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by

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Welded Continuous Frames and Their Components

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PLASTIC DESIGN OF PINNED-BASE GABLE FRAMES

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S Y N O P S I S

Present day analysis and design of continuous structures as defined by specifications and design codes are based on an assumed elastic behavior of the structure. The criterion of the design in most cases is the attainment of an allowable extreme fiber stress. While a design that results from using such a procedure will be safe, the actual degree of safety is unknown and may vary between extreme limits.

Recently, a different type of analysis based on the maximum carrying capacity of a structure as a whole has "come of age". This new procedure known as "plastic analysis" or "plastic design" gives a clearer insight into the actual strength of structures and therefore promises a more economic usage of materials. It should also be noted that the procedure is rational and has proven to be time saving.

After listing the basic assumptions of plastic analysis, this paper presents a method whereby complex multiple span frames can be readily designed. Several design examples are carried out. The problem of economy in main member is also discussed and procedures are presented whereby the design of a "least weight" structure can be approached.

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I. INTRODUCTION

During the past several years much attention has been directed toward the method of structural analysis and design known as Plastic Design. In essence, these procedures have as their end objective the determination of the load at which a structure becomes a mechanism through the development of "yield" (more often referred to as plastic) hinges at points of maximum moment throughout the structure. This reduction to a mechanism corresponds to the maximum carrying capacity of the structure provided certain conditions are met. According to the simple plastic theory, these are:*

(a) The moment-curvature relationship for the material and cross-section in question is as shown in Figure (1); that is, as moment approaches its full plastic value, M_p, curvature increases at an ever increasing rate;

(b) The strength of the structure can be sufficiently defined by considering a first order theory; that is, equilibrium is formulated in the undeformed position;

(c) No instability will occur before the attainment of the full plastic load;

(d) No influence of axial thrust or shear is considered;

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*The degree to which "practical structures" meet these requirements and the methods of modification for including the influence of certain of these factors have been discussed in several papers. See for example References 5 and 10.

(e) Continuity is assumed at connections; that
     is, there is a known amount of maximum
     moment that can be transmitted through the
     connection; and

(f) All loads are increased proportionally (see
     Figure 2).

As shown in References (1) and (2), the necessary and
sufficient conditions for a plastic analysis solution are as
follows:

(a) The structure must be in equilibrium,

(b) The moment at any section must be less
     than or equal to the fully plastic moment,
     that is, $|M| \leq M_p$, and

(c) A mechanism must be formed.

Several approaches or procedures could be used to arrive
at a solution that will satisfy these conditions. The more
noteworthy among these are (a) the Statical Method, (b) the
Mechanism Method, (c) the Method of Inequalities, and (d) the
Moment Balancing Method. Consider each of these individually.

(a) Statical Method[3],[4]

For continuous beams and certain other problems, it is
possible to visualize from the outset the general pattern that
the ultimate carrying capacity moment diagram must take. A
plastic analysis solution could therefore be obtained by ad-
justing the magnitudes of the maximum moment values of this
diagram always keeping $|M| \leq M_p$ until a sufficient number of
plastic hinges had been developed to reduce the structure to a
mechanism. This method is a simple and relatively fast means

of solving continuous beam problems. It can also be effectively used in the solution of certain types of frame problems where only a few redundants exist. The solution to more complex problems by this method, however, becomes extremely complicated.

(b) <u>Mechanism Method</u>[5],[6]

The mechanism method of solution approaches the problem from an entirely different point of view. Since the structure will fail at its first opportunity, a systematic investigation of each of the possible failure configurations and a determination of the corresponding critical loads will enable one to select the lowest of these and thereby the correct solution. Since a procedure of this type gives a upper limit (or bound) to the true carrying capacity of the structure[5], it is necessary to determine a lower limit in order that one may be certain of the correctness of the assumed answer. This is accomplished by the establishment of the moment diagram (Plasticity check). If the moment value nowhere exceeds M_p the assumed solution is the correct one, since each of the three necessary conditions will have been fulfilled.

This type of procedure is very general and lends itself readily to the solution of extremely complicated problems. It will be used in the development of the solution to the gable frame problems that will be discussed later.

(c) <u>Method of Inequalities</u>[7]

Since it is known that a member can sustain a moment equal to or less than its full plastic value, a set of linear inequalities could be written for each of the points of possible plastic hinge formation within the structure. By combining and

eliminating these inequalities the correct solution can be obtained. While this type of procedure is elegant, a computer is recommended for the solution of the more complex problems.

(d) <u>Moment Balancing</u>[8],[9]

As in the case of elastic design a successive relaxation of moment values could be carried out for plastic design taking into account the plasticity condition, $|M| \leq M_p$. For plastic analysis or design by this method a much greater degree of freedom is allowed the designer than in the elastic case.

In this report the mechanism method will be used to plastically design single and multiple span gable frames. The results will be given in curve form and design examples will be carried out to illustrate their use.

II. <u>S I N G L E   S P A N ,   P I N N E D - B A S E   F R A M E S</u>

Since the mechanism method assumes a possible failure configuration from the outset, one of the three necessary conditions for a plastic analysis solution is automatically fulfilled if this method is used. If in addition a virtual displacement type of procedure is employed to relate the external loads to the internal strengths of the various members, then equilibrium is also satisfied.* By investigating <u>all</u> of the possible modes in which the structure may fail, the third remaining condition can be satisfied. As was pointed out earlier,

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* It should be pointed out that such a procedure assumes that the structure and the applied loads are in equilibrium at the instant the mechanism is formed. Therefore, the increase in internal work associated with the virtual displacement will equal the corresponding external work. Furthermore, the increase in internal work will take place only at points of plastic hinge formation since only at these points will increased rotations occur.

this can also be checked by computing for the assumed correct solution the moment diagram of the structure. If it nowhere exceeds the full plastic value, (that is, $-M_p \leqslant M \leqslant M_p$), then the correct answer has been obtained.

Consider the pinned-base, gable frame shown in Figure (3). The span length is "L", the height of the columns is "aL" and the total rise of the rafters is "bL". There is a uniformly distributed vertical load of "w" lbs/ft. acting on each of the rafters as well as a concentrated horizontal load P acting at the eave. It is assumed that both the rafters and the columns deliver a given M_p value in the presence of whatever axial thrust may be acting.[10] As shown in Reference (11), such an assumption will result in a minimum total weight of structure for a majority of the cases found in practice.

To ascertain the possible failure configurations, it is first of all necessary to locate the points of possible plastic hinge formation. Since these can occur only at points of zero shear, at corners or where more than two members join; the possible plastic hinge locations for this problem are as shown in Figure (4). They have been numbered (1) through (5). It should be noted that the exact locations of hinges (2) and (4) have not been specified. These would be determined by mini- mizing the resulting expression for critical load, or in the case of design maximizing the required M_p value for the given loading.

Since for the case in question, only two plastic hinges are needed to reduce the structure to a mechanism, there are ten (10) combinations of these five possible hinges that could

result in failure.   This follows from

$$_RC_N = \frac{(N)(N-1)(N-2)\cdots(N-R+1)}{R!} \quad ----------- (1)$$

where $_RC_N$ is the number of possible combinations of hinges, N
the number of possible plastic hinges and R the number of hinges
required for "failure".   For the problem in question

$$_2C_5 = \frac{(5)(4)}{(2)} = 10$$

The combinations of hinges would be 1-2, 1-3, 1-4, 1-5, 2-3, 2-4,
2-5, 3-4, 3-5 and 4-5.

By determining the critical load corresponding to each
of these ten possible failure configurations, all possibilities
of failure will have been examined and the lowest critical load
will be the correct solution.   It is possible for the problem
in question, however, to exclude certain of the combinations as
being virtually impossible from the outset.   For example, for
hinges to form at locations (1) and (2) a part of the applied
external load would have to do negative work; that is, the
structure would have to move against the load.   Since the
structure will in general fail in as easy a manner as possible,
it would seem more probable that the external loads should do
positive work during failure, and for the majority of single
span cases, this is found to be true.

A somewhat different approach for determining the possible
failure mechanisms is that illustrated in References (5), (6).

If it is assumed that the structure will fail as whole*, then it can be shown that there will be (N-X) <u>independent modes</u> in which the structure may fail. The condition can therefore be written

Number of Independent Mechanisms = (N - X) .... (2)

As to what is termed independent, full freedom of choice is left to the individual. For example, for the problem in question, there are five (5) points of possible plastic hinge formation. The structure is one (1) time redundant. Therefore,

Number of Independent Mechanisms = (5 - 1) = 4

Four possible independent mechanisms are shown in Figure (5). It is obvious that these are independent since in each case a "new" hinge is involved.

To complete the solution of the problem it is necessary to determine the critical load corresponding to each of these assumed independent mechanisms. Furthermore, all of their comb- incations must also be examined. As an illustration of what is meant by a combination, consider a failure configuration con- sisting of a combination of the mechanisms shown in Figures 5(a) and 5(b). If the structure shown in Figure 5(a) is assumed to "sway" to the right and that shown in Figure 5(b) to the left, one can conceive of a composite mechanism of these two assumed independent cases which has hinges only at locations (1) and (2).

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*For the single span structure in question, if failure occurs in any manner whatsoever, it will fail as whole. For other struc- tures, for example, a two-span frame; it is possible for a part to fail and the remainder to be below its maximum carrying capa- city. It should be recognized that such a condition does not represent the most efficient use of material.

This then is a possible failure configuration that must be investigated. It should be noted, however, that for such a failure to occur the horizontal force P will have to do negative work. Such will also be found to be the case for the other combinations of the chosen independent modes. Therefore, an investigation of only the four cases shown in Figure (5) should give the correct solution. To be absolutely certain of the result, however, it is advisable to plot the moment diagram for the assumed solution. If it nowhere exceeds M_p, then there is no question; all three of the necessary conditions for a plastic analysis solution will have been fulfilled.

The question is immediately raised as to what happens if more hinges develop than the minimum number required to produce a mechanism. For example, consider the case where hinges form at (1), (2) and (5). Since only two are required to reduce the structure to a mechanism, the structure at failure is "over-determinate". In considering the seriousness of this situation, assume first of all that instead of having an M_p value at location (1) that there was a value of $1.1M_p$. For such a case, it is obvious that failure would be of the (b) type shown in Figure (5), since a plastic hinge would not develop at (1). If, however, location (2) had the increased strength, failure would have been of the (a) type. Since this line of reasoning holds regardless of how small the increase in strength above the M_p value; and since it is inconceivable to think that an infinitesimal increase in member size at a given location can materially change the carrying capacity of the structure; the critical loads computed using either mechanisms (a) or (b) should give

identical answers.   This is found to be the case.   Furthermore,
when polotting the moment diagram, M_p values will be observed
at each of the points (1), (2) and (5) regardless of which
assumption of failure was originally made.

Proceeding then to determine the critical loads for each
of the chosen independent mechanisms shown in Figure (5), con-
sider the mechanism shown in Figure (6).   If the columns are
assumed to be subjected to a virtual rotation about their base
equal to Θ, the horizontal force P will move through a horizontal
distance (Θ)(aL).   The distributed vertical load, on the other
hand, will remain at a fixed vertical height (assume a first
order movement).   It therefore does no work during failure.
Internally, each of the plastic hinges at the tops of the columns
rotates through a virtual angle Θ.   The work expression associated
with the assumed virtual displacement is therefore

$$W_{ext} = W_{int}$$

$$P (aL)(\Theta) = M_p (\Theta) + M_p (\Theta),$$

or the critical load corresponding to this failure configuration
is

$$P = \frac{2 M_p}{aL} \qquad --------------- \quad (3)$$

For the solution of the mechanism shown in Figure (7), it
is desirable to use the concept of the Instantaneous Center of
Rotation to aid in the definition of the geometry.   The reader
is referred to Lecture (6) of Reference (10) for a detailed dis-
cussion of this procedure.

Assuming that the horizontal distance from the left hand
(windward) column to the plastic hinge in the left hand rafter
is αL, it can be shown that the vertical distance from the top

of the right hand column to the instantaneous center of rotation
of the rigid bar connecting plastic hinges (2) and (5) is

$$L(\frac{a}{\alpha} - a + 2b)$$

If now the structure is subjected to a virtual displacement,
all angles may readily be determined from simple geometrical
relationships. Assuming that $\theta_B = \theta$, linkage (2)-(5) will
rotate about I.C. with a value

$$\theta_{I.C.} = \theta\left[\frac{aL}{L(\frac{a}{\alpha} - a + 2b)}\right] = \theta\left[\frac{\alpha}{1 - \alpha + \frac{2b}{a}\alpha}\right] \quad ---- \quad (4)$$

Since the top of the right hand column must move through the
same horizontal distance as point (5) of linkage (2)-(5).
Similarly,

$$\theta_A = \theta\left[\frac{1 - \alpha}{1 - \alpha + \frac{2b}{a}\alpha}\right] \quad ------------ \quad (5)$$

Since at plastic hinge (2) the member must rotate through both
the virtual angles θ_A and $\theta_{I.C.}$, and since at hinge (5) it must
rotate through $\theta_{I.C.}$ and θ_B

$$\theta_2 = \theta\left[\frac{1}{1 - \alpha + \frac{2b}{a}\alpha}\right] \quad ------------ \quad (6)$$

and

$$\theta_5 = \theta\left[\frac{1 + 2\frac{b}{a}\alpha}{1 - \alpha + \frac{2b}{a}\alpha}\right] \quad ---------------- \quad (7)$$

At each of these locations the full plastic hinge value, M_p, acts. Therefore, the total internal work associated with this assumed virtual displacement equals

$$W_{int} = M_p (\Theta_2 + \Theta_5)$$

Externally, the loads must be considered in parts. For example, the horizontal force P acts on the linkage (A)-(2). Since the instantaneous center of this linkage is at (A), the horizontal distance through which the load moves during the virtual disturbance equals the rotation at A, that is, Θ_A, times the vertical distance from (A) to the load. That part of the vertical load to the left of hinge (2) also rotates about point (A). Therefore, its external work is computed as $\frac{1}{2}w(\alpha L)^2 \Theta_A$. Correspondingly, the vertical load to the right of hinge (2) rotates about I.C., since it acts on linkage (2)-(5), which has its center of rotation at I.C.

The total virtual work expression for this assumed failure condition is therefore

$$W_{ext} = W_{int}$$

$$P(aL)\theta\left[\frac{1-\alpha}{1-\alpha+\frac{2b}{a}\alpha}\right] + \frac{wL^2}{2}(\alpha)^2\theta\left[\frac{1-\alpha}{1-\alpha+\frac{2b}{a}\alpha}\right] + \frac{wL^2}{2}(1-\alpha)^2\theta\left[\frac{\alpha}{1-\alpha+\frac{2b}{a}\alpha}\right]$$

$$= M_P\,\theta\left[\frac{1}{1-\alpha+\frac{2b}{a}\alpha} + \frac{1+\frac{2b}{a}\alpha}{1-\alpha+\frac{2b}{a}\alpha}\right] \quad ----- \quad (8)$$

Noting that each of the external work terms contain a $wL^2/2$ except that for horizontal force, equation (8) can be

reduced to

$$\frac{M_P}{wL^2} = \frac{1}{4}\left[\frac{(1-\alpha)(A+\alpha)}{1+\frac{b}{a}\alpha}\right] \quad ------- \quad (9)$$

providing the parameter "A" is chosen according to the equation

$$A = 2a\left(\frac{P}{wL}\right) \quad ----------- \quad (10)$$

That is,

$$P(aL) = (A)(wL^2/2) \quad ------------ \quad (10a)$$

It will be noted that equation (9) is in terms of the unknown distance αL to the plastic hinge in the left rafter. Since α is an independent variable and since the structure will fail at its first opportunity, the correct α distance will be determined from the expression

$$\frac{\partial M_P}{\partial \alpha} = 0 \quad -------------- \quad (11)$$

Equation (9) is of the general form "u/v". Therefore, the differentiation will be according to the formula

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

But since this expression will be set equal to zero,

$$v\,du - u\,dv = 0 \quad ------------ \quad (12)$$

Performing this operation on Equation (9), the correct "α" distance is then

$$\alpha = \frac{1}{b/a}\left[\sqrt{1 - \frac{b}{a}\left[A\left(1 + \frac{b}{a}\right) - 1\right]} - 1\right],$$

$$\text{for } \quad b/a > 0$$

and

$$\alpha = \left[\frac{1 - A}{2}\right], \qquad \text{for } \quad b/a = 0$$

$$\left.\right\} \,-\!-\!-\, (13)$$

The mechanism shown in Figure (8) is a special case of that given in Figure (7), where α is set equal to 1/2. The solution for this case is therefore

$$\frac{M_p}{wL^2} = \frac{1}{4}\left[\frac{A + \frac{1}{2}}{2 + \frac{b}{a}}\right] \quad -\!-\!-\!-\!-\!-\!-\!-\!-\!- \quad (14)$$

Similarly, the failure configuration shown in Figure (6) is a special case of Figure (7) with $\alpha = 0$. The resulting expression for M_p/wL^2 would be

$$\frac{M_p}{wL^2} = \left[\frac{A}{4}\right] \quad -\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!- \quad (15)$$

This corresponds to the solution given in Equation (3).

Going through the same process for the mechanism shown in Figure (9) as was done in the case of Figure (7), it is found that

$$\frac{M_p}{wL^2} = \frac{1}{4} \left[\frac{\alpha'(1-\alpha'+A)}{1 + \frac{b}{a}\alpha'} \right] \; - - - - - - - - \; (16)$$

where

$$\alpha' = \frac{1}{b/a} \left[\sqrt{1 + \frac{b}{a}\left[1+A\right]} - 1 \right] ,$$

$$\text{for } \frac{b}{a} > 0$$

and

$$\alpha' = \left[\frac{1+A}{2} \right] , \qquad \text{for } \frac{b}{a} = 0$$

$$\left. \right\} \; - - - \; (17)$$

The problem now is to determine which of equations (9), (14), (15) or (16) requires (for a given loading condition) the largest plastic value[*]. This will in general depend on the value of b/a and A under consideration. By assuming various values for these parameters and solving each of the equations for the corresponding M_p/wL^2 value, ranges of applicability for each equation can be determined. Carrying out such a procedure it is observed that only equation (9) and its special case, equation (15), govern the solutions. Plotting the resulting values of b/a versus A versus M_p/wL^2, the design curves shown in Figure (10) are obtained. Below the dashed line in this figure, Equation (9) governs. Above the line, Equation (15) defines the solution. The corresponding α values are shown in Figure (11).

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[*]Since Equations (14) and (15) are special cases of Equations (9) and/or (16), only these two equations need be considered.

It should be noted that for a value of A=0; that is, no horizontal load; the structure may "sway" either to the right or to the left with equal ease. As a consequence, plastic hinges will develop symmetrically at each of the locations (1), (2), (4) and (5). To show that such a failure mode will result in the same value of M_p/wL^2, consider the failure mechanism shown in Figure (12). Noting that the center part of the rafters; that is, linkage (2)-(4); will move vertically downward, the instant-aneous center of linkage (1)-(2) will be vertically above (1) at a height equal to the height of hinge (2). A similar condition will exist for linkage (4)-(5). Going through the process of equating external work to internal work as described earlier, it will be found that the equation governing this failure mech-anism is

$$\frac{M_p}{wL^2} = \frac{1}{4} \left[\frac{\alpha(1-\alpha)}{1 + \frac{b}{a}\alpha} \right] \quad ----------- \quad (18)$$

where

$$\alpha = \frac{1}{\frac{b}{a}} \left[\sqrt{1 + (\frac{b}{a})} - 1 \right], \quad \text{for} \quad \frac{b}{a} > 0 \Bigg\}$$

and

$$\alpha = \frac{1}{2}, \qquad\qquad \text{for} \quad \frac{b}{a} = 0 \Bigg] \quad --- \quad (19)$$

This solution is the same as Equations (9), (14) and (16) for the case where A = 0.

If the horizontal load acting on the structure in question is distributed rather than concentrated, a conservative M_p prediction (suitable for design) can be obtained by selecting an effective value of P for the concentrated load problem (Design Curve - Figure 10) which has an "over-turning" moment about the base of the structure equal to that of the distributed horizontal load. (This will be true providing a hinge would not have developed in the left hand column; a condition which will not occur for the majority of practical cases.) For the notation shown in Figure (13),

$$P_{eff}(aL) = \frac{\gamma w L^2}{2}(a+b)^2 \quad - - - - - \quad (20)$$

But for use in the design curves of Figures (10) and (11), it is not necessary to explicitly solve for P_{eff} and then determine the corresponding "A" value. This can be done in one operation.

$$P_{eff}(aL) = A\frac{wL^2}{2} = \frac{\gamma w L^2}{2}(a+b)^2$$

or

$$A = \gamma(a+b)^2 \quad - - - - - - \quad (21)$$

To show that such a prediction will be on the safe side, consider Figure (14). Assuming that the location of the hinge in the left rafter is the same as it was in the concentrated load case (see Figure 7), the internal work at each of the corresponding hinges will be identical. Therefore, a qualitative comparison of only the external work due to the two assumptions of horizontal forces will give a means of proving the above statement. By making the "over-turning" moment of

the P_{eff} system the same as that of the distributed load case,
it is assumed that there is a linear increase in virtual hori-
zontal deflection from zero at the base of the left hand column
to a maximum value at the upmost point of the rafter. But as
seen in Figure (14) this is not true for the distributed case;
above the plastic hinge in the rafter the rate of virtual de-
formation decreases. Therefore, by making the overturning
moments equal, more external work is introduced into the system
than will actually occur. (Note the heavily cross-hatched re-
gion of deformation shown in Figure 14). Thus, a design based
on this assumption will be conservative.

III. M U L T I - S P A N,   P I N N E D - B A S E   F R A M E S

1. Direct Procedure

Having solved the pinned-base, single-span, gable frame
problem for the assumptions listed; and having found that for
a major range of variables the mechanism that will control the
design is the one where hinges develop in the windward rafter
and at the top of the right hand column; a logical first at-
tempt at a mechanism for the multi-span problem might be that
shown in Figure (15). For the two-span problem shown the lengths
of span, heights of columns and total rise of rafters have been
chosen equal. Furthermore, the plastic strengths, M_p, of each
of the spans are also asssumed to be equal. It cannot be
assumed, however, that the distances to the hinges in the wind-
ward rafters will correspond. The resulting expression for M_p
will therefore contain the variables α and β; and since each
of these are independent, two separate differentiations (one of

the form $\frac{\partial M_p}{\partial \alpha} = 0$ and the other $\frac{\partial M_p}{\partial \beta} = 0$) will be needed to solve for the correct α and β values.

The consistent virtual rotations as determined from a consideration of the instantaneous centers shown in Figure (16) are as follows:

$$\theta_C = \theta$$

$$\theta_B = \theta \varphi$$

$$\theta_A = \theta \varphi \left[\frac{1-\alpha}{1-\alpha+\frac{2b}{a}\alpha} \right]$$

$$\theta_{I.C._1} = \theta \varphi \left[\frac{\alpha}{1-\alpha+\frac{2b}{a}\alpha} \right]$$

$$\theta_{I.C._2} = \theta \left[\frac{\beta}{1-\beta+\frac{2b}{a}\beta} \right]$$

$$\left. \right\} \quad - - - - - (22)$$

Where

$$\varphi = \left[\frac{1-\beta}{1-\beta+\frac{2b}{a}\beta} \right]$$

Using the same notation as previously described (that is, $PaL = A \frac{wL^2}{2}$), the resulting expression for M_p/wL^2 is

$$\frac{M_p}{wL^2} = \frac{1}{4} \left[\frac{-\alpha^2 - 2\alpha\beta - \beta^2 + \beta + \alpha + \beta^2\alpha + \beta\alpha^2 + \frac{2b}{a}\alpha\beta + \frac{2b}{a}\beta^2\alpha + A - A\beta - A\alpha - A\alpha\beta}{2 - \beta + \frac{4b}{a}\alpha - \frac{2b}{a}\alpha\beta - \alpha + \frac{b}{a}\beta + 2(\frac{b}{a})^2\alpha\beta} \right] \quad - - -(23)$$

Needless to say, the differentiation of this expression and subsequent solution for α and β is somewhat involved. Moreover, even if an explicit solution of α and β were obtained, it is questionable if such an equation as (23) could be used in design.

Another possibility, however, exists. Assuming that the loading and geometry of the frame are given, the variables involved are α, β and M_p. An implicit differentiation of the work expression rather than the explicit one considered above may lead to an easier formulation of the solution.

As shown on page 138 of Reference (12); if a function

$$F(M_p, \alpha, \beta) = 0 \quad --------------- \quad (24)$$

is given; and if it is known that $\frac{\partial M_p}{\partial \alpha} = 0$ and $\frac{\partial M_p}{\partial \beta} = 0$, then it can be shown that

$$\left. \begin{array}{c} \frac{\partial F}{\partial \alpha} = 0 \\[2em] \frac{\partial F}{\partial \beta} = 0 \end{array} \right\} \quad ---------- \quad (25)$$

Consider now, for the problem in question, the work expression in the implicit form. That is,

$$F = W_{int} - W_{ext} = 0$$

Noting that all of the angles, and thus the work expressions, for the left hand span are multiplied by φ, F can be written as

$$F(M_p, \alpha, \beta) = \varphi \left[R(M_p, \alpha) \right] + g(M_p, \beta) = 0$$

or dividing through by φ (which is a function of β alone)

$$F(M_p, \alpha, \beta) = R(M_p, \alpha) + S(M_p, \beta) = 0 \quad ----- \quad (26)$$

This indicates that the function F is made up of two separate parts: the first which is a function of M_p and $\propto$ alone, and the second which contains only M_p and β as variables. Differentiation of this new expression (Equation 26) according to Equation (25) is more easily obtained than the explicit differentiation of Equation (23) discussed above.

It should be remembered that only one possibility of failure has thus far been considered for this two-span problem. To solve a particular structure in question other modes would also need to be examined to determine the one that would actually develop.

2. Solution by Separation

Based on the preceeding discussion, it can be reasoned that since for the problem under consideration the variables separate into two groups (one having to do with the loading and resistance of only the left hand portion, the other concerned with the right hand part alone) a solution might be more readily obtained by mentally dividing the structure into two parts. A solution to the multiple span case could then be realized by solving each of these separate parts in terms of the loading parameters at the cut section and then in the final stage equating these parameters. If this division is made at the junction of the right hand rafter of the left structure and the center column, the loading condition will be as shown in Figure (17).

For the left hand structure (Figure 17a), the equation for M_p will be of the form

$$M_p = f(P, w, \propto, H, \text{dimensions}) \quad ----- \quad (27)$$

It should be noted that the structure does not move in the vert-
ical direction at the cut section for the assumed virtual dis-
placement (first order movement). Therefore, V will not appear
in the solution. The equation for the right hand sub-structure
will be

$$M_p = g(H,\ w,\ \beta,\ \text{dimensions}) \quad ----\ -\ -\ -\ - \quad (28)$$

Had the plastic strength of the right hand part differed from
that of the left, there would have been an additional term in
Equation (28) relating these two.

As the structure deforms consistent with an assumed
virtual displacement, both M_p and H do work. However, for the
two parts to be in equilibrium when the structure is "put to-
gether" it is necessary that the right hand structure be loaded
with the same amount of work that it transfers to the left hand
portion as a result of its resistance to movement. Since it
is "work done" that is important, the actual moment and force
that develop at the cut section are not of primary concern and
can be replaced by some hypothetical moment assumed to act
about the base of the structure*. Furthermore, noting from
Figure (18) that these two moments, Q_L and Q_R, rotate through the
same angles during the virtual deformation; all that is re-
quired for the two works to be equal is that the two hypothet-
ical moment values be equal.

For further ease of solution it will be assumed that the

---

*It should be pointed out that such a procedure is true regard-
less of the division of individual terms. What essentially is
being done is to take one expression (the work equation for the
structure as a whole) and through the introduction of an addi-
tional parameter (Q) rewrite it as two parametric equations.
For example, if a function is given as

$$x + y + \eta + \nu = 0,$$

it can be rewritten as two equations by introducing the addi-
tional parameter, z. That is

$$x + y + z = 0$$
$$\eta + \eta' - z = 0$$

total amount of internal work done at the plastic hinge in the
rafter adjacent to the center column is done on the left hand
structure. This has the advantage that the right hand structure
will then be subjected to only the loading on the right hand
span, the resistance of this span and the hypothetical moment
Q_L; whereas, the left hand structure will be subjected to only
those things occurring in its span and the hypothetical moment
Q_R. The strength of each of these structures could be determined
in terms of these "Q" moment parameters which could then be
equated to solve the problem in question.

If a structure and loading as shown in Figure (19) are now
assumed, either of the two possibilities shown in Figure (18) can
be represented. For case (a) (left hand sub-structure), Q_L would
be chosen equal to (P) (aL). For case (b), Q_R would be equal to
zero. A general solution for the strength of this structure
which includes the variable moment and loading terms and which
considers all possible modes of failure will therefore afford a
means of solving the general multiple span design problem.

When using such a procedure as described above no possi-
bility of the development of a plastic hinge in the center column
is considered. For relatively large horizontal thrust, however,
it will be found that the moment at the top of this column will
exceed the M_p value. For such a case, it is obvious that the
investigated mechanism is not the correct one and the corres-
ponding value of M_p is too small. The actual failure mode that
would develop would more than likely be the one where hinges
form at the tops of the right hand and center columns, in the
right hand rafter of the left hand span adjacent to the center

column and in the left hand rafter of the left hand span. Since
the solution of such a failure pattern would result in a greater
M_p value for all of the members of the structure, the new design
would in almost all cases be less economical (in terms of least
total weight of structure) than that design based on the assump-
tion that the center column can supply whatever is needed. (Note
the relative length of the center column in comparison to the
total lengths of the remaining members of the structure). From
economic considerations, then, the failure mode having a hinge
at the top of the center column will be excluded from consider-
ation. This does not, however, exclude the possibility of
selecting the size of this "center type column" such that a
hinge develops at the same load which produces failure in the
remainder of the structure. In such a case the exact size of
this member would be determined from a moment diagram for the
structure as a whole.

3. Development of Design Charts

To be able to solve all types of multiple span problems
by this method, it is necessary to ascertain all of the various
possible sub-structures (or assemblages) that can occur. For
example; if a three span symmetrical gable frame were subjected
to only vertical loads as shown in Figure (20a), the two types
of sub-structure failures shown in Figure (20b) could occur.
(The exact location of the hinges is not critical at this
stage. What is important to note is that for the center span
each of the columns spread an equal amount away from each other
during the failure. The outside span, on the other hand, fails
as assumed in the preceeding problem with both columns moving
in the same direction.)

Had the structure under consideration been a four span symmetrical frame as shown in Figure (20c), the center two spans would have failed with their outside columns spreading. Due to symmetry, the center column would remain vertical. A fourth type of failure condition results when a three span, unsymmetrical frame fails. For such a case, the center two columns may spread through different angles. Therefore, this condition must also be investigated.

The five types of sub-structures and loadings that must be considered for the solution to pinned-base, gable frame problems are therefore as shown in Figure (21). By selecting a value of the left hand "Q-moment" in case (b) equal to the moment produced by the concentrated horizontal force of case (a), these two problems reduce to one. The equations governing their solution are tabulated in Appendix A as equations (1) through (4) and (8) through (12). For cases (c), (d) and (e) of Figure (21), it can be shown that each reduces to the same solution[13]. Furthermore, it can be shown that in each case the moment to the left, Q_L, must equal that to the right, Q_R. The governing equations are given as Equations (5), (6) and (7) of Appendix A.

As in the single span case, non-dimensional parameters have been introduced to relate the overturning moments, Q_L, and Q_R, to the vertical loads and span lengths. These have been chosen according to the relationships

$$\text{and} \quad \left. \begin{aligned} Q_L &= A \left(-\tfrac{1}{2} w L^2\right) \\ Q_R &= D \left(\tfrac{1}{2} w L^2\right) \end{aligned} \right\} \quad \text{---------} \quad (29)$$

It will be noted that the equations in Appendix A are in terms of these parameters.

The resulting design curves of M_p/wL^2 versus A versus D for various values of b/a (0, 0.2, 0.4, 0.6, 0.8 and 1.0) are given as Figures (22) through (28). The corresponding values of "α" versus these same parameters are shown in Figures (29) - (35). Also shown on these curves are the ranges of applicability of each of the types of failure.

IV. D I S C U S S I O N

The discussion will be divided into four parts: a general consideration of the design charts for multiple span structures, the size of "center-type" columns, the question of safety and the load factors for use in plastic design and the problem of economical designs.

1. The Design Curves (Figures 22-35)

It should first of all be noted that when D=A (Figures 22-27) the structures can "sway" with equal ease to the left or to the right. Therefore, hinges will develop symmetrically and the solutions will equal those shown in Figure (28) for the case where the columns move away from each other. It should further be observed that this condition (D=A) represents the smallest M_p value for the span in question. No solutions exist below these values unless ties are used connecting the eaves which prevent their spreading. (For such cases a "beam type" mechanism would control the design and the design values would be for

$$b/a > 0, \quad M_p/wL^2 = 0.0156$$
$$b/a = 0, \quad M_p/wL^2 = 0.0625$$

These are the absolute lowest values of M_p possible.) Even for

such a situation, certain values of A=D and b/a can cause this
same condition to be realized without the use of ties. See, for
example, the lowest design point on Figure (27).

For the major range of variables encountered in practice,
the mode of failure is the one where hinges develop at the top
of the leeward column and in the windward rafter. It should be
noted, however, that in a multiple span structure, this hinge at
the top of the leeward interior columns would actually develop in
the leeward rafter adjacent to the column in question rather than
in the column itself.

As noted earlier, Figures (29)-(35) give the location of the
plastic hinge in the rafter as a function of A, D and b/a. This
information is useful in constructing the moment diagram for the
chosen solution.

2. Determination of the Size of Interior Columns

The solution to the multiple span design problem as out-
lined in this report assumes that the interior columns will be
chosen such that they provide the strength needed to keep the
structure in equilibrium. Their size will therefore be deter-
mined from the moment diagram. Since in all cases of design
using these charts the structure will be determinate at failure,
this presents no difficulty. It is possible, however, to shorten
the amount of time required to determine the size of these mem-
bers by relating their maximum moment values to the A and D
parameters discussed earlier.

As shown in Figure (36), if it is assumed that the top of
the columns move to the right in forming the mechanism, a plastic
hinge will in most practical cases develop in the leeward rafter

adjacent to this column. If it is possible to determine the
moment in the windward rafter at this same section (M^Q in Figure
36), then the moment required in the column will be

$$M_C = kM_p + M^Q \qquad\qquad\qquad (30)$$

where kM_p is the fully plastic moment of the span to the left of
the one in question. See Figure (36) for the assumed positive
directions.

For the situation where hinges form in the leeward rafter
adjacent to the leeward column of the span in consideration (the
span shown bold in Figure 36) and in the windward rafter of this
span, it can be shown that

$$\frac{M^Q}{M_p} = \left[\left(\frac{wL^2}{M_p} \right)\left(\frac{A-D}{2} \right) - 1 \right] \qquad - - - - - - - - \quad (31)$$

A,D, and M_p/wL^2 refer to conditions occurring in the span in
question.

3. Factor of Safety

Since plastic design results in a structure that will
just sustain the imposed loading, there must be included in the
design load a certain margin of safety above the anticipated
working value. Accepting this philosophy, the next step is the
selection of a criterion for determining the numerical value of
this safety factor.

If it is assumed that it is desirable to have the load
factor of safety of a continuous structure equal to that of a
statically determinate one, and if it is further assumed that
an average wide-flange, simple beam designed according to the
present AISC Specification[14] has an adequate reserve in strength,

then it can be shown that the load factors should be as
follows:[15]

    1. Load factor for Vertical Load only   1.88

    2. Load Factor for Vertical Load, Wind,
       Earthquake, etc.   1.41

The design requiring the greater member size will be the one
governing.

      A somewhat different appraoch to the general question of
safety can be based on the philosophy that a structure is no
better than the load analysis. Therefore, this factor should
play a major part in the determination of the factor of safety.
Furthermore, the ability to predict loads is dependent on the
type of loading. The uncertainty in each of the loads making
up the total could also be taken into account.

      While the question of safety is important, it is not
unique to plastic analysis. It is therefore considered that
further discussion in this paper is not warranted. For the
design examples that follow the load factors listed above will
be used.

4. Economical Designs

      Many factors enter into the selection of an "economical
design". The criterion used in this paper will be "least weight".
As will be illustrated, such a design can be determined in a
straight-forward manner through use of the design charts.

      Since in plastic design the quantity most often en-
countered is the fully plastic moment value, M_p, it would be
desirable to have an expression relating this property (or the
plastic modulus which is equal to M_p/σ_y) and the unit weight of
the member. Two designs could then be compared by summing the
M_p values times the lengths of the various members.

Unfortunately the plastic modulus not only takes into account the area of the section but also the moment of this area. The relationship will therefore not be linear. Assuming that it will be one of a power, the plastic modulus values for rolled wide-flange shapes have been plotted versus unit weight on a log-log scale in Figure (37).[16] A straight line on this plot would correspond to an equation of the form

$$W = C \, Z^n \quad — \, — \, — \, — \, — \, — \, — \, — \, — \, — \, — \, — \, — \quad (32)$$

where W is the weight per unit length of the member. It is noted that within a given nominal size of member a straight line relationship does hold. The corresponding n values are approximately 0.90 with C varying between 5 and 16 lbs/ft. of member.

If one equation is to represent the entire range of member sizes, n must lie between 0.5 and 0.9. Had all members been geometrically similar it can be shown that a value of 0.67 would be the value of the exponent.[11] As seen from the figure, this is also a reasonable value for the entire range of WF shapes.

While the absolute difference between an exponent of 0.5 and say 1.0 is extremely large, the net effect on the isolation of the more economical choice of member size is rather small. In addition the assumption of equal rafter sizes in a given span, etc., will often over shadow the difference. Therefore a one-to-one correspondence between weight and plastic modulus (or M_p) will be assumed in the remainder of this discussion.

V. DESIGN EXAMPLES

1. Design Example No.1

As a first design example consider the single-span, gable frame loaded as shown in Figure (38). The b/a ratio for this structure equals 9/15 = 0.6. Assuming that a load factor of safety of at least 1.88 against vertical loads and at least 1.41 against combinations of vertical and wind loads is desired, two separate designs must be considered.

For the case of vertical load alone P equals zero. Therefore the "A" loading parameter (Equation 10) is also equal to zero. From the design curve of Figure (10) (for b/a = 0.6, A=0)

$$\frac{M_p}{wL^2} = 0.0488^* \ — — — — — — \ (33)$$

This gives as a required fully plastic moment value

$$M_p = (0.0488) \left[(1)(1.88) \right] (40)^2$$
$$= 147 \text{ ft. kips}$$
$$= 1764 \text{ inch kips} \ — — — — — — — \ (33a)$$

Assume now that the structure is subjected to both vertical and horizontal loads. For this case

$$A = (2a)(P/wL)$$
$$= \frac{(0.75)(10.7)}{(1.0)(40)} = 0.20 \ — — — — — — \ (34)$$

From Figure (10)

$$\frac{M_p}{wL^2} = 0.0742 \ — — — — — — — — — \ (35)$$

or

$$M_p = (0.0742) \left[(1)(1.41) \right] (40)^2$$
$$= 167 \text{ ft. kips}$$
$$= 2004 \text{ inch kips} \ — — — — — — — \ (35a)$$

---
*These values were obtained from enlarged versions of the design curves shown herein.

Since the required M_p value for the case including wind is greater than that for vertical loads alone, Equation (35a) controls the design. The corresponding moment diagram is shown in Figure (38b). The distance to the hinge in the rafter is obtained from Figure (11).

2. Design Example No. 2

As a second example consider the two span gable frame shown in Figure (39a). Here as in Example No. 1, two loading cases will be examined.

Excluding the horizontal force, the structure and loading are symmetrical. The center column will therefore remain vertical. Since D=0 (no external horizontal loads applied to the outside columns) and since for this type of failure A = D, the required value of M_p determined from Figure (28) is

$$\frac{M_p}{wL^2} = 0.0488 \quad\text{—} \quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—} \quad (36)$$

This is the same value as for the single span case (Design Example No. 1). The required M_p value is therefore

$$M_p = 1764 \text{ inch kips} \quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—} \quad (36a)$$

and the moment diagram is as shown in Figure (39b).

Assuming that the wind force could develop from either the left or the right, the member sizes should be equal in each span. The sub-structures and loadings are then as shown in Figure (40a).

For span (1), A_1 will be determined from the expression

$$(P) \ (aL_1) = A_1 \ (\tfrac{1}{2} wL^2)$$

or

$$A_1 = 0.20 \quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—}\quad\text{—} \quad (37)$$

Since no external horizontal loads act on the leeward side of span (2),

$$D_2 = 0 \quad — — — — — — — — — — — — \quad (37a)$$

The condition for a solution is that at the center column

$$D_1 \left(\frac{1}{2} wL^2\right) = A_2 \left(\frac{1}{2} wL^2\right)$$

Since $L_1 = L_2$, this reduces to

$$D_1 = A_2 \quad — — — — — — — — — — — \quad (38)$$

Consider span (1): Since $b/a=0.6$ and since A_1 is known to be equal to 0.20, the relation between D_1 and M_p/wL^2 can be determined from the design curve for $b/a=0.6$ (Figure 25). This relationship has been plotted as the solid line in the left hand graph of Figure (40b).

For the right hand span, it is known that $b/a=0.6$ and that $D_2 = 0$. The relationship between A_2 and M_p/wL^2 therefore corresponds to the case $D = 0$ of Figure (25). It is reproduced as the right hand graph of Figure (40b).

The two conditions for solution as previously stated are that the two M_p values must be equal and that $D_1 = A_2$. Since the coordinate axes of these two graphs (Figure 40b) are identical, the graphs can be superimposed one on the other as shown by the dashed line in the left hand graph. The solution is therefore

$$\frac{M_p}{wL^2} = 0.0575 \quad — — — — — — — \quad (39)$$

or

$$M_p = (0.0575)\left[(1)(1.41)\right](40)^2$$
$$= 130 \text{ ft-kips}$$
$$= 1560 \text{ inch-kips} \quad — — — — — — \quad (39a)$$

Since the larger value of required M_p corresponds to the case where wind is neglected, this condition controls the design. None-the-less, Figure (40c) is the moment diagram for the case including wind.

3. Design Example No. 3

The third design example is concerned with the two-span unsymmetrical frame shown in Figure (41a). Three loading conditions will need to be investigated: vertical load alone as shown in Figure (41b), vertical load plus wind from the left as shown in Figure (41c) and vertical load plus wind from the right as shown in Figure (41d).

Case (a): Vertical load alone (Figure 41b)

Since the D's are zero (assuming that the outside columns spread) and since A will equal D for such failures; that is,

$$A_1 = 0$$

$$D_2 = 0$$

$$D_1 = A_1 \text{ and } A_2 = D_2 \text{ (Failure type (e) of Figure 21);}$$

the solutions as determined from Figure (28) are

$$\left. \begin{array}{l} \text{span (1), } b/a = 0.6, \quad \dfrac{M_p}{wL_1^2} = 0.0488 \\[2em] \text{span (2), } b/a = 1.0, \quad \dfrac{kM_p}{wL_2^2} = 0.0428 \end{array} \right\} \text{— — — — — (40)}$$

Using a load factor of 1.88, the corresponding required M_p values are

$$\left. \begin{array}{l} M_p = 1760 \text{ inch kips} \\[1em] kM_p = 3480 \text{ inch kips} \end{array} \right\} \text{— — — — — — — (40a)}$$

Case (b): Vertical load plus wind from the left (Figure 41c)

For this loading case the structure must be divided into two sub-assemblages. For span (1)

$$A_1 \left(\frac{1}{2} wL_1^2 \right) = P (aL_1)$$

or

$$A_1 = 0.20 \text{ — — — — — — — (41)}$$

For the right hand span

$$D_2 = 0 \text{ — — — — — — — (41a)}$$

The condition for a solution is that at the center column

$$(D_1) \left(\frac{1}{2} wL_1^2 \right) = (A_2) \left(\frac{1}{2} wL_2^2 \right)$$

or

$$A_2 = D_1 (L_1/L_2)^2 = 0.444 \, D_1 \quad — — — — — — \quad (41b)$$

Since solutions can exist for all value of D_1 from zero to 0.20 (the value of A_1), it is helpful to set up the solution in tabular form. Table I gives the solution for various selected values of D_1.

TABLE I

b/a = 0.6			b/a = 1.0				
A_1	D_1	$\frac{M_p}{wL_1^2}$	A_2	D_2	$\frac{kM_p}{wL_2^2}$	M_p^*	kM_p^*
0.20	0	0.0742	0	0	0.0428	2010	2610
	0.05	0.0630	0.022		0.0451	1705	2740
	0.10	0.0522	0.044		0.0475	1412	2890
	0.15	0.0406	0.067		0.0502	1100	3050
	0.20	0.0298	0.089		0.0528	806	3220

Case (c): Vertical load plus wind from the right (Figure 41d) (Note: since the wind is from the right and since in the derivations it was assumed that the "A" parameter is always on the windward side of the span in question, the location of the moment parameters "A" and "D" will be the opposite to that of Case (b).)

$$A_2 \left(\frac{1}{2} wL_2^2 \right) = P (aL_2)$$

or

$$A_2 = \frac{(13.4)(15)(2)}{(1)(60)^2} = 0.112 \quad — — — — — — \quad (42)$$

---
*A load factor of 1.41 was used in calculating these values.

For span (1) (no horizontal load to the left of the span)

$$D_1 = 0 \qquad — — — — — — — — — \quad (42a)$$

The condition for a solution is therefore

$$(D_2) \left(\tfrac{1}{2} wL_2{}^2 \right) = (A_1) \left(\tfrac{1}{2} wL_1{}^2 \right)$$

or

$$A_1 = D_2 (L_2/L_1)^2 = 2.250 \, D_2 \quad — — — — — — \quad (42c)$$

The solution for various selected values of D_2 is tabulated in Table II.

TABLE II

b/a = 0.6			b/a = 1.0				
A_1	D_1	$\dfrac{M_p}{wL_1^2}$	A_2	D_2	$\dfrac{k M_p}{wL_2^2}$	$M_p{}^*$	$kM_p{}^*$
0	0	0.0488	0.112	0	0.0558	1320	3400
0.112		0.0621		0.05	0.0430	1681	2620
0.225		0.0780		0.10	0.0305	2110	1860
0.252		0.0820		0.112	0.0277	2220	1690

Figure (42) is a plot showing the solution to these three loading conditions: vertical load alone, wind from the left and wind from the right. Since the structure must provide the greatest M_p and correspond kM_p values, the condition with vertical load alone (load factor = 1.88) governs. The required values are therefore as given in Equation (40a).

Suppose that case (b) and (c) above had not been solved and that it was desired to check to see if case (a) was adequate. Since the load factor for combined wind and vertical loading is 1.41, the value of $M_p/wL_1{}^2$ for the first span would be

----

*A load factor of 1.41 was used in calculating these values.

$$\frac{M_p}{wL_1{}^2} = \frac{1760}{(1)(1.41)(1600)(12)} = 0.0650$$

The corresponding value for the second span is

$$\frac{k_1 M_p}{wL_2{}^2} = 0.0571$$

From Equation (41) it is known that $A_1 = 0.20$. The needed value of D_1 must therefore be 0.04 (from Figure 25). From Equation (41b), this means that $A_2 = (0.444)(0.040) = 0.0178$ or entering Figure (27) with this value of A and D = 0, it is seen that $k_1 M_p/wL_2{}^2 = 0.0448$. Since the member required by case (a) is greater than this value, the case including wind will not be critical.

With regard to Figure (42) an observation can be made that will prove beneficial in the next example. For each of the cases including the influence of wind, different solutions were obtained by varying the "D" value of the windward span from zero to a maximum value equal to the "A" value of that span. Since the function is continuous, only the two end points (D = 0 and D = A) need be considered to determine the range of influence. Furthermore, the relationship is almost linear.

4. Design Example No. 4

As a final design example, consider the three span un-symmetrical structure loaded as shown in Figure (43). To illustrate the procedure, only two cases will be examined. The first of these will be the case of vertical load alone. Wind from the left in combination with the vertical load will be the second. For a "real" problem wind from the right in combination with the vertical load would also need to be considered.

Case (a): Vertical load alone.

For the case excluding the influence of wind, both columns of the two outside spans will tend to move away from the center. The design curves for each of the three spans will therefore be as follows:

Span (1), b/a = 0.6 - Figure (25)

Span (2), b/a = 1.0 - Figure (28)

Span (3), b/a = 0.8 - Figure (26)

The known conditions are

$$\left. \begin{array}{l} D_1 = 0 \\ D_3 = 0 \end{array} \right\} \quad \text{------------} \quad (43)$$

Since the two interior columns will tend to spread, (failure mechanism "e" of Figure 21) it is also known that

$$A_2 = D_2 \quad \text{--------------} \quad (43a)$$

At the two interior columns it is necessary that the following conditions be met:

$$A_1 \ (\tfrac{1}{2} wL_1{}^2) = D_2 \ (\tfrac{1}{2} wL_2{}^2)$$

which gives

$$A_1 = D_2 \ (L_2/L_1)^2 = 2.780 \ D_2 \quad \text{--------} \quad (44)$$

and

$$A_3 \ (\tfrac{1}{2} wL_3{}^2) = D_2 \ (\tfrac{1}{2} wL_2{}^2)$$

or

$$A_3 = D_2(L_2/L_3)^2 = 1.562 \ D_2 \quad \text{-------} \quad (44a)$$

The tabulated solution for various selected values of D_2 is shown in Table III. The values of A_1 and A_3 were determined from a consideration of Equations (44) and (44a).

TABLE III

b/a = 1.0		b/a = 0.6		b/a = 0.8		Span(1)	Span(2)	Span(3)
D_2	$\dfrac{k_1 M_p}{wL_2^2}$	A_1	$\dfrac{M_p}{wL_1^2}$	A_3	$\dfrac{k_2 M_p}{wL_3^2}$	$\dfrac{M_p}{wL^2}$	$\dfrac{k_1 M_p}{wL^2}$	$\dfrac{k_2 M_p}{wL^3}$
0	0.0428	0	0.0488	0	0.0456	0.440	1.070	0.730
0.05	0.0358	0.139	0.0656	0.070	0.0545	0.591	0.895	0.872
0.10	0.0292	0.278	0.0888	0.156	0.0645	0.795	0.730	1.032
0.15	0.0232	0.417	0.1105	0.234	0.0758	0.995	0.580	1.212
0.20	0.0175	0.556	0.1412	0.312	0.0884	1.272	0.437	1.415
0.22	0.0156	0.612	0.1550	0.344	0.0939	1.395	0.390	1.502

A plot of these values is given in Figure (44). As would be expected the function is continuous with extremes corresponding to the cases where (a) the center span is as small as possible and (b) the outside spans have their smallest M_p values.

Assuming that this loading condition (vertical load alone) is the critical one for design, the question is immediately raised as to what will be the better choice of M_p values for the various members of the frame. As pointed out earlier, this report will consider "least total weight of structure" as the criterion.

From the discussion on Economical Design of the preceding section, it is assumed that

$$W = C (M_p) \quad\text{——————————————— (45)}$$

where W is the weight per unit length of the member in question and C is a constant. The total weight of any given beam is therefore

$$W L_i = C (M_p) L_i \quad\text{————————————— (45a)}$$

where L_i is the length of the considered member. Since only

relative comparisons are required, the C of Equation (45a) could

just as well be taken to the other side of the equation, or

$$\frac{W\,L_i}{C} = \text{Weight Function} = \rho = (M_p)\,L_i \quad - - - - \quad (45b)$$

For a structure consisting of a number of members, the total

weight function would be

$$\rho = \sum_{i=1}^{i=n} (M_p)\,(L_i) \quad - - - - - - - \quad (45c)$$

where n is equal to the number of different member sizes within

the structure.

Neglecting for a moment the influence of the interior

columns, the weight function for the three span structure (Case a)

would be

$$\rho = 4.23\,L\,(M_p) + 5.39\,L\,(k_1 M_p) + 5.30\,L\,(k_2 M_p) \quad - - (46)$$

Dividing through by wL^3 to have the plastic moment values in

terms of the non-dimensional parameters computed earlier

$$\frac{\rho}{wL^3} = 4.23\left(\frac{M_p}{wL^2}\right) + 5.39\left(\frac{k_1 M_p}{wL^2}\right) + 5.30\left(\frac{k_2 M_p}{wL^2}\right) \quad - - - (46a)$$

Figure (45) is a plot of this function versus $k_1 M_p/wL^3$. It should

be noted that for failure of the structure as a whole the defini-

tion of any one of the three M_p values automatically fixes the

other two. Therefore, a two-dimensional plot is sufficient.

There is also shown as a dashed line in this graph the

relationship between the weight function and $k_1 M_p$ including the

influence of the varying size of the interior columns. These

were determined from a consideration of Equations (30) and (31).

The sizes of the columns are tabulated in Table IV. As noted

from Figure (45), the inclusion of the size of these members

does not change the selection of the "least-weight" design.
While no general rule concerning this condition can be formu-
lated, this seems to be true for most practical structures.
The "least-weight" design (for this one loading condition)
would therefore be the one where

$$\frac{M_p}{wL^2} = 0.440$$

$$\frac{k_1 M_p}{wL^2} = 1.070$$

$$\frac{k_2 M_p}{wL^2} = 0.730$$

TABLE IV

$\frac{M_p}{wL^2}$	$\frac{k_1 M_p}{wL^2}$	$\frac{k_2 M_p}{wL^2}$	M/wL^2 Column(A)	M/wL^2 Column(B)
0.440	1.070	0.730	0.630	0.340
0.591	0.895	0.872	0.929	0.647
0.795	.730	1.032	1.186	0.947
0.995	.580	1.212	1.462	1.240
1.272	.437	1.415	1.667	1.520
1.395	.390	1.502	1.749	1.640

Case (b): Vertical load plus wind from the left.
Since for this condition of loading all spans tend to sway to
the right, the design curves governing the solution are as
follows:

Span (1), b/a = 0.6   Figure (25)

Span (2), b/a = 1.0   Figure (27)

Span (3), b/a = 0.8   Figure (26)

The known conditions and the requirements for solving the problem are:

a) $A_1 (\frac{1}{2}wL_1^2) = P (L)$ or $A_1 = 0.20$ . . . . (47)

b) $A_2 (\frac{1}{2}wL_2^2) = D_1 (\frac{1}{2}wL_1^2)$ or $A_2 = 0.36 D_1$ . . . (48)

c) $A_3 (\frac{1}{2}wL_3^2) = D_2 (\frac{1}{2}wL_2^2)$ or $A_3 = 1.56 D_2$ . . . (49)

d) $D_3 = 0$ . . . . . . (50)

In setting up this type of a problem, it is recalled that the solution will be a continuous function in the three variables, M_p, k_1M_p and k_2M_p. Furthermore, if A is given for any one particular span, D in that span can vary from a value of zero to the full A value. The "least-weight" solution (for that one span in question) will correspond to the case where A = D.

Since the total structure in question contains three spans, three limiting cases are apparent: (1) spans 1 and 2 as small as possible, with span 3 providing what is needed for equilibrium; (2) spans 1 and 3 as small as possible, with span 2 making up the difference; and (3) spans 2 and 3 small, with span 1 as large as need be. The solutions for these three cases are tabulated in Table V.

TABLE V

	1	2	3	4	5	6
	A_1	$\dfrac{M_p}{wL_1^2}$	D_1	A_2	$\dfrac{k_1 M_p}{wL_2^2}$	D_2
1	0.2	0.0298	0.2	0.072	0.0330	0.072
2	0.2	0.0298	0.2	0.072	0.0508	0
3	0.2	0.0742	0	0	0.0428	0

	7	8	9	10	11	12
	A_3	$\dfrac{k_2 M_p}{wL_3^2}$	D_3	$\dfrac{M_p}{wL^2}$	$\dfrac{k_1 M_p}{wL^2}$	$\dfrac{k_2 M_p}{wL^2}$
1	0.113	0.0588	0	0.268	0.825	0.940
2	0	0.0455	0	0.268	1.270	0.728
3	0	0.0455	0	0.668	1.070	0.728

These results (plotted as points 1, 2, and 3) as well as intermediate values, are shown in Figure (46). For a solution to exist which causes the structure to fail as a whole, the design must fall within the region shown. As pointed out earlier, the boundaries of this region are almost straight lines (one is a straight line). They each represent the case where one of the spans is maintained in its minimum M_p condition (i.e., with A = D).

Proceeding now to determine the particular values of M_p, $k_1 M_p$, and $k_2 M_p$ that result in a least total weight of structure solution, the weight function neglecting the size of the interior columns is

$$\frac{\rho}{wL^3} = 4.23(\frac{M_p}{wL^2}) + 5.39 (\frac{k_1 M_p}{wL^2}) + 5.30(\frac{k_2 M_p}{wL^2}) \ \circ \ \circ \ \circ \ \circ \ (51)$$

The contour lines of equal weight functions are shown in Figure (47), and indicate that the least total weight of struc- ture results when spans 1 and 2 are held at their minimum values.

A generalization regarding the selection of the var- ious member sizes that comprise the least total weight of structure solution can be made. Since the boundaries defin- ing the region of permissible design are for all practical purposes straight lines, and since this is also the case for the weight function, the least weight solution must occur at one of the corners of the design region. For a three-span structure then, only three solutions need be examined. For a four-span problem, four cases must be investigated: spans 1, 2, and 3 minimum; spans 1, 2, and 4 minimum; spans 1, 3, and 4 minimum and spans 2, 3, and 4 minimum.

5. Further Considerations Regarding Use Of The Design Curves

It should be reemphasized that in plastic design super- position does not hold. Each loading condition must be inves- tigated separately. The actual selection of member sizes will be determined by the loading condition which imposes the most severe requirement.

While the design examples shown in this report cover a variety of situations, other types of problems could equally well be solved. For example, in each of the cases illustrated, the column heights were equal throughout the structure. This is not a requirement of the method of solution. All that is

needed to use the design charts is that in any one given span
the rafters must join to the columns at the same elevation.
Adjacent spans may have different column heights. It should
be pointed out, however, that in all cases it has been assumed
that the size of the interior columns must be sufficient to
cause the rafters to participate in the failure mechanism.
Fortunately, for most practical structures, this situation
results in greater economy (in terms of least weight).

     Solutions similar to the ones given herein for the
pinned-base, gable frame problem have also been developed
for the fixed-base, gable frame and for the "lean-to" type
structure (Reference 13). Due to space limitation, these
are not included in this report.

VI. S U M M A R Y

     In this paper the following have been considered:

a. The assumptions of the simple plastic theory and a
    short description of the various methods whereby
    solutions to problems in plastic analysis can be
    obtained were first presented.

b. The mechanism method was then used to solve the single-
    span gable frame problem, and the results were given
    in the form of design charts (Figures 10 and 11).

c. This was followed by an attempted extension of the pro-
    cedures used for the single-span case to the multiple-
    span problem. The difficulties of using such an
    approach were discussed.

d. Next, a different approach to the plastic analysis and
    design of multiple-span structures was presented. It
    was based on the concept of dividing the structure
    into sub-structures (single-span structures) for the
    purposes of analysis.

e. The equations governing the solution of each of these
    sub-structures were obtained and design charts were
    presented.

f. To aid in the determination of the "least-weight" de-
    sign, the relationship between the fully plastic
    moment value and the unit weight of rolled wide-
    flange shapes was considered.

g. Finally, four design problems, typical of those found
    in practice, were solved to illustrate the methods
    developed.

VII. A C K N O W L E D G E M E N T S

VIII. N O M E N C L A T U R E

a   non-dimensional parameter, relating the
                  height of a column to the span length

b   non-dimensional parameter, relating the
                  total rise of the rafter to the span length

f, g function values

$k, (k_1, k_2)$ non-dimensional parameter, relating the fully
                  plastic moment values of two spans

w   distributed vertical load per unit length

w_w distributed vertical working load
                  per unit length

$A, (A_1, A_2)$ non-dimensional parameter, relating the hori-
                  zontal force acting on a structure (or the
                  hypothetical "overturning" moment of one
                  part of a structure on the adjacent part)
                  to the vertical loads. (See Equations 10
                  and 29). It is assumed that "A" results in
                  positive work being done as the structure
                  fails

C   constant

${}_R C_N$ number of possible combinations of hinges
                  which result in failure of the structure

$D, (D_1, D_2)$ non-dimensional parameter, relating the hori-
                  zontal resisting force or hypothetical
                  "over-turning" moment acting on a structure
                  to its vertical loading. It is assumed
                  that "D" results in negative work being
                  done as the structure fails

F, R, S function values

H   concentrated horizontal reaction (see Fig. 17)

$L, (L_1, L_2, L_3)$ length measurement. Can be total span length
                  or fractional part of it

M   bending moment

M_p fully plastic moment value

M^c moment at the top of interior column
                  (see Fig. 36)

VIII. Nomenclature (cont'd.)

M^Q moment in the windward rafter adjacent to
                  the windward column (see Fig. 36)

N   number of possible plastic hinges

P   concentrated load

P_W concentrated working load

Q_L, Q_R hypothetical "over-turning" or resisting
                  moment assumed acting about the base
                  of a structure

V   vertical reaction (see Fig. 17)

W   weight per unit length of a structural member

W_{ext} external work associated with a virtual
                  displacement of an assured mechanism

W_{int} internal work associated with a virtual
                  displacement of an assured mechanism

X   number of redundancies

Z   plastic modulus

$\alpha, (\alpha_1, \alpha_2)$ non-dimensional parameters, defining the
β distance to the plastic hinge in the
                  rafter of a structure

γ non-dimensional parameters, relating the
                  distributed horizontal load per unit
                  length to the distributed vertical load
                  per unit length

$\theta, (\theta_1, \theta_2)$ virtual rotation

ζ non-dimensional parameter, relating two
                  special virtual rotations (Equation 22)

ϕ curvature

ρ weight function (see Equation 45)

IX. R E F E R E N C E S

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1.

$$\frac{M_p}{wL^2} = \frac{1}{4}\left[\frac{(1-\alpha)(A+\alpha)}{1+\frac{b}{a}\alpha}\right] \text{ , where } A = (2a)\left(\frac{P}{wL}\right)$$

$$\alpha = \frac{1}{\frac{b}{a}}\left[\sqrt{1-\frac{b}{a}\left[A\left(1+\frac{b}{a}\right)-1\right]}-1\right]\ldots\ldots \text{ for } \frac{b}{a} > 0$$

$$\alpha = \left[\frac{1-A}{2}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ for } \frac{b}{a} = 0$$

2.
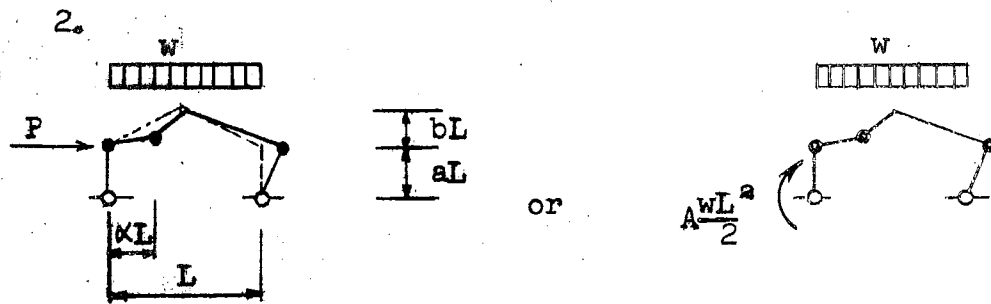
$$\frac{M_p}{wL^2} = \frac{1}{4}\left[\frac{\alpha(1-\alpha)}{1+\frac{b}{a}\alpha}\right] \text{ , where } A = (2a)\left(\frac{P}{wL}\right)$$

$$\alpha = \frac{1}{\frac{b}{a}}\left[\sqrt{1+\frac{b}{a}}-1\right] \quad \ldots\ldots\ldots\ldots \text{for } \frac{b}{a} > 0$$

$$\alpha = \frac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{for } \frac{b}{a} = 0$$

3.

$$\frac{M_p}{wL^2} = \left[\frac{A}{4}\right] \quad \text{where } A = (2a)\left(\frac{P}{wL}\right)$$

4.
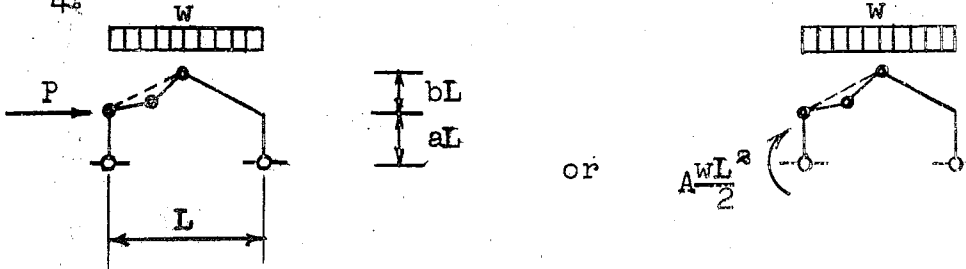
$$\frac{M_p}{wL^2} = 0.0156 \text{ for } \frac{b}{a} > 0, \quad \frac{M_p}{wL^2} = 0.0625 \text{ for } \frac{b}{a} = 0$$

5.
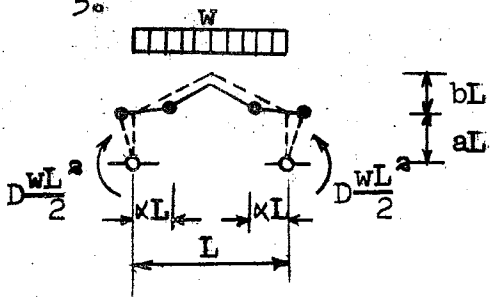
$$\frac{M_p}{wL^2} = \frac{1}{4}\left[\frac{\propto(1-\frac{2b}{a}D-\propto)}{1+\frac{b}{a}\propto}\right]$$

$$\propto = \frac{1}{\frac{b}{a}}\left[\sqrt{1-(\frac{b}{a})(2\frac{b}{a}D-1)} -1\right] \dots\dots\dots\text{for } \frac{b}{a} > 0$$

$$\propto = \frac{1}{2} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\text{for } \frac{b}{a} = 0$$

6.
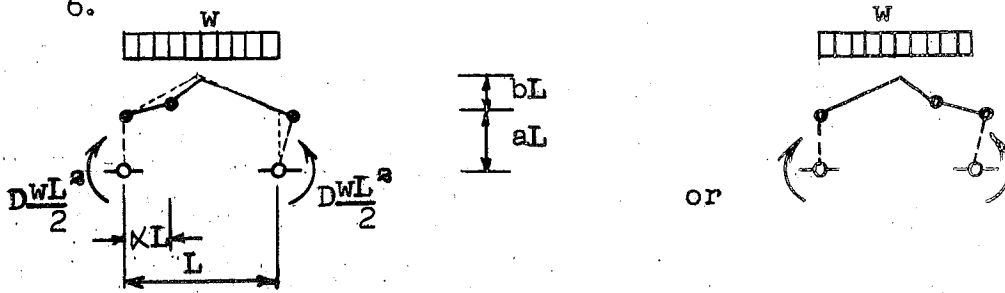
$$\frac{M_p}{wL^2} = \frac{1}{4}\left[\frac{\alpha(1-\frac{2b}{a}D-\alpha)}{1+\frac{b}{a}\alpha}\right]$$

$$\alpha = \frac{1}{\frac{b}{a}}\left[\sqrt{1-\left(\frac{b}{a}\right)\left(2\frac{b}{a}D-1\right)}\;-1\right] \quad\ldots\ldots\ldots \text{for } \frac{b}{a}>0$$

$$\alpha = \frac{1}{2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{for } \frac{b}{a}=0$$

7.   NOTE: D_0 will be equal to D.
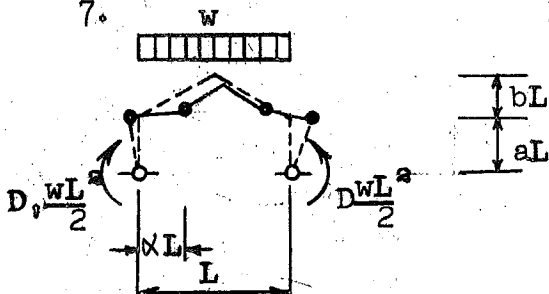
$$\frac{M_p}{wL^2} = \frac{1}{4}\left[\frac{\alpha(1-\frac{2b}{a}D-\alpha)}{1+\frac{b}{a}\alpha}\right]$$

$$\alpha = \frac{1}{\frac{b}{a}}\left[\sqrt{1-\left(\frac{b}{a}\right)\cdot\left(2\frac{b}{a}D-1\right)}\;-1\right] \quad\ldots\ldots\ldots \text{for } \frac{b}{a}>0$$

$$\alpha = \frac{1}{2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{for } \frac{b}{a}=0$$

8.

$$\frac{M_p}{wL^2} = \left[\frac{A-D}{4}\right]$$

9.

$$A\frac{wL^2}{2} \qquad D\frac{wL^2}{2}$$
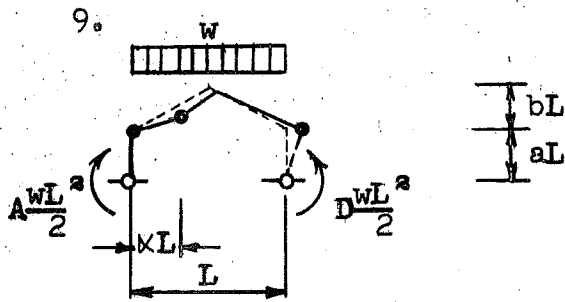
$$\frac{M_p}{wL^2} = \frac{1}{4}\left[\frac{\varkappa\left(1-D\;\dfrac{2b}{a}-\varkappa\right)}{1+\dfrac{b}{a}\varkappa}\right]$$

$$\varkappa = \frac{1}{\dfrac{b}{a}}\left[\sqrt{1+\frac{b}{a}\left[1-D\left(\frac{2b}{a}\right)\right]}\; -1\right] \quad \ldots\ldots\ldots \text{for } \frac{b}{a} > 0$$

$$\varkappa = \frac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{for } \frac{b}{a} = 0$$

10.

$$A\frac{wL^2}{2} \qquad D\frac{wL^2}{2}$$
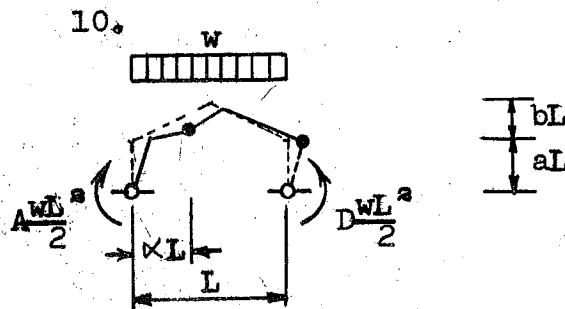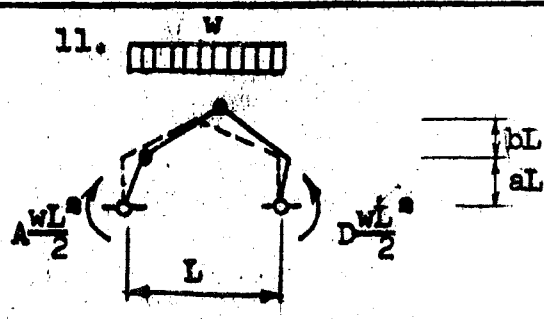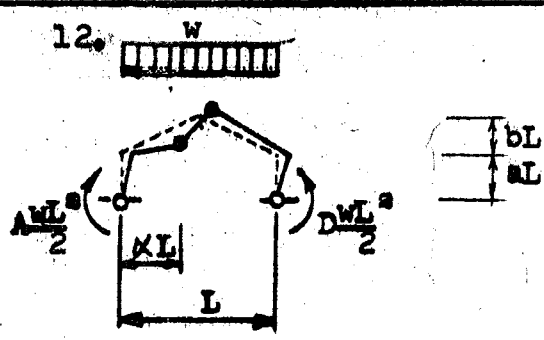
$$\frac{M_p}{wL^2} = \frac{1}{4}\left[\frac{(1-\varkappa)(A+\varkappa-D)-D\left(\dfrac{2b}{a}\right)\varkappa}{1+\dfrac{b}{a}\varkappa}\right]$$

$$\varkappa = \frac{1}{\dfrac{b}{a}}\left[\sqrt{1-\frac{b}{a}\left[A\left(1+\frac{b}{a}\right)-D\left(1-\frac{b}{a}\right)-1\right]}\; -1\right] \quad \ldots \text{for } \frac{b}{a} > 0$$

$$\varkappa = \left[\frac{1-A+D}{2}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{for } \frac{b}{a} = 0$$

11.

$$\frac{M_p}{WL^2} = \frac{1}{4}\left[\frac{A\left(1+\frac{2b}{a}\right)-D-\frac{1}{2}}{2+\frac{b}{a}}\right]$$

12.

$$\frac{M_p}{WL^2} = \frac{1}{4}\left[\frac{\left\{A\left(1+2\frac{b}{a}\right)-D-\frac{1}{2}\right\}-2\alpha\left\{A\left(1+2\frac{b}{a}\right)-D-1-\frac{b}{2a}\right\}-2\alpha^2\left\{1+\frac{b}{a}\right\}}{2+\alpha\left(\frac{2b}{a}\right)+\left(\frac{b}{a}\right)}\right]$$

$$\alpha = \frac{\sqrt{\left(2+\frac{b}{a}\right)^2-4\left(\frac{b}{a}\right)\left[A+A\frac{2b}{a}-D-1-\frac{b}{4a}\right]}-\left(2+\frac{b}{a}\right)}{2\frac{b}{a}} \quad \ldots \text{for } \frac{b}{a}>0$$
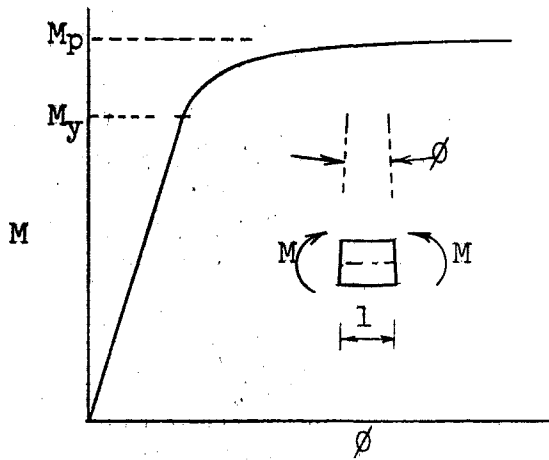
Fig. 1
MOMENT-CURVATURE RELATIONSHIP
ASSUMED IN PLASTIC DESIGN
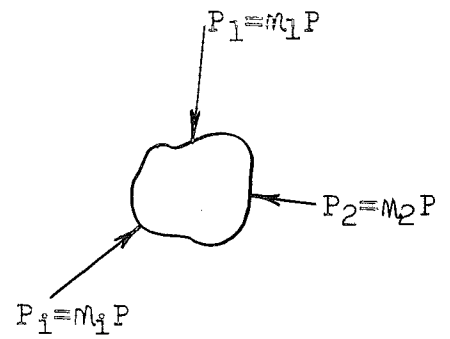
Fig. 2
PROPORTIONAL LOADING
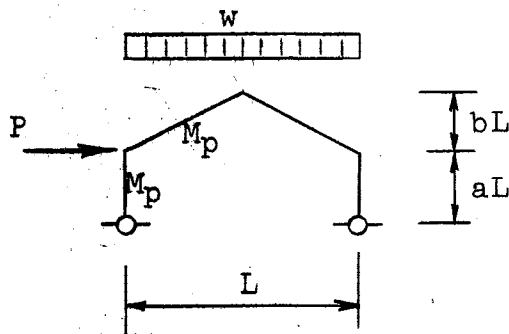
Fig. 3
SINGLE SPAN RIGID FRAME

Fig. 4
LOCATION OF
POSSIBLE PLASTIC HINGES
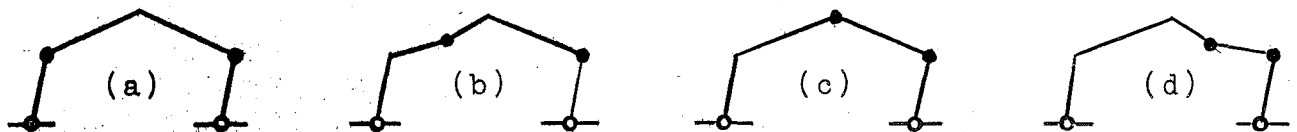
Fig. 5
FOUR ASSUMED, INDEPENDENT FAILURE MECHANISMS
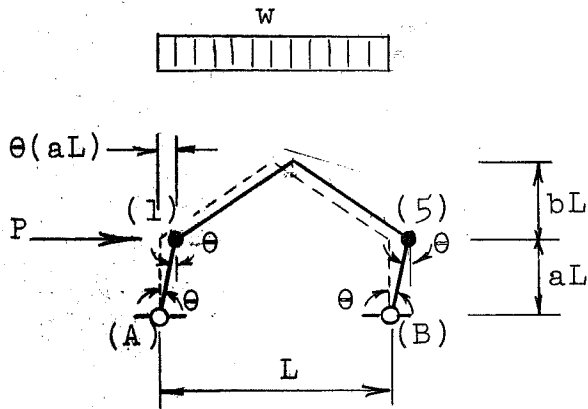
Fig. 6
FAILURE MECHANISM (a)
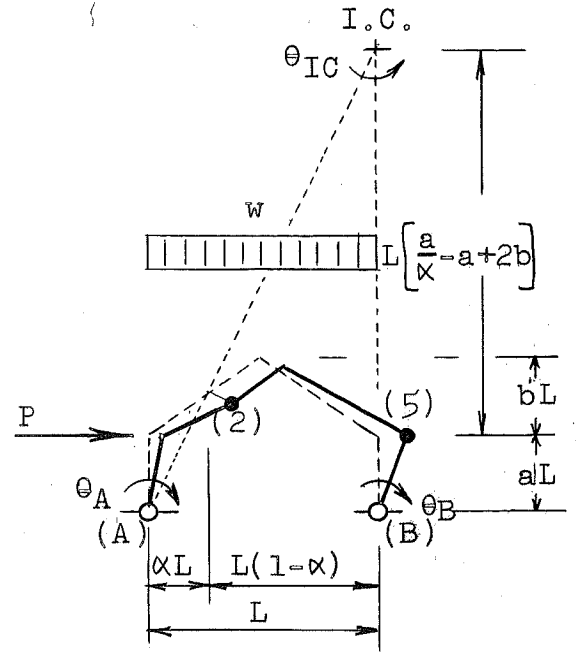
Fig. 7
FAILURE MECHANICS (b)
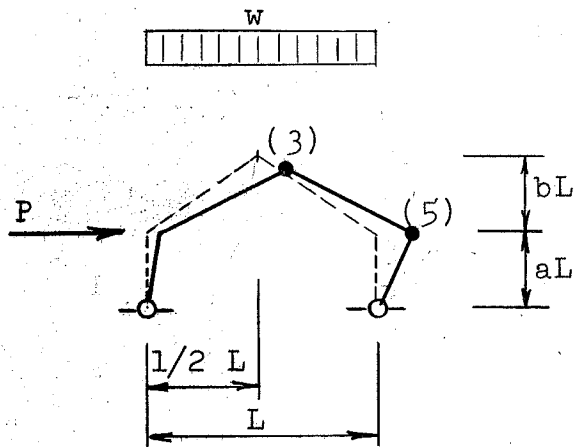
Figure 8
FAILURE MECHANISM (c)
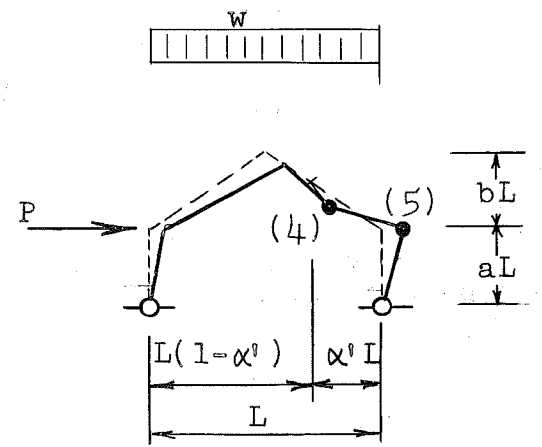
Fig. 9
FAILURE MECHANICS (d)

205.56
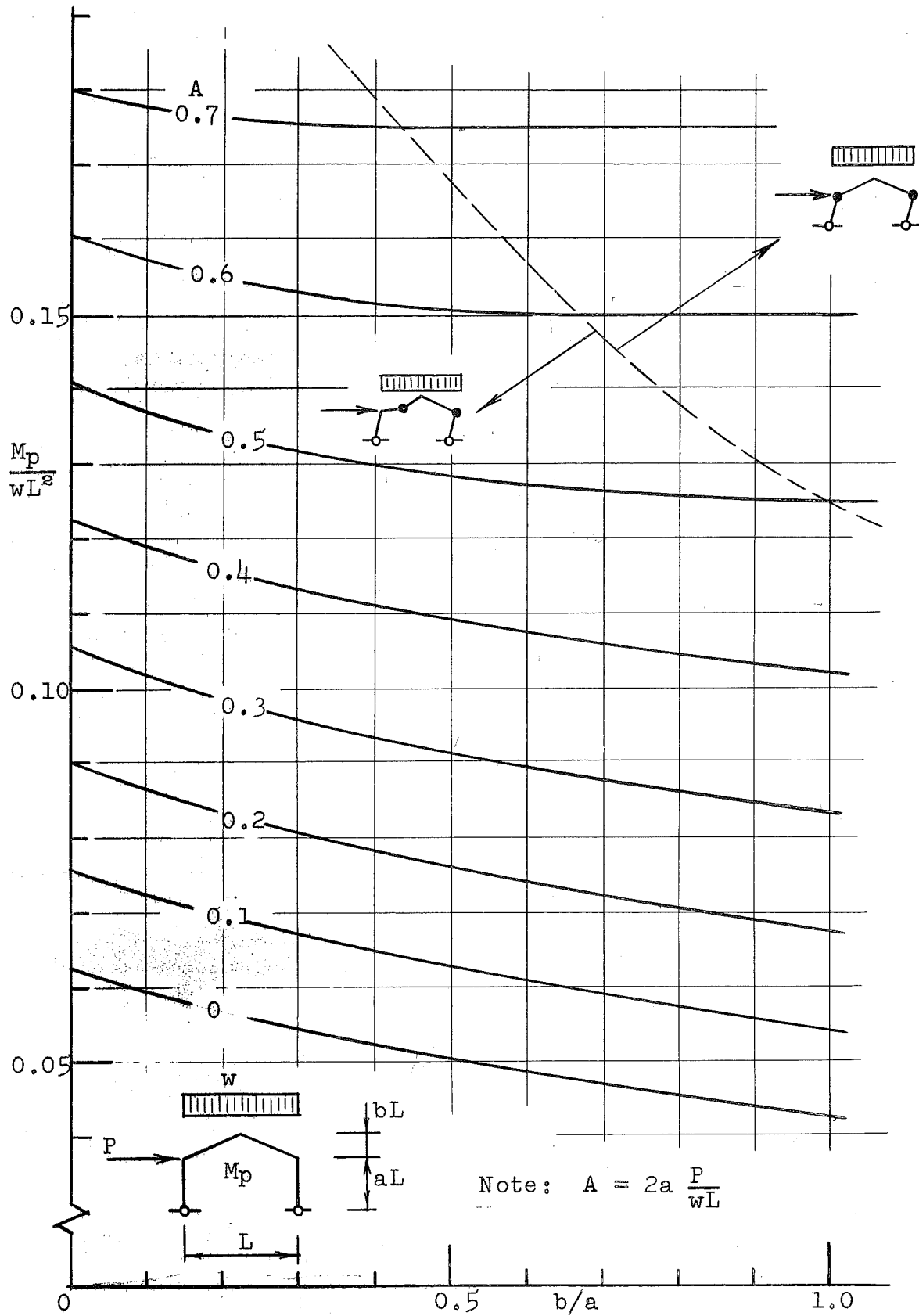
Note: $A = 2a \dfrac{P}{wL}$

Fig. 10 - DESIGN CURVES FOR PINNED-BASE, SINGLE-SPAN,
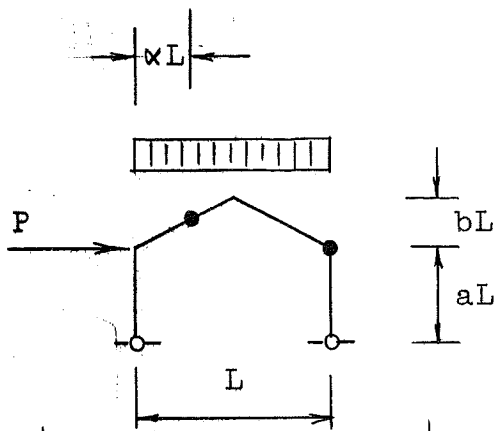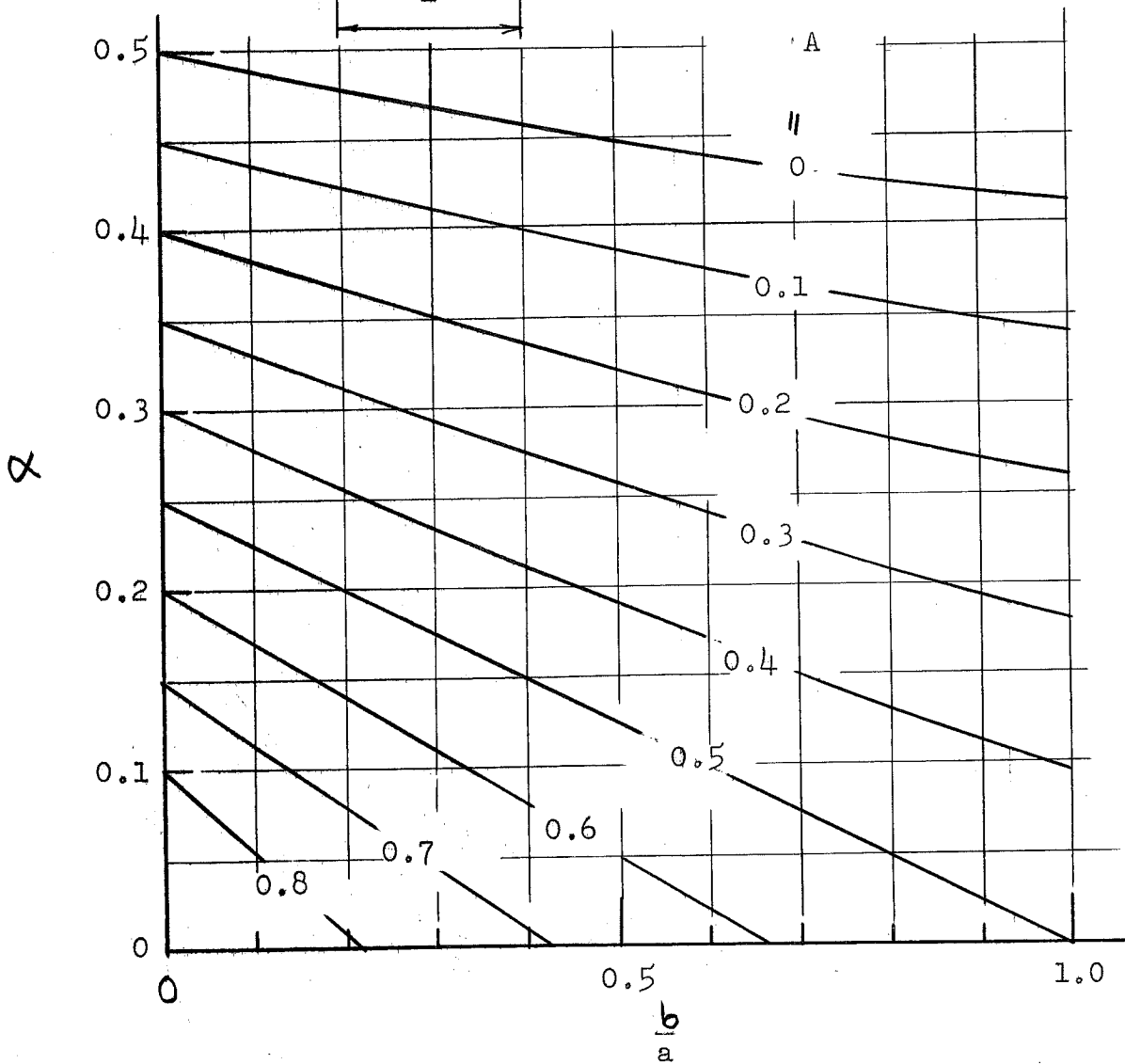
GABLE FRAMES

Fig. 11 - DESIGN CURVES FOR SINGLE-SPAN, PINNED-BASE, GABLE FRAMES. LOCATION OF PLASTIC HINGE
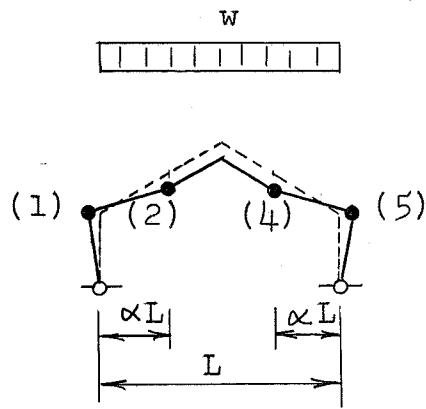
Fig. 12 - "OVER DETERMINATE"
FAILURE MECHANISM
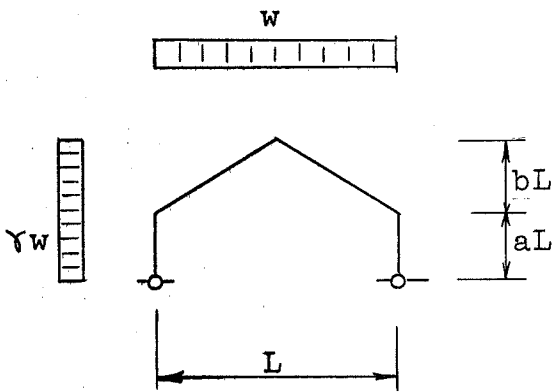
Fig. 13 - STRUCTURE SUBJECTED
TO DISTRIBUTED
WIND LOAD
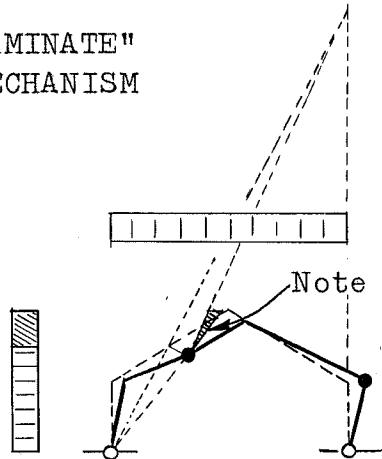
Fig. 14 - FAILURE MECHANISM
FOR DISTRIBUTED
WIND LOAD CONDITION
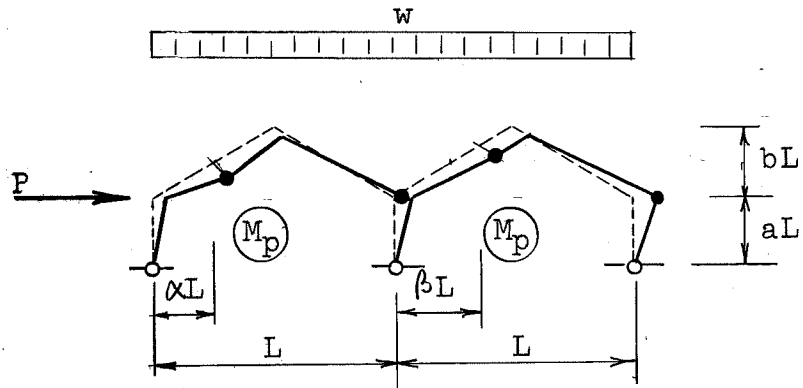
Fig. 15 - POSSIBLE FAILURE MECHANISM
FOR TWO-SPAN STRUCTURE
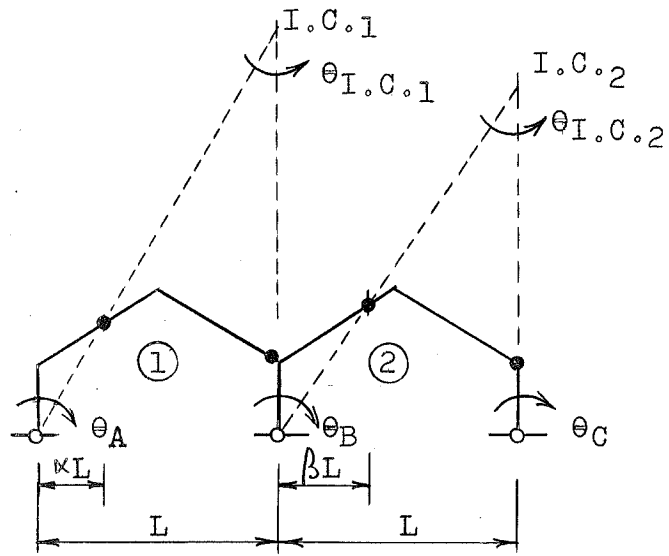
Fig. 16 - TWO-SPAN STRUCTURE
SHOWING INSTANTANEOUS CENTERS
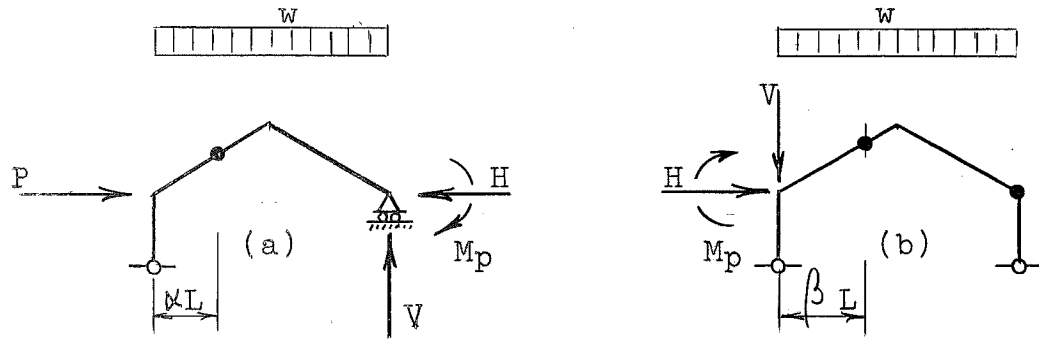
Fig. 17 - "SUB-STRUCTURES" WITH FORCES
AT CUT SECTION SHOWN
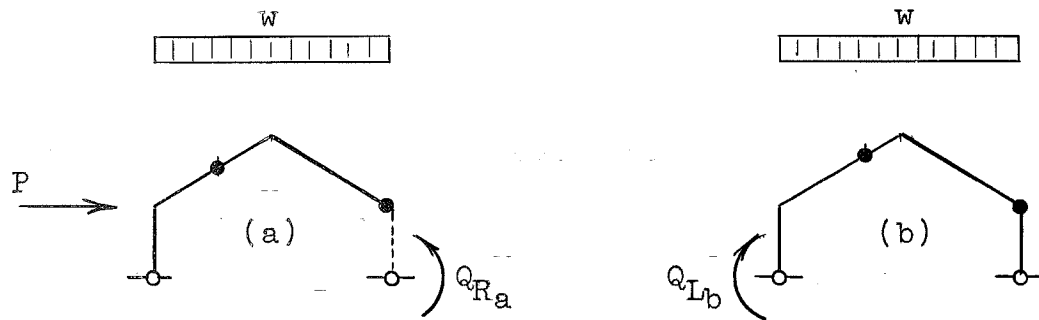
Fig. 18 - "SUB-STRUCTURES" WITH HYPOTHETICAL MOMENTS
AT CUT SECTION SHOWN
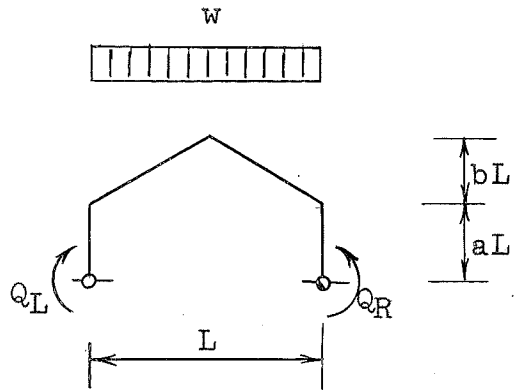
Fig. 19 - PINNED-BASE GABLE FRAME WITH
GENERAL HYPOTHETICAL MOMENTS SHOWN
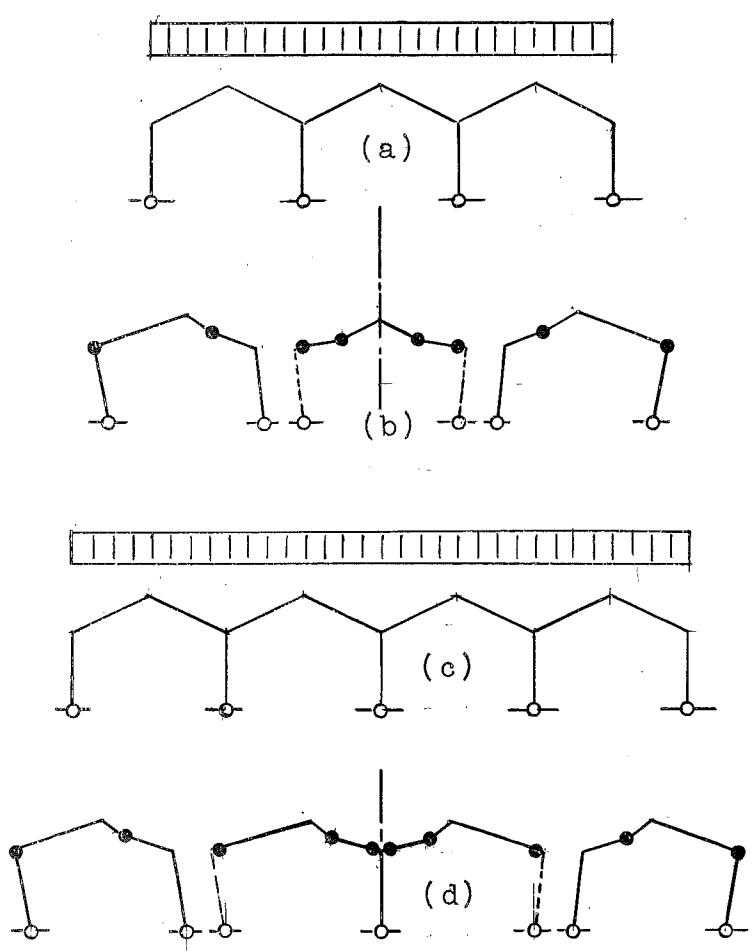
Fig. 20 - POSSIBLE FAILURE SITUATIONS THAT COULD OCCUR
FOR MULTIPLE-SPAN, PINNED-BASE, GABLE FRAMES
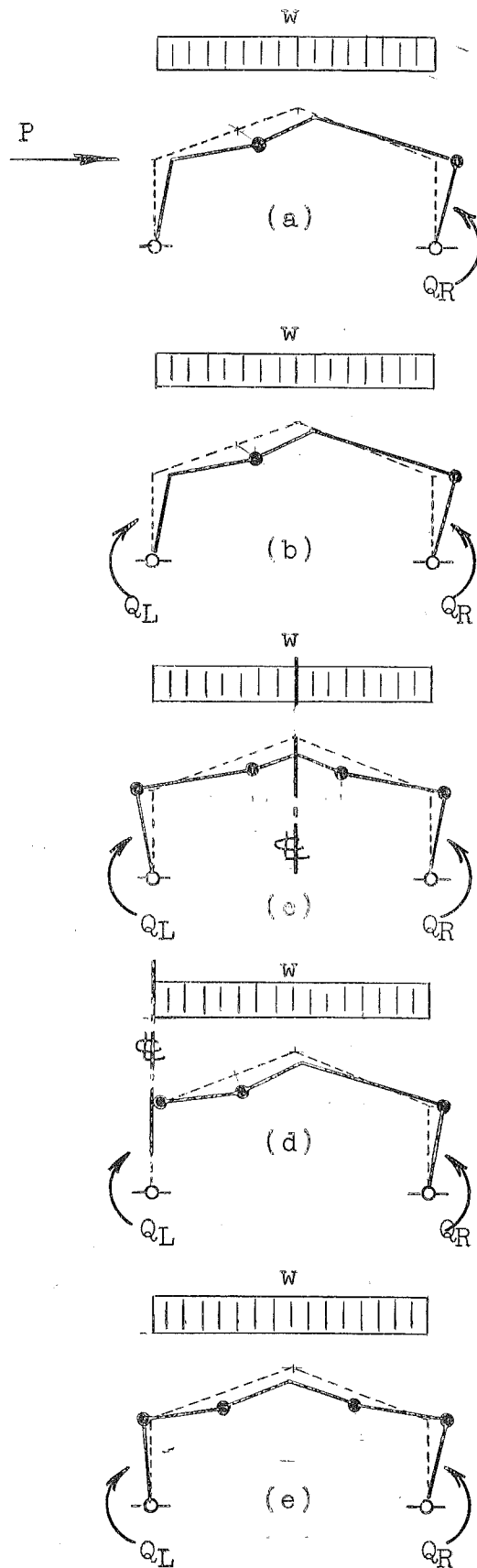
Fig. 21 – POSSIBLE FAILURE SITUATIONS
FOR MULTIPLE-SPAN, PINNED-BASE,
GABLE FRAME
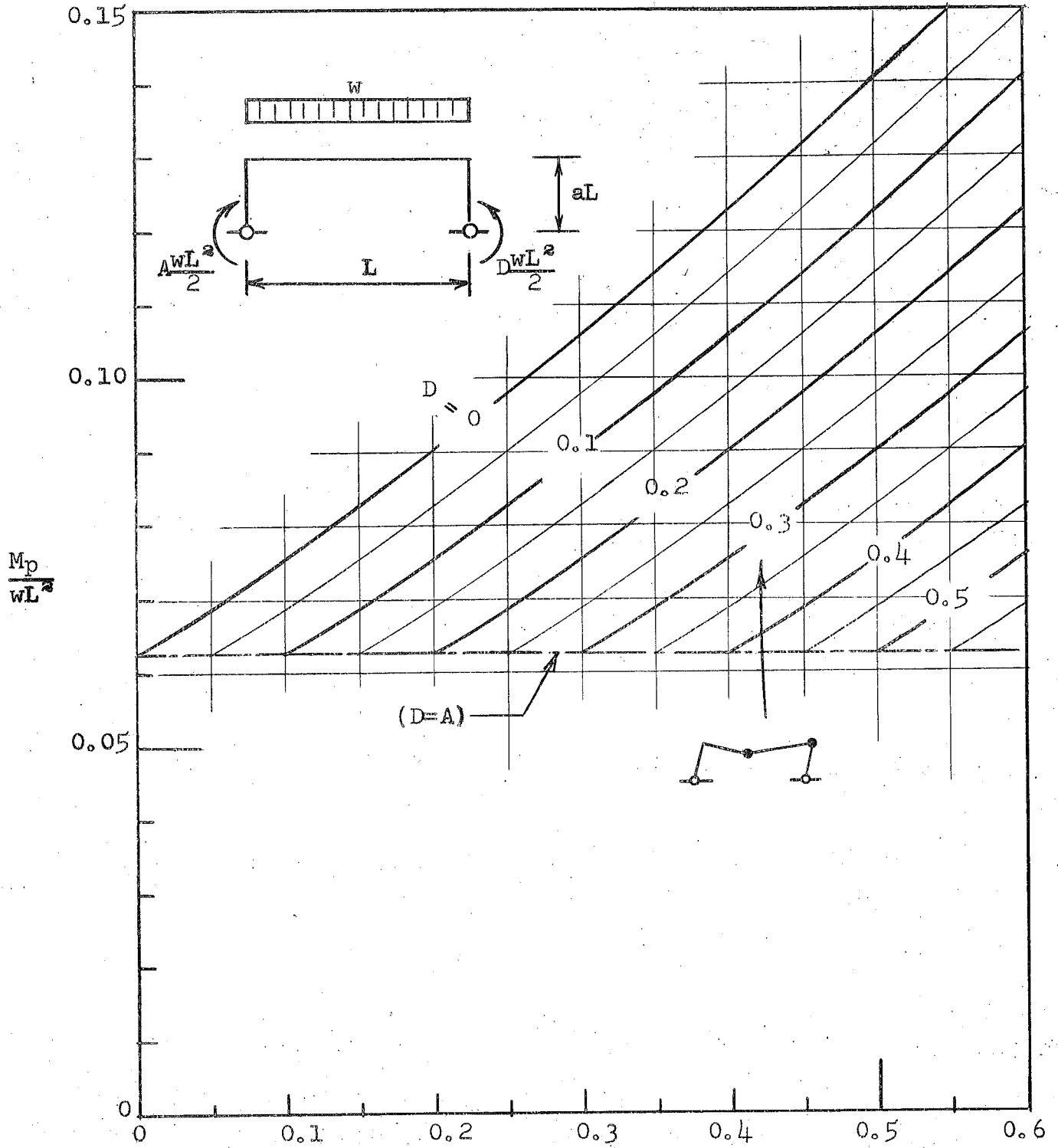
Fig. 22 – DESIGN CORNERS FOR PINNED-BASE, GABLE FRAMES
DETERMINATION OF MEMBER SIZE
$(b/s = 0)$
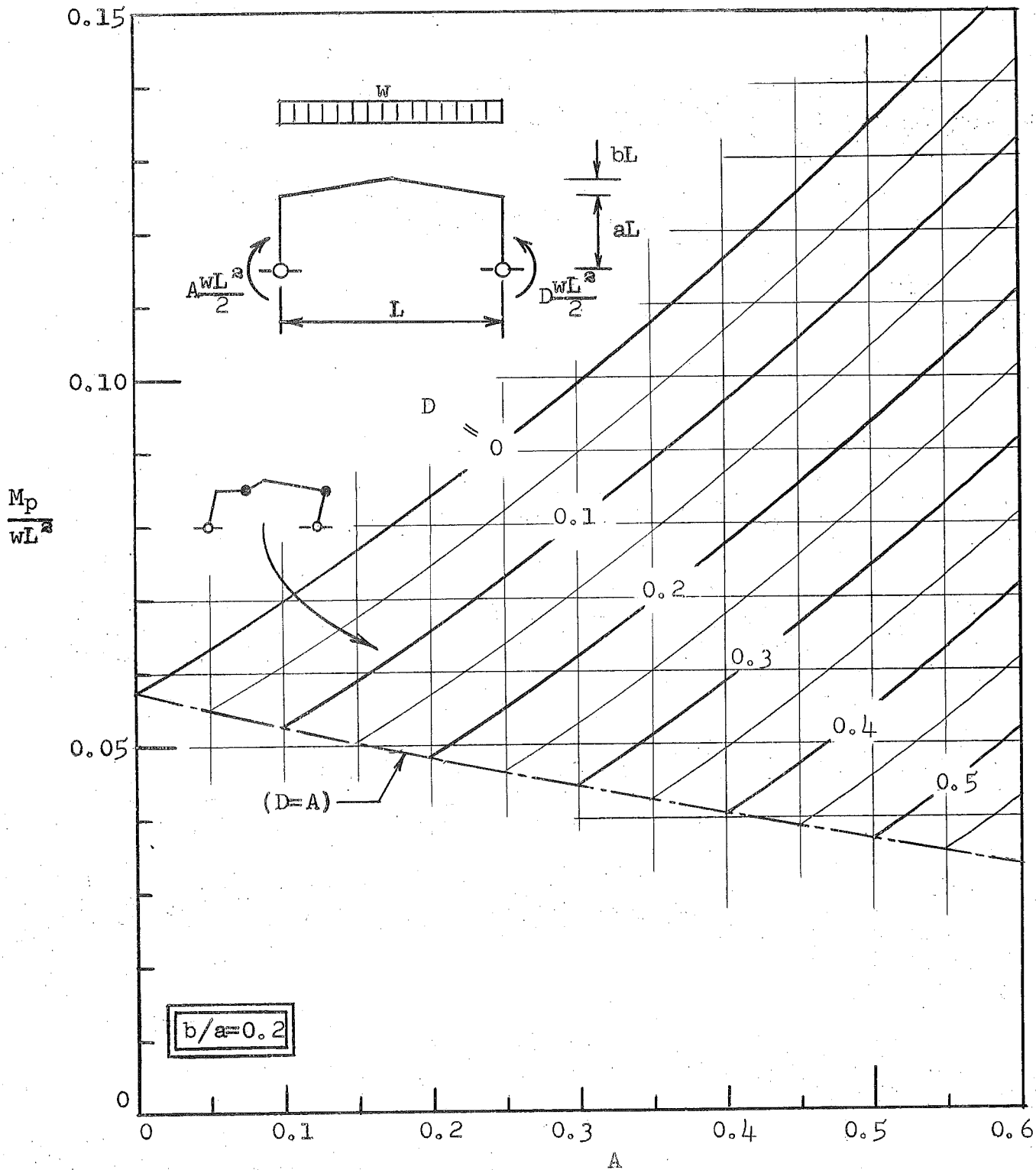
Fig. 23 - DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES

DETERMINATION OF MEMBER SIZE

$(b/a = 0.2)$
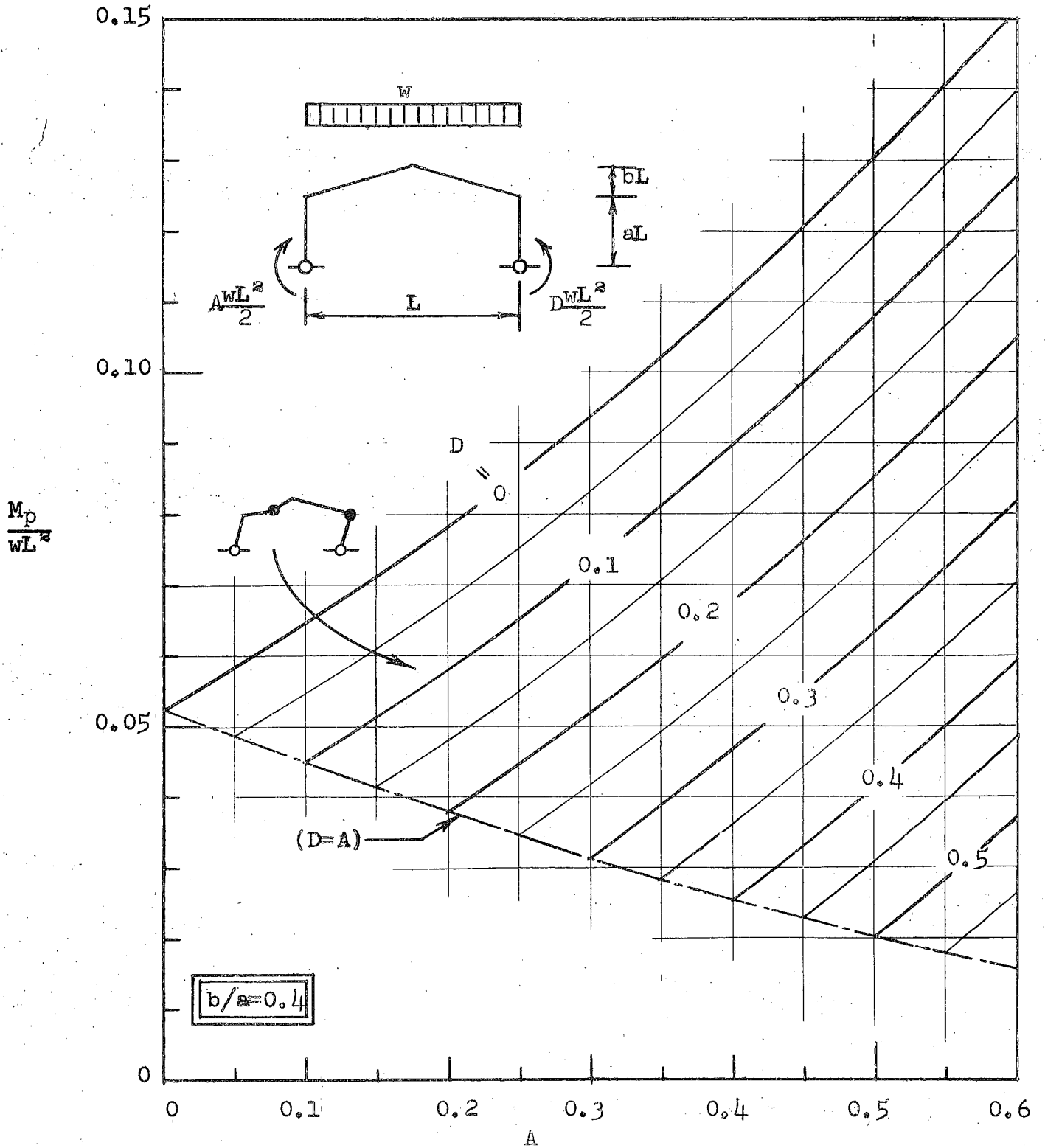
Fig. 24 - DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES
DETERMINATION OF MEMBER SIZE
(b/a = 0.4)
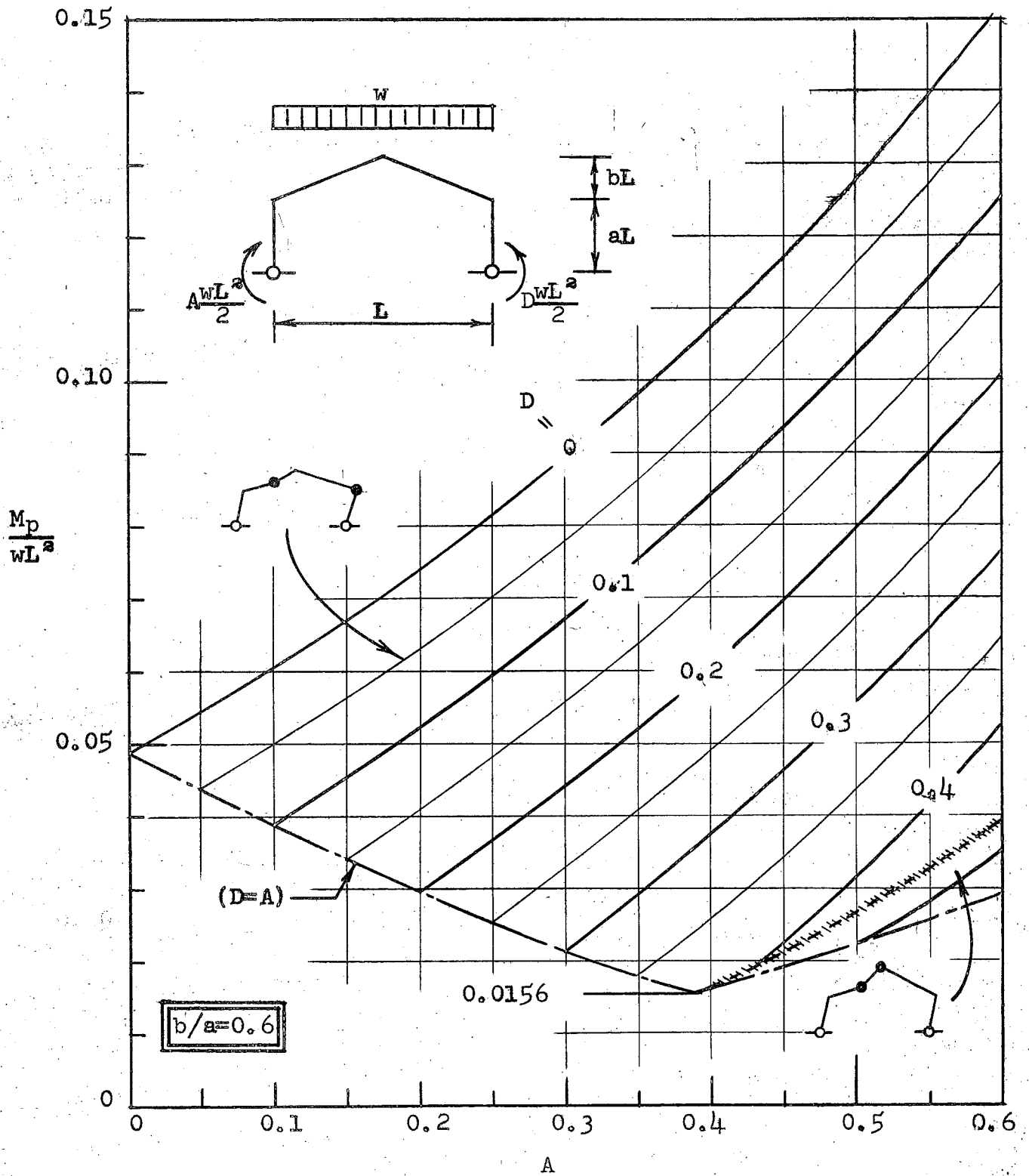
Fig. 25 - DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES
DETERMINATION OF MEMBER SIZE
(b/a = 0.6)
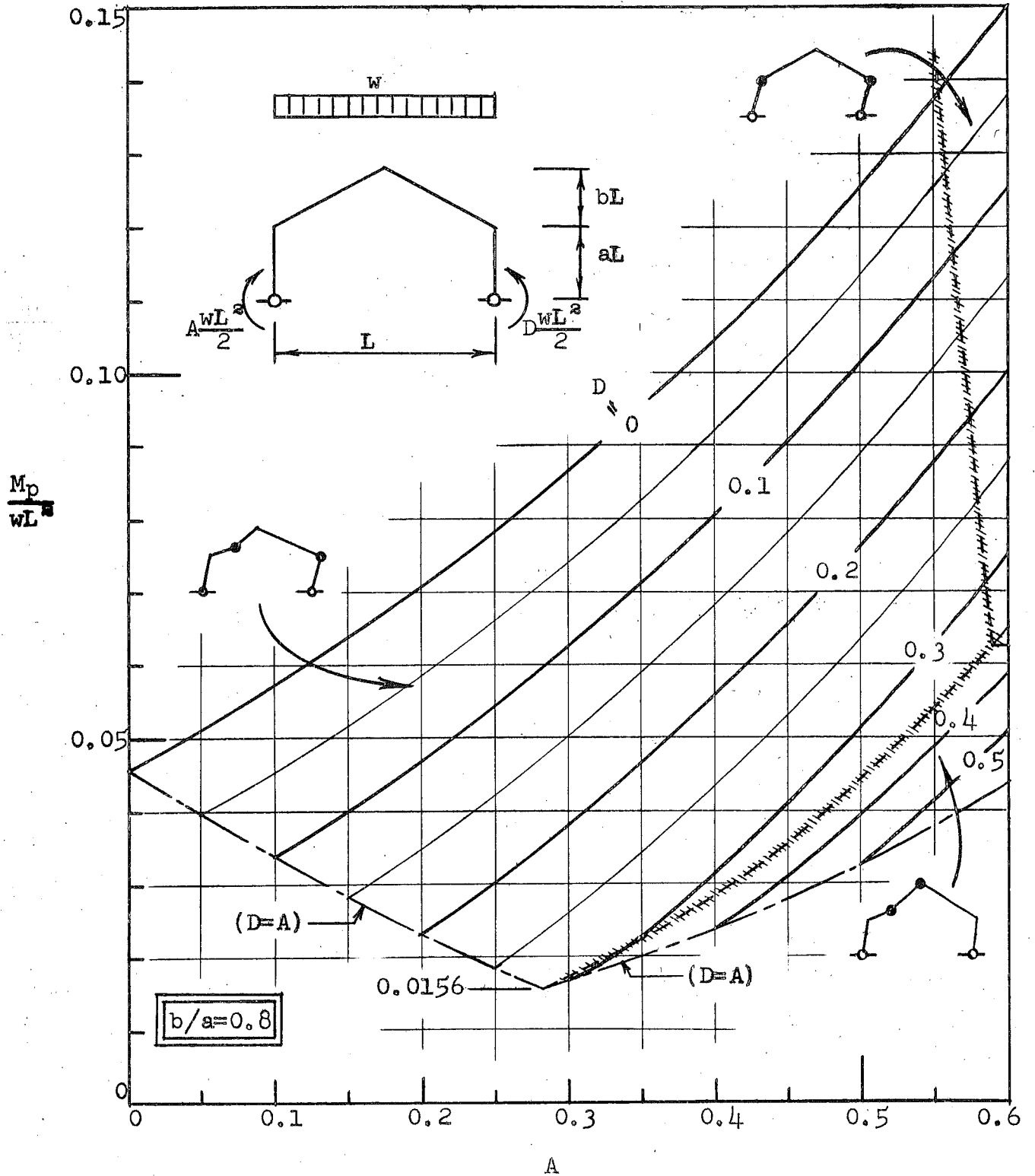
Fig. 26 - DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES

DETERMINATION OF MEMBER SIZE

(b/a = 0.8)
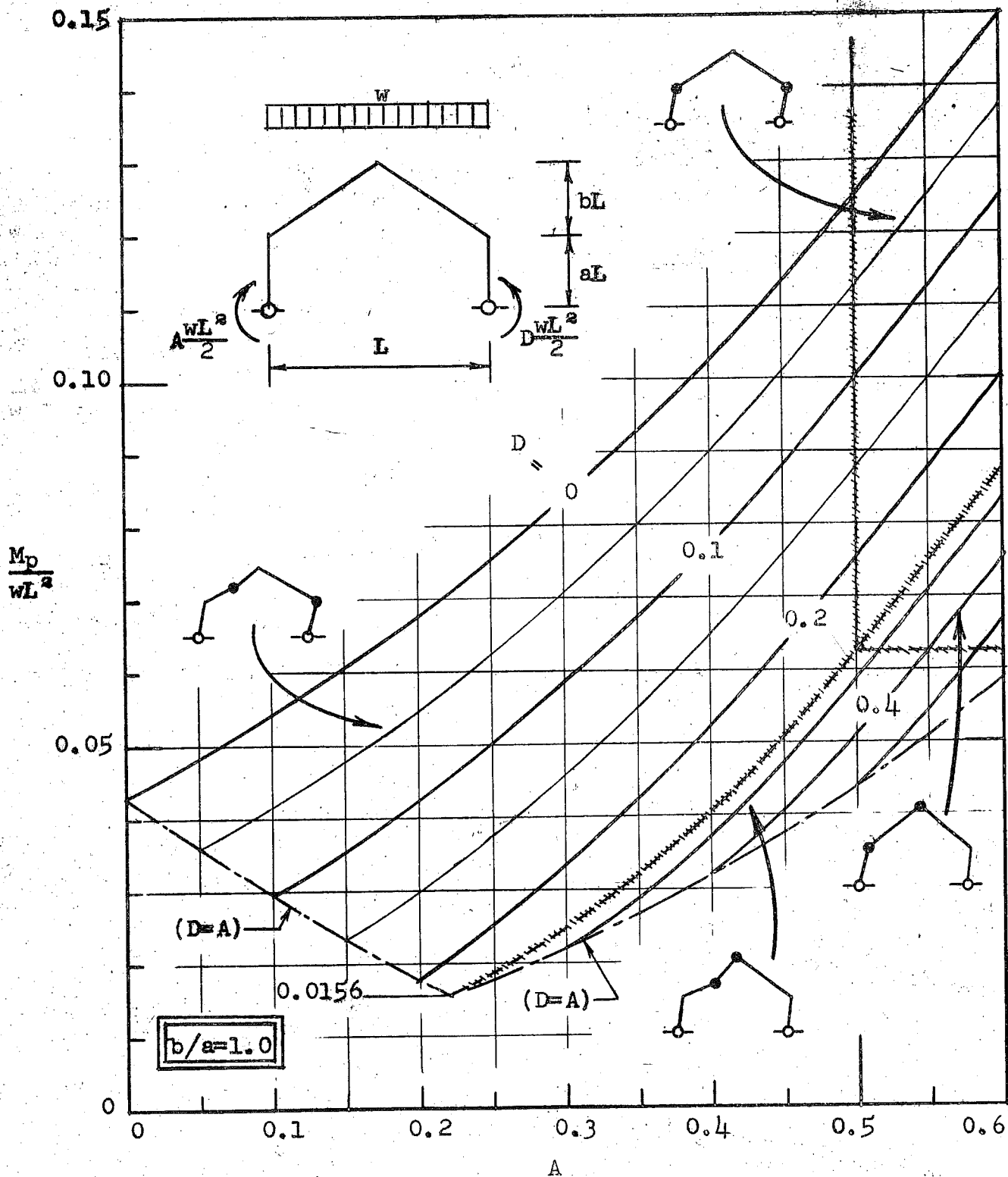
Fig. 27 - DESIGN CURVES FOR PINNED-BASED, GABLE FRAMES

DETERMINATION OF MEMBER SIZE

$(b/a = 1.0)$
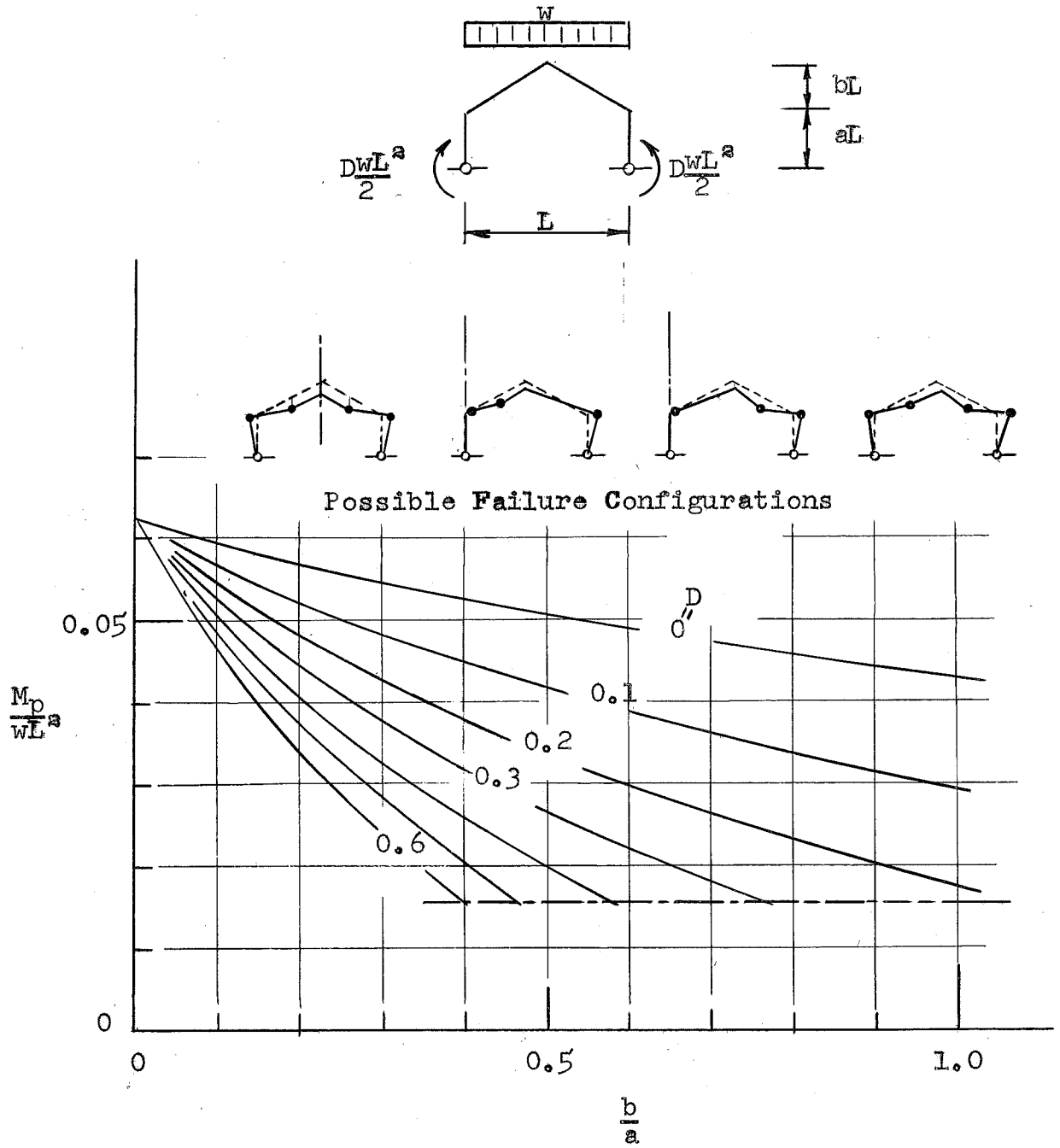
FIG. 28 DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES
DETERMINATION OF MEMBER SIZE
(A=D)
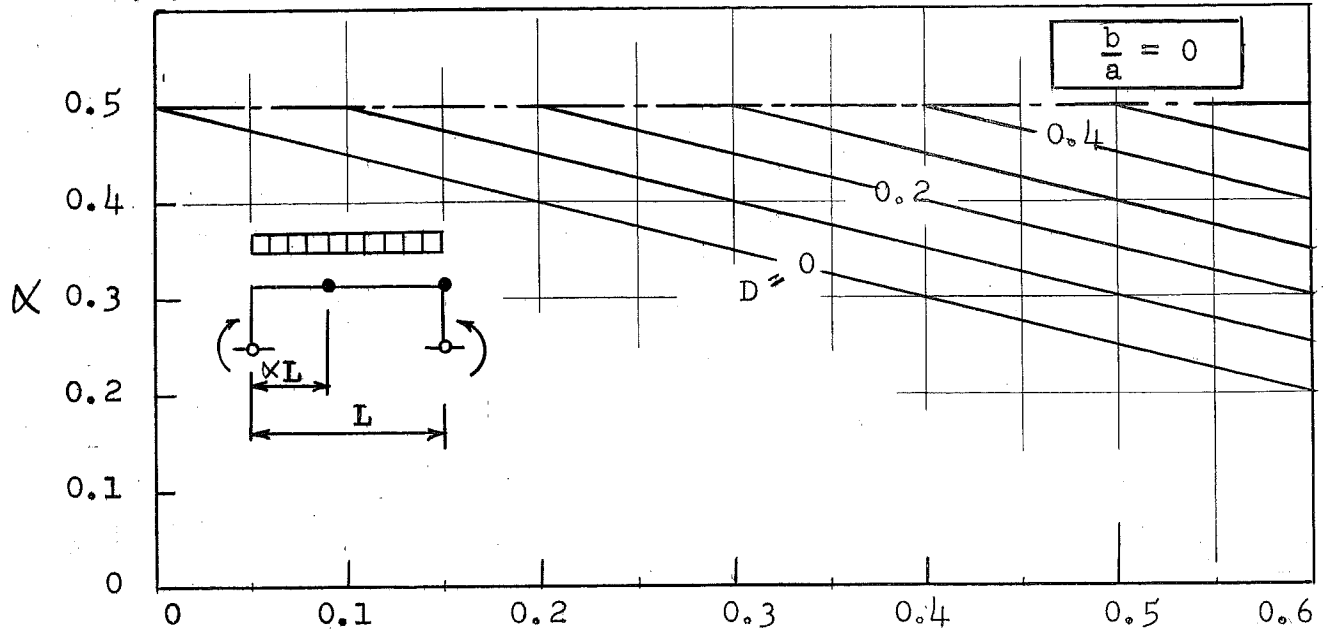
Fig. 29 = DESIGN CURVES FOR PINNED=BASE, GABLE FRAMES
LOCATION OF PLASTIC HINGE

(b/a = 0)
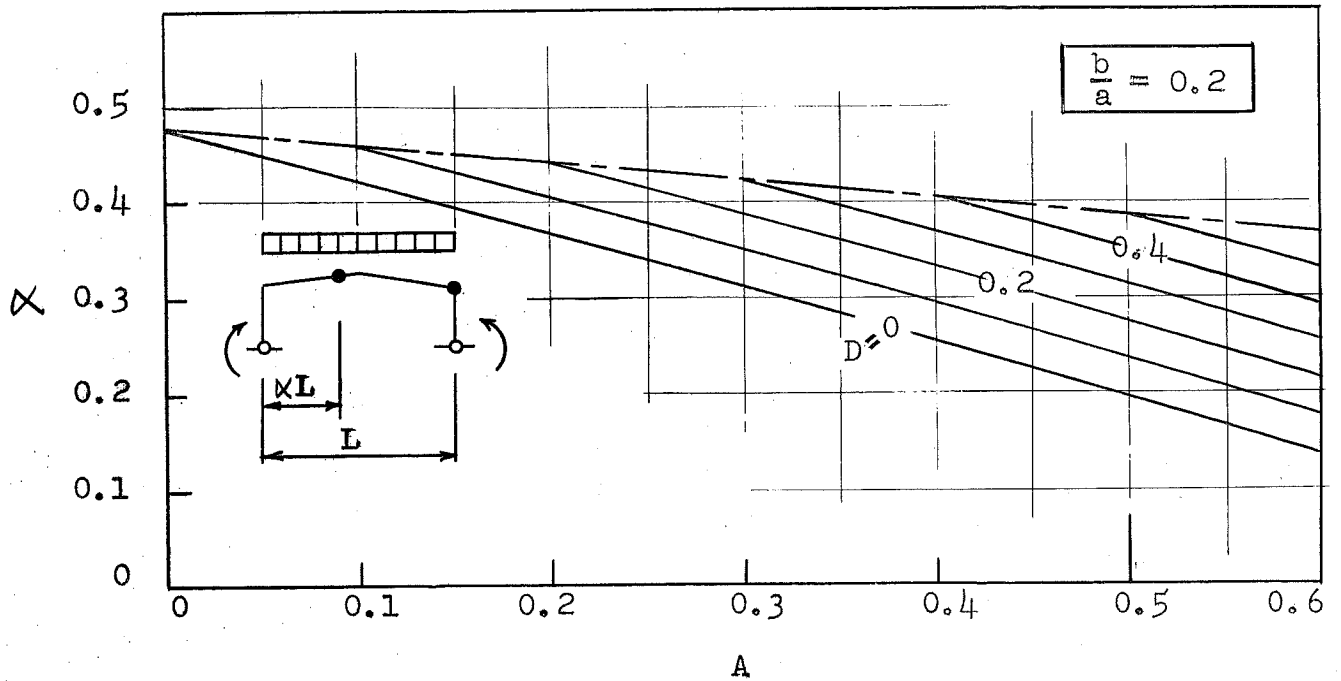
Fig. 30 - DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES
LOCATION OF PLASTIC HINGE

(b/a = 0.2)
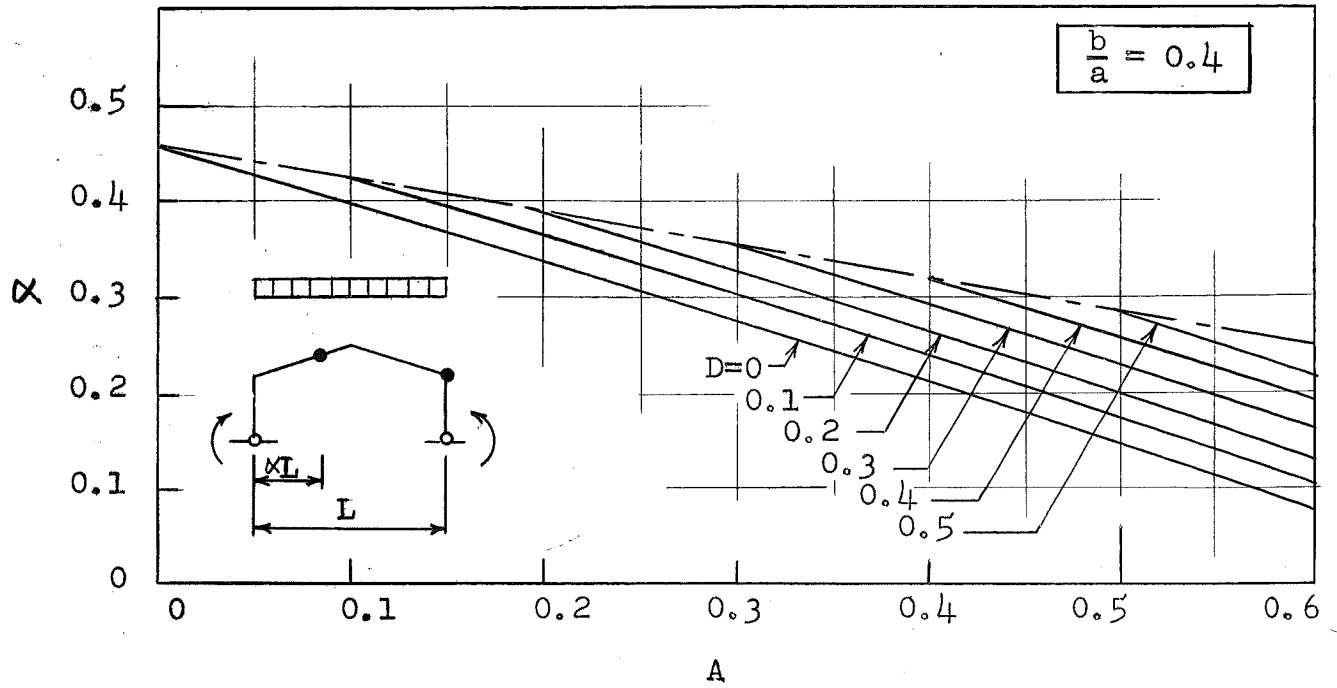
Fig. 31 - DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES
LOCATION OF PLASTIC HINGE
(b/a = 0.4)
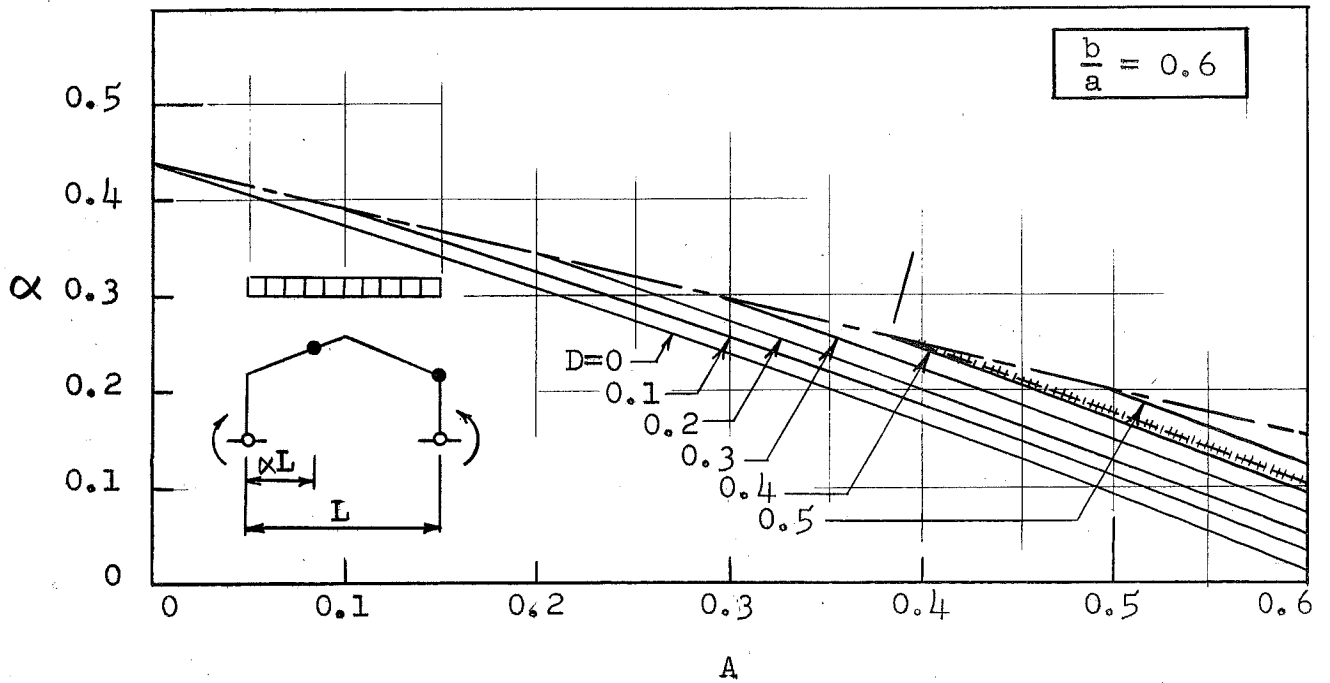
Fig. 32 - DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES
LOCATION OF PLASTIC HINGE
(b/a = 0.6)
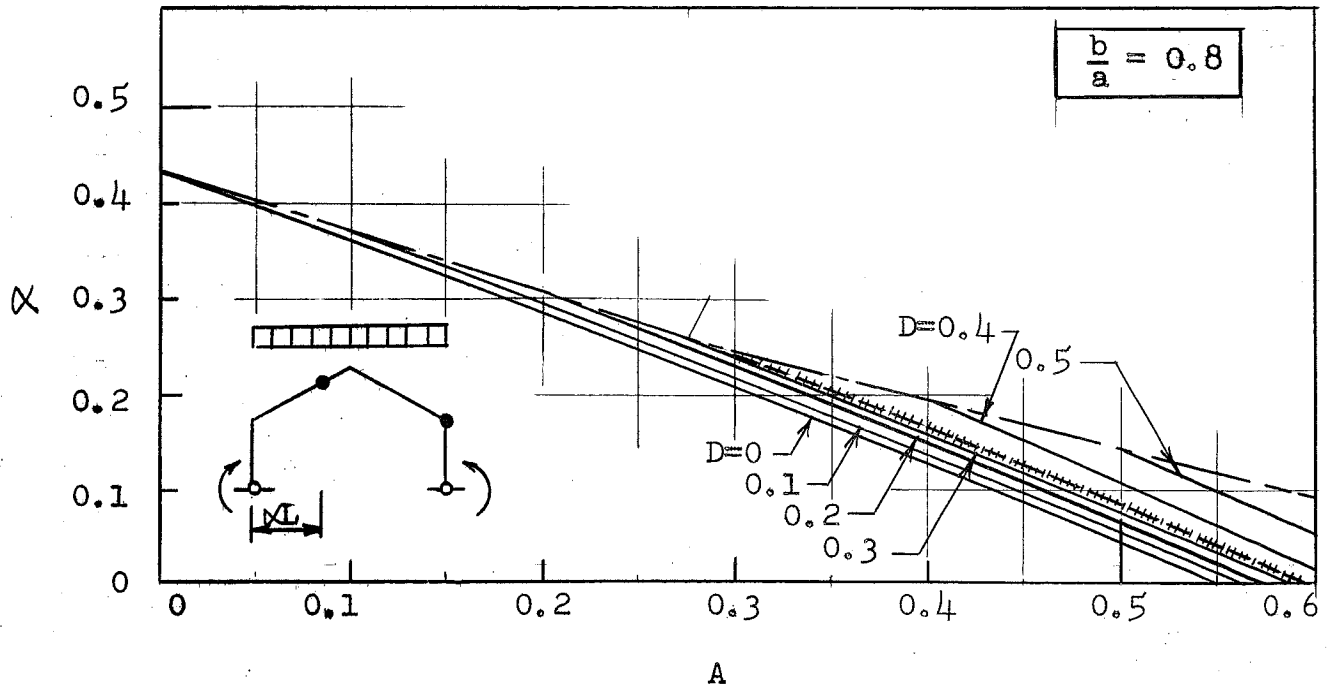
Fig. 33 - DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES
LOCATION OF PLASTIC HINGE
(b/a = 0.8)
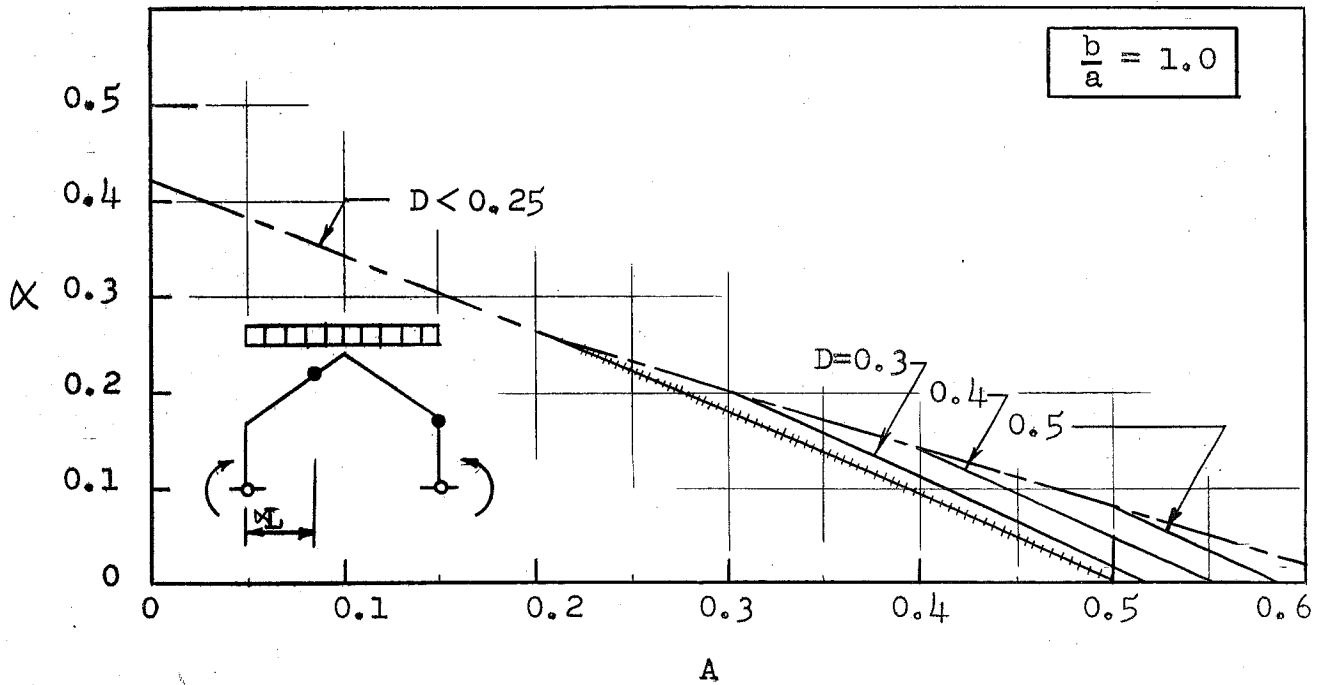
Fig. 34 - DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES
LOCATION OF PLASTIC HINGE
(b/a = 1.0)
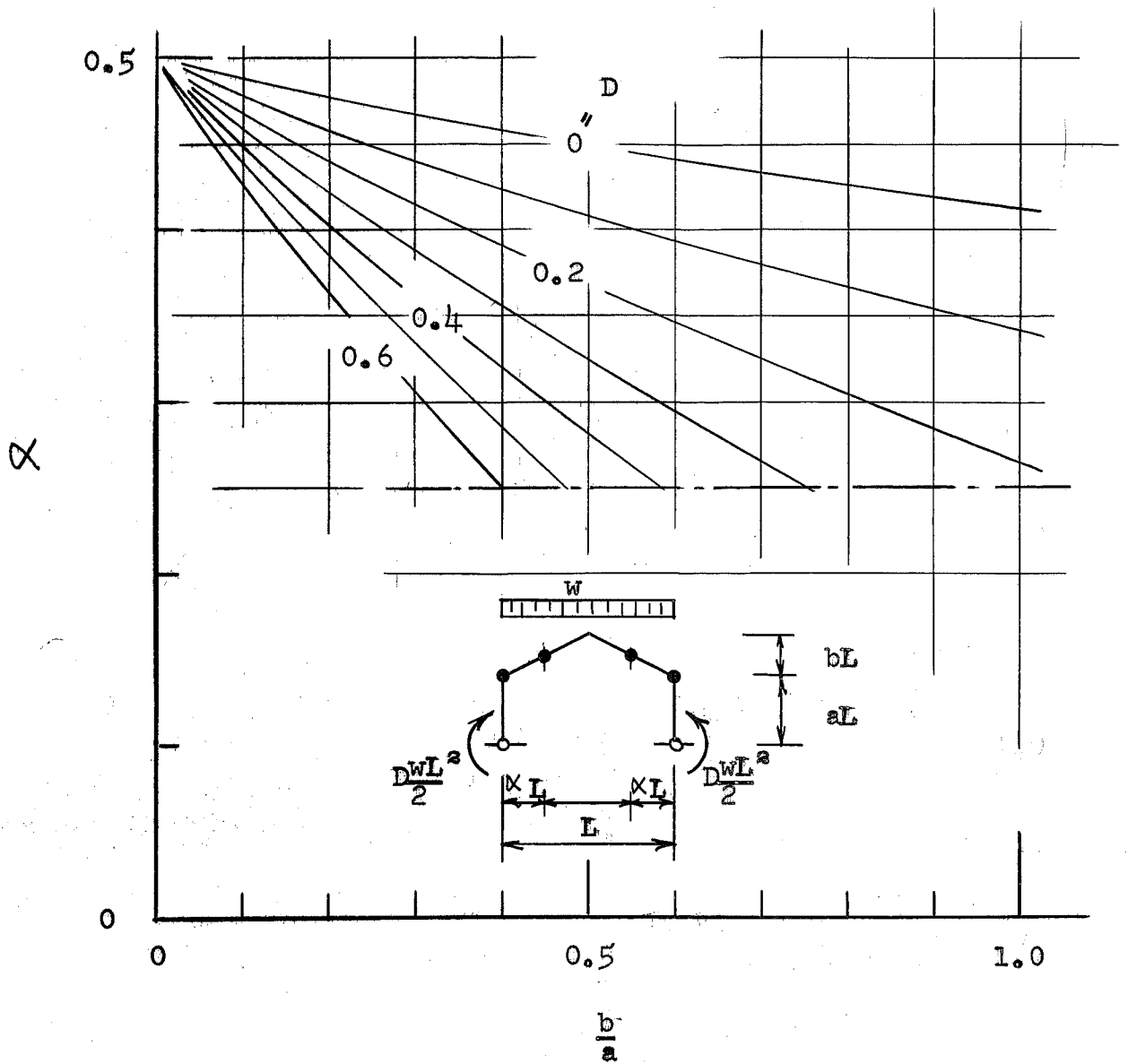
FIG. 35 DESIGN CURVES FOR PINNED-BASE, GABLE FRAMES LOCATION OF PLASTIC HINGE

(A = D)
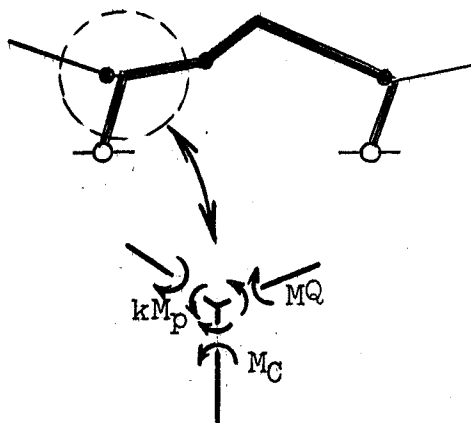
Fig. 36 – DETERMINATION OF
SIZE OF INTERIOR COLUMNS

$w_W = 1^k/\text{ft.}$
(working load)

$P_W = 10.7^k$

M_p

M_p

$bL = 9'$

$aL = 15'$

(a)

$L = 40'$

(b)

$\alpha = 0.31$

Fig. 38 – DESIGN EXAMPLE NO. 1

$w_W = 1^k/\text{ft.}$ (working load)

$P_W = 10.7^k$

(1)   (2)

$bL = 9'$

$aL = 15'$

$L = 40'$ $L = 40'$

(a)

$\alpha = 0.44$ $\alpha = 0.44$

αL | αL | αL | αL

(b)
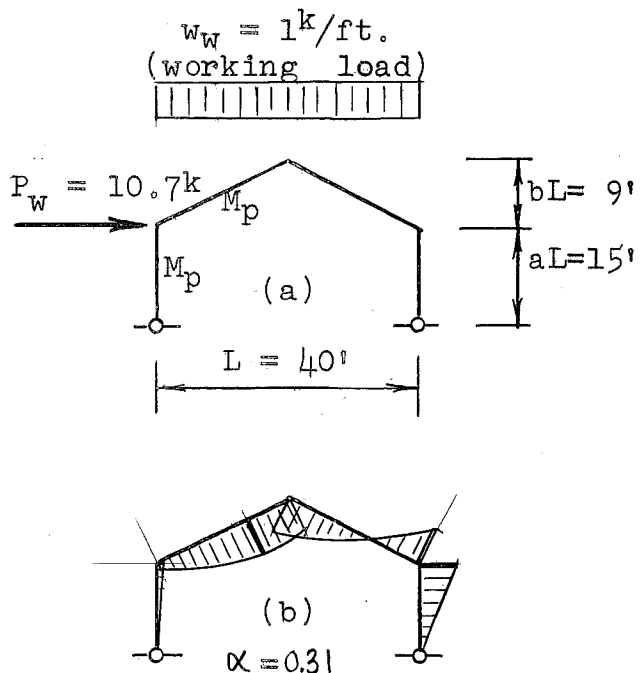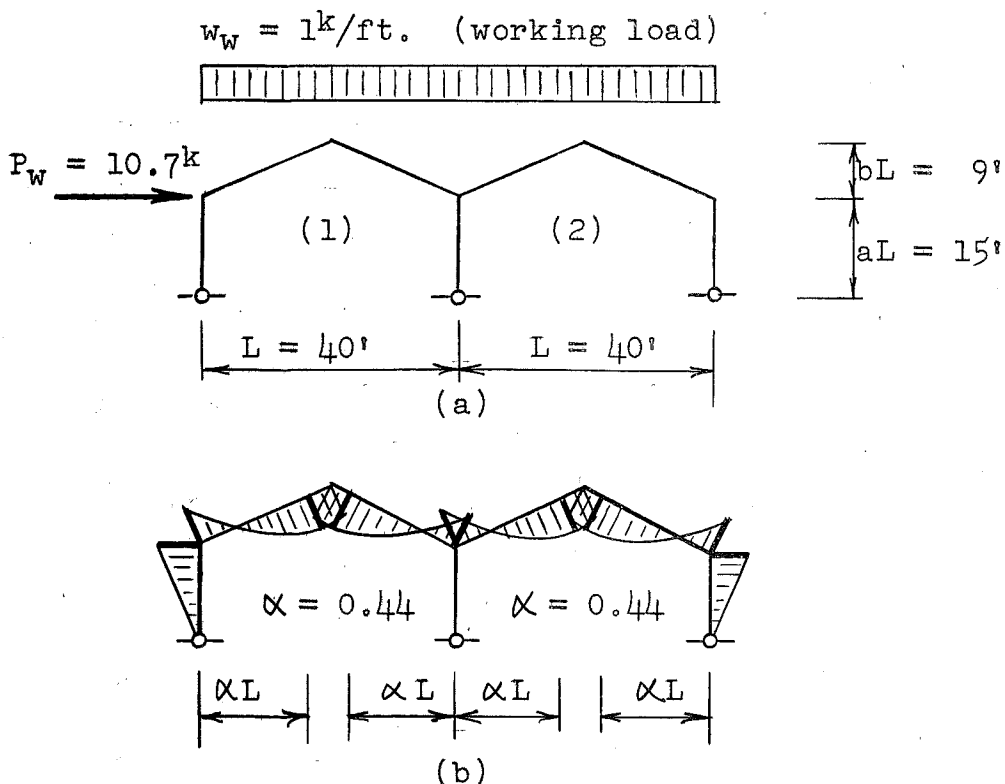
Fig. 39 – DESIGN EXAMPLE NO. 2, INDICATING:
(a) LOADING;   AND
(b) MOMENT DIAGRAM FOR SOLUTION –
NEGLECTING WIND FORCE
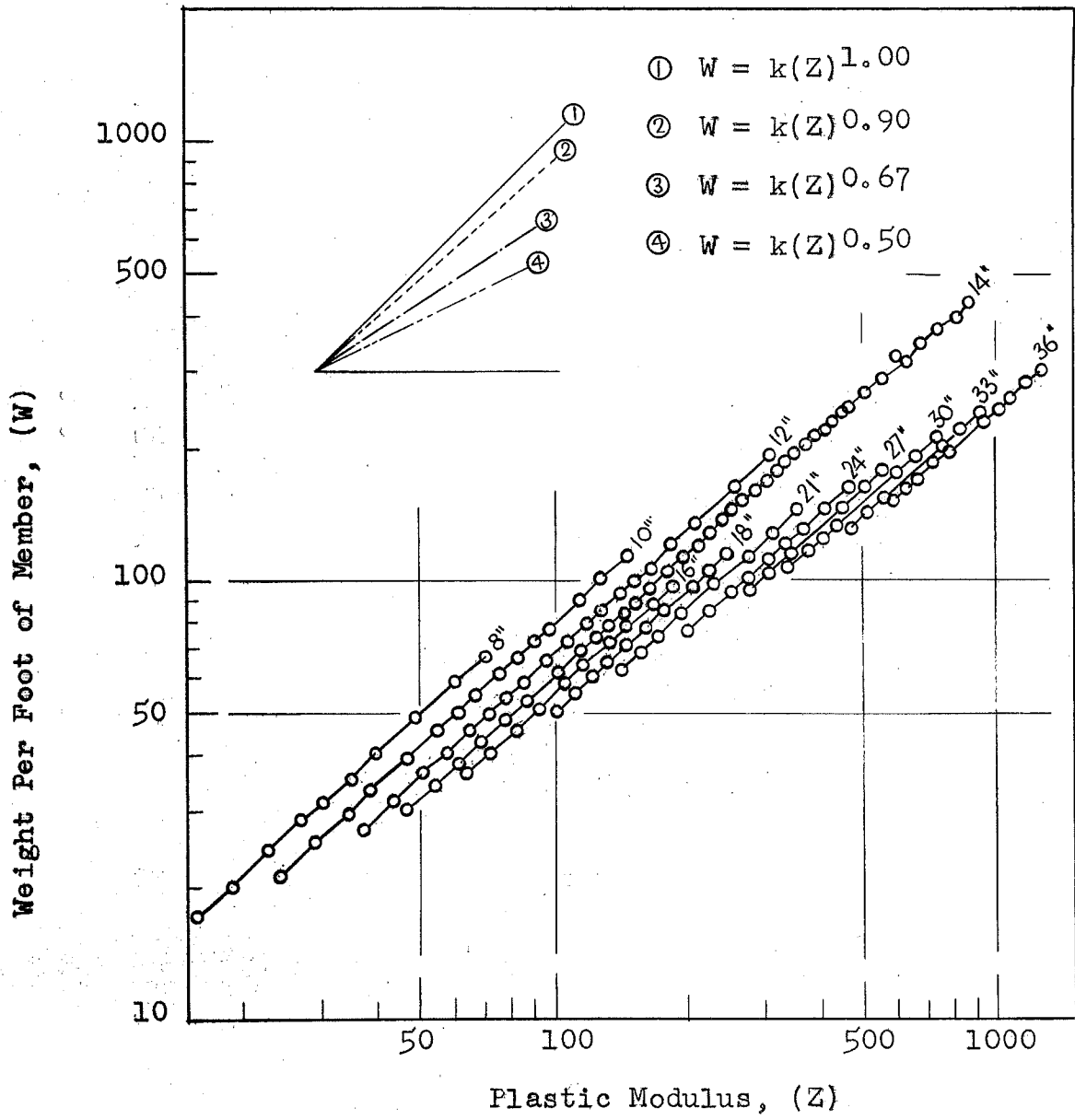
Fig. 37 - WEIGHT OF ROLLED SECTION AS A FUNCTION
OF THE PLASTIC MODULUS

$w_W = 1^k/ft.$

$w_W = 1^k/ft.$

$P_W = 10.7^k$

M_p M_p ① M_p

② M_p M_p

$(1/2)D_1wL^2$ $(1/2)A_2wL^2$

(a)

SOLUTION

$\dfrac{M_p}{wL^2} = (0.0575), D_1 = A_2 = (0.075)$

$\dfrac{M_p}{wL^2}$

SPAN 2

$A_1 = 0.20$

0.05

(b)

Span 1

0 0.1 0.2

D_1

$\dfrac{M_p}{wL^2}$

$D_2 = 0$

0.05

Span 2

0 0.1 0.2

A_2

$\alpha_1 = 0.325$ $\alpha_2 = 0.395$

Moment
Diagram

(c)
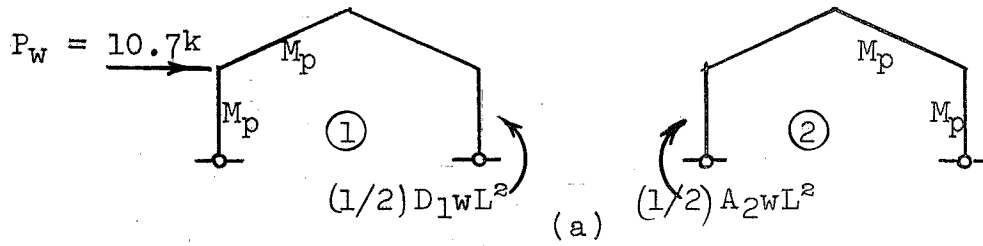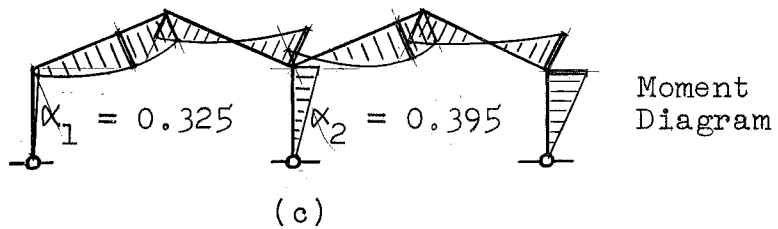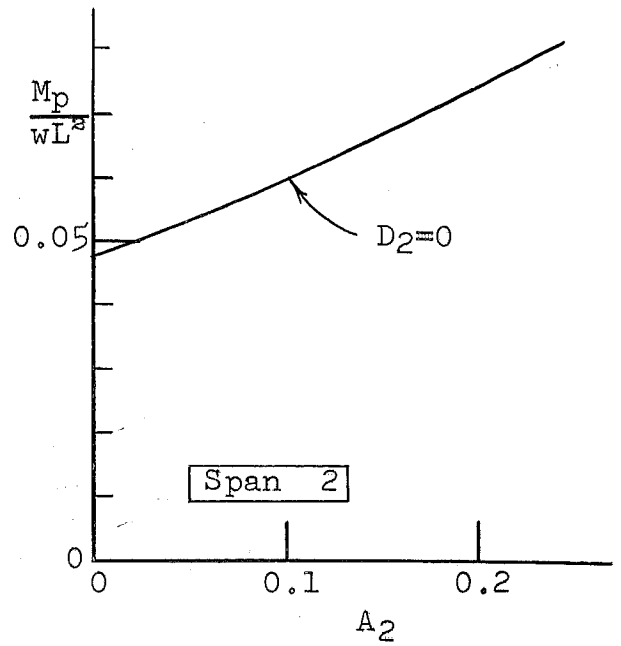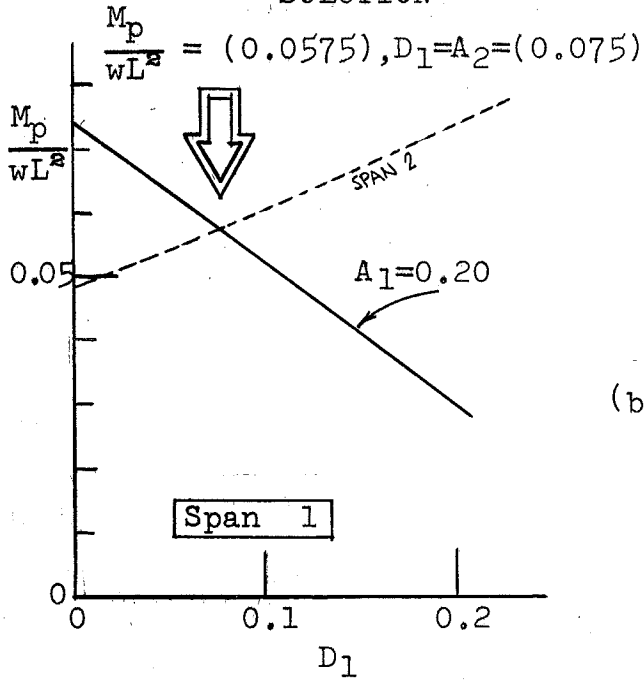
Fig. 40 - DESIGN EXAMPLE NO. 2 - SOLUTION
INCLUDING THE INFLUENCE OF WIND
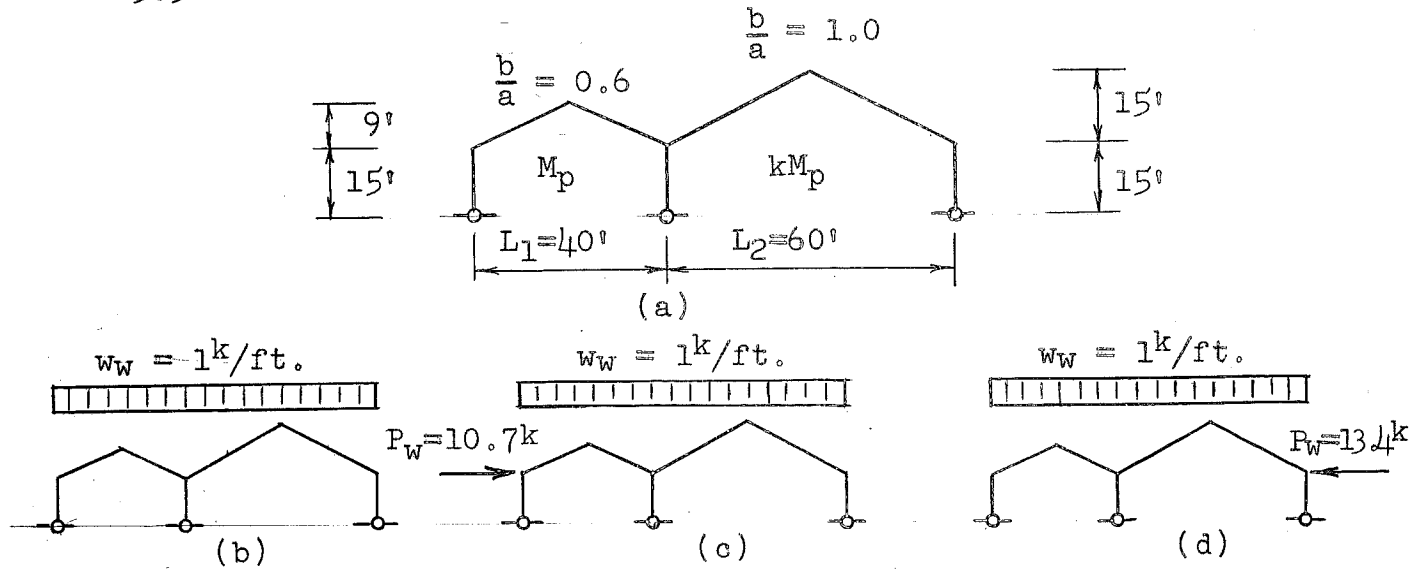
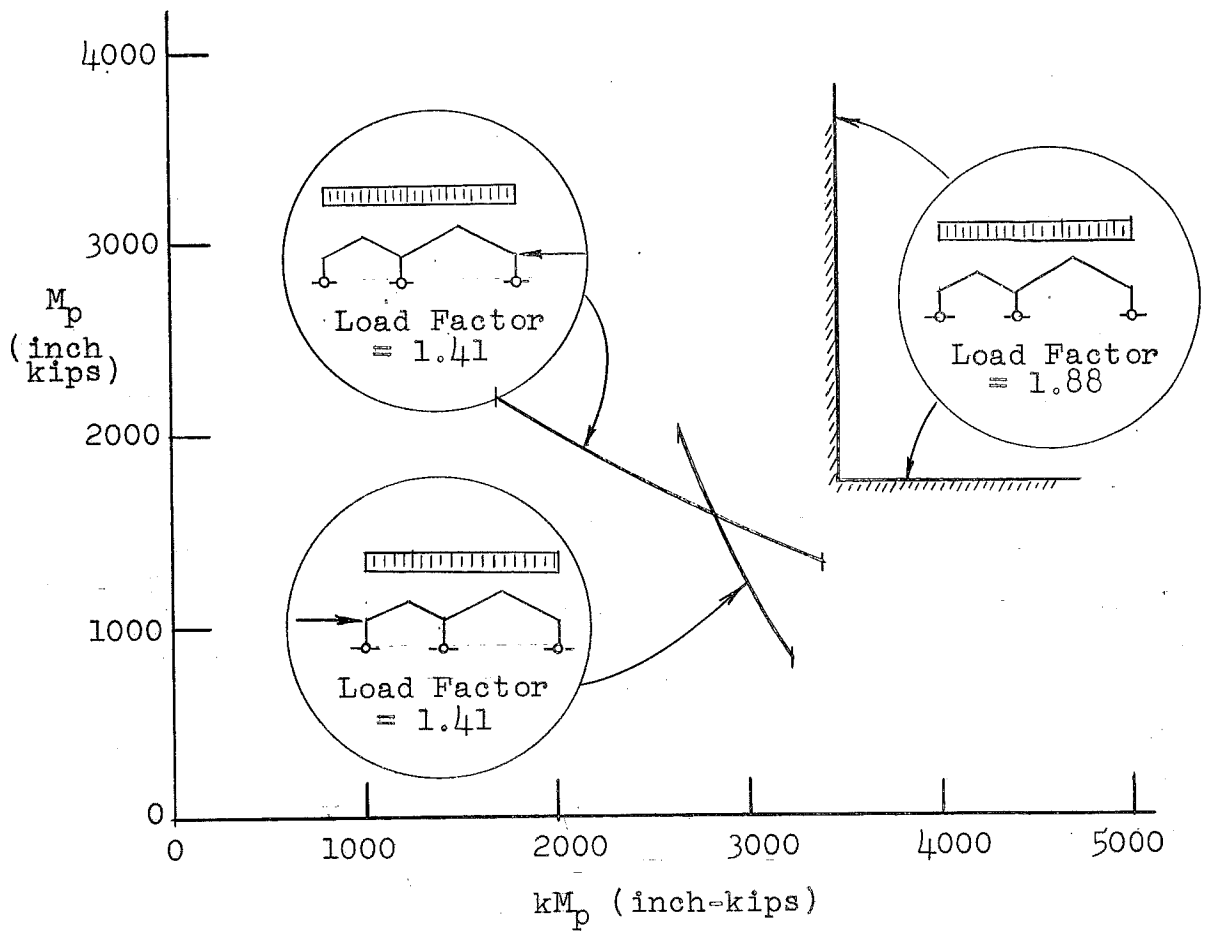Fig. 41 - DESIGN EXAMPLE NO. 3, INDICATING LOADING CONDITIONS

Fig. 42 - SOLUTIONS FOR DESIGN EXAMPLE NO. 3

Span(1)   Span(2)   Span(3)

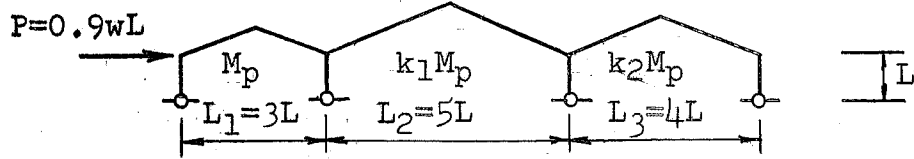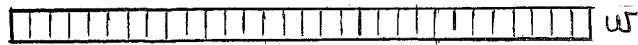$\dfrac{b}{a} = 0.6$ $\dfrac{b}{a} = 1.0$ $\dfrac{b}{a} = 0.8$

$P=0.9wL$

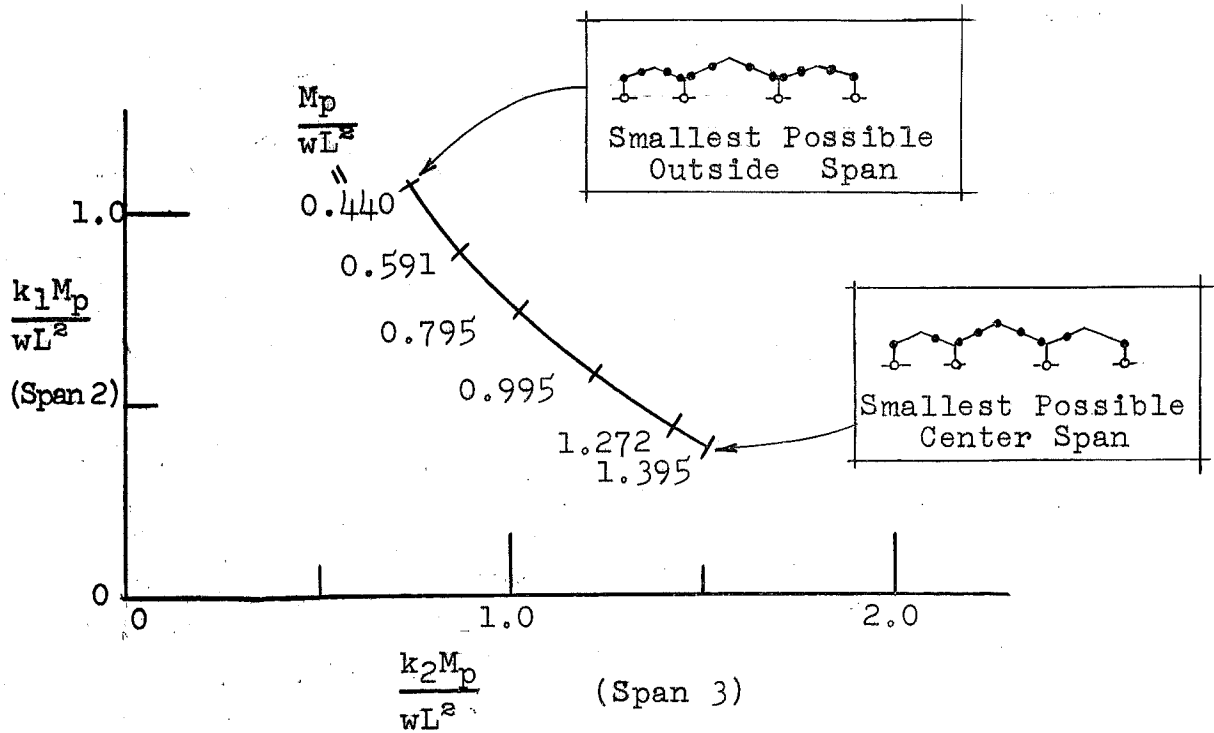M_p $k_1 M_p$ $k_2 M_p$

$L_1=3L$ $L_2=5L$ $L_3=4L$

Fig. 43 - DESIGN EXAMPLE NO. 4

$\dfrac{M_p}{wL^2}$

1.0

$\dfrac{k_1 M_p}{wL^2}$

(Span 2)

0.440

0.591

0.795

0.995

1.272
1.395

Smallest Possible
Outside Span

Smallest Possible
Center Span

0

0   1.0   2.0

$\dfrac{k_2 M_p}{wL^2}$ (Span 3)

Fig. 44 - SOLUTION FOR DESIGN EXAMPLE NO. 4 - NEGLECTING WIND

20

Weight
Function,

$\left(\dfrac{\rho}{wL^3}\right)$

10

Including Influence
Of Center Columns

Neglecting Influence
of Center Columns

Least Weight
Design
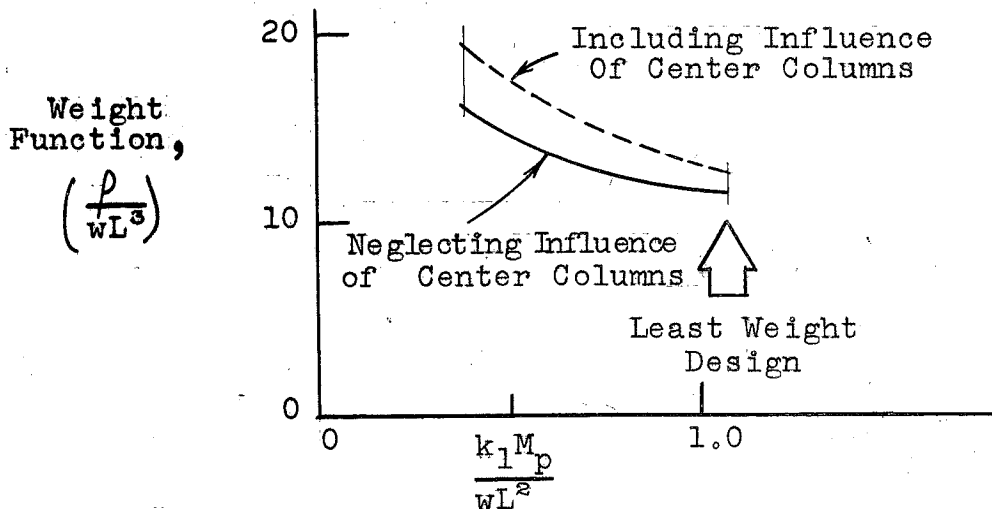
0

0    $\dfrac{k_1 M_p}{wL^2}$ 1.0

Fig. 45 - DESIGN EXAMPLE NO. 4, (NEGLECTING WIND)
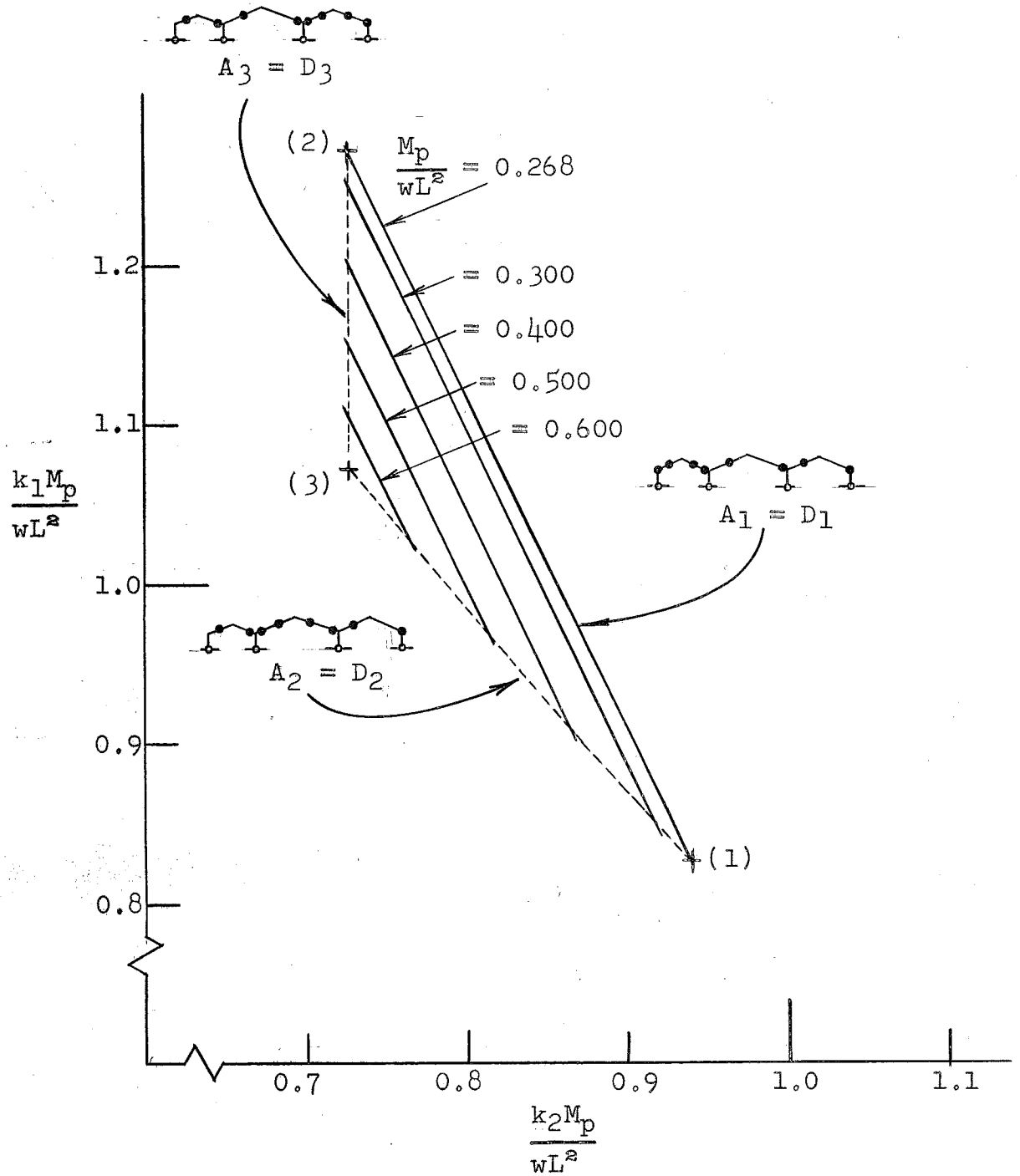WEIGHT FUNCTION VERSUS MEMBER SIZE
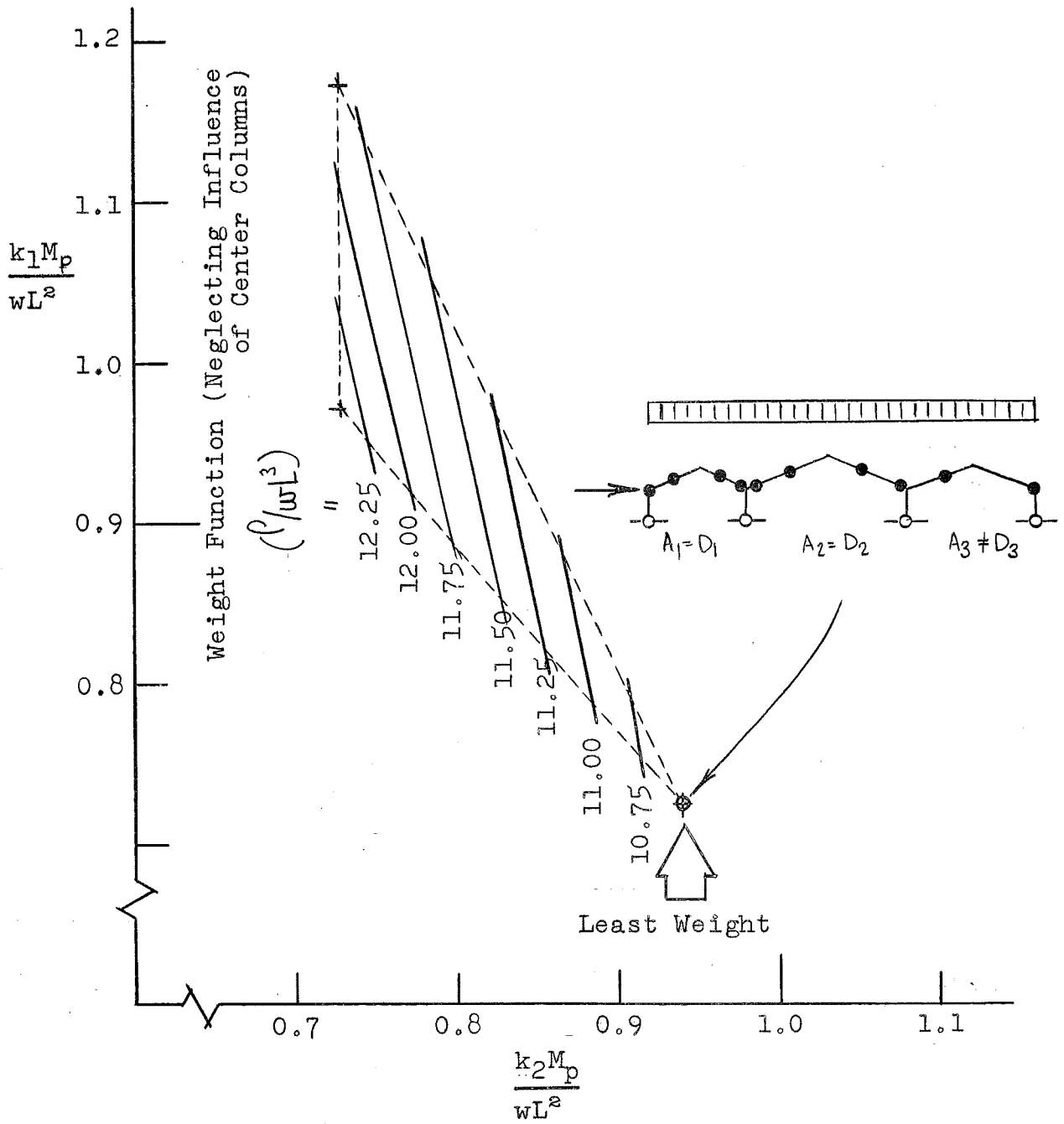
Fig. 46 - DESIGN EXAMPLE NO. 4 - MEMBER SIZES

Fig. 47 - DESIGN EXAMPLE NO. 4 - TOTAL WEIGHT OF STRUCTURE

AS A FUNCTION OF MEMBER SIZES