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BUCKLING OF STIFFENED PANELS IN ELASTIC AND STRAIN-HARDENING RANGE

by

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ABSTRACT

The general instability of a stiffened flat plate under axial compression is a problem of primary buckling consisting of a bending type of deformation of the stiffener about an axis parallel to the plane of the sheet and a twisting type of deformation in which the stiffener rotates about an axis in the plane of the sheet. The dissertation treats the primary buckling of stiffened plates in the strain-hardening range as well as in the elastic range. The secondary or local buckling of the stiffener or plate such as crippling of the stiffener or interfastener buckling of plate are excluded.

From consideration of the yielding process of structural steel it has been shown that the material becomes anisotropic when strained into the strain-hardening range. Introducing appropriate moduli the problem may then be treated as buckling of an orthotropic plate.

Buckling in the elastic range, then, is a particular case of the orthotropic solution with the elastic moduli of steel. The application of the integral equation to the buckling of a stiffened plate simplifies the problem and saves labor and time for computation. The single basic integral equation covers the whole domain of the plate. The

stiffener reactions enter as transverse or twisting loads but do not introduce new boundary conditions.

The required minimum bending rigidity of a stiffener is obtained in both elastic and strain-hardening range. The effect of torsional resistance of a stiffener on buckling strength is investigated for the stiffener with a thin-walled open cross-section.

1. INTRODUCTION

1.1 Field of Application and Reasons for Study

Presently used specifications for ship structures (for example, Rules of American Bureau of Shipping, Rules of Nippon Kaiji Kyokai, Japan, and Rules of Lloyd's Register of Shipping, Great Britain), are based partly on conventional elastic design and mostly on experiences obtained by trial and error. Theoretical studies on ship structures have been carried out especially on the buckling strength of stiffened panels within the elastic range. Consequently the specifications for such stiffened panels as deck or bottom construction of the hull can safely be applied to structures in which the design is based upon theoretical first yield as the limiting condition. However, the wide application of welding caused new problems, such as the corrugation of the bottom shell plating. This type of failure has become extermely common in ships having transversely framed bottom of welded construction since World War II. It has been experienced in almost every ship building country of the world, demanding changes in the specifications for welded panels as well as a change in the type of construction from a transverse to longitudinal stiffening system. The use of longitudinal stiffeners, therefore, becomes more and more important for the simple reason to reduce the thickness of ship plating and still obtain sufficient material to resist the required forces. The stiffeners are generally of Tee-shape. No solutions of stiffened panels considering the torsional and warping rigidity of such stiffeners are available. Hence, there is no definite conclusion on the comparison between the stiffeners with Tee-shaped cross-section and inverted angle stiffeners.

Considerable uncertainty seems to exist concerning the computation of the appropriate cross-sectional properties such as area, moment of inertia, torsional rigidity and warping rigidity. A study of these problems will be made by investigating the interactions between stiffeners and plates.

A second problem of importance is the inelastic buckling of such stiffened panels. Considerable experimental data on the strength of ship plating is available, especially from work done at the Taylor Model Basin. Frankland(1) studied simply supported plates under edge compression. The results showed that ship plating of the usual dimensions will buckle in the inelastic range where the influence of residual stresses is an important factor. Due to the uncertainty of the distribution and magnitude of these residual stresses a solution of the problem becomes practically prohibitive. Therefore, a new approach to the design of a stiffened panel is proposed in this dissertation.

1.2 The Importance of the Plastic Design Method for Stiffened Panel Design

The design of ship plating is presently based on essentially empirical rules. The prescribed dimensions are such that buckling will generally occur in the inelastic range. Therefore an elastic analysis of the buckling strength of stiffened panels is inadequate for the determination of the appropriate cross-sectional properties of stiffeners.

The buckling strength of stiffened panels in the inelastic range depends considerably upon the magnitude and distribution of the residual stresses due to welding. These residual stresses depend on many factors such as plate thickness, type of electrode, welding conditions, welding sequence, magnitude of restraint and so on. There is no hope to control all these factors sufficiently to produce a well defined residual stress state required for a more rigorous analysis. However, yielding of steel wipes out these residual stresses such that at the point of strain-hardening their influence can be neglected. Therefore, it may be advantageous to design the geometric properties of ship plating such that no buckling will occur prior to the point of strain-hardening of the material. In addition, such a design would allow sufficiently large plastic deformations to take place. The structure would be able to redistribute the internal forces under extreme loading conditions. It is also expected that the welding distortions could be better controlled for stiffened panels designed according to such a criterion. Recently such an approach--namely designing the plate elements of structural members such that no buckling will occur prior to strain-hardening--has been found useful in formulating rules controlling the geometric properties of wide flange beams in plastically designed structures(2).

The stiffened panels can carry a considerable amount of loads after buckling in the elastic range. This post-buckling strength of thin plates is well known since many investigators, such as von Kármán, Marguerre and Wagner, studied these problems. The linear buckling theory is no longer useful to analyse the post-buckling behavior of a thin plate and the buckling strength cannot be used as a design criterion for such structures. Since the problem is too complicated to be treated mathematically, in general several empirical formulae have been put forward based on GALCIT and MIT tests(3). On the other hand, plastically designed stiffened panels will fail almost at the point of the buckling. There will be little reserved strength in the post-buckling range because of the rapid decrease in tangent moduli in the strain-hardening range. For such comparatively thick plates as encountered in ship deck constructions, therefore, the linear buckling theory of the plate in the strain-hardening range based on the effective moduli of the material after yielding can furnish the appropriate design criterion which predicts the ultimate strength of the panels. The optimum panel design for a given compressive loading will be obtained when instability of the plate occurs simultaneously with the instability of the stiffeners. Therefore, if the cross-sectional dimensions of a longitudinal stiffener are chosen such that no local buckling of its web or flange will occur prior to the point of strain-hardening, the whole panel will fail simultaneously at the point of strain-hardening. This condition satisfies the mimimum weight design in plastic analysis.

It is therefore proposed to use an approach based on the "Plastic Design Method" in the investigation of the inelastic buckling of longitudinally and transversely stiffened ship plates. Geometric properties will be derived such that the panel will not buckle in any type of instability such as primary or secondary (local or crippling) buckling prior to the point of strain-hardening. A ship designed under such an assumption would be able to sustain considerable inelastic deformations without buckling and hence be able to redistribute the forces by inelastic action.

1.3 Approach to the Problem of Inelastic Instability

According to Timoshenko[4] a plate cannot sustain any load above the yield point of the material: "Experiments show that, when the compressive stress reaches the yield point of the material, ...the plate buckles for any value of the ratio $b/_h$": Reference (4), p. 385. Essentially the same conclusions must be drawn from the equations for inelastic buckling presented by Bleich[5]. Both the column and plate buckling equations .....Equations (21) and (653) in Reference (5) ....contain the factor $\tau = Et/_E$. When the yield stress is reached, the tangent modulus E_t reduces to zero and hence buckling seems to be unavoidable.

Recent investigations at Fritz Engineering Laboratory, Lehigh University[2],[6] are in opposition to this statement. In the elastic range the material exhibits homogeneous and isotropic behavior. At the yielding stress, σ_o, considerable straining takes place without an increase in stress so that the tangent modulus, E_t, seems to reduce to zero. However, the mechanism of yielding is discontinuous, taking place in small slip bands by a sudden jump of strain from the proportional limit to strain-hardening. The slip bands form successively, starting at a weak point and then spread out into the specimen[7]. Therefore, during yielding some of the material is still elastic while the yielded zones have reached the point of strain-hardening. The material within the coupon is therefore heterogeneous. Once all material has been strain-hardened, the stress starts to increase again.

Again, all of the material has identical physical properties and hence is homogeneous. However, slip produces such changes that the material is no longer isotropic, i.e., its properties are now direction dependent. If proper consideration is given to these physical facts the behavior of compression elements of structural steel in the yield and strain-hardening range can be explained and predicted.

Considering the effective moduli of the material after yielding has taken place a solution based on the theory of orthotropic plates has been developed by Haaijer and Thürlimann[(2),(6)] at Fritz Engineering Laboratory, Lehigh University. Tests on the buckling of angles, flanges of WF beams, etc. are in fair agreement with this theory. It has been shown that for certain dimensions such elements can be stressed into the strain-hardening range without buckling.

1.4 Parameters Affecting General Instability of Stiffened Plates

When a stiffened plate buckles the stiffener undergoes bending about an axis parallel to the plane of the sheet and/or twisting around an axis in the plane of the sheet. The term "General Instability" means here a primary failure of a stiffened panel either in the strain-hardening range or in the elastic range. It is assumed that the stiffener cross-section and the compressive stress are constant throughout the length of a plate. Primary failure is defined as any type of buckling in which the cross-sections of stiffeners are translated, rotated, or translated and rotated but not distorted in their own planes as far as the stiffeners are concerned.

When a stiffener is attached to a sheet, the great stiffness of the sheet in its own plane causes the axis of rotation to lie in the plane of the sheet. Therefore the torsional properties of a stiffener with thin-walled open cross-section such as Tee-shape or inverted angle must be modified by the enforcement imposed by the plate. The torsional resistance of a stiffener consists of two parts,

(a) St. Venant's Torsion (G_tK)

(b) Warping Torsion (E_tW)

where G_t is effective shear modulus, K is St. Venant's torsion constant, E_t is tangent modulus and W is the warping constant.

The bending of a stiffener is resisted by the membrane action of the sheet. The condition of the continuity of stress along the intersection of stiffeners and plate gives a certain amount of effective width of plate, which depends on the side ratio $\alpha = a/_b$ of the plate and the boundary conditions of the plate for membrane action.

The effective moduli of the material in the strain-hardening range consist of three effective moduli (E_x, E_y, G_t) and two coefficients of dilation (ν_x, ν_y) in the x- and y-direction respectively. These constants depend on the type of steel and its history.

The cross-sections of stiffeners may generally be classified as

(a) Solid

(b) Thin-walled

(i) Open section

(ii) Closed section

In this dissertation only thin-walled open cross-sections will be considered, however, the results can be easily modified by taking proper values for the torsional rigidities for thin-walled closed cross-sections. For solid stiffeners with rectangular cross-section the torsional rigidities are negligible. For thin-walled closed cross-sections such as hat-shaped stiffeners, the warping rigidity can be neglected as compared to St. Venant's torsional rigidity. The combined type of cross-sections, consisting of thin-walled open and closed cross-sections, such as inverted Y-shaped stiffeners as in airplane structures can also be obtained by taking their own section properties.

1.5 Historical Review of Investigation on Buckling of Stiffened Plates

A considerable number of references[8] are available on the buckling strength of a stiffened plate since Timoshenko solved the problem by using his energy method in 1921[4]. The concept of the required minimum bending rigidity of the stiffener was also considered first by Timoshenko. Barbre[9],[10] investigated the effect of the position of a longitudinal stiffener by using the differential equation. The buckling strength of rectangular plates with boundary conditions different from simply supported edges have been studied by Melan, Rendulic and Miles. The effect of the torsional rigidity of the supporting elastic flanges was discussed first by Chwalla[11], who showed the considerable influence of this torsional rigidity upon the critical stress of the plate. Windenburg[12] solved the buckling of a panel, simply supported along one longitudinal edge and elastically restrained by a flange resisting lateral deflection and twist along the other edge. The results apply to the stability of web plate of a Tee-shaped stiffener.

Zahorski[13] was first in suggesting that the optimum panel stress for a given compressive-loading was obtained when instability of the plate occurred simultaneously with the instability of the stiffeners. The crippling or local buckling of stiffener sections were summarized in charts and tables by Becker[14] for elastic buckling. The local buckling of wide flange shapes in the strain-hardening range was studied by Haaijer and Thürlimann[2] and the geometric conditions of flange and web plates were proposed in order to prevent local buckling of wide flange shapes prior to the point of strain-hardening.

The flexural-torsional buckling of column with thin-walled open cross-sections was initiated by Wagner[15], followed by Kappus[16]. The effect of an enforced axis of rotation on the flexural-torsional buckling of columns with symmetric thin-walled open cross-section was studied by Lundquist[17]. Goodier[18],[19] clarified the scope of Wagner's formula for torsional buckling and found a general theory simpler than that of Kappus. Simple results were derived for further problems of a bar attached to a flexible sheet, and a bar with an enforced axis of rotation. The basic differential equations for a stiffener with arbitrary thin-walled open cross-section under axial compression derived by Goodier will be used in this dissertation with the modification for an enforced axis of rotation in order to describe the behavior of a longitudinal stiffener.

2. PRELIMINARY STUDIES FOR INTEGRAL EQUATION ON BUCKLING STRENGTH OF AN ORTHOTROPIC PLATE

2.1 The Advantage in Application of Integral Equations to the Stability of Stiffened Plates

Among different mathematical approaches to engineering problems the integral equation is seldom used compared to such methods as differential equation, calculus of variation, etc. The reason for this situation may be due to the fact that many problems can be solved by means of differential equations without any knowledge of integral equations. Furthermore, mathematically the integral equation seems to offer no definite advantages. However, for some problems the method by integral equations can save much time and labor as for example, the buckling of stiffened plates. The problem can be treated by solving the differential equation for each isolated panel, and connecting the solutions by using the boundary conditions along the stiffeners. However, the boundary conditions along the stiffeners consist of

(a) Continuity of deflections of plate and stiffener,

$$w_i = w_{i+1} = w_{is}$$

(b) Continuity of slopes of plate,

$$\frac{\partial w_i}{\partial y} = \frac{\partial w_{i+1}}{\partial y}$$

(c) Equilibrium of shear forces,

$$EI_i \frac{\partial^4 w_{is}}{\partial x^4} = - D_i \left\{ \frac{\partial^3 w_i}{\partial y^3} + (2-\nu) \frac{\partial^3 w_i}{\partial x^2 \partial y} \right\}$$

$$+ D_{i+1} \left\{ \frac{\partial^3 w_{i+1}}{\partial y^3} + (2-\nu) \frac{\partial^3 w_{i+1}}{\partial x^2 \partial y} \right\} - \sigma x A i \frac{\partial^2 w_{is}}{\partial x^2}$$

(d) Equilibrium of moments,

$$EW_i \frac{\partial^5 w_{is}}{\partial x^4 \partial y} - (GK_i - \sigma_x A_i R_i^2) \frac{\partial^3 w_{is}}{\partial x^2 \partial y}$$

$$= - D_i \left(\frac{\partial^2 w_i}{\partial y^2} + \nu \frac{\partial^2 w_i}{\partial x^2} \right) + D_{i+1} \left(\frac{\partial^2 w_{i+1}}{\partial y^2} + \nu \frac{\partial^2 w_{i+1}}{\partial x^2} \right)$$

where

w_i = Deflection of plate in i-th bay

w_{is} = Deflection of i-th stiffener

EI_i, EW_i, GK_i, and A_i = Bending rigidity, Warping rigidity, St. Venant's torsional rigidity and Area of i-th stiffener respectively

D_i = Bending rigidity of plate in i-th bay

$R_i^2 = I_{pi}/A_i + d_i^2$

I_{pi} = Polar moment of inertia of the stiffener cross-section

di = Distance between centroid and enforced axis of rotation

σ_x = Compressive stress in the direction of longitudinal stiffeners (Fig. 1)

For transverse stiffeners, similar boundary conditions can be formulated along each one of stiffeners excluding the terms on the effect of axial compression σ_x in conditions (c) and (d) (Fig. 2). The solution of the differential equation on buckling of plate,

$$D\nabla^4 w_i + N_x \frac{\partial^2 w_i}{\partial x^2} = 0$$

with

$$N_x = \sigma_x h$$

together with the above boundary conditions becomes very involved. Therefore, most solutions were obtained by neglecting the effect of torsional rigidity of the stiffener. As seen from this example, if the problem is piece-wise continuous, the method by differential equation becomes practically unmanageable. However, the method by integral equation can treat the whole region of a stiffened plate in one equation. It is not necessary to connect the individual panel by continuity conditions. Moreover, the solution by integral equation can satisfy all boundary conditions along the stiffeners automatically provided Green's functions (or influence functions) for the deflection produced by a vertical load and a concentrated moment are known. For the effect of a stiffener can be expressed by its vertical reaction and its twisting resistance. Therefore, the exact solution of the buckling strength of longitudinally and transversely stiffened plates with stiffeners having an arbitrary thin-walled open cross-section can be obtained in a practical manner only by using the integral equation.

2.2 Green's Functions for Deflection of an Orthotropic Plate due to Unit Load and/or Concentrated Unit Moment

The differential equation of bending of an orthotropic plate under transverse loads can be written in form of Cartesian coordinates as follows[20]

$$K_x \frac{\partial^4 w}{\partial x^4} + 2K_{xy} \frac{\partial^4 w}{\partial x^2 \partial y^2} + K_y \frac{\partial^4 w}{\partial y^4} = q\ (x,y) \qquad (1)$$

where K_x, K_{xy} and K_y are the rigidities of the orthotropic plate. Green's function for deflection due to a unit load corresponds to the solution of Equation (1) with the condition $qdA=1$, where dA is an infinitesimally small area of the plate surface, $dA=d\xi . d\eta$ (Fig. 3-1). The load intensity q is assumed constant over the small area dA. Therefore $q\ (x.y)$ in the Equation (1) is expressed as

$$q(x.y) = \frac{1}{dA} \text{ for } \quad \xi - \frac{d\xi}{2} \leqq x \leqq \xi + \frac{d\xi}{2}$$

$$\eta - \frac{d\eta}{2} \leqq y \leqq \eta + \frac{d\eta}{2} \qquad (2)$$

$$= 0 \text{ for entire plate surface except } dA$$

For the plate with four edges simply supported the partially distributed load $q\ (x.y)$ can be expanded into a double

Fourier series according to Navier's method

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{3}$$

where

$$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \tag{4}$$

Substituting Equation (2) into Equation (4), the Fourier coefficient Amn yields

$$a_{mn} = \frac{4}{ab}\cdot\frac{1}{dA}\int_{\xi-\frac{d\xi}{2}}^{\xi+\frac{d\xi}{2}} \sin\frac{m\pi x}{a}\,dx \int_{\eta-\frac{d\eta}{2}}^{\eta+\frac{d\eta}{2}} \sin\frac{n\pi y}{b}\,dy$$

$$= \frac{4}{ab}\cdot\frac{\sin\frac{m\pi\xi}{a}\sin\frac{m\pi d\xi}{2a}}{\frac{m\pi d\xi}{2a}}\cdot\frac{\sin\frac{n\pi\eta}{b}\sin\frac{n\pi d\eta}{2b}}{\frac{n\pi d\eta}{2b}}$$

where $\lim\limits_{d\xi\to 0}\dfrac{\sin\frac{m\pi d\xi}{2a}}{\frac{m\pi d\xi}{2a}} = 1$, $\lim\limits_{d\eta\to 0}\dfrac{\sin\frac{n\pi d\eta}{2b}}{\frac{n\pi d\eta}{2b}} = 1$

since for a concentrated unit load $\lim\limits_{dA\to 0} q\,dA = 1$.

Therefore

$$a_{mn} = \frac{4}{ab}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b} \tag{5}$$

The deflection w in Equation (1) can also be assumed as

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6}$$

for the plate with four edges simply supported. Substituting Equations (3), (5) and (6) into Equation (1), the coefficient bmn yield

$$b_{mn} = \frac{\frac{4}{ab}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}}{K_x\left(\frac{m\pi}{a}\right)^4 + 2K_{xy}\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + K_y\left(\frac{n\pi}{b}\right)^4} \tag{7}$$

From Equations (6) and (7), therefore, Green's function for the deflection due to unit load is expressed as

$$G(x,y;\xi,\eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varphi_{mn}(x,y)\cdot\varphi_{mn}(\xi,\eta)}{\lambda_{mn}^2} \tag{8}$$

where

$$\varphi_{mn}(x,y) = \frac{2}{\sqrt{ab}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\varphi_{mn}(\xi,\eta) = \frac{2}{\sqrt{ab}}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

$$\lambda_{mn}^2 = K_x\left(\frac{m\pi}{a}\right)^4 + 2K_{xy}\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + K_y\left(\frac{n\pi}{b}\right)^4 \tag{9}$$

Green's function for deflection of an orthotropic plate due to unit moment can be formulated by superimposing two deflections w_1 and w_2, where w_1 is the deflection of plate under concentrated load P acting upwards at the point $(\xi-\frac{\varepsilon}{2}, \eta)$ and w_2 is the deflection due to P acting downwards at the point $(\xi+\frac{\varepsilon}{2},\eta)$ as shown in Fig. 3-2. The moment M_x in x-direction is expressed by

$$M_x = \lim_{\varepsilon\to 0} P\cdot\varepsilon = 1 \tag{10}$$

From the definition of Green's function for a unit load the deflections w_1 and w_2 are written as

$$w_1 = -P\cdot G(x,y;\xi-\frac{\varepsilon}{2},\eta)$$

$$w_2 = P\cdot G(x,y;\xi+\frac{\varepsilon}{2},\eta) \tag{11}$$

By superposition of these two deflections and from Equations (8), (9) and (10), the total deflection w (x,y) due to moment M_x yields

$$w(x,y) = M_x\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{2\varphi_{mn}(x,y)}{\sqrt{ab}\,\lambda_{mn}^2}\sin\frac{n\pi\eta}{b}\lim_{\varepsilon\to 0}\frac{2\cos\frac{m\pi\xi}{a}\sin\frac{m\pi\varepsilon}{2a}}{\varepsilon}$$

where

$$\lim_{\varepsilon\to 0}\frac{2\cos\frac{m\pi\xi}{a}\sin\frac{m\pi\varepsilon}{2a}}{\varepsilon} = \frac{m\pi}{a}\cos\frac{m\pi\xi}{a}$$

Therefore

$$w(x,y) = M_x\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m\pi}{a}\cdot\frac{2\varphi_{mn}(x,y)\cos\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}}{\sqrt{ab}\,\lambda_{mn}^2}$$

For unit moment $M_x = 1$ at the point (ξ, η)

$$w(x,y) = \frac{\partial G}{\partial \xi}(x,y;\xi,\eta) \equiv \overline{G}_\xi(x,y;\xi,\eta) \tag{12}$$

Similarly for unit moment in y-direction at the point (ξ,η),

$$w(x,y) = \frac{\partial G}{\partial \eta}(x,y;\xi,\eta) \equiv \overline{G}_\eta(x,y;\xi,\eta) \tag{13}$$

where

$$\overline{G}_\xi(x,y;\xi,\eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varphi_{mn}(x,y)\cdot\overline{\varphi}_{mn}(\xi,\eta)_\xi}{\lambda^2_{mn}}$$

$$\overline{G}_\eta(x,y;\xi,\eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varphi_{mn}(x,y)\cdot\overline{\varphi}_{mn}(\xi,\eta)_\eta}{\lambda^2_{mn}} \tag{14}$$

and

$$\overline{\varphi}_{mn}(\xi,\eta)_\xi = \frac{\partial \varphi_{mn}}{\partial \xi}(\xi,\eta)$$

$$\overline{\varphi}_{mn}(\xi,\eta)_\eta = \frac{\partial \varphi_{mn}}{\partial \eta}(\xi,\eta)$$

Therefore Green's functions for deflection of an orthotropic plate due to a unit load and/or unit moment in x- and y-direction are defined by Equations (8), (12) and (13) respectively.

2.3 Effective Width of an Orthotropic Plate with Stiffeners under Bending

At the instant of buckling of a plate in the strain-hardening range, the plate has orthotropic properties characterized by the effective moduli in the range of strain-hardening. The stress-strain relations are given by the incremental theory of plasticity with a modification of the moduli as presented in Reference(2). Therefore, the effect of an orthotropic plate on the bending rigidity of a stiffener can be analysed by using the following assumptions for infinitesimal deformations.

(a) Ordinary bending theory of beam for stiffeners.

(b) Transverse bending rigidity of plate neglected as compared to rigidity of stiffener.

(c) Transverse stiffeners remain elastic, longitudinal stiffeners are compressed into strain-hardening range.

(d) Over the effective width, b_e, as shown in Fig. 4, the stress increment σ^* is constant such that a stiffener with an effective flange width b_e will be equivalent to the actual stiffener if

$$2 b_e \sigma^* h = h \int_{-a/2}^{a/2} (d\sigma_y)\, dx \tag{15}$$

The incremental stress-strain relations are given as follows[6].

$$d\varepsilon_x = \frac{1}{E_x} d\sigma_x - \frac{\nu_y}{E_y} d\sigma_y$$

$$d\varepsilon_y = \frac{1}{E_y} d\sigma_y - \frac{\nu_x}{E_x} d\sigma_x$$

$$d\gamma_{xy} = \frac{1}{G_t} d\tau_{xy} \tag{16}$$

The equations of equilibrium for stress increments are

$$\frac{\partial}{\partial x}(d\sigma_x) + \frac{\partial}{\partial y}(d\tau_{xy}) = 0$$

$$\frac{\partial}{\partial x}(d\tau_{xy}) + \frac{\partial}{\partial y}(d\sigma_y) = 0 \tag{17}$$

From Equation (17) the stress function ϕ is introduced such that:

$$d\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$$

$$d\sigma_y = \frac{\partial^2 \phi}{\partial x^2}$$

$$d\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \tag{18}$$

The compatibility condition for strain increments is

$$\frac{\partial^2}{\partial x^2}(d\varepsilon_y) + \frac{\partial^2}{\partial y^2}(d\varepsilon_x) - \frac{\partial^2}{\partial x \partial y}(d\gamma_{xy}) = 0 \tag{19}$$

Substituting Equations (16) and (18) into Equation (19), the differential equation for the stress function ϕ can be expressed as

$$D_x \frac{\partial^4 \phi}{\partial x^4} - 2F \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad (20)$$

where

$$D_x = \frac{E_x}{1 - \nu_x \nu_y}$$

$$D_y = \frac{E_y}{1 - \nu_x \nu_y}$$

$$2F = D_x \left(\nu_y - \frac{E_y}{G_t} \right) + \nu_x D_y \qquad (21)$$

Transverse Stiffener:

The bending moment acting on a transverse stiffener can be expressed by a general term of Fourier series as

$$M_y = M \sin \frac{n \pi y}{b}$$

The corresponding bending stress of stiffener σ_y is given by

$$\sigma_y = f(M) \sin \frac{n \pi y}{b}$$

The condition of continuity for normal stress along a stiffener is

$$d\sigma_y \,(\text{stiffener}) = d\sigma_y \,(\text{Plate}) = \sigma^*$$

Hence $d\sigma_y$, stress increment of plate in y-direction, should also vary as sine function, therefore, the stress function ϕ in Equation (20) can be expressed as

$$\phi = X(x) \sin \frac{n\pi y}{b} \tag{22}$$

Substituting Equation (22) into Equation (20),

$$D_x \frac{d^4 X}{dx^4} + 2F\left(\frac{n\pi}{b}\right)^2 \frac{d^2 X}{dx^2} + D_y\left(\frac{n\pi}{b}\right)^4 X = 0 \tag{23}$$

The solution of Equation (23) is given by

$$\begin{aligned} X = &\left[A \cosh \frac{1}{\sqrt{2}}\left(\frac{n\pi}{b}\right)\sqrt{q^2 - p^2}\, x \right. \\ &\qquad \left. + B \sinh \frac{1}{\sqrt{2}}\left(\frac{n\pi}{b}\right)\sqrt{q^2 - p^2}\, x \right] \\ &\times \left[C \cos \frac{1}{\sqrt{2}}\left(\frac{n\pi}{b}\right)\sqrt{q^2 + p^2}\, x \right. \\ &\qquad \left. + D \sin \frac{1}{\sqrt{2}}\left(\frac{n\pi}{b}\right)\sqrt{q^2 + p^2}\, x \right] \end{aligned} \tag{24}$$

where

$$p^2 = \frac{F}{D_x}, \quad q^2 = \sqrt{\frac{D_y}{D_x}} \quad ; \quad q^2 > p^2$$

The boundary conditions for the plate are given as

$$\begin{aligned} x = 0 \ ; &\quad u = 0, \quad d\sigma_y = \sigma^* \sin \frac{n\pi y}{b} \\ x = \pm a/2 \ ; &\quad u = 0, \quad d\tau_{xy} = 0 \end{aligned} \tag{25}$$

where u is the incremental displacement in x-direction.

From Equations (16), (22), (24) and (25) the integral constants of integration follow:

$$AC = 2\,\frac{\sigma^*}{\mu}\cdot\left(\frac{b}{n\pi}\right)^2$$

$$AD = -2\,\frac{\omega_1}{\omega_2}\cdot\frac{1}{\psi}\cdot\frac{\rho\,\sigma^*}{\mu}\cdot\left(\frac{b}{n\pi}\right)^2$$

$$BC = -2\,\frac{\rho\,\sigma^*}{\mu}\cdot\left(\frac{b}{n\pi}\right)^2$$

$$BD = 2\,\frac{\omega_1}{\omega_2}\cdot\frac{1}{\psi}\cdot\frac{\rho^2\sigma^*}{\mu}\cdot\left(\frac{b}{n\pi}\right)^2 \qquad (26)$$

where

$$\mu = \left(1 + 2\rho^2\frac{1}{\psi}\right)\cdot\left(g^2 - p^2\right) - \left(g^2 + p^2\right)$$

$$\rho = \frac{\sinh\frac{\omega_1 a}{2}\cosh\frac{\omega_1 a}{2} + \frac{\omega_2}{\omega_1}\cdot\psi\cdot\sin\frac{\omega_2 a}{2}\cos\frac{\omega_2 a}{2}}{\cosh^2\frac{\omega_1 a}{2}\sin^2\frac{\omega_2 a}{2} + \sinh^2\frac{\omega_1 a}{2}\cos^2\frac{\omega_2 a}{2}}$$

$$\omega_1 = \frac{1}{\sqrt{2}}\left(\frac{n\pi}{b}\right)\sqrt{g^2 - p^2}$$

$$\omega_2 = \frac{1}{\sqrt{2}}\left(\frac{n\pi}{b}\right)\sqrt{g^2 + p^2}$$

$$\psi = \frac{g^2 - \nu_y}{g^2 + \nu_y} = \frac{\sqrt{D_x D_y} - \nu_y D_x}{\sqrt{D_x D_y} + \nu_y D_x} \qquad (27)$$

Substituting the stress increment $d\sigma_y$ obtained from Equations (18), (22), (24), (26) and (27) into Equation (15), the final result for the effective width for a transverse stiffener can be expressed for n=1 as

$$\frac{b_e}{b} = \frac{2\,b\,\omega_1}{\pi^2 \mu\, \psi}\Big[(\psi+\rho^2)\sinh\frac{\omega_1 a}{2}\cos\frac{\omega_2 a}{2}$$

$$-\left(\frac{\omega_1}{\omega_2}-\frac{\omega_2}{\omega_1}\psi\right)\rho\sinh\frac{\omega_1 a}{2}\sin\frac{\omega_2 a}{2}$$

$$-(1+\psi)\,\rho\left(\cosh\frac{\omega_1 a}{2}\cos\frac{\omega_2 a}{2}-1\right)$$

$$+\left(\frac{\omega_1}{\omega_2}\rho^2-\frac{\omega_2}{\omega_1}\psi\right)\cosh\frac{\omega_1 a}{2}\sin\frac{\omega_2 a}{2}\Big] \qquad (28)$$

For structural steel, for example A-7 steel, the following values are available(2)

$$D_x = 3{,}000 \text{ ksi}$$

$$D_y = 32{,}800 \text{ ksi}$$

$$G_t = 2{,}400 \text{ ksi}$$

$$\nu_x D_y = \nu_y D_x = 8{,}100 \text{ ksi} \qquad (29)$$

Therefore from Equation (29) the nondimensional parameters in Equation (28) depend only on side ratio α of the plate, that is,

$$\psi = 0.100983$$

$$\frac{\omega_1 a}{2} = 1.8856\,\alpha$$

$$\frac{\omega_2 a}{2} = 2.1425\,\alpha$$

$$\frac{\omega_1}{\omega_2} = 0.879007$$

$$\frac{\omega_2}{\omega_1}\psi = 0.114883$$

$$\frac{2b\omega_1}{\pi^2\mu\psi} = \frac{1}{7.542438\,\rho^2 - 0.112055} \tag{30}$$

Equation (28) is plotted in Fig. 5 for various values of the side ratio $\alpha = a/b$ together with the elastic case. For the case of an infinitely long strip with width b the effective width b_e yields

$$\frac{b_e}{b} = \frac{b\omega_1}{\pi^2}\cdot\frac{1+\psi}{\frac{\omega_1}{\omega_2}\sqrt{q^4 - p^4} - p^2\psi} \tag{31}$$

Substituting the values of Equation (29) into Equation (31),

$$\frac{b_e}{b} = 0.14817$$

Longitudinal Stiffener:

Similarly the effective width b_e for a longitudinal stiffener can be expressed as

$$\frac{b_e}{a} = \frac{2a\bar{\omega}_1}{\pi^2 \bar{\mu} \bar{\psi}} \Big[(\bar{\psi} + \bar{\rho}^2) \sinh \frac{\bar{\omega}_1 b}{2} \cos \frac{\bar{\omega}_2 b}{2}$$

$$- \left(\frac{\bar{\omega}_1}{\bar{\omega}_2} - \frac{\bar{\omega}_2}{\bar{\omega}_1} \bar{\psi} \right) \bar{\rho} \sinh \frac{\bar{\omega}_1 b}{2} \sin \frac{\bar{\omega}_2 b}{2}$$

$$+ (1 + \bar{\psi}) \bar{\rho} \left(\cosh \frac{\bar{\omega}_1 b}{2} \cos \frac{\bar{\omega}_2 b}{2} - 1 \right)$$

$$+ \left(\frac{\bar{\omega}_1}{\bar{\omega}_2} \bar{\rho}^2 - \frac{\bar{\omega}_2}{\bar{\omega}_1} \bar{\psi} \right) \cosh \frac{\bar{\omega}_1 b}{2} \sin \frac{\bar{\omega}_2 b}{2} \Big] \tag{32}$$

where

$$\bar{p}^2 = \frac{F}{D_y}, \qquad \bar{q}^2 = \sqrt{\frac{D_x}{D_y}}$$

$$\bar{\rho} = \frac{\sinh \frac{\bar{\omega}_1 b}{2} \cosh \frac{\bar{\omega}_1 b}{2} + \frac{\bar{\omega}_2}{\bar{\omega}_1} \bar{\psi} \sin \frac{\bar{\omega}_2 b}{2} \cos \frac{\bar{\omega}_2 b}{2}}{\cosh^2 \frac{\bar{\omega}_1 b}{2} \sin^2 \frac{\bar{\omega}_2 b}{2} + \sinh^2 \frac{\bar{\omega}_1 b}{2} \cos^2 \frac{\bar{\omega}_2 b}{2}}$$

$$\bar{\omega}_1 = \frac{1}{\sqrt{2}} \left(\frac{n\pi}{a} \right) \sqrt{\bar{q}^2 - \bar{p}^2}$$

$$\bar{\omega}_2 = \frac{1}{\sqrt{2}} \left(\frac{n\pi}{a} \right) \sqrt{\bar{q}^2 + \bar{p}^2}$$

$$\bar{\mu} = (1 + 2\bar{\rho}^2 / \bar{\psi})(\bar{q}^2 - \bar{p}^2) - (\bar{q}^2 + \bar{p}^2)$$

$$\bar{\psi} = \frac{\sqrt{D_x D_y} - \nu_x D_y}{\sqrt{D_x D_y} + \nu_x D_y}$$

For the case of an infinitely wide strip with length a, the effective width b_e yields

$$\frac{b_e}{a} = \frac{a\bar{\omega}_1}{\pi^2} \cdot \frac{1 + \bar{\psi}}{\frac{\bar{\omega}_1}{\bar{\omega}_2}\sqrt{\bar{q}^4 - \bar{p}^4} - \bar{p}^2\bar{\psi}} = 0.500$$

The Equation (32) is plotted in Fig. 6 with the elastic case.

3. STIFFENED PLATE WITH TRANSVERSE AND/OR LONGITUDINAL STIFFENERS HAVING A SYMMETRIC THIN-WALLED OPEN CROSS-SECTION

3.1 Behavior of a Stiffener with Symmetric Thin-Walled Open Cross-Section

3.1.1 Longitudinal Stiffeners

The deformation of a centrally compressed bar attached to a plate is considered. If the bar is symmetric about an axis perpendicular to the plate passing through the point of attachment, it is convenient to formulate the equations of equilibrium of the bar by using centroidal principal axes of the system as shown in Fig. 7. Denoting the components of displacements of the shear center S by v_s and w_s in y- and z-direction respectively, the coordinates of shear center by y_s^L and z_s^L, the coordinates of the point of attachment by y_A^L and z_A^L and the angle of rotation with respect to the axis of connection by φ^L, (clockwise positive), the equation of equilibrium of a stiffener can be written as[21] (Reference, p. 598 Equation (67) and (69))

$$E_t I_z^L \frac{d^4 v_s}{dx^4} + \sigma_x A^L \left[\frac{d^2 v_s}{dx^2} + z_s^L \frac{d^2 \varphi^L}{dx^2} \right] + k_y \left[v_s + (z_s^L - z_A^L) \varphi^L \right] = 0 \quad (33)$$

$$E_t I_y^L \frac{d^4 w_s}{dx^4} + \sigma_x A^L \left[\frac{d^2 w_s}{dx^2} - y_s^L \frac{d^2 \varphi^L}{dx^2} \right] + k_z \left[w_s - (y_s^L - y_A^L) \varphi^L \right] = 0 \quad (34)$$

$$E_t W^L \frac{d^4 \varphi^L}{dx^4} - \left[G_t K^L - \sigma_x A^L (r_o^L)^2 \right] \frac{d^2 \varphi^L}{dx^2} - \sigma_x A^L \left[y_s^L \frac{d^2 w_s}{dx^2} - z_s^L \frac{d^2 v_s}{dx^2} \right]$$
$$+ k_y \left[v_s + (z_s^L - z_A^L) \varphi^L \right] (z_s^L - z_A^L)$$
$$- k_z \left[w_s - (y_s^L - y_A^L) \varphi^L \right] (y_s^L - y_A^L)$$
$$+ k_\varphi \cdot \varphi^L = 0 \quad (35)$$

where

I_z^L = Moment of inertia of longitudinal stiffener about z - axis.

I_y^L = Moment of inertia of longitudinal stiffener about y - axis including the effective width of plate.

W^L = Warping constant of longitudinal stiffener (Warping Torsion).

K^L = Torsion constant for longitudinal stiffener (St. Venant's Torsion).

A^L = Cross-sectional area of longitudinal stiffener

$$(r_o^L)^2 = \frac{I_P^L}{A^L} + (y_S^L)^2 + (z_S^L)^2$$

$I_P^L = I_y^L + I_z^L$ = Polar moment of inertia with effective width of plate for I_y^L.

k_y, k_z = Deflectional spring constants of elastic medium in y- and z-direction respectively.

k_φ = Rotational spring constant of elastic medium.

For the symmetric stiffener, $y_S^L = y_A^L = 0$. The displacement of the point A in y-direction is zero, that is,

$$v_A = v_S + (z_S^L - z_A^L)\varphi^L = 0 \tag{36}$$

However, the spring constant k_y of the plate is infinitive and the product $k_y \cdot v_A$ is finite and equal to horizontal reaction H.

For a Tee-stiffener

$$z_S^L - z_A^L = -d_w^L$$

where d_w^L = the depth of web of longitudinal Tee-stiffener. Therefore Equation (36) yields

$$v_S = d_w^L \cdot \varphi^L \tag{37}$$

The displacement of the point A in -direction is

$$w_A = w_S - (y_S^L - y_A^L)^L = w_S \tag{38}$$

Substituting these relations of Equations (37), (38) and $k_y v_A = H$ into Equations (33), (34) and (35),

$$E_t I_z^L d_W^L \frac{d^4\varphi^L}{dx^4} + \sigma_x A^L [d_W^L + z_S^L] \frac{d^2\varphi^L}{dx^2} + H = 0 \tag{33a}$$

$$E_t I_y^L \frac{d^4 w_A}{dx^4} + \sigma_x A^L \frac{d^2 w_A}{dx^2} + k_z w_A = 0 \tag{34a}$$

$$E_t W^L \frac{d^4\varphi^L}{dx^4} + [\sigma_x A^L (r_o^L)^2 - G_t K^L] \frac{d^2\varphi^L}{dx^2} + \sigma_x A^L z_S^L d_W^L \frac{d^2\varphi^L}{dx^2}$$

$$- H \cdot d_W^L + k_\varphi \cdot \varphi^L = 0 \tag{35a}$$

Eliminating the horizontal reaction H from Equations (33a) and (35a),

$$[E_t W^L + E_t I_z^L (d_W^L)^2] \frac{d^4\varphi^L}{dx^4} - [G_t K^L - \sigma_x A^L (R^L)^2] \frac{d^2\varphi^L}{dx^2} + k_\varphi \cdot \varphi^L = 0 \tag{39}$$

where

$$(R^L)^2 = \frac{I_P^L}{A^L} + (z_A^L)^2$$

As shown in Fig. 7, z_S^L has a negative value.

Therefore $(d_W^L + z_s^L)$ in Equation (33a) equals the distance between the centroid and the enforced axis on the plate, that is, z_A^L. For a Tee-stiffener itself the warping constant W^L vanishes. Therefore from Equations (34a) and (39),

$$k_z w_A = -E_t I_y^L \frac{d^4 w_A}{dx^4} - \sigma_x A^L \frac{d^2 w_A}{dx^2}$$

$$k_\varphi \varphi^L = -E_t I_z^L (d_w^L)^2 \frac{d^4 \varphi^L}{dx^4} + \left[G_t K^L - \sigma_x A^L (R^L)^2\right] \frac{d^2 \varphi^L}{dx^2} \qquad (40)$$

where $I_z^L \cdot (d_W^L)^2$ is the warping constant $\overline{W}^L$ of Tee-section with an enforced axis of rotation.

$$\overline{W}^L = I_z^L \cdot (d_W^L)^2 = \frac{1}{12} b_f^3 \cdot t_f \cdot (d_W^L)^2 \qquad (41)$$

and b_f = Flange width of Tee-section

t_f = Flange thickness of Tee-section

The warping constant of Tee-section with an enforced axis of rotation, given by Equation (41), can also be derived directly by the following geometric considerations: Denoting the translation of the flange plate in y-direction by v_F and the angle of rotation in clockwise direction by φ as shown in Fig. 8, the benging moment M_F in flange plate is expressed as

$$M_F = -EI_F \frac{d^2 v_F}{dx^2}$$

where I_F = Moment of inertia of the flange plate about

z - axis $= \frac{1}{12} b_f^3 t_f$

The shear force in the flange V_F is

$$V_F = \frac{dM_F}{dx} = - EI_F \frac{d^3 v_F}{dx^3}$$

and the warping moment Mw is given by

$$M_W = V_F\ d_W = - EI_F\ d_W \frac{d^3 v_F}{dx^3}$$

Since the translation of enforced point in y-direction is zero, the translation of the flange v_F is given by

$$v_F = d_W \cdot \varphi$$

Therefore

$$M_W = - EI_F\ d_W^2 \frac{d^3\varphi}{dx^3}$$

Introducing the moment due to St. Venant's torsion

$$M_{ST} = GK \frac{d\varphi}{dx}$$

The total twisting moment M_T is

$$M_T = - EI_F\ d_W^2 \frac{d^3\varphi}{dx^3} + GK \frac{d\varphi}{dx}$$

Differentiating this expression with respect to x furnishes the distributed twisting moment along the stiffener

$$m_T = - EI_F \cdot d_w^2 \frac{d^4 \varphi}{dx^4} + GK \frac{d^2 \varphi}{dx^2} \tag{42}$$

The first term represents the warping resistance. Introducing the warping constant $\overline{W}$ for a Tee-section with an enforced axis of rotation

$$\overline{W} = I_F \cdot d_w^2 = \frac{1}{12} b_f^3 \; t_f \; d_w^2 = I_z^L \cdot (d_w^L)^2$$

Equation (42) checks with the previously derived Equation (41).

3.1.2 Transverse Stiffeners

Longitudinal compression of the plate does not produce any transverse stresses. Hence a transverse stiffener remains elastic, even after the plate has yielded. Therefore, its vertical reaction $k_z w_A$ and rotational resistance $k_\varphi \cdot \varphi$ can easily be derived by using the coordinate system in Fig. 9.

$$k_z w_A = - EI_x^T \frac{d^4 w_A}{dy^4}$$

$$k_\varphi \cdot \varphi^T = - EI_z^T \cdot (d_w^T)^2 \frac{d^4 \varphi^T}{dy^4} + GK^T \frac{d^2 \varphi^T}{dy^2} \tag{43}$$

where I_x^T = Moment of inertia of transverse stiffener about x-axis including the effective width of plate

I_z^T = Moment of inertia of transverse stiffener about z-axis

d_w^T = Depth of web of transverse Tee-stiffener

φ^T = Angle of rotation of transverse stiffener about y-axis

K^T = Torsion constant of transverse stiffener

3.2 Integral Equation for Buckling of an Orthotropic Plate with Symmetric Type of Stiffeners

According to the definition of Green's function the deflection of a plate can be obtained by integrating the product of Green's function and the distributed loads over the whole plate. If the distributed loads are expressed in differential form as a function of the deflection, this leads to a linear homogeneous integro-differential equation[22] in the following form.

$$w(x,y) = \int_0^a \int_0^b G(x,y;\xi,\eta) \cdot N\left[w(\xi,\eta)\right] d\xi \, d\eta$$

where G is Green's function and N is a differential operator.

The same procedure can be applied to the case of loading by moments. For the case under investigation the distributed vertical reactions and twisting resistances of the i-th longitudinal and j-th transverse stiffener are given by Equations (40) and (43) respectively. The deflection of a stiffener w_A along the line of attachment must be equal to the deflection of the plate. Therefore

$$w_{Ai}(x) = w(\xi, \eta_i)$$

$$\varphi_i^L(x) = \frac{\partial w}{\partial \eta}(\xi, \eta_i)$$

for i-th longitudinal stiffener in Equation (40), and

$$w_{Aj}(y) = w(\xi_j, \eta)$$

$$\varphi_j^T(y) = \frac{\partial w}{\partial \xi}(\xi_j, \eta)$$

for j-th transverse stiffener in Equation (43). Then Equation (40) yields

$$k_{zi} w_{Ai} = -E_t I_{yi}^L \frac{\partial^4 w}{\partial \xi^4}(\xi, \eta_i) - \sigma_x A_i^L \frac{\partial^2 w}{\partial \xi^2}(\xi, \eta_i)$$

$$k_{\varphi i} \cdot \varphi_i^L = -E_t I_{zi}^L (d_{wi}^L)^2 \frac{\partial^5 w}{\partial \xi^4 \partial \eta}(\xi, \eta_i)$$

$$+ \left[G_t K_i^L - \sigma_x A_i^L (R_i^L)^2 \right] \frac{\partial^3 w}{\partial \xi^2 \partial \eta}(\xi, \eta_i) \qquad (44)$$

Similarly Equation (43) for the case of transverse stiffeners follows:

$$k_{zj} \cdot w_{Aj} = -E I_{xj}^T \frac{\partial^4 w}{\partial \eta^4}(\xi_j, \eta)$$

$$k_{\varphi j} \cdot \varphi_j^T = -E I_{zj}^T (d_{wj}^T)^2 \frac{\partial^5 w}{\partial \xi \partial \eta^4}(\xi_j, \eta) + G K_j^T \frac{\partial^3 w}{\partial \xi \partial \eta^2}(\xi_j, \eta) \qquad (45)$$

Now the twisting resistance $k_{\varphi i}\varphi_i^L$ of i-th longitudinal stiffener equals the resisting bending moment in y-direction acting on the plate along the line of attachment. Therefore, the contribution of twisting resistance of the i-th longitudinal stiffener to the deflection of the plate is given by

$$\Delta w_{1i} = \int_0^a (k_{\varphi i}\cdot\varphi_i^L)\cdot\overline{G}_\eta(x,y;\xi,\eta_i)\,d\xi$$

The contribution of the bending resistance to the deflection of plate is similarly:

$$\Delta w_{2i} = \int_0^a (k_{zi}\cdot w_{Ai})\cdot G(x,y;\xi,\eta_i)\,d\xi$$

The total influence of n longitudinal stiffeners upon the deflection of plate is

$$w^L = \sum_{i=1}^{n}\int_0^a (k_{zi}\cdot w_{Ai})\cdot G(x,y;\xi,\eta_i)\,d\xi$$

$$+\sum_{i=1}^{n}\int_0^a (k_{\varphi i}\varphi_i^L)\cdot\overline{G}_\eta(x,y;\xi,\eta_i)\,d\xi \tag{46}$$

Similarly from m transverse stiffeners

$$w^{T} = \sum_{j=1}^{m} \int_{0}^{b} (k_{z_j} \cdot w_{A_j})\, G(x, y; \xi_j, \eta)\, d\eta$$

$$+ \sum_{j=1}^{m} \int_{0}^{b} (k_{\varphi_j} \cdot \varphi_j^{T}) \cdot \overline{G}_{\xi}(x, y; \xi_j, \eta)\, d\eta \qquad (47)$$

Considering the influence of the axial compression of the plate toward its deflection the corresponding contribution toward the deflection is

$$w^{P} = \int_{0}^{a}\int_{0}^{b} (k_z \cdot w) \cdot G(x, y; \xi, \eta)\, d\xi\, d\eta \qquad (48)$$

where

$$k_z \cdot w = - N_x \frac{\partial^2 w}{\partial \xi^2}(\xi, \eta)$$

The total deflection of plate equals the sum of all these individual contributions.

$$w = w^{P} + w^{L} + w^{T} \qquad (49)$$

Substituting Equations (44), (45), (46), (47) and (48) into Equation (49), the following linear homogeneous integro-differential equation is obtained.

$$
\begin{aligned}
w(x, y) &= -\int_0^a\int_0^b N_x \frac{\partial^2 w}{\partial \xi^2}(\xi,\eta)\cdot G(x,y;\xi,\eta)\,d\xi\,d\eta \\
&\quad -\sum_{i=1}^{n}\int_0^a \left[E_t I^L_{yi}\frac{\partial^4 w}{\partial \xi^4}(\xi,\eta_i) + \sigma_x A^L_i \frac{\partial^2 w}{\partial \xi^2}(\xi,\eta_i)\right] \\
&\qquad \times G(x,y;\xi,\eta_i)\,d\xi \\
&\quad -\sum_{i=1}^{n}\int_0^a \Big[E_t I^L_{zi}(d^L_{wi})^2 \frac{\partial^5 w}{\partial \xi^4\partial\eta}(\xi,\eta_i) - \left\{G_t K^L_i - \sigma_x A^L_i (R^L_i)^2\right\} \\
&\qquad \times \frac{\partial^3 w}{\partial \xi^2\partial\eta}(\xi,\eta_i)\Big]\cdot \overline{G}_\eta(x,y;\xi,\eta_i)\,d\xi \\
&\quad -\sum_{j=1}^{m}\int_0^b E I^T_{xj}\frac{\partial^4 w}{\partial \eta^4}(\xi_j,\eta)\cdot G(x,y;\xi_j,\eta)\,d\eta \\
&\quad -\sum_{j=1}^{m}\int_0^b \left[E I^T_{zj}(d^T_{wj})^2 \frac{\partial^5 w}{\partial \xi\partial\eta^4}(\xi_j,\eta) - G K^T_j \frac{\partial^3 w}{\partial \xi\partial\eta^2}(\xi_j,\eta)\right] \\
&\qquad \times \overline{G}_\xi(x,y;\xi_j,\eta)\,d\eta
\end{aligned}
\tag{50}
$$

3.3 Solution of Integral Equation and its Secular Equation for Eigenvalues

The linear homogeneous integro-differential Equation (50) can be integrated by making use of the orthogonality relations of Green's function in Equations (8), (9) and (14). The deflection of the plate w in Equation (50) can be assumed as

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_{mn}\, \varphi_{mn}(x,y) \tag{51}$$

where

$$\varphi_{mn}(x,y) = \frac{2}{\sqrt{ab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

This expression fulfills all boundary conditions of a simply supported plate. According to the characteristic of the normalized orthogonal function, Equation (50) yields the following expressions by multiplying both sides of Equation (50) with $\varphi_{rs}(x,y)$ and integrating over the whole plate. The left side of Equation (50) becomes

$$\int_0^a\int_0^b \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_{mn}\, \varphi_{mn}(x,y)\, \varphi_{rs}(x,y)\, dx\, dy$$

$$= \alpha_{rs}$$

The first term of the right side yields

$$-\int_0^a\int_0^b\Big[\int_0^a\int_0^b N_x\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)\left(\frac{m\pi}{a}\right)^2\alpha_{mn}\varphi_{mn}(\xi,\eta)\Big\}$$

$$\times\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\varphi_{mn}(x,y)\,\varphi_{mn}(\xi,\eta)}{\lambda_{mn}^2}\Big\}\varphi_{rs}(x,y)\,dx\,dy\Big]\,d\xi\,d\eta$$

$$=\int_0^a\int_0^b N_x\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m\pi}{a}\right)^2\alpha_{mn}\varphi_{mn}(\xi,\eta)\frac{\varphi_{rs}(\xi,\eta)}{\lambda_{rs}^2}\,d\xi\,d\eta$$

$$=N_x\left(\frac{r\pi}{a}\right)^2\alpha_{rs}\cdot\frac{1}{\lambda_{rs}^2}$$

Similar integrations can be carried out for the remaining terms in Equation (50). The final result is given by

$$\left[N_x\left(\frac{r\pi}{a}\right)^2-\lambda_{rs}^2\right]\alpha_{rs}$$

$$=\frac{2}{b}\left(\frac{r\pi}{a}\right)^4\sum_{i=1}^{n}E_tI_{y_i}^L\sin\frac{s\pi\eta_i}{b}\sum_{p=1}^{\infty}\alpha_{rp}\sin\frac{p\pi\eta_i}{b}$$

$$-\frac{2}{b}\left(\frac{r\pi}{a}\right)^2\sum_{i=1}^{n}\sigma_xA_i^L\sin\frac{s\pi\eta_i}{b}\sum_{p=1}^{\infty}\alpha_{rp}\sin\frac{p\pi\eta_i}{b}$$

$$+\frac{2}{b}\left(\frac{r\pi}{a}\right)^2\left(\frac{s\pi}{b}\right)\sum_{i=1}^{n}\Big[E_tI_{z_i}^L(d_{w_i}^L)^2\left(\frac{r\pi}{a}\right)^2+G_tK_i^L$$

$$-\sigma_xA_i^L(R_i^L)^2\Big]\cos\frac{s\pi\eta_i}{b}\sum_{p=1}^{\infty}\alpha_{rp}\left(\frac{p\pi}{b}\right)\cos\frac{p\pi\eta_i}{b}$$

$$+ \frac{2}{a}\left(\frac{s\pi}{b}\right)^4 \sum_{j=1}^{m} EI_{x_j}^{T} \sin\frac{r\pi\xi_j}{a} \sum_{q=1}^{\infty} \alpha_{qs} \sin\frac{q\pi\xi_j}{a}$$

$$+ \frac{2}{a}\left(\frac{s\pi}{b}\right)^2\left(\frac{r\pi}{a}\right) \sum_{j=1}^{m}\left[EI_{z_j}^{T}(d_{w_j}^{T})^2\left(\frac{s\pi}{b}\right)^2 + GK_j^{T}\right]$$

$$\times \cos\frac{r\pi\xi_j}{a} \sum_{q=1}^{\infty} \alpha_{qs}\left(\frac{q\pi}{a}\right) \cos\frac{q\pi\xi_j}{a} \qquad (52)$$

Equation (52) furnishes an infinite number of simultaneous linear homogeneous equations with respect to the coefficients α_{rs}, where $r = 1, 2, 3 \ldots \infty$ and $s = 1, 2, 3 \ldots \infty$. The determinant of these simultaneous linear homogeneous equations is the secular equation for the eigenvalues of buckling of the stiffened plate with longitudinal and transverse stiffeners having symmetric thin-walled open cross-sections.

4. BUCKLING STRENGTH OF AN ORTHOTROPIC PLATE WITH A SYMMETRIC TYPE OF STIFFENER TRANSVERSELY PLACED ON A PLATE

As an introduction the buckling of a plate without stiffeners is discussed. In this case all cross-sectional constants of stiffeners in Equation (52) such as I_{yi}^{L}, I_{zi}^{L}, A_{i}^{L}, K_{i}^{L}, I_{xj}^{T}, I_{zj}^{T} and K_{j}^{T} reduce to zero. Therefore Equation (52) yields

$$N_x \left(\frac{r\pi}{a}\right)^2 - \lambda_{rs}^2 = 0$$

where λ_{rs}^2 is given by Equation (9) as

$$\lambda_{rs}^2 = K_x \left(\frac{r\pi}{a}\right)^4 + 2\ K_{xy} \left(\frac{r\pi}{a}\right)^2 \left(\frac{s\pi}{b}\right)^2 + K_y \left(\frac{s\pi}{b}\right)^4$$

For the elastic case, $K_x = K_{xy} = K_y = DI$ and

$$D = \frac{E}{1 - \nu^2}$$

$$I = \frac{h^3}{12} = \text{Moment of inertia per unit width of plate}$$

The buckling stress σ_{cr} is defined by

$$\sigma_{cr} = \frac{N_x}{h}$$

Therefore

$$\sigma_{cr} = \frac{1}{h} \cdot \left(\frac{a}{r\pi}\right)^2 \lambda_{rs}{}^2$$

$$= \frac{\pi^2}{12} \frac{E}{1-\nu^2} \cdot \left(\frac{h}{b}\right)^2 \cdot \left(\frac{s^2}{r} \alpha + \frac{r}{\alpha}\right)^2$$

The smallest buckling stress is given by taking $S = 1$ for any values of side ratio α, therefore,

$$\sigma_{cr} = \frac{\pi^2}{12} \cdot \frac{E}{1-\nu^2} \cdot \left(\frac{h}{b}\right)^2 \cdot \left(\frac{\alpha}{r} + \frac{r}{\alpha}\right)^2 \tag{53}$$

This is the buckling stress of a plate in the elastic range determined by Bryan(4), r being the number of half waves in the loading direction.

In the strain-hardening range the material properties are direction dependent. The bending rigidities of an orthotropic plate K_x, K_{xy}, and K_y in Equation (1) are given by the following quantities(6).

$$K_x = \frac{E_x I}{1-\nu_x \nu_y} = D_x I$$

$$K_y = \frac{E_y I}{1-\nu_x \nu_y} = D_y I$$

$$2\, K_{xy} = \nu_x D_y I + \nu_y D_x I + 4 G_t I = 2\, HI$$

$$I = \frac{h^3}{12}$$

Therefore the eigenvalue λ_{rs}^2 in Equation (9) yields

$$\lambda_{rs}^2 = I\left[D_x \left(\frac{r\pi}{a}\right)^4 + 2H \left(\frac{r\pi}{a}\right)^2 \left(\frac{s\pi}{b}\right)^2 + D_y \left(\frac{s\pi}{b}\right)^4\right] \tag{54}$$

The smallest buckling stress in the strain-hardening range occurs when r = S = 1 and

$$\sigma_{cr} = \frac{N_x}{h} = \frac{\pi^2}{12} \left(\frac{h}{b}\right)^2 \left[D_x \left(\frac{b}{a}\right)^2 + D_y \left(\frac{a}{b}\right)^2 + 2H\right] \tag{55}$$

where

$$2H = \nu_x D_y + \nu_y D_x + 4G_t$$

The Equation (55) is identical with Equation (3.10) in Reference (6) when the elastic restraint $\beta = \frac{\psi b}{2D_y I}$ along unloaded edges is equal to zero.

4.1 Convergence of a Secular Equation for Buckling in the Elastic Range

If a plate has one transverse stiffener in the middle of the plate, the elastic buckling stress can be obtained from Equation (52) by substituting the proper indices for the stiffeners, i.e., i = 0 and j = 1, $\xi_j = \frac{a}{2}$ and the elastic constants

$$K_x = K_{xy} = K_y = \frac{Eh^3}{12(1-\nu^2)} = DI$$

The smallest buckling stress is obtained by taking the number of half waves in the transverse direction S = 1. Therefore the Equation (52) yields

$$r^2 [k - k_{r,1}] \alpha_{r1} = 2\alpha \gamma_B^T \sin \frac{r\pi}{2} \sum_{q=1}^{\infty} \alpha_{q1} \sin \frac{q\pi}{2}$$

$$+ 2 \frac{r}{\alpha} (\gamma_s^T + \pi^2 \gamma_w^T) \cos \frac{r\pi}{2} \sum_{q=1}^{\infty} \alpha_{q1} \cdot q \cdot \cos \frac{q\pi}{2} \qquad (56)$$

where

k = Buckling coefficient of stiffened plate

$k_{r,1}$ = Buckling coefficient of plate without stiffeners in the mode of r and 1, $= (\frac{r}{\alpha} + \frac{\alpha}{r})^2$

α = Side ratio of plate, $\frac{\text{Length}}{\text{Width}} = a/_b$

$\gamma_B{}^T$ = Bending rigidity coefficient of the transverse stiffener including the effective width of plate to the bending rigidity of plate $= \frac{EI_x^T}{DIb}$

$\gamma_s{}^T$ = Torsional rigidity coefficient of the transverse stiffener to the bending rigidity of plate $= \frac{GK^T}{DIb}$

$\gamma_w{}^T$ = Warping rigidity coefficient of the transverse stiffener to the bending rigidity of plate $= \frac{EI_z^T}{DIb} \left(\frac{d_w{}^T}{b} \right)^2$

The symmetric type of buckling with respect to the transverse stiffener, that is, $r = 1, 3, 5, \ldots$, is separated from the antisymmetric type of buckling, $r = 2, 4, 6, \ldots$, as can be seen from Equation (56). Therefore, these two modes of buckling, symmetric and antisymmetric, are independent of each other. By gradually increasing γ_B^T, we finally arrive at the condition where the plate buckles into the antisymmetric mode and the nodal line coincides with the stiffener of the plate. At this stage an increase in the bending resistance of the stiffener will increase the buckling stress of the plate no further, but the torsional resistance becomes effective. This limiting values of γ_B^T depends on the side ratio of plate. However, if the side ratio α is greater than $\sqrt{2}$, one transverse stiffener in the middle of plate is no longer effective[4] except for its torsional resistance. In the case of symmetric buckling Equation (56) yields

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cdot \frac{1}{k - k_{(2n-1),1}} = \frac{1}{2\alpha\gamma_B^T} \tag{57}$$

where $k_{(2n-1),1}$ = Buckling coefficient of plate without stiffener in the mode of $(2n-1)$ and 1 half waves in x- and y-direction respectively.

$$k_{(2n-1),1} = \left[\frac{(2n-1)}{\alpha} + \frac{\alpha}{(2n-1)}\right]^2$$

If only the first term $n=1$ in Equation (57) is considered, the buckling coefficient k in the first approximation yields

$$k = k_{11} + 2\alpha\gamma_B^T = (\frac{1}{\alpha} + \alpha)^2 + 2\alpha\gamma_B^T$$

or
$$\sigma_{cr} = \frac{\pi^2}{12}(\frac{h}{b})^2 \cdot \frac{E}{1-\nu^2}\,\frac{(1+\alpha^2)^2+2\gamma_B^T\alpha^3}{\alpha^2}$$

This is the same expression as Equation (229) in Reference (4). For the antisymmetric mode of buckling in the longitudinal direction, Equation (56) yields

$$\sum_{n=1}^{\infty} \frac{1}{K - K_{2n,1}} = \frac{1}{2} \cdot \frac{\alpha}{\gamma_S^T + \pi^2 \gamma_W^T} \qquad (58)$$

where

$k_{2n,1}$ = Buckling coefficient of plate without stiffener in the mode of (2n) and 1 half waves in x- and y-direction respectively.

$$= \left[\frac{(2n)}{\alpha} + \frac{\alpha}{(2n)}\right]^2 .$$

The convergence of Equations (57) and (58) with respect to the numbers of half waves r taken into account--that is, r = 2n-1, (n = 1, 2, 3, ...) for Equation (57) and r = 2n, (n = 1, 2, 3, ...) for Equation (58)--is rather remarkable as shown in Fig. 10. For this figure the side ratios $\alpha = 0.5$ with $\gamma_B^T = 10$ and $\alpha = 1.0$ with $\gamma_B^T = 12.7$ were chosen* for the case of the symmetric buckling mode and $\alpha = 1.0$ with $\gamma_s^T + \pi^2 \gamma_w^T = 0.085$* and 1.0 for the anti-symmetric buckling mode in order to examine the convergence of the solution under extreme conditions. The buckling coefficient k for each approximation and its error in comparison to the 5th approximation which is supposed to be very close to the accurate value are listed in Table 1. Since the error in the 2nd approximation for k is less than one percent except for the case $\gamma_s^T + \pi^2 \gamma_w^T = 1.0$ which corresponds to an extremely large torsional resistance for an open cross-section, the 2nd approximation gives adequate results for practical design purpose.

4.2 Effect of Torsional Resistance of a Stiffener on Buckling Strength in Elastic Range

Since the buckling coefficient k in Equation (58) converges rapidly, the 2nd approximation obtained from two terms in Equation (58) can be used for design purpose. Hence

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* These values are obtained in Appendix 11.1 Equation (A-4)

$$k = \frac{1}{32}\left[\frac{320}{\alpha^2} + 64 + 5\alpha^2 + 64\frac{(\gamma_s^T + \pi^2\gamma_w^T)}{\alpha}\right]$$

$$x\left[1 - \sqrt{1 - \frac{16\left\{(\frac{4}{\alpha} + \alpha)^2 \cdot (\frac{16}{\alpha} + \alpha)^2 + 8 \cdot \frac{(\gamma_s^T + \pi^2\gamma_w^T)}{\alpha}(\frac{320}{\alpha^2} + 64 + 5\alpha^2)\right\}}{\left\{\frac{320}{\alpha^2} + 64 + 5\alpha^2 + 64\frac{(\gamma_s^T + \pi^2\gamma_w^T)}{\alpha}\right\}^2}}\right] \tag{59}$$

Equation (59) is plotted in Fig. 11 for values of the torsional resistance coefficient $(\gamma_s^T + \pi^2\gamma_w^T)$ and the side ratio α. The effect of the torsional resistance on buckling coefficient k is approximately linear. In other words, k is proportional to the torsional resistance for any value of α. The difference in buckling strength due to the torsional resistance is approximately 21% for $\alpha = 0.5$, 25% for $\alpha = 0.75$ and 27% for $\alpha = 1.0$ at the value of torsional resistance $\gamma_s^T + \pi^2\gamma_w^T = 1$.

4.3 Required Minimum Bending Rigidity of a Stiffener for Elastic Buckling of Plate

The 2nd approximate solution of Equation (57) for the symmetric mode of buckling is given by

$$k = \frac{1}{9}\left[\frac{45}{\alpha^2} + 18 + 5\alpha^2 + 10\alpha\gamma_B^T\right]$$

$$\times\left[1 - \sqrt{1 - \frac{9\left\{\left(\frac{1}{\alpha} + \alpha\right)^2\left(\frac{9}{\alpha} + \alpha\right)^2 + 4\alpha\gamma_B^T\left(\frac{41}{\alpha^2} + 10 + \alpha^2\right)\right\}}{\left(\frac{45}{\alpha^2} + 18 + 5\alpha^2 + 10\alpha\gamma_B^T\right)^2}}\;\right] \qquad (60)$$

The buckling coefficient k in Equation (60) is plotted in Fig. 12 for various values of γ_B^T and α. As seen from this figure the buckling strength of a plate with a transverse stiffener is not proportional to the bending rigidity of the stiffener, but reaches a maximum value depending upon the value of the side ratio α. Although the buckling coefficient k in Fig. 12 increases with the bending rigidity γ_B^T of the stiffener until it reaches the maximum value, the buckling mode will change at specific value of γ_B^T from a symmetric to an antisymmetric mode. Beyond this value the bending rigidity of the stiffener has no further influence on the buckling strength because the stiffener lies on the nodal line of the buckled plate. Therefore this value defines the required minimum bending rigidity of the stiffener. According to this definition the required minimum

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$$\times\left[1 - \sqrt{1 - \frac{9\left\{\frac{1}{(\alpha+\alpha)^2}\,\frac{9}{(\alpha+\alpha)^2} + 4\alpha\gamma_B^T\left(\frac{41}{\alpha^2} + 10 + \alpha^2\right)\right.}{\left(\frac{45}{\alpha^2} + 18 + 5\alpha^2 + 10\alpha\gamma_B^T\right)^2}}\right] \qquad (60)$$

The buckling coefficient k in Equation (60) is plotted in Fig. 12 for various values of γ_B^T and α. As seen from this figure the buckling strength of a plate with a transverse stiffener is not proportional to the bending rigidity of the stiffener, but reaches a maximum value depending upon the value of the side ratio α. Although the buckling coefficient k in Fig. 12 increases with the bending rigidity γ_B^T of the stiffener until it reaches the maximum value, the buckling mode will change at specific value of γ_B^T from a symmetric to an antisymmetric mode. Beyond this value the bending rigidity of the stiffener has no further influence on the buckling strength because the stiffener lies on the nodal line of the buckled plate. Therefore this value defines the required minimum bending rigidity of the stiffener. According to this definition the required minimum bending rigidity is obtained by equating the buckling coefficient in Equation (59) and (60), or (57) and (58). The result is shown in Fig. 11 for different values of torsional resistance and side ratio α . If the torsional resistance of a stiffener is neglected, the maximum possible buckling strength of plate corresponds to the minimum value of an even mode in the loading direction. The required minimum bending rigidity of the stiffener is obtained as shown in Table 2 for each value of the side ratio α by taking 2 or 3 terms in Equation (57) into account, where k is given by $(\frac{2}{\alpha} + \frac{\alpha}{2})^2$. The results are compared with the numerical values by Timoshenko[4] and Fröhlich[23]. The values obtained from the integral equation by using 3 terms in Equation (57) give errors of less than 0.5% in comparison to the value by Fröhlich obtained from his solution in closed form. The German specification on buckling of structures[24] give the following equation for the minimum bending rigidity ("Mindeststeifigkeit").

$$\gamma = \frac{4\left(\frac{4}{\alpha^2} - \frac{\alpha^2}{4}\right)}{\pi^2 \alpha \left(1 - \frac{\pi^2 \alpha^4}{12\alpha^4 - 48}\right)}$$

This equation is based on the result by Fröhlich. The values of the required minimum bending rigidity γ_B^T by Timoshenko give larger errors than that by integral equation method.

Especially, the value for $\alpha = 1.0$ is too high.

4.4 Convergence of a Secular Equation for Buckling in Strain-Hardening Range

When a plate with a transverse stiffener in the middle of the plate buckles in the strain-hardening range, the bending rigidities of the strain-hardened plate are given by

$$K_x = D_x I$$

$$K_y = D_y I$$

$$K_{xy} = HI$$

and the eigenvalue λ_{rs}^2 is given by Equation (54). The secular equation for this case is also given by Equation (56). It can be separated into Equations (57) and (58) for the symmetric and antisymmetric mode of buckling respectively. The differences from the elastic buckling are in the coefficients k_{r1}, γ_B^T, γ_s^T and γ_w^T since the plate is compressed into strain-hardening range. These coefficients are defined as follows:

$$k_{(2n-1),1} = \left(\frac{2n-1}{\alpha}\right)^2 + 2\lambda + \mu\left(\frac{\alpha}{2n-1}\right)^2$$

$$k_{2n,1} = \left(\frac{2n}{\alpha}\right)^2 + 2\lambda + \mu\left(\frac{\alpha}{2n}\right)^2$$

$$\lambda = \frac{H}{D_x}$$

$$\mu = \frac{D_y}{D_x}$$

$$\gamma_B^T = \frac{EI_x^T}{D_x Ib}$$

$$\gamma_s^T = \frac{GK^T}{D_x Ib}$$

$$\gamma_w^T = \frac{EI_z^T}{D_x Ib}\left(\frac{d^T{}_w}{b}\right)^2$$

Then the secular equations of the buckling coefficient k are obtained by replacing the coefficients in Equations (57) and (58) by the above coefficients.

For symmetric buckling,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \quad \frac{1}{k - k_{(2n-1),1}} = \frac{1}{2\alpha\gamma_B^T} \tag{57a}$$

For antisymmetric buckling,

$$\sum_{n=1}^{\infty} \frac{1}{k-k_{2n,1}} = \frac{1}{2}\,\frac{\alpha}{\gamma_s^T + \pi^2\gamma_w^T} \tag{58a}$$

The buckling coefficients k in Equations (57a) and (58a) converge as rapidly as those in Equations (57) and (58). This is shown in Fig. 13 and Table 3, where the coefficients for the orthotropy of the plate λ and μ for structural steel (A-7 Steel)(2) were taken:

$$\lambda = 4.30$$

$$\mu = 10.93$$

4.5 Effect of Torsional Resistance of a Stiffener on Buckling Strength in the Strain-Hardening Range

The buckling coefficient k for a transversely stiffened plate in the strain-hardening range is given by using the 2nd approximation of Equation (58a) as

$$k = \frac{1}{32}\left[\frac{320}{\alpha^2} + 64\lambda + 5\mu\alpha^2 + 64\,\frac{(\gamma_S^T + \pi^2\gamma_W^T)}{\alpha}\right]\cdot\left[1 - \sqrt{1 - \frac{16\left\{\left(\frac{16}{\alpha^2}+8\lambda+\mu\alpha^2\right)\left(\frac{256}{\alpha^2}+32\lambda+\mu\alpha^2\right)+8\cdot\frac{(\gamma_S^T+\pi^2\gamma_W^T)}{\alpha}\left(\frac{320}{\alpha^2}+64\lambda+5\mu\alpha^2\right)\right\}}{\left\{\frac{320}{\alpha^2}+64\lambda+5\mu\alpha^2+64\cdot\frac{(\gamma_S^T+\pi^2\gamma_W^T)}{\alpha}\right\}^2}}\,\right] \tag{59a}$$

The effect of the torsional resistance $(\gamma_S^T + \pi^2\gamma_W^T)$ on the buckling coefficient k in Equation (59a) is shown in Fig. 14 for the side ration $\alpha = 0.5$. Since the minimum value of buckling coefficient k in the strain-hardening range is obtained for the simply supported plate with no stiffener at a side ratio $\alpha = \sqrt[4]{\frac{D_x}{D_y}}$ (2), a transverse stiffener placed in the middle of plates of structural steel (A-7) is no longer effective to prevent the buckling in the strain-hardening range if the side ratio α is greater than 0.777.

4.6 Required Minimum Bending Rigidity of a Transverse Stiffener for Buckling of Plate in the Strain-Hardening Range

Equations (57a) and (59a) determine the required minimum bending rigidity γ_B^T of a transverse stiffener in the strain-hardening range by equating the value k in these two equations. The value γ_B^T is plotted in Fig. 14 for $\alpha = 0.5$. It is approximately proportional to the torsional reistance as seen in the figure.

5. BUCKLING STRENGTH OF AN ORTHOTROPIC PLATE WITH A SYMMETRIC TYPE OF STIFFENER LONGITUDINALLY PLACED ON A PLATE

5.1 Convergence of a Secular Equation for Buckling of Longitudinally Stiffened Plate in the Elastic Range

In the previous chapters it has just been shown that for a plate with a transverse stiffener, the buckling mode in one half wave in the transverse or unloaded direction always gives the smallest value of buckling strength. However, this is no longer the case for a plate with a longitudinal stiffener. Therefore the fundamental buckling coefficient of plate without stiffener must be derived from a general mode with the number of half waves r and s in x- and y-direction respectively, that is,

$$k_{rs} = \left(\frac{r}{\alpha} + s^2 \frac{\alpha}{r}\right)^2$$

This relation follows directly from the elastic buckling stress of plate with no stiffener as given for instance in Reference(4):

$$\sigma_{cr} = \frac{1}{h} \cdot \frac{\pi^2 \, Eh^3}{12(1-\nu^2)} \cdot \frac{a^2}{r^2} \cdot \left(\frac{r^2}{a^2} + \frac{s^2}{b^2} \right)^2$$

By proper rearrangement of the terms one obtains:

$$\sigma_{cr} = \frac{\pi^2}{12} \cdot \frac{E}{1-\nu^2} \cdot \left(\frac{h}{b}\right)^2 \cdot \left(\frac{r}{\alpha} + s^2 \frac{\alpha}{r}\right)^2 = \frac{\pi^2}{12} \frac{E}{1-\nu^2} \cdot \left(\frac{h}{b}\right)^2 \cdot k_{rs}$$

Introducing the following non-dimensional parameters for longitudinal stiffeners

$$\gamma_B^L = \frac{EI_y^L}{DIb} \qquad D = \frac{E}{1-\nu^2}$$

$$\gamma_s^L = \frac{GK^L}{DIb} \qquad I = \frac{h^3}{12}$$

$$\gamma_w^L = \frac{EI_z^L}{DIb} \cdot \left(\frac{d_w^L}{b}\right)^2$$

$$\delta = \frac{A^L}{bh} = \frac{\text{Cross-sectional area of longitudinal stiffener}}{\text{Cross-sectional area of plate (width x thickness)}}$$

Equation (52) can be expressed for an elastic plate with one longitudinal stiffener in the middle of the plate by substituting $i = 1$, $j = 0$ and $\eta_i = \frac{b}{2}$ into this equation:

$$\begin{aligned}(k - k_{rs})\alpha_{rs} = {} & 2\frac{r^2}{\alpha^2}\gamma_B^L \sin\frac{s\pi}{2}\sum_{p=1}^{\infty}\alpha_{rp}\sin\frac{p\pi}{2} \\ & + 2s\left[\gamma_s^L + \frac{r^2}{\alpha^2}\pi^2\gamma_w^L - \pi^2\delta\left(\frac{R^L}{b}\right)^2 k\right]\cos\frac{s\pi}{2} \\ & \times \sum_{p=1}^{\infty} p\,\alpha_{rp}\cos\frac{p\pi}{2} \\ & - 2\delta k \sin\frac{s\pi}{2}\sum_{p=1}^{\infty}\alpha_{rp}\sin\frac{p\pi}{2} \end{aligned} \qquad (61)$$

Inspection of this equation shows that the symmetric mode of buckling with respect to the longitudinal stiffener, that is, s = 1, 3, 5 .... is separated from the anti-symmetric type, s = 2, 4, 6 .... .

The secular equation for the symmetric type is expressed in the form:

$$\sum_{n=1}^{\infty} \frac{1}{k - k_{\bar{r},(2n-1)}} = \frac{1}{2\frac{\bar{r}^2}{\alpha^2}\gamma_B^L - 2\delta k} \tag{62}$$

For the antisymmetric mode the secular equation yields

$$\sum_{n=1}^{\infty} \frac{n^2}{k - k_{r,(2n)}} = \frac{1}{8}\,\frac{1}{\gamma_s^L + \frac{r^2}{\alpha^2}\pi^2\gamma_W^L - \pi^2\delta\left(\frac{R^L}{b}\right)^2 k} \tag{63}$$

The numbers of half waves in the x- or loading direction, $\bar{r}$ in Equation (62) and r in Equation (63), must be determined such that the buckling coefficient k becomes a minimum for each case respectively.

If the torsional resistance of a longitudinal stiffener can be neglected, then from Equation (63)

$$k = k_{r,(2n)} \quad \text{since} \quad \gamma_s^L = \gamma_W^L = R^L = 0.$$

The minimum buckling coefficient k is obtained for n = 1, that is,

$$k = k_{r2} = \left(\frac{r}{\alpha} + 4\frac{\alpha}{r}\right)^2 \tag{64}$$

The number of half waves r in x-direction is determined such that k becomes a minimum for a given value of α in Equation (64). By equating

$$k_{r2} = k_{(r+1),2}$$

the ratios α for which a transition from r to (r + 1) takes place can be determined. Upon substitution it follows:

$$\left(\frac{r}{\alpha} + 4\frac{\alpha}{r}\right)^2 - \left(\frac{r+1}{\alpha} + 4\frac{\alpha}{r+1}\right)^2 = 0$$

hence:

$$\alpha = \frac{\sqrt{r(r+1)}}{2} \tag{65}$$

The side ratios α are obtained by introducing values for r:

$$r = 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6$$

$$\alpha = \frac{\sqrt{2}}{2}, \quad \frac{\sqrt{6}}{2}, \quad \frac{\sqrt{12}}{2}, \quad \frac{\sqrt{20}}{2}, \quad \frac{\sqrt{30}}{2}, \quad \frac{\sqrt{42}}{2}$$

This means that the number of half waves r in x-direction for the antisymmetric mode in y-direction is determined for each range of side ratio α as follows:

$$r = 1 \quad \text{for} \quad 0 < \alpha \leq \frac{\sqrt{2}}{2}$$

$$r = 2 \quad \text{for} \quad \frac{\sqrt{2}}{2} \leq \alpha \leq \frac{\sqrt{6}}{2}$$

$$r = 3 \quad \text{for} \quad \frac{\sqrt{6}}{2} \leq \alpha \leq \frac{\sqrt{12}}{2}$$

$$r = 4 \quad \text{for} \quad \frac{\sqrt{12}}{2} \leq \alpha \leq \frac{\sqrt{20}}{2}$$

$$r = 5 \quad \text{for} \quad \frac{\sqrt{20}}{2} \leq \alpha \leq \frac{\sqrt{30}}{2}$$

$$r = 6 \quad \text{for} \quad \frac{\sqrt{30}}{2} \leq \alpha \leq \frac{\sqrt{42}}{2}$$

The number of half waves $\bar{r}$ in x-direction for the symmetric mode in y-direction can be obtained from Equation (62) by equating the buckling coefficient k in the mode of $\bar{r}$ and $(\bar{r} + 1)$. This process requires a trial and error solution because the deformed shape of the plate involving bending of the longitudinal stiffener is no longer a simple sine curve in y-direction. It consists of a Fourier series involving the odd half waves, s = 1, 3, 5 .... Therefore the buckling coefficient k cannot be obtained in an explicit form in terms of the side ratio α, number of half waves $\bar{r}$, stiffener area coefficient δ and stiffness coefficient γ^{L}_{B}. As a first approximation the first term in Equation (62) can be used to obtain a relation between α and $\bar{r}$, then

$$k = \frac{k_{\bar{r},1} + 2\frac{\bar{r}^2}{\alpha^2}\gamma\frac{L}{B}}{1 + 2\delta} = \frac{k_{(\bar{r}+1),1} + 2\frac{(\bar{r}+1)^2}{\alpha^2}\gamma\frac{L}{B}}{1 + 2\delta}$$

Therefore

$$\alpha = \sqrt{\bar{r}\,(\bar{r} + 1)} \cdot \sqrt[4]{1 + 2\gamma\frac{L}{B}} \qquad (66)$$

For example, assuming value for $\gamma\frac{L}{B}$, the critical side ratios α for which the number of half waves $\bar{r}$ changes from 1 to 2 are as follows:

$\alpha = 3.40$ for $\gamma\frac{L}{B} = 16$

$\alpha = 3.58$ for $\gamma\frac{L}{B} = 20$

$\alpha = 3.74$ for $\gamma\frac{L}{B} = 24$

If two or more terms are taken into account in Equation (62) the critical side ratios α can be determined more precisely. Nevertheless Equation (66) gives a fair approximation in spite of its simple form as a comparison with results obtained by Barbré[9] will show. According to his findings:

$\alpha = 3.30$ for $\gamma\frac{L}{B} = 16$ and $\delta = 0$

$\alpha = 3.50$ for $\gamma\frac{L}{B} = 20$ and $\delta = 0.05$

$\alpha = 3.75$ for $\gamma\frac{L}{B} = 24$ and $\delta = 0.1$

Therefore Equation (66) is used to find the numbers of half waves $\bar{r}$ for the given α and $\gamma \frac{L}{B}$.

The torsional resistance of the stiffener, previously neglected, has a certain influence upon the critical side ratio α in the case of an antisymmetric transverse mode. The maximum influence can be derived from the following consideration. The minimum buckling stress of a plate with three edges simply supported and one unloaded edge fixed is obtained at $\alpha = 0.79$ [25], whereas in the case of a plate with four edges simply supported the minimum buckling stress occurs at $\alpha = 1$ [4]. This tendency of α to decrease with increasing torsional resistance was also obtained by Okuda[26]. He computed the case of a plate with three edges simply supported and one unloaded edge elastically restrained, the resisting moment being proportional to the edge rotation.

On the basis of these considerations it must be concluded that the number of half waves r for a given value of the side ratio α depends upon the magnitude of the torsional rigidity of the longitudinal stiffener. However, the influence of the torsional resistance of a stiffener with an open cross-section upon the critical side ratio α is comparatively small compared to the fixed-end case. Therefore if a value of α is chosen halfway between the limits for α established by Equation (65), neglecting the torsional resistance, the number of half waves r will be unchanged. For example, a square plate has the value of $\alpha = 1$ in the middle of two critical points $\alpha = \frac{\sqrt{2}}{2}$ and $\alpha = \frac{\sqrt{6}}{2}$, therefore r is given by 2.

The convergence of Equation (63) is shown in Fig. 15 and Table 4 by using the following example:

$$\alpha = 1, \quad (r = 2)$$

$$\gamma_s^L + \frac{r^2}{\alpha^2} \pi^2 \gamma_w^L = 0.292^*$$

$$\pi^2 \delta \left(\frac{R^L}{b}\right)^2 = 0.0025^*$$

The convergence for the buckling coefficient k of a longitudinally stiffened plate is slightly slower than that of a transversely stiffened plate. However, it is sufficiently fast such that the 2nd approximation gives an error of less than 1 percent.

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* See Apprneix 11.1, Equations (A-5) and (A-6).

5.2 Effect of Torsional Resistance of a Longitudinal Stiffener on Buckling Strength in the Elastic Range

A stiffener of rectangular shape has little torsional resistance. Hence the buckling coefficient k for a plate with a longitudinal rectangular stiffener is k = 16[10] for $\alpha = 1.0$ if the bending rigidity of a stiffener is greater than the required minimum stiffness. A stiffener of Tee-shape whose proportions are chosen such that the local instability in neither flange nor web plate will occur prior to the point of strain-hardening has a certain amount of torsional rigidity. Taking as an example,

$$\gamma_s^L + \frac{r^2}{\alpha^2} \pi^2 \gamma_w^L = 0.292$$

$$\pi^2 \delta \left(\frac{R^L}{b}\right)^2 = 0.0025$$

as previously used in Section 5.1, the buckling coefficient k for the same plate, $\alpha = 1$, increases to k = 17.656 as shown in Table 4, which was obtained by using 5 terms in Equation (63). The increase amounts to 10.35 percent compared to Barbre's value neglecting torsional resistance.

It is of interest to recall the increase of the buckling coefficient k due to the torsional resistance of a similar transverse stiffener. For $\alpha = 1.0$ the corresponding values are k = 6.25 with no torsional resistance, and

$k = 6.417$ with $\gamma_s^T + \pi^2 \gamma_w^T = 0.085$ as shown in Table 1. The percentage of increase in k is 2.67 for this case, where the total weight of the stiffener equals the one of the longitudinal stiffener investigated above. This means that the effect of the torsional resistance of a longitudinal stiffener is greater than that of a transverse stiffener with equal cross-section. The reason can be found in the fact that the warping resistance increases considerably when a stiffener has both positive and negative twist along its span as shown in Fig. 16. However, in order to obtain this increase in the buckling coefficient k the required minimum bending rigidity of the stiffener must also be raised to enforce a nodal line along its length. In the above example, the required minimum bending rigidities of the stiffeners for the case $\alpha = 1.0$ are given as follows:

Transverse stiffener,

$\gamma_B^T = 1.191$ with no torsional resistance

$\gamma_B^T = 1.294$ with torsional resistance

Longitudinal stiffener,

$\gamma_B^L = 8.809$ with no torsional resistance

$\gamma_B^L = 9.443$ with torsional resistance

5.3 Required Minimum Bending Rigidity of a Longitudinal Stiffener for the Buckling of Plate in the Elastic Range

As discussed in the previous Section 5.2 the required minimum bending rigidity of a longitudinal stiffener depends upon the torsional resistance of the stiffener. When the torsional resistance is negligible as in the case of a rectangular stiffener, the buckling coefficient k is given by Equation (64). Substituting this value of k into Equation (62) and using proper number of r and $\bar{r}$ for each case of α, the required minimum bending rigidity of a longitudinal stiffener is given for $\delta = 0.1$ in Fig. 17 and Table 5 together with values from Barbré[9], Bleich's approximation and the German specification. Bleich's approximation is given[5] by

$$\gamma = 11.4\,\alpha + (1.25 + 16\,\delta)\,\alpha^2 - 5.4\,\sqrt{\alpha}$$

and the German specification[24] prescribe

$$\gamma = \frac{\alpha^2}{2}\left[16\,(1 + 2\,\delta) - 2\right] - \frac{\alpha^4}{2} + \frac{1 + 2\delta}{2}$$

The differences from Barbré's values are also tabulated in percentage. Barbré obtained the solution in the closed form starting from the differential equation and connecting the solutions of the individual plate panels by introducing the appropriate boundary conditions. As shown in Table 5 the results using 5 terms in Equation (62) have errors of less 1% -- less than 0.2% when $\alpha > 1$. The advantage of the much simpler computation by means of the recurrence formula, Equation (62), over the solution of Barbré's transcendental equation are evident, since r, $\bar{r}$, k, $k_{\bar{r},(2n-1)}$, α and δ in Equation (62) are all given and the value of the left hand series of Equation (62) can be computed as accurately as desired in a simple manner.

In order to compute the required minimum bending rigidity of a longitudinal stiffener with Tee-shape, the following torsional rigidities are used as an example.*

$$\delta = 0.05$$

$$\gamma \frac{L}{s} = 0.015$$

$$\pi^2 \gamma \frac{L}{w} = 0.070$$

$$\pi^2 \delta \left(\frac{R^L}{b}\right)^2 = 0.0025$$

---

* See Appendix 11.1, Equations (A-4) and (A-6).

As discussed in the previous Section 5.1, the 2nd approximation of Equation (63) gives an error less than 1%. Therefore the buckling coefficient k can be expressed by using 2 terms in Equation (63),

$$k = \frac{(1 + 32\Omega)\, k_{r2} + (1 + 8\Omega)\, k_{r4} + 40\,\Theta_r}{2\,(1 + 40\Omega)} \left[1 - \sqrt{1 - \frac{4(1 + 40\Omega)\cdot\{ 8\,\Theta_r\,(4\,k_{r2} + k_{r4}) + k_{r2}\,k_{r4}\}}{\{(1 + 32\,\Omega)\,k_{r2} + (1 + 8\,\Omega)\,k_{r4} + 40\,\Theta_r\}^2}}\;\right] \tag{67}$$

where

$$\Omega = \pi^2\,\delta\left(\frac{R^L}{b}\right)^2$$

$$\Theta_r = \gamma_s^L + \frac{r^2}{\alpha^2}\,\pi^2\,\gamma_w^L$$

Using the above values for the torsional coefficients γ_s^L, γ_w^L, and R^L the buckling coefficient k of the antisymmetric mode and the corresponding required minimum bending rigidity of the longitudinal stiffener γ_B^L can be obtained from Equation (67) and (63) respectively for values of side ratio by using the proper numbers for r and $\bar{r}$. The results are shown in Table 6 and in Fig. 17 as a dotted line, where the values of γ_B^L with no torsional resistance of the stiffener are taken from the result by Barbré[9]. As previously seen in Table 6, the required increase in the bending rigidity γ_B^L

due to the torsional resistance of a stiffener seems to be proportional to the value $\gamma\frac{L}{B}$ with no torsional resistance except for the case of small values of α. In this example the buckling strength of a plate increases by 11.4% with an increase in the bending rigidity of the stiffener equal to 18% compared to the case with no torsional resistance.

5.4 Convergence of a Secular Equation for Buckling in the Strain-Hardening Range

In the strain-hardening range the effective width b_e of a plate contributing to the bending rigidity of a longitudinal stiffener is given by

$$\frac{be}{b} = 0.392 \quad \text{for} \quad \alpha = 1$$

$$\frac{be}{b} = 0.468 \quad \text{for} \quad \alpha = 2$$

$$\frac{be}{b} = 0.486 \quad \text{for} \quad \alpha = 4$$

For long plates, therefore,

$$0.4 < \frac{be}{b} < 0.5$$

such that $\frac{be}{b}$ approaches the whole width, $\frac{be}{b} = 0.5$, assymptotically. For these cases, the following values* may be used.

$$\delta = \frac{A^L}{bh} = 0.1$$

$$\gamma_s^L = \frac{G_t K^L}{D_x Ib} = 0.07$$

$$\pi^2 \gamma_w^L = \frac{\pi^2 E_t \ I_z^L}{D_x Ib} \cdot \left(\frac{d_w^L}{b} \right)^2 = 0.18$$

$$\pi^2 \delta \left(\frac{R^L}{b} \right)^2 = 0.026$$

and
$$\gamma_B^L = \frac{E_t \ I_y^L}{D_x \ Ib}$$

Since the stiffener is compressed into the strain-hardening range prior to buckling, its moduli are the ones corresponding to the beginning of strain-hardening.

The number of half waves r for the antisymmetric mode in the transverse direction is obtained for this case by modifying Equation (65):

$$\alpha = \frac{\sqrt{r \cdot (r + 1)}}{2} \ \frac{1}{\sqrt[4]{\mu}} \tag{68}$$

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* Appendix 11.2, Equation (A-7).

where $\mu = D_y/D_x = 10.93$ for A-7 steel, therefore

$$\alpha = 0.55 \frac{\sqrt{r\cdot(r+1)}}{2}$$

This means that the critical side ratio α decreases by 55% compared to the elastic case, that is,

$$
\begin{array}{lll}
r = 1 & \text{for} & \alpha \leqq 0.389 \\
r = 2 & \text{for} & 0.389 \leqq \alpha \leqq 0.674 \\
r = 3 & \text{for} & 0.674 \leqq \alpha \leqq 0.953 \\
r = 4 & \text{for} & 0.953 \leqq \alpha \leqq 1.230 \\
r = 5 & \text{for} & 1.230 \leqq \alpha \leqq 1.506 \\
r = 6 & \text{for} & 1.506 \leqq \alpha \leqq 1.782 \\
r = 7 & \text{for} & 1.782 \leqq \alpha \leqq 2.058 \\
r = 8 & \text{for} & 2.058 \leqq \alpha \leqq 2.333
\end{array}
$$

The number of half waves $\bar{r}$ for the symmetric mode in the transverse direction must be modified similarly by the strain-hardening moduli. The convergence of the secular equation for buckling is nearly as fast as in the elastic case.

5.5 Effect of Torsional Resistance of a Longitudinal Stiffener on Buckling Strength in the Strain-Hardening Range

The increase in the buckling coefficient k due to the torsional resistance of a longitudinal stiffener is given in Table 7 by using the values of torsional properties in Section 5.4. The buckling coefficient k is given by Equation (67) with the plate coefficients k_{rs} and the stiffener coefficients γ^L_s, γ^L_w, and R^L computed for the strain-hardening range. The buckling coefficient k with no torsional resistance of a stiffener is simply k_{r2} of the strain-hardened plate with the proper value of r for each side ratio α. The increase in k for this example due to the torsional resistance of the stiffener amounts from 8 to 13 percent.

5.6 Required Minimum Bending Rigidity of a Longitudinal Stiffener for Buckling of a Stiffened Plate in the Strain-Hardening Range

The required minimum bending rigidity of a longitudinal stiffener in the strain-hardening range is obtained by substituting the values of k in Table 7 into Equation (62) with proper coefficients $k_{\bar{r},(2m-1)}$ for the buckling of the plate in the strain-hardening range. The values of γ^L_B for the case with or without torsional resistance of the stiffener and their differences are also given in Table 7 and in Fig. 18. For the case in which the side ratio α is greater than 1.8 the value of $\gamma\frac{L}{B}$ equals 46.8 without torsional resistance and 58.8 with torsional resistance. The increase in the required bending rigidity is approximately 25 percent in this example.

6. STIFFENED PLATE WITH TRANSVERSE AND/OR LONGITUDINAL STIFFENERS HAVING AN UNSYMMETRIC THIN-WALLED OPEN CROSS-SECTION

6.1 Behavior of a Stiffener with an Unsymmetric Thin-Walled Open Cross-Section

For any shape of thin-walled open cross-section attached to a plate, the equation of equilibrium of a stiffener can be written in the same form as in the previous Section 3.1 if proper consideration is given to the mutual interaction between plate and stiffener. This influence of plate can be divided into two parts, one concerning the bending and the other the twisting of the stiffener. If a beam with an arbitrary cross-section is subjected to transverse loads, the beam is bent and twisted simultaneously unless the line of loading passes through the shear center. In the case of a stiffener attached to the plate, the stiffener is subjected to transverse loads passing through the junction between the plate and the stiffener and twisting moments due to a rotation of the plate along the junction at the moment of buckling. Along the line of junction with the plate the stiffener cannot deflect in the plane of the plate but only in a transverse direction. This enforced axis causes twisting of the stiffener when it is bent or the other way around, bending of the stiffener when it is twisted. Therefore the bending and the twisting of the stiffener are no longer separable for such a stiffener with an unsymmetric cross-section. Bending of such a stiffener can therefore occur only if it is associated with the twisting[27] producing warping deformations (Fig. 19.1). In the case of an inverted angle stiffener, for example, the axis of flange bending which causes warping, passes through the junction at which the center of rotation of the stiffener is located[28]. Therefore the resistance to the lateral movement of its centroid is provided by flange bending only.

Bending of a stiffener about an axis parallel to the plane of the plate causes stresses in the plate. This influence can be considered by determining the effective width of the plate as discussed in the previous Section 2.3. The position of an axis parallel to the plate, then, can be obtained by considering this effective width as part of the stiffener. The intersection of this axis with the z-axis considering the stiffener only as shown in Fig. 19.2 determines the position of the apparent centroid of the cross-section. In Fig. 19.2 the effective width b_e appears only in the computation of the moment of inertia about the y-axis, and the moment of inertia about z-axis has no effective width of plate. Thus I_z is given by

$$I_z = \frac{1}{3} b_f^3 \cdot t_f$$

and

$$I_{yz} = \frac{1}{2} (d_w - z_A)\, b_f^{2} . t_f$$

The inclination of the principal axes $\bar{y}$, $\bar{z}$ from the axes y, z , which are parallel and perpendicular to the plane of plate, is given by

$$\tan 2\theta = \frac{2 I_{yz}}{I_y - I_z}$$

6.1.1 Longitudinal Stiffeners

The centroidal principal axes $\bar{y}$ and $\bar{z}$ of the cross-section of a longitudinal stiffener are chosen as shown in Fig. 20. The spring constants $\bar{k}_y$ and $\bar{k}_z$ of an elastic medium are also acting in the directions of the principal axes $\bar{y}$ and $\bar{z}$. The equation of equilibrium of a longitudinal stiffener with respect to the centroidal principal axes is given by[21]

$$E_t \bar{I}_z^L \frac{d^4 \bar{v}_s}{dx^4} + \sigma_x A^L \left[\frac{d^2 \bar{v}_s}{dx^2} + \bar{z}_s^L \frac{d^2 \varphi^L}{dx^2} \right] + \bar{k}_y \left[\bar{v}_s + \varphi^L (\bar{z}_s^L - \bar{z}_A^L) \right]$$

$$= 0 \qquad (69a)$$

$$E_t \bar{I}_y^L \frac{d^4 \bar{w}_s}{dx^4} + \sigma_x A^L \left[\frac{d^2 \bar{w}_s}{dx^2} - y_s^L \frac{d^2 \varphi^L}{dx^2} \right] + \bar{k}_z \left[\bar{w}_s - \varphi^L (y_s^L - y_A^L) \right]$$

$$= 0 \tag{69b}$$

$$E_t W^L \frac{d^4 \varphi^L}{dx^4} - \frac{d^2 \varphi^L}{dx^2} \left[G_t K^L - \sigma_x A^L (\bar{r}_o^L)^2 \right]$$

$$- \sigma_x A^L \left[\bar{y}_s^L \frac{d^2 \bar{w}_s}{dx^2} - \bar{z}_s^L \frac{d^2 \bar{v}_s}{dx^2} \right]$$

$$+ \bar{k}_y \left[\bar{v}_s + \varphi^L (\bar{z}_s^L - \bar{z}_A^L) \right] (\bar{z}_s^L - \bar{z}_A^L)$$

$$- \bar{k}_z \left[\bar{w}_s - \varphi^L (\bar{y}_s^L - \bar{y}_A^L) \right] (\bar{y}_s^L - \bar{y}_A^L)$$

$$+ k_\varphi \cdot \varphi^L$$

$$= 0 \tag{69c}$$

where subscript s refers to shear center and superscript L pertains to longitudinal stiffener as in Section 3.1. The displacements $\bar{v}_s$ and $\bar{w}_s$ of the shear center can be expressed by the displacements $\bar{v}_A$ and $\bar{w}_A$ of enforced point A as follows

$$\begin{aligned} \bar{v}_A &= \bar{v}_s + \varphi^L (\bar{z}_s^L - \bar{z}_A^L) \\ \bar{w}_A &= \bar{w}_s - \varphi^L (\bar{y}_s^L - \bar{y}_A^L) \end{aligned} \tag{70}$$

The transformation of the components of the displacement is given by

$$\bar{v}_A = v_A \cos\theta + w_A \sin\theta$$

$$\bar{w}_A = w_A \cos\theta - v_A \sin\theta \qquad (71)$$

where v_A, the displacement of the point A in the direction of the plane of plate, must be zero, and w_A is the displacement of the point A in the direction perpendicular to the plate. Then Equation (71) yields

$$\bar{v}_A = w_A \sin\theta$$

$$\bar{w}_A = w_A \cos\theta \qquad (72)$$

From Equations (70) and (72), the displacement of the shear center can be obtained in terms of w_A and φ^L.

Therefore

$$\bar{v}_s = w_A \sin\theta - \varphi^L(\bar{z}_s^L - \bar{z}_A^L)$$

$$\bar{w}_s = w_A \cos\theta + \varphi^L(\bar{y}_s^L - \bar{y}_A^L) \qquad (73)$$

The spring constants $\bar{k}_y$ and $\bar{k}_z$ in the direction of the principal axes are also transformed similarly:

$$\bar{k}_y = k_z \sin \theta$$

$$\bar{k}_z = k_z \cos \theta \qquad (74)$$

because the spring constant in the direction of the plane of plate is assumed to be infinitive.

$$k_y = \infty \quad \text{for} \quad v_A = 0$$

However, the product of k_y and v_A is finite and equal to the horizontal reaction H.

$$k_y \cdot v_A = H \qquad (75)$$

Therefore the last term in the Equation (69a), which represents the distributed load in the direction of the $\bar{\bar{y}}$-axis, acting from the plate to the stiffener along the connection between them, at the instant of the plate buckling, can be expressed by using Equations (73), (74) and (75) as

$$\bar{\bar{q}}_y = - \bar{\bar{k}}_y \left[\bar{\bar{v}}_S + \varphi^L (\bar{\bar{z}}_S^L - \bar{\bar{z}}_A^L) \right.$$

$$= - k_z \cdot w_A \sin^2 \theta - H \cos \theta \qquad (76)$$

Similarly in Equation (69b)

$$\bar{q}_z = -\bar{k}_z [\bar{w}_S - \varphi^L (\bar{y}_S^L - \bar{y}_A^L)]$$

$$= -k_z \cdot w_A \cos^2\theta - H \sin\theta \tag{77}$$

Substituting Equations (73) and (76) into Equation (69a),

$$E_t \bar{I}_z^L \left[\sin\theta \frac{d^4 w_A}{dx^4} - (\bar{z}_S^L - \bar{z}_A^L) \frac{d^4\varphi^L}{dx^4} \right] + \sigma_x A^L \left[\sin\theta \frac{d^2 w_A}{dx^2} + \bar{z}_A^L \frac{d^2\varphi^L}{dx^2} \right] + k_z \cdot w_A \sin^2\theta + H \cos\theta = 0 \tag{78a}$$

From Equations (73), (77) and (69b),

$$E_t \bar{I}_y^L \left[\cos\theta \frac{d^4 w_A}{dx^4} + (\bar{y}_S^L - \bar{y}_A^L) \frac{d^4\varphi^L}{dx^4} \right] + \sigma_x A^L \left[\cos\theta \frac{d^2 w_A}{dx^2} - \bar{y}_A^L \frac{d^2\varphi^L}{dx^2} \right] + k_z w_A \cos^2\theta + H \sin\theta = 0 \tag{78b}$$

Similarly Equation (69c) yields

$$E_t W^L \frac{d^4\varphi^L}{dx^4} - \frac{d^2\varphi^L}{dx^2}\left[G_t K^L - \sigma_x A^L (\bar{r}_0^L)^2 \right]$$

$$- \sigma_x A^L \left[\bar{y}_s^L \left\{ \cos\theta \frac{d^2 w_A}{dx^2} + (\bar{y}_s^L - \bar{y}_A^L) \frac{d^2\varphi^L}{dx^2} \right\} \right.$$

$$\left. - \bar{z}_s^L \left\{ \sin\theta \frac{d^2 w_A}{dx^2} - (\bar{z}_s^L - \bar{z}_A^L) \frac{d^2\varphi^L}{dx^2} \right\} \right]$$

$$+ (\bar{z}_s^L - \bar{z}_A^L) \cdot (k_z w_A \sin^2\theta + H \cos\theta)$$

$$- (\bar{y}_s^L - \bar{y}_A^L) \cdot (k_z w_A \cos^2\theta + H \sin\theta)$$

$$+ k_\varphi \cdot \varphi^L$$

$$= 0 \qquad (78c)$$

The horizontal force H and the vertical force $k_z \cdot w_A$ can be obtained from Equations (78a) and (78b). The twisting moment $k_\varphi \cdot \varphi^L$ can also be determined by substituting these forces H and $k_z \cdot w_A$ into Equation (78c).

For an inverted angle stiffener, (Fig. 19.2)

$$\bar{y}_s^L = -(d_w^L - z_A^L)\sin\theta$$

$$\bar{y}_A^L = z_A \sin\theta$$

$$\bar{z}_s^L = -(d_w^L - z_A^L)\cos\theta$$

$$\bar{z}_A^L = z_A^L \cos\theta$$

$$W^L = 0 \text{ (Warping Constant)}$$

Therefore

$$\bar{y}_s^L - \bar{y}_A^L = -d_w^L \sin\theta$$

$$\bar{z}_s^L - \bar{z}_A^L = -d_w^L \cos\theta \tag{79}$$

Substituting the values in Equation (79) into Equation (78), the final result yields

$$k_z \cdot w_A = -\left[A_1^L \frac{d^4 w_A}{dx^4} + B_1^L \frac{d^2 w_A}{dx^2} - C_1^L \frac{d^4 \varphi^L}{dx^4} - D_1^L \frac{d^2 \varphi^L}{dx^2}\right] \tag{80a}$$

$$k_\varphi \cdot \varphi^L = -\left[A_2^L \frac{d^4 \varphi^L}{dx^4} - B_2^L \frac{d^2 \varphi^L}{dx^2} + C_2^L \frac{d^4 w_A}{dx^4}\right] \tag{80b}$$

where

$$A_1^L = \frac{\cos 2\theta}{\cos^3\theta - \sin^3\theta} E_t \left[I_y^L + I_{yz}^L \tan 2\theta \right]$$

$$B_1^L = \frac{\cos 2\theta}{\cos^3\theta - \sin^3\theta} \cdot \sigma_x A^L$$

$$C_1^L = \frac{1}{2} \cdot \frac{\sin 2\theta}{\cos^3\theta - \sin^3\theta} \cdot E_t I_p^L \cdot d_w^L$$

$$D_1^L = \frac{\sin 2\theta}{\cos^3\theta - \sin^3\theta} \cdot \sigma_x A^L z_A^L$$

$$A_2^L = E_t (d_w^L)^2 \left[\frac{1}{2} I_p^L \sin^2 2\theta + (I_z^L - I_{yz}^L \tan 2\theta) \cos^2 2\theta \right]$$

$$B_2^L = G_t K^L - \sigma_x A^L \left[\frac{I_p^L}{A^L} + (z_A^L)^2 \right] = G_t K^L - \sigma_x A^L (R^L)^2$$

$$C_2^L = E_t d_w^L \sin 2\theta \cos 2\theta \left[\frac{1}{2} I_p^L - (I_y^L + I_{yz}^L \tan 2\theta) \right]$$

$$I_p^L = \text{Polar moment of inertia} = I_y^L + I_z^L \qquad (81)$$

For the case of symmetric stiffeners about the z-axis,

$$\theta = 0 \quad \text{and} \quad I_{yz}^L = 0$$

therefore,

$$A_1^L = E_t I_y^L$$

$$B_1^L = \sigma_x A^L$$

$$C_1^L = 0$$

$$D_1^L = 0$$

$$A_2^L = E_t I_z^L (d_w^L)^2$$

$$B_2^L = G_t K^L - \sigma_x A^L (R^L)^2$$

$$C_2^L = 0$$

Then the force $k_z\, w_A$ and the moment $k_\varphi \cdot \varphi^L$ yield

$$k_z \cdot w_A = -\left[E_t I_y^L \frac{d^4 w_A}{dx^4} + \sigma_x A^L \frac{d^2 w_A}{dx^2} \right]$$

$$k_\varphi \cdot \varphi^L = -\left[E_t I_z^L (d_w^L)^2 \frac{d^4 \varphi^L}{dx^4} - \left\{ G_t K^L - \sigma_x A^L (R^L)^2 \right\} \frac{d^2 \varphi^L}{dx^2} \right]$$

These are exactly the same expressions which were obtained in the previous Section 3.1.1.

6.1.2 Transverse Stiffeners

The transverse stiffeners lie in the direction of y-axis of the plate and no axial force acts on it. Therefore

Equations (80a) and (80b) change into the form:

$$k_z \cdot w_A = -\left[A_1^T \frac{d^4 w_A}{d y^4} - C_1^T \frac{d^4 \varphi^T}{d y^4} \right] \tag{82a}$$

$$k_\varphi \cdot \varphi^T = -\left[A_2^T \frac{d^4 \varphi^T}{d y^4} - B_2^T \frac{d^2 \varphi^T}{d y^2} + C_2^T \frac{d^4 w_A}{d y^4} \right] \tag{82b}$$

where

$$A_1^T = \frac{\cos 2\theta}{\cos^3\theta - \sin^3\theta} E \left[I_x^T + I_{xy}^T \tan 2\theta \right]$$

$$C_1^T = \frac{1}{2} \cdot \frac{\sin 2\theta}{\cos^3\theta - \sin^3\theta} E I_p^T d_w^T$$

$$A_2^T = E (d_w^T)^2 \left[\frac{1}{2} I_p^T \sin^2 2\theta + (I_z^T - I_{xz}^T \tan 2\theta) \cos^2 2\theta \right]$$

$$B_2^T = G K^T$$

$$C_2^T = E d_w^T \sin 2\theta \cos 2\theta \left[\frac{1}{2} I_p^T - (I_x^T + I_{xz}^T \tan 2\theta) \right] \tag{83}$$

Since the transverse stiffeners are subject to no axial forces, therefore the moduli E and G remain always elastic. For the symmetric transverse stiffeners, $\theta = 0$ and $I_{xz}^T = 0$, therefore

$$A_1^T = E I_x^T$$

$$C_1^T = 0$$

$$A_2^T = EI_z^T \, (d_w^T)^2$$

$$B_2^T = GK^T$$

$$C_2^T = 0$$

Then

$$k_z \cdot w_A = - EI_x^T \frac{d^4 w_A}{d y^4}$$

$$k_\varphi \cdot \varphi^T = -\left[EI_z^T (d_w^T)^2 \frac{d^4 \varphi^T}{d y^4} - GK^T \frac{d^2 \varphi^T}{d y^2} \right]$$

These are also the same results as in the previous Section 3.1.2.

6.2 Integral Equation for Buckling of an Orthotropic Plate with Unsymmetric Type of Stiffeners

When the plate starts to buckle, the distributed resistance due to the i-th longitudinal stiffener is given by Equations (80a) and (80b) as

$$k_{zi} w_{Ai} = -\left[A_{1i}^L \frac{d^4 w_{Ai}}{d x^4} + B_{1i}^L \frac{d^2 w_{Ai}}{d x^2} - C_{1i}^L \frac{d^4 \varphi_i^L}{d x^4} - D_{1i}^L \frac{d^2 \varphi_i^L}{d x^2} \right]_{y=\eta_i}$$

$$k_{\varphi i} \varphi_i^L = -\left[A_{2i}^L \frac{d^4 \varphi_i^L}{d x^4} - B_{2i}^L \frac{d^2 \varphi_i^L}{d x^2} + C_{2i}^L \frac{d^4 w_{Ai}}{d x^4} \right]_{y=\eta_i}$$

where

$$w_{Ai}^{L} = w(\xi, \eta_i)$$

$$\varphi_i^{L} = \frac{\partial w}{\partial y}(\xi, \eta_i)$$

Therefore

$$k_{zi} w_{Ai} = -\left[A_{1i}^{L} \frac{\partial^4 w}{\partial \xi^4} + B_{1i}^{L} \frac{\partial^2 w}{\partial \xi^2} - C_{1i}^{L} \frac{\partial^5 w}{\partial \xi^4 \partial \eta} - D_{1i}^{L} \frac{\partial^3 w}{\partial \xi^2 \partial \eta} \right]_{\eta=\eta_i}$$

$$k_{\varphi i} \varphi_i^{L} = -\left[A_{2i}^{L} \frac{\partial^5 w}{\partial \xi^4 \partial \eta} - B_{2i}^{L} \frac{\partial^3 w}{\partial \xi^2 \partial \eta} + C_{2i}^{L} \frac{\partial^4 w}{\partial \xi^4} \right]_{\eta=\eta_i} \quad (84)$$

Similarly for the j-th transverse stiffener, from Equation (82)

$$k_{zj} w_{Aj} = -\left[A_{1j}^{T} \frac{\partial^4 w}{\partial \eta^4} - C_{1j}^{T} \frac{\partial^5 w}{\partial \xi \partial \eta^4} \right]_{\xi=\xi_j}$$

$$k_{\varphi j} \varphi_j^{T} = -\left[A_{2j}^{T} \frac{\partial^5 w}{\partial \xi \partial \eta^4} - B_{2j}^{T} \frac{\partial^3 w}{\partial \xi \partial \eta^2} + C_{2j}^{T} \frac{\partial^4 w}{\partial \eta^4} \right]_{\xi=\xi_j} \quad (85)$$

Using Equations (84) and (85), the integral equation for the buckling of a longitudinally and transversely stiffened plate can be formulated as in Section 3.2 as follows:

$w(x, y)$

$$= -\int_0^a\int_0^b N_x \frac{\partial^2 w}{\partial \xi^2}(\xi, \eta) \cdot G(x, y; \xi, \eta)\, d\xi\, d\eta$$

$$- \sum_{i=1}^{n} \int_0^a A_{1i}^L \frac{\partial^4 w}{\partial \xi^4}(\xi, \eta_i) \cdot G(x, y; \xi, \eta_i)\, d\xi$$

$$- \sum_{i=1}^{n} \int_0^a B_{1i}^L \frac{\partial^2 w}{\partial \xi^2}(\xi, \eta_i) \cdot G(x, y; \xi, \eta_i)\, d\xi$$

$$+ \sum_{i=1}^{n} \int_0^a C_{1i}^L \frac{\partial^5 w}{\partial \xi^4 \partial \eta}(\xi, \eta_i) G(x, y; \xi, \eta_i)\, d\xi$$

$$+ \sum_{i=1}^{n} \int_0^a D_{1i}^L \frac{\partial^3 w}{\partial \xi^2 \partial \eta}(\xi, \eta_i) G(x, y; \xi, \eta_i)\, d\xi$$

$$- \sum_{i=1}^{n} \int_0^a A_{2i}^L \frac{\partial^5 w}{\partial \xi^4 \partial \eta}(\xi, \eta_i) \cdot \bar{G}_\eta(x, y; \xi, \eta_i)\, d\xi$$

$$+ \sum_{i=1}^{n} \int_0^a B_{2i}^L \frac{\partial^3 w}{\partial \xi^2 \partial \eta}(\xi, \eta_i) \cdot \bar{G}_\eta(x, y; \xi, \eta_i)\, d\xi$$

$$- \sum_{i=1}^{n} \int_0^a C_{2i}^L \frac{\partial^4 w}{\partial \xi^4}(\xi, \eta_i) \cdot \bar{G}_\eta(x, y; \xi, \eta_i)\, d\xi$$

$$- \sum_{j=1}^{m} \int_0^b A_{1j}^T \frac{\partial^4 w}{\partial \eta^4}(\xi_j, \eta) \cdot G(x, y; \xi_j, \eta)\, d\eta$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\int_0^b C^T_{1j}\,\frac{\partial^5 w}{\partial\xi\partial\eta^4}(\xi_j,\eta)\cdot G(x,y;\xi_j,\eta)\,d\eta \\
&-\sum_{j=1}^{m}\int_0^b A^T_{2j}\,\frac{\partial^5 w}{\partial\xi\partial\eta^4}(\xi_j,\eta)\cdot \bar{G}_{\xi}(x,y;\xi_j,\eta)\,d\eta \\
&+\sum_{j=1}^{m}\int_0^b B^T_{2j}\,\frac{\partial^3 w}{\partial\xi\partial\eta^2}(\xi_j,\eta)\cdot \bar{G}_{T\xi}(x,y;\xi_j,\eta)\,d\eta \\
&-\sum_{j=1}^{m}\int_0^b C^T_{2j}\,\frac{\partial^4 w}{\partial\eta^4}(\xi_j,\eta)\cdot \bar{G}_{\xi}(x,y;\xi_j,\eta)\,d\eta
\end{aligned}
\tag{86}
$$

where m and n are numbers of transverse and longitudinal stiffeners respectively.

6.3 Solution of Integral Equation and Its Secular Equation for Eigenvalues

Equation (86) can be integrated by using the orthogonality of the deflection w and Green's function $G(x,y;\xi,\eta)$, $\bar{G}_{\xi}(x,y;\xi,\eta)$ and $\bar{G}_{\eta}(x,y;\xi,\eta)$ as one assumes the deflection w as follows:

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn}\,\varphi_{mn}(x,y)$$

and $$\varphi_{mn}(x,y)=\frac{2}{\sqrt{ab}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

The result of the integration of Equation (86) is obtained as

$$\left[N_x \left(\frac{r\pi}{a}\right)^2 - \lambda_{rs}^2 \right] \alpha_{rs}$$

$$= \frac{2}{b}\left(\frac{r\pi}{a}\right)^4 \sum_{i=1}^{n} A_{1i}^L \sin\frac{s\pi\eta_i}{b} \sum_{p=1}^{\infty} \alpha_{rp} \sin\frac{p\pi\eta_i}{b}$$

$$- \frac{2}{b}\left(\frac{r\pi}{a}\right)^2 \sum_{i=1}^{n} B_{1i}^L \sin\frac{s\pi\eta_i}{b} \sum_{p=1}^{\infty} \alpha_{rp} \sin\frac{p\pi\eta_i}{b}$$

$$- \frac{2}{b}\left(\frac{r\pi}{a}\right)^4 \sum_{i=1}^{n} C_{1i}^L \sin\frac{s\pi\eta_i}{b} \sum_{p=1}^{\infty} \alpha_{rp}\left(\frac{p\pi}{b}\right) \cos\frac{p\pi\eta_i}{b}$$

$$+ \frac{2}{b}\left(\frac{r\pi}{a}\right)^2 \sum_{i=1}^{n} D_{1i}^L \sin\frac{s\pi\eta_i}{b} \sum_{p=1}^{\infty} \alpha_{rp}\left(\frac{p\pi}{b}\right) \cos\frac{p\pi\eta_i}{b}$$

$$+ \frac{2}{b}\left(\frac{r\pi}{a}\right)^4\left(\frac{s\pi}{b}\right) \sum_{i=1}^{n} A_{2i}^L \cos\frac{s\pi\eta_i}{b} \sum_{p=1}^{\infty} \alpha_{rp}\left(\frac{p\pi}{b}\right) \cos\frac{p\pi\eta_i}{b}$$

$$+ \frac{2}{b}\left(\frac{r\pi}{a}\right)^2\left(\frac{s\pi}{b}\right) \sum_{i=1}^{n} B_{2i}^L \cos\frac{s\pi\eta_i}{b} \sum_{p=1}^{\infty} \alpha_{rp}\left(\frac{p\pi}{b}\right) \cos\frac{p\pi\eta_i}{b}$$

$$+ \frac{2}{b}\left(\frac{r\pi}{a}\right)^4\left(\frac{s\pi}{b}\right) \sum_{i=1}^{n} C_{2i}^L \cos\frac{s\pi\eta_i}{b} \sum_{p=1}^{\infty} \alpha_{rp} \sin\frac{p\pi\eta_i}{b}$$

$$+ \frac{2}{a}\left(\frac{s\pi}{b}\right)^4 \sum_{j=1}^{m} A_{1j}^T \sin\frac{r\pi\xi_j}{a} \sum_{q=1}^{\infty} \alpha_{qs} \sin\frac{q\pi\xi_j}{a}$$

$$- \frac{2}{a}\left(\frac{s\pi}{b}\right)^4 \sum_{j=1}^{m} C_{1j}^T \sin\frac{r\pi\xi_j}{a} \sum_{q=1}^{\infty} \alpha_{qs}\left(\frac{q\pi}{a}\right) \cos\frac{q\pi\xi_j}{a}$$

$$+ \frac{2}{a}\left(\frac{r\pi}{a}\right)\left(\frac{s\pi}{b}\right)^4 \sum_{j=1}^{m} A_{2j}^{T} \cos\frac{r\pi\xi_j}{a} \sum_{q=1}^{\infty} \alpha_{qs}\left(\frac{q\pi}{a}\right)\cos\frac{q\pi\xi_j}{a}$$

$$+ \frac{2}{a}\left(\frac{r\pi}{a}\right)\left(\frac{s\pi}{b}\right)^2 \sum_{j=1}^{m} B_{2j}^{T} \cos\frac{r\pi\xi_j}{a} \sum_{q=1}^{\infty} \alpha_{qs}\left(\frac{q\pi}{a}\right)\cos\frac{q\pi\xi_j}{a}$$

$$+ \frac{2}{a}\left(\frac{r\pi}{a}\right)\left(\frac{s\pi}{b}\right)^4 \sum_{j=1}^{m} C_{2j}^{T} \cos\frac{r\pi\xi_j}{a} \sum_{q=1}^{\infty} \alpha_{qs} \sin\frac{q\pi\xi_j}{a} \tag{87}$$

The determinant of the coefficients α_{rs} in Equation (87) gives the secular equation for eigenvalues and the solution of the determinant gives the buckling strength of the stiffened plate with longitudinal and transverse stiffeners having an inverted angle section.

7. BUCKLING STRENGTH OF AN ORTHOTROPIC PLATE WITH AN UNSYMMETRIC TYPE OF STIFFENER TRANSVERSELY PLACED ON A PLATE

7.1 Convergence of a Secular Equation for Buckling in the Elastic Range

For the elastic buckling of a stiffened plate with one transverse stiffener in the middle of the plate, having an unsymmetric cross-section, for example an inverted angle stiffener, i = 0 and j = 1 in Equation (87), therefore Equation (87) yields

$$
\begin{aligned}
&\left[N_x \left(\frac{r\pi}{a}\right)^2 - \lambda_{rs}^2 \right] \alpha_{rs} \\
&= \frac{2}{a}\left(\frac{s\pi}{b}\right)^4 A_1^T \sin\frac{r\pi}{2} \sum_{q=1}^{\infty} \alpha_{qs} \sin\frac{q\pi}{2} \\
&\quad - \frac{2}{a}\left(\frac{s\pi}{b}\right)^4 C_1^T \sin\frac{r\pi}{2} \sum_{q=1}^{\infty} \alpha_{qs}\left(\frac{q\pi}{a}\right) \cos\frac{q\pi}{2} \\
&\quad + \frac{2}{a}\left(\frac{r\pi}{a}\right)\left(\frac{s\pi}{b}\right)^4 A_2^T \cos\frac{r\pi}{2} \sum_{q=1}^{\infty} \alpha_{qs}\left(\frac{q\pi}{a}\right) \cos\frac{q\pi}{2} \\
&\quad + \frac{2}{a}\left(\frac{r\pi}{a}\right)\left(\frac{s\pi}{b}\right)^2 B_2^T \cos\frac{r\pi}{2} \sum_{q=1}^{\infty} \alpha_{qs}\left(\frac{q\pi}{a}\right) \cos\frac{q\pi}{2} \\
&\quad + \frac{2}{a}\left(\frac{r\pi}{a}\right)\left(\frac{s\pi}{b}\right)^4 C_2^T \cos\frac{r\pi}{2} \sum_{q=1}^{\infty} \alpha_{qs} \sin\frac{q\pi}{2}
\end{aligned}
\tag{88}
$$

For the transversely stiffened plate the minimum buckling stress can be obtained when the number of half waves in the transverse direction S equals to 1. The critical stress σ_{cr} is expressed by using a non-dimensional coefficient of buckling k as

$$\sigma_{cr} = \frac{N_x}{h} = \frac{\pi^2}{12}\frac{E}{1-\nu^2}\left(\frac{h}{b}\right)^2 k$$

The plate buckling coefficient k_{rs} with no stiffener is expressed by

$$k_{r1} = \left(\frac{r}{\alpha} + \frac{\alpha}{r}\right)^2 \quad \text{for } s = 1$$

The following parameters are defined to simplify the expression of Equation (88).

$$\gamma_{B1}^T = \frac{A_1^T}{DIb} = \frac{\cos 2\theta}{\cos^3\theta - \sin^3\theta}\,\frac{EI_e^T}{DIb}$$

$$\gamma_{B2}^T = \frac{\pi C_1^T}{DIb^2} = \frac{1}{2}\cdot\frac{\sin 2\theta}{\cos^3\theta - \sin^3\theta}\cdot\frac{EI_P^T}{DIb}\cdot\left(\frac{\pi d_w^T}{b}\right)$$

$$\gamma_{T1}^T = \frac{\pi^2 A_2^T}{DIb^3} = \left(\frac{\pi d_w^T}{b}\right)^2 \frac{E}{DIb}\left[I_e^{T'}\cos^2 2\theta + \frac{I_P^T}{2}\sin^2 2\theta\right]$$

$$\gamma_{T2}^T = \frac{B_2^T}{DIb} = \frac{GK^T}{DIb}$$

$$\gamma_{T3}^T = \frac{\pi C_2^T}{DIb^2} = \left(\frac{\pi d_w^T}{b}\right)\frac{E}{DIb}\left[\frac{I_P^T}{2} - I_e^T\right]\sin 2\theta\cdot\cos 2\theta$$

(89)

where

$$D = \frac{E}{1-\nu^2}$$

$$I = \frac{h^3}{12}$$

$$I_e^T = I_x^T + I_{xz}^T \tan 2\theta$$

$$I_e^{T\prime} = I_z^T - I_{xz}^T \tan 2\theta$$

$$I_p^T = I_x^T + I_z^T = I_e^T + I_e^{T\prime}$$

For a symmetric stiffener such as the Tee-shape, $\theta = 0$, then

$$\gamma_{B_1}^T = \frac{EI_x^T}{DIb} = \gamma_B^T$$

$$\gamma_{B_2}^T = 0$$

$$\gamma_{T_1}^T = \left(\frac{\pi d^T{}_w}{b}\right)^2 \cdot \frac{EI_z^T}{DIb} = \pi^2 \gamma_w^T$$

$$\gamma_{T_2}^T = \frac{GK^T}{DIb} = \gamma_s^T$$

$$\gamma_{T_3}^T = 0$$

Using the abbreviation in Equation (89), the Equation (88) yields

$$2\alpha\gamma^T_{B1}\Phi + 2\frac{(\gamma^T_{T1}+\gamma^T_{T2})}{\alpha}\chi - 4\left[\gamma^T_{B1}\cdot(\gamma^T_{T1}+\gamma^T_{T2}) + \gamma^T_{B2}\cdot\gamma^T_{T3}\right]\cdot\Phi\cdot\chi = 1 \tag{90}$$

where

$$\Phi = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cdot\frac{1}{k - k_{(2n-1),1}}$$

$$\chi = \sum_{n=1}^{\infty}\frac{1}{k - k_{(2n),1}}$$

$$k_{(2n-1),1} = \left(\frac{2n-1}{\alpha} + \frac{\alpha}{2n-1}\right)^2$$

$$k_{(2n),1} = \left(\frac{2n}{\alpha} + \frac{\alpha}{2n}\right)^2$$

For the Tee stiffener, $\gamma^T_{B2} = \gamma^T_{T3} = 0$, therefore

$$\left[1 - 2\alpha\gamma^T_B\Phi\right]\left[1 - 2\frac{(\gamma^T_S + \pi^2\gamma^T_W)}{\alpha}\chi\right] = 0$$

Then

$$\Phi = \frac{1}{2\alpha\gamma^T_B} \tag{91a}$$

$$\chi = \frac{\alpha}{2(\gamma^T_S + \pi^2\gamma^T_W)} \tag{91b}$$

These Equations (91a) and (91b) are identical to Equations (57) and (58) respectively, in the previous Section 4.1. The convergence of the buckling coefficient k in Equation (90) is comparatively rapid as shown in Table 8 and Fig. 21. In this example the following values* are used.

$$\theta = 0.419 \text{ radian} = 24 \text{ degrees}$$

$$\gamma^{T}_{B_1} = 13.876$$

$$\gamma^{T}_{B_2} = -1.023$$

$$\gamma^{T}_{T_1} = 0.0687$$

$$\gamma^{T}_{T_2} = 0.0606$$

$$\gamma^{T}_{T_3} = 0.760$$

$$\alpha = 1.0 \text{ (square plate)}$$

$$b/h = 100$$

---

* See Appendix 11.3, Equation (A-7).

7.2 Comparison of the Buckling Strength to that of a Plate with Tee Stiffener

The example in Section 7.1 is chosen such that the area A^T, the moment of inertia about an axis parallel to the plate I_x^T and the side ratio α are the same as in the example in Section 4.1. Therefore the buckling strength of a transversely stiffened plate with an inverted angle stiffener can be compared to that of a Tee stiffener with the same weight of material. The buckling strength is given by

$$k = 6.417 \quad \text{for Tee stiffener (Fig. 10)}$$

$$k = 6.383 \quad \text{for Angle stiffener (Table 8)}$$

The difference in the buckling strength k for the angle stiffener is 0.5 percent of Tee stiffener. As it is shown in Table 8, the error in the buckling coefficient k for the 2nd approximation is less than 1 percent, therefore the 2nd approximation can be used to compute the value k for different side ratios α as follows:

$$k = \frac{1}{2}\left[\frac{5}{\alpha^2} + 4 + \frac{5}{4}\alpha^2 + 2\frac{(\gamma_{T1}^T + \gamma_{T2}^T)}{\alpha} + 2\alpha\gamma_{B1}^T\right]\left[1 - \sqrt{1 - \frac{4\left\{\left(\frac{1}{\alpha}+\alpha\right)^2\left(\frac{2}{\alpha}+\frac{\alpha}{2}\right)^2 + 2\frac{(\gamma_{T1}^T+\gamma_{T2}^T)}{\alpha}\left(\frac{1}{\alpha}+\alpha\right)^2 + 2\alpha\gamma_{B1}^T\left(\frac{2}{\alpha}+\frac{\alpha}{2}\right)^2 + 2\left(1+\frac{\gamma_{B2}^T}{\gamma_{B1}^T}\cdot\frac{\gamma_{T3}^T}{\gamma_{T1}^T+\gamma_{T2}^T}\right)\right\}}{\left\{\frac{5}{\alpha^2} + 4 + \frac{5}{4}\alpha^2 + 2\frac{(\gamma_{T1}^T+\gamma_{T2}^T)}{\alpha} + 2\alpha\gamma_{B1}^T\right\}^2}}\,\right] \tag{92}$$

In order to compare the efficiency of a transverse angle stiffener to that of a Tee stiffener having the same cross-sectional area and moment of inertia as the angle stiffener, the cross-sectional properties of the Tee stiffener are chosen to satisfy the required minimum bending rigidity for the severest case of side ratio. If the side ratio α varies from 0.5 to 1.0, the required minimum bending rigidity of the Tee stiffener for $\alpha = 0.5$ is greater than that for $\alpha > 0.5$ and satisfies the condition of the second mode of buckling of plate for $\alpha > 0.5$. Therefore the cross-sectional properties of Tee stiffener are given by

$$\gamma_B^T = 12.7$$

and

$$\gamma_s^T + \pi^2 \gamma_w^T = 0.085$$

The corresponding cross-sectional properties of the angle stiffener are

$$\gamma_{T_1}^T + \gamma_{T_2}^T = 0.1$$

$$\gamma_{B_1}^T = 15$$

$$1 + \frac{\gamma_{B_2}^T}{\gamma_{B_1}^T} \cdot \frac{\gamma_{T_3}^T}{\gamma_{T_1}^T + \gamma_{T_2}^T} = 0.55$$

Using these values of section property the buckling coefficients k for the Tee and the angle stiffeners are given by Equation (59) and (92) as shown in Table 9. The result shows that the efficiency of the Tee stiffener is superior to that of the angle stiffener for any value of side ratio α, however, the difference is comparatively small.

7.3 Convergence of a Secular Equation for Buckling in the Strain-Hardening Range

The following values of cross-sectional properties of a transverse angle stiffener* are used to examine the convergence of the buckling coefficient k of a stiffened plate in the strain-hardening range.

$$\gamma^T_{B1} = \frac{\cos 2\theta}{\cos^3\theta - \sin^3\theta} \cdot \frac{E(I^T_x + I^T_{xz}\tan 2\theta)}{D_x I b} = 17.490$$

$$\gamma^T_{B2} = \frac{1}{2} \cdot \frac{\sin 2\theta}{\cos^3\theta - \sin^3\theta} \cdot \frac{E I^T_P}{D_x I b}\left(\frac{\pi d^T_w}{b}\right) = -1.340$$

$$\gamma^T_{T1} = \left(\frac{\pi d^T_w}{b}\right)^2 \frac{E}{D_x I b}\left[I^{T'}_e \cos^2 2\theta + \frac{1}{2} I^{T'}_P \sin^2 2\theta \right] = 0.0985$$

$$\gamma^T_{T2} = \frac{G K^T}{D_x I b} = 0.0271$$

---

* Appendix 11.4, Equation (A-9).

$$\gamma_{T3}^{T} = \left(\frac{\pi d_w^T}{b}\right)\frac{E}{D_x I b}\left[\frac{1}{2} I_p^T - I_e^T\right]\sin 2\theta \cos 2\theta = 0.8716$$

$$\theta = 0.445^{rad.} = 25^{\circ}\,30'$$

where the coefficients of orthotropy λ and μ for A-7 steel are the same as in the previous section, that is,

$$\lambda = \frac{H}{D_x} = 4.30$$

$$\mu = \frac{D_y}{D_x} = 10.93$$

Substituting these constants in Equation (90) with the buckling coefficients of the plate itself in the strain-hardening range, the convergence of k is shown in Table 10, where

$$k_{(2n-1),1} = \frac{(2n-1)^2}{\alpha^2} + 2\lambda + \mu\frac{\alpha^2}{(2n-1)^2}$$

$$k_{(2n),1} = \frac{(2n)^2}{\alpha^2} + 2\lambda + \mu\frac{\alpha^2}{(2n)^2}$$

7.4 Comparison of the Buckling Strength in the Strain-Hardening Range to that of a Plate with a Tee Stiffener

The corresponding Tee stiffener with the same cross-sectional area and moment of inertia about the strong axis as the inverted angle stiffener, whose cross-sectional properties are given in the Section 7.3, is given by

$$\gamma_B^T = 15$$

$$\gamma_s^T + \pi^2 \gamma_w^T = 0.063$$

The required minimum bending rigidity of a Tee stiffener which leads a plate to the second mode of buckling in the strain-hardening range for the side ratio $\alpha = 0.5$ is given by 10.9 as shown in Fig. 14. Therefore the corresponding Tee stiffener is strong enough to cause the second mode of buckling of a plate. The buckling coefficient k for this anti-symmetric buckling is given in Fig. 14 by

$$k = 25.48$$

Therefore the Tee stiffener in the strain-hardening range is more effective than the inverted angle stiffener as well as in the elastic range.

The buckling strength of a strain-hardened plate with no stiffener is given by

$$k_{11} = 15.33 \quad \text{for} \quad \alpha = 0.5$$

and for the second mode of buckling

$$k_{21} = 25.28$$

where the buckling coefficient k of a strain-hardened plate with the transverse angle stiffener is

$$k = 25.12$$

This means that the angle stiffener increases the buckling strength of the plate from $k = 15.33$ to $k = 25.12$; however, it cannot keep its position as a nodal line of a buckled plate, therefore the buckling strength may be less than that of a plate itself in the second mode of buckling. This is due to the characteristics of an angle stiffener in which the bending deformation is always accompanied by the twisting of the stiffener. The difference in the buckling coefficient between $k = 25.48$ and $k_{21} = 25.28$ is due to the torsional resistance of the Tee stiffener.

8. BUCKLING STRENGTH OF AN ORTHOTROPIC PLATE WITH AN UNSYMMETRIC TYPE OF STIFFENER LONGITUDINALLY PLACED ON A PLATE

8.1 Convergence of a Secular Equation for Buckling in the Elastic Range

When a plate has one longitudinal stiffener in the middle of the plate, i = 1, j = 0 and $\eta_i = \frac{b}{2}$ in Equation (87), Equation (87) yields

$$
\begin{aligned}
&\left[N_x \left(\frac{r\pi}{a}\right)^2 - \lambda_{rs}^2 \right] \alpha_{rs} \\
&= \frac{2}{b}\left(\frac{r\pi}{a}\right)^4 A_1^L \sin\frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} \sin\frac{p\pi}{2} \\
&\quad - \frac{2}{b}\left(\frac{r\pi}{a}\right)^2 B_1^L \sin\frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} \sin\frac{p\pi}{2} \\
&\quad - \frac{2}{b}\left(\frac{r\pi}{a}\right)^4 C_1^L \sin\frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp}\left(\frac{p\pi}{b}\right) \cos\frac{p\pi}{2} \\
&\quad + \frac{2}{b}\left(\frac{r\pi}{a}\right)^2 D_1^L \sin\frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp}\left(\frac{p\pi}{b}\right) \cos\frac{p\pi}{2} \\
&\quad + \frac{2}{b}\left(\frac{r\pi}{a}\right)^4 \left(\frac{s\pi}{b}\right) A_2^L \cos\frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp}\left(\frac{p\pi}{b}\right) \cos\frac{p\pi}{2} \\
&\quad + \frac{2}{b}\left(\frac{r\pi}{a}\right)^2 \left(\frac{s\pi}{b}\right) B_2^L \cos\frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp}\left(\frac{p\pi}{b}\right) \cos\frac{p\pi}{2} \\
&\quad + \frac{2}{b}\left(\frac{r\pi}{a}\right)^4 \left(\frac{s\pi}{b}\right) C_2^L \cos\frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} \sin\frac{p\pi}{2}
\end{aligned}
\tag{93}
$$

where A_1^L, B_1^L, C_1^L, D_1^L, A_2^L, B_2^L, and C_2^L are given by Equation (81). The following non-dimensional parameters are similarly defined as in the case of a transverse stiffener.

$$\gamma_{B1}^L = \frac{A_1^L}{DIb} = \frac{\cos 2\theta}{\cos^3\theta - \sin^3\theta} \cdot \frac{E}{DIb} \cdot \left[I_y^L + I_{yz}^L \tan 2\theta \right]$$

$$\gamma_{B2}^L = \frac{\pi C_1^L}{DIb^2} = \frac{1}{2} \cdot \frac{\sin 2\theta}{\cos^3\theta - \sin^3\theta} \cdot \frac{EI_P^L}{DIb} \cdot \left(\frac{\pi d_w^L}{b}\right)$$

$$\gamma_{T1}^L = \frac{\pi^2 A_2^L}{DIb^3} = \left(\frac{\pi d_w^L}{b}\right)^2 \cdot \frac{E}{DIb} \cdot \left[\frac{I_P^L}{2} \sin^2 2\theta + (I_z^L - I_{yz}^L \tan 2\theta) \cos^2 2\theta \right]$$

$$\gamma_{T2}^L = \frac{GK^L}{DIb}$$

$$\gamma_{T3}^L = \frac{\pi C_2^L}{DIb^2} = \left(\frac{\pi d_w^L}{b}\right) \frac{E}{DIb} \left[\frac{I_P^L}{2} - (I_y^L + I_{yz}^L \tan 2\theta) \right] \sin 2\theta \cos 2\theta$$

$$\delta = \frac{A^L}{bh}$$

$$\beta = \frac{\cos 2\theta}{\cos^3\theta - \sin^3\theta}$$

$$\bar{\beta} = \frac{\sin 2\theta}{\cos^3\theta - \sin^3\theta}$$

$$\rho = \frac{\pi R^L}{b}$$

$$i = \frac{\pi z_A^L}{b} \tag{94}$$

Using these notations in Equation (94), Equation (93) can be expressed as

$$
\begin{aligned}
&\left[k - k_{rs} \right] \alpha_{rs} \\
&= 2 \frac{r^2}{\alpha^2} \gamma^L_{B1} \sin \frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} \sin \frac{p\pi}{2} \\
&\quad - 2 \beta \delta k \sin \frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} \sin \frac{p\pi}{2} \\
&\quad - 2 \frac{r^2}{\alpha^2} \gamma^L_{B2} \sin \frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} (p) \cos \frac{p\pi}{2} \\
&\quad + 2 \bar{\beta} \delta i k \sin \frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} (p) \cos \frac{p\pi}{2} \\
&\quad + 2 \frac{r^2}{\alpha^2} s \gamma^L_{T1} \cos \frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} (p) \cos \frac{p\pi}{2} \\
&\quad + 2 s \gamma^L_{T2} \cos \frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} (p) \cos \frac{p\pi}{2} \\
&\quad - 2 s \rho^2 \delta k \cos \frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} (p) \cos \frac{p\pi}{2} \\
&\quad + 2 \frac{r^2}{\alpha^2} s \gamma^L_{T3} \cos \frac{s\pi}{2} \sum_{p=1}^{\infty} \alpha_{rp} \sin \frac{p\pi}{2}
\end{aligned} \tag{95}
$$

Using the following abbreviations

$$S_{\bar{r}k} = 2\frac{\bar{r}^2}{\alpha^2}\gamma^L_{B_1} - 2\beta\delta k$$

$$T_{rk} = 2\frac{r^2}{\alpha^2}\gamma^L_{B_2} - 2\bar{\beta}i\delta k$$

$$U_{rk} = 2\frac{r^2}{\alpha^2}\gamma^L_{T_1} + 2\gamma^L_{T_2} - 2\rho^2\delta k$$

$$V_{\bar{r}} = 2\frac{\bar{r}^2}{\alpha^2}\gamma^L_{T_3}$$

$$\Phi^L_{\bar{r}} = \sum_{n=1}^{\infty}\frac{1}{k - k_{\bar{r},(2n-1)}}$$

$$\chi^L_r = \sum_{n=1}^{\infty}\frac{(2n)^2}{k - k_{r,(2n)}} \tag{96}$$

the secular equation of Equation (95) yields

$$S_{\bar{r}k}\cdot\Phi^L_{\bar{r}} + U_{rk}\cdot\chi^L_r = 1 + \left[S_{\bar{r}k}U_{rk} + T_{rk}V_{\bar{r}}\right]\Phi^L_{\bar{r}}\chi^L_r \tag{97}$$

For the case of a symmetric stiffener, for example the Tee stiffener, $\theta = 0$, then

$$\gamma^L_{B_1} = \gamma^L_B$$

$$\gamma^L_{B_2} = 0$$

$$\gamma^L_{T_1} = \pi^2 \gamma^L_w$$

$$\gamma^L_{T_2} = \gamma^L_s$$

$$\gamma^L_{T_3} = 0$$

$$\beta = 1$$

$$\bar{\beta} = 0$$

and

$$S_{\bar{r}k} = 2\frac{\bar{r}^2}{\alpha^2}\gamma^L_B - 2\delta k$$

$$T_{rk} = 0$$

$$U_{rk} = 2\frac{r^2}{\alpha^2}\pi^2\gamma^L_w + 2\gamma^L_s - 2\rho^2\delta k$$

$$V_{\bar{r}} = 0$$

The Equation (97) yields

$$\left[1 - U_{rk}\chi^L_r\right]\left[1 - S_{\bar{r}k}\Phi^L_{\bar{r}}\right] = 0$$

or

$$\Phi^L_{\bar{r}} = \frac{1}{S_{\bar{r}k}} = \frac{1}{2\frac{\bar{r}^2}{\alpha^2}\gamma^L_B - 2\delta k}$$

and

$$\chi_r^L = \frac{1}{U_{rk}} = \frac{1}{2\left(\frac{r^2}{\alpha^2}\pi^2\gamma_W^L + \gamma_S^L - \rho^2\delta k\right)}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{k - k_{\bar{r},(2n-1)}} = \frac{1}{\frac{2\bar{r}^2}{\alpha^2}\gamma_B^L - 2\delta k} \tag{98a}$$

or

$$\sum_{n=1}^{\infty} \frac{1}{k - k_{r,(2n)}} = \frac{1}{8}\,\frac{1}{(\gamma_S^L + \frac{r^2}{\alpha^2}\pi^2\gamma_W^L) - \pi^2\delta(\frac{R^L}{b})^2 k} \tag{98b}$$

The Equations (98a) and (98b) are identical to Equations (62) and (63) respectively in Section 5.1. The cross-sectional properties are chosen*, for example, as

$$\gamma_{B_1}^L = 14.532$$

$$\gamma_{B_2}^L = -2.3297$$

$$\gamma_{T_1}^L = 0.06901$$

$$\gamma_{T_2}^L = 0.03085$$

---

* Appendix 11.3, Equation (A-8).

$$\gamma \frac{L}{T_3} = 0.76222$$

$$\delta = 0.055$$

$$\alpha = 1.0$$

For $\alpha = 1.0$, $\bar{\bar{r}} = 1$ and $r = 2$ where $\bar{\bar{r}}$ is the number of half waves in x-direction for the symmetric component of the buckled shape in the y-direction and r is the number of half waves in x-direction for the antisymmetric component of the buckled shape in the y-direction. The convergence of the buckling coefficient k for this example is shown in Table 11. For the case of a longitudinally placed angle stiffener, it is necessary to use six terms in Equation (97) in order to obtain the accuracy within 1% of the error.

8.2 Comparison of Buckling Strength to that of a Plate with a Tee Stiffener

The buckling coefficient k in Table 4 and Table 11 give the comparison in the efficiency of the stiffener. The stiffeners in both tables have the same area and the same moment of inertia about their strong axis. From Table 4, the buckling coefficient k of a plate with a longitudinal Tee stiffener in the middle of the plate is given by

$$k = 17.656$$

and for a plate with a longitudinal angle stiffener in the middle of the plate

$$k = 13.517$$

where the buckling coefficient k of plate alone is

$$k = 4.000$$

since the side ratio $\alpha = 1$. Therefore the increase in k due to a longitudinal stiffener is

$$\Delta k = \frac{17.656-4}{4} = 314\% \text{ for the Tee stiffener}$$

$$\Delta k = \frac{13.517-4}{4} = 238\% \text{ for the Angle stiffener}$$

The difference in the efficiency Δk is 76%. On the other hand, the buckling coefficient of a plate with a transverse stiffener is given by Section 7.2 as

$$k = 6.417 \text{ for the Tee stiffener}$$

$$k = 6.383 \text{ for the Angle stiffener}$$

Then the increase in k due to a stiffener is

$$\Delta k = \frac{6.417-4}{4} = 60.4\% \text{ for the Tee stiffener}$$

$$\Delta k = \frac{6.383-4}{4} = 59.5\% \text{ for the Angle stiffener}$$

The difference in the efficiency Δk is only 0.9%. From these computations it may be concluded that a longitudinal stiffener is tolerably effective to prevent the buckling of a plate compared to a transverse stiffener, and the efficiency of an inverted angle stiffener is considerably lower than that of a Tee stiffener when the stiffener is subject to axial loads as in a longitudinal stiffener. In other words, the existence of axial load weakens the torsional resistance of a stiffener with an unsymmetric open cross-section as well as the bending resistance of the stiffener, and its influence for an unsymmetric cross-section is considerably greater than that for a symmetric one.

8.3 Convergence of a Secular Equation for Buckling in the Strain-Hardening Range

For the buckling of a plate in the strain-hardening range with a longitudinal angle stiffener in the middle of the plate, the following values of the section properties are chosen* as an example:

$$\delta = 0.1$$

$$\gamma \frac{L}{B_1} = 20.119$$

---

* Appendix 11.4, Equation (A-10).

$$\gamma^{L}_{B_2} = -1.0590$$

$$\gamma^{L}_{T_1} = 0.2836$$

$$\gamma^{L}_{T_2} = 0.1109$$

$$\gamma^{L}_{T_3} = 0.8044$$

For the side ratio $\alpha = 0.8$, the numbers of half waves in the loaded direction (x-direction) are given from Fig. 18 by

$$\bar{r} = 1 \quad \text{and} \quad r = 3$$

The convergence of the buckling coefficient k in Equation (97) with the plate coefficient $k_{\bar{r},(2n-1)}$ and $k_{r,(2n)}$ in the strain-hardening range is shown in Table 12, and it is comparatively slower than that of the Tee stiffener. It may be necessary to use at least four terms to get the accuracy within the error of one percent.

8.4 Comparison of Buckling Strength in the Strain-Hardening Range to that of a Plate with a Tee Stiffener

The examples of Tee stiffeners, in the previous Section 5.4 and angle stiffeners in Section 8.3 have the same area $\delta = 0.1$ and the same moment of inertia about the strong axis. The side ratio α is equal to 0.8 for both cases. Therefore the difference in the buckling coefficient k gives the efficiency of these stiffeners in the strain-hardening range.

$$k = 67.144 \quad \text{for the Tee stiffener}$$

$$k = 54.951 \quad \text{for the Angle stiffener}$$

$$k = 14.639 \quad \text{for the plate alone}$$

The increase in k due to a stiffener, therefore, is given by

$$\Delta k = \frac{67.144-14.639}{14.639} = 358\% \quad \text{for the Tee stiffener}$$

$$\Delta k = \frac{54.951-14.639}{14.639} = 275\% \quad \text{for the Angle stiffener}$$

The difference in the efficiency between the Tee and the angle stiffeners is 83% of buckling strength of the plate alone.

Therefore the Tee stiffener is considerably effective in both the elastic and the strain-hardening range compared to that of the inverted angle stiffener.

9. NONDIMENSIONAL EXPRESSION OF PLATE CURVE

It is advantageous to use nondimensional expressions for the buckling strength of a plate in order to eliminate the variations in the yield stress of the material. The behavior of plates which buckle in the intermediate range between the proprtional limit (sum of applied and residual stress equal to yield stress) and the strain-hardening range, is governed by the magnitude and distribution of residual stresses. No direct solution of this problem has yet been developed, however, a reasonable transition curve was proposed by Haaijer and Thürlimann[2]. It can therefore be assumed that the similar expression may hold true for stiffened plates.

The elastic buckling stress, σ_e, of a perfectly plane plate of isotropic material with no stiffener, subjected to forces acting in its plane, is given by

$$\sigma_e = k_e \frac{\pi^2}{12} \frac{E}{(1-\nu^2)} \left(\frac{h}{b}\right)^2$$

where k_e = plate buckling coefficient in the elastic range. The nondimensional expression for this elastic buckling may be

$$\frac{\sigma_e}{\sigma_o} = \frac{k_e}{\zeta^2} \qquad (99)$$

where

$$\sigma_o = \text{yield stress}$$

$$\zeta = \frac{b}{\pi h}\sqrt{\frac{12\,\sigma_o\,(1-\nu^2)}{E}} \qquad (100)$$

The proportional limit may be assumed to be equal to $\sigma_o/2$(2). Then the value of the nondimensional parameter at the proportional limit ζ_p is given $\zeta_p = \sqrt{2k_e}$. At the point of strain-hardening, the ratio of the buckling stress σ_{cr} to the yield stress σ_o is unity, that is,

$$\frac{\sigma_{cr}}{\sigma_o} = k_s\,\frac{\pi^2}{12}\cdot\left(\frac{h}{b}\right)^2\cdot\frac{D_x}{\sigma_o} = \frac{k_s}{\zeta^2}\cdot\frac{(1-\nu^2)\;D_x}{E} = 1$$

where k_s = plate buckling coefficient in the strain-hardening range. Therefore the value of ζ at the point of strain-hardening ζ_s is given by

$$\zeta_s = \sqrt{k_s\cdot\frac{(1-\nu^2)\;D_x}{E}}$$

For A-7 steel,

$$D_x = 3,000 \text{ ksi}$$

$$E = 30,000 \text{ ksi}$$

$$\nu = 0.3$$

The plate buckling coefficient k_s in the strain-hardening range depends upon the boundary conditions and the side ratio α. For a simply supported plate with the side ratio $\alpha = 0.5$, the value of the plate buckling coefficient k_s is given by

$$k_s = 15.33^{*}$$

Therefore

$$\zeta_s = 1.181$$

From the definition of ζ in Equation (100) the corresponding ratio b/h of the plate is given by

$$\frac{b}{h} = 34$$

where the yield stress σ_o is assumed as

$$\sigma_o = 33^{ksi}$$

9.1 Plates with a Transverse Tee Stiffener

For the case of a plate with a transverse stiffener of the Tee-shape in the middle of the plate, the same consideration can be applied with proper modification to the buckling coefficient k. For example, for $\alpha = 0.5$ and

---

* $k_s = \frac{1}{\alpha^2} + 2\lambda + \mu\alpha^2$ from Equation (55).

$\gamma_s^T + \pi^2 \gamma_w^T = 0.1$ the required minimum bending rigidity in the elastic range is given by $\gamma_B^T = 13.20$* and the buckling coefficient of the plate is

$$k_e = 18.46\text{*}$$

Therefore the value of ζ at the point of proportional limit is given by

$$\zeta_p = 6.076$$

The buckling coefficient k in the strain-hardening range with the required minimum bending rigidity $\gamma_B^T \doteq 11.08$** for $\gamma_s^T + \pi^2 \gamma_w^T = 0.1$ and $\alpha = 0.5$ is $k_s = 25.68$**. Therefore the point where strain-hardening starts is given by

$$\zeta_s = 1.529$$

or

$$\frac{b}{h} = 44 \text{ for A-7 steel}$$

9.2 <u>Plates with a Longitudinal Tee Stiffener</u>

In order to compare the efficiency of a longitudinal stiffener with that of a transverse one, the same side ratio $\alpha = 0.5$ is chosen for a plate with a longitudinal Tee

---

* See Fig. 11
** See Fig. 14

stiffener in the middle of the plate. The buckling coefficient k in the elastic range is given by

$$k_e = 17.825^*$$

with the required minimum bending rigidity of the stiffener $\gamma_B^L = 3.492^*$ for $\delta = 0.05$, $\gamma_s^L = 0.015$ and $\pi^2 \gamma_W^L = 0.070$.

Then

$$\zeta_p = 5.971$$

For the buckling in the strain-hardening range, the required minimum bending rigidity is given by

$$\gamma_B^L = 11.003^{**}$$

for $\delta = 0.1$, $\gamma_s^L = 0.07$ and $\pi^2 \gamma_W^L = 0.18$.

The buckling coefficient k for this case is

$$k_s = 69.368^{**}$$

Therefore

$$\zeta_s = 2.513$$

---

* See Table 6
** See Table 7

or

$$\frac{b}{h} = 72 \text{ for A-7 steel}$$

The results of these computations are shown in Fig. 22 with proper transition curves between the proportional limit and the strain-hardening range. The starting points of strain-hardening, therefore, can be summarized for A-7 steel as

$b/h = 34$ for the plate alone

$b/h = 44$ for the plate with a transverse stiffener

$b/h = 72$ for the plate with a longitudinal stiffener

where each stiffener is of Tee-shape and fulfills the condition of the required minimum bending rigidity.

The test results on inelastic buckling of simply supported plates[1] show that the buckling stress exceeds yield stress approximately at

$$B = \frac{b}{h}\sqrt{\frac{\sigma_Y}{E}} = 1.15 \qquad \text{(Reference: Fig. 3)}$$

which corresponds to

$$\frac{b}{h} = 34.7$$

for A-7 steel.

The side ratio α of the test specimen was chosen as

$$\alpha = 1$$

The theoretical value of k_s for $\alpha = 1$ is given by

$$k_s = (\frac{2}{\alpha})^2 + 2\lambda + \mu(\frac{\alpha}{2})^2 = 15.34^*$$

for simply supported plate with no stiffener. For this case the critical ratio $b/_h$ at which the strain-hardening starts is also given by

$$b/_h = 34$$

---

* For $0.78 \leqq \alpha \leqq 1.35$, the number of half waves in the loaded direction $r = 2$ gives the smallest buckling stress.

10. SUMMARY AND DISCUSSION

The results of the investigation presented in this dissertation can be divided into two parts: 1) the influence of torsional resistance of the stiffeners with thin-walled open cross-section, for example the Tee shape and the Angle shape, on the buckling strength of stiffened plates and 2) to investigate certain geometric conditions of stiffened panels which can be compressed beyond the yield point and even into the strain-hardening range without buckling.

The application of integral equation to the buckling strength of the stiffened panel simplifies the computation by the aid of Green's function for the deflection of the plates. The theory of an orthotropic plate can be applied to investigate the buckling of stiffened panels in the strain-hardening range with proper moduli of materials.

The findings are summarized in the following numbered paragraphs.

1. Cases Solved

The buckling strength of stiffened plates are studied in both the elastic and the strain-hardening range, considering the effect of St. Venant's and warping tosrional rigidity of stiffeners for the following cases:

1.1 Transverse stiffener with Tee shape

1.2 Longitudinal stiffener with Tee shape

1.3 Transverse stiffener with Angle shape

1.4 Longitudinal stiffener with Angle shape

2. Convergence of Eigenvalue

In both the elastic and the strain-hardening range the convergence of eigenvalues in the secular equation for the buckling of the longitudinally or transversely stiffened plate is considerably rapid and only two or three terms in the secular equation give fair result for design purposes.

3. Required Minimum Bending Rigidity of Stiffener

The required minimum bending rigidity of the Tee stiffener is obtained under the consideration of the torsional rigidity of the stiffener and compared to Barbré's results which are obtained by neglecting the torsional effects.

4. Effect of Torsional Resistance

The torsional resistance of longitudinal stiffener with a thin-walled open cross-section cannot be neglected in the estimation of the buckling strength of the stiffened plate since the torsional rigidity increases the critical strength about 10 percent of Barbré's results.

5. Warping Torsion

The warping resistance is the main portion of the torsional resistance of the stiffener and it increases in proportion to $1/L^2$ where L is the length of half wave of buckled shape, therefore the warping torsion of longitudinal stiffener becomes an important factor in the buckling of the stiffened plate in the strain-hardening range, because the half wave length L in the strain-hardening range is approximately half of that in the elastic range.

6. Shape of Stiffener

The symmetric stiffener, for example, the Tee shape is more profitable than the unsymmetric stiffener, like the angle stiffener, especially for the longitudinal stiffener. The existance of axial compression considerably reduces the torsional resistance of the unsymmetric stiffener as well as the bending resistance as compared to those of the symmetric stiffener.

The results of this dissertation are used to specify the proper geometric conditions of the stiffened plates such that each panel may develop large plastic deformation without buckling and a consequent fall-off in load. These requirements are essential for a successful application of Plastic Design Methods to plate structures like box girders of bridge construction or deck panels of ship structures.

11. NOMENCLATURE

Common Notation:

Superscript	L	=	Longitudinal stiffener
"	T	=	Transverse stiffener
Subscript	B	=	Bending
"	S	=	St. Venant's torsion
"	W	=	Warping torsion
"	i	=	i-th longitudinal stiffener
"	j	=	j-th transverse stiffener
"	x	=	x-axis
"	y	=	y-axis
"	z	=	z-axis

Notations:

A = Area of stiffener

a = Length of plate

b = Width of plate

b_e = Effective width of plate

b_f = Width of flange plate of stiffener

c = be/b

$$D = \frac{E}{1-\nu^2}$$

$$D_x = \frac{E_x}{1-\nu_x \nu_y}$$

$D_y = \dfrac{E_y}{1-\nu_x\nu_y}$

d_i = Distance between centroid and enforced axis of rotation

d_w = Depth of web plate of stiffener

E = Young's modulus

E_t = Tangent modulus

E_x = Tangent modulus in the x-direction

E_y = Tangent modulus in the y-direction

$2F = D_x\left(\nu_y - \dfrac{E_y}{G_t}\right) + \nu_x D_y$

G = Shear modulus

G_t = Effective shear modulus

$G(x,y;\xi,\eta)$ = Green's function for deflection of plate under unit load

$\bar{G}_\xi(x,y;\xi,\eta)$ = Green's function for deflection of plate under unit couple in the x-direction

$\bar{G}_\eta(x,y;\xi,\eta)$ = Green's function for deflection of plate under unit couple in the y-direction

$2H = \nu_x D_y + \nu_y D_x + 4G_t$

H = Horizontal reaction of stiffener

h = Plate thickness

$I = \dfrac{h^3}{12}$

$I_e^T = I_x^T + I_{xz}^T \cdot \tan 2\theta$

$I_e^{T\prime} = I_z^T - I_{xz}^T \cdot \tan 2\theta$

I_p = Polar moment of inertia of the cross-section of the stiffener

I_x = Moment of inertia of stiffener cross-section about the x-axis

I_y = Moment of inertia of stiffener cross-section about the y-axis

I_z = Moment of inertia of stiffener cross-section about the z-axis

$i = \pi z_A^L / b$

K = St. Venant's torsional constant

K_x, K_{xy}, K_y = Bending rigidities of orthotropic plate

k = Buckling coefficient of stiffened plate

k_{rs} = Buckling coefficient of plate in r- and s-mode

k_y = Spring constant of elastic medium in the y-direction

k_z = Spring constant of elastic medium in the z-direction

M = Moment

M_y = Moment in the y-direction

m = Number of transverse stiffener

N_x = Edge compression of plate in the x-direction

n = Number of longitudinal stiffener

$q(x,y)$ = Distributed load on plate

$$(R^L)^2 = \frac{I_P^L}{A^L} + (z_A^L)^2$$

$\bar{r}$, r = Numbers of half waves of buckled plate in the x-direction (symmetric and antisymmetric mode in y-direction respectively)

$S_{\bar{r}k} = 2\frac{\bar{r}^2}{\alpha^2}\cdot\gamma\frac{L}{B_1} - 2\beta\delta k$

s = Number of half waves of buckled plate in y-direction

$T_{rk} = 2\frac{r^2}{\alpha^2}\cdot\gamma\frac{L}{B_2} - 2\bar{\beta}i\delta k$

t_f = Thickness of the flange plate of the stiffener

t_w = Thickness of the web plate of the stiffener

$U_{rk} = 2\frac{r^2}{\alpha^2}\cdot\gamma\frac{L}{T_1} + 2\gamma\frac{L}{T_2} - 2\rho^2\delta k$

u = Displacement of the plate in the x-direction

$V_{\bar{r}} = 2\frac{r^2}{\alpha^2}\gamma\frac{L}{T_3}$

v = Displacement of the plate in the y-direction

W = Warping constant of the stiffener

w = Deflection of the plate

w_i = Deflection of the plate in the i-th panel

w_{is} = Deflection of the i-th stiffener

x, y, z = Coordinate system

α = Side ratio of plate

$\beta = \frac{\cos 2\theta}{\cos^3\theta - \sin^3\theta}$, $\bar{\beta} = \frac{\sin 2\theta}{\cos^3\theta - \sin^3\theta}$

γ = Ratio of bending or torsional rigidity of the stiffener to the plate

$$\gamma_B^T = \frac{EI^T{}_x}{DIb} \quad \text{in the elastic range}$$

or $$\gamma_B^T = \frac{EI^T{}_x}{D_x Ib} \quad \text{in the strain-hardening range}$$

$$\gamma_s^T = \frac{GK^T}{DIb} \quad \text{or} \quad \frac{GK^T}{D_x Ib}$$

$$\gamma_w^T = \frac{EI_z^T}{DIb}\left(\frac{d_w^T}{b}\right)^2 \quad \text{or} \quad \frac{EI_z^T}{D_x Ib}\left(\frac{d_w^T}{b}\right)^2$$

$$\gamma_B^L = \frac{EI_y^L}{DIb} \quad \text{or} \quad \frac{E_t I_y^L}{D_x Ib}$$

$$\gamma_s^L = \frac{GK^L}{DIb} \quad \text{or} \quad \frac{G_t K^L}{D_x Ib}$$

$$\gamma_w^L = \frac{EI_z^L}{DIb}\left(\frac{d_w^L}{b}\right)^2 \quad \text{or} \quad \frac{E_t I_z^L}{D_x Ib}\left(\frac{d_w^L}{b}\right)^2$$

$$\gamma_{B_1}^T = \frac{\beta EI^T{}_e}{DIb} \quad \text{or} \quad \frac{\beta EI^T{}_e}{D_x Ib}$$

$$\gamma_{B_2}^T = \frac{1}{2}\,\frac{\bar{\beta} EI_p^T}{DIb} \cdot \left(\frac{d_w^T}{b}\right)$$

$$\gamma_{T_1}^T = \left(\frac{\pi d_w^T}{b}\right)^2 \cdot \frac{E}{DIb}\left(\frac{1}{2} I_p^T \sin^2 2\theta + I_e^{T\prime} \cos^2 2\theta\right)$$

or $$\left(\frac{\pi d_w^T}{b}\right)^2 \cdot \frac{E}{D_x Ib} \cdot \left(\frac{1}{2} I_p^T \sin^2 2\theta + I_e^{T\prime} \cos^2 2\theta\right)$$

$$\gamma_{T_2}^T = \frac{GK^T}{DIb} \quad \text{or} \quad \frac{GK^T}{D_x Ib}$$

$$\gamma^{T}_{T_3} = \left(\frac{\pi d_w^T}{b}\right)\cdot \frac{E}{DIb}\left(\frac{1}{2}\, I_p^T - I_e^T\right)\sin 2\,\theta\cdot\cos 2\,\theta$$

$$\text{or}\left(\frac{\pi d_w^T}{b}\right)\cdot \frac{E}{D_x Ib}\left(\frac{1}{2}\, I_p^T - I_e^T\right)\sin 2\,\theta\cdot\cos 2\,\theta$$

$$\gamma^{L}_{B_1} = \frac{\beta E I_e^L}{DIb} \quad \text{or} \quad \frac{\beta E_t I_e^L}{D_x Ib}$$

$$\gamma^{L}_{B_2} = \frac{1}{2}\,\frac{\bar{\beta} E I_p^L}{DIb}\left(\frac{\pi d_w^L}{b}\right) \quad \text{or} \quad \frac{1}{2}\,\frac{\bar{\beta} E_t I_p^L}{D_x Ib}\left(\frac{\pi d_w^L}{b}\right)$$

$$\gamma^{L}_{T_1} = \left(\frac{\pi d_w^L}{b}\right)^2\cdot \frac{E}{DIb}\left(\frac{1}{2}\, I_p^L \sin^2 2\,\theta + I_e^{L\prime}\cos^2 2\,\theta\right)$$

$$\text{or}\left(\frac{\pi d_w^L}{b}\right)^2\cdot \frac{E_t}{D_x Ib}\left(\frac{1}{2}\, I_p^L \sin^2 2\,\theta + I_e^{L\prime}\cos^2 2\,\theta\right)$$

$$\gamma^{L}_{T_2} = \frac{GK^L}{DIb} \quad \text{or} \quad \frac{G_t K^L}{D_x Ib}$$

$$\gamma^{L}_{T_3} = \left(\frac{\pi d_w^L}{b}\right)\cdot \frac{E}{DIb}\left(\frac{1}{2}\, I_p^L - I_e^L\right)\sin 2\,\theta\cdot\cos 2\,\theta$$

$$\text{or}\left(\frac{\pi d_w^L}{b}\right)\cdot \frac{E_t}{D_x Ib}\left(\frac{1}{2}\, I_p^L - I_e^L\right)\sin 2\,\theta\cdot\cos 2\,\theta$$

where the first one of each coefficient γ is used for the elastic range and the last one for the strain-hardening range.

$\delta = \frac{A^L}{bh}$

ε_x = Strain in the x-direction

ε_y = Strain in the y-direction

$\zeta = \frac{b}{\pi h}\sqrt{\frac{12\,\sigma_o(1-\nu^2)}{E}}$

ζ_p = Value of ζ at the proportional limit

ζ_s = Value of ζ at the strain-hardening point

ξ, η = Coordinate system

θ = Angle between the principal axis of the stiffener and the x- or y-axis

$\lambda = \frac{H}{D_x}$ = coefficient of orthotropic plate

λ^2_{rs} = Eigenvalue of orthotropic plate

$\mu = \frac{D_y}{D_x}$ = coefficient of orthotropic plate

ν = Poisson's ratio

ν_x = Dilation coefficient in the x-direction

ν_y = Dilation coefficient in the y-direction

$\rho = \frac{\pi R^L}{b}$

σ_{cr} = Buckling stress of plate

σ_o = Yield stress

σ_x = Stress in the x-direction

σ_y = Stress in the y-direction

$\tau = \frac{E_t}{E}$ = Tangent modulus coefficient

τ_{xy} = Shearing stress

$$\Phi = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cdot \frac{1}{k - k(2n-1),1}$$

$$\Phi^L_{\bar{r}} = \sum_{n=1}^{\infty} \frac{1}{k - k\bar{r},(2n-1)}$$

ϕ = Stress function

φ_{mn} = Normalized orthogonal function

$$\chi = \sum_{n=1}^{\infty} \frac{1}{k - k_{2n,1}}$$

$$\chi^L_r = \sum_{n=1}^{\infty} \frac{(2n)^2}{k - k_{r,(2n)}}$$

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13. APPENDICES

13.1 Section Properties of a Symmetric Stiffener in the Elastic Range

The geometric proportions of WF sections are proposed by Haaijer and Thürlimann(2) such that no local buckling of the flanges and the web plates will occur prior to strain-hardening. The recommendations are following:

$$b_f/t_f \leqq 17$$

$$d_w/t_w \leqq 34 \qquad \text{(Reference ; Fig. 8)}$$

where

b_f = Width of Flange

t_f = Thickness of Flange

d_w = Depth of Web

t_w = Thickness of Web

For the Tee stiffener the web plate is subject to axial force and bending, furthermore high axial force accompanied by bending moment may affect these ratios. Therefore, in this dissertation the following ratios are used for the numerical examples.

$$b_f/t_f = 17$$

$$d_w/t_w = 20$$

$$t_f/t_w = 2$$

$$t_f/h = m$$

Therefore the section properties of a transverse Tee stiffener are given by

$$\gamma_s^T = 25.55\ m^4\ (\frac{h}{b}) \tag{A-1}$$

$$\pi^2 \gamma_w^T = 4.413\ m^6\ (\frac{h}{b})^3 \cdot 10^6 \tag{A-2}$$

$$\gamma_B^T = 3.64 \Big[17m(10m-\alpha)^3(\frac{h}{b}) - 16.5m(9m-\alpha)^3(\frac{h}{b}) + 2c(1+\alpha)^2 - \Big\{ 2c - \frac{m}{2}(\frac{h}{b}) \Big\} \cdot \alpha^3 \Big] \tag{A-3}$$

where

$$c = \frac{b_e}{b}$$

b_e = Effective width of plate

For example, for $b/h = 100$, $m = 1/2$ and $\alpha/2 \geq 0.3$ ($c = 0.193$), the section properties in Equations (A-1), (A-2) and (A-3) yield

$$\gamma_B^T = 12.7$$

$$\gamma_s^T = 0.015$$

$$\pi^2 \gamma_W^T = 0.070$$

$$\text{or} \quad \gamma_s^T + \pi^2 \gamma_W^T = 0.085 \tag{A-4}$$

For a longitudinal stiffener the effective width of the plate is related to the length of the plate a as

$$b_e = c\ a$$

$$\text{or} \qquad b_e = c\,\alpha\, b$$

Then $\qquad c = 0.193 \quad \text{for} \quad a/_b \leqq 5/_3$

Therefore for $\alpha = 1$, $b/_h = 100$ and $m = 1/2$ the values of γ_B^L, γ_s^L and $\pi^2\gamma_W^L$ are the same as those in the transverse stiffener in Equation (A-4), however, the number of half waves in the loaded direction $r = 2$ for $\alpha = 1$. Therefore,

$$\gamma_s^L + \frac{r^2}{\alpha^2}\pi^2 \gamma_W^L = 0.292 \tag{A-5}$$

The value of $\pi^2 \delta\left(\frac{R^L}{b}\right)^2$ is obtained as follows:

$$(R^L)^2 = \frac{1}{A}L\ (I_Y^L + I_Z^L) + (\bar{Z}_A^L)^2$$

$$\delta = \frac{A^L}{bh}$$

Using the same values for α , $b/_h$ and m,

$$\delta = 0.05$$

$$\pi^2 \delta \left(\frac{R^L}{b}\right)^2 = 0.0025 \qquad \text{(A-6)}$$

13.2 Section Properties of a Symmetric Stiffener in the Strain-Hardening Range

The coefficient c of the effective width of the plate is assumed to be

$$c = \frac{b_e}{b} = 0.4 \quad \text{for} \quad \alpha > 1$$

Then $\quad I_y^L = 272.4\ h^4$

for $\quad \alpha = 1,\ m = 1/2,$ and $b/_h = 50$

Therefore

$$\gamma_B^L = \frac{E_t I_y^L}{D_x I b} = 19.61$$

$$\gamma_s^L = \frac{G_t K^L}{D_x I b} = 0.07$$

$$\pi^2 \gamma_w^L = \frac{\pi^2 E_t I_z^L}{D_x I b} \cdot \left(\frac{d_w^L}{b}\right)^2 = 0.18$$

$$\delta = \frac{A}{bh} = 0.1$$

$$\pi^2 \delta \left(\frac{R^L}{b} \right)^2 = 0.026 \qquad \text{(A-7)}$$

where

$$E_t = 900^{ksi}$$

$$G_t = 2,400^{ksi}$$

$$D_x = 3,000^{ksi}$$

for A-7 steel.

13.3 Section Properties of an Unsymmetric Stiffener in the Elastic Range

Transverse Stiffener:

In order to compare the angle stiffener to the Tee stiffener with the same area and the same moment of inertia about the strong axis, the angle stiffener is assumed to have half width of flange b_f and twice the thickness of the flange t_f, that is,

$$b_f \text{ (Angle)} = \frac{1}{2} b_f \text{ (Tee)}$$

$$t_f \text{ (Angle)} = 2\, t_f \text{ (Tee)}$$

Therefore the position of the neutral axis z_A^T is equal to each other and so are I_x^T, I_z^T and A^T of the transverse stiffener. The product moment inertia I_{xz}^T is given by

$$I_{xz}^T = \frac{1}{2}\,(d_w^T - z_A^T)\cdot\left(\frac{b_f}{2}\right)^2\cdot(2t_f) = 44.45h^4$$

Then $\quad \tan 2\,\theta = 1.108$

$$\theta = 0.419^{rad} = 24 \text{ degree}$$

$$I_e^T = I_x^T + I_{xz}^T \tan 2\,\theta = 155.08h^4$$

$$I_e^{T'} = I_z^T - I_{xz}^T \tan 2\,\theta = -\,23.65h^4$$

$$I_p^T = I_x^T + I_z^T = 131.42h^4$$

Therefore

$$\gamma_{B_1}^T = 13.876$$

$$\gamma_{B_2}^T = 1.023$$

$$\gamma_{T_1}^T = 0.0687$$

$$\gamma_{T_2}^T = 0.0606$$

$$\gamma^{T}_{T_3} = 0.760$$

for $\alpha = 1$ and $b/_h = 100.$ (A-7)

Longitudinal Stiffener:

The following relations between the Tee and the angle stiffener are assumed:

$$b_f \text{ (Angle)} = \frac{1}{\sqrt{2}} b_f \text{ (Tee)}$$

$$t_f \text{ (Angle)} = \sqrt{2}\, t_f \text{ (Tee)}$$

The web plates are of the same proportion to each other.

Then $b_f \text{ (Angle)}/t_{f\text{(Angle)}} = 8.5$

$$\theta = 0.580^{\text{rad.}} = 33°13'$$

$$\gamma^{L}_{B_1} = 14.532$$

$$\gamma^{L}_{B_2} = -2.3297$$

$$\gamma^{L}_{T_1} = 0.06901$$

$$\gamma^{L}_{T_2} = 0.03085$$

$$\gamma^{L}_{T_3} = 0.76222$$

$$\delta = 0.055$$

for $\alpha = 1$ and $b/_h = 100$.

13.4 Section Properties of an Unsymmetric Stiffener in the Strain-Hardening Range

Transverse Stiffener:

The required proportions of the Tee stiffener is given by

$$b_f \text{ (Tee)} = 3.808h$$

$$t_f \text{ (Tee)} = 0.224h$$

$$d_w^T \text{ (Tee)} = 2.24h$$

$$t_w \text{ (Tee)} = 0.112h$$

$$A_F = 0.853h^2$$

$$A_w = 0.251h^2$$

$$b_f/t_f = 17$$

$$d_w/t_w = 20$$

The corresponding angle stiffener is

$$b_f \text{ (Angle)} = 2.7h$$

$$t_f \text{ (Angle)} = 0.316h$$

$$b_f/t_f = 8.5$$

$$A_F = 0.853h^2$$

The web plate is of the same proportion as the Tee shape.

Then $\theta = 0.445$ radian $= 25^\circ\ 30'$

$$\gamma^T_{B_1} = 17.490$$

$$\gamma^T_{B_2} = -1.340$$

$$\gamma^T_{T_1} = 0.0985$$

$$\gamma^T_{T_2} = 0.0271$$

$$\gamma^T_{T_3} = 0.8716$$

for $\alpha = 0.5$ and $b/h = 50$. (A-9)

<u>Longitudinal Stiffener:</u>

$$b_f/t_f = 8.5$$

$$d_w^L / t_w = 20$$

$$b/_h = 50$$

Assume that

$$b_f = 5.65h$$

$$t_f = 0.664h$$

$$d_w^L = 5h$$

$$t_w = 0.25h$$

$$\delta = 0.1 \quad \text{and} \quad \alpha = 0.8$$

Then $\theta = 0.1478 \text{ radian} = 8°28'$

$$\gamma_{B_1}^L = 20.119$$

$$\gamma_{B_2}^L = -1.0590$$

$$\gamma_{T_1}^L = 0.2836$$

$$\gamma_{T_2}^L = 0.1109$$

$$\gamma_{T_3}^L = 0.8044 \qquad \text{(A-10)}$$

The corresponding Tee stiffener satisfies the required minimum bending rigidity.

TABLE 1

CONVERGENCE OF BUCKLING COEFFICIENT k

FOR TRANSVERSELY STIFFENED PLATE WITH T-STIFFENER

SYMMETRIC TYPE						
α	γ_B^T	Using 1 term	Using 2 terms	Using 3 terms	Using 4 terms	Using 5 terms
0.5	10	16.250	15.695	15.646	15.624	15.620
Error in %		4.036	0.484	0.172	0.031	--
1.0	12.7	29.400	10.205	10.185	10.183	10.180
Error in %		188.79	0.242	0.045	0.020	--

ANTISYMMETRIC TYPE						
α	$\gamma_s^T+\pi^2\gamma_w^T$	Using 1 term	Using 2 terms	Using 3 terms	Using 4 terms	Using 5 terms
1.0	0.085	6.420	6.418	6.417	6.417	6.417
Error in %		0.047	0.016	0	0	---
1.0	1.0	8.250	7.922	7.870	7.827	7.800
Error in %		5.766	1.560	0.888	0.338	---

TABLE 2

REQUIRED MINIMUM BENDING RIGIDITY γ_B^T
OF TRANSVERSE STIFFENER WITH NO TORSIONAL RESISTANCE
AND ERRORS IN COMPARISON TO FRÖHLICH'S RESULTS
(Elastic Buckling)

α	Using 2 terms	Using 3 terms	Timoshenko	Fröhlich
0.5	12.64	12.72	12.60	12.75
0.6	7.18	7.23	7.18	7.24
0.7	4.38	4.41	4.39	4.42
0.8	2.79	2.81	2.80	2.82
0.9	1.81	1.83	1.82	1.84
1.0	1.17	1.19	1.26	1.19
0.5	-0.86%	-0.23%	-1.20%	
0.6	-0.83	-0.14	-0.83	
0.7	-0.90	-0.23	-0.68	
0.8	-1.07	-0.35	-0.71	
0.9	-1.63	-0.54	-1.09	
1.0	-1.68	0	+5.88	

TABLE 3

BUCKLING COEFFICIENT k OF PLATE WITH TRANSVERSE T-STIFFENER AND ERROR IN COMPARISON TO THE 5th APPROXIMATION

(Symmetric Mode - Strain-Hardening Range)

α	γ^T_B	Using 1 term	Using 2 terms	Using 3 terms	Using 4 terms	Using 5 terms
0.5	10	25.333	24,810	24.768	24.758	24.756
		2.331%	0.218%	0.048%	0.008%	--

TABLE 4

BUCKLING COEFFICIENT k OF PLATE WITH LONGITUDINAL T-STIFFENER AND ERROR IN COMPARISON TO THE 5th APPROXIMATION

(Antisymmetric Mode - Elastic Range)

α	$\gamma^L_s + \pi^2\frac{r^2}{\alpha^2}\gamma^L_w$	$\pi^2\delta(\frac{R^L}{b})^2$	Using 1 term	Using 2 terms	Using 3 terms	Using 4 terms	Using 5 terms
1.0	0.292	0.0026	17.959	17.792	17.721	17.681	17.656
			1.70%	0.77%	0.37%	0.14%	--

TABLE 5

COMPARISON OF REQUIRED MINIMUM BENDING RIGIDITY

OF LONGITUDINAL STIFFENER WITH NO TORSIONAL RESISTANCE

($\delta = 0.1$, Elastic Range)

Method / α	Bleich	German Spec.	Integral Eq.	Barbre
1/2	2.594 (-0.42%)	2.719 (+4.4%)	2.582 (-0.88%)	2.605 ---
$1/\sqrt{2}$	4.945 (-21%)	4.775 (-24%)	6.267 (-0.27%)	6.284 ---
1	8.850 (+0.27%)	8.700 (+1.4%)	8.809 (-0.19%)	8.826 ---
3/2	16.899 (-2.4%)	17.419 (+0.57%)	17.303 (-0.098%)	17.320 ---

TABLE 6

EFFECT OF TORSIONAL RESISTANCE ON BUCKLING AND REQUIRED MINIMUM BENDING RIGIDITY OF LONGITUDINAL T-STIFFENER

($\delta = 0.05$, Elastic Range)

Item	Buckling Coefficient k			Bending Rigidity γ_B^L		
α	No Torsion	With Torsion	Increase %	No Torsion	With Torsion	Increase %
1/2	16	17.825	11.4	2.405	3.492	45
1				8.026	9.498	18
3/2				15.52	18.14	18
2				23.42	27.73	18

TABLE 7

EFFECT OF TORSIONAL RESISTANCE ON BUCKLING AND REQUIRED MINIMUM BENDING RIGIDITY OF LONGITUDINAL T-STIFFENER

($\delta = 0.1$, Strain-Hardening Range)

Item	Buckling Coefficient k			Bending Rigidity γ_B^L		
α	No Torsion	With Torsion	Increase %	No Torsion	With Torsion	Increase %
0.3	61.235	65.993	7.73	3.880	4.881	25.80
0.5	61.333	69.368	13.10	8.946	11.003	22.99
0.8	60.902	67.144	10.25	19.245	22.097	14.82
1.0	61.333	69.368	13.10	27.678	33.026	14.32
1.4	60.870	65.777	8.06	42.265	48.254	14.17
1.6	60.902	67.144	10.25	46.589	56.402	21.11
1.9	60.861	66.616	9.46	44.660	57.238	28.16
2.2	60.853	66.250	8.87	31.455	43.031	36.80

TABLE 8

BUCKLING COEFFICIENT k OF TRANSVERSELY STIFFENED PLATE WITH ANGLE STIFFENER

(Elastic Range)

α	$\gamma^T_{B1}=13.876$ $\gamma^T_{B2}=-1.023$ $\gamma^T_{T1}=0.0687$ $\gamma^T_{T2}=0.0606$ $\gamma^T_{T3}=0.760$	Using 1 term	Using 2 terms	Using 3 terms	Using 4 terms	Using 5 terms	Using 6 terms	Using 7 terms	Using 8 terms	Using 9 terms
1.0	k	31.751	6.340	6.383	6.376	6.384	6.383	6.383	6.383	6.383
	Error %	--	-0.67	0	-0.11	+0.02	0	0	0	0

TABLE 9

BUCKLING COEFFICIENT k OF TRANSVERSELY STIFFENED PLATE

(Elastic Range)

Type of Stiffener \ Side Ratio α	0.5	0.6	0.7	0.8	0.9	1.0
Tee	18.40	13.48	10.53	8.62	7.33	6.42
Angle	17.24	13.05	10.25	8.41	7.16	6.27
No Stiffener	6.25	5.14	4.53	4.20	4.04	4,00

TABLE 10

BUCKLING COEFFICIENT k OF TRANSVERSELY STIFFENED PLATE WITH ANGLE STIFFENER

(Strain-Hardening Range)

α	$\gamma^T_{B1} = 17.490$ $\gamma^T_{B2} = -1.340$ $\gamma^T_{T1} = 0.0985$ $\gamma^T_{T2} = 0.8716$	Using 1 Term	Using 2 Terms	Using 3 Terms	Using 4 Terms	Using 5 Terms	Using 6 Terms	Using 7 Terms	Using 8 Terms
0.5	k	32.8230	25.1746	25.1164	25.1185	25.1158	25.1157	25.1157	25.1157
	Error in %	---	+0.234	+0.0028	+0.0111	+0.0004	0	0	0

TABLE 11

BUCKLING COEFFICIENT k OF LONGITUDINALLY STIFFENED PLATE WITH ANGLE STIFFENERS

($\delta = 0.055$, Elastic Range)

α	$\gamma^L_{B1} = 14.532$ $\gamma^L_{B2} = -2.3297$ $\gamma^L_{T1} = 0.06901$ $\gamma^L_{T2} = 0.03085$ $\gamma^L_{T3} = 0.76222$	Using 2 terms	Using 3 terms	Using 4 terms	Using 5 terms	Using 6 terms	Using 7 terms	Using 8 terms	Using 9 terms	Using 10 terms
1.0	k	14.975	13.844	13.714	13.661	13.583	13.569	13.532	13.517	13.517
	Error in %	10.8	2.4	1.4	1.1	0.5	0.4	0.04	0	0

TABLE 12

BUCKLING COEFFICIENT k OF LONGITUDINALLY STIFFENED PLATE WITH ANGLE STIFFENER

($\delta = 0.1$, Strain-Hardening Range)

α	$\gamma^L_{B1} = 10.202$ $\gamma^L_{B2} = 8.754$ $\gamma^L_{T1} = 2.874$ $\gamma^L_{T2} = 0.111$ $\gamma^L_{T3} = 2.549$	Using 2 terms	Using 3 terms	Using 4 terms	Using 5 terms	Using 6 terms	Using 7 terms	Using 8 terms	Using 9 terms	Using 10 terms
0.8	k	56.948	55.409	55.324	55.122	55.075	55.017	54.988	54.969	54.951
	Error in %	3.63	0.83	0.69	0.31	0.23	0.12	0.07	0.03	--

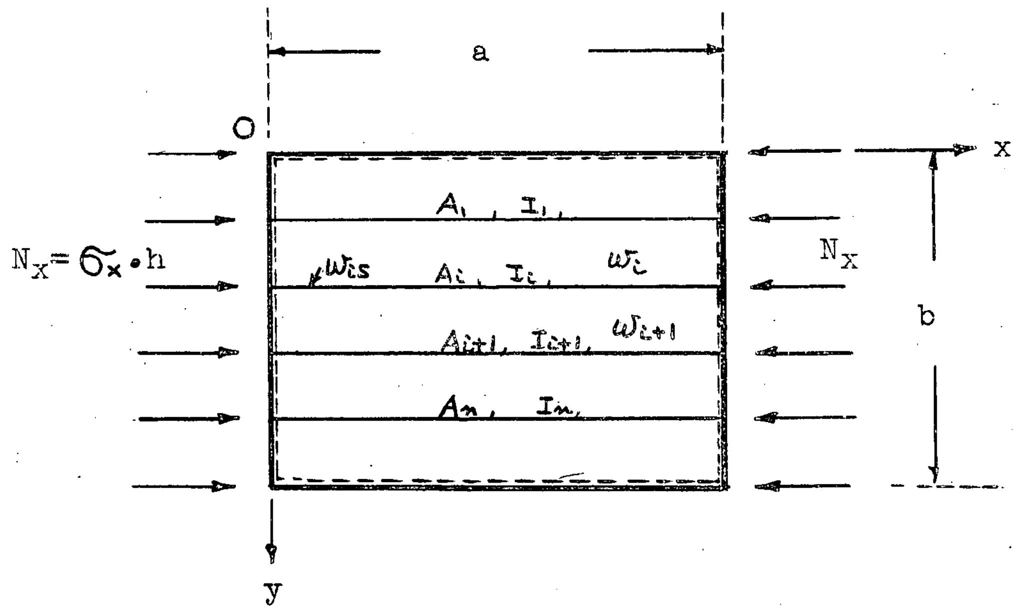


Fig. 1 - LONGITUDINALLY STIFFENED PLATE

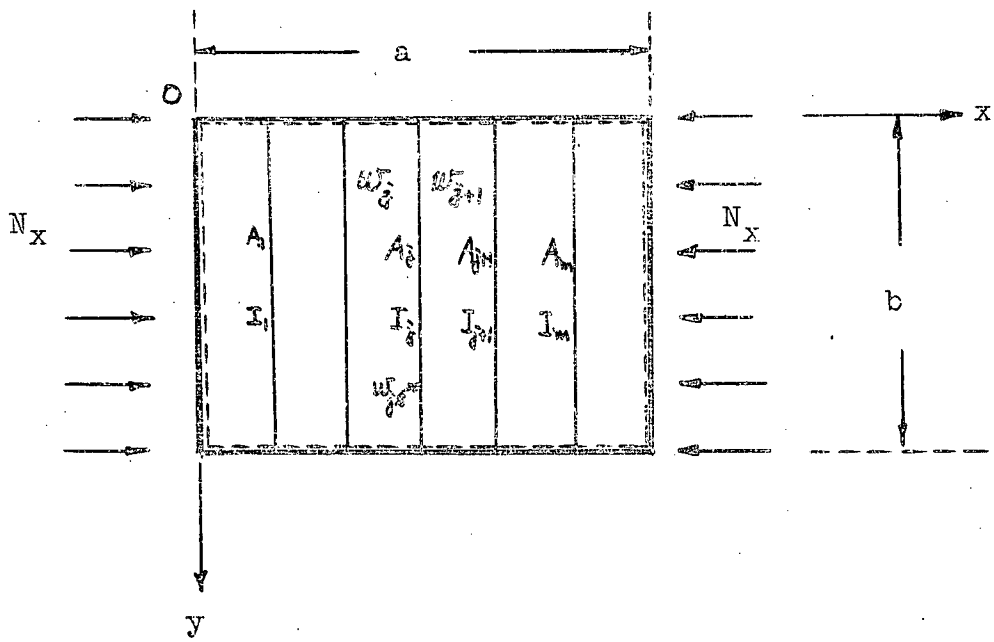


Fig. 2 - TRANSVERSELY STIFFENED PLATE

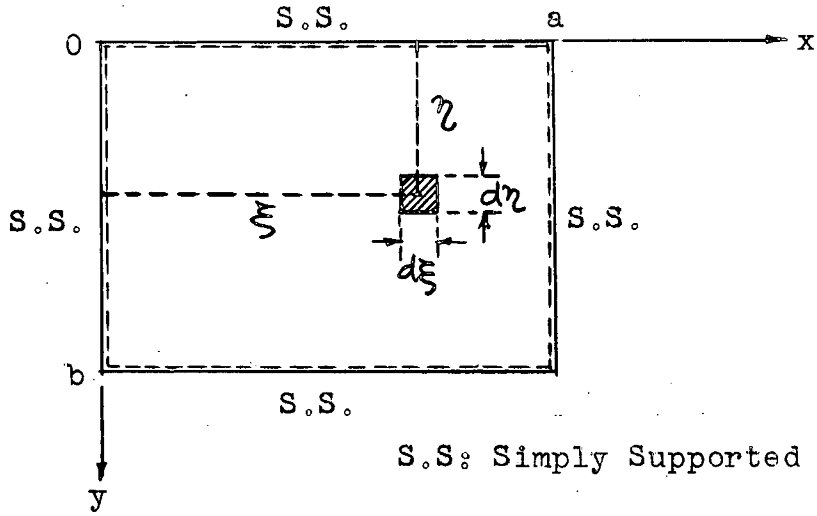
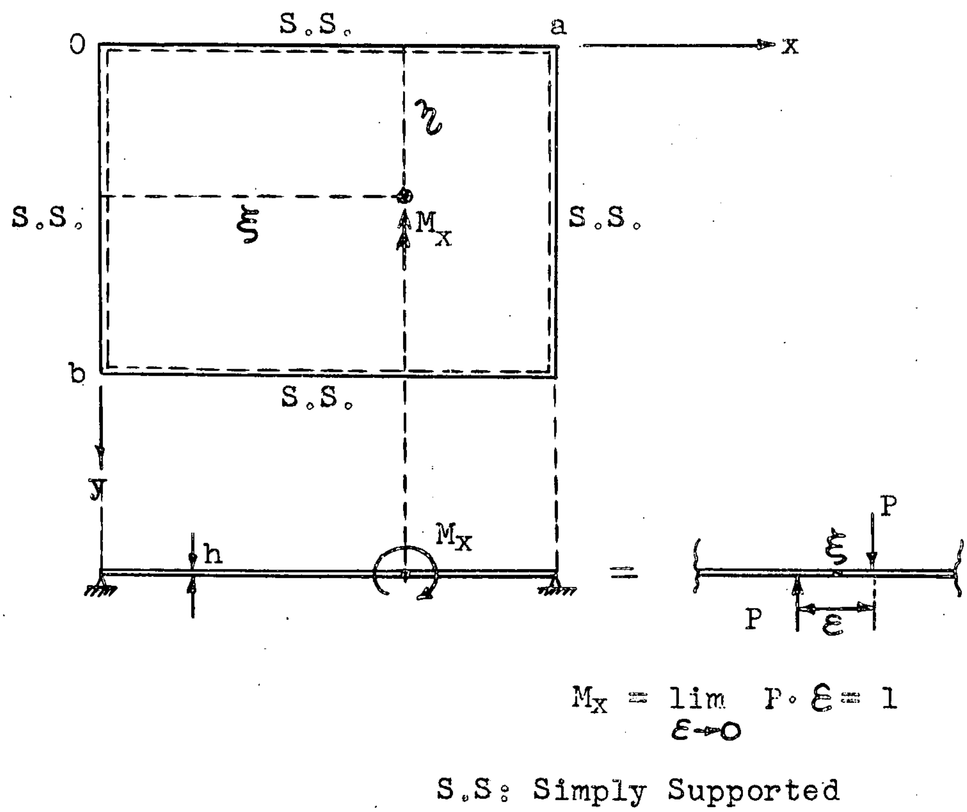


Fig. 3-1 - PARTIALLY DISTRIBUTED LOAD ON PLATE



v Fig. 3-2 - CONCENTRATED MOMENT ON PLATE

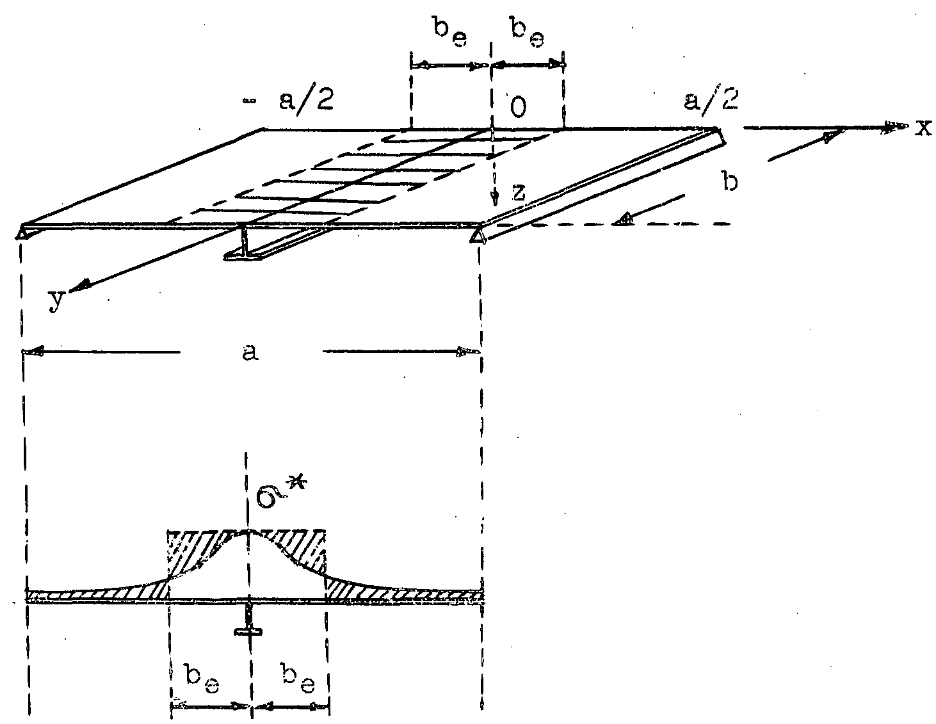


Fig. 4 - EFFECTIVE WIDTH OF PLATE

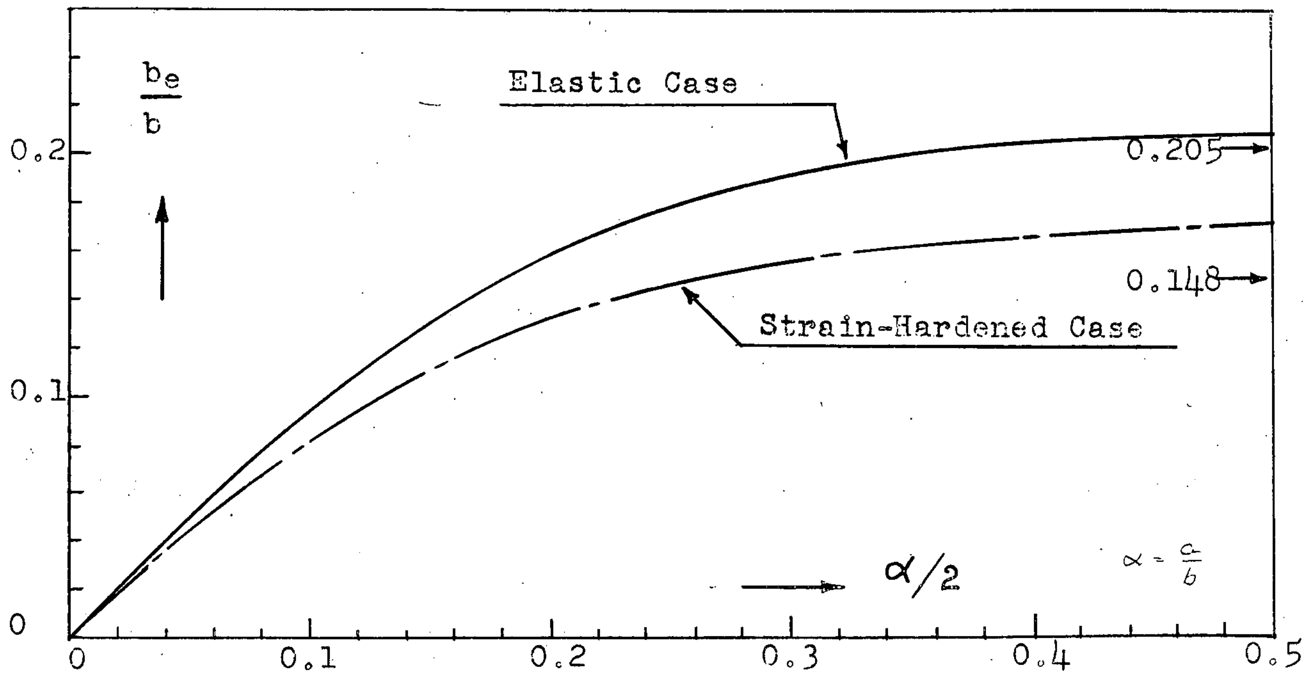


Fig. 5 - EFFECTIVE WIDTH OF PLATE FOR TRANSVERSE STIFFENER

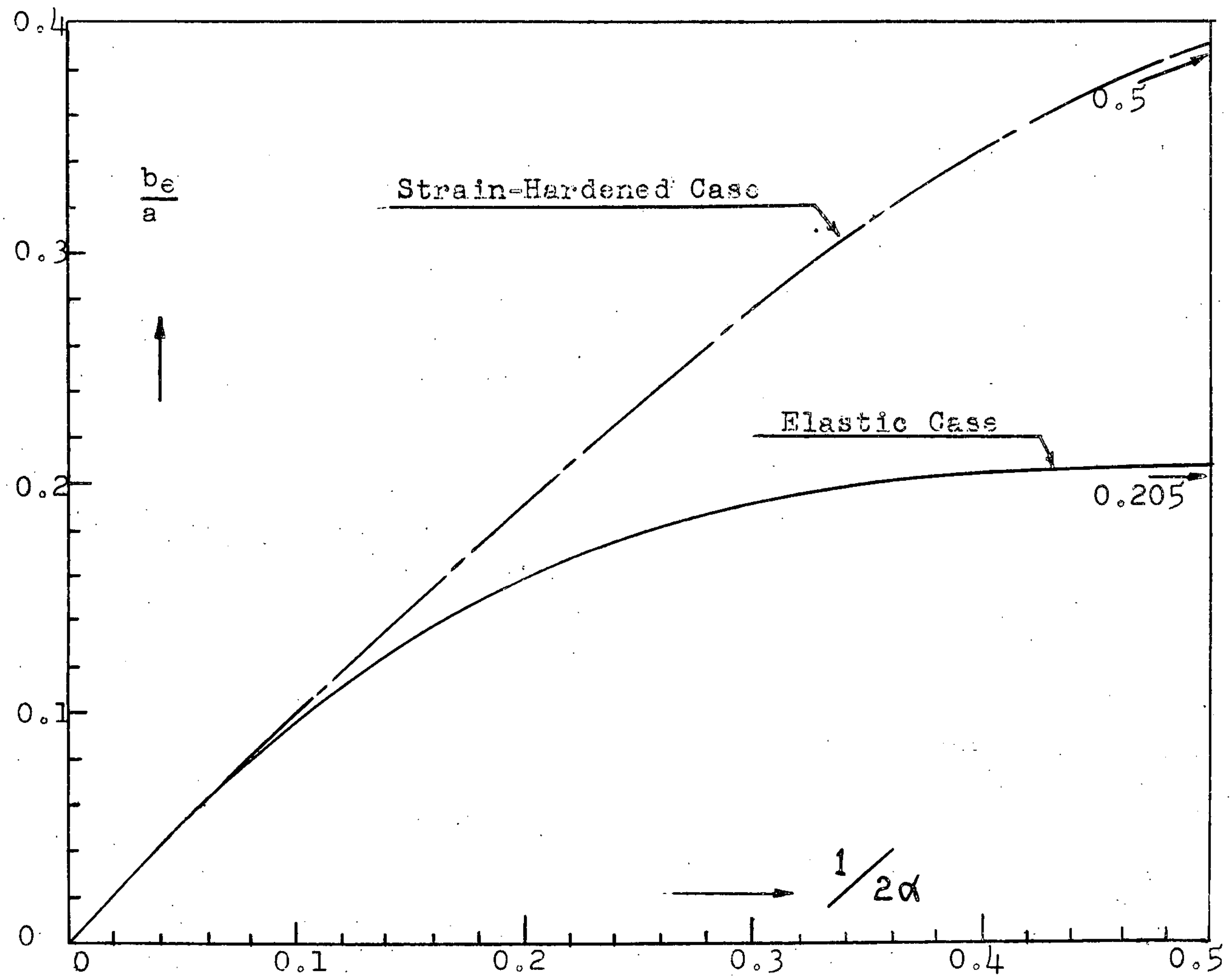


Fig. 6 - EFFECTIVE WIDTH OF PLATE FOR LONGITUDINAL STIFFENER

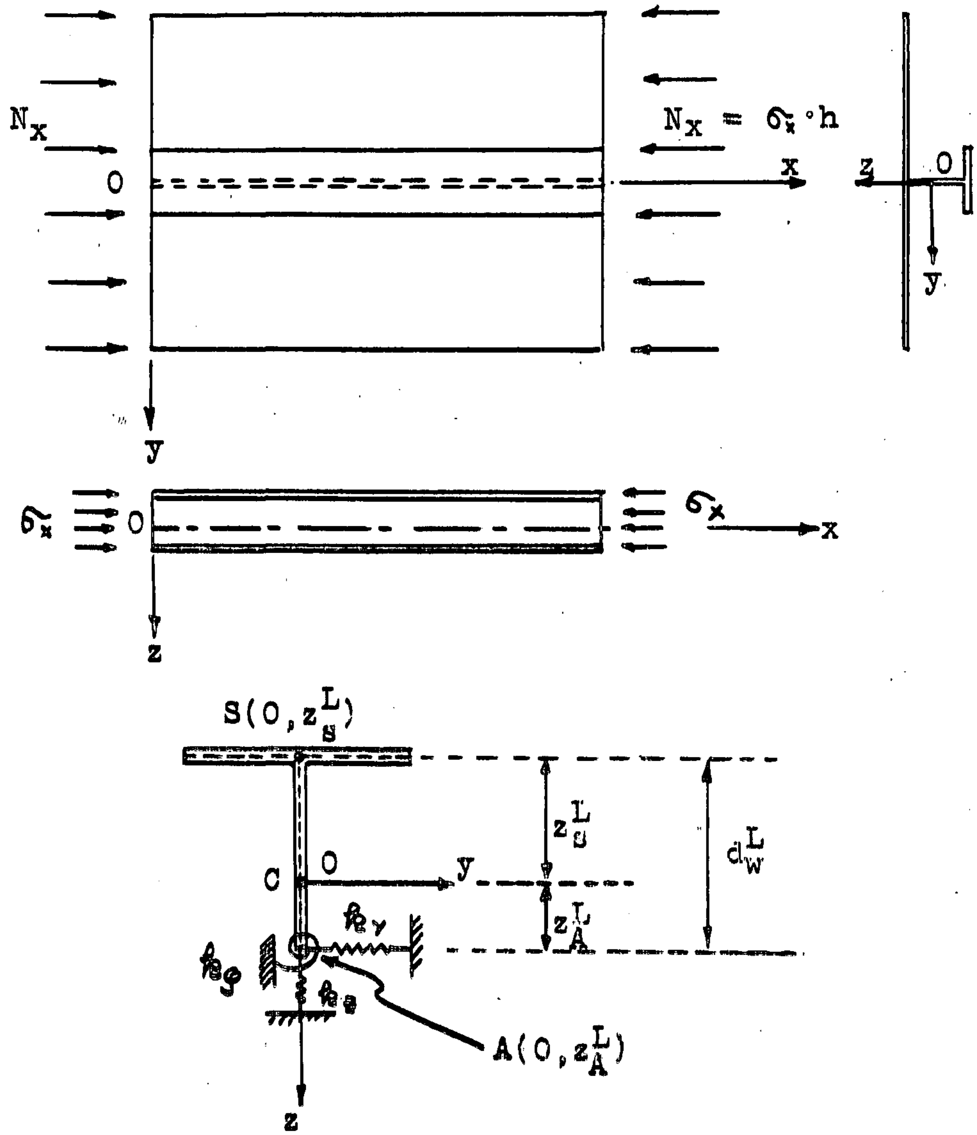


Fig. 7 - COORDINATES OF LONGITUDINAL STIFFENER
(Symmetric Cross-Section)

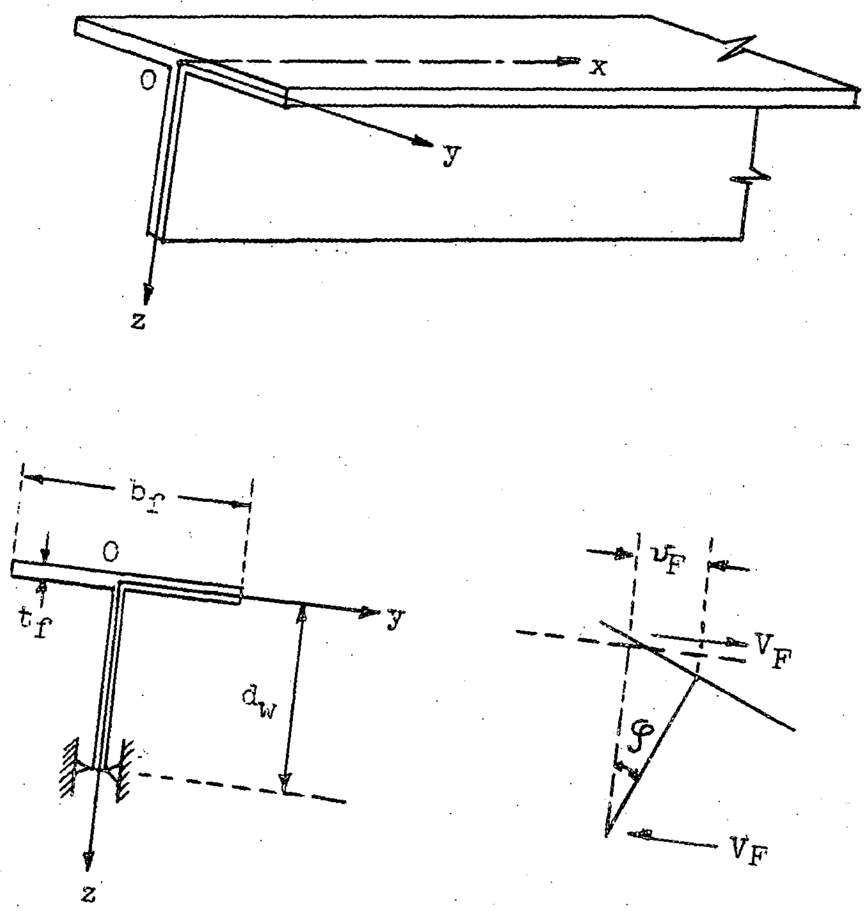


Fig. 8 - WARPING TORSION OF T-STIFFENER WITH ENFORCED AXIS OF ROTATION

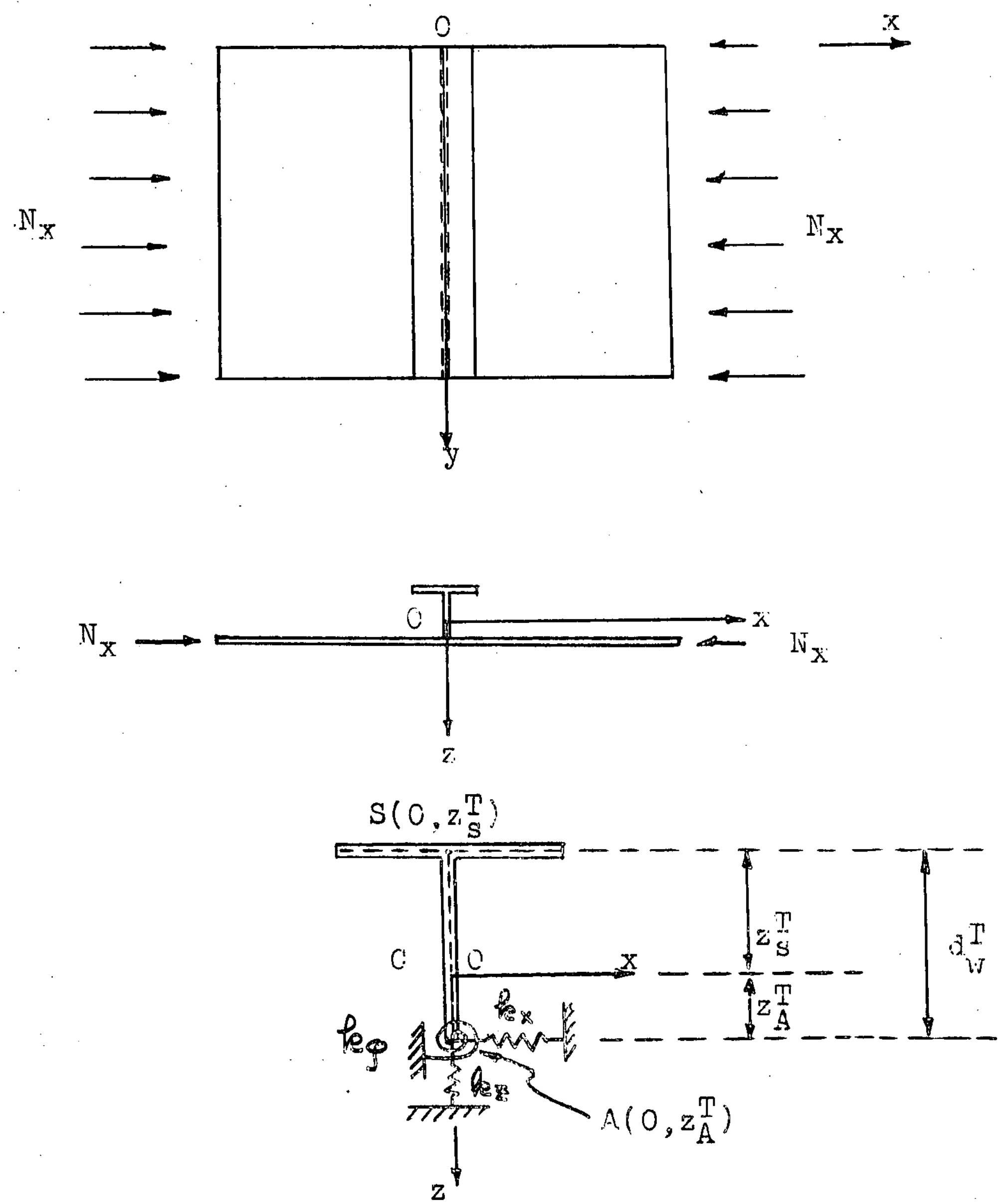


Fig. 9 - COORDINATES OF TRANSVERSE STIFFENER
(Symmetric Cross-Section)

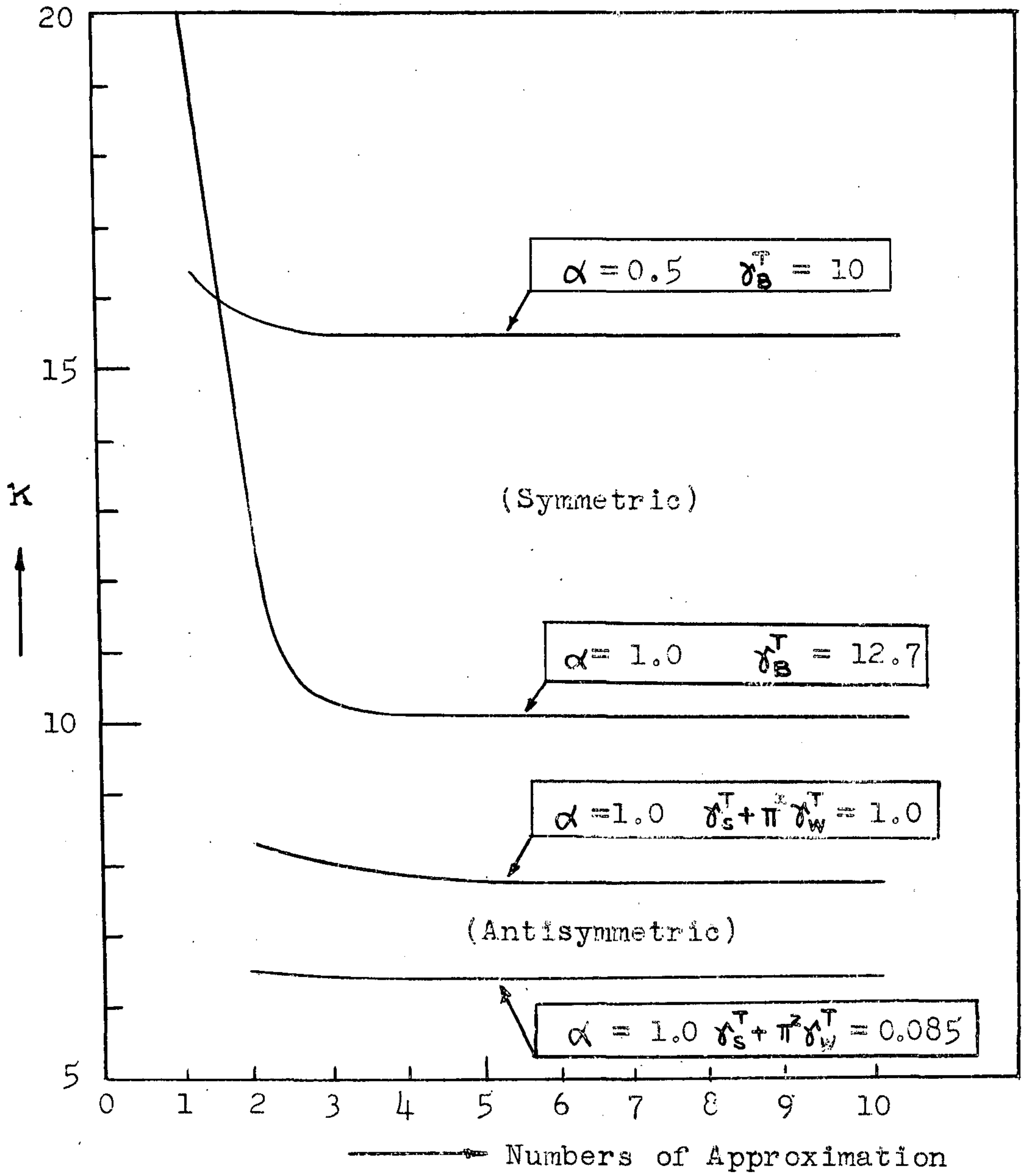


Fig. 10 - CONVERGENCE OF BUCKLING COEFFICIENT κ OF TRANSVERSELY STIFFENED PLATE (Elastic)

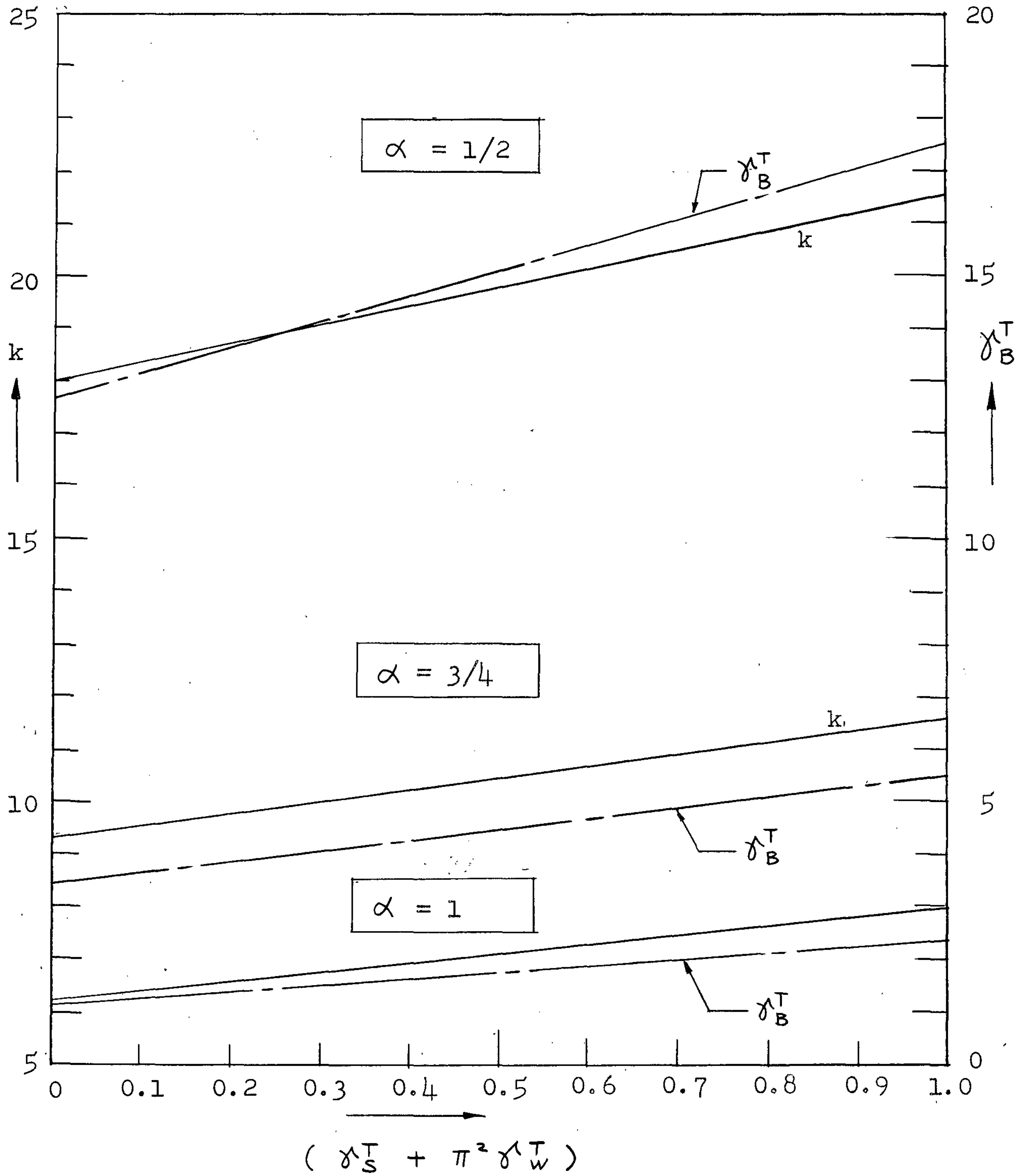


Fig. 11 - EFFECT OF TORSIONAL RESISTANCE OF STIFFENER ON BUCKLING OF TRANSVERSELY STIFFENED PLATE (Elastic)

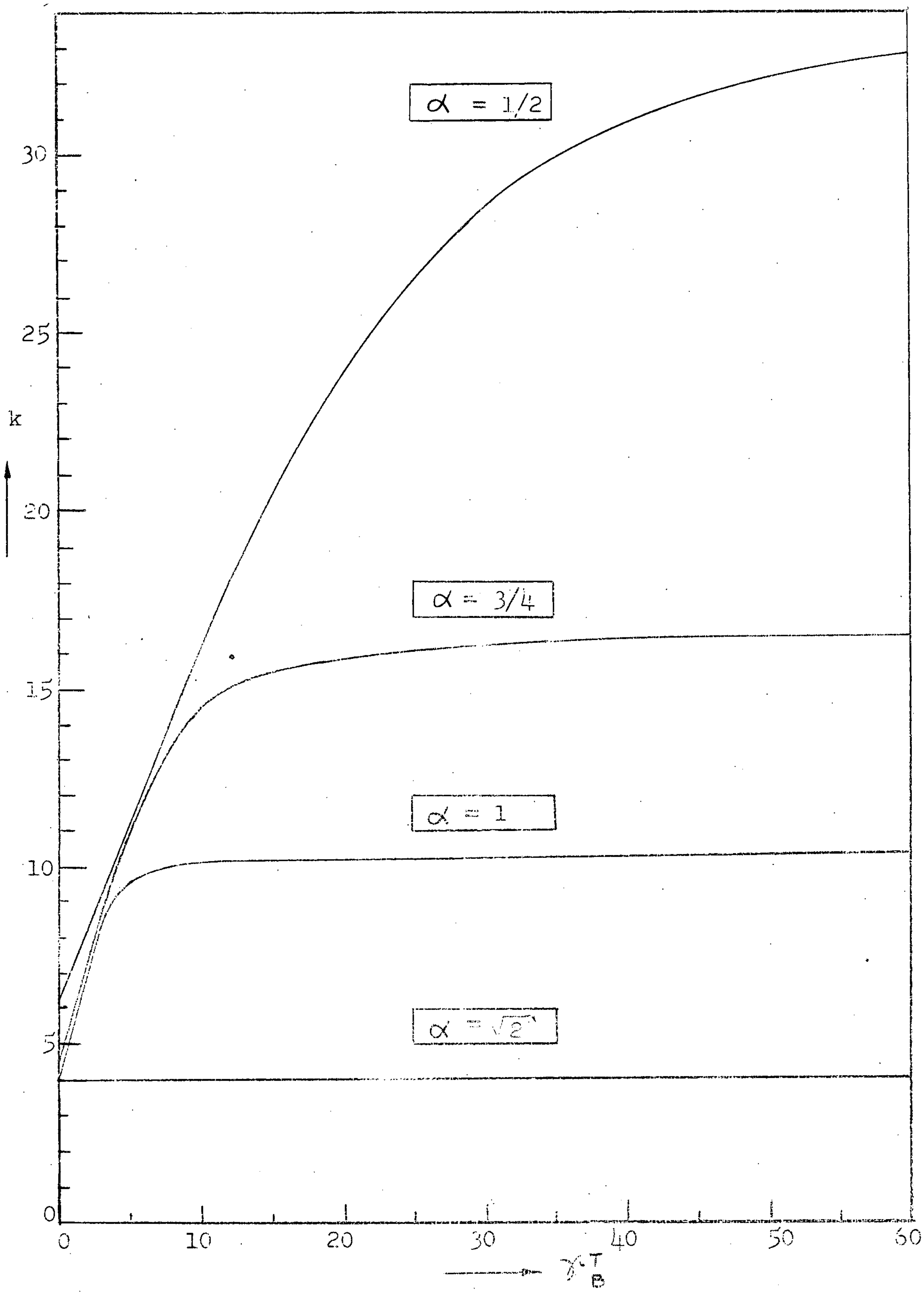


Fig. 12 - EFFECT OF BENDING RIGIDITY OF STIFFENER ON BUCKLING OF TRANSVERSELY STIFFENED PLATE (Elastic)

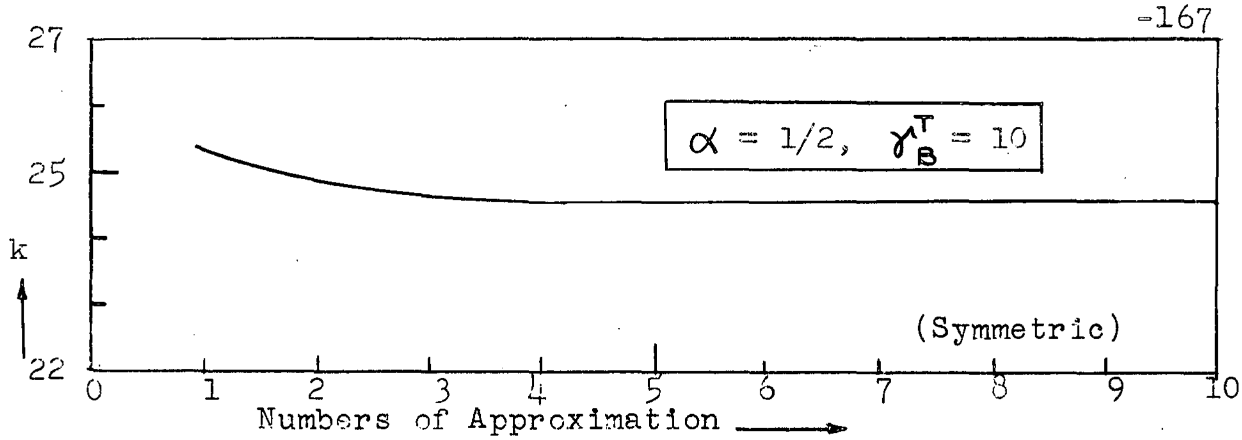


Fig. 13 - CONVERGENCE OF BUCKLING COEFFICIENT OF TRANSVERSELY STIFFENED PLATE (Strain-Hardening)

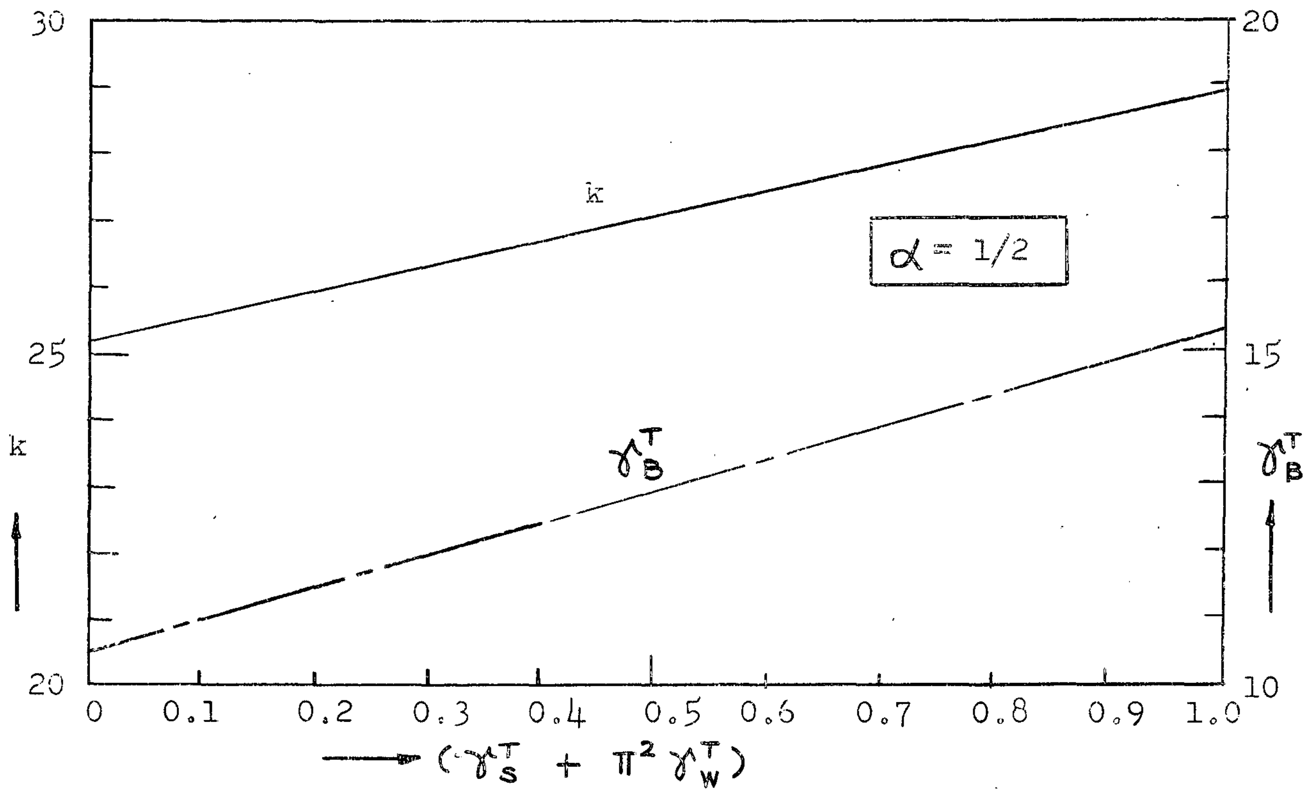


Fig. 14 - EFFECT OF TORSIONAL RESISTANCE OF STIFFENER ON BUCKLING OF TRANSVERSELY STIFFENED PLATE (Strain-Hardening)

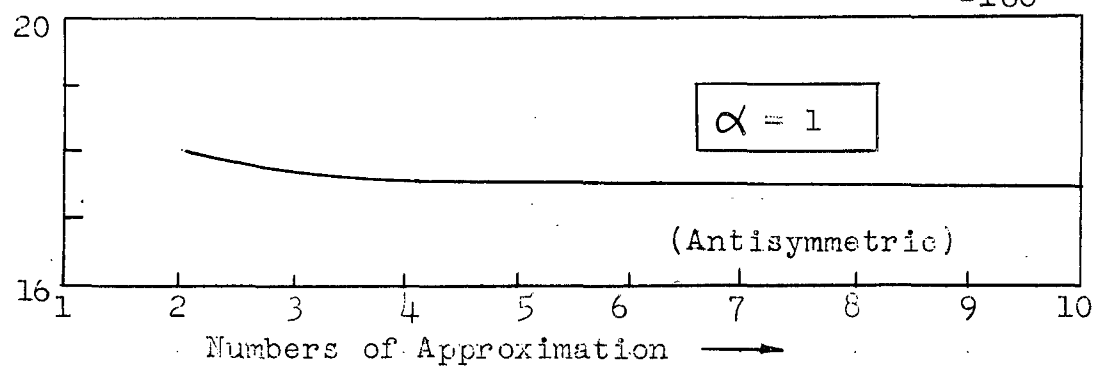


Fig. 15 - CONVERGENCE OF BUCKLING COEFFICIENT k OF LONGITUDINALLY STIFFENED PLATE (Elastic)

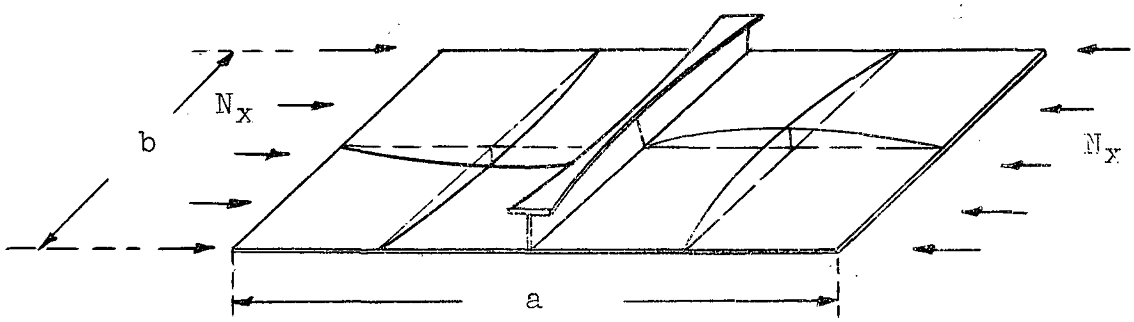


Fig. 16-1 - WARPING DEFORMATION OF TRANSVERSE STIFFENER

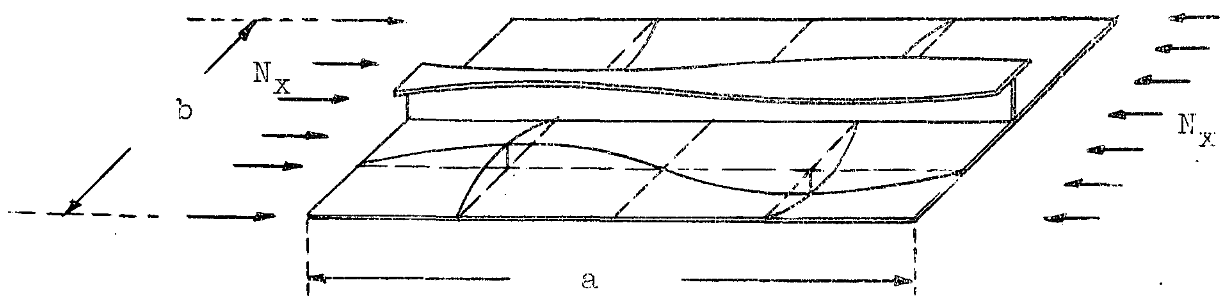


Fig. 16-2 - WARPING DEFORMATION OF LONGITUDINAL STIFFENER

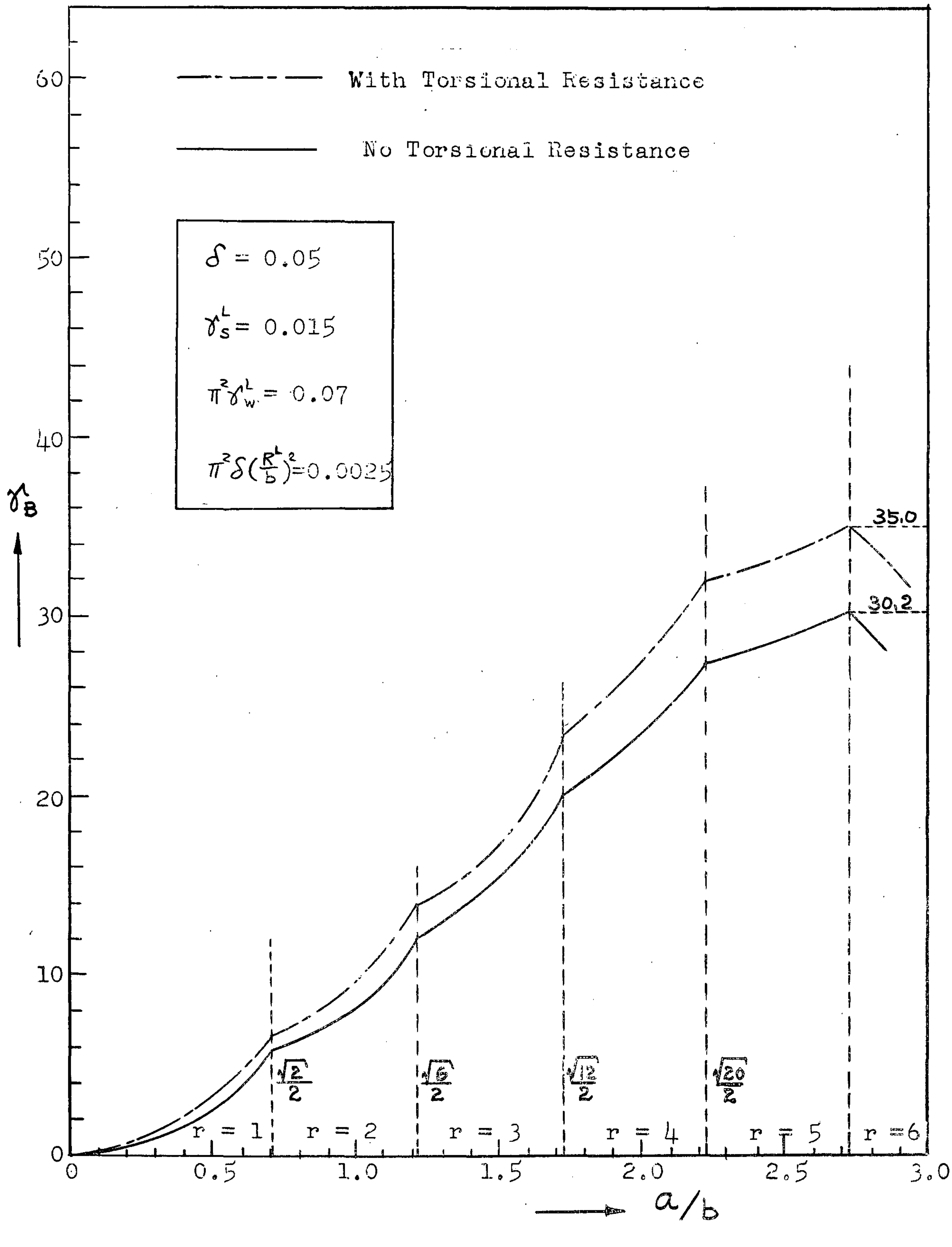


Fig. 17 - REQUIRED BENDING RIGIDITY
OF LONGITUDINAL T-STIFFENER
(Elastic)

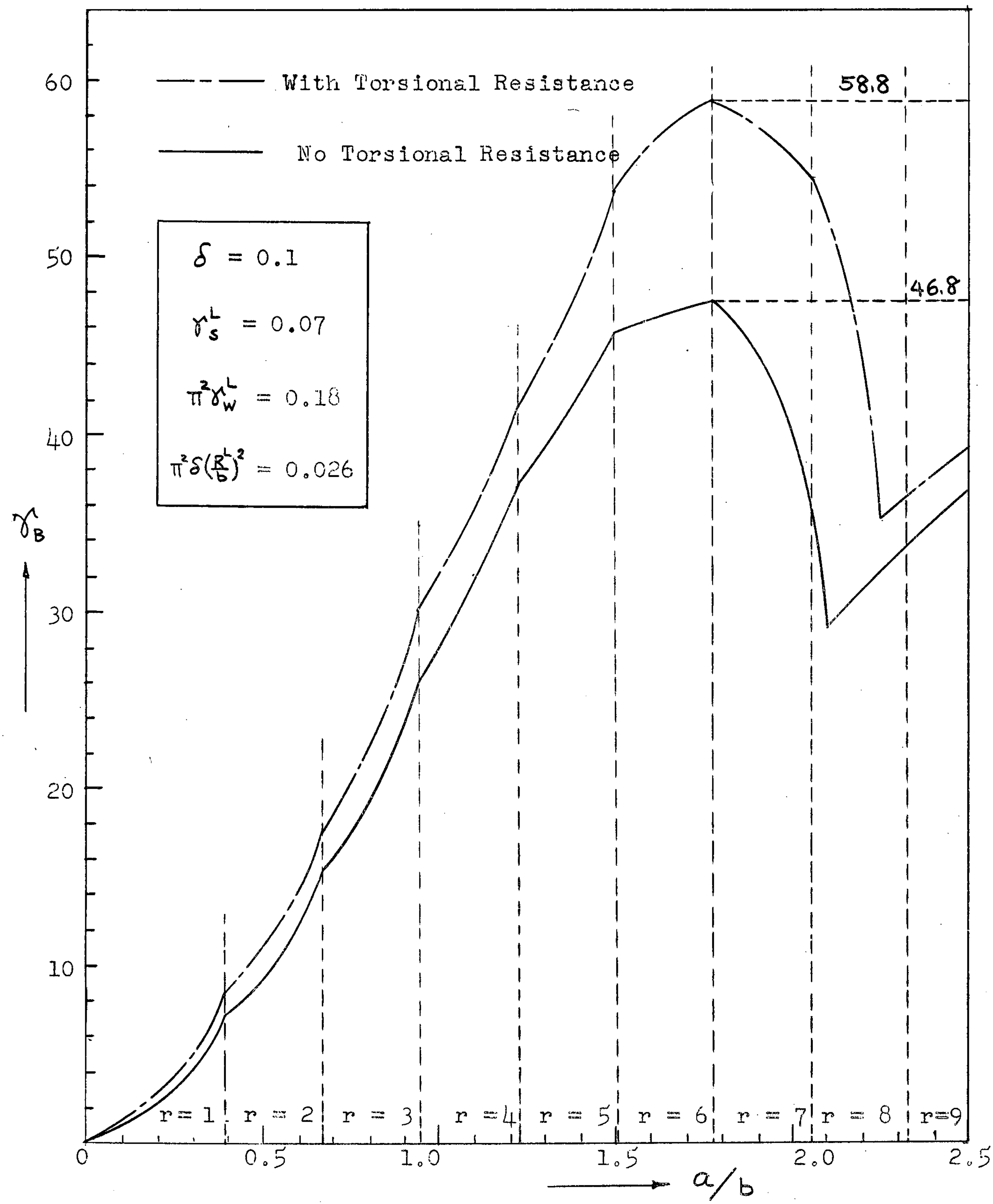


Fig. 18 - REQUIRED BENDING RIGIDITY OF LONGITUDINAL T-STIFFENER (Strain-Hardening)

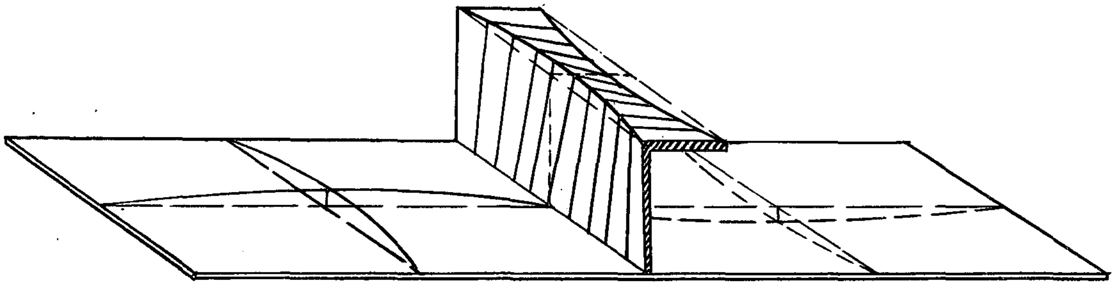


Fig. 19-1 - WARPING DEFORMATION OF INVERTED ANGLE STIFFENER

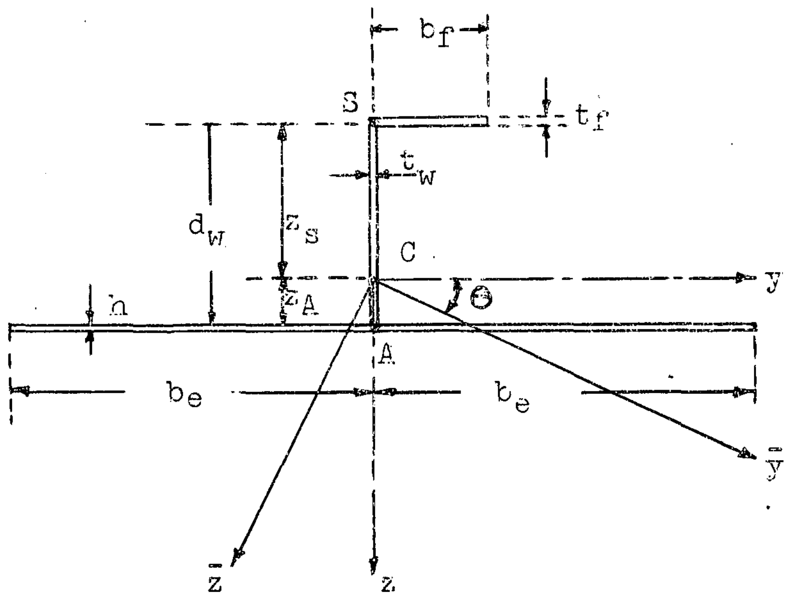


Fig. 19-2 - COORDINATES OF INVERTED ANGLE STIFFENER

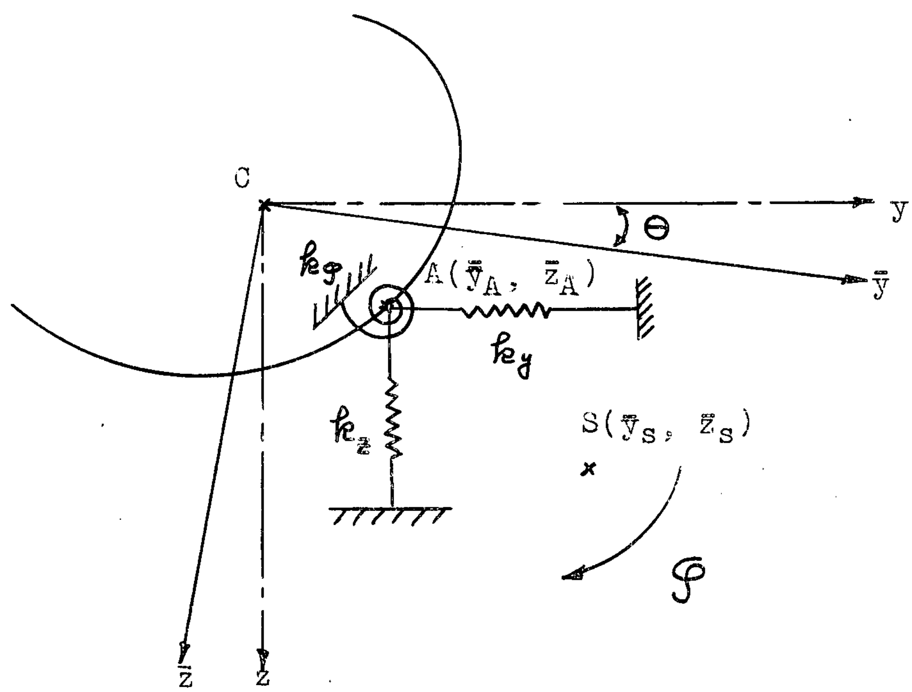


Fig. 20 - COORDINATES OF THIN-WALLED OPEN CROSS-SECTION WITH ENFORCED AXIS OF ROTATION

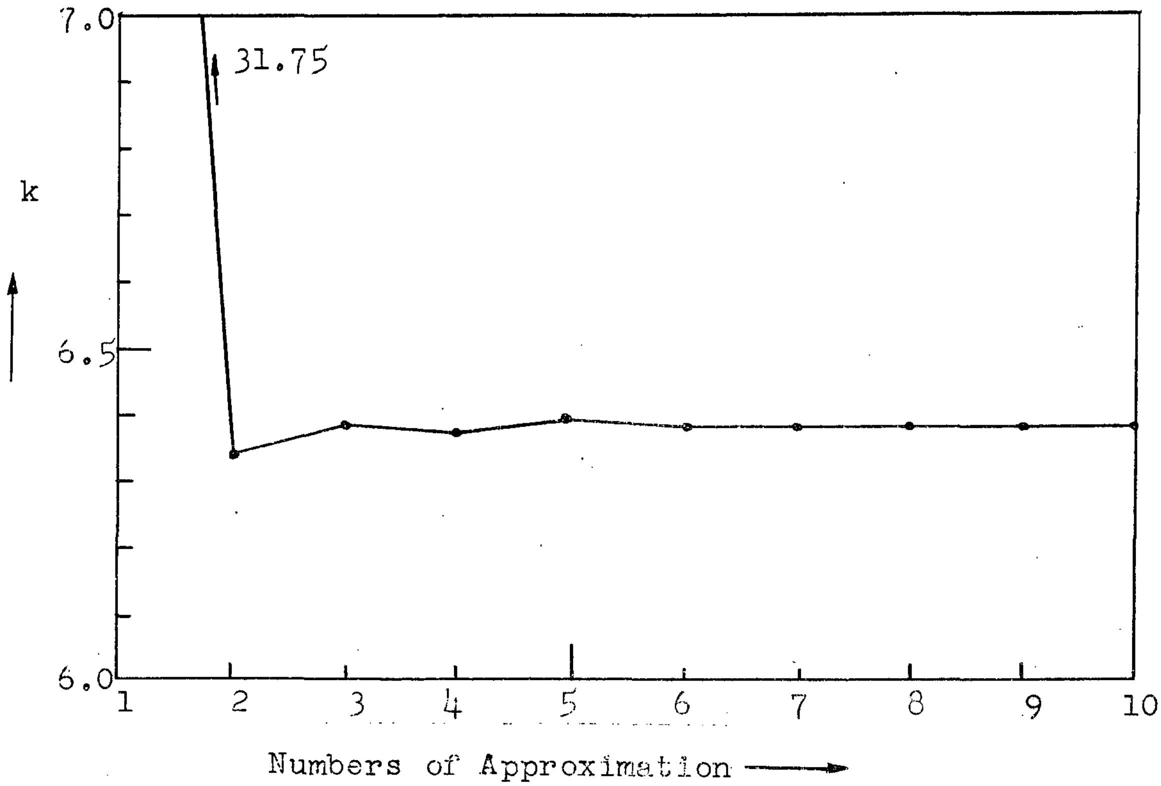


Fig. 21 - CONVERGENCE OF BUCKLING COEFFICIENT k OF TRANSVERSELY STIFFENED PLATE WITH INVERTED ANGLE STIFFENER (Elastic)

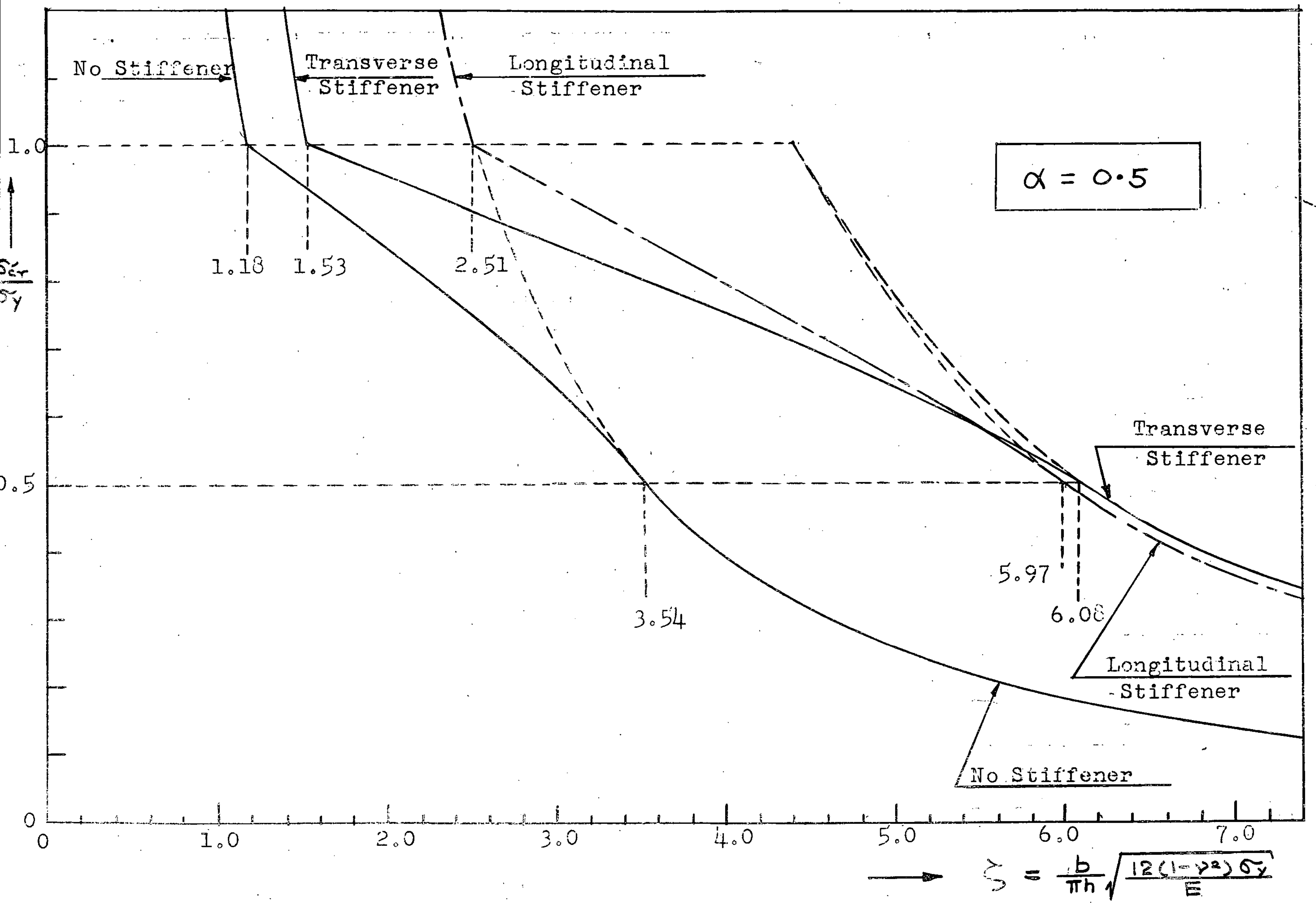


Fig. 22 - PLATE BUCKLING CURVES

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