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Plastic Design of Multi-Story Frames

SWAY SUBASSEMBLAGE ANALYSIS FOR UNBRACED FRAMES

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SWAY SUBASSEMBLAGE ANALYSIS FOR UNBRACED FRAMES[a]

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INTRODUCTION

The plastic methods of structural analysis based on simple plastic theory
are limited to the design of frames in which the deformations are so small that
the equilibrium equations can be formulated for the undeformed state, and where
no instability will occur prior to the attainment of the ultimate load (3).[3]
These limitations imply that the effects of axial forces can be ignored. Such
methods therefore are not suitable for the design of tall unbraced multi-story
frames in which the strength and behavior are considerably influenced by axial
force effects due to gravity loads (7). This is especially so when the frames
are subjected to combined gravity and wind (or seismic) loads.

The chief concern for gravity loads is the magnitude of additional over-
turning moment ($P\Delta$ moment) that causes the frame to fail by instability. As the
frame sways under the action of the combined loads, the total gravity load, P,
above a given story acts through the relative sway displacement, Δ, of the story
to produce an overturning moment which the frame is required to resist.

Much research effort is being directed towards the development of plastic
methods which will be suitable for the analysis and design of unbraced multi-
story frames. Typical of the results being obtained are the second-order elastic-
plastic methods of analysis (15). They can include $P\Delta$ moments and can predict
the lateral-load versus sway-deflection behavior of the frame or portions of it
up to failure. Such methods however find their major application only for the

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final design stages since a complete, albeit preliminary, design of the frame must already be available. Relatively simple design procedures have been developed which are suitable for the preliminary design stage of unbraced frames (7,8,9,10). The most successful is plastic moment balancing, because it can include an estimated $P\Delta$ moment in each story of the frame (7,9). The $P\Delta$ moments are estimated on the basis of the expected sway deflection corresponding to either the formation of a mechanism in each story or to the attainment of the maximum load.

Following the preliminary design by moment balancing, a lateral-load versus sway-deflection analysis is required in order to check the estimated sway deflection used to compute the $P\Delta$ moments, and to determine the sway deflection at the working load level of the combined loads. This paper presents the theoretical basis for an approximate method of analysis which can be used for this purpose (4). It is particularly applicable to the middle and lower stories of an unbraced frame where the $P\Delta$ moments become significant. The method is based on the concept of sway subassemblages (12), and uses directly the results of recent studies of restrained columns permitted to sway (11). It accounts for $P\Delta$ moments (in columns and frames) as well as plastification and residual stresses in columns and plastic hinges in beams. In this method, a story with known member sizes is analyzed by subdividing it into sway subassemblages. Each sway subassemblage is then analyzed either manually or with a computer for its load-deflection behavior (1,5). Specially prepared design charts can be used to assist the manual analysis (4,6,14). The resulting load-deflection curves are then combined to give the complete load-deflection curve for the story up to and beyond the maximum load. The adequacy of the preliminary design can then be determined on the basis of maximum strength, maximum deflection (at working or factored load) or any other suitable load or deflection criteria. If a revision of member sizes is necessary the previous analysis will assist in the selection of revised members. The final design step using a second-order elastic-plastic analysis, for example, could proceed once the sway subassemblage analysis has indicated that the preliminary design is satisfactory.

Restrictions and Assumptions.- The following restrictions and assumptions are applied to the frame and its members:

1. Unbraced frames considered are regular, plane, rectangular multi-story steel frames of approximately equal story height, one or more bays in width, having rigid connections and no bracing or cladding in the plane of the frame to resist sway. The number of stories considered is less than that for which column shortening effects become significant (15).

2. Steel members must have one axis of symmetry. They may be rolled or welded shapes such as wide-flange or I sections and may include welded hybrid sections. These members can be made of different steels with different yield stress levels.

3. Out-of-plane deformations of the frame as well as lateral-torsional, local buckling of the members is prevented. Biaxial bending of the columns does not occur. The effect of axial and curvature shortening of the beams and columns is neglected.

4. Lengths of beams and columns will be taken as the distance between centroidal axes of the members. The centroidal axes of the beams and columns are each assumed to be collinear. The minor axis of each member coincides with the plane of the frame which is a plane of symmetry.

Loading Conditions.- The sway subassemblage method is applicable only for the combined gravity and wind load condition. All loads are assumed either horizontal or vertical and acting in the plane of the frame. Gravity loads may be concentrated or distributed. Lateral wind loads are distributed loads, but are assumed concentrated at the floor levels. It is unlikely that the gravity and lateral loads would increase proportionally in a practical frame. It is more likely that the gravity loads will remain virtually unchanged as the lateral loads are applied. Therefore, it is assumed that the factored gravity loads are applied first, followed by monotonically increasing lateral loads. The choice of load factor is arbitrary, although, where indicated will be taken in accordance with that established in Ref. 7, namely 1.30. The probability that the full gravity loads are not likely to be present at all times can be accounted for by live load reduction factors (7).

Application of the Sway Subassemblage Method.- The method is limited to the analysis of those stories in an unbraced multi-story frame where the columns are bent into nearly symmetrical double curvature. In general this will comprise the middle and lower stories. The member design in the upper several stories is normally controlled by the gravity load condition alone. Therefore a separate analysis is required in this region. It is assumed that the unbraced frame and its gravity loads are sufficiently symmetrical that sidesway under gravity loads alone is small and can be neglected.

Sign Convention.- The sign convention adopted in the analysis is as follows (11):

1. External moments acting at a joint are positive when clockwise.

2. Moments and rotations at the ends of members are positive when clockwise, and

3. Horizontal shear is positive if it causes a clockwise moment about a joint.

THE SWAY SUBASSEMBLAGES IN AN UNBRACED FRAME

One-Story Assemblage.- In a well-proportioned regular unbraced multi-story frame such as shown in Fig. 1, the member sizes in a region containing the middle and lower stories will likely increase at a relatively small and uniform rate with increasing distance from the top of the frame. Although the factored gravity loads (1.3 w) within each bay will likely be constant in this region, the beams and columns will increase in size due to the increasing wind and P∆ moments which must be carried by the lower stories. The columns will also increase in size due to the accumulation of gravity loads on the beams. In addition, although the wind loads may not be uniformly distributed over the height of the frame, and the sway deflection may not be uniform for each story, the variation over two or three stories is probably small. As a result, if level n is within this region (Fig. 1) the load deflection behavior at levels n + 1, n, and n - 1, could be expected to be nearly the same. Therefore, it is assumed that the behavior of one story of the frame containing level n can be represented by the behavior of a one-story assemblage of beams and columns which contains level n.
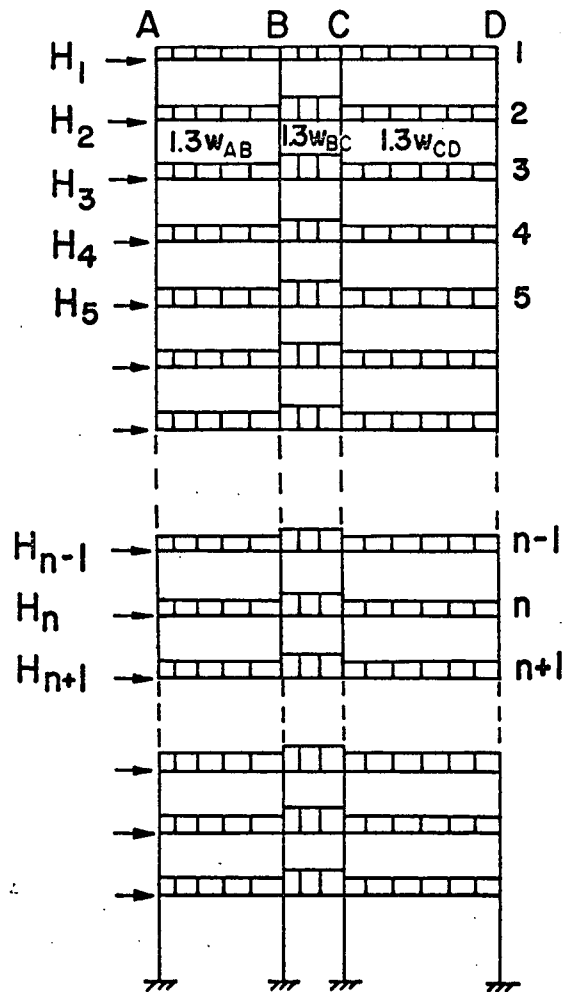
FIG. 1   UNBRACED FRAME AND LOADING

-5-

On the basis of recent studies (5,15) it is assumed that a point of inflection will occur at the mid-height of each column above and below level n. A one-story assemblage can now be isolated from the frame by passing horizontal cuts through the assumed inflection points. The resulting assemblage is shown in Fig. 2 together with the forces acting on the members and the resulting deformations. The total shear between levels n and n - 1 is ΣH_{n-1}. Similarly, the total shear between levels n and n + 1 is ΣH_n where $\Sigma H_n = \Sigma H_{n-1} + H_n$. The constants λ_A, λ_B, ---etc. define the distribution of the total wind shear to the columns and is assumed to be the same above and below level n. The axial forces in the columns above and below level n are designated as P_{n-1} and P_n respectively, and are assumed to remain constant in the analysis. For a particular column, P_{n-1} is calculated as the algebraic sum of the estimated shear forces at the two column faces resulting from:

1. The factored gravity loads on the tributary length of each beam connected to the column above level n (7), and,

2. The moments present in each beam connected to the column above level n, due to the factored wind loads and the estimated $P\Delta$ effect. These moments are computed for the assumed mechanism or instability load of each story and are easily obtained from the moment balance process (9).

The axial loads P_n are calculated in a similar manner but include the shear forces from the beams at level n.

Half-Story Assemblage.- In Figs. 1 and 2, the shear forces ΣH were calculated from the design ultimate value of lateral load. In the analysis ΣH is to be compared with the lateral load, ΣQ, versus sway deflection, Δ, response of the one-story assemblage. Therefore in the sway subassemblage analysis, the shear forces ΣH_{n-1} and ΣH_n acting on the one-story assemblage (Fig. 2) will be replaced by shear forces ΣQ_{n-1} and ΣQ_n where ΣQ is a function of Δ. The one-story assemblage may also be simplified by replacing each column above level n with the equivalent joint forces. Such a simplification is shown in Fig. 3. Each column above level n applies a vertical and horizontal force and a bending moment to the joint at level n. The horizontal force ΣQ_{n-1} may be combined with the force
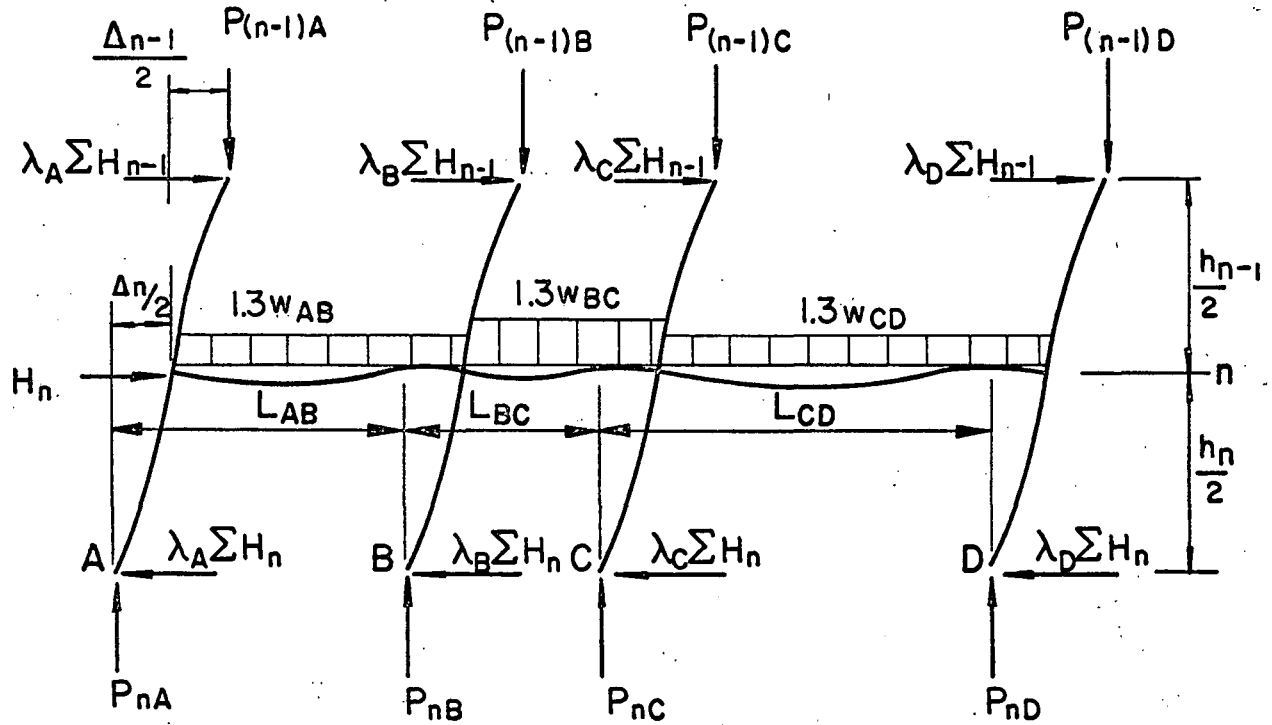
$\dfrac{\Delta_{n-1}}{2}$ $P_{(n-1)A}$ $P_{(n-1)B}$ $P_{(n-1)C}$ $P_{(n-1)D}$

$\lambda_A \Sigma H_{n-1}$ $\lambda_B \Sigma H_{n-1}$ $\lambda_C \Sigma H_{n-1}$ $\lambda_D \Sigma H_{n-1}$

$\dfrac{\Delta_n}{2}$ $1.3 w_{AB}$ $1.3 w_{BC}$ $1.3 w_{CD}$

$\dfrac{h_{n-1}}{2}$

H_n L_{AB} L_{BC} L_{CD} n

$\dfrac{h_n}{2}$

A $\lambda_A \Sigma H_n$ B $\lambda_B \Sigma H_n$ C $\lambda_C \Sigma H_n$ D $\lambda_D \Sigma H_n$

P_{nA} P_{nB} P_{nC} P_{nD}

FIG. 2 ONE-STORY ASSEMBLAGE

-7-

Q_n to give a total force at level n of ΣQ_n where

$$\Sigma Q_n = \Sigma Q_{n-1} + Q_n \qquad (1)$$

Also the bending moment at each joint will have a magnitude of

$$M_{n-1} = - (\lambda \Sigma Q_{n-1}) \frac{h_{n-1}}{2} - P_{n-1} \frac{\Delta_{n-1}}{2} \qquad (2)$$

As shown in Fig. 3, from statics, the internal moments, M_n, in the columns imme-diately below level n will be given by

$$M_n = - (\lambda \Sigma Q_n) \frac{h_n}{2} - P_n \frac{\Delta_n}{2} \qquad (3)$$

Now, the factor λ was assumed to have the same value for a column above and below level n. In addition, h_n was assumed to be approximately equal to h_{n-1}. Since $\Sigma Q_n > \Sigma Q_{n-1}$ and $P_n > P_{n-1}$ then if Δ_n is approximately equal to Δ_{n-1}, M_n will likely be greater than M_{n-1}. In fact, if $h_n \geq h_{n-1}$ and $\Delta_n \geq \Delta_{n-1}$, then M_n will always be greater than M_{n-1}. With the ideal behavior of the middle and lower stories assumed of a well proportioned frame it can then be conservatively assumed that M_{n-1} is equal to M_n and acts in the same sense.

The Sway Subassemblages.- To facilitate the load-deflection analysis of the half-story assemblage (Fig. 3), it will be subdivided into smaller units called sway subassemblages. A sway subassemblage will consist of one restrained column plus the adjacent beams at the column top. Three types of sway subassemblages which are possible in any multi-bay unbraced frame are shown in Fig. 4. In each sway subassemblage the beams are considered as the restraining members which provide rotational restraint to the column top.

Rotational restraints are also imposed at the free ends of the beams in each sway subassemblage to account for the restraining effects of the members either side of a sway subassemblage. These restraints are represented as springs in Fig. 4.

THE RESTRAINED COLUMN IN A SWAY SUBASSEMBLAGE

The method of analysis developed in this paper for the sway subassemblages described above requires an understanding of the behavior of restrained columns.
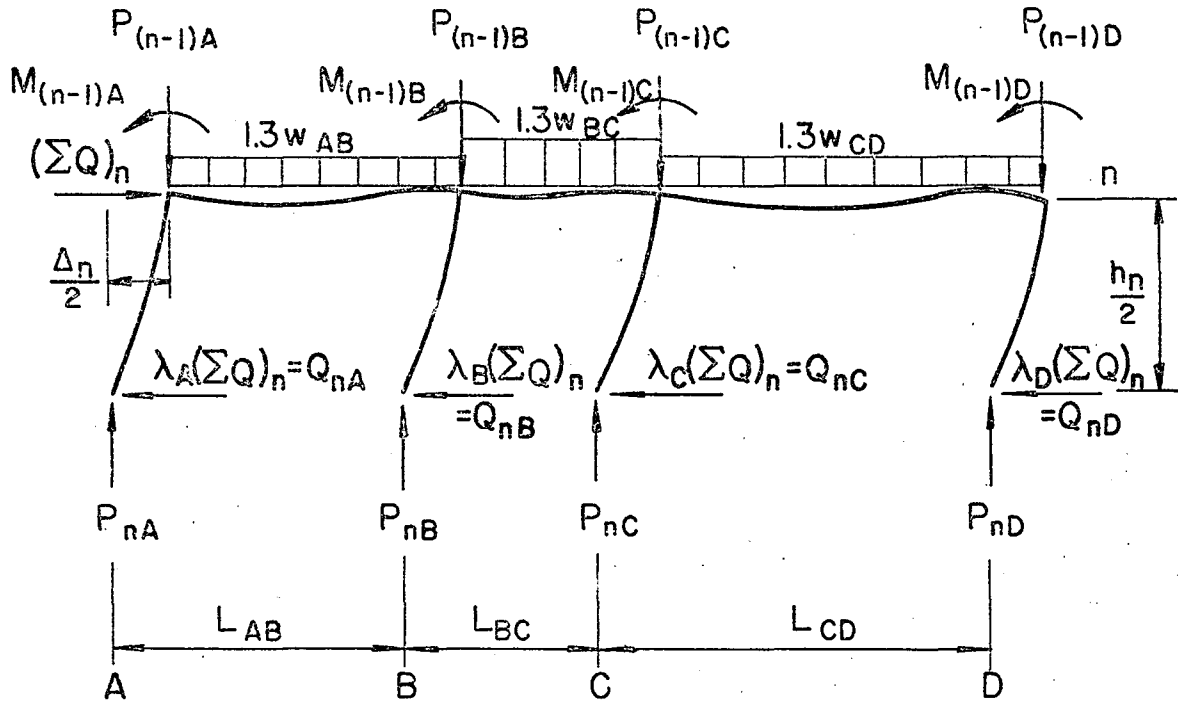
$$M_{(n-1)A} = - (\lambda_A \Sigma Q_{n-1}) \frac{h_{n-1}}{2} - P_{(n-1)A} \frac{\Delta_{n-1}}{2}$$

$$M_{(n-1)B} = - (\lambda_B \Sigma Q_{n-1}) \frac{h_{n-1}}{2} - P_{(n-1)B} \frac{\Delta_{n-1}}{2}$$

$$M_{(n-1)C} = - (\lambda_C \Sigma Q_{n-1}) \frac{h_{n-1}}{2} - P_{(n-1)C} \frac{\Delta_{n-1}}{2}$$

$$M_{(n-1)D} = - (\lambda_D \Sigma Q_{n-1}) \frac{h_{n-1}}{2} - P_{(n-1)D} \frac{\Delta_{n-1}}{2}$$
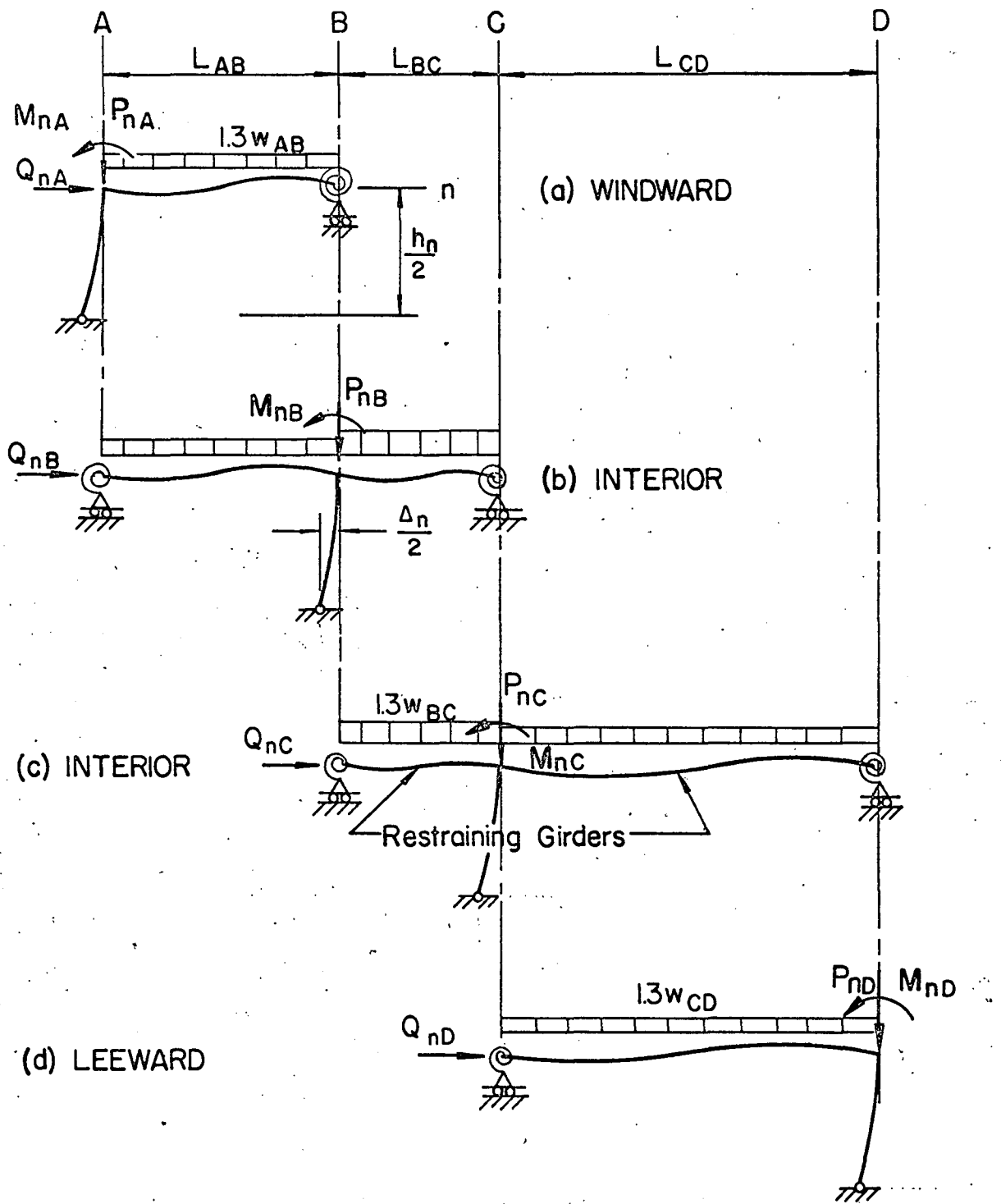
FIG. 3   HALF-STORY ASSEMBLAGE

FIG. 4 THE SWAY SUBASSEMBLAGES

A restrained column depends on the rotational restraint provided by the adjacent beams to resist the applied shear forces. For the columns in the half-story assemblage, this restraint is a function of the stiffnesses of all the beams and columns. In this study, two types of rotational restraint will be considered: (1) linearly elastic restraint for all values of sway deflection, and (2) linearly elastic restraint which decreases abruptly at discrete intervals of sway deflection. The results of this study will then be applied to determine the load-deflection behavior of a sway subassemblage where the beams initially provide linear elastic rotational restraint which decreases abruptly at discrete intervals of sway deflection with the formation of plastic hinges in the beams.

Equilibrium and Compatibility.- Figure 5(a) shows a typical restrained column to be treated in this study. It is subjected to a constant vertical load, P_n, and to varying lateral force, Q_n, and moment, M_n. The resulting deformation configuration is shown in Fig. 5(b). A linearly elastic rotational restraint at the column top provides a restraining moment of M_r. The three rotations, Δ_n/h_n, θ, and γ are measured from the reference lines shown in Fig. 5(b) and are positive when clockwise (γ as shown is therefore negative). From statics the moment at the upper end of the column will be given by

$$M_n = -\left[Q_n \frac{h_n}{2} + P_n \frac{\Delta_n}{2}\right]$$ (4)

Equilibrium of moments at the joint then requires that

$$2M_n + M_r = 0$$ (5)

For small deformations, the rotations θ, γ, and Δ_n/h_n in Fig. 5(b) are related by the compatibility condition:

$$\frac{\Delta_n}{h_n} = \theta - \gamma$$ (6)

The load-deflection relationship, Q_n versus $\Delta_n/2$ of the restrained column with linear restraint stiffness can be determined by solving Eqs. 4, 5 and 6 together with the known moment-rotation relationship, M_n versus γ, for the column (7).
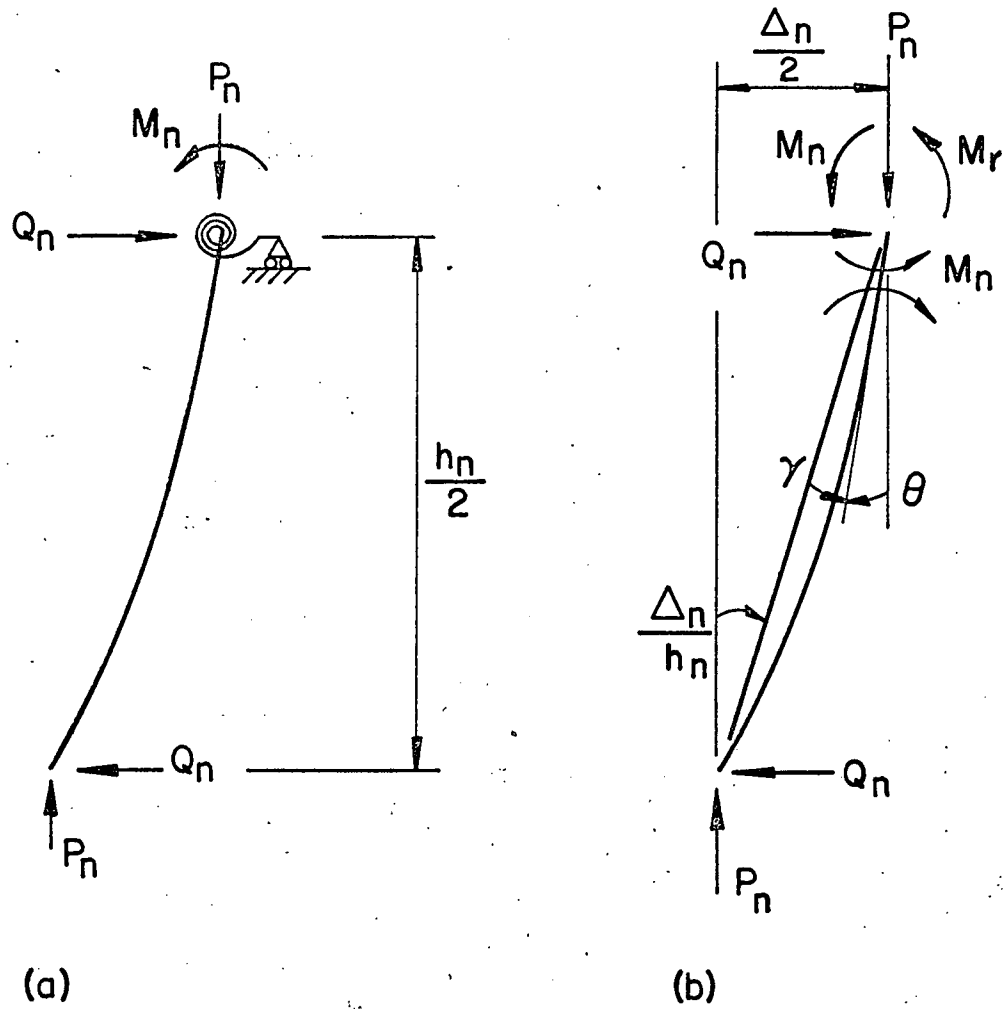
FIG. 5 THE RESTRAINED COLUMN IN A SWAY SUBASSEMBLAGE

<u>Moment-Rotation Relationship.</u>- For small values of sway deflection Δ_n, it can be assumed that the restrained column is axially loaded by the vertical forces P_n. The restrained column will then be subjected to unsymmetrical single curvature bending under the system of forces shown in Figure 5(b). However, this column is actually the upper half of a column of length h_n (Fig. 3) which will be subjected to symmetrical double curvature bending. The moment-rotation relationship for such a column can be determined following the procedures described in Chapters 4 and 9 of Ref. 7 or it can be obtained from curves such as those shown in Charts III-1 to III-7 of Ref. 14 for specified values of the axial load ratio P_n/P_y and slenderness ratio h_n/r_x. P_y is the axial load corresponding to full yielding of the column cross-section and r_x is the radius of gyration of the column for strong axis bending. The charts shown in Ref. 14 give the moment-rotation curves for columns which have moment applied at one end, are pinned at the other end and are of length h_n. These curves can be used for columns which are bent in symmetrical double curvature where the half-length is $h_n/2$, by using an equivalent slenderness ratio equal to one-half of the actual slenderness ratio of the column.

An examination of the curves in Charts III-1 to III-7 of Ref. 14 indicates that for symmetrical double curvature bending and for P_n/P_y less than or equal to 0.90, the maximum value of M_n will be equal to or slightly below the reduced plastic moment M_{pc}, (reduced due to P_n) for columns with h_n/r_x less than 40. Therefore it is assumed in this paper that a plastic hinge can develop at the top of the column.

<u>Load-Deflection Equation for Constant Restraint Stiffness.</u>- Since Eqs. 4, 5 and 6 are valid for any restrained column in the story below any level n, the subscript n can then be deleted and Eq. 4 re-arranged and written

$$\frac{Qh}{2} = - \left[M + \frac{P\Delta}{2} \right] \qquad (7)$$

or equivalently

$$\frac{Qh}{2M_{pc}} = - \left[\frac{M}{M_{pc}} + \frac{P\Delta}{2M_{pc}} \right] \qquad (8)$$

For $0.15 < P/P_y < 1.0$, can be approximated by (2)

$$M_{pc} = 1.18 \ (1-P/P_y) \ M_p \tag{9}$$

in which M_p is the full plastic moment of the column section. If M_p is taken as

$$M_p = \sigma_y fS = 2P_y f \frac{r_x^2}{d} \tag{10}$$

in which σ_y is the yield stress of the column, f the shape factor, S the section modulus and d the column depth, then

$$\frac{P \Delta}{2M_{pc}} = \frac{\frac{P}{P_y} \frac{h}{r_x} \frac{d}{2r_x} \frac{\Delta}{h}}{2.36 \ f(1 - \frac{P}{P_y})} \tag{11}$$

When Eq. 11 is substituted into Eq. 8, the following non-dimensional load-deflection relationship is obtained

$$\frac{Qh}{2M_{pc}} = -\left[\frac{M}{M_{pc}} + \frac{\frac{P}{P_y} \frac{h}{r_x} \frac{d}{2r_x} \frac{\Delta}{h}}{2.36 \ f(1 - \frac{P}{P_y})}\right] \tag{12}$$

The above equation may be simplified by noting that f and $d/2r_x$ can be approximated by their average values of 1.11 and 1.15 respectively for wide-flange shapes normally used for columns (13). With these substitutions, Eq. 12 becomes

$$\frac{Qh}{2M_{pc}} = -\left[\frac{M}{M_{pc}} + \frac{\frac{P}{P_y} \frac{h}{r_x} \frac{\Delta}{h}}{2.28(1 - \frac{P}{P_y})}\right] \tag{13}$$

This equation can be further simplified to become

$$\frac{Qh}{2M_{pc}} = -\left[\frac{M}{M_{pc}} + C \frac{\Delta}{h}\right] \tag{14}$$

in which C is a constant given by

$$C = \frac{\frac{P}{P_y} \frac{h}{r_x}}{2.28(1 - \frac{P}{P_y})} \tag{15}$$

Load-Deflection Behavior for Constant Restraint Stiffness.- For all values of θ

$$M_r = \bar{k} \ \theta$$

-14-

or equivalently

$$M_r = k\ \theta M_{pc} \qquad\qquad (17)$$

in which $\bar{k}$ is the stiffness of the restraint and $k = \bar{k}/M_{pc}$. Since the moment M, at the top of a restrained column cannot be expressed in terms of the chord rotation, γ, except in the elastic range, Eq. 14 cannot be solved explicitly. However a tabular form of solution is possible (11).

The non-dimensional load-deflection relationship, $Qh/2M_{pc}$ versus Δ/h, for a particular restrained column with slenderness ratio, h/r_x, constant axial load ratio, P/P_y, and constant restraint stiffness, k, is shown by curve 0-a-b-c-e in Fig. 6.

The maximum value of restraining moment will be reached when a plastic hinge forms at the top of the column, that is

$$M = - \frac{M_r}{2} = - M_{pc} \qquad\qquad (18)$$

The load-deflection relationship for the restrained column after the formation of this plastic hinge (and thus a mechanism) can be found from Eqs. 14 and 18 as

$$\frac{Qh}{2M_{pc}} = 1 - C\frac{\Delta}{h} \qquad\qquad (19)$$

Equation 19 appears in Fig. 6 as the straight line segment d-e. Curves 0-a-b-c-e and d-c-e intersect at point c when a plastic hinge forms at the top of the column. The restraining moment corresponding to point c can be found from Eq. 18 as

$$M_r = 2M_{pc} \qquad\qquad (20)$$

The angle, θ_p, corresponding to the formation of the plastic hinge may be found by equating Eqs. 17 and 20

$$\theta_p = \frac{2}{k} \qquad\qquad (21)$$

It should be apparent from the derivation of Eq. 14 that lines 0-d and d-c-e in Fig. 6 define the second-order, rigid-plastic load-deflection curves for the column shown in the figure. It is evident then that the restraining moment, M_r will have a constant value, M'_r, everywhere on d-c-e, which is equal to
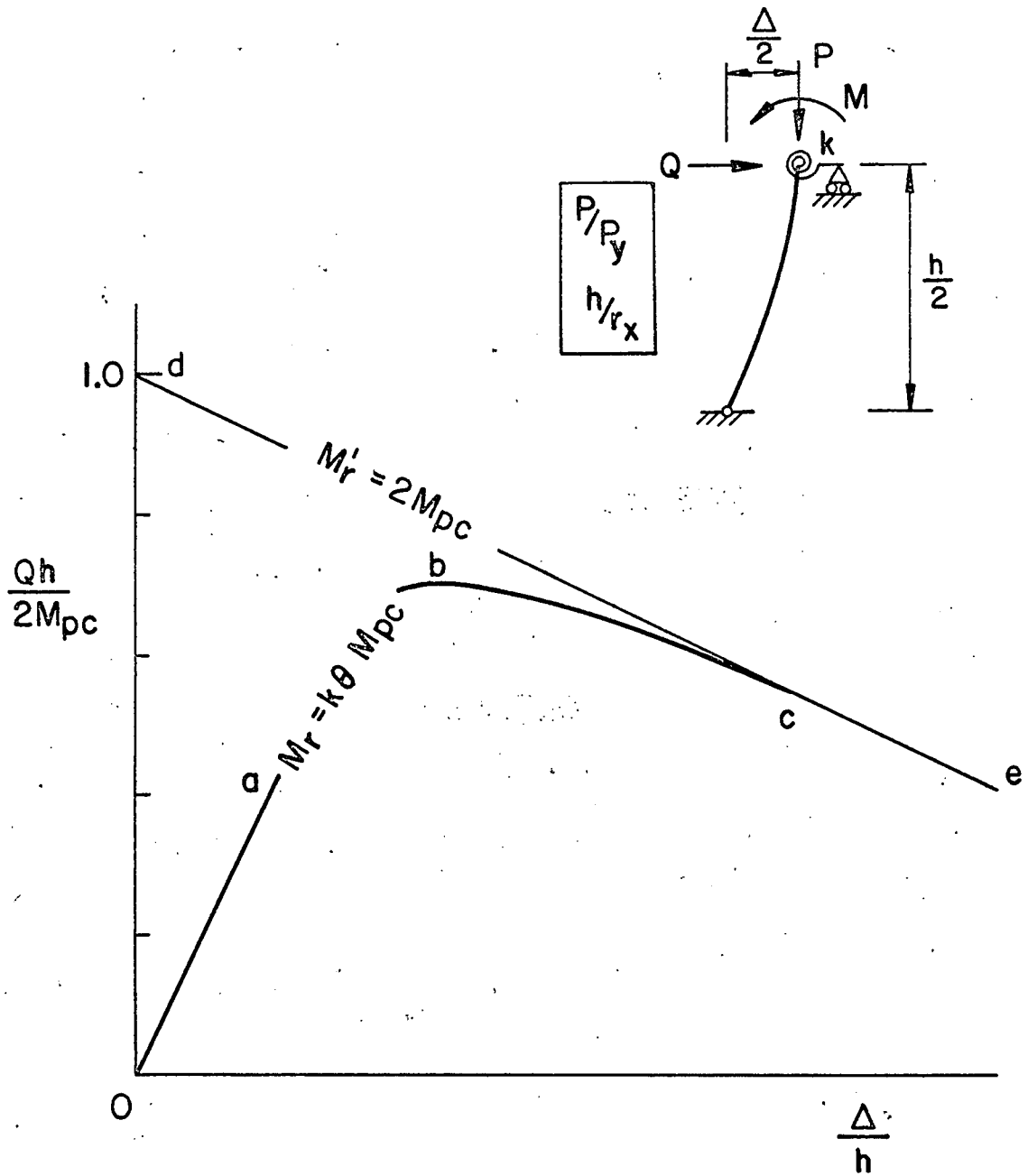
FIG. 6   LOAD-DEFLECTION CURVE OF A RESTRAINED COLUMN
WITH CONSTANT RESTRAINT STIFFNESS

-16-

$$M'_r = 2M_{pc} \qquad\qquad (22)$$

Additional load-deflection curves may also be obtained for the column shown in Fig. 6. Each curve would correspond to a different value of restraint stiffness, k, $0 \leq k \leq \infty$. All curves would be similar in shape to 0-a-b-c-e and all would pass through point 0. In addition, all curves would intersect the line d-e (or its extension for greater values of Δ/h), since the maximum restraining moment, M'_r for all curves is independent of the restraint stiffness k.

Load-Deflection Behavior for Variable Restraint Stiffness.- In general, the restraint stiffness, k, will not remain constant for all values of joint rotation, θ, but will decrease abruptly at discrete intervals as θ increases due to the successive formation of plastic hinges in the various members. Of the infinite number of k - θ relationships possible, only two of them are fundamental to the sway subassemblage method of analysis. These two may be described as follows:

    1. <u>Constant - Zero Restraint Stiffness:</u>

$$k = k_1 \qquad\qquad (0 \leq \theta \leq \theta_1) \qquad\qquad (23)$$

$$k = k_2 = 0 \qquad\qquad (\theta_1 < \theta \leq \infty) \qquad\qquad (24)$$

where $\theta_1 < \theta_p$ (Eq. 21).

    2. <u>Constant - Constant Restraint Stiffness:</u>

$$k = k_1 \qquad\qquad (0 \leq \theta \leq \theta_1) \qquad\qquad (25)$$

$$k = k_2 \qquad\qquad (\theta_1 < \theta \leq \infty) \qquad\qquad (26)$$

where $k_2 < k_1$.

Constant - Zero Restraint Stiffness.- The restraining moment at the column top will be defined by the equations

$$M_r = k_1 \theta M_{pc} \qquad\qquad (0 \leq \theta \leq \theta_1) \qquad\qquad (27)$$

$$M_r = M'_r = k_1 \theta_1 M_{pc} = p_1 M_{pc} \qquad\qquad (\theta_1 < \theta \leq \infty) \qquad\qquad (28)$$

in which p_1 is a constant and by the definition of θ_1, $0 \leq p_1 < 2$, (Eq. 21). The solution of Eq. 14 for the restraining moment defined by Eq. 27 will give
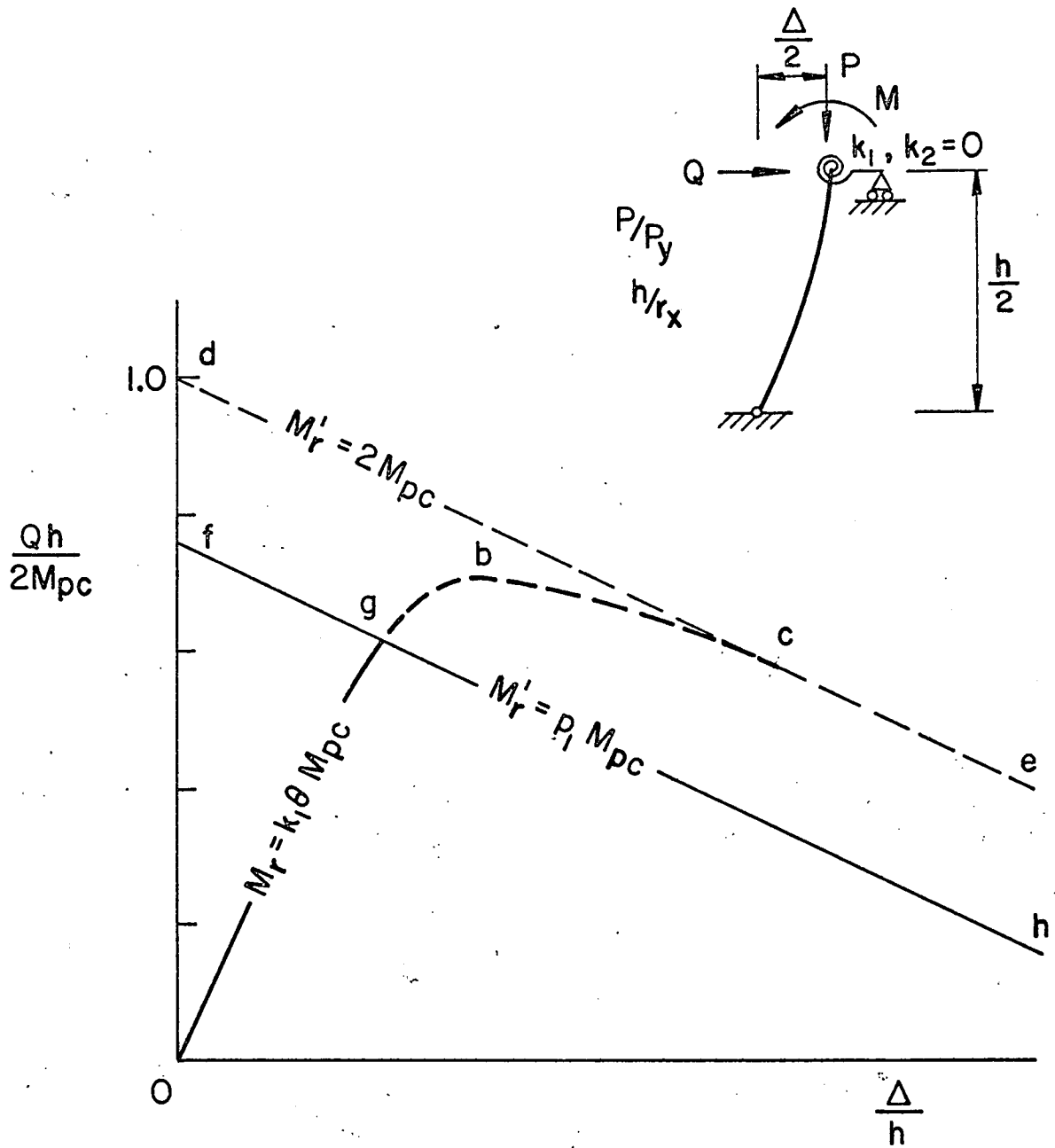
FIG. 7   LOAD-DEFLECTION CURVE OF A RESTRAINED COLUMN
WITH CONSTANT - ZERO RESTRAINT STIFFNESS

the load-deflection curve 0 - g in Fig. 7. For equal values of P/P_y, h/r_x and $k = k_1$ curve 0 - g would be the initial segment of the complete load-deflection curve shown as 0-b-c-e in both Figs. 6 and 7. At point g, however, the restraint stiffness becomes zero. Therefore additional restraining moment cannot be generated and a mechanism condition results. The restraining moment after the formation of the mechanism becomes constant (M'_r) and is given by Eq. 28. Using Eqs. 5 and 28, the maximum column moment is given by

$$M = - \frac{p_1}{2} M_{pc} \tag{29}$$

The load-deflection curve following the formation of the mechanism can be obtained by substituting Eq. 29 into Eq. 14 to give

$$\frac{Qh}{2M_{pc}} = \frac{p_1}{2} - C \frac{\Delta}{h} \tag{30}$$

Equation 30 is shown in Fig. 7 as the straight line f-g-h. Since the derivatives with respect to Δ/h of Eqs. 19 and 30 are equal, f-g-h will be parallel to d-c-e. From the previous discussion, 0 - f and f-g-h in Fig. 7 will define the second-order rigid-plastic load-deflection curves for the rigid-plastic column with maximum restraining moment given by Eq. 28. The moment, M, at the top of the column, and the column chord rotation, γ, will be constant everywhere on f-g-h.

Constant - Constant Restraint Stiffness.- The restraining moment at the column top will now be defined by the equations

$$M_r = k_1 M_{pc} \qquad (0 \le \theta \le \theta_1) \tag{31}$$

$$M_r = k_2 M_{pc} \qquad (\theta_1 < \theta \le \infty) \tag{32}$$

The solution of Eq. 14 for M_r defined by Eq. 31 gives curve segment 0 - g in Fig. 8. For equal values of P/P_y and h/r_x, 0 - g in Figs. 7 and 8 are the same. However, at point g in Fig. 8 the restraint stiffness reduces to k_2. Additional restraining moment, M_r, can be developed after point g but at a smaller rate than before (Eq. 26). The resulting load-deflection curve is shown as curve g-j-m in Fig. 8, which intersects the line d-e at point m with the formation of a plastic hinge in the column top.
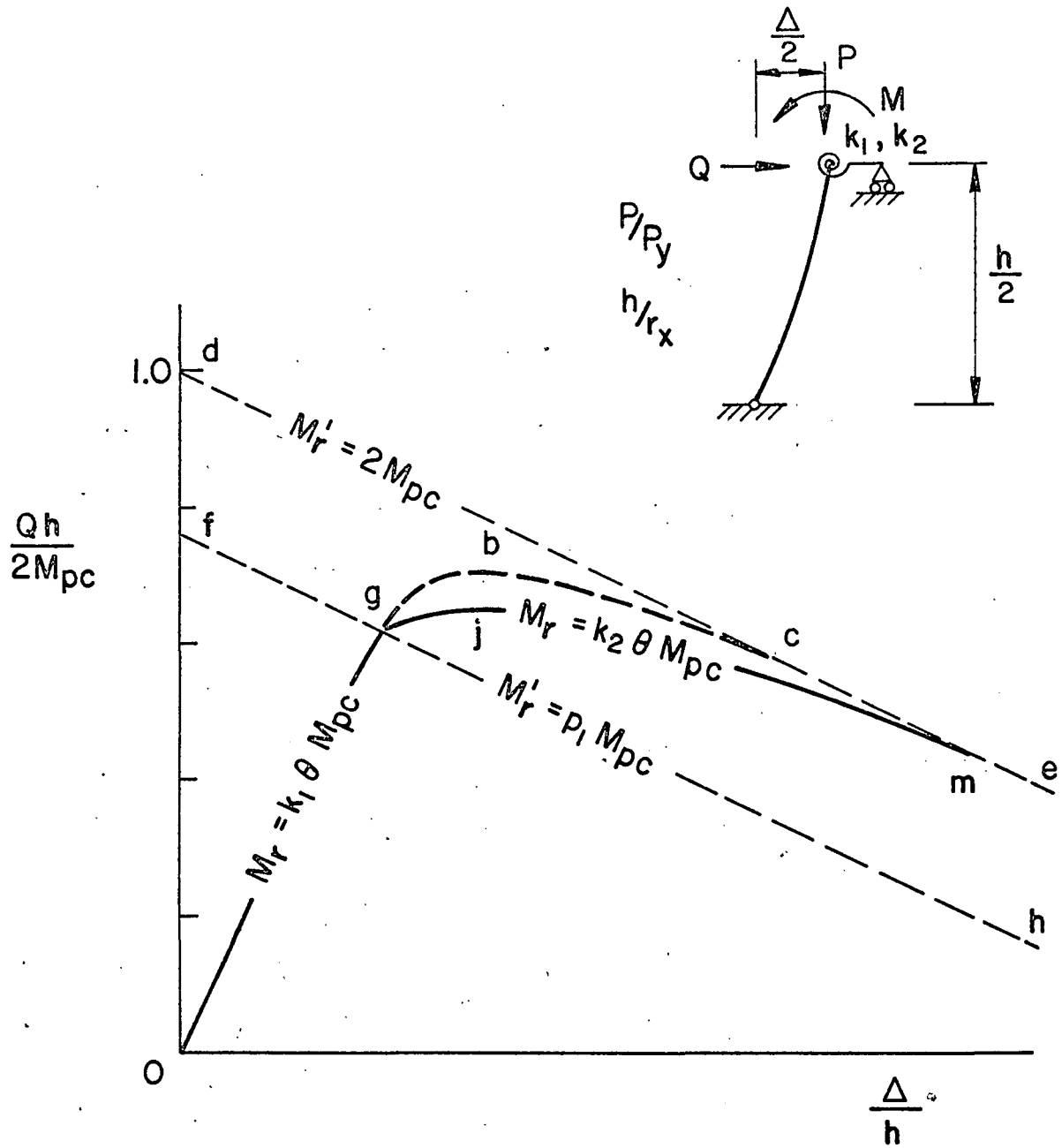
FIG. 8 LOAD-DEFLECTION CURVE OF A RESTRAINED COLUMN
WITH CONSTANT - CONSTANT RESTRAINT STIFFNESS
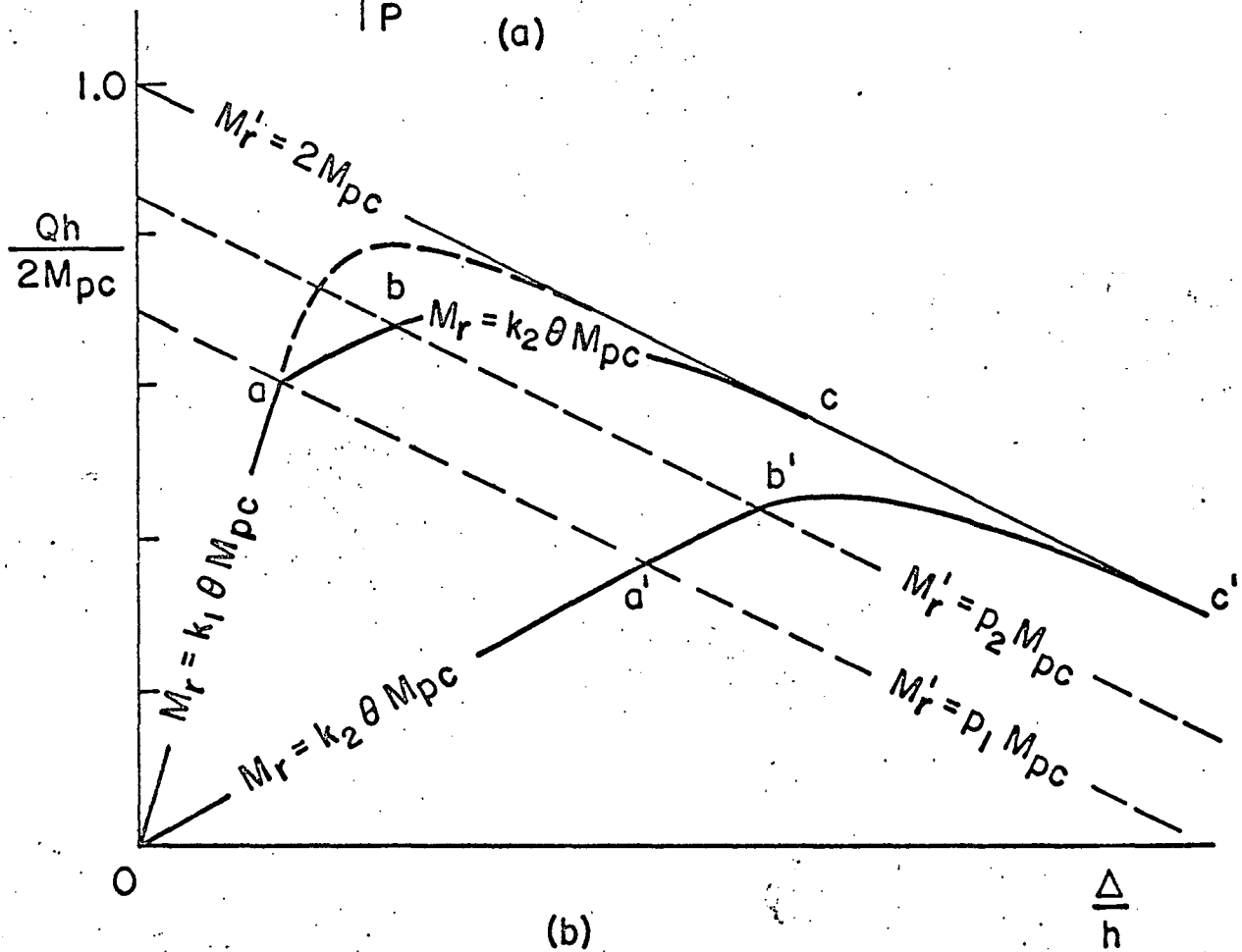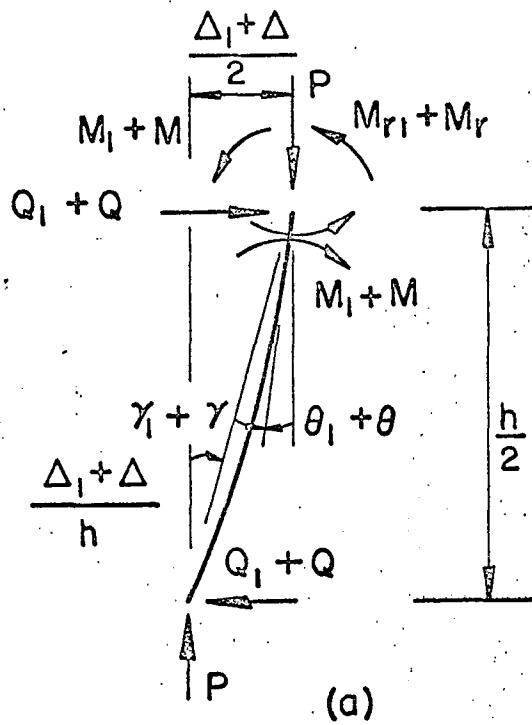
-20-

(a)

(b)

FIG. 9 SUPERPOSITION OF LOAD-DEFLECTION CURVES

Superposition of Load-Deflection Curves.- Consider the two load-deflection curves shown in Fig. 9(b). Curve 0-a-b-c represents the load-deflection curves for a column whose restraint stiffness decreases from k_1 to k_2 at point a. Curve 0-a'-b'-c', however, represents the load-deflection curve for the same column but with constant restraint stiffness k_2. Segment 0-a of curve 0-a-b-c and the complete curve 0-a'-b'-c' may both be obtained by solving Eq. 14, where the restraining moments M_r, are defined by $k_1 \theta M_{pc}$ and $k_2 \theta M_{pc}$ respectively. The load-deflection equation for segment a-b-c of curve 0-a-b-c can be derived considering the restrained column shown in Fig. 9(a). The forces acting on the restrained column are shown together with the resulting deformations. The initial conditions (point a in Fig. 9(b)) are given by Q_1, Δ_1, θ_1, γ_1, M_1 and M_{r1}. Following the same procedure used to derive Eq. 14, the load-deflection relationships for curve segment a-b-c of Fig. 9(b) is given by

$$\frac{(Q_1 + Q)h}{2M_{pc}} = -\left[\frac{(M_1 + M)}{M_{pc}} + C \frac{(\Delta_1 + \Delta)}{h} \right] . \qquad (33)$$

A linear transformation of axes in Fig. 9(b) with the new origin at point a will reduce Eq. 33 to Eq. 14. Equation 14 also applies to the segment a'-b'-c' with origin at point a'. It can be shown that curve segments a-b-c and a'-b'-c' are identical since the intersection of the two curves with the straight line defined by $M'_r = p_2 M_{pc}$, where $p_1 \leq p_2 \leq 2$ (such as points b and b') have identical slopes (5). Therefore it is not necessary to derive the load-deflection equation corresponding to each reduced value of restraint stiffness, k. Instead the load-deflection curve may be built up from segments of complete load-deflection curves which are given by Eq. 14 for the appropriate values of k.

Design Charts:- The solutions of Eqs. 14 and 30 have been presented in Ref. 6 in the form of 78 design charts to assist the manual analysis. A typical chart for $P = 0.45 P_y$ and $h = 22 r_x$ is shown in Fig. 10. Such charts can be prepared using a tabular solution similar to those shown in Ref. 11. The charts in Ref. 6 have been prepared for use only with ASTM A36 Steel wide-flange column shapes with a nominal yield stress level of 36 ksi. However they can be applied to steels of other yield stress levels by substituting an equivalent slenderness ratio
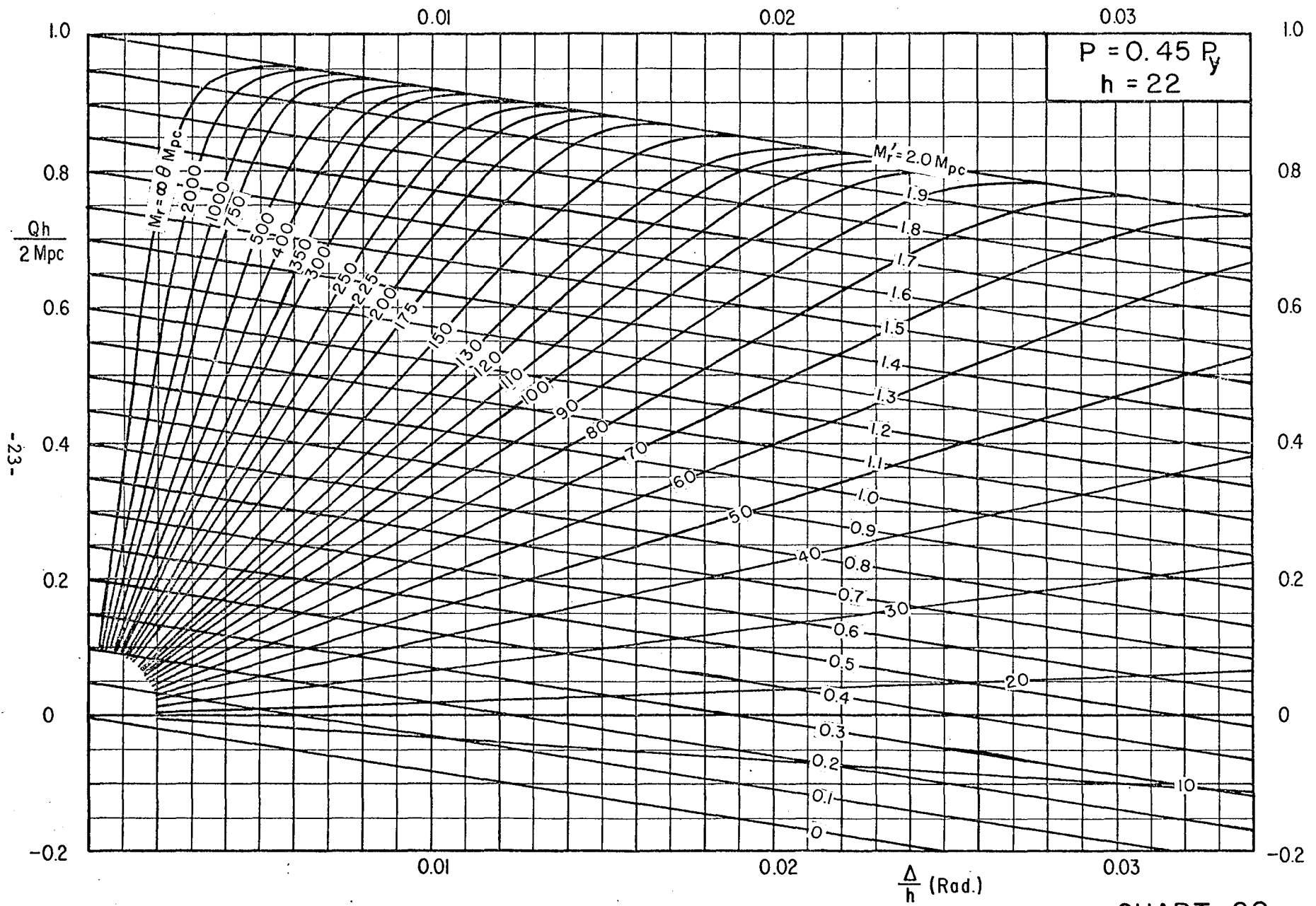
P = 0.45 P_y
h = 22

$\frac{Qh}{2\,Mpc}$

$M_r = \infty\ \theta\ Mpc$
2000
1000
750
500
400
350
300
250
225
200
175
150
130
120
110
100
90
80
70
60
50
40
30
20
10

$M_r' = 2.0\ M_{pc}$
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

$\frac{\Delta}{h}$ (Rad.)

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CHART 20

FIG. 10 TYPICAL DESIGN CHART

$$\left(\frac{L}{r}\right)_{equ} = \left(\frac{L}{r}\right)_{\sigma_y} \frac{36}{\sigma_y} \tag{34}$$

This substitution will yield exact results providing the residual stress has the same distribution pattern over the cross-section and the same proportion of the yield stress for the different steels. The ranges of P/P_y and h/r_x covered by the charts are

$$0.30 \le \frac{P}{P_y} \le 0.90 \qquad \text{intervals of } 0.05 \frac{P}{P_y}$$

$$20 \le \frac{h}{r_x} \le 30 \qquad \text{intervals of } 2 \frac{h}{r_x}$$

Each load-deflection curve was constructed for a constant value of restraining moment, M_r, which is defined by Eq. 17. Also shown on each chart are the straight lines representing constant values of maximum restraining moment, M'_r. These lines have been constructed for values of p varying from 0 to 2 at intervals of 0.1.

RESTRAINING CHARACTERISTICS OF BEAMS AND COLUMNS

Initial Restraint.- The term "initial restraint" will be used in this paper to denote the rotational restraint provided to the top of a restrained column by the restraining system prior to the formation of the first plastic hinge in the restraining system. The restraining system will consist of all the beams and columns on both sides of the column.

Initial Restraint Coefficients.- The interior region of a half-story assemblage is shown in Fig. 11(a) together with the vertical forces, P, and joint moments, M, which were previously determined. The deflected configuration is consistent with a relatively small applied shear force, Q, acting towards the right. The behavior of all the beams and columns is assumed to be elastic. The sway deflec-tion, Δ/h, will also be relatively small.

Consider the restrained column at joint i. It is desired to calculate the initial elastic value of restraint stiffness, k_i, which is provided by the beams and columns of the half-story assemblage. The restraining moment, M_r, at joint i will be the sum of the restraining moments on either side of the joint and can be written
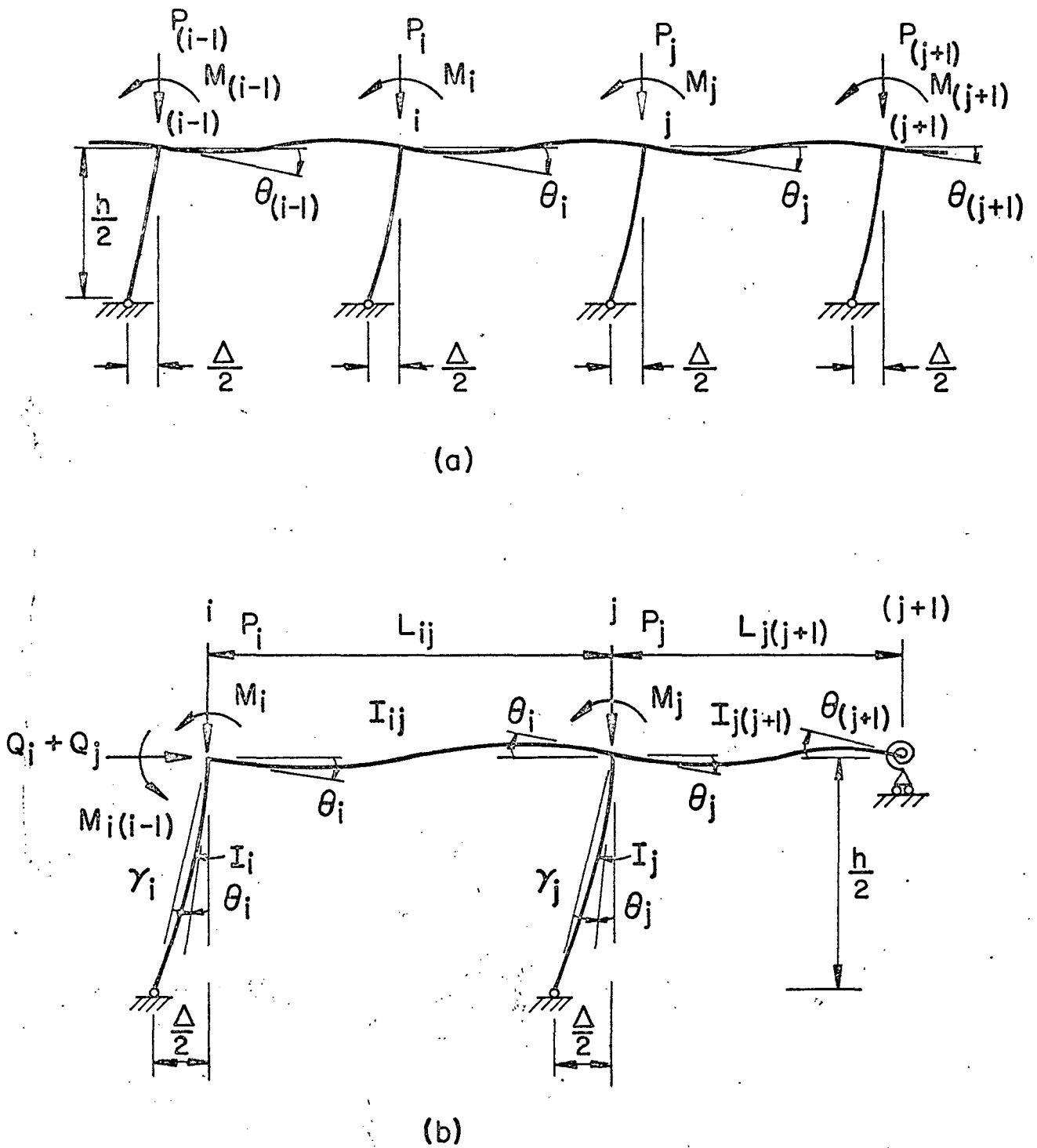
(a)

(b)

FIG. 11 DERIVATION OF INITIAL RESTRAINT COEFFICIENTS

$$M_r = M_{i(i-1)} + M_{ij} = \left[K_{i(i-1)} \frac{E \, I_{i(i-1)}}{L_{i(i-1)}} + K_{ij} \frac{E \, I_{ij}}{L_{ij}} \right] \theta_i \qquad (35)$$

in which $M_{i(i-1)}$ and M_{ij} are the moments at i for beams i(i-1) and ij respectively and $K_{i(i-1)}$ and K_{ij} are the initial restraint coefficients for the same beams. Also $I_{i(i-1)}$ and I_{ij} are the moments of inertia of beams i(i-1) and ij; θ_i is the rotation of joint i and E is the modulus of elasticity.

Equation 35 may also be written in non-dimensional form as follows:

$$M_r = \left[K_{i(i-1)} \frac{E \, I_{i(i-1)}}{L_{i(i-1)} M_{pci}} + K_{ij} \frac{E \, I_{ij}}{L_{ij} M_{pci}} \right] \theta_i M_{pci} \qquad (36)$$

where M_{pci} is the reduced plastic moment capacity of the restrained column at joint i corresponding to the axial load ratio P/P_y of column i. Equating Eqs. 17 and 36 yields:

$$k_i = K_{i(i-1)} \frac{E \, I_{i(i-1)}}{L_{i(i-1)} M_{pci}} + K_{ij} \frac{E \, I_{ij}}{L_{ij} M_{pci}} \qquad (37)$$

The solution of Eq. 37 requires only the determination of the initial restraint coefficients $K_{i(i-1)}$ and K_{ij} since all other terms are known.

The initial restraint coefficients can be closely evaluated by considering only the members shown in Fig. 11(b). The following simplifying assumptions will be made:

1. The restraining moment $M_{i(i-1)}$ on the windward side of joint i is known.

2. The restraining effect of the members to the right of joint (j+1) will be approximated by taking $\theta_{(j+1)} = \theta_j$.

3. For $P_i > 0.70 \, P_y$ yielding of the restrained column at i will occur due to residual stresses. Thus, the moment of inertia, I_j, of the column will be calculated using the remaining elastic core. Similarly for the column at joint j.

4. No gravity loads exist on beams ij and j(j+1).

The initial restraint coefficient K_{ij} may be determined if the relationship between M_{ij} and θ_i can be found when joint i, j and (j+1) each undergo a

small sway displacement equal to $\Delta/2$. At joint i, the moment, M_i, the joint rotation, θ_i, and the sway deflection, Δ/h, are related as follows:

$$M_i = \frac{6E\ I_i}{h}\left[\theta_i - \frac{\Delta}{h}\right] \tag{38}$$

The restraining moments, M_r, at joint i can be obtained from Eq. 35. Equilibrium of moments yields:

$$2M_i + M_{ij} + M_{i(i-1)} = 0 \tag{39}$$

so that

$$12\ \frac{E\ I_i}{h}\left[\theta_i - \frac{\Delta}{h}\right] + M_{ij} + K_{i(i-1)}\ \frac{E\ I_{i(i-1)}}{L_{i(i-1)}}\ \theta_i = 0 \tag{40}$$

At joint j the stress-resultants can be expressed as

$$M_j = \frac{6E\ I_{ij}}{h}\left[\theta_j - \frac{\Delta}{h}\right] \tag{41}$$

$$M_{ji} = \frac{E\ I_{ij}}{L_{ij}}\left[4\theta_j + 2\theta_i\right] \tag{42}$$

$$M_{j(j+1)} = \frac{6E\ I_{j(j+1)}}{L_{j(j+1)}}\ \theta_j \tag{43}$$

Equilibrium of moments at joint j then requires that

$$12\ \frac{E\ I_i}{h}\left[\theta_j - \frac{\Delta}{h}\right] + \frac{E\ I_{ij}}{L_{ij}}\left[4\theta_j + 2\theta_i\right] + \frac{6E\ I_{j(j+1)}}{L_{j(j+1)}}\ \theta_j = 0 \tag{44}$$

The sway deflection, Δ/h, can be evaluated from Eqs. 40 and 44 as:

$$\frac{\Delta}{h} = \left[1 + \frac{\alpha'}{12}\ K_{i(i-1)}\right]\theta_i + \frac{M_{ij}h}{12E\ I_i} \tag{45}$$

and

$$\frac{\Delta}{h} = \frac{\beta}{6}\ \theta_i + \left[1 + \frac{\beta}{3} + \frac{\eta}{2}\right]\theta_j \tag{46}$$

in which

$$\alpha' = \frac{h\ I_{i(i-1)}}{L_{i(i-1)}I_i} \tag{47}$$

$$\beta = \frac{h\ I_{ij}}{L_{ij}I_j} \tag{48}$$

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$$\eta = \frac{h \, I_{i(j+1)}}{L_{j(j+1)} I_j} \qquad (49)$$

Now θ_j can be expressed as a function of θ_i

$$\theta_j = \frac{M_{ij} L_{ij}}{2E \, I_{ij}} - 2\theta_i \qquad (50)$$

Equating Eqs. 45 and 46 and using Eq. 50 yields the moment M_{ij} as

$$M_{ij} = 6 \left[\frac{3 + 0.5\beta + \eta + \frac{\alpha'}{12} K_{i(i-1)}}{3 - 0.5\alpha + \beta + 1.5\eta} \right] \frac{E \, I_{ij}}{L_{ij}} \, \theta_j \qquad (51)$$

where

$$\alpha = \frac{h \, I_{ij}}{L_{ij} I_i} \qquad (52)$$

Hence

$$K_{ij} = 6 \left[\frac{3 + 0.5\beta + \eta + \frac{\alpha'}{12} K_{i(i-1)}}{3 - 0.5\alpha + \beta + 1.5\eta} \right] \qquad (53)$$

The initial restraint coefficient to the left of joint j, K_{ji}, is related to K_{ij} as follows:

$$K_{ji} = 4 \left[\frac{K_{ij} - 3}{K_{ij} - 4} \right] \qquad (54)$$

Similarly

$$K_{i(i-1)} = 4 \left[\frac{K_{(i-1)i} - 3}{K_{(i-1)i} - 4} \right] \qquad (55)$$

where joint i is an interior joint.

A more accurate expression for K_{ij} can be obtained by including one more bay to the right of joint (j+1) in Fig. 11(b) and assuming that $\theta_{(j+2)} = \theta_{(j+1)}$. The resulting expression for K_{ij} is:

$$K_{ij} = 6 \left[\frac{3 + \frac{\beta}{12}(6 + 2\tau + 3\xi) + \frac{\eta}{12}(6 + 2\tau + 4\xi) + 1.5\xi + \tau}{3 - \frac{\alpha}{12}(6 - \eta + 2\tau + 3\xi) + \frac{\beta}{12}(12 + 4\tau + 6\xi)} \right.$$
$$\left. \frac{+ \frac{\alpha'}{72} K_{i(i-1)}(6 - \eta + 2\tau + 3\xi)}{+ \frac{\eta}{12}(12 + 3\tau + 6\xi) + \tau + 1.5\xi} \right] \qquad (56)$$

in which

$$\tau = \frac{h\ I_{j(j+1)}}{L_{j(j+1)}I_{(j+1)}} \qquad (57)$$

$$\xi = \frac{h\ I_{(j+1)(j+2)}}{L_{(j+1)(j+2)}I_{(j+1)}} \qquad (58)$$

It has been found that for many unbraced frames the values of the initial restraint coefficients, K_{ij}, K_{ji}, etc. may be approximated by using the constant value of 6.0 (2). The resulting loss of accuracy may be relatively small for many frames (1).

Reduced Restraint Coefficients.- As the lateral shear force, ΣQ on a half-story assemblage increases in magnitude from zero, the successive formation of plastic hinges in the beams and columns will reduce the restraint stiffness at the top of each restrained column. For a particular sway subassemblage, it has been shown (5) that the formation of plastic hinges within the sway subassemblage will have the greatest effect on the reduction of the restraint stiffness. Figure 12 shows the locations of the possible plastic hinges within an interior sway subassemblage. Also shown are the plastic hinges which can form at the top of columns (i-1) and j. The sequence in which these plastic hinges form will be a function of the relative member stiffnesses, plastic moment capacities and the intensity of the factored gravity loads. Plastic hinges 1,3,4,6 and 7 will usually be the first to form and will occur at the ends of the members. In certain cases, plastic hinges 2 and 5 may also form at the windward ends of the beams. A method of determining the positions of plastic hinges 2 and 5 with distributed gravity loads is discussed in Ref. 7 . Although all the plastic hinges shown are possible for interior sway subassemblages only 4,5,6 and 7 can occur in windward sway sub-assemblages while 1,2,3 and 4 are possible plastic hinge locations for leeward sway subassemblages.

Referring to the numbered locations of plastic hinges shown in Fig. 12, and assuming that 3 and 6 will form before 2 and 5, respectively, the reduced restraint coefficients can be determined as follows:

FIG. 12   POSSIBLE PLASTIC HINGE LOCATIONS

1. **1 occurs before 3:** Since additional moment cannot be developed at joint (i-1), beam i(i-1) may be considered pinned at (i-1). Thus $K_{i(i-1)}$ reduces to 3.0.

2. **3 occurs after 1:** $K_{i(i-1)}$ reduces from 3.0 to 0.

3. **3 occurs before 1:** $K_{i(i-1)}$ reduces to zero.

4. **6 or 7 occurs:** K_{ij} reduces to 3.0.

5. **5 occurs after 6 or 7:** K_{ij} reduces from 3.0 to 0.

6. **4 occurs:** $K_{i(i-1)}$ and K_{ij} remain unchanged from their values at the time 4 develops.

LOAD-DEFLECTION BEHAVIOR OF A STORY

Evaluation of M'_r.- The distribution of bending moments in an interior sway subassemblage for distributed gravity loads on the beams is shown dashed in Fig. 13. In general, an initial lateral shear force, ΣQ, will be required to maintain $\Delta/h = 0$. The effect of the gravity loads on the evaluation of M'_r will be discussed later.

Consider now the effect of a small increment of lateral shear force δQ acting from the left. The moments at the leeward ends of the beams will be increased by $\delta M_{i(i-1)}$ and δM_{ji} while at the windward ends the moments will decrease by $\delta M_{(i-1)i}$ and δM_{ji}. These small increments in moment will be related to the small increments of joint rotation as follows:

$$\delta M_{ji} = K_{ji} \frac{E\,I_{ij}}{L_{ij}}\,\delta\theta_j \tag{59}$$

$$\delta M_{ij} = K_{ij} \frac{E\,I_{ij}}{L_{ij}}\,\delta\theta_i \tag{60}$$

$$\delta M_{i(i-1)} = K_{i(i-1)} \frac{E\,I_{i(i-1)}}{L_{i(i-1)}}\,\delta\theta_i \tag{61}$$

$$\delta M_{(i-1)i} = K_{(i-1)i} \frac{E\,I_{i(i-1)}}{L_{i(i-1)}}\,\delta\theta_{(i-1)} \tag{62}$$
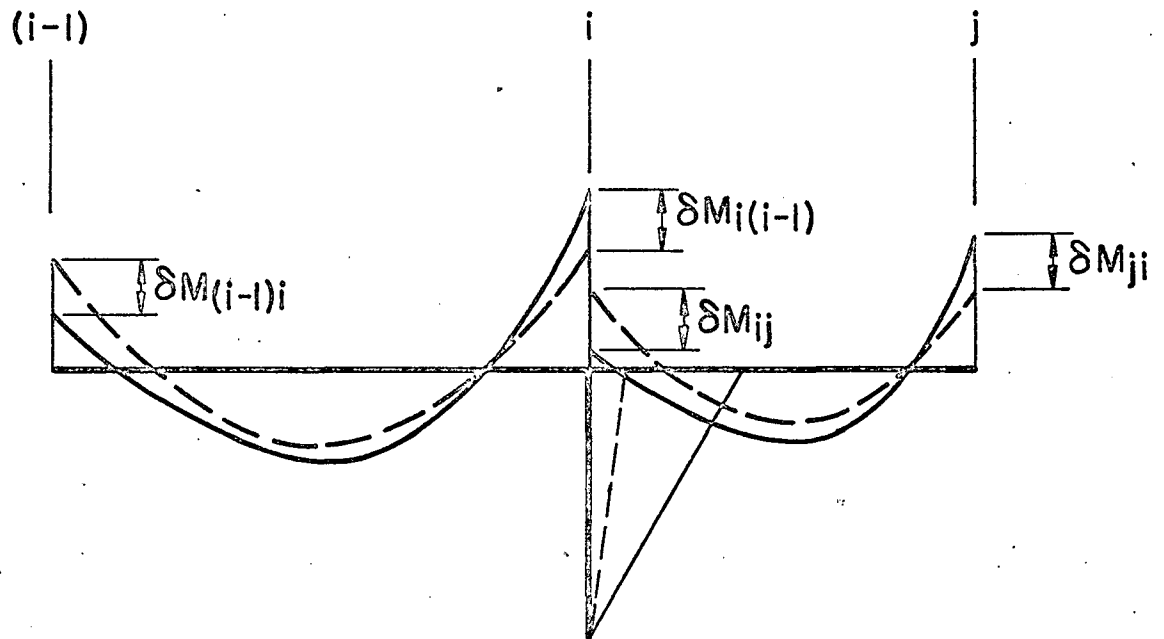
FIG. 13   DISTRIBUTION OF BENDING MOMENTS

in which K_{ji}, K_{ij}, etc., are the initial restraint coefficients, assuming that initial sway is completely elastic. It can be shown that the joint rotations $\delta\theta_j$ and $\delta\theta_i$ are related as follows (5),

$$\delta\theta_j = \left[\frac{K_{ij} - 4}{2}\right]\delta\theta_i \qquad (63)$$

Similarly

$$\delta\theta_{(i-1)} = \left[\frac{2}{K_{(i-1)} - 4}\right]\delta\theta_i \qquad (64)$$

Substitution into Eqs. 59 and 62 gives

$$\delta M_{ji} = K_{ji}\frac{E\,I_{ij}}{L_{ij}}\left[\frac{K_{ij} - 4}{2}\right]\delta\theta_i \qquad (65)$$

$$\delta M_{ij} = K_{ij}\frac{E\,I_{ij}}{L_{ij}}\delta\theta_i \qquad (66)$$

$$\delta M_{i(i-1)} = K_{i(i-1)}\frac{E\,I_{i(i-1)}}{K_{i(i-1)}}\delta\theta_i \qquad (67)$$

$$\delta M_{(i-1)i} = K_{(i-1)i}\frac{E\,I_{i(i-1)}}{L_{i(i-1)}}\left[\frac{2}{K_{(i-1)i} - 4}\right]\delta\theta_i \qquad (68)$$

Referring again to Fig. 12 and assuming that plastic hinges 3 and 6 will form before plastic hinges 2 and 5, respectively, then δM_{ji} and $\delta M_{i(i-1)}$ can be taken equal to the increments in moment required to form the plastic hinges at 6 and 3, respectively. The solutions of Eqs. 65 and 67 will yield two values of moment for the increment of joint rotation $\delta\theta_i$; the minimum value will correspond to the formation of the first plastic hinge. The corresponding value of M'_r will then be given by

$$M_r' = pM_{pc} \qquad (69)$$

where

$$p = k_i\delta\theta_i \qquad (0 \le p \le 2.0) \qquad (70)$$

and k_i is given by Eq. 37 for the initial values of $K_{i(i-1)}$ and K_{ij}. Substitution of the minimum value of $\delta\theta_i$ found above into Eqs. 65 to 68 will determine a set of moment increments which when added to the initial moments will yield the distirbution of moments corresponding to the formation of the first plastic hinge.

One or more values of initial restraint coefficient, K, can now be reduced as discussed earlier. The reduced values are now used in Eqs. 65 and 68 when determining the second plastic hinge in the sway subassemblage and so forth until a mechanism has been formed.

The final result after the sway subassemblage has been reduced to a mechanism will be a set of values of M_r and M_r'. The load-deflection curve of the restrained column and thus of the sway subassemblage can then be determined from the appropriate design chart given in Refs. 6 and 14.

Construction of a Typical Load-Deflection Curve.- Figure 14 illustrates the method of constructing a typical load-deflection curve for an interior sway subassemblage. It is assumed that a mechanism occurs with the formation of three plastic hinges at a, b and c, in that order. An analysis determined that the initial restraint stiffness was k_1 and that the first plastic hinge formed at a joint rotation $\delta\theta_1 = \theta_1$ so that $p_1 = k_1\theta_1$. Similarly, prior to the second and third plastic hinges the restraint stiffness was found to be k_2 and k_3 respectively, and it was found that the second and third plastic hinges formed at joint rotations of θ_2 and θ_3. Therefore $p_2 = k_2\theta_2$ and $p_3 = k_3\theta_3$.

A design chart can now be selected which will correspond to the axial load ratio P/P_y, and slenderness ratio, h/r, of the restrained column. A simplified chart is shown in Fig. 14. The set of M_r values previously determined will define the three load-deflection curves O-e, O-f, and O-g. Similarly, the set of M_r' values will define the three sloping straight lines in Fig. 14 which intersect the vertical axis at $p_1/2$, $p_2/2$, and $p_3/2$. The initial segment of the load-deflection curve is O-a. This segment terminates with the formation of the first plastic hinge at point a. The second segment of the load-deflection curve is shown as a-b, where point b corresponds to the formation of the second plastic hinge. This segment is obtained by translating segment a'-b' of curve O-f parallel to the lines M_r'. Similarly, segment b-c is obtained by translating segment b"-c" of curve O-g and point c corresponds to the formation of the third plastic hinge and a mechanism. The final segment c-d of the load-deflection curve is the second-order plastic mechanism curve and follows the straight line $M_{r3}' = p_3 M_{pc}$.
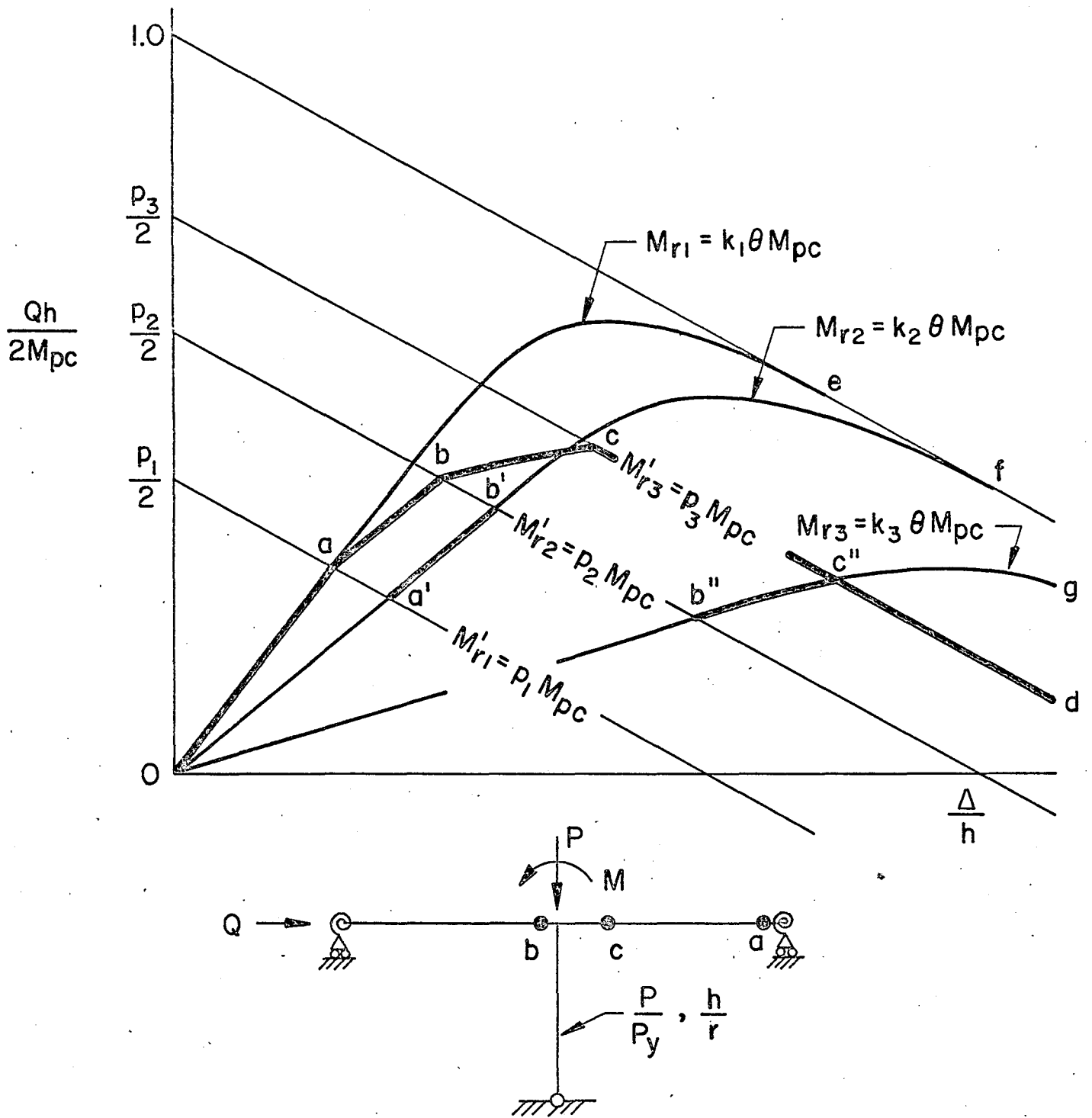
FIG. 14 CONSTRUCTION OF LOAD-DEFLECTION CURVE

Non-dimensional load-deflection curves must be constructed for each sub-assemblage in the one-story assemblage. Before combining these curves to obtain the load-deflection curve for the one-story assemblage, it is necessary to transform them to Q versus Δ/h curves by multiplying the ordinates of each curve by the appropriate values of $2M_{pc}/h$.

Effect of Gravity Loads on Beams.- Gravity loads on the beams will result in initial joint moments when $\Delta/h = 0$. These moments may be obtained by a Hardy Cross moment distribution analysis on the one-story assemblage, or approximated by using fixed-end moment values. If for a particular joint the net moment from the beams is M_e, then the initial value of M_r' will be given by $M_r' = p_o M_{pc}$ where

$$p_o = \frac{M_e}{M_{pc}} \qquad (0 \le p_o \le 2) \qquad (71)$$

Assuming that no plastic hinges form in the beams (that is k_i = constant), Fig. 15 shows three restrained column curves that are possible depending on whether M_e is positive, negative or zero. Curve 1 is for $M_e = 0$ and passes through point 0. This is the same as the curve shown in Fig. 6. Curves 2 and 3 are for positive and negative values of M_e respectively. From the previous discussion in this paper it is apparent that all three curves shown in Fig. 15 are identical above the line $M_r' = 0$, but displaced from each other parallel to the lines M_r'. For Curve 3 it is extended backwards from 0" to b to obtain the initial portion of the load-deflection behavior.

Load-Deflection Curve of a Story.- Figure 16 illustrates the method of combining the load-deflection curves of each of the sway subassemblages in a one-story assemblage. The method requires that the ordinates, Q, corresponding to a constant value of sway deflection Δ/h be added algebraically to determine the total shear resistance ΣQ of the one-story assemblage.
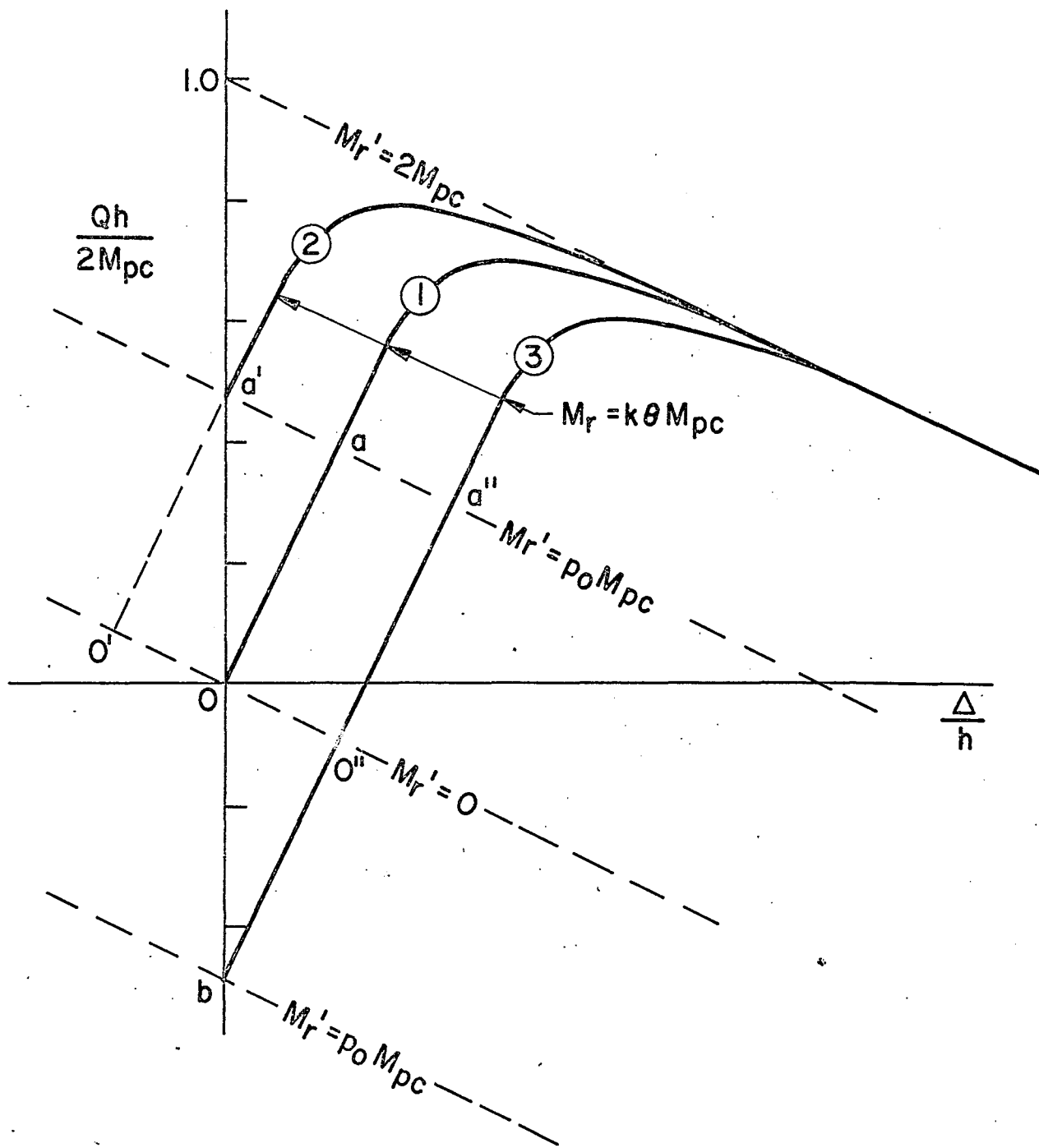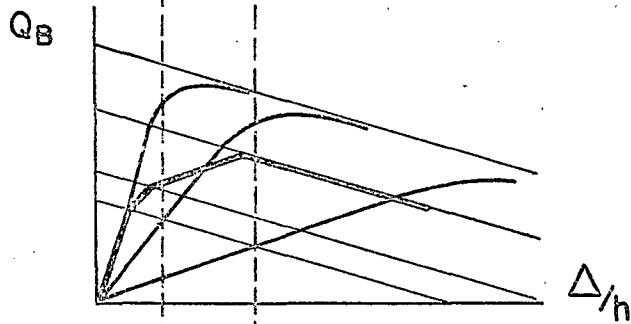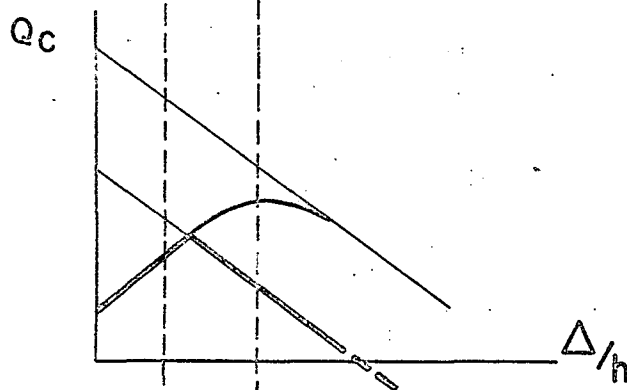
FIG. 15 EFFECT OF INITIAL MOMENTS
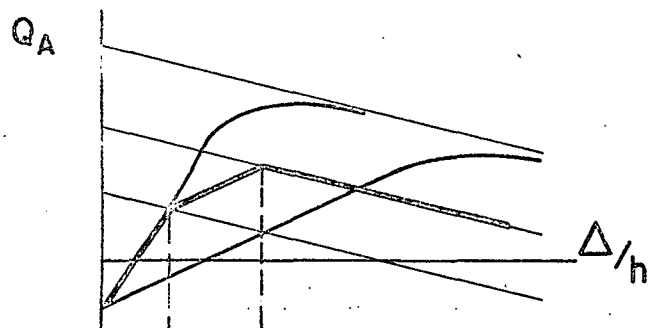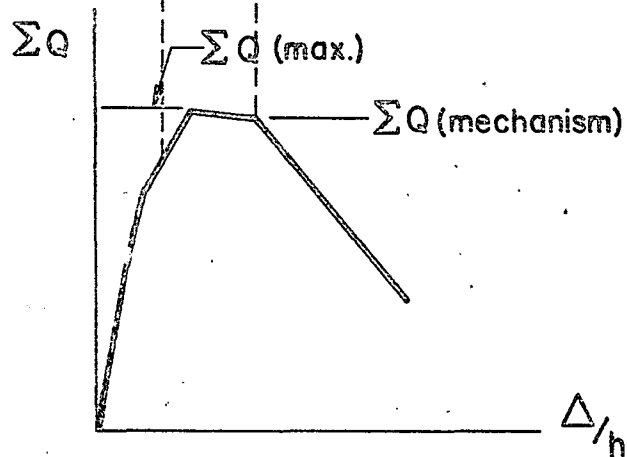
FIG. 16   CONSTRUCTION OF LOAD-DEFLECTION CURVE FOR A ONE-STORY ASSEMBLAGE.

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SUMMARY

The sway subassemblage method of analysis described in this paper will enable the determination of the approximate lateral-load versus sway-deflection curve of a story in the middle and lower stories of an unbraced frame which is subjected to combined loads. Such a curve will allow a check to be made on the sway deflection estimates used in the preliminary design of the frame. In addition, the adequacy of the preliminary design may be checked on the basis of the sway deflection at working loads, the maximum lateral load capacity and the mechanism load or any other load or deflection criterion.

The method of analysis is based on the concept of sway subassemblages and uses directly the results of previous research on the strength and behavior of restrained columns permitted to sway. In the analysis a one-story assemblage is isolated from the frame by passing cuts through the assumed inflection points of the columns, located at mid-height of a story. The one-story assemblage is further subdivided into a number of sway subassemblages, each consisting of a restrained column and either one or two adjacent restraining beams. The lateral load versus sway-deflection relationship of each sway subassemblage is determined either manually with the aid of specially prepared charts, or with a computer. These individual load-deflection curves are then combined to produce the load-deflection curve of the one-story assemblage. The story of the frame at the level of this one-story assemblage is assumed to have identical behavior.

The sway subassemblage method of analysis accounts for the reduction in strength of a frame due to $P\Delta$ effects. It also considers plastification of the columns including residual stresses as well as plastic hinges in the beams.

The sway subassemblage method does not consider unbraced frames with significantly large initial sway deflections under factored gravity loads alone. The effect of differential column shortening on the strength and deflection of the frame is also not considered.

ACKNOWLEDGMENTS

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