a $(4, 4, 3)$-imperfect grid

base cases. The manipulations are similar to above, except we use the full rung property between rows as opposed to between columns.

□

The next 4 auxiliary claims prove to be very helpful. We summarize the differences between these claims now.

Claim 4.2.17 through Claim 4.2.20 all assume that d_1 is as small as possible, that is, $d_1 = 0$ if d_3 is even and $d_1 = 1$ if d_3 is odd. Claim 4.2.17 shows that a factor of a grid is possible when d_4 is as large as possible, that is, when $d_4 = (n-2)(m-2)$. On the other hand, Claim 4.2.18 through Claim 4.2.20 consider factors when d_3 is as large as desired. As a result, d_4 may be small and so we must remove edges appropriately from the interior of a grid or an imperfect grid to account for this. Furthermore, in order to allow d_3 to be as large as possible, Claim 4.2.18 through Claim 4.2.20 assume that d_2 is small, i.e., $d_2 = 4$ or $d_2 = 5$. This contrasts Claim



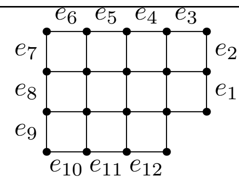
d_2	d_1	d_3	edges to remove
7	0	6	e_7
7	1	5	e_7, e_{11}
7	2	4	e_1, e_4, e_7
7	3	3	e_1, e_4, e_7, e_{11}
7	4	2	$e_1, e_4, e_7, e_{10}, e_{11}$
7	5	1	$e_1, e_4, e_6, e_7, e_{10}, e_{12}$
5	6	0	$e_1, e_3, e_4, e_6, e_7, e_{10}, e_{12}$
5	0	8	
5	1	7	e_{11}
5	2	6	e_1, e_4
5	3	5	e_1, e_4, e_{11}
5	4	4	e_1, e_4, e_{10}, e_{11}
5	5	3	$e_1, e_4, e_8, e_{10}, e_{11}$
5	6	2	$e_1, e_4, e_7, e_8, e_{10}, e_{11}$
5	7	1	$e_1, e_3, e_4, e_7, e_8, e_{10}, e_{11}$
5	8	0	$e_1, e_3, e_4, e_5, e_7, e_8, e_{10}, e_{12}$

Table 4.6: Factors of a $(4, 5, 2)$-imperfect grid

4.2.17 which allows d_2 to be as large as possible.

Claim 4.2.17. *Let G be an $n \times m$ grid where $n \geq 3$ and $m \geq 3$. Let d_1, d_2, d_3 be non-negative integers where $d_1 = 1$ and d_3 is odd. If d_1, d_2, d_3 sum to $2n + 2m - 4$ and $d_3 \geq 3$ and $d_2 \geq 4$, then G has a $[0, 1, d_2, d_3, (n-2)(m-2)]$-factor H where a corner vertex of G is degree 1 in H.*

Proof. Let $r = n - 1$ and let G' be an (n, m, r')-imperfect grid. Then G' has 1 less vertex than G. Per Theorem 4.2.16, G' has a $[0, 0, d_2 + 1, d_3 - 1, d_4]$-factor H'. Since H' has no degree 1 vertices, the fifth corner v of G' is degree 2 in H'. Add a pendant adjacent to v in H'. Doing so decreases the number of degree 2 vertices by 1 and also increases both the number of degree 3 vertices and the number of degree



d_2	d_1	d_3	edges to remove
7	0	6	e_8
7	1	5	e_1, e_8
7	2	4	e_1, e_3, e_8
7	3	3	e_1, e_3, e_4, e_8
7	4	2	e_1, e_3, e_4, e_5, e_8
7	5	1	$e_1, e_3, e_4, e_5, e_8, e_{12}$
5	6	0	$e_1, e_3, e_4, e_5, e_7, e_8, e_{12}$
5	0	8	
5	1	7	e_1
5	2	6	e_1, e_3
5	3	5	e_1, e_3, e_4
5	4	4	e_1, e_3, e_4, e_5
5	5	3	e_1, e_3, e_4, e_5, e_7
5	6	2	$e_1, e_3, e_4, e_5, e_7, e_8$
5	7	1	$e_1, e_3, e_4, e_5, e_7, e_8, e_{12}$
5	8	0	$e_1, e_3, e_4, e_5, e_7, e_8, e_{11}, e_{12}$

Table 4.7: Factors of a $(4, 5, 3)$-imperfect grid

1 vertices each by 1, thus yielding a $[0, 1, d_2, d_3, d_4]$-factor of G. $\square$

Claim 4.2.18. *Let G be an $n \times m$ grid where $n \geq 3$ and $m \geq 3$. Let d_1, d_2, d_3, d_4 be nonnegative integers that sum to nm where $d_1 + d_2 + d_3 \geq 2n + 2m - 4$. Let $d_1 = 1$ if d_3 is odd and $d_1 = 0$ otherwise. Then G has a $[0, d_1, 4, d_3, d_4]$-factor H where any corner vertex of G is degree 1 in H if $d_1 > 0$. Otherwise, all corners of G are degree 2 in H and are adjacent to 2 degree 3 vertices in H. Also, if $d_4 > 0$, then there exists a degree 4 vertex in H adjacent to a degree 3 vertex.*

Proof. The number of degree 3 vertices on the border of G is $2n+2m-8$ and is thus even. Removing an edge between 2 degree 4 vertices yields 2 degree 3 vertices and our goal is to remove edges between pairs of degree 4 vertices so as to obtain a factor with d_3 degree 3 vertices. Let P be the Hamiltonian path with endpoints through the interior of G'. If d_3 is even (and thus $d_1 = 0$), then the first $d_3 - (2n + 2m - 8)$ vertices of P yields a subpath with even order and thus has a perfect matching. Remove this matching for the desired $[0, 0, 4, d_3, d_4]$-factor of G. Otherwise, if d_3 is odd (and thus $d_1 = 1$), then use the previous argument to get a $[0, 0, 4, d_3 + 1, d_4]$-factor H of G. All corners are degree 2 vertices adjacent to a degree 3 vertex. Remove the edge in H between a corner vertex and one of its neighbors. This yields the desired $[0, 1, 4, d_3, d_4]$-factor of G. Note that if $d_4 > 0$, then one of the endpoints of P is degree 4 in H and is adjacent to a vertex on a wall of G with degree 3 in H. □

Claim 4.2.19. *Let G be an $n \times m$ grid where $n \geq 3$ and $m \geq 3$. Let d_1, d_2, d_3, d_4 be nonnegative integers that sum to nm where $d_1 + d_2 + d_3 \geq 2n + 2m - 4$. Let $d_1 = 1$ if d_3 is odd and $d_1 = 0$ otherwise. Then G has a $[0, d_1, 5, d_3, d_4]$-factor H where any corner vertex of G is degree 1 in H if $d_1 > 0$. Otherwise, all corners of G are degree 2 in H and are adjacent to at least 1 degree 3 vertices in H.*

Proof. Per Claim 4.2.18, G has a $[0, d_1, 4, d_3, d_4{+}1]$-factor H where any corner vertex of G is degree 1 in H if $d_1 > 0$. Also, since $d_4 + 1 > 0$, there exists a degree 4 vertex u in H adjacent to a degree 3 vertex v. Remove the edge uv from H to obtain the desired factor of G. □

Claim 4.2.20. *Let G be an (n, m, r)-imperfect grid where $n \geq 3, m \geq 3$, and $2 \leq r < n$. Let d_1, d_3, d_4 be nonnegative integers that sum to the order of G and $d_1{+}d_2{+}d_3 \geq 2n + 2m - 5$. Let $d_1 = 1$ if d_3 is odd and $d_1 = 0$ otherwise. Then G has a $[0, d_1, 5, d_3, d_4]$-factor H where the fifth corner of G is degree 1 in H if $d_1 > 0$ unless $r = 2$. Otherwise, all corners of G are degree 2 in H. Also, the cutout vertex is degree 3 in H and is adjacent in H to the fifth corner of G if $d_1{+}d_2{+}d_3 > 2n{+}2m{-}5$.*

Proof. Per Claim 4.1.7, there are $2n + 2m - 10$ degree 3 vertices on the border of G. Let v be the cutout vertex. Let P be the Hamiltonian path with endpoint through the degree 4 vertices of G. Then one of the endpoints of P is v. If d_3 is odd (and thus $d_1 = 1$), then remove the edge vv_N if $r > 2$. If $r = 2$, remove an edge adjacent to the SE corner of G, that is, the vertex in row n and column $m-1$. Then the border of the resulting factor has $2n+2m-11$ degree 3 vertices on the border of G. The first $d_3-(2n+2m-11)$ vertices of P yields a subpath with even order and thus has a perfect matching. Remove this matching for the desired $[0,1,5,d_3,d_4]$-factor of G. When d_3 is even, remove a perfect matching from the subpath of P of size $d_3-(2n+2m-10)$. This yields the desired $[0,0,5,d_3,d_4]$-factor of G. Note that if $d_1+d_2+d_3 > 2n+2m-5$ then the fifth corner is adjacent to v which is degree 3 id $d_1+d_2+d_3 > 2n+2m-5$. □

4.3 Grid Factors when $d_3 = d_4 = 0$

The main result of this section is Theorem 4.3.2 which characterizes $[d_0, d_1, d_2, d_3, d_4]$-factors of grids when $d_3 = d_4 = 0$. Hence, Theorem 4.3.2 answers the Factor Problem for Grids when the desired factor has no degree 3 or 4 vertices. Because this is a straightforward case, we prove it first for ease of future proofs. Claim 4.3.1 proves a pathological case.

Claim 4.3.1. *No grid has a $[d_0, 0, d_2, 0, 0]$-factor where d_2 is odd or $d_2 = 1$.*

Proof. A $[d_0, 0, d_2, 0, 0]$-factor is a realization of the sequence with d_2 2's. If $d_2 = 1$, the sequence $< 2 >$ is not realizable. If $d_2 \geq 3$ and odd, then any realization of a sequence with d_2 2's must have an odd cycle and so is not the factor of a grid per Corollary 4.2.6. □

Theorem 4.3.2. *Let G be an $n \times m$ grid where $m, n > 1$. Let d_0, d_1, d_2 be non-negative integers that sum to nm and assume d_1 is even. Then except in the following cases, G has a $[d_0, d_1, d_2, 0, 0]$-factor.*

1. $[d_0, 0, 2, 0, 0]$

2. $[d_0, 0, d_2, 0, 0]$, *d_2 is odd.*

Furthermore, if $d_1 > 0$ and G has a $[d_0, d_1, d_2, 0, 0]$-factor, then there exists such a factor with a degree 1 vertex at one of the corner vertices in G.

Proof. Claim 4.3.1 proves the given pathological cases are truly pathological. First assume $d_1 > 0$. Per Claim 4.2.8, G has a Hamiltonian path P with endpoints u, v where u and v are corner vertices in G. By Claim 4.2.1, d_1 is even. Then per Claim 2.1.1, P (and so G) has the desired factor where u or v is a degree 1 vertex and so G.

If $d_1 = d_2 = 0$, then nm isolated vertices yield the desired sequence. If $d_1 = 0$ and $d_2 > 0$ then d_2 is even and at least 4 or we are in a pathological case. If $d_2 \leq 2n$, then the desired factor is an even cycle within the first 2 rows of G and a set of d_0 additional isolated vertices. See Figure 4.5 for clarification. If $d_2 \leq 2m$, an even cycle within the first 2 columns yields the claim. Now assume $d_2 > 2n$ and $d_2 > 2m$. There exists m', r such that $d_2 = nm' + r$ where $0 \leq r < n$. Since $d_2 > 2n$ and $d_2 > 2m$, the hypotheses force that $n \geq 3$ and $m' \geq 3$. If $r = 0$, then an $n \times m'$ grid G' has even order because $d_2 = nm'$ is even. Per Claim 4.2.10, G' has a Hamiltonian cycle. Now if $r \geq 2$, an (n, m', r)-grid G' has even order and per Corollary 4.2.12, G' has a Hamiltonian cycle. These Hamiltonian cycles with an additional d_0 isolated vertices yield the desired factor of G. If $r = 1$, then $d_2 = nm' + 1$ implies $d_2 - 2 = nm' - 1 = n(m' - 1) + (n - 1)$. An $(n, m', n - 1)$-grid G' has even order and per Corollary 4.2.12, G' has a Hamiltonian cycle on $d_2 - 2$ vertices. The NE corner of G' is adjacent to v_S and v_W in this cycle. Add $d_0 + 2$ vertices to this cycle to yield a $[d_0 + 2, 0, d_2 - 2, 0, 0]$-factor H of G. Let x be the E neighbor of v_S in G. Delete the edge vv_S add the edges vv_E and $v_S w$ and $v_E w$ to yield a $[d_0, 0, d_2, 0, 0]$-factor of G.

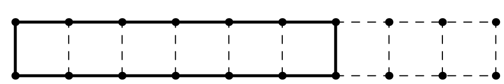


Figure 4.5: An even cycle factor within 2 rows of G

□

4.4 Grid Factors when $d_1 + d_2 < 4$

Theorem 4.4.4 characterizes $[d_0, d_1, d_2, d_3, d_4]$-factors when $d_1 + d_2 < 4$ and is the main result of this section. This case is rather trivial because of the following logic. Every factor of a grid with degree 3 and 4 vertices must have at least 4 'corners' and these corners must be degree 1 or 2. Claim 4.4.1 formalizes this concept by showing that any factor of a grid with degree 3 or 4 vertices must have at least 4 degree 1 and degree 2 vertices except in the specific case when $d_4 = 0$ and $d_3 = 1$. Hence, when $d_1 + d_2 < 4$, the list of possible $[d_0, d_1, d_2, d_3, d_4]$-factors is short.

Claim 4.4.1. *If a grid has a $[d_0, d_1, d_2, d_3, d_4]$-factor where $d_3+d_4 > 0$, then $d_1+d_2 \geq 4$ except possibly when $d_3 = 1$ and $d_4 = 0$.*

Proof. First assume $d_4 > 0$. Consider any degree 4 vertex v in H. From v_N, walk along edges in H to vertices that are N or W of the current vertex until this is not possible. (If we have a choice between N or W, it does not matter which we pick.) Let a be the final vertex in this walk. Then a cannot have a N or W neighbor. Hence, the degree of a in the factor is 2 or 1. Similarly, from v_E, walk N or E until doing so is no longer possible. The final vertex in this walk is also degree 2 or 1 and is distinct from a because these walks never intersect. By walking S or E from v_S and by walking S or W from v_W until it is no longer possible to do so, we also arrive at two more distinct vertices of degree 1 or 2. Hence, $d_1 + d_2 \geq 4$.

Now assume that $d_4 = 0$ and $d_3 \geq 2$. Let v and u be two distinct degree 3 vertices in H. Let G' be the subgraph induced by v, u, and the neighbors of v and u. G' may or may not be connected, but nonetheless, G' must have at least 4 vertices distinct from v and u. These vertices may be degree 1, 2, or 3 in the factor H, but nonetheless, we can again define NE, SE, SW, or NW walks from these vertices to argue that $d_1 + d_2 \geq 4$, as before. See Figure 4.6 for examples.

□

We now give two auxiliary claims which prove pathological cases before proving our main result of this section, Theorem 4.4.4.

Claim 4.4.2. *No grid has a $[d_0, d_1, d_2, 1, 0]$-factor where $d_1 + d_2 < 3$.*

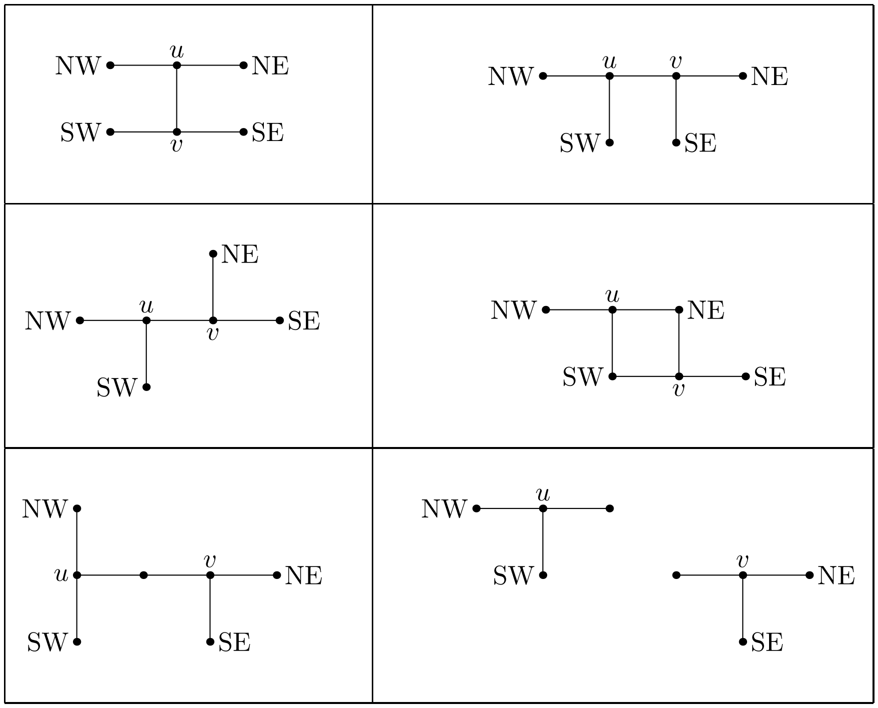


Figure 4.6: Assigning NE, SE, NW, SW walks

Proof. A sequence with d_i entries of the integer i is not realizable if $d_3 = 1$ and $d_1 + d_2 < 3$. $\square$

Claim 4.4.3. *No grid has a* $[d_0, 1, 2, 1, 0]$*-factor.*

Proof. A $[d_0, 1, 2, 1, 0]$-factor is realization of the sequence $< 3, 2, 2, 1 >$ plus isolated vertices. However, $< 3, 2, 2, 1 >$ is uniquely realizable and this unique realization has an odd cycle and so is not the factor of a grid per Corollary 4.2.6. $\square$

Theorem 4.4.4. *Let G be an $n \times m$ grid where $n \geq 2$ and $m \geq 2$. Let d_0, d_1, d_2, d_3, d_4 be nonnegative integers that sum to nm where d_1 and d_3 have the same parity and $d_3 + d_4 \leq nm - 4$. If $d_1 + d_2 < 4$, then the $[d_0, d_1, d_2, d_3, d_4]$-factors of G are exactly the following.*

1. $[d_0, 0, 0, 0, 0]$

2. $[d_0, 2, 0, 0, 0]$

3. $[d_0, 2, 1, 0, 0]$

4. $[d_0, 3, 2, 1, 0]$

Proof. Per Claim 4.4.1, if $d_1 + d_2 < 4$, then $d_3 = 0$ or $d_3 = 1$. First assume $d_3 = 0$. Then the claim holds per Theorem 4.3.2. Now assume $d_3 = 1$ and so by parity d_1 is odd. Since $1 = d_3 + d_4 \leq nm - 4$, it follows that $nm \geq 5$ and so at least n or m, say m, is at least 3. By Claim 4.4.2, $d_1 + d_2 \geq 3$, and by hypothesis, $d_1 + d_2 < 4$. Hence, $d_1 + d_2 = 3$. If $d_1 = 1$, then $d_2 = 2$ and this case is a pathological case by Claim 4.4.3. If $d_1 = 3$, then $d_2 = 1$ and Figure 4.7 with d_0 isolated vertices is a factor of any $n \times m$ grid where $n \geq 2$ and $m \geq 3$. $\square$

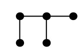
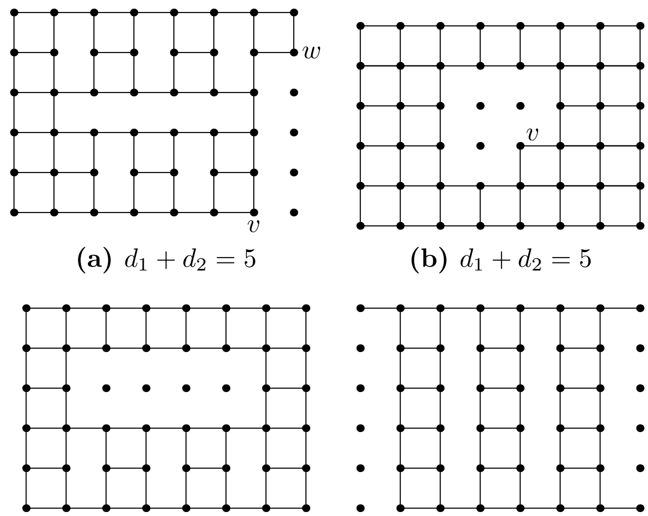


Figure 4.7: A $[d_0, 3, 1, 1, 0]$-factor

4.5 Grid Factors when $d_1 + d_2 = 4$

In this section, we briefly discuss factors of grids when $d_1 + d_2 = 4$. If $d_3 = d+4 = 0$, Theorem 4.3.2 characterizes when such factors are possible. If $d_3 + d_4 > 0$, then Claim 4.4.1 shows that it is impossible for a grid to have a $[d_0, d_1, d_2, d_3, d_4]$-factor when $d_1 + d_2 < 4$. Therefore, in a sense, any factor where $d_1 + d_2 = 4$ and $d_3 + d_4 > 0$ comes very close to being impossible. As a result, it is reasonable that the case when $d_1 + d_2 = 4$ is restrictive. We exemplify this now.

When searching for a factor H of a grid G where $d_0 > 0$, we often wish to 'deal with' degree 0 vertices by removing rows and columns from G until d_0 is small, that is, until $d_0 < \min\{n, m\}$. However, when $d_0 < \min\{n, m\}$, degree 0 vertices on the border increase the required number of degree 1 and 2 vertices elsewhere in the factor. Consider the factor shown in Figure 4.8 (a). Since $d_0 < \min\{n, m\}$, any placement of the degree 0's on the E wall forces more 'corners' in the factor which cannot be degree 3 or 4. The vertices v and w in Figure 4.8 (a) are examples of such 'corners.' Even if we place the degree 0 vertices in the interior of the grid, as shown in Figure 4.8 (b), we can still create more 'corners.' These 'corners' therefore force that the value of $d_1 + d_2$ becomes greater than 4. Now consider the factors in Figure 4.8 (c)-(d). Because $d_1 + d_2 = 4$ in both factors, the degree 0 vertices must be carefully arranged to not cause any additional 'corners.' This therefore greatly restricts the final shape of the factor.



(a) $d_1 + d_2 = 5$ **(b)** $d_1 + d_2 = 5$

(c) $d_1 + d_2 = 4$ **(d)** $d_1 + d_2 = 4$

Figure 4.8: Factors when $d_1 + d_2$ is small

Based on this discussion, it is sensible that the case when $d_1 + d_2 = 4$ reduces to a case-by-case analysis. We make the following conjecture about the structure of factors when $d_1 + d_2 = 4$ and $d_3 + d_4 > 0$.

Conjecture 4.5.1. *Let H be a $[d_0, d_1, d_2, d_3, d_4]$-factor of an $n \times m$ grid G where $d_1 + d_2 = 4$ and $d_3 + d_4 > 0$. Let H^+ be the subgraph of H that remains when all isolated vertices of H are removed. Then H^+ is one of the following:*

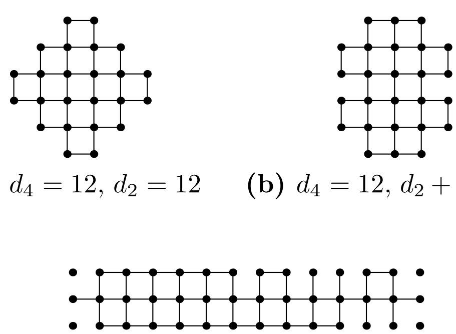
1. *A factor of a grid G' where possibly G' has subgrids deleted. All edges on the border of G' are in H^+. [See Figure 4.8 (c).]*

2. *A factor of a grid G' with additional pendants where possibly G' has subgrids deleted. All the edges on the border of G' are in H^+. [See Figure 4.8 (d).]*

4.6 Grid Factors when $d_4 > 0$ and $d_1 + d_2 \geq 5$

In this section, we summarize what we know concerning factors with degree 4 vertices. As indicated in Claim 4.2.3, an $n \times m$ grid when $n \leq 2$ or $m \leq 2$ has no degree 4 vertices. Hence, all theorems in this section assume $n \geq 3$ and $m \geq 3$. The main result of this section is Theorem 4.6.6, which shows that when $d_1 + d_2 + d_3$ are 'large enough', a $[d_0, d_1, d_2, d_3, d_4]$-factor of a grid when $d_4 > 0$ is possible. We now formalize what me mean by 'large enough.'

Let H be a factor of a grid and let H_4 be the set of vertices of G which are degree 4 in H. If a vertex v is in the neighborhood of H_4 in G, then v must have positive degree in H. Furthermore, v must have degree 1, 2, or 3 in H since $v \notin H_4$. Thus, the vertices in H_4 force degree 1, 2, and 3 vertices in $H - H_4$. We wish to know the minimum number of degree 1, 2, and 3 vertices that are forced by H_4. Depending on the layout of H_4, this number changes. For example, Figure 4.9 shows 3 factors of a grid (minus isolated vertices) where $d_4 = 12$ but the values of $d_1 + d_2 + d_3$ differ.



(a) $d_4 = 12$, $d_2 = 12$ **(b)** $d_4 = 12$, $d_2 + d_3 = 14$

(c) $d_4 = 12$, $d_1 + d_2 + d_3 = 26$

Figure 4.9: Factors where $d_4 = 12$

We define $B(n, m, d_4)$ to capture the minimum number of degree 1, 2, and 3 vertices in the neighborhood of H_4. We use the variable B to remind us that the vertices on the *border* of H_4, that is, the vertices in H_4 which are adjacent to vertices in $H - H_4$, force degree 1, 2, and 3 vertices elsewhere in H.

Definition 4.6.1. *Given a factor H of an $n \times m$ grid G, let H_4 be the set of vertices of G which are degree 4 in H. Let $|N(H_4)|$ denote the number of vertices in the neighborhood of H_4 in H. For a fixed d_4 value, let $\mathcal{S}(n, m, d_4)$ be the set of all $[d_0, d_1, d_2, d_3, d_4]$-factors of G. Define $B(n, m, d_4)$ as such:*

$$B(n, m, d_4) = \min_{\forall H \in \mathcal{S}(n,m,d_4)} |N(H_4)|$$

.

It follows immediately from the Definition 4.6.1 that $d_1 + d_2 + d_3$ must be at least as large as $B(n, m, d_4)$ in any $[d_0, d_1, d_2, d_3, d_4]$-factor of a grid. Claim 4.6.2 captures this.

Claim 4.6.2. *In any $[d_0, d_1, d_2, d_3, d_4]$-factor of an $n \times m$ grid G where $d_4 > 0$, $d_1 + d_2 + d_3 \geq B(n, m, d_4)$.*

The reader may ask why the value of $B(n, m, d_4)$ is dependent on n and m. Note that the factor in Figure 4.9(a) has the smallest $d_1 + d_2 + d_3$ value of those shown. However, this factor clearly does not 'fit' in an 3×14 grid whereas the factor shown in Figure4.9(c) does. The configuration of any factor is constrained by n and m and thus $B(n, m, d_4)$ is as well.

In Claim 4.6.3, we give a lower bound on $B(n, m, d_4)$. We can interpret the quantities n_4 and m_4 in Claim 4.6.3 to be the least number of rows and columns, respectively, of G which must have a degree 4 vertex in any $[d_0, d_1, d_2, d_3, d_4]$-factor of G.

Claim 4.6.3. *Let G be an $n \times m$ grid. Let $n_4 = \lceil \frac{d_4}{n-2} \rceil$. Let $m_4 = \lceil \frac{d_4}{m-2} \rceil$. Define $B(n, m, d_4)$ as in Definition 4.6.1. If $d_4 > 0$, then $B(n, m, d_4) \geq \max\{2n_4 + 2, 2m_4 + 2\}$.*

Proof. By definition, n_4 indicates the least number of rows of G which must have a degree 4 vertex in any $[d_0, d_1, d_2, d_3, d_4]$-factor of G. Similarly, m_4 indicates the least number of columns of G which must have a degree 4 vertex in any $[d_0, d_1, d_2, d_3, d_4]$-factor of G. Let H be any $[d_0, d_1, d_2]$-factor of G. Consider any column i of G with a vertex that is degree 4 in H. Column i has northmost and southmost vertices v and w, respectively, which are degree 4 in H. Then v_N and v_S are neighbors of H_4. Since there are at least m_4 such columns, this yields at least $2m_4$ vertices in $N(H_4)$. Now consider the northwest-most vertex x with degree 4 in H and the northeast-most vertex y in G with degree 4 in H. Then x_W and y_E are 2 additional vertices in $N(H_4)$. Thus, any factor H must have at least $2m_4+2$ vertices in $N(H_4)$. A similar argument yields that $B(n, m, d_4) \geq 2m_4 + 2$ as well. □

Corollary 4.6.4. *Let G be an $n \times m$ grid and let $n_4 = \lceil \frac{d_4}{n-2} \rceil$ and $m_4 = \lceil \frac{d_4}{m-2} \rceil$. In any $[d_0, d_1, d_2, d_3, d_4]$-factor of G where $d_4 > 0$, $d_1+d_2+d_3 \geq \max\{2n_4+2, 2m_4+2\}$.*

Proof. Per Claim 4.6.2 and Claim 4.6.3, $d_1 + d_2 + d_3 \geq B(n, m, d_4) \geq \max\{2n_4 + 2, 2m_4 + 2\}$. □

The bound in Corollary 4.6.4 is tight. An example is the $[4, 6, 8, 12, 12]$-factor of a 3×14 grid shown in Figure 4.9(c). Here $m_4 = \lceil \frac{d_4}{n-2} \rceil = \lceil \frac{12}{3-2} \rceil = 12$. Also, $n_4 = \lceil \frac{d_4}{m-2} \rceil = \lceil \frac{12}{14-2} \rceil = 1$. Note that $d_1 + d_2 + d_3 = 26 = \max\{2n_4 + 2, 2m_4 + 2\}$.

As previously illustrated, the values of n and m may prevent a factor from having a desired $[d_0, d_1, d_2]$-factor. In an effort to obtain results despite this issue, we assume $d_1 + d_2 + d_3 \geq 2n + 2m_4 - 1$. Note that a grid with m_4 columns of degree 4 vertices in its interior has $m' = m_4 + 2$ columns in total. With this in mind, recall per Claim 4.1.7, an (n, m', r)-imperfect grid has $2n + 2m' - 5 = 2n + 2m_4 - 1$ degree 1, 2, and 3 vertices on its border. Hence, if $d_1 + d_2 + d_3 \geq 2n + 2m_4 - 1$, we have a possibility of creating factors H where H_4 is the shape of an $(n, m_4 + 2, r)$-imperfect grid. Theorem 4.6.6, the main result of this section, therefore assumes that $d_1 + d_2 + d_3 \geq \min\{2n_4 + 2m - 1, 2n + 2m_4 - 1\}$. Before proving Theorem 4.6.6, we capture a special case of this theorem, namely, when $n = 3$ or $m = 3$, in Theorem 4.6.5. We mention that there is much similarity between the proof of these theorems. However, organizing the proofs into two separate theorems prevents having a single theorem with a long list of cases that are complicated to verify.

Theorem 4.6.5. *Let G be an $n \times m$ grid where n or m equals 3 and the other is at least 3. Let d_0, d_1, d_2, d_3, d_4 be nonnegative integers that sum to nm where d_1 and d_3 have the same parity and $d_4 \leq (n-2)(m-2)$. Let $n_4 = \lceil \frac{d_4}{m-2} \rceil$ and $m_4 = \lceil \frac{d_4}{n-2} \rceil$. Assume $d_4 > 0$. If $d_1 > 0, d_2 \geq 5, d_1 + d_3 \geq 4$, and $d_1 + d_2 + d_3 \geq \min\{2n_4 + 2m - 1, 2n + 2m_4 - 1\}$, then G has a $[d_0, d_1, d_2, d_3, d_4]$-factor.*

Proof. Without loss of generality, assume $n = 3$. Note that $m_4 = d_4$ when $n = 3$. We argue now that $\min\{2n_4 + 2m - 1, 2n + 2m_4 - 1\} = 2n + 2m_4 - 1$. Since $2n + 2m_4 - 1 = 6 + 2d_4 - 1 = 2d_4 + 5$, it suffices to show $2n_4 + 2m - 1 \geq 2d_4 + 5$. Since $d_4 \leq (n-2)(m-2) = m - 2$, we see that $n_4 = \lceil \frac{d_4}{m-2} \rceil \leq 1$. Then $2n_4 + 2m - 1 \geq 2 + 2m - 1 = 2m + 1 \geq 2(d_4 + 2) + 1 \geq 2d_4 + 5$. Thus, we may assume $d_1 + d_2 + d_3 \geq 2n + 2m_4 - 1 = 2d_4 + 5$ for the rest of the claim.

Let $m' = d_4 + 2$ so that $2n + 2d_4 - 1 = 2n + 2m' - 5$. Let $r = 2$. Note that there are $(n-2)(m'-2) - (n - r - 1) = d_4$ degree 4 vertices in an (n, m', r)-imperfect grid G^* per Claim 4.1.7. Claim 4.1.7 also yields that G^* has $2n + 2m' - 5$ vertices which are not degree 4 on its border.

Our strategy is as follows. We find a $[d'_0, d'_1, d'_2, d_3, d_4]$-factor H' of a grid G' or a $[d'_0, d'_1, d'_2, d_3, d_4]$-factor H'' an imperfect grid G'' where G' and G'' are subgraphs of G. Oftentimes, the imperfect grid of interest is G^*. Note that H and H'' have the desired number of degree 3 and degree 4 vertices in the factor that we seek. If $d'_1 > 0$, we take care to ensure that a corner vertex of G' is degree 1 in H' or the fifth corner of G'' is degree 1 in H''. This property allows us to use Theorem 4.2.13 to show that a $[d_0, d_1, d_2, d_3, d_4]$-factor of G exists.

Let $d'_2 = d_2$ and $d'_1 = d_1$. Decrease d'_2 by 1 and d'_1 by 2 until doing so would violate one of the following inequalities: $d'_2 \geq 5, d'_1 + d_3 \geq 4$, and $d'_1 + d'_2 + d_3 \geq 2n + 2m_4 - 1$. Note that d_1 and d'_1 have the same parity when this process is done.

Case I - $d'_2 \geq 6$: Since $d'_2 \geq 6$, we see that $d'_1 + d'_2 - 1 + d_3 < 2n + 2m_4 - 1$ because otherwise we can decrease d'_2 again. Since $2n + 2m_4 - 1 \leq d'_1 + d'_2 + d_3 < 2n + 2m_4$ we see that $d'_1 + d'_2 + d_3 = 2n + 2m_4 - 1$. By Theorem 4.2.16, G^* has a $[0, d'_1, d'_2, d_3, d_4]$-factor H^* where the fifth corner v is degree 1 in H^* if $d'_1 > 0$ unless $d'_1 = 1$. If $d_1 - d'_1 > 0$ or v is degree 1 in H^*, then G has the desired factor per Theorem 4.2.13. Otherwise, $d_1' = d_1 = 1$ and thus d_3 is odd and at least 3 by hypothesis. If $d_2 = d_2'$, then add isolated vertices to H^* to obtain the desired factor of G. Else $d_2 \geq d_2' + 1$. Let $\hat{G}$ be an $n \times m'$ grid. Note that $\hat{G}$ has one more vertex than G^* since $r = 2$ and $n = 3$. Per Claim 4.2.17, $\hat{G}$ has a $[0, 1, d_2' + 1, d_3, d_4]$-factor where a corner vertex is degree 1. Then G again has the desired factor per Theorem 4.2.13.

Case II - $d_2' = 5$, $d_1' + d_3 = 4$, $d_1' \leq 1$: If $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$, follow the argument of Case I. Otherwise, $2d_4 + 5 = 2n + 2m_4 - 1 < d_1' + d_2' + d_3 = 9$ and so $2d_4 < 4$ or equivalently, $d_4 < 2$ and so $d_4 = 1$. If $d_1' = 1$, then $d_3 = 3$ since $d_1' + d_3 = 4$. Figure 4.10(a) shows a $[0, d_1', 4, d_3', 1]$-factor $\hat{G}$ of a 3×3 grid $\hat{G}$ with a degree 1 vertex in the corner and so G has the desired factor per Theorem 4.2.13. If $d_1' = 0$, then $d_3 = 4$ since $d_1' + d_3 = 4$. Then the 3×3 grid $\hat{G}$ is a $[0, 0, 4, d_3', 1]$-factor of itself. Since $d_1 > 0$ by hypothesis, we see that $d_1 - d_1' > 0$. Then G again has the desired factor per Theorem 4.2.13.

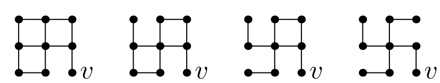


Figure 4.10: Factors of 3×3 grids

Case III - $d_2' = 5$, $d_1' + d_3 = 4$, $d_1' \geq 2$: If $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$, then by Theorem 4.2.16, G^* has a $[0, d_1', d_2', d_3, d_4]$-factor H^* where the fifth corner v is degree 1 since $d_1' \geq 2$. Use Theorem 4.2.13 to obtain the desired factor of G. If $d_1' + d_2' + d_3 > 2n + 2m_4 - 1$, then as in case II, we can argue that $d_4 = 1$. Figure 4.10(b)-(d) shows $[0, d_1', 4, d_3', 1]$-factor $\hat{H}$ of a 3×3 grid $\hat{G}$ when $d_1' \geq 2$ with a degree 1 vertex in the corner. Due to this degree 1 vertex in the corner, G again has the desired factor per Theorem 4.2.13.

Case IV - $d_2' = 5$, $d_1' + d_3 > 4$, $d_1' \geq 2$: By parity, $d_1' + d_3 > 4$ implies $d_1' + d_3 \geq 6$. Since $d_1' \geq 2$, we see that $d_1' - 2 + d_2' + d_3 < 2n + 2m_4 - 1$ because otherwise we can decrease d_1' by 2 again. Since $2n + 2m_4 - 1 \leq d_1' + d_2' + d_3 < 2n + 2m_4 + 1$

we see that $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$ or $d_1' + d_2' + d_3 = 2n + 2m_4$. However, $d_1' + d_3$ is even by parity and $d_2' = 5$ by assumption so $d_1' + d_2' + d_3$ is odd and thus $d_1' + d_2' + d_3 \neq 2n + 2m_4$. As a result, $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$. By Theorem 4.2.16, G^* has a $[0, d_1', d_2', d_3, d_4]$-factor H^* where the fifth corner v is degree 1 since $d_1' \geq 2$. Since v is degree 1 in H^*, G again has the desired factor per Theorem 4.2.13.

Case V - $d_2' = 5$, $d_1' + d_3 > 4$, $d_1' = 1$: Then $d_3 > 3$. If $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$, follow the argument of Case I. Otherwise, $d_1' + d_2' + d_3 > 2n + 2m_4 - 1$. Since $d_1' + d_2' = 6$, we see that $d_3 > 2n + 2m_4 - 7$, or equivalently, $d_3 \geq 2n + 2m_4 - 6$.

Since $n = 3$, n divides either $d_3 + d_4 + 6$, $d_3 + d_4 + 5$, or $d_3 + d_4 + 6$. In each case, it suffices to find a $[0, d_1', d_2', d_3, d_4]$-factor H of a subgrid of G where a corner of the subgrid is degree 1 in H. Then Theorem 4.2.13 yields that G has a $[d_0, d_1, d_2, d_3, d_4]$-factor .

If n divides $d_3 + d_4 + 6$, then $d_3 + d_4 + 6 = nk = 3k$ for some integer k. Since $d_3 > 3$ and $d_4 > 0$, we see that $3k = d_3 + d_4 + 6 \geq 11$ and so $k \geq 3$. Then by Claim 4.2.19, an $n \times k$ grid G' has a $[0, d_1', 5, d_3, d_4]$-factor H' where the NE corner v of G' is degree 1.

If n divides $d_3 + d_4 + 5$, then $d_3 + d_4 + 5 = nk$ for some integer k. The argument that $k \geq 3$ follows as above. Then by Claim 4.2.18, an $n \times k$ grid G' has a $[0, 0, 4, d_3 + 1, d_4]$-factor H' where the NE corner v of G' is degree 2 in H' and is adjacent to a degree 3 vertex w in H'. Delete the edge vw for a $[0, 1, 4, d_3, d_4]$-factor of G' in which v is degree 1.

Otherwise, n divides $d_3 + d_4 + 4$. Thus, $d_3 + d_4 + 4 = nk$ for some k. A similar argument as above shows that $k \geq 3$. Per Claim 4.2.19, an $n \times k$ grid G' with corner u has a $[0, 0, 5, d_3 - 1, d_4, 0]$-factor H'. In H', add a pendant v to u to yield a $[0, 1, 4, d_3, d_4, 0]$-factor of an $(n, k + 1, 1)$-imperfect grid.

$\square$

Theorem 4.6.6. *Let G be an $n \times m$ grid where $n \geq 3$ and $m \geq 3$. Let d_0, d_1, d_2, d_3, d_4 be nonnegative integers that sum to nm where d_1 and d_3 have the same parity and $d_4 \leq (n-2)(m-2)$. Let $n_4 = \lceil \frac{d_4}{m-2} \rceil$ and $m_4 = \lceil \frac{d_4}{n-2} \rceil$. Assume $d_4 > 0$. If $d_1 > 0, d_2 \geq 5, d_1 + d_3 \geq 4,$ and $d_1 + d_2 + d_3 \geq \min\{2n_4 + 2m - 1, 2n + 2m_4 - 1\}$, then G has a $[d_0, d_1, d_2, d_3, d_4]$-factor.*

Proof. Without loss of generality, assume $\min\{2n_4 + 2m - 1, 2n + 2m_4 - 1\} = 2n + 2m_4 - 1$. If $n = 3$ or $m = 3$, then the claim holds by Theorem 4.6.5 so assume $n \geq 4$ and $m \geq 4$. Note that $2n + 2m_4 - 1 = 2n + 2m' - 5$. Let $r = d_4 - (m_4 - 1)(n-2) + 1$. This value is of interest because there are d_4 degree 4 vertices in an (n, m', r)-imperfect grid G^*.

Our strategy is that of Theorem 4.6.5. To summarize, we find factors H of grids and imperfect grids which are subgraphs of G. We are careful to verify that an appropriate corner is degree 1 in H. We then use Theorem 4.2.13 to show that a $[d_0, d_1, d_2, d_3, d_4]$-factor of G exists. Let $d_2' = d_2$ and $d_1' = d_1$. Decrease d_2' by 1 and d_1' by 2 until doing so would violate one of the following inequalities: $d_2' \geq 5, d_1' + d_3 \geq 4$, and $d_1' + d_2' + d_3 \geq 2n + 2m_4 - 1$. Note that d_1 and d_1' have the same parity.

Case I - $d_2' \geq 6$: Since $d_2' \geq 6$, we see that $d_1' + d_2' - 1 + d_3 < 2n + 2m_4 - 1$ because otherwise we can decrease d_2' again. Since $2n + 2m_4 - 1 \leq d_1' + d_2' + d_3 < 2n + 2m_4$ we see that $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$. By Theorem 4.2.16, G^* has a $[0, d_1', d_2', d_3, d_4]$-factor H^* where the fifth corner v is degree 1 if $d_1 > 0$ unless $d_1' = 1$ and $r = 2$. If $d_1 - d_1' > 0$ or v is degree 1 in H^* then Theorem 4.2.13 yields the claim. Otherwise, $d_1' = d_1 = 1$ and $r = 2$ and so d_3 is odd by parity. If $d_2 = d_2'$, then add isolated vertices to H^* to obtain the desired factor of G. Otherwise, $d_2 - d_2' \geq 1$. Let $\hat{G}$ be an $(n, m', r+1)$-imperfect grid. Since $r + 1 = 3$ and $n \geq 4$, we see that $\hat{G}$ is indeed an imperfect grid and not a grid. Then $\hat{G}$ has a $[0, 0, d_2' + 2, d_3 - 1, d_4 + 1]$-factor $\hat{H}$ by Theorem 4.2.16. Also, Theorem 4.2.16 yields that the fifth corner u of $\hat{G}$ is degree 2 in $\hat{G}$ since $\hat{H}$ has no degree 1 vertices. Finally, Theorem 4.2.16 also implies that u is adjacent to the degree 4 vertex u_W in $\hat{H}$. Delete the edge uu_W for a $[0, 1, d_2' + 1, d_3, d_4]$-factor of $\hat{G}$ where the fifth corner v is degree 1 and so Theorem 4.2.13 again yields that the desired factor exists.

Case II - $d_2' = 5$, $d_1' + d_3 = 4$, $d_1' \leq 1$: If $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$, follow the argument of Case I. Otherwise, $9 = d_1' + d_2' + d_3 > 2n + 2m_4 - 1$ and so $10 > 2n + 2m_4$ or equivalently, $5 > n + m_4$. Since we assumed $n \geq 4$ and $m_4 \geq 1$, this is a contradiction.

Case III - $d_2' = 5$, $d_1' + d_3 = 4$, $d_1' \geq 2$: If $d_1' + d_2' + d_3 > 2n + 2m_4 - 1$, we obtain the same contradiction as in Case II. So assume that $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$. Then by Theorem 4.2.16, G^* has a $[0, d_1', d_2', d_3, d_4]$-factor H^* where the fifth corner v is degree 1 since $d_1' \geq 2$. Because v is degree 1, Theorem 4.2.13 yields that the desired factor exists.

Case IV - $d_2' = 5$, $d_1' + d_3 > 4$, $d_1' \geq 2$: By parity, $d_1' + d_3 > 4$ implies $d_1' + d_3 \geq 6$. Since $d_1' \geq 2$, we see that $d_1' - 2 + d_2' + d_3 < 2n + 2m_4 - 1$ because otherwise we can decrease d_1' by 2 again. Since $2n + 2m_4 - 1 \leq d_1' + d_2' + d_3 < 2n + 2m_4 + 1$ we see that $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$ or $d_1' + d_2' + d_3 = 2n + 2m_4$. However, $d_1' + d_3$ is even by parity and $d_2' = 5$ by assumption so $d_1' + d_2' + d_3$ is odd and thus $d_1' + d_2' + d_3 \neq 2n + 2m_4$. As a result, $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$. By Theorem 4.2.16, G^* has a $[0, d_1', d_2', d_3, d_4]$-factor H^* where the fifth corner v is degree 1 since $d_1' \geq 2$. Since v is degree 1, Theorem 4.2.13 again yields that the desired factor exists.

Case V - $d_2' = 5$, $d_1' + d_3 > 4$, $d_1' = 1$: If $d_1' + d_2' + d_3 = 2n + 2m_4 - 1$, follow the argument of Case I. Otherwise, $d_1' + d_2' + d_3 > 2n + 2m_4 - 1$. Since $d_1' + d_2' = 6$, we see that $d_3 > 2n + 2m_4 - 7$, or equivalently, $d_3 \geq 2n + 2m_4 - 6$.

If n divides $d_3 + d_4 + 6$, then $d_3 + d_4 + 6 = nk$ for some integer k. We argue now that $k \geq 3$. Since $m_4 \geq 1$ and $d_3 \geq 2n + 2m_4 - 6$, we see that $d_3 \geq 2n - 4$. Then $nk = d_3 + d_4 + 6 \geq 2n - 4 + d_4 + 6 = 2n + d_4 + 2 > 2n$. Hence $nk > 2n$ and so $k > 2$, or equivalently, $k \geq 3$. Then by Claim 4.2.19, an $n \times k$ grid G' has a $[0, 0, 5, d_3 + 1, d_4]$-factor H' where the NE corner v of G' is degree 1. Since v is degree 1, Theorem 4.2.13 then yields the desired factor.

If n divides $d_3 + d_4 + 5$, then $d_3 + d_4 + 5 = nk$ for some integer k. The argument that $k \geq 3$ follows as above. Then by Claim 4.2.18, an $n \times k$ grid G' has a $[0, 0, 4, d_3 + 1, d_4]$-factor H' where the NE corner v of G' is a degree 2 in H' and is adjacent to a degree 3 vertex w in H'. Delete the edge vw for a $[0, 1, 4, d_3, d_4]$-factor of G' in which v is degree 1. As expected, Theorem 4.2.13 yields the desired factor.

If n divides $d_3 + d_4 + 4$, then $d_3 + d_4 + 4 = nk$ for some k. A similar argument as above shows that $k \geq 3$. Per Claim 4.2.20, an imperfect $(n, k, n-1)$-grid G' with fifth corner u has a $[0, 0, 5, d_3 - 1, d_4, 0]$-factor H'. In H', add a pendant v adjacent to u in H' to yield a $[0, 1, 4, d_3, d_4, 0]$-factor of G'. Note that v is the SE corner of an $n \times k$ grid, and so Theorem 4.2.13 yields the desired factor.

Finally, assume n does not divide $d_3 + d_4 + 5$ or $d_3 + d_4 + 5$ or $d_3 + d_4 + 6$. Let $d_3 + d_4 + 6 = nk + r$ where $0 \leq r < n$. Then $r \neq 0, 1$, or 2 since n does not divide $d_3 + d_4 + 6$, $d_3 + d_4 + 5$, or $d_3 + d_4 + 4$. Hence, $3 \leq r < n$. Using the same arguments as before, we can argue that $k \geq 2$ in this case. Let G' be an imperfect $(n, k + 1, r)$-grid G' with fifth corner v. Then by Claim 4.2.20, G' has a $[0, d_1', d_2', d_3, d_4]$-factor H' where v is degree 1 in H' if $d_1 > 0$ since $r \neq 2$. Since $d_1 - d_1' > 0$ or v is degree 1 in H', Theorem 4.2.13 yields the desired factor.

□

Due to Theorem 4.6.6, we know that when $d_4 > 0$ and $d_1 + d_2 + d_3 \geq \min\{2n_4 + 2m - 1, 2m_4 + 2n - 1\}$ and a few other weak conditions hold, an $n \times m$ grid G has a $[d_0, d_1, d_2, d_3, d_4]$-factor. We conclude this section by illustrating a $[d_0, d_1, d_2, d_3, d_4]$-factor of a $n \times m$ grid G which satisfies all hypotheses of Theorem 4.6.6 except that $\max\{2n_4 + 2, 2m_4 + 2\} \leq d_1 + d_2 + d_3 < \min\{2n_4 + 2m - 1, 2m_4 + 2n - 1\}$. We also give an example of a grid which does not have a $[d_0, d_1, d_2, d_3, d_4]$-factor in this same scenario. Thus, we know of cases when G does and does not have a $[d_0, d_1, d_2, d_3, d_4]$-factorwhen $\max\{2n_4 + 2, 2m_4 + 2\} \leq d_1 + d_2 + d_3 < \min\{2n_4 + 2m - 1, 2m_4 + 2n - 1\}$, and this range of d_4 values is left for future work.

To illustrate a possible factor in this range, add isolated vertices to the factor show in Figure 4.9(a) so that it is a $[12, 0, 12, 0, 12]$-factor H of a 6×6 grid. Then $d_1 + d_2 + d_3 = 12$ and so $8 = \max\{2n_4 + 2, 2m_4 + 2\} \leq 12 < \min\{2n_4 + 2m - 1, 2m_4 + 2n - 1\} = 17$. To illustrate an impossible factor in this range, let G be a $n \times m$ grid where $n = m = 6$. G does not have a $[6, 2, 10, 2, 16]$-factor and we explain why. Note that $d_4 = (n-2)(m-2) = 16$. Hence, if such a factor H exists, then all interior vertices must be degree 4 in H. Hence, all vertices on the border of G except for the corners must have positive degree in H. G has $2n + 2m - 8 = 16$ non-corner vertices on its border. But $d_1 + d_2 + d_3 = 14 < 16$ and so $d_1 + d_2 + d_3$ is not big enough for these non-corner border vertices to all have positive degree. Hence, H cannot exist.

4.7 Grid Factors when $d_4 = 0$ and $d_1 + d_2 \geq 5$

In this section, we assume that $n \geq 3$ and $m \geq 3$ because all results concerning $n = 2$ or $m = 2$ are in Section 4.8. We identify a list of impossible factors when $d_4 = 0$. Identifying these factors required a case-by-case analysis. We first conjecture that this list is complete when $d_1 + d_2 \geq 5$. We then show that each of the pathological cases in Conjecture 4.7.1 are truly impossible. In the most extreme cases, that is, when either $d_1 = 5$ and $d_2 = 0$ or when $d_1 = 0$ and $d_2 = 5$, Claim 4.7.8 and Claim 4.7.4, respectively, show that the conjecture holds true.

Conjecture 4.7.1. *Let G be an $n \times m$ grid and let d_0, d_1, d_2, d_3 be non-negative integers whose sum is nm where $n \geq 3, m \geq 3$ and d_1 and d_3 have the same parity. If $d_1 + d_2 \geq 5$, then G has a $[d_0, 0, d_2, d_3, 0]$-factor except in the following cases.*

1. *$[0, d_1, 0, d_3, 0], d_1$ and d_3 are odd (Claim 4.7.2)*

2. *$[d_0, 0, d_2, 0, 0]$ and d_2 odd (Claim 4.3.1)*

3. *$[d_0, 0, 5, 0, 0]$ (Claim 4.7.4)*

4. *$[d_0, 0, 5, 2, 0]$ (Claim 4.7.4)*

5. $[n-1, 0, 5, n^2-n-4, 0]$, $d_0 = n-1$, $n = m \leq 6$ *(Claim 4.7.7)*

6. $[d_0, 3, 2, d_3, 0], d_0 < \min\{n, m\} - 3$ *(Claim 4.7.7)*

7. $[d_0, 4, 1, d_3, 0], d_0 < \min\{n, m\} - 2$ *(Claim 4.7.7)*

8. $[d_0, 5, 0, d_3, 0], d_0 < 2\min\{n, m\} - 5$ *(Claim 4.7.7)*

9. $[d_0, 6, 0, d_3, 0], \min\{n, m\} > 3$ *and* $d_0 < \min\{n, m\} - 2$ *(Claim 4.7.7)*

10. $[d_0, 7, 0, d_3, 0], d_0 < \min\{n, m\} - 3$ *(Claim 4.7.7)*

In Claim 4.7.2, we show that no grid has a $[0, d_1, 0, d_3, 0]$-factor where d_1 and d_3 are odd. We remark that there are bipartite graphs with degree sequences which consist of d_1 1s and d_3 3's, where d_1 and d_3 are odd. Thus, this pathological case is specific to grids and not bipartite graphs in general. For example, $< 3, 3, 3, 3, 3, 1, 1, 1 >$ is the degree sequence of the bipartite graph shown in Figure 4.11. Note that in Figure 4.11, the partite sets X and Y do not have the same number of vertices. The proof of Claim 4.7.2 shows that a $[0, d_1, 0, d_3, 0]$-factor, where d_1 and d_3 are odd, requires that $|X| = |Y|$, which causes a contradiction. This explanation sheds light on why such a factor is a pathological case for grids but possibly not for bipartite graphs in general.

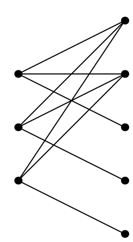


Figure 4.11: A bipartite graph with degree sequence $< 3, 3, 3, 3, 3, 1, 1, 1 >$

Claim 4.7.2. *No grid has a* $[0, d_1, 0, d_3, 0]$*-factor where* d_1 *and* d_3 *are odd.*

Proof. Per Claim 4.2.1, d_1 and d_3 have the same parity and so $d_1 + d_3$ is even. Also, in any $n \times m$ grid G with a $[0, d_1, 0, d_3, 0]$-factor, $nm = d_1 + d_3$ and thus nm is even.

Let H be a $[0, d_1, 0, d_3, 0]$-factor of G. Then $< 3, 3, \ldots, 3, 1, 1, \ldots 1 >$ is the degree sequence of H. By Claim 4.2.4, H is a bipartite graph with partite sets X and Y where $|X| = |Y|$. Let x_1, y_1 be the number of degree 1 vertices in X, Y, respectively. Let x_3, y_3 be the number of degree 3 vertices in X, Y, respectively. Then $x_1 + y_1 = d_1$ and $x_3 + y_3 = d_3$. Because partite sets in a bipartite graph have equal degree sum, $x_1 + 3x_3 = y_1 + 3y_3$. Since $|X| = x_1 + x_3$ and $|Y| = y_1 + y_3$, we see that $x_1 + 3x_3 = y_1 + 3y_3$ implies $|X| + 2x_3 = |Y| + 2y_3$ and so $|X| - |Y| = 2(y_3 - x_3)$. Since $|X| = |Y|$, we see that $y_3 = x_3$. Then $d_3 = y_3 + x_3$ is even. Therefore, d_1 is even by Claim 4.2.1. Hence, if an $n \times m$ grid has a $[0, d_1, 0, d_3, 0]$-factor H, d_1 and d_3 are not odd. $\square$

Claim 4.7.3. *If $n \leq 6$, then a $n \times n$ grid G does not have a $[n-1, 0, 5, n^2-n-4, 0]$-factor. If $n \geq 7$, then G has a $[n-1, 0, 5, n^2-n-4, 0]$-factor H with the following properties:*

Proof. The reader can verify that G does not have a $[n-1, 0, 5, n^2-n-4, 0]$-factor when $n \leq 6$. The reader should first try to find such a factor H by making the border vertices of G degree 0 in the factor. This is problematic since no degree 1 vertices are permitted. Thus, the reader should try making the interior vertices of G degree 0 in the factor. However, because the interior is smallish when $n \leq 6$, factors in which an interior vertex of G is degree 0 in H tend to force some of its N, S, E, and W neighbors to be degree 2 in H. This is problematic since only 5 degree 2 vertices are permitted.

For $n \geq 7$, we show the claim by induction on n. The base cases are $n = 7$ and $n = 8$. For these cases, Figure 4.12(a)-(b) shows $[n-1, 0, 5, n^2-n-4, 0]$-factors of an $n \times n$ grid where each factor is really a factor H' of an (n, n, r)-imperfect grid G' with $n - r$ additional isolated vertices, where $3 \leq r \leq n - 3$. All other desired properties hold as well.

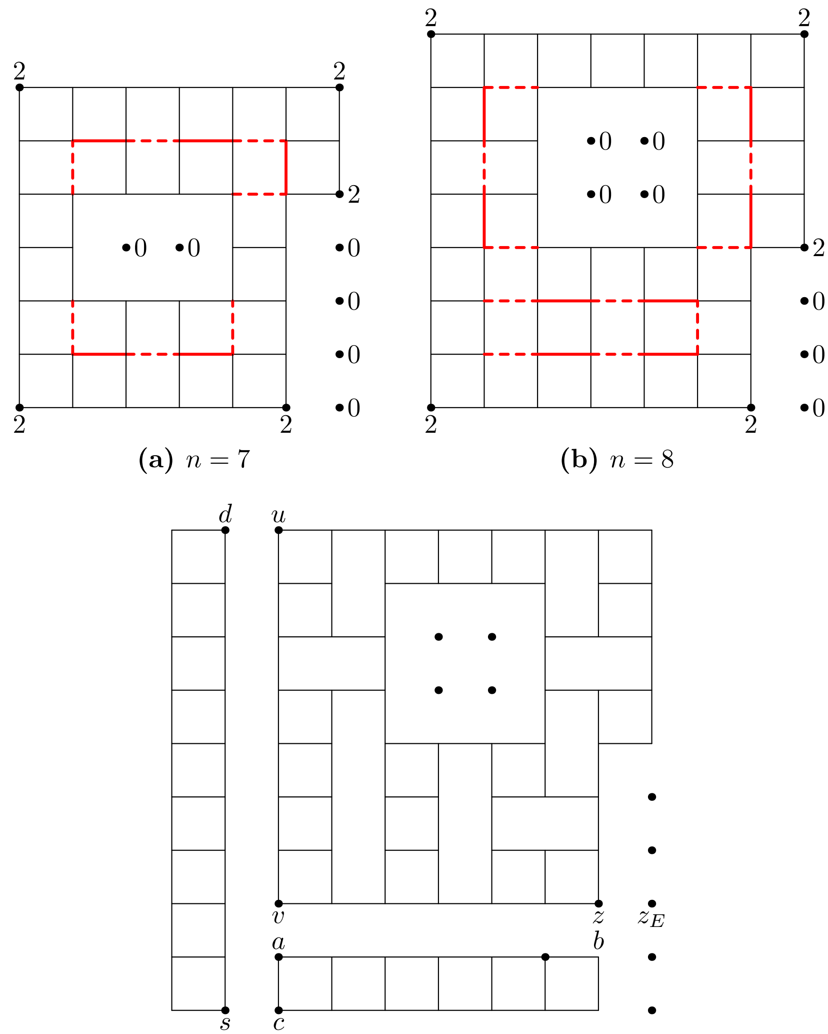
Now assume an $n \times n$ grid has such a $[n-1, 0, 5, n^2-n-4, 0]$-factor when $n \geq 9$. We obtain such an $[n, 0, 5, (n+1)^2-(n+1)-4, 0]$-factor of an $(n+1) \times (n+1)$ in the following manner. By induction, there exists an $[n-2, 0, 5, (n-1)^2-(n-1)-4, 0]$-factor H of an $(n-1)\times(n-1)$ grid where H is a factor H' of an $(n-1, n-1, r)$-imperfect grid G' with $n-1-r$ additional isolated vertices where $3 \leq r \leq n-4$. Let u, v, z be the NW, SW, SE corners, respectively, of G', and by induction, note that z_E is degree 0 in H. To obtain the desired factor, perform the following process and see Figure 4.12(c) for clarification. Consider a $2\times(n-2)$ grid where a, b, c are the NW, NE, SW corners, respectively. Consider also a $(n+1)\times 2$ grid with NE corner d and SE corner s. Add edges av, bz, du, sc and two additional isolated vertices south of z_E in H. This yields a $[n, 0, 5, (n+1)^2-(n+1)-4, 0]$-factor of an $(n+1)\times(n+1)$ grid with the desired properties. □

Claim 4.7.4 gives necessary and sufficient conditions for when a $n\times m$ grid has a $[d_0, 0, 5, d_3, 0]$-factor.

Claim 4.7.4. *No grid has a $[d_0, 0, 5, 0, 0]$-factor or a $[d_0, 0, 5, 2, 0]$-factor. Let G be an $n\times m$ grid and let d_0 and d_3 be non-negative integers whose sum is $nm-5$ where $n \geq 2, m \geq 2$, and d_0 is even. Then G has a $[d_0, 0, 5, d_3, 0]$-factor H except in the following cases.*

1. $[d_0, 0, 5, 0, 0]$
2. $[d_0, 0, 5, 2, 0]$
3. $[d_0, 0, 5, d_3, 0]$, *$n = 2$ or $m = 2$*
4. $[n-1, 0, 5, n^2-n-4, 0]$, *$d_0 = n-1$, $n = m \leq 6$*

Proof. Any graph with d_0 degree 0 vertices and 5 degree 2 vertices has an odd cycle and so cannot be a subgraph of a bipartite graph. Thus, no grid has a $[d_0, 0, 5, 0, 0]$-factor. The partite sets of any bipartite graph have equal sums. As a result, a $[d_0, 0, 5, 0, 2]$-factor must then have partite sets X, Y where $X = \{2, 2, 2, 2\}$ with additional 0's and $Y = \{3, 3, 2\}$ with additional 0's. However, the reader can check that there is a unique realization of a bipartite graph with partite sets X, Y and that this realization has an odd cycle. Hence, no grid has a $[d_0, 0, 5, 2, 0]$-factor. If



(c) Building from the inductive step

Figure 4.12: $[n-1, 0, 5, n^2-n-4, 0]$-factors of $n \times n$ grids

d_3 is even, $d_0 = n - 1$, and $n = m$, then Claim 4.7.3 yields that an $n \times m$ grid has a $[d_0, 0, 5, d_3, 0]$-factor if and only if $n = m \geq 7$, and so case 4 is indeed a pathological case.

To show pathological case 3, we now argue that $2 \times m$ grid G does not have a $[d_0, 0, 5, d_3, 0]$-factor when $d_3 \geq 4$. Note that $d_2 = 5$ and $d_3 \geq 4$ implies $m \geq 5$. Assume such a factor H of G does exist. If there are columns in G in which both vertices are degree 0 in H, then remove these columns from G and H and apply the following argument to the resulting grid and factor. Hence, we may assume each column of G has a vertex of positive degree in H. Therefore, the NW or SW corner of G has positive degree in H. Because corners cannot be degree 3, the NW or SW corner must then be degree 2, but this forces that *both* the NW and SW corners are degree 2 in H. Similarly, the NE and SE corners are degree 2 in H as well. Thus, all corners of G have degree 2 in H. Then the corners are adjacent in H to the vertices in columns 2 and $m-1$ and so the vertices in columns 2 and $m-1$ have positive degree in H. Because the vertices columns 2 and $m-1$ have positive degree in H, all degree 0 vertices in H are between columns 3 and $m-2$. Note that by hypothesis, $d_0 + d_3 = 2m - 5$. Since d_3 is even, this implies d_0 is odd and so at least one vertex in H must be degree 0.

Because the 4 corners are degree 2 in H, exactly one non-corner vertex has degree 2 in H. Since $d_0 \geq 1$, we may consider the westmost degree 0 vertex v in H. Note that v is not in columns $1, 2, m-1, m$ since these columns have vertices only of positive degree in H. Without loss of generality, assume v is in row 1. Since v is not in columns $1, 2, m-1, m$, we see that v_S and v_W are not corner vertices of G. Also, because v has degree 0, v_S and v_W are not adjacent to v in H and so cannot have degree 3 in H. By assumption, no column has 2 degree 0 vertices and so v_S has positive degree and thus must be degree 2 in H. Also, since v is the westmost degree 0 vertex in H, we see that v_W also has positive degree in H and must also be degree 2 in H. The four corner vertices plus v_S and v_W yield that H has at least 6 degree 2 vertices, a contradiction. Hence, a $2 \times m$ grid G does not have a $[d_0, 0, 5, d_3, 0]$-factor when $d_3 \geq 4$

Now assume $n \geq 3$, $m \geq 3$, $d_3 \geq 4$, and d_3 is even. We argue that an $n \times m$ grid G has the desired factor except when $d_0 = n - 1$ and $n = m \leq 6$. If d_0 is large, our strategy at times requires that we remove rows and columns from G to find a factor of a smaller grid. In doing so, we must take care to ensure that the smaller grid has at least 3 rows and 3 columns because we just showed that a $2 \times m$ or an $n \times 2$ grid does not have a $[d_0, 0, 5, d_3, 0]$-factor otherwise.

The number of vertices with positive degree in the desired factor is $d_3 + 5$ which is at least 9. If d_3+5 vertices cannot fill more than two columns or more than two rows, ie, if $d_3+5 \leq 2\max\{n,m\}$, then we perform the *zooming transformation*, which we define below. Without loss of generality, assume $n \leq m$. We let $m' = \lceil \frac{d_3+5}{3} \rceil$.

Definition 4.7.5. *To perform the **zooming transformation** means to remove $n-3$ rows and $m-m'$ columns from the grid and to decrease d_0 by the number of vertices removed.*

Perform the zooming transformation and note that a $3 \times m'$ grid results. Find the desired factor of this resulting grid and then add $n-3$ rows and $m-m'$ columns of isolated vertices for the desired factor. Note that since $d_3+5 \geq 9$ by hypothesis, we see that $m' \geq 3$. Since $d_3+5 \geq 9$, d_3+5 vertices fills at least 3 rows and at least 3 columns in this $3 \times m$ grid. For the remainder of this proof, we thus assume $d_3+5 > 2\max\{n,m\}$. Please note a subtle point that is important later. The zooming transformation results in a $3 \times m$ grid where $m \geq 3$. We now argue that the zooming transformation cannot place us in pathological case (4) and we call this Fact (1). If it did, then the shrinking transformation results in a 3×3 grid and $d_0 = 2$. However, because $d_3+5 \geq 9$, we see that $d_3 = 4, d_2 = 9$ and so $d_0 = 0$, a contradiction.

Let $d_3+5 \equiv r \mod n$ and $d_3+5 \equiv s \mod m$ where $0 \leq r < n$ and $0 \leq s < n$. We break the proof into 3 cases: r or s is 0, $r \geq 2$ or $s \geq 2$, and finally, $r = s = 1$.

Assume first that r or s is 0 and so either n or m, say n, divides d_3+5. Then $d_3+5 = nm'$ where $m' \geq 3$ since $d_3+5 > 2\max\{n,m\}$. Since d_3+5 is odd, nm' is odd too. Claim 4.2.19 implies that a $[0,0,5,d_3,0]$-factor H of an $n \times m'$ grid G' exists. Adding an additional $m-m'$ columns of isolated vertices to H yields the desired factor of G.

Now assume $r \geq 2$ or $s \geq 2$. Without loss of generality, we assume $r \geq 2$ and we proceed as follows. Then $d_3+5 = nm'+r$ for some m'. Also, $m' \geq 2$ since $d_3+5 > 2\max\{n,m\}$. Consider an $(n,m'+1,r)$-imperfect grid G'. Per Claim 4.2.20 G' has a $[0,0,5,d_3,0]$-factor H. Add d_0 isolated vertices to H to yield the desired factor of G.

Finally, assume $r = s = 1$. Without loss of generality, assume $n \leq m$. If $d_0 \geq n$, then because $d_0 + 5 + d_3 = nm$, it follows that $d_3 + 5 \leq (m-1)n$. We remove a column from G and we consider an $n \times (m-1)$ grid G' and we argue that $m-1 > 3$ which is required for G' to have the desired factor. Since $d_3 + 5 \equiv 1 \mod n$, we see that $d_3 + 5 = mk + 1$ where $0 < k < m$. Also, $mk + 1 = d_3 + 5 > 2\max\{n, m\} = 2m$ implies $k > 2$. Also, $mk + 1 = d_3 + 5 \leq (m-1)n \leq (m-1)m$, implies $k < m - 1$. Then $2 < k < m-1$ and so $m-1 > 3$, as desired. Also, $d_3+5 = (m-1)k+(k+1)$ and so $d_3 + 5 \equiv k+1 \mod m-1$. Because $2 < k < m-1$ implies $3 \leq k+1 < m$, we see that $d_3 + 5 \not\equiv 1 \mod m-1$. Then by the previous cases, G' has a $[d_0 - n, 0, 5, d_3, 0]$-factor H. Add a column of isolated vertices to H to yield the desired factor of G.

Otherwise, $d_0 < n$. Recall $r = s = 1$ and so $d_3 + 5 \equiv 1 \mod n \equiv 1 \mod m$. Since $d_0 + 5 + d_3 = nm$, this implies that $d_0 \equiv -1 \mod n \equiv -1 \mod m$. $d_0 < n \leq m$, we see that $d_0 = n - 1 = m - 1$ and so $n = m$. Then $d_0 + 5 + d_3 = nm = n^2$ and so $d_3 = n^2 - d_0 - 5 = n^2 - n - 4$. Then Claim 4.7.3 yields that G has a $[n-1, 0, 5, n^2-n-4, 0]$-factor if and only if $n = m \geq 7$. If $n = m \leq 6$, then we just argued G does not have the desired factor. If G were a grid that resulted from the zooming transformation, then the reader may wonder if the original un-transformed grid could have the desired factor. However, since $d_0 = n - 1 = m - 1$ and $n = m$, Fact (1) indicates that the zooming transformation cannot place us in pathological case (4). $\square$

In Claim 4.7.7, we use the term *chain of 3's on the border*, which we define now. As an example, in the factor shown in Figure 4.13, $v_1v_2v_3$ is a chain of $3's$ on the border with endpoints a and b. Note that the endpoints of a chain of 3's on the border are always degree 1 or 2 in the factor.

Definition 4.7.6. *A* ***chain of 3's on the border*** *in a factor H of a grid G is a maximally connected set of degree 3 vertices in H all of which are on the border of G. Given a chain of 3's $\mathcal{S}$ on the border in H, let P be the path in H whose internal vertices are $\mathcal{S}$, the* ***endpoints of a chain of 3's on the border*** *are the endpoints of P.*

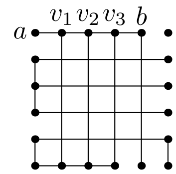


Figure 4.13: A chain of 3's on the border

Claim 4.7.7. *Let G be an $n \times m$ grid where $n \geq 3$ and $m \geq 3$ and let d_0, d_1, d_2, d_3 be non-negative integers which sum to nm where d_1 and d_3 have the same parity. G does not have a $[d_0, d_1, d_2, d_3, 0]$-factor in the following cases.*

1. $[d_0, 3, 2, d_3, 0], d_0 < \min\{n, m\} - 3$
2. $[d_0, 4, 1, d_3, 0], d_0 < \min\{n, m\} - 2$
3. $[d_0, 4, 0, d_3, 0], d_0 < 2\min\{n, m\} - 4$
4. $[d_0, 5, 0, d_3, 0], d_0 < 2\min\{n, m\} - 5$
5. $[d_0, 6, 0, d_3, 0], d_0 < \min\{n, m\} - 2$
6. $[d_0, 7, 0, d_3, 0], d_0 < \min\{n, m\} - 3$

Proof. Without loss of generality, assume $n \leq m$. Recall per Claim 4.1.7, there are $2n + 2m - 4$ vertices on the border of G. Because corners cannot be degree 3, note that a wall of size n can have at most $n - 2$ degree 3 vertices in a factor. Consider any chain of 3's on the border of a factor H. The endpoints to this chain are either degree 1 or degree 2. Also, a degree 1 vertex in H can be the endpoint to at most 1 chain of 3's on the border. A degree 2 vertex in H can be the endpoint to at most 2 chain of 3's on the border.

In a $[d_0, 3, 2, d_3, 0]$-factor H, the 2 degree 2 vertices are the endpoints to at most 4 chains of 3's on the border and each of the 3 degree 1 vertices is the the endpoint to at most 1 chain. Thus, there are at most 7 endpoints to chains of 3's on the border in H. Then there are at most 3 chain's of 3's on the border and so at most 3 walls of G have vertices which are degree 3 in H. If all non-corner vertices on the N, S, and W (or E) walls are degree 3 in H, then there are $2m+n-6$ vertices on the border of G with degree 3 in H. Since $d_1+d_2=5$, there are thus at most $2m+n-1$ vertices on the border of G with positive degree in H. Thus, there must be at least $2n+2m-4-(2m+n-1)=n-3$ vertices of degree 0 on the border in H.

Now consider a $[d_0,4,1,d_3,0]$-factor. As in the previous case, we can argue that there are at most 3 chain's of 3's on the border and so at most 3 walls of G have vertices which are degree 3 in H. Consider any 3 walls, say the N, S, and W walls. If all non-corner vertices are degree 3 in H on these walls, then the NW and SW corners both must have degree 2 in H. However, $d_2=1$, a contradiction. So if 3 walls of G have vertices which are degree 3 in H, then at least 1 non-corner vertex on one of these walls is not degree 3 in H. Thus, there are at most $2m+n-7$ vertices (1 less than in the previous case) on the border of G with degree 3 in H. Since $d_1+d_2=5$, there are then at most $2m+n-2$ vertices on the border of G with positive degree in H. Thus, there must be at least $2n+2m-4-(2m+n-2)=n-2$ vertices of degree 0 on the border in H.

Consider a $[d_0,5,0,d_3,0]$-factor H. Because H has no degree 2 vertices, any chain of 3's on the border of H has exactly 2 degree 1 endpoints. Since $d_1=5$, there are therefore at most 5 vertices in H which can be endpoints to a chain of 3's on the border. Thus, there are at most 2 chain of 3's on the border and so at most 2 border walls with a degree 3 vertex in H. The maximum number of vertices on the border of G with degree 3 in H is therefore $2m-4$ and the maximum number vertices on the border of G with degree 1 in H is $d_1=5$. Hence, the maximum number of vertices on the border of G with positive degree in H is $2m+1$. Thus, $d_0 \geq 2n+2m-4-(2m+1)=2n-5$. A similar argument shows a $[d_0,4,0,d_3,0]$-factor requires $d_0 \geq 2n-4$.

Finally, consider a $[d_0,7,0,d_3,0]$-factor. As in the previous arguments, at most 3 walls of G have vertices which are degree 3 in H. No 2 chains share an endpoint since $d_2=0$. So if 3 walls have a chain of 3's, there are exactly 6 endpoints to these chains on the 3 walls. Thus, at least 2 of these endpoints are not corner vertices. Hence, the maximum number of vertices on the border of G with degree 3 in H is therefore

$2m+n-6-2 = 2m+n-8$ and the maximum number vertices on the border of G with degree 1 in H is $d_1 = 7$. Thus, $d_0 \geq 2n+2m-4-(2m+n-8+7) = n-3$. A similar argument shows that a $[d_0, 6, 0, d_3, 0]$-factor requires $d_0 \geq n-2$.

$\square$

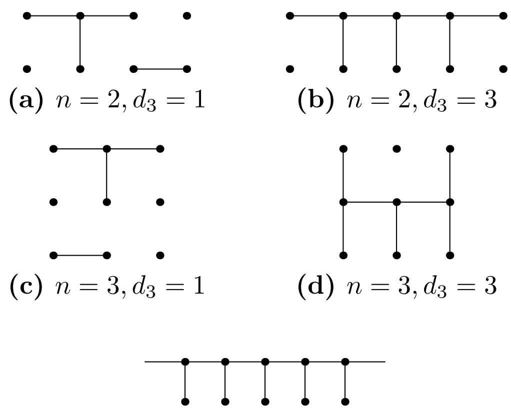
Similar to Claim 4.7.4, Claim 4.7.8 gives necessary and sufficient conditions for when a $n \times m$ grid has a $[d_0, 0, 5, d_3, 0]$-factor.

Claim 4.7.8. *Let G be an $n \times m$ grid and let d_0, d_3 be non-negative integers where d_3 is odd and $d_0 + d_3 = nm - 5$. Then G has a $[d_0, 5, 0, d_3, 0]$-factor except in the following cases.*

1. $[0, 5, 0, d_3, 0]$
2. $[d_0, 5, 0, d_3, 0], d_0 < 2\min\{n, m\} - 5$
3. $[d_0, 5, 0, d_3, 0], n = 2$ *or* $m = 2$

Proof. Per claim 4.7.2 no grid has a $[0, 5, 0, d_3, 0]$-factor. Per Claim 4.7.7 no grid has a $[d_0, 5, 0, d_3, 0]$-factor where $d_0 < 2\min\{n, m\} - 5$.

Assume for a moment that $n = 2$ or $m = 2$. Without loss of generality, assume $n = 2$. Let $d_0 > 0$ since no grid has a $[0, 5, 0, d_3, 0]$-factor. If $d_3 = 1$, then Figure 4.14(a) with additional columns of 0's is a $[d_0, 5, 0, 1, 0]$-factor of a $2 \times m$ grid. If $d_3 = 3$, then Figure 4.14(b) with additional columns of 0's is a $[d_0, 5, 0, 3, 0]$-factor of a $2 \times m$ grid. We now show a $2 \times m$ grid G does not have a $[d_0, 5, 0, d_3, 0]$-factor where $d_3 \geq 5$ and $d_3 > 0$. Assume such a factor H exists. No two degree 1 vertices u, v can be adjacent in H because then $H - \{u, v\}$ is a factor with 3 degree 1 vertices and at least 5 degree 3 vertices, thus contradicting Claim 4.4.1. A degree 3 vertex adjacent to 3 degree 1 vertices also contradicts Claim 4.4.1, as does 2 or more degree 3 vertices each adjacent to 2 degree 1 vertices. If each degree 1 vertex is adjacent to a distinct degree 3 vertex, then removing the degree 1 vertices, we obtain a $[d_0+5, 0, 5, d_3-5, 0]$-factor of G, thus contradicting Theorem 4.7.4. Hence, exactly 1 degree 3 vertex u, say in row 1, is adjacent to 2 degree 1 vertices a, b and a degree 3 vertex w, and the rest of the degree 3 vertices are adjacent to exactly 1 degree 1 vertex. See Figure 4.14 (e) Then w_S is degree 1, forcing w_E to be degree 3. Since $d_1 = 5$, this process can only continue for 3 degree 3 vertices, but $d_3 \geq 5$, a contradiction.



(a) $n = 2, d_3 = 1$ **(b)** $n = 2, d_3 = 3$

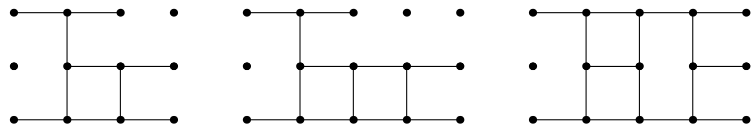
(c) $n = 3, d_3 = 1$ **(d)** $n = 3, d_3 = 3$

(e)

Figure 4.14: $[d_0, 5, 0, d_3, 0]$-factors when $d_3 = 1$ or 3

Now assume that $3 \leq n \leq m$ for an $n \times m$ grid G. If $d_3 = 1$ or $d_3 = 3$, then Figure 4.14 (c)-(d) plus additional 0's is a $[d_0, 5, 0, d_3, 0]$-factor of G. Assume $d_3 \geq 5$ for the rest of the proof. We now show the claim holds for $n = 3$ and $n = 4$.

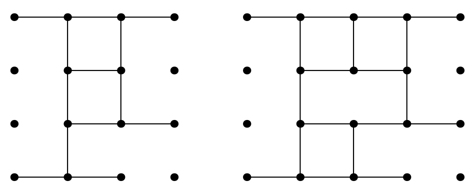
Let $n = 3$. Figure 4.15 shows the desired $[d_0, 5, 0, d_3, 0]$-factors for a 3×4 grid and a 3×5 grid for all d_i values which satisfy the hypotheses. Notice that all factors in Figure 4.15 have the full rung property between columns 2 and 3. To show the desired claim for all $3 \times m$ grids where $m \geq 6$, we use induction on m. Note that $2n - 5 = 1$ and so $d_0 \geq 1$ by hypothesis. If $d_0 \geq 4$, by induction a $[d_0 - 3, 5, 0, d_3, 0]$-factor H of an $3 \times (m - 1)$ grid exists. Add a column of 0's to this factor for the desired factor. Otherwise, $1 \leq d_0 \leq 3$. Then $d_3 = 3m - 5 - d_0 \geq 18 - 5 - 3 = 10$. Hence by parity, $d_3 \geq 11$ so $d_3 - 6 \geq 5$. Then by induction, a $[d_0, 5, 0, d_3 - 6, 0]$-factor H of an $3 \times (m - 1)$ grid exists with the full rung property between columns 2 and 3. Subdivide each edge between columns 2 and 3 twice. Let a_i, b_i for $1 \leq i \leq n$ denote the new vertices added to each row. Also add the edges $a_i a_{i+1}$ and $b_i b_{i+1}$ for $1 \leq i < n$. This preserves the full rung property between these columns.



(a) $[2,5,0,5,0]$ **(b)** $[3,5,0,7,0]$ **(c)** $[1,5,0,9,0]$

Figure 4.15: $[d_0,5,0,d_3,0]$-factors of 3×4 and 3×5 grids

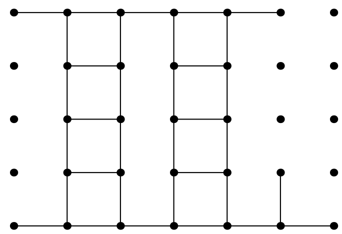
Now let $n=4$. Figure 4.16 shows H for a 4×4 grid and a 4×5 grid for all d_i values which satisfy the hypotheses. Notice that all factors again have the full rung property between columns 2 and 3. We must show the claim for a $4\times m$ grid when $m\geq 6$. Note now that $2n-5=3$ and so $d_0\geq 3$. If $d_0\geq 7$, then use the $n=3$ case to obtain a $[d_0-m,5,0,d_3,0]$-factor of an $3\times m$ grid and add a row of 0's. If $3\leq d_0<7$, use induction on m as in the $n=3$ case.



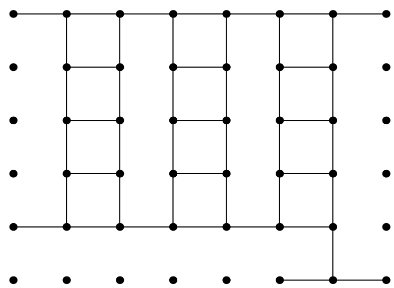
(a) $[4,5,0,7,0]$ **(b)** $[4,5,0,11,0]$

Figure 4.16: $[d_0,5,0,d_3,0]$-factors of 4×4 and 4×5 grids

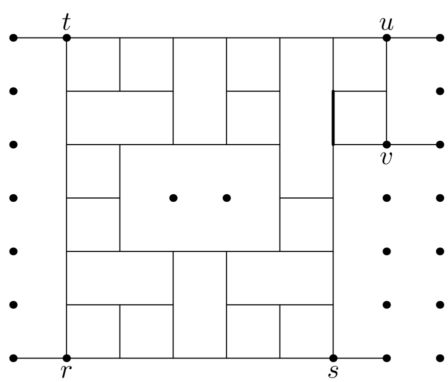
Finally, assume $m\geq n\geq 5$. If $d_3=5$ or $d_3=7$, then the factors in Figure 4.15(a)-(b) plus additional 0's yields the claim. So assume $d_3\geq 9$. Let G' be an $n\times(m-2)$ grid. Then by Claim 4.7.4, there exists a $[d_0+5-2n,0,5,d_3-5,0]$-factor H' of G' unless $n=m-2\leq 6$ and $d_0+5-2n=n-1$, or equivalently, $d_0=3n-6$. Since $n\geq 5$, this pathological case can only occur if $n=5, m=7, d_0=9$ or if $n=6, m=8, d_0=12$ and Figures 4.17(a)-(b) show the desired factors in these cases. Otherwise, H' exists, and also by Claim 4.7.4, we may assume that after removing rows and columns of isolated vertices from H', the resulting factor is either an imperfect grid within G' where the degree 2 vertices are at the five corners or the resulting factor is a subgrid within G' where the corners are degree 2 and the fifth degree 2 vertex is on any wall of the subgrid we desire. Add a column of 0's to the E and W of H' for a $[d_0 + 5, 0, 5, d_3 - 5, 0]$-factor of G. Add an edge between each degree 2 vertex in H' and an isolated vertex to obtain a $[d_0, 5, 0, d_3, 0]$-factor of G, as is exemplified at vertices r, s, t, u, v in Figure 4.17(c).



(a) $[9, 5, 0, 21, 0]$-factor of a 5×7 grid



(b) $[12, 5, 0, 31, 0]$-factor of a 6×8 grid



(c) $[d_0+5-2n, 0, 5, d_3-5, 0]$ becomes $[d_0, 5, 0, d_3, 0]$

Figure 4.17: Factors when $m \geq n \geq 5$ and $d_3 \geq 9$

$\square$

4.8 Grid Factors when $n = 2$ or $m = 2$

Recall per Claim 4.2.3 that an $n \times m$ grid does not have any degree 4 factors when $n = 2$ or $m = 2$. Hence, in this section we always assume $d_4 = 0$ and the factors we seek are $[d_0, d_1, d_2, d_3]$-factors. The main result of this section is Theorem 4.8.1, which characterizes $[d_0, d_1, d_2, d_3]$-factors of a $2 \times m$ or $n \times 2$ grid.

Theorem 4.8.1. *Let G be a $n \times m$ grid where n or m is 2 and the other is at least 2. Let d_0, d_1, d_2, d_3 be non-negative integers that sum to nm. Assume $d_3 \leq 2m - 4$ and assume that the sequence consisting of d_i entries of the integer i is realizable. Then G has a $[d_0, d_1, d_2, d_3]$-factor except in the following pathological cases.*

1. $[d_0, 1, 2, 1]$ *(Claim 4.4.2)*

2. $[d_0, 0, d_2, 0]$ *where d_2 is odd (Claim 4.3.1)*

3. $[d_0, d_1, d_2, d_3]$ *where $d_3 \geq 2$ and $d_1 + d_2 < 4$ (Claim 4.4.1)*

4. $[d_0, 0, 5, d_3]$ *where d_3 is even (Claim 4.7.4)*

5. $[d_0, 5, 0, d_3]$ *where d_3 is odd and $d_3 \geq 5$ (Claim 4.7.8)*

6. $[0, d_1, 0, d_3]$ *where d_1 and d_3 are odd (Claim 4.7.2)*

Proof. ($\Rightarrow$) The Pathological cases (PC) follow from the claims listed next to each. ($\Leftarrow$) This proof is constructive. Without loss of generality, we assume $n = 2$ and $m \geq 2$. If $d_3 = 0$, then since we are not in pathological case 2, Theorem 4.3.2 yields that G has a $[d_0, d_1, d_2, 0]$-factor. Let G' be a subgrid or an imperfect grid within G. Our strategy is to find a factor H of G' that contains d_3 degree 3 vertices and a minimal number of degree 1 and 2 vertices. Furthermore, if $d_1 > 0$, we construct H so that the 5th corner of G' (if G' is an imperfect grid) or a corner of G' (if G' is a subgrid) is degree 1 in H. Then by Theorem 4.2.13, G has the desired factor.

Assume $d_3 = 1$ and so by parity d_1 is odd and at least 1. Then $d_1 + d_2 \geq 3$ since otherwise the sequence consisting of d_i entries of the integer i is not realizable.

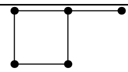
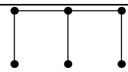
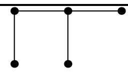
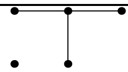
$d_2 \geq 3$		
$d_2 = 2$		$d_1 + d_2 \geq 3$ and PC 1 imply $d_1 \geq 3$
$d_2 = 1$		$d_1 + d_2 \geq 3$ and the parity of d_1 imply $d_1 \geq 3$
$d_2 = 0$		$d_1 + d_2 \geq 3$ implies $d_1 \geq 3$. PC 6 implies $d_0 > 0$.

Table 4.8: Theorem 4.8.1 Case: $d_3 = 1$, d_1 odd

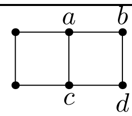
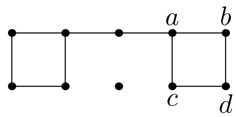
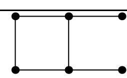
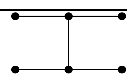
$d_1 = 0$, d_2 even		$d_1 + d_2 \geq 4$ implies $d_2 \geq 4$. Subdivide the edges ab and cd $\frac{d_2-4}{2}$ times each. Add $\frac{d_0}{2}$ columns of degree 0 vertices.
$d_1 = 0$, d_2 odd		d_2 odd implies d_0 is odd, so $d_0 \geq 1$. Also, $d_1 + d_2 \geq 4$ and the parity of d_2 implies. $d_2 \geq 5$. Then $d_2 \geq 7$ or else we are in PC 4. Subdivide ab and cd $\frac{d_2-7}{2}$ times each. Add $\frac{d_0}{2}$ columns of degree 0 vertices.
$d_1 = 2$		$d_1 + d_2 \geq 4$ implies $d_2 \geq 2$.
$d_1 \geq 4$		

Table 4.9: Theorem 4.8.1 Case: $d_3 = 2$, d_1 even

Table 4.8 shows the desired factor H of G'. Note that each factor has a degree 1 vertex where desired.

For the rest of the proof, we will assume $d_3 \geq 2$. Since we are not in pathological case 3, we assume that $d_1 + d_2 \geq 4$ for the rest of the proof. Also, note that since $d_0 + d_1 + d_2 + d_3 = 2m$ and d_1 and d_3 have the same parity, d_0 and d_2 must have the same parity. Now let $d_3 = 2$. Then by parity d_1 is even. Table 4.9 describes how to obtain the desired factor when $d_1 = 0$. Otherwise, Table 4.9 shows a factor with a degree 1 vertex where desired.

Let $d_3 = 3$. Then by parity d_1 is odd and so at least 1. Table 4.10 shows the desired factor H of G' where the desired vertex is degree 1.

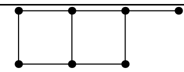
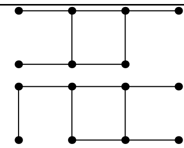
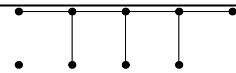
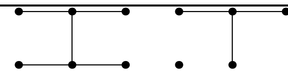
$d_1 = 1$		$d_1 + d_2 \geq 4$ implies $d_2 \geq 3$.
$d_1 = 3$		$d_1 + d_2 \geq 4$ implies $d_2 \geq 1$.
$d_1 \geq 5, d_2 > 0$		same as previous case
$d_1 = 5, d_2 = 0$		By PC 6, $d_0 > 0$ so $d_0 \geq 2$ by parity.
$d_1 \geq 7, d_2 = 0$		Again, $d_0 > 0$ so $d_0 \geq 2$ by parity.

Table 4.10: Theorem 4.8.1 Case: $d_3 = 3$, d_1 odd

Notice that in Table 4.9 and Table 4.10 except when $d_1 = 5$, and $d_2 = 0$, the given factor has the full rung property between columns 1 and 2. Also, note that when $d_3 > 0$, except in the case $d_1 = d_3 = 3$ and $d_2 = 1$, the NW corner in the factor is a degree 2 corner vertex adjacent to a degree 3 vertex.

Now let $d_3 \geq 4$. Note that if $d_3 \geq 5$, then because we are not in pathological case 5, it is not the case that $d_1 = 5$, and $d_2 = 0$. If d_3 is even, use Table 4.9 to get a $[d_0, d_1, d_2, 2, 0]$-factor H of a $2 \times (m - \frac{d_3-2}{2})$ grid. Because we are not in the case that $d_1 = 5$ and $d_2 = 0$, H has the full rung property between columns 1 and 2. Subdivide each edge between these columns in H $\frac{d_3-5}{2}$ times. Let a_1 through $a_{\frac{d_3-2}{2}}$ be the new vertices added to row 1. Let b_1 through $b_{\frac{d_3-2}{2}}$ be the new vertices added to row 2. Add the edges $a_i b_i$ for $1 \leq i \leq \frac{d_3-2}{2}$ to yield $d_3 - 2$ additional degree 3 vertices in H for the desired factor. If d_3 is odd, use Table 4.10 to get a $[d_0, d_1, d_2, 3, 0]$-factor H of a $2 \times (m - \frac{d_3-3}{2})$ grid and perform a similar procedure. $\square$

4.9 Conclusion

We remind the reader that Table 4.1 includes a summary of our results concerning factors of grids. This table also includes the cases which remain open. Besides considering these open questions, future work includes characterizing factors of hypercubes and of grids on a torus.

Chapter 5

Degree Sequence of Partial 2-trees

In this chapter, our goal is to characterize the degree sequences of partial 2-trees. A tree can be formed by repeatedly adding degree 1 vertices to an isolated vertex. A k-tree generalizes this notion. *Graph Classes: A Survey* [1] gives the following definition for k-tree.

Definition 5.0.1. *A k-**tree** is recursively defined as follows.*

1. *A complete graph on k vertices is a k-tree.*
2. *If G is a k-tree and vertices $v_1, \ldots, v_k$ form a k-clique in G, then the graph obtained by adding a vertex to G and connecting it by an edge to each of $v_1, \ldots, v_k$ is a k-tree.*

A 1-tree is therefore a tree. We adhere to Definition 5.0.1, but it is not uncommon to see alternative definitions of k-tree. Some places in literature (such as [13] and [16]) define k-tree similarly, but replace "A complete graph on k vertices" in Definition 5.0.1 with a "A complete graph on $k+1$ vertices." We refer to this as *alternative definition A*. Note that if we add a vertex adjacent to all vertices in a k-clique, we obtain a $k+1$ clique, and so the base k-tree of Definition 5.0.1 starts the recursion one step earlier than the base k-tree of the alternative definition. Thus, the only graph which is a k-tree by Definition 5.0.1 but not alternative definition A is a complete graph on k vertices.

Besides Definition 5.0.1 and alternative definition A, other definitions exist for k-trees. For example, in [7] and [6], Duke and Winkler work towards characterizing degree sets of k-trees, which they define as k-uniform hypergraphs that are constructed in a recursive style similar to 5.0.1. A degree set of a graph is the set of elements in its degree sequence. In other words, a degree set ignores the multiplicity of the elements in the degree sequence of a graph. If we translate the work of Duke and Winkler into the terminology of Definition 5.0.1, then Duke and Winkler's results show the following. In [6], they show that for every $k \geq 1$, the list of sets which are not the degree set of any k-tree is finite. More specifically, in [7], they show that every degree set $\mathcal{D}$ which contains a k for $k = 1, 2, 3$, is the degree set of some k-tree. For $k = 4$, they also show that there is exactly one set $\{4, 7, 9\}$ which is not the degree set of any k-tree.

In [1], the following is given as the definition of a partial k-tree.

Definition 5.0.2. *A* ***partial*** $\boldsymbol{k}$***-tree*** *is a spanning subgraph of a* k*-tree.*

Note that Definition 5.0.2 does not force a partial k-tree to be connected. Also, Definition 5.0.2 implies that deleting edges from a k-tree yields a partial k-tree. Thus, a k-tree is a partial k-tree. As a result, it immediately follows from Duke and Winkler's results that for every $k \geq 1$, there is a finite list of sets which are not the degree sets of any partial k-tree.

For the duration of this chapter, we concentrate specifically on degree sequences of 2-trees and partial 2-trees, also known as series-parallel graphs. Bose, et. al., were able to characterize the degree sequences of 2-trees [13] and we generalize their results to partial k-trees. These authors use the terminology *attaching a vertex to an edge* to describe building 2-trees.

Definition 5.0.3. *To* ***attach a vertex*** v ***to an edge*** uw *in a graph* G *means to add a new vertex* v *to* G *and to make* v *adjacent to both* u *and* w*.*

With this terminology, Bose, et. al., in [13] state that a 2-tree is either a K_3 or is a graph that results from repeatedly attaching vertices to edges. Hence, they define 2-trees according to alternative definition A instead of Definition 5.0.1. However, a single edge is not a partial 2-tree under alternative definition A because a single edge is not a spanning subgraph of any graph with three or more vertices. Both definitions imply that any tree on 3 or more vertices is a partial 2-tree. (See Corollary 5.1.11.) If we accept that a single edge is a partial 2-tree, then we can more simply say that any non-trivial tree is a partial 2-tree. Thus, we opt to adhere to Definition 5.0.1 throughout this chapter, and so we adopt the convention that a single edge is a partial 2-tree. This convention only affects work done in Section 5.5, which is the only section to consider partial 2-trees with degree 1 vertices.

We adopt the following notation from [13]: n_k denotes the multiplicity of k in a given sequence of integers, $d(w)$ denotes the degree of vertex w, and $a^{<b>}$ denotes the sequence $< a, \ldots, a >$ of length b. Additionally, we define an *even sequence* of integers in the same manner as the authors of [13] have.

Definition 5.0.4. *A sequence of integers is* ***even*** *if each element of the sequence is an even integer.*

The following Theorem appears as Theorem 1 in the paper of Bose, et. al [13]. We comment that using the other hypotheses, we can show that the quantity d in condition (d) must equal $\frac{n+1}{2}$. (See Claim 5.3.4.) Thus, specifying that $d = \frac{n+1}{2}$ would not change the correctness of Theorem 5.0.5.

Theorem 5.0.5 ([13]). *Let D be a sequence of n positive integers. Then D is the degree sequence of a 2-tree if and only if the following conditions are satisfied:*

(a) *The sum of D is $4n - 6$,*

(b) *The maximum element of D is at most $n - 1$,*

(c) *The minimum element of D is 2 and $n_2 \geq 2$,*

(d) *D is not of the form $< d, d, d, d, 2^{<n_2>} >$ where $d \geq 5$,*

(e) *$n_2 \geq \frac{n+3}{3}$ whenever D is even.*

Moreover, if D satisfies Conditions (a)-(e) above, then given any $r \in D$ where $r > 2$, there exists a 2-tree whose degree sequence is D which has a vertex of degree r adjacent to a vertex of degree 2.

Our main result is Theorem 5.6.2 which is stated below and is proven in Section 5.6. Recall that per Definition 5.0.2, a partial k-tree need not be connected. Theorem 5.6.2 includes connectedness results. In specific, Theorem 5.6.2 shows that if a degree sequence D is realizable by some partial 2-tree, D is realizable as a *connected* partial 2-tree *unless* no graph which realizes D is connected. Finally, we point out that condition (e) of Theorem 5.6.2 is the analog of condition (d) of Theorem 5.0.5. The proof of Theorem 5.6.2 relies on the proof of 5.0.5.

Theorem 5.6.2. *Let D be a sequence of n positive integers. Then D is realizable as a partial 2-tree if and only if the following conditions hold:*

(a) *The sum of D is $4n - 6 - 2g$ where g is a non-negative integer.*

(b) *The maximum element of D is at most $n - 1$.*

(c) $n_1 + n_2 \geq 2$

(d) $n_2 \geq \frac{n+3-2g}{3}$ *whenever D is even.*

(e) *If $g = 0$ then D is not of the form $< \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, 2^{<n-4>} >$ where $n \geq 4$.*

(f) $n_1 \leq g$ *or D is the degree sequence of a star.*

If the above conditions hold and the sum of the entries in D is at least $2n - 2$, then there exists a connected partial 2-tree which realizes D. If the above conditions hold and $n_1 = 0$, then given any $r \in D$ where $r > 2$, there exists a connected partial 2-tree which whose degree sequence is D which has a vertex of degree r adjacent to a vertex of degree 2. If the sum of entries in D is less than $2n - 2$ and even, then D is realizable as a forest and thus a union of partial 2-trees.

We break down the proof of Theorem 5.6.2 in the following manner. In Section 5.1, we discuss certain properties of partial 2-trees and how to obtain new partial 2-trees from existing ones. In Section 5.2, we discuss the characterizations of partial 2-trees when d_1 is 'large'. This allows us to assume d_1 is small when applying induction in later theorems. In Section 5.3, we discuss the characterizations of partial 2-trees when D is an even sequence. In Section 5.4, we discuss the characterizations when D has no degree 1 elements regardless of whether the sequence is even or not. In Section 5.5, we incorporate degree 1 elements. Finally, in Section 5.6, we combine the results of the previous sections to prove Theorem 5.6.2.

5.1 Properties of Partial 2-trees

We begin by defining a minor, a concept intrinsic to the partial 2-trees graph class.

Definition 5.1.1. *[1] A graph H is a minor of a graph G if H can be obtained from G by a series of zero or more vertex deletions, edge deletions, and/or edge contractions (i.e., replacing two adjacent vertices v and w by a vertex that is adjacent to all neighbors of v and w).*

A K_4 is a complete graph on 4 vertices. Per Theorem 5.1.2, partial 2-trees are those graphs with no K_4-minor. Then clearly no graph with a K_4 is a partial 2-tree. As another example, the graph G in Figure 5.1 is not a partial 2-tree because it has a K_4-minor (but no K_4). The reader can check that removing any vertex from G yields a graph with no K_4-minor, and thus, a partial 2-tree.

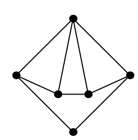


Figure 5.1: A graph with a K_4-minor

Let G be a partial 2-tree. Then G has no K_4-minor, and thus, no subgraph of G has a K_4-minor. Thus, subgraphs of partial 2-trees are again partial 2-trees. This implies that being a partial 2-tree is not only a hereditary property, meaning every *induced* subgraph is also a partial 2-tree, but even more generally, *any* subgraph of a partial 2-tree is a partial 2-tree.

In Theorem 5.1.2, we summarize the aforementioned facts about partial 2-trees as well as additional well-known facts that are noteworthy or useful later.

Theorem 5.1.2. *The following are well-known facts about partial 2-trees:*

1. *2-trees are 2-connected [13].*
2. *Partial 2-trees are those graphs with no K_4 minor. (p. 174 of [1])*
3. *Partial 2-trees are equivalent to series-parallel graphs. (p. 174 of [1])*
4. *Subgraphs of partial 2-trees are partial 2-trees.*

Because a partial k-tree is defined to be a spanning subgraph of a k-tree, there exists a set of edges which when removed from a 2-tree G yields a partial 2-tree G'. We now define the term *gap* to capture how many edges must be removed from a 2-tree to obtain a partial 2-tree.

Definition 5.1.3. *Let G' be a partial k-tree and assume G' is a spanning subgraph of the k-tree G. The* ***gap*** *g of the partial k-tree G' is a non-negative integer which indicates the number of edges which must be removed from G in order to obtain G'.*

Note that if the gap of a partial k-tree is 0, then the partial k-tree is a k-tree. Claim 5.1.4 shows that a partial 2-tree with gap g must have degree sum $4n-6-2g$.

Claim 5.1.4. *The degree sum of a partial 2-tree with n vertices and gap g is $4n-6-2g$.*

Proof. Consider a partial 2-tree T with gap g. Then T is a spanning subgraph of a 2-tree G, which has degree sum $4n-6$ by Theorem 5.0.5. Since we remove g edges from G to obtain T, T has degree sum $4n-6-2g$. $\square$

In the upcoming proofs, we strategically form one partial 2-tree and then modify it to obtain a second partial 2-tree. Definitions 5.1.5 through 5.1.9 are relied upon often while performing these modifications.

Definition 5.1.5. *Consider a graph G with an alternating cycle $abcd$, meaning ab and cd are edges but bc and da are non-edges. To* ***2-switch*** *ab and cd is to remove edges ab and cd and add edges bc and da.*

Definition 5.1.6. *To* ***subdivide*** *an edge xy in a graph means to replace xy with the path xzy where z is a new vertex.*

Below we define *gluing* vertices. In this paper, we only glue together vertices from different components and so gluing never creates multi-edges.

Definition 5.1.7. *To* ***glue*** *together two vertices v, w in a graph means to add an edge between v and all neighbors of w and then to remove w from the graph.*

Definition 5.1.8. *A* ***pendant*** *of a graph is a degree 1 vertex. An* ***ear*** *of a graph is a degree 2 vertex.*

Definition 5.1.9. *Let w be an ear with neighbors x and y. To* ***splice an ear*** *means to subdivide the edge xw to create the path xaw, to subdivide the edge yw to create the path ybw, and finally, to add the edge ab. See Figure 5.2 for the result of splicing an ear.*

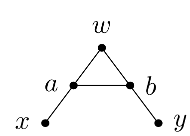


Figure 5.2: The result of splicing an ear w

Theorem 5.1.10 proves that many of the modifications we rely on preserve partial 2-trees.

Theorem 5.1.10. *Given a partial 2-tree G, performing any of the following procedures to G yields a new partial 2-tree.*

1. *Deleting an edge or removing a vertex*

2. *Attaching a vertex to an edge*

3. *Adding a pendant adjacent to any vertex*

4. *Subdividing an edge*

5. *Splicing an ear*

6. *Gluing together two vertices, one from G and another from a distinct partial 2-tree*

Proof. Let G be a partial 2-tree. Then there exists some 2-tree G' of which G is a spanning subgraph.

1. Deleting an edge or removing a vertex from G yields a subgraph of G' and so the resulting graph is a partial 2-tree by Theorem 5.1.2.

2. Recall that attaching a vertex to a 2-tree yields a second 2-tree. Thus, attaching a vertex to an edge in G yields a subgraph which spans the 2-tree that results from attaching a vertex to the same edge in G'.

3. Consider a vertex v in G to which we wish to add a pendant. In the 2-tree G', v must be incident to some edge vw since every vertex in a 2-tree has degree 2 or higher. Attach a vertex to vw in G'. This yields a 2-tree which contains as a spanning subgraph the graph G with an additional pendant adjacent to v.

4. It is well known that subdividing an edge in a series-parallel graph yields another series-parallel graph, so by Theorem 5.1.2, subdividing an edge in a partial 2-tree yields a partial 2-tree.

5. To splice an ear w with neighbors x and y, first subdivide the edge xw into xaw. Rename w to b and then attach a vertex w to the edge ab. These procedures preserve partial 2-trees and so the resulting graph is a partial 2-tree.

6. Gluing together vertices from distinct partial 2-trees cannot create a K_4 minor where one did not previously exist. So by Theorem 5.1.2, gluing together two vertices, each from a distinct partial 2-tree, yields a new partial 2-tree.

□

Because adding pendants preserves partial 2-trees, Corollary 5.1.11 follows immediately from Theorem 5.1.10.

Corollary 5.1.11. *A tree with at least two vertices is a partial 2-tree.*

Claim 5.3 is helpful in Section 5.5. See Figure 5.3 to clarify the details of the claim.

Claim 5.1.12. *Let G be a connected partial 2-tree with an edge ab on a cycle. Let G' be a second connected partial 2-tree with an edge vz where v is a pendant. Then 2-switching ab and vz in $G \cup G'$ yields a connected partial 2-tree with the same degree sequence as $G \cup G'$.*

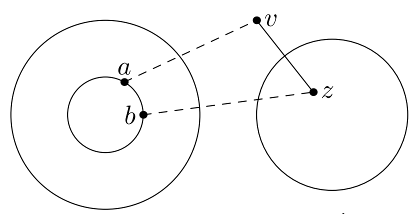


Figure 5.3: Details of Claim 5.3

Proof. It is clear that 2-switching ab and vz in $G \cup G'$ yields a connected graph with the same degree sequence as $G \cup G'$. To show that the resulting graph is indeed a partial 2-tree, note that 2-switching ab and vz is equivalent to performing the following procedures. Glue together the vertices b and v. Delete the edge ab. Add a pendant adjacent to a. Since all of these procedures preserve partial 2-trees by Theorem 5.1.10, the resulting graph is indeed a partial 2-tree. □

5.2 Partial 2-Tree Degree Sequences When d_1 Is Large

When the maximum element in the degree sequence D of a partial 2-tree is close to $n-1$, we can obtain a realization of D by creating a forest F and then adding a vertex v adjacent to most of F. This is the re-occurring strategy in this section. Claim 5.2.1 shows that adding a vertex adjacent to all vertices in a forest yields a partial 2-tree.

Claim 5.2.1. *Given a forest G, adding a vertex v to G and all edges between v and G yields a connected partial 2-tree H. Furthermore, if G is a tree, then H is a 2-tree.*

Proof. We first show that adding a vertex adjacent to all vertices in a tree is a 2-tree. This is a well-known fact but we prove it here for completeness. If $n = 1$, then G is an isolated vertex. Adding a vertex adjacent to this vertex yields an edge, which is a 2-tree by Definition 5.0.1. For the inductive hypothesis, assume that adding a vertex adjacent to all vertices in a tree with n vertices yields a 2-tree. Consider any leaf v and it's neighbor w in a tree G with $n+1$ vertices. Remove v to obtain the tree $T\backslash\{v\}$. Add a vertex y adjacent to all vertices in the tree $T\backslash\{v\}$. The resulting graph is a 2-tree by induction. Now attach the vertex v to the edge wy. The resulting graph is still a 2-tree. Also the resulting graph is simply the graph obtained by adding a vertex adjacent to all vertices in G.

Now we show the claim holds true for a forest G. Add a set of edges E' to G to obtain G', a tree. By the claim in the previous paragraph, adding a vertex adjacent to all vertices in G' is a 2-tree H. Delete the edges E' from H to obtain G', a subgraph of H and so a partial 2-tree. G' is precisely the forest G with an additional vertex adjacent to all vertices in G. $\square$

Corollary 5.2.2. *A cycle on $n \geq 3$ vertices is a partial 2-tree.*

Proof. Consider a path $P = v_1 \ldots v_{n-1}$ on at least two vertices. By Claim 5.2.1, adding all edges between a new vertex v_n and P yields a 2-tree G. Delete all edges incident to v_n except v_1v_n and $v_{n-1}v_n$, thus yielding a partial 2-tree by Theorem 5.1.10. Note that this partial 2-tree is simply a cycle. $\square$

Claim 5.2.3 obtains a partial 2-tree realization of a sequence of integers D whose largest element is exactly $n-1$ by using the following strategy. First we create a forest and then we add a vertex adjacent to all the vertices in the forest. Note that the hypotheses of Claim 5.2.3 require that all elements of D are at least 2. This is because our strategy can fail if D contains a 1.

Claim 5.2.3. *Let D be a sequence of integers $d_1 \geq d_2 \geq \ldots \geq d_n \geq 2$. Assume that $d_1 = n-1$ and $\sum d_i = 4n-6-2g$ where g is a non-negative integer. Then D is the degree sequence of a connected partial 2-tree. Moreover, if d_i and d_j, $i \neq j$, are not both 2, then there exists a connected partial 2-tree with degree sequence D which has a vertex of degree d_i is adjacent to a vertex of degree d_j.*

Proof. Remove d_1 and subtract 1 from d_2 through d_n, thus decreasing the sum of D by $n-1-d_1 = 2(n-1)$. Call the new sequence D'. Then D' has $n' = n-1$ nonzero elements because $d_n \geq 2$. D' has even sum since D does. Below we show that the sum of D' is at most $2n'-2$ and thus is realizable as a forest by Claim 1.0.2.

$$\sum_{i=2}^{n}(d_i - 1) = 4n-6-2g-2(n-1) = 2n-4-2g \leq 2(n-1)-2 = 2n'-2$$

Thus, D' is realizable as a forest. Given d_i and d_j, $i \neq j$, not both 2, we see that $d_i - 1$ and $d_j - 1$ are not both 1. Thus, by Claim 1.0.2, there exists a realization of D' with a vertex v of degree $d_i - 1$ adjacent to a vertex w of degree $d_j - 1$. Add a vertex adjacent to each vertex in G. This is a connected partial 2-tree by Claim 5.2.1, and also, v and w are adjacent with degrees d_i and d_j.

$\square$

As mentioned prior to Claim 5.2.3, the strategy employed by Claim 5.2.3 can fail if a sequence of integers D includes a 1. For example, consider the sequence $D = < 7, 5, 3, 3, 3, 3, 1, 1 >$, which has $n = 8$ elements and maximum element equal to $n-1$. In any graph whose degree sequence is D, the vertex of degree 7 must be adjacent to all other vertices. Removing the vertex of degree 7, the resulting graph has degree sequence $< 4,2,2,2,2,0,0 >$. If we tried to use the strategy of Claim 5.2.3, we would need to create a forest F with degree sequence $< 4,2,2,2,2,0,0 >$ and then add a vertex adjacent to every vertex in F to obtain a partial 2-tree realization of D. However, $< 4,2,2,2,2,0,0 >$ has too large of a sum to be the degree sequence of a forest by Claim 1.0.1. However, if we assume that $n_1 \leq g$, this issue can be prevented. The necessity of condition $n_1 \leq g$ is explained in more detail in Section 5.5 where we characterize degree sequences of partial 2-trees with at least one vertex of degree 1 and where we use Corollary 5.2.4 below.

Corollary 5.2.4. *Let D be a sequence of integers $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$ with sum $4n-6-2g$ where g is a non-negative integer. Also assume that $d_1 = n-1$ and $n_1 + n_2 \geq 2$. If $n_1 \leq g$ and $n \geq 4$, then D is realizable as a connected partial 2-tree.*

Proof. If $d_2 > n-1-n_1$, then below we show that $4n-6-2g > 4n-6-2n_1$. Rearranging this inequality, we obtain $n_1 > g$, a contradiction.

$$\begin{aligned}
4n-6-2g = \sum_{i=1}^{n} d_i &\geq d_1 + d_2 + 2(n-n_1-2) + n_1 \\
&> (n-1) + (n-1-n_1) + 2(n-n_1-2) + n_1 \\
&= 4n-6-2n_1
\end{aligned}$$

Hence, $d_2 \leq n-1-n_1$. Remove the n_1 1's from D and replace $d_1 = n-1$ with $n-1-n_1$. Since $d_2 \leq n-1-n_1$, $n-1-n_1$ is the maximum element in the resulting sequence D'. Also, D' has $n-n_1$ elements, the smallest of which is 2, and has sum exactly $4(n-n_1)-6-2(g-n_1)$. Then by Claim 5.2.3, D' is realizable as a connected partial 2-tree G'. Add n_1 pendants adjacent to a vertex of degree $n-1-n_1$ in G' to obtain G, a realization of D. Adding pendants preserves partial 2-trees by Theorem 5.1.10 and so G is a connected partial 2-tree.

□

2-trees must have at least two vertices of degree 2. Therefore, a partial 2-tree must have at least two vertices of degree at most 2, i.e., $n_1 + n_2 \geq 2$. For example, $< 5, 3, 3, 3, 3, 3 >$ cannot be realizable as a partial 2-tree. The hypothesis that $d_1 = n - 1$ of Claim 5.2.3 together with the other hypotheses of Claim 5.2.3 imply that $n_2 \geq 2$. Therefore, even though $n_2 \geq 2$ is not included as a hypothesis in Claim 5.2.3, this property does indeed hold. In the following claim, however, we relax the hypothesis that $d_1 = n - 1$, and in doing so, we introduce the possibility that $n_2 < 2$. As a result, the following claim must include the hypothesis that $n_2 \geq 2$ in D.

Theorem 5.2.5. *Let D be a sequence of integers $d_1 \geq \ d_2 \ \ldots \ \geq \ d_n \geq 2$ with $\sum d_i = 4n - 6 - 2g$ where g is a non-negative integer. If $n - 1 - g \leq d_1 \leq n - 1$ and $n_2 \geq 2$, then D is realizable as a connected partial 2-tree. Moreover, if d_i and d_j, $i \neq j$, are not both 2, then there exists a connected partial 2-tree with degree sequence D which has a vertex of degree d_i adjacent to a vertex of degree d_j.*

Proof. We first show that if $n_2 = 2$, then $g = 0$. If $n_2 = 2$ then all other d_i are at least 3. Below we show that this fact and the hypotheses imply that $g = 0$ as when $n_2 = 2$, as desired.

$$4n - 6 - 2g = \sum_{i=1}^{n} d_i \geq (n - 1 - g) + 3(n - 3) + 2 + 2 = 4n - 6 - g$$
$$\implies 4n - 6 - 2g \geq 4n - 6 - g \implies g = 0$$

We now proceed to prove the theorem by induction on g. If $g = 0$, then $d_1 = n-1$ and Claim 5.2.3 yields the desired result. For the inductive hypothesis, assume the claim is true for a sequence D with sum $4n - 6 - 2(g - 1)$ where $g > 0$. Consider a sequence D with sum $4n - 6 - 2g$. If $d_1 = n - 1$, then Claim 5.2.3 again yields the desired result so assume $n - 1 - g \leq d_1 \leq n - 2$. Since $g > 0$, we know that $n_2 > 2$ because at the start of this proof we showed that if $n_2 = 2$, then $g = 0$. Remove a 2 from D to get D', a sequence with $n' = n - 1$ elements, whose maximum element is $d_1 \leq n' - 1$, and whose smallest two elements are both 2. Also, let $g' = g - 1$

and notice that the sum of D' is $4n - 6 - 2g - 2 = 4n' - 6 - 2g'$. Furthermore, $d_1 \geq n - 1 - g = n' - 1 - g'$. Then by induction, D' is realizable as a connected partial 2-tree and there exists a connected realization G' where a vertex v of degree d_i is adjacent to a vertex w of degree d_j when $i \neq j$ and when both d_i and d_j are not both 2. Finally, let xy be any edge in G' except vw. Subdivide xy and the resulting graph has degree sequence D, is connected, has vertices of degree d_i and d_j which are adjacent, and is still a partial 2-tree by Theorem 5.1.10. □

5.3 Partial 2-Trees with Even Degree Sequences

The main result of this section is Theorem 5.3.7 which characterizes the degree sequences of partial 2-trees with even degree sequences. Condition (e) of Theorem 5.0.5 shows that even degree sequences of 2-trees require a lower bound on n_2. As with 2-trees, even degree sequences of partial 2-trees require a lower bound on n_2. The next theorem proves the necessity of this bound.

Theorem 5.3.1. *Let D be the degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n \geq 2$ of a partial 2-tree. If D is even, then $n_2 \geq \frac{n+3-2g}{3}$.*

Proof. We proceed by induction on g, the gap of any partial 2-tree realizing D. If $g = 0$, then condition (e) of Theorem 5.0.5 indicates that $n_2 \geq \frac{n+3}{3}$, as desired. Assume the claim holds for D with degree sum $4n - 6 - 2(g - 1)$, where $g \geq 1$. Consider D with degree sum $4n - 6 - 2g$.

Let G be any partial 2-tree realization of D. Because $g \geq 1$, G has gap at least one and so there exists some edge xy which we can add to G and the result is still a partial 2-tree. Now add an ear adjacent to this edge. Thus, we increased the degrees of both x and y by 2 and so all degrees remain even. Therefore, the resulting graph G' is a partial 2-tree which realizes an even degree sequence D' with $n' = n + 1$ vertices and with gap $g' = g - 1$. Both, neither, or exactly one of x or y might be degree 2 vertices in G, but in any case, due to the additional ear and edge xy, G' has at most $n_2' \leq n_2 + 1$ degree 2 vertices. By induction, $n_2' \geq \frac{n'+3-2g'}{3}$. Below we show that this implies $n_2 \geq \frac{n+3-2g}{3}$.

$$n_2 \geq n_2' - 1 \geq \frac{n' + 3 - 2g'}{3} - 1 = \frac{n + 6 - 2g}{3} - 1 = \frac{n + 3 - 2g}{3}$$

□

When a sequence consists of only 2's and 4's, the condition $n_2 \geq \frac{n+3-2g}{3}$ is equivalent to $n_2 \geq \frac{n+9}{5}$ as shown in Claim 5.3.2.

Claim 5.3.2. *Let D be a sequence of n integers where $D =< 4^{<n-n_2>}, 2^{<n_2>} >$. Assume D has sum $4n - 6 - 2g$ where g is a non-negative integer. Then $n_2 \geq \frac{n+3-2g}{3}$ if and only if $n_2 \geq \frac{n+9}{5}$.*

Proof. Since $n = n_2 + n_4$, we see that $4n - 6 - 2g = 4(n_4 + n_2) - 6 - 2g$. Since D consists of only 4's and 2's, D also must sum to $4n_4 + 2n_2$. Thus, $4n_4 + 2n_2 = 4(n_4 + n_2) - 6 - 2g$. Solving for g, we see that $g = n_2 - 3$. Then $n_2 \geq \frac{n+3-2g}{3} = \frac{n+3-2n_2+6}{3} = \frac{n+9}{3} - \frac{2n_2}{3} \iff \frac{5n_2}{3} = \frac{n+9}{3} \iff n_2 \geq \frac{n+9}{5}$. □

Theorem 5.3.3 proves that when a sequence D consists only of 2's and 4's, the bound $n_2 \geq \frac{n+9}{5}$ is sufficient for knowing that there exists some partial 2-tree with degree sequence D. Therefore, the equivalent bound $n_2 \geq \frac{n+3-2g}{3}$ from Theorem 5.3.1 is tight. Theorem 5.3.3 is helpful in the proof of Theorem 5.3.7.

Theorem 5.3.3. *Consider a sequence D of the form $< 4^{<n_4>}, 2^{<n_2>} >$ where $n_4 \geq 1$. Let $n = n_2 + n_4$. Then D is realizable as a partial 2-tree if and only if $n \geq 5$ and $n_2 \geq \frac{n+9}{5}$. Moreover, if these conditions hold, then there exists a connected partial 2-tree with degree sequence D which has a degree 4 vertex is adjacent to a degree 2 vertex.*

Proof. ($\Rightarrow$) If D is realizable as a partial 2-tree G, then the maximum degree in G is $d_1 = 4$ since $n_4 \geq 1$. Since $d_1 \leq n - 1$ is necessary for realizability, we see that $n \geq 5$. Also, Theorem 5.3.1 shows that $n_2 \geq \frac{n+3-2g}{3}$, which by Claim 5.3.2, is equivalent to $\frac{n+9}{5}$. Finally, given any realization of D, let U be a component with a degree 4 vertex in the realization. U is a partial 2-tree and thus has at least two degree 2 vertices. Because U is connected, one of these degree 2 vertices must be adjacent to a degree 4 vertex.

($\Leftarrow$) We point out that because $n = n_2 + n_4$,the hypothesis $\frac{n+9}{5} \leq n_2$ is equivalent to $\frac{n_4+9}{4} \leq n_2$. We proceed to show by induction on n_2 that if $n \geq 5$ and $\frac{n_4+9}{4} \leq n_2$, a sequence D of the form $< 4^{<n_4>}, 2^{<n_2>} >$ where $n_4 \geq 1$ is realizable as a connected partial 2-tree. Since $n_4 \geq 1$ and $n_2 \geq \frac{n_4+9}{4}$, we see that $n_2 \geq 3$.

If $n_2 = 3$, then $n \geq 5$ and $\frac{n_4+9}{4} \leq n_2$ force that D is either $< 4, 4, 2, 2, 2 >$ or $< 4, 4, 4, 2, 2, 2 >$, both of which are realizable as 2-trees as shown in Figure 5.4.

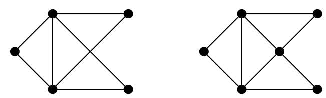


Figure 5.4: Base case of Theorem 5.3.3 when $n_2 = 3$

If $n_2 = 4$ then the bounds $n \geq 5$ and $\frac{n_4+9}{4} \leq n_2$ force that $n_4 \leq 7$. For $n_4 = 2$ or 3, consider the realization of $< 4, 4, 2, 2, 2 >$ or $< 4, 4, 4, 2, 2, 2 >$ shown in Figure 5.4. Repeatedly subdivide an edge in this realization so as to introduce the appropriate number of degree 2 vertices required for a realization of D. This realization is a partial 2-tree since subdividing an edge of a partial 2-tree yields another partial 2-tree by Theorem 5.1.10. For $n_4 = 1, 4, 5, 6$ or 7, Figure 5.5 shows a partial 2-tree realization of D. Notice that in each of these figures, two 2-trees are glued together at a vertex. Theorem 5.1.10 shows that gluing vertices preserves partial 2-trees.

For our inductive hypothesis, we assume that if $\frac{n_4+9}{4} \leq n_2 - 1$ and $n_2 \geq 5$, then $< 4^{<n_4>}, 2^{<n_2-1>} >$ is realizable as a connected partial 2-tree. Consider the sequence $< 4^{<n_4>}, 2^{<n_2>} >$.

If $n_4 \leq 4$, then let $n_2' = 4$ and $n_4' = n_4$. The base cases show a partial 2-tree realization for $< 4^{<n_4'>}, 2^{<n_2'>} >$. Repeatedly subdivide any edge xy in this realization in order to replace xy with a path with $n_2 - 4$ internal vertices. By Theorem 5.1.10, this yields a partial 2-tree realization of $< 4^{<n_4>}, 2^{<n_2>} >$.

Otherwise, $n_4 \geq 5$. Let $n_2' = n_2 - 1$ and $n_4' = n_4 - 4$. Then $n_2' \geq 4$ and the fact that $\frac{n_4+9}{4} \leq n_2$ implies that $\frac{n_4'+9}{4} \leq n_2'$. By induction, $< 4^{<n_4'>}, 2^{<n_2'>} >$ is realized by some connected partial 2-tree G'. Let v be any degree 2 vertex in G'. Glue v to an ear w in the partial 2-tree H shown in Figure 5.6 to obtain a graph G which is a

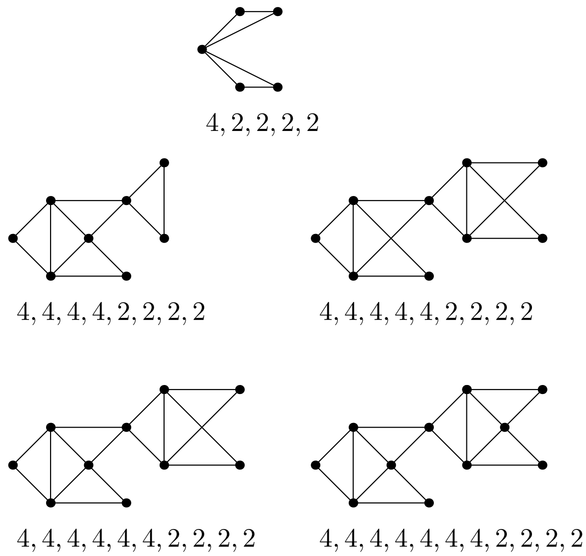


Figure 5.5: Base case of Theorem 5.3.3 when $n_2 = 4$

partial 2-tree by Theorem 5.1.10. Because v becomes a degree 4 vertex in G, G is a partial 2-tree consisting of four more degree 4 vertices than G' and one more degree 2 vertex. Thus, G is a connected partial 2-tree realization of $< 4^{<n_4>}, 2^{<n_2>} >$.

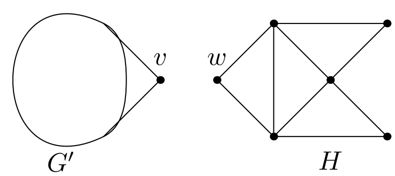


Figure 5.6: Inductive step of Theorem 5.3.3

□

Claim 5.3.4 to Claim 5.3.6 simplify the proof of Theorem 5.3.7 and the proofs of theorems in future sections.

Claim 5.3.4. *Let D be the sequence of n integers $< d, d, d, d, 2^{<n_2>} >$ where $d \geq 5$. Assume D has sum $4n - 6 - 2g$ where g is a non-negative integer. If $g = 0$, then D is not realizable as a partial 2-tree and thus not as a 2-tree. If $g > 0$, then D is realizable as a connected partial 2-tree. Moreover, there exists a connected partial*

2-tree with degree sequence D which has a vertex of degree d adjacent to a vertex of degree 2.

Proof. Note that $n = n_2 + 4$. Because the sum of D is $4n - 6 - 2g$ as well as $4d + 2n_2 = 4d + 2(n-4)$, we see that $4n - 6 - 2g = 4d + 2(n-4)$ and so $d = \frac{n+1-g}{2}$. If $g = 0$, then the claim holds by Theorem 5.0.5. Assume $g > 0$. It is helpful to note that $n_2 \geq 2d - 4$, which we show now. The hypotheses imply that D has sum $4d+2n_2 = 4n-6-2g = 4(n_2+4)-6-2g$. Solving for n_2, we see that $n_2 = 2d+g-5$. However, since $g > 0$, we see that $n_2 \geq 2d - 4$, as desired.

Since $n_2 \geq 2d-4$, the connected graph shown in Figure 5.7 realizes D. Furthermore, the graph in Figure 5.7 is indeed a partial 2-tree because it can be constructed in the following method. Create a cycle $abcd$ on 4 vertices, which is a partial 2-tree by Corollary 5.2.2. The graph in Figure 5.7 can be constructed from $abcd$ by attaching $d-2$ vertices to edge ab, attaching another $d-2$ vertices to edge cd, and subdividing edge bc so that it becomes a path with $n_2 - (2d-4)$ internal vertices. After doing so, vertices a, b, c, d of the original cycle become the degree d vertices that are shown in Figure 5.7. By Theorem 5.1.10, these operations preserve partial 2-trees and so the graph in Figure 5.7 is indeed a partial 2-tree.

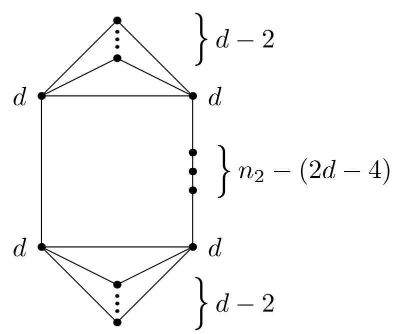


Figure 5.7: Partial 2-tree realization of $< d, d, d, d, 2^{<n_2>} >$ when $d = \frac{n+1}{2} \geq 5$ and $g > 0$

$\square$

Claim 5.3.5. *Let D be a sequence of integers $d_1 \geq d_2 \ \dots \ \geq d_n \geq 2$ such that $\sum d_i = 4n-6-2g$ where $g = 1, d_1 = n-3$, and $n \geq 7$. If D is even and $n_2 \geq \frac{n+3-2g}{3}$,* *then D is realizable as a connected partial 2-tree. Moreover, if $r > 2$ and $r \in D$, then there exists a connected partial 2-tree with degree sequence D which has a vertex of degree r adjacent to to a vertex of degree 2.*

Proof. Since $n \geq 7$ and $g = 1$, the assumption that $n_2 \geq \frac{n+3-2g}{3}$ implies that $n_2 \geq 3$. Create a new integer sequence D' with n elements as follows. Add 1 to d_1 and replace a 2 in D with a 3. D' has sum $4n - 6 - 2g + 2 = 4n - 6$ and so has gap 0. We now show that all hypotheses of Theorem 5.0.5 hold now. The new multiplicity of 2 in D' is $n_2 - 1 \geq 2$. Since $d_1 = n - 3$ the maximum element in D' is $n - 2$. Furthermore, since D' includes a 3, D' cannot be even and D' cannot be of the form $< d, d, d, d, 2^{<n_2>} >$ where $d \geq 5$. By Theorem 5.0.5, D' is realizable as a 2-tree. Consider $r \in D$ where $2 < r < n - 3$. Then $r \in D'$ and Theorem 5.0.5 also implies that there exists a 2-tree realization of D' in which a vertex s of degree r is adjacent to a vertex t of degree 2. Let G' be this realization of D'. Recall that 2-trees are 2-connected (see Theorem 5.1.2) and so G' is connected.

Our strategy is to argue that the unique vertex v of degree $d_1 = n - 2$ must be adjacent to the unique vertex w of degree 3 in G'. Then we can delete the edge vw to obtain a realization of D. Assume that v and w are not adjacent. Then w has neighbors a, b, c, and moreover, v is adjacent to all vertices but w. If there exists a path P between a and b that does not go through v and w, then the cycle $wavbw$, the path P, and the path wcv yields a K_4 minor, thus contradicting Theorem 5.1.2. For the same reason, all paths between any two of a, b, or c must go through v and w. Thus, $G - \{v, w\}$ has components U_a, U_b, U_c, which contain a, b, c, respectively. Furthermore, if $G - \{v, w\}$ has any other components, then v is a cut vertex in G. Since Theorem 5.1.2 states that a 2-tree is 2-connected, this is a contradiction. Thus, U_a, U_b, U_c are the only components of $G - \{v, w\}$. See Figure 5.8.

Remove w and consider the resulting subgraph H of G, which is partial 2-tree by Theorem 5.1.2. The vertex v is a cut vertex in H. The degree sum of H is $4n - 6 - 6 = 4(n - 1) - 6 - 2$ and so H has gap 1. Thus, there is some edge e which when added to H yields a 2-tree H'. However, e is an edge between at most two of U_a, U_b, and U_c. Thus, v is still a cut vertex in H'. This contradicts Theorem 5.1.2

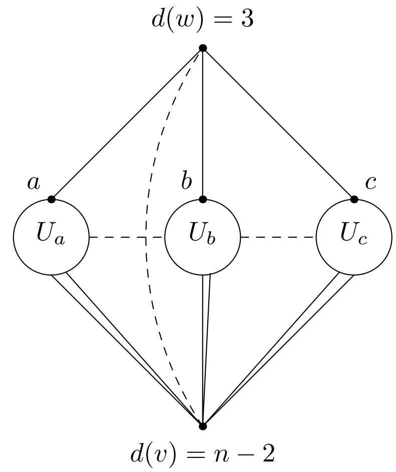


Figure 5.8: Components U_a, U_b, U_c

which states that a 2-tree is 2-connected.

We now know that v is adjacent to w in G'. Delete vw to yield a realization G of D. Deleting edges preserves partial 2-trees. We argue now that deleting vw cannot disconnect the graph. After deleting vw, every vertex except two are still adjacent to v since $d(v) = n-3$ in G. Because all vertices have degree at least 2 by hypothesis, these two vertices each must be adjacent to at least one neighbor of v. Hence, there is a path between any vertex and v and so G' is connected.

Recall that G' has a vertex s with degree $2 < r < n-3$ where $r \in D$ is adjacent to a vertex t of degree 2. Since v has degree $n-3$, $s \neq v$. Also, since 3 is not a degree in D, $s \neq w$. Thus, the edge st is distinct from vw and the degrees of s and t do not change when vw is deleted. Hence, s and t are still adjacent and so G still has a vertex of degree $2 < r < n-3$ adjacent to a vertex of degree 2, as desired. Finally, note that v must be adjacent to a degree 2 vertex in G since $d_1 = n-3$ and $n_2 \geq 3$. Hence, there is also a vertex of degree $n-3$ adjacent to a degree 2 vertex in G. We have just shown that G is a connected partial 2-tree realization of D in which a vertex of degree $r \in D$ where $r > 2$ is adjacent to a vertex of degree 2. $\square$

Claim 5.3.6. *Let D be a sequence of n positive integers with sum $4n-6-2g$ where g is a non-negative integer. Then the following inequalities hold.*

1. $3n_1 + 2n_2 + n_3 \geq 6 + 2g$

2. *If* $n_1 = 0$ *and* $g > 0$*, then* $2n_2 + n_3 \geq 8$.

3. *If* D *is even, then* $2n_2 \geq g + 3 + n - n_4$.

Proof. Note that $n = n_1 + n_2 + n_3 + n_4 + \sum_{k\geq 5} n_k$. Then we see that:

$$4n - 6 - 2g = \sum d_i = n_1 + 2n_2 + 3n_3 + 4n_4 + \sum_{k\geq 5} kn_k \implies$$
$$4(n_1 + n_2 + n_3 + n_4 + \sum_{k\geq 5} n_k) - 6 - 2g = n_1 + 2n_2 + 3n_3 + 4n_4 + \sum_{k\geq 5} kn_k \implies$$
$$3n_1 + 2n_2 + n_3 = 6 + 2g + \sum_{k\geq 5}(k-4)n_k$$

Thus, $3n_1+2n_2+n_3 = 6+2g+\sum_{k\geq 5}(k-4)n_k$. This implies $3n_1+2n_2+n_3 \geq 6+2g$. So if $n_1 = 0$ and $g > 0$, it immediately follows that $2n_2 + n_3 \geq 8$. If D is even, then $n_k = 0$ when k is odd. So if D is even, $3n_1 + 2n_2 + n_3 = 6 + 2g + \sum_{k\geq 5}(k-4)n_k$ implies $2n_2 = 6 + 2g + \sum_{k\geq 6}(k-4)n_k$, or equivalently, $n_2 = 3 + g + \sum_{k\geq 6} \frac{(k-4)(n_k)}{2}$. The following string of inequalities uses the fact that $\frac{k-4}{2} \geq 1$ when $k \geq 6$ to show that $n_2 \geq 3 + g + n - n_2 - n_4$, as desired.

$$n_2 = 3 + g + \sum_{k\geq 6} \frac{(k-4)(n_k)}{2} \geq 3 + g + \sum_{k\geq 6} n_k = 3 + g + n - n_2 - n_4$$

□

We now have the tools to prove the main result of this section.

Theorem 5.3.7. *Let* D *be the sequence of integers* $d_1 \geq d_2 \geq \ldots \geq d_n \geq 2$ *with sum* $4n - 6 - 2g$ *where* g *is a non-negative integer. Assume* D *is even. Then* D *is realizable as a partial 2-tree if and only if the following conditions hold:*

1. $d_1 \leq n - 1$

2. $n_2 \geq \frac{n+3-2g}{3}$

3. If $g = 0$, then D is not of the form $< d, d, d, d, 2^{<n_2>} >$ where $d = \frac{n+1}{2} \geq 5$.

Furthermore, if the above conditions hold, then D is realizable as a connected partial 2-tree. Moreover, if the above conditions hold and $r \in D$ where $r > 2$, then there exists a connected partial 2-tree with degree sequence D which has a vertex of degree r adjacent to a vertex of degree 2.

Proof. ($\Rightarrow$) This direction is clear by Theorem 5.0.5 and Theorem 5.3.1.
($\Leftarrow$) We proceed by induction on n. We take care to show that at each step of the induction, D is realizable as a *connected* partial 2-tree. The base cases are $3 \leq n \leq 6$. The integer sequences that fit the hypotheses are shown below.

$$\begin{array}{cccc} <2,2,2> & <2,2,2,2> & <2,2,2,2,2> & <2,2,2,2,2,2> \\ <4,2,2,2,2> & <4,4,2,2,2> & <4,2,2,2,2,2> & <4,4,2,2,2,2> \end{array}$$

If the sequence consists only of 2's, then a cycle of size n is a partial 2-tree (by Corollary 5.2.2) which realizes the sequence. Otherwise, the sequence consists of 4's and 2's, in which case, Theorem 5.3.3 indicates that the sequence is realizable as a connected partial 2-tree and that there exists a connected realization in which a degree 4 vertex is adjacent to a degree 2 vertex.

Assume that when $n \geq 7$ and the hypotheses of the theorem hold, an even sequence D with $n-1$ positive integers is realizable as a connected partial 2-tree, and additionally, for $r \in D$ where $r > 2$, there exists a connected partial 2-tree with degree sequence D which has a vertex of degree r adjacent to a vertex of degree 2. Let D be an even sequence of n integers with sum $4n - 6 - 2g$, where $d_1 \leq n-1$ and $n_2 \geq \frac{n+3-2g}{3}$.

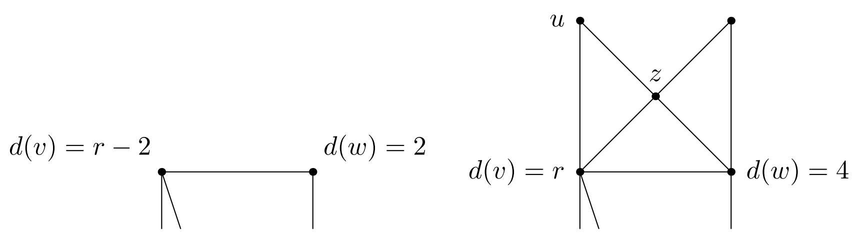
If $g = 0$, then the claim holds for D by Theorem 5.0.5. We therefore assume that $g > 0$. If D is a sequence such that $d_1 \geq n - 1 - g$, then the desired claim holds for D by Theorem 5.2.5. We therefore also assume that $d_1 < n - 1 - g$. Since $g > 0$, $d_1 < n - 1 - g$ implies $d_1 < n - 2$.

Case: $n_4 \geq 2$

If $d_1 = 4$, then the desired claim holds by Theorem 5.3.3. Thus, assume $d_1 > 4$. Let $r > 4$ be an arbitrary element in D. Remove two 4's and one 2 from D and replace r with $r-2$. This decreases the sum of D by 12, and since $r > 4$, the resulting sequence which we call D' has exactly one less 2. Then D' is an even sequence with $n' = n-3$ elements, with a multiplicity of 2 equal to $n'_2 = n_2 - 1$, and with sum $4n - 6 - 2g - 12 = 4n' - 6 - 2g$. Hence, D' has gap $g' = g$. We now show that $n'_2 \geq \frac{n'+3-2g'}{3}$. By hypothesis, $n_2 \geq \frac{n+3-2g}{3}$. Thus, $n'_2 = n_2 - 1 \geq \frac{n+3-2g}{3} - 1 = \frac{n'+3-2g'}{3}$.

We let d'_1 be the maximum element in D'. Then $d'_1 \leq d_1$. (Note that if $d'_1 < d_1$, then $r = d_1$.) In order to apply the inductive hypothesis, we must show d'_1 to be at most $n' - 1 = n - 4$.

So if $d'_1 \leq n-4$, then by induction, the claim holds and there exists a connected partial 2-tree G' of D' in which a vertex v of degree $r-2$ is adjacent to a vertex w of degree 2. Add an ear z to the edge vw and then proceed to add an ear u to the edge vz and another ear to the edge wz. See Figure 5.9. By Theorem 5.1.10, adding ears to edges preserves partial 2-trees, and so the resulting graph G is still a partial 2-tree. Furthermore, v becomes a degree r vertex in G, w becomes a degree 4 vertex in G, and z is an additional degree 4 vertex in G. Thus, G is a connected realization of D in which a degree $r > 4$ vertex is adjacent to a degree 2 vertex. Finally, G is also a realization in which a degree 4 vertex, namely z, is adjacent to a degree 2 vertex, namely u.



(a) G' **(b)** G

Figure 5.9: Insert $< 4, 4, 2 >$ and increase r to $r - 2$

Now assume $d'_1 \geq n-3$. Prior to this case, we assumed that $d_1 < n-2$, or equivalently, $d_1 \leq n-3$. Then $n-3 \leq d_1' \leq d_1 \leq n-3$ and so $d_1 = n-3$. Also prior to this case, we assumed that $n \geq 7, g > 0$, and $d_1 < n-1-g$. Since $d_1 = n-3$, we see that $n-3 < n-1-g$ and so $g < 2$. Then $g = 1$ since $g > 0$. Because $g = 1$, $d_1 = n-3$, and $n \geq 7$, the claim holds for D by Claim 5.3.5.

Case: $n_4 \leq 1$

Recall that prior to any cases, we assumed that $d_1 < n-2$. Let D' be the same sequence as D but with one less 2. Thus, D' is an even sequence with $n' = n-1$ elements, with a multiplicity of 2 equal to $n_2' = n_2 - 1$ with sum $4n-6-2g-2 = 4n'-6-2(g-1)$. Hence, D' has gap $g' = g-1$. Also, the maximum element in D' is $d_1' = d_1 < n-2$. Thus, $d_1' < n'-1$. Now, if $g' = 0$ and D' is of the form $< d, d, d, d, 2^{<n_2>} >$ where $d \geq 5$, then the desired claim holds for the original sequence D by Claim 5.3.4. Thus, we may assume it is not the case that $g' = 0$ and D' is of the form $< d, d, d, d, 2^{<n_2>} >$ where $d \geq 5$. In order to apply the inductive hypothesis, we need only show $n_2' \geq \frac{n'+3-2g'}{3}$. Below, we first show an auxiliary inequality, namely, $2n_2 > n+1$.

$$\begin{aligned} 2n_2 &\geq g+3+n-n_4 \quad \text{by Claim 5.3.6} \\ &\geq g+3+n-1 \quad \text{since } n_4 \leq 1 \\ &> n+1 \end{aligned}$$

Since $2n_2 > n+1$, subtracting 2 from both sides yields that $2(n_2-1) \geq n-1$, and thus, $2n_2' \geq n'$. Then $n_2' \geq \frac{n'}{2}$. But $\frac{n'}{2} \geq \frac{n'+3}{3}$ as long as $n' \geq 6$, which holds since $n \geq 7$. Thus, we can apply induction, and so D' is realizable as a connected partial 2-tree and there exists a connected realization G' with a vertex of degree $r \geq 4$ adjacent to a vertex of degree 2. Subdivide any edge xy in G' to obtain the desired realization of D. Subdividing edges preserves partial 2-trees by Theorem 5.1.10.

□

5.4 Degree Sequences of Partial 2-Trees When $d_n = 2$

Theorem 5.4.1 generalizes Theorem 5.3.7 and characterizes all partial 2-tree degree sequences (even or not) with no degree 1 vertices.

Theorem 5.4.1. *Let D be the sequence of integers $d_1 \geq d_2 \geq \ldots \geq d_n > 0$ with $d_n \geq 2$. Let D have sum $4n - 6 - 2g$ where g is a non-negative integer. Then D is realizable as a partial 2-tree if and only if the following conditions hold:*

1. $n_2 \geq 2$
2. $d_1 \leq n - 1$
3. *If D is even, then $n_2 \geq \frac{n+3-2g}{3}$.*
4. *If $g = 0$, then D is not of the form $< d, d, d, d, 2^{<n_2>} >$ where $d \geq 5$.*

Furthermore, if the above conditions hold, then D is realizable as a connected partial 2-tree. Moreover, if the above conditions hold and $r \in D$ where $r > 2$, then there exists a connected partial 2-tree with degree sequence D which has a vertex of degree r is adjacent to to a vertex of degree 2.

Proof. (⇒) By hypothesis, all elements in D are at least 2. Since a partial 2-tree is a subgraph of a 2-tree and any 2-tree has at least 2 ears, n_2 must be at least 2. Furthermore, in any graphical sequence, $d_1 \leq n - 1$. Condition (3) holds by Theorem 5.3.7. Condition (4) holds by Theorem 5.0.5.
(⇒) We proceed by induction on n. We take care to show that at each step of the induction, D is realizable as a *connected* partial 2-tree. If $n = 3$ or 4, the only integer sequences that fit the hypotheses are $< 2, 2, 2 >$, $< 3, 3, 2, 2 >$, and $< 4, 2, 2, 2 >$. The reader can verify that all are uniquely realizable as connected partial 2-trees

and that in these realizations, any vertex with degree higher than 2 is adjacent to a degree 2 vertex.

For the inductive hypothesis, assume that if $n \geq 5$ and if the hypotheses of the theorem hold, a sequence D with $n-1$ positive integers is realizable as a connected partial 2-tree, and additionally, for $r \in D$ where $r > 2$, there exists a connected partial 2-tree with degree sequence D which has a vertex of degree r adjacent to a vertex of degree 2.

Now consider an integer sequence D of n elements which satisfies the hypotheses. If D is even, then Theorem 5.3.7 yields the desired claim. If $g = 0$, then Theorem 5.0.5 yields the desired claim. If $d_1 \geq n-1-g$, then Theorem 5.2.5 yields the desired claim. Thus, assume that D is not even, that $g > 0$, and that $d_1 < n-1-g$. The assumptions $g > 0$ and $d_1 < n-1-g$ imply that $d_1 \leq n-3$.

We first observe that either n_2 or n_3 must be at least 3. Otherwise, $2n_2 + n_3 < 8$. This contradicts Claim 5.3.6 which indicates that $2n_2 + n_3 \geq 8$.

If $n_3 \geq 3$, remove two 3's from D thus decreasing the sum of D by 6 and call the resulting sequence D'. Then D' is a sequence $n' = n-2$ elements, has has sum $4n - 6 - 2g - 6 = 4n' - 6 - 2(g-1)$, and so has gap $g - 1$. D' has maximum element $d_1 \leq n - 3 = n' - 1$ and the multiplicity of 2 in D' is $n_2 \geq 2$. Since D' still has at least one 3, D' is not even and is not of the form $< d, d, d, d, 2^{<n_2>} >$ where $d \geq 5$. Thus, all conditions hold. Let r be any element in D' or D. (Although the multiplicities of elements in D and D' differ, the set of elements in each are the same.) By induction, there exists a connected realization G' of D' with a vertex u of degree r adjacent to a degree 2 vertex v. Since G' has at least two ears, there is an ear $w \neq v$. Splice the ear w to obtain a new graph G. (Recall Definition 5.1.9 describes splicing and Figure 5.2 demonstrates this procedure.) Since $w \neq v$, uv is still an edge in G and so a vertex of degree r is adjacent to a vertex of degree 2 in G. Since G has two additional degree 3 vertices, G is a realization of D. Finally, splicing an ear preserves partial 2-trees by Theorem 5.1.10 and so G is the desired partial 2-tree realization of D.

If $n_2 \geq 3$, remove a 2 from D to create D', a sequence with $n' = n - 1$ elements and $n_2' = n_2 - 1$ 2's and with sum $4n - 6 - 2g - 2 = 4n' - 6 - 2(g-1)$. Then D' has gap $g' = g - 1 \geq 0$ and is not even since D is not. Also, D' has maximum element $d_1 \leq n - 3 < n' - 1$. If $g' = 0$ and D' is of the form $< d, d, d, d, 2^{<n'-4>} >$ where $d \geq 5$, then the original sequence D has form $< d, d, d, d, 2^{<n_2>} >$ where $d \geq 5$ and $g > 0$. In this case, the result holds by Claim 5.3.4. Otherwise, all conditions required for inductive hypothesis hold, and so there exists a connected realization G' of D' with a vertex of degree r adjacent to a degree vertex v. Subdivide any edge xy in G' with edge. Subdividing an edge preserves partial 2-trees by Theorem 5.1.10 and so the resulting graph is the desired realization of D.

□

5.5 Degree Sequences of Partial 2-Trees When $d_n = 1$

The main result of this section is Theorem 5.5.1, which characterizes the degree sequences of partial 2-trees with at least one degree 1 vertex. Recall that Theorem 5.4.1 characterizes degree sequences of partial 2-trees with no degree 1 vertices. Furthermore, Theorem 5.4.1 also includes an adjacency result, namely, that a degree 2 vertex can be made adjacent to any vertex of higher degree. However, such an adjacency result does not hold true for partial 2-trees with at least one degree 1 vertex. For example, $< 4, 3, 3, 3, 3, 1, 1 >$ is realizable as a partial 2-tree but the reader can check that a degree 4 vertex is not adjacent to degree 1 vertex in any realization. Thus, Theorem 5.5.1 does not include adjacency results.

We present an example now to help the reader understand condition (3) in Theorem 5.5.1 which states that $n_1 \leq g$ in a partial 2-tree with degree 1 vertices. This condition is intuitive. Consider any partial 2-tree which is not simply a single edge. Because there are no degree 1 vertices in such a 2-tree, every degree 1 vertex in such a partial 2-tree must be 'missing an edge.' Thus, each degree 1 vertex in a partial 2-tree forces the gap to increase by 1, that is, $n_1 \leq g$. The exceptional case is a star, in which $n_1 = g + 1$.

While the explanation in the previous paragraph gives intuition for why $n_1 \leq g$, it does not suffice as a proof. In order to help the reader understand the proof of necessity of $n_1 \leq g$ as given in Theorem 5.5.1, we give the following example. Consider the sequence $D =< 5, 5, 5, 3, 1, 1, 1, 1, 1, 1 >$, which has sum equal to $4n - 6 - 2g = 24$ where $n = 10, n_1 = 6, g = 5$ and so clearly $n_1 > g$. D is realizable but not realizable as a partial 2-tree. If there were a partial 2-tree G which realized D, then removing the pendants from G leaves the subgraph H on $n_H = 4$ vertices. By Theorem 5.1.2, H is also a partial 2-tree and so the edges of H can contribute at most $4n_H - 6 = 10$ to the degree sum of G. Also, the $n_1 = 6$ pendants in G contribute at most $2n_1$ more to the degree sum of G. Thus, the sum of D can be at most $(4n_H - 6) + 2n_1 = 10 + 12 = 22$, which contradicts that D has sum 24.

In summary, the degree sum of the subgraph H of G induced by vertices of degree higher than 1 was simply too large for D to be realizable as a partial 2-tree. Although D may appear to have a large gap, this gap is introduced by the large number of 1's in D. Then the gap is not well-distributed, and so in any realization of D, too many edges must exist between the vertices of degree 5 and 3, thus making the degree sum of these vertices too large.

Unlike theorems in previous sections, Theorem 5.5.1 does not and cannot guarantee that any degree sequence D which is realizable as a partial 2-tree is also realizable as a *connected* partial 2-tree. In general, any *connected* graph must have degree sum at least $2n - 2$ in order to have a spanning tree. In previous sections, we assumed that $d_n \geq 2$, i.e., that all elements of a given sequence D were at least 2, and so the sum of D can be no smaller than $2n - 2$. This allows for connectivity. However, if we allow D to include a 1 as we do in this section, then the sum of D can be less than $2n - 2$, in which case there is no *connected* partial 2-tree which realizes D. Hence, in Theorem 5.5.1, we must assume D has sum at least $2n - 2$ for connectedness.

We remind the reader that in Section 5, we explained that we adhere to Definition 5.0.2, and thus we adopted the convention that a single edge is indeed a partial 2-tree.

Theorem 5.5.1. *Let D be the integer sequence $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ with* *sum $4n - 6 - 2g$ where g is a non-negative integer. Assume $n_1 \geq 1$. Then D is realizable as a partial 2-tree if and only if the following conditions hold:*

1. $n_1 + n_2 \geq 2$
2. $d_1 \leq n - 1$
3. $n_1 \leq g$ *or D is the degree sequence of a star.*

If the above conditions hold and the sum of the entries in D is at least $2n - 2$, then there exists a connected partial 2-tree which realizes D. If the sum of entries in D is even and less than $2n - 2$, then D is realizable as a forest and thus a union of partial 2-trees.

Proof. ($\Rightarrow$) Since a 2-tree has at least two ears, a partial 2-tree must have at least 2 vertices of degree less than 3. Hence, $n_1 + n_2 \geq 2$. In any graphical sequence, $d_1 \leq n - 1$. We now prove condition (3). Any realization of D is or is not a star. Assume D is not realizable as a star. Then $n > 2$ because if $n = 2$, the only partial 2-tree is a single edge, which is a star. Since $n > 2$ and D is not realizable as a star, then $n_1 \leq n - 2$. Let H be the subgraph induced by all vertices of degree greater than 1. Note that H has at least two vertices since $n_1 \leq n - 2$. Also, since H is a subgraph of a partial 2-tree, H is also a partial 2-tree by Theorem 5.1.2. Thus, we can view any realization G of D as the partial 2-tree H with n_1 additional pendants, some of which may be adjacent to vertices in H and some of which may be adjacent to each other. See Figure 5.10.

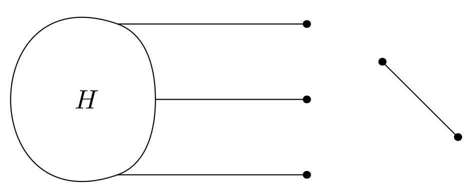


Figure 5.10: Partial 2-tree H with n_1 pendants

Consider the degree sequence D_H of H, which has $n - n_1 \geq 2$ vertices. Because D_H is realizable as a partial 2-tree, the sum of D_H is at most $4(n - n_1) - 6$. Due to the pendants of $G-H$, G has degree sum at most $(4(n-n_1)-6)+2n_1 = 4n-2n_1-6$. Then $4n-6-2g \leq 4n-2n_1-6$. Rearranging this inequality, we obtain $n_1 \leq g$, as desired.

($\Leftarrow$) If the sum of entries in D is less than $2n-2$, then because the sum of D is $4n-6-2g$ and is thus even, D is realizable as a forest G by Claim 1.0.1. Every component of G is a tree on at least two vertices and is thus a partial 2-tree by Corollary 5.1.11. Hence, D is realizable as a union of partial 2-trees. We assume that the sum of D is at least $2n-2$ for the rest of this proof.

If $n = 2$ or $n = 3$, then the only sequences which satisfy all hypotheses are $< 1, 1 >$ and $< 2, 1, 1 >$. These sequences are realizable as trees, which are connected and are partial 2-trees, again by Corollary 5.1.11. For the duration of the proof, we assume that $n \geq 4$. We proceed by induction on n_1. Assume $n_1 = 1$ for the base case. Since $n \geq 4$ and $n_1 = 1$, D is not realizable as a star. Thus, $n_1 \leq g$ by condition (3). Our strategy is to create a second sequence D' and to use Theorem 5.4.1 to obtain a connected partial 2-tree realization of D'.

Base Case A: $n_1 = 1$ and $n_2 \geq 2$

> Replace the 1 in D with a 3, thus increasing the sum of D by 2. Let D' be the resulting sequence and we verify now that D' satisfies the hypotheses of Theorem 5.4.1. D' has zero degree 1 vertices, n vertices, and $n_2 \geq 2$ degree 2 vertices. If 3 is the maximum element in D', then since $n \geq 4$, the maximum element of D' is at most $n-1$. Otherwise, the maximum element in D' is that of D and so is $d_1 \leq n-1$. Also, D' has sum equal to $4n-6-2g+2 = 4n-6-2(g-1)$ and so the gap of D' is now $g-1$. Since $1 \leq n_1 \leq g$, we see that $g-1 \geq 0$ and thus the gap pf D' is non-negative. Since D' has a 3, D' is not even and is not of the form $< d, d, d, d, 2^{<n_2>} >$ where $d \geq 5$. Hence, by Theorem 5.4.1, D' is realizable as a connected partial 2-tree with a degree 3 vertex u adjacent to a degree 2 vertex v. Delete the edge uv. If uv is not a cut edge, this yields a connected partial 2-tree which realizes D. If uv is a cut edge, then no component of $G-uv$ can be a tree

since $n_1 = 1$ and every tree has at least two leaves. Thus, each component has a cycle. Let C be a cycle in the component containing u. Let ab be any edge on C. Let vz be the edge incident to v. 2-switch ab and vz. By Claim 5.1.12, this procedure yields a connected partial 2-tree which realizes D.

Base Case B: $n_1 = 1$ and $n_2 < 2$

Because $n_2 < 2$, the hypothesis $n_1 + n_2 \geq 2$ implies $n_2 = 1$. If $d_1 = n - 1$, then Corollary 5.2.4 yields the desired claim, so assume $d_1 \leq n - 2$. Since $n_1 = n_2 = 1$ and $n \geq 4$ by assumption, we know $d_1 \geq d_2 > 2$. If there exists an even $d_i > 2$ in D, let D' be the sequence with d_i replaced by $d_i + 1$ and 1 replaced by 2, thus increasing the sum of D by 2 and n_2 by 1 so that the multiplicity of 2 in D' is $n_2' = 2$. Then D' is not even and has maximum element $d_1 + 1$ which is at most $n - 1$ since $d_1 \leq n - 2$. If D' is of the form $< d, d, d, d, 2^{<n_2'>} >$ where $d \geq 5$, then D is $< d, d, d, d-1, 2, 1 >$. However, this is impossible since the reader can check that condition (3) does not hold for $< d, d, d, d-1, 2, 1 >$ when $d \geq 5$. By Theorem 5.4.1, D' is then realizable as a connected partial 2-tree with a degree $d_1 + 1$ vertex u adjacent to a degree 2 vertex v. Delete the edge uv. If uv is not a cut edge, then this yields a connected partial 2-tree realization of D. If uv is a cut edge, follow the reasoning of Base Case A to obtain the desired connected partial 2-tree realization of D.

Now, if there does not exist an even $d_i > 2$ in D, then d_1 and d_2 must both be odd since $d_1 \geq d_2 > 2$. Form D'' by replacing d_1 with $d_1 + 1$ and replacing 1 by 2. The argument then follows the argument for D'.

We now assume the following. When conditions (1)-(3) hold for a degree sequence D whose sum $4n - 6 - 2g$ is at least $2n - 2$ and in which 1 has multiplicity $n_1 - 1$ for $n_1 \geq 2$, then there exists a connected partial 2-tree realization of D. Consider a sequence D where 1 has multiplicity n_1. If D is realizable as a star, then D is a tree which is connected and which is a partial 2-tree by Corollary 5.1.11. We now assume that D is not realizable as a star and so by condition (3), we assume $n_1 \leq g$. Since $2 \leq n_1 \leq g$, the gap is nonzero.

If $d_1 = n-1$, then Corollary 5.2.4 yields the desired claim, so assume $d_1 \leq n-2$. If $n_3 \geq 1$, then replace a 3 with a 2 and remove a 1 from D to create the sequence D' with even sum equal to 2 less than that of D. D' is a sequence with $n' = n-1$ elements, $n'_1 = n_1 - 1$ degree 1 elements, $n'_2 = n_2 + 1$ degree 2 elements, and a maximal element of at most $n - 2 = n' - 1$ (since $d_1 \leq n - 2$). Notice that $n_1 + n_2 \geq 2$ implies $n'_1 + n'_2 \geq 2$. Also, since $n \geq 4$, $n' \geq 3$. The sum of D is $4n - 6 - 2g - 2 = 4(n-1) - 6 - 2(g-1)$ and so the new gap is $g' = g - 1$. Then g' is non-negative and $n'_1 \leq g'$. By induction, D' is realizable by a connected partial 2-tree. Add a pendant adjacent to the vertex with degree 2 to yield a connected realization of D which is still a partial 2-tree by Theorem 5.1.10.

Otherwise, $n_3 = 0$. If $d_1 = 2$, then D is a sequence of 2's and two 1's and since D has sum at least $2n - 2$, D is a sequence of n_2 2's and exactly two 1's. Thus, D can be realized by a path on $n_2 + 2$ vertices. This is connected and is a partial 2-tree by Corollary 5.1.11. So we assume that $d_1 > 2$. Replace d_1 with $d_1 - 1$ and remove a 1 from D to create a sequence D' with even sum two less than that of D. D' is a sequence with $n' = n - 1$ elements, $n'_1 = n_1 - 1$ degree 1 elements, and a maximal element of at most $n - 2 = n' - 1$. The sum of D is $4n - 6 - 2g - 2 = 4(n - 1) - 6 - 2(g - 1)$ and so the gap has again decreased by 1. Then the new gap is $g' = g - 1$ and is non-negative and $n'_1 \leq g'$. We must now show that D' has at least two elements of degree 1 or 2. If not, then since $n_1 + n_2 \geq 2$ and $n_1 \geq 2$, we know that $n_1 = 2$ and $n_2 = 0$ in D. Claim 5.3.6 yields that $3n_1 + 2n_2 + n_3 \geq 6 + 2g$. Since $g > 0$ and $n_3 = 0$, the values $n_1 = 2$ and $n_2 = 0$ contradict this inequality. Thus, it must be true that D' has at least two elements of degree 1 or 2. By induction, D' is realizable by a connected partial 2-tree. Add a pendant adjacent to the vertex with degree $d_1 - 1$ to yield a connected realization of D, which is still a partial 2-tree by Theorem 5.1.10.

□

5.6 Combined Results

We can now prove our main result, Theorem 5.6.2. We combine the results of the previous section, specifically Theorem 5.4.1 and Theorem 5.5.1, to do so. We first prove Claim 5.6.1 because both directions of the proof of Theorem 5.6.2 rely on this claim.

Claim 5.6.1. *Let D be a sequence of n positive integers with sum $4n-6-2g$ where g is a non-negative integer. If D has the form $< d, d, d, d, 2^{<n_2>} >$, then $d = \frac{n+1-g}{2}$. Also, if $g = 0$, then D has the form $< d, d, d, d, 2^{<n_2>} >$ for some integer $d \geq 5$ if and only if D has the form $< \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, 2^{<n_2>} >$ for some $n \geq 9$.*

Proof. Assume D has the form $< d, d, d, d, 2^{<n_2>} >$. Per Claim 5.3.4, $d = \frac{n+1-g}{2}$. Assume now that $g = 0$. Then $d = \frac{n+1-g}{2} = \frac{n+1}{2}$, and rearranging this for n, we see $n = 2d - 1$. Thus, the rest of the claim follows since $d \geq 5$ if and only if $n \geq 9$. □

Theorem 5.6.2. *Let D be a sequence of n positive integers. Then D is realizable as a partial 2-tree if and only if the following conditions hold:*

(a) *The sum of D is $4n - 6 - 2g$ where g is a non-negative integer.*

(b) *The maximum element of D is at most $n - 1$.*

(c) $n_1 + n_2 \geq 2$

(d) $n_2 \geq \frac{n+3-2g}{3}$ *whenever D is even.*

(e) *If $g = 0$ then D is not of the form $< \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, 2^{<n-4>} >$ where $n \geq 4$.*

(f) $n_1 \leq g$ *or D is the degree sequence of a star.*

If the above conditions hold and the sum of the entries in D is at least $2n - 2$, then there exists a connected partial 2-tree which realizes D. If the above conditions hold and $n_1 = 0$, then given any $r \in D$ where $r > 2$, there exists a connected partial 2-tree which whose degree sequence is D which has a vertex of degree r adjacent to a vertex of degree 2. If the sum of entries in D is less than $2n - 2$ and even, then D is realizable as a forest and thus a union of partial 2-trees.

Proof. ($\Rightarrow$) Except for condition (e), the forward direction follows immediately from Theorem 5.4.1 and Theorem 5.5.1. By Theorem 5.0.5, if $g = 0$, D is not of the form $< d, d, d, d, 2^{<n_2>} >$ for $d \geq 5$. By Claim 5.6.1, we thus know that D is not of the form $< \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, 2^{<n_2>} >$ where $n \geq 9$. We show now that D is not of this form for $4 \leq n \leq 8$ either. If n is any even number, specifically, if $n = 4, 6, 8$, then $d = \frac{n+1}{2}$ is not an integer and D is not a sequence of integers, a contradiction. If $n = 7$, then $D =< 4, 4, 4, 4, 2, 2, 2 >$ and so D is an even sequence. But $n_2 < \frac{n+3}{3}$, thus contradicting condition (d). If $n = 5$, then $D =< 3, 3, 3, 3, 2 >$, which contradicts (c). Thus, the form is not realizable for $n \geq 4$.

($\Leftarrow$) For the backward direction, we point out that condition (e) cannot fail if $n_1 > 0$ and that condition (f) cannot fail if $n_1 = 0$. Also, note that $d_n \geq 2$ if and only if $n_1 = 0$, in which case D has sum at least $2n$. So if the sum of D is less than $2n - 2$, it must be true that $n_1 > 0$.

If $n_1 > 0$, the claim follows from Theorem 5.5.1. Now assume that $n_1 = 0$. If $g = 0$, condition (e) and claim 5.6.1 imply D is not of the form $< d, d, d, d, 2^{<n_2>} >$ for $d \geq 5$. Because D has sum at least $2n$, we must show that D can be realized by a connected partial 2-tree. This follows from Theorem 5.4.1. Furthermore, Theorem 5.4.1 also yields that if there exists $r \in D$ where $r > 2$, there exists a connected partial 2-tree which whose degree sequence is D which has a vertex of degree r adjacent to a vertex of degree 2. $\square$

Chapter 6

Conclusion and Future Work

In summary, via this dissertation, my original contributions to the field of mathematics include the following items.

1. Characterization of degree vector sequences of k-edge-colored unicyclic graphs (Chapter 1)

2. Characterization of degree vector sequences of factors of fixed DUPs, fixed DUCs, and fixed graphs with maximum degree at most 2 (Chapter 2)

3. Characterization of degree sequences of 2-edge-colored fixed DUPs and fixed DUCs and proof that one restricted case for each is NP-Complete (Chapter 3)

4. Characterization of degree sequences of partial 2-trees (Chapter 4)

5. Characterization of degree sequences of grid factors in a subset of cases (Chapter 5)

There are several clear pathways for future work. For unicyclic graphs, it is interesting to consider whether or not the results for k-edge-colored unicylic graphs from Chapter 1 generalize to k-edge-colored graphs in which each component has at most 1 cycle. As discussed in Chapter 3, our characterization of degree vector sequences of 2-edge-colored fixed DUPs and fixed DUCs yields necessary conditions

(Claim 3.5.2) for characterizing the degree vector sequences of k-edge-colored fixed DUPs and fixed DUCs when $k \geq 3$. It is natural to continue our analysis of whether or not these necessary conditions are sufficient for the case $k \geq 3$. Regarding our results on grid factors, Table 4.1 gives open cases for which we have no results. It would be interesting to consider these open cases.

Finally, as mentioned in the Introduction, it may be possible to answer the k-Edge-Coloring Problem or the related Factor Problem for cacti graphs, Halin graphs, and edge-maximal outerplanar graphs. We leave the exploration of these graph families for future work.

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