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Frame stability by energy method, November 1962

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FRAME INSTABILITY BY ENERGY METHOD

by

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1. ASSUMPTIONS

a) The structure is conservative. Therefore bending moment is single-valued function of curvature.

b) Deflection curves of a beam can be represented by three-term sine curves and column deflection curves by the third degree polynominals.

c) The effect of shearing force is negligible.

2. BENDING STRAIN ENERGY

Figure 1 shows the non-dimensional moment vs. curvature relations of the 8WF31 section. The area under the curve represents the bending strain energy corresponding to the given thrust and curvature.

The area integration can best be obtained by Simpson's Rule. The results are plotted as non-dimenstional strain energy $\bar{\bar{u}}$ vs. non-dimensional curvature $\bar{\Phi}$ in Fig. 2. The curves can be fitted in second degree and third degree polynominals.

$$\bar{u} = f_2'(\bar{P})\,\Phi^2 + f_3'(\bar{P})\,\Phi \quad \text{------------- (1)}$$

or

$$\bar{u} = f_1(\bar{P})\,\Phi^3 + f_2(\bar{P})\,\Phi^2 + f_3(\bar{P})\,\Phi \quad \text{------------- (2)}$$

where

$$f_1(\bar{P}) = -0.3980\bar{P}^2 + 0.0960\bar{P} - 0.1620 \quad \text{------- (3)}$$

$$f_2(\bar{P}) = 0.5380\bar{P}^2 - 0.4930\bar{P} + 0.7100 \quad \text{------- (4)}$$

$$f_3(\bar{P}) = -0.3650\bar{P}^2 + 0.3070\bar{P} - 0.0300 \quad \text{------- (5)}$$

$$f_2'(\bar{P}) = -0.3320\bar{P}^2 - 0.1384\bar{P} + 0.2500 \quad \text{------- (6)}$$

$$f_3'(\bar{P}) = 0.0188\bar{P}^2 + 0.0664\bar{P} + 0.2500 \quad \text{------- (7)}$$

3. POTENTIAL ENERGY OF A BEAM

Three term sine curves have three amplification constants. The second and the third constants are, however, eliminated by imposing the boundary conditions. Therefore the deflection curve can be expressed in terms of ψ_i, ψ_j and a_K as shown in Fig. 3.

Deflection equation:

$$\omega(x) = a_K L \sin\frac{\pi x}{L} + \frac{(\psi_i+\psi_j)L}{4\pi}\sin\frac{2\pi x}{L} + \frac{L}{6\pi}(\psi_i-\psi_j-2\pi a_K)\sin\frac{3\pi x}{L} \quad ----- (8)$$

Slope equation:

$$\frac{d(\omega x)}{d(x)} = \pi a_K \cos\frac{\pi x}{L} + \frac{(\psi_i+\psi_j)}{2}\cos\frac{2\pi x}{L} + \frac{(\psi_i-\psi_j-2\pi a_K)}{2}\cos\frac{3\pi x}{L} \quad ---------- (9)$$

Curvature equation:

$$\Phi = \frac{d^2\omega(x)}{dx^2} = -\frac{\pi^2}{L}a_K \sin\frac{\pi x}{L} - \frac{\pi}{L}(\psi_i+\psi_j)\sin\frac{2\pi x}{L} - \frac{3\pi}{2L}(\psi_i-\psi_j-2\pi a_K)\sin\frac{3\pi x}{L} \quad ----- (10)$$

Substitute eq. (10) in eq. (2) and perform the integration throughout the length of the beam, strain energy of a beam.

$$\begin{aligned} U_B = {} & \frac{\pi^2 M_y}{4\Phi_y^2 L^2} f_1(\bar{H}_K)\Big[-\frac{69}{14}\psi_i^3 + \frac{69}{14}\psi_j^3 - \frac{2048}{105}\pi^3 a_K^3 + \frac{15}{14}\psi_i^2\psi_j + \frac{40}{7}\pi\psi_j^2 a_K \\ & + \frac{384}{35}\pi^2\psi_i a_K^2 - \frac{15}{14}\psi_i\psi_j^2 - \frac{384}{35}\pi^2\psi_j a_K^2 + \frac{40}{7}\pi\psi_i^2 a_K + \frac{112}{35}\pi\psi_i\psi_j a_K\Big] \\ & + \frac{\pi^2 M_y}{8\Phi_y L} f_2(\bar{H}_K)\big[13\psi_i^2 + 13\psi_j^2 + 40\pi^2 a_K^2 - 10\psi_i\psi_j + 36\pi\psi_j a_K - 36\pi\psi_i a_K\big] \\ & + M_y f_3(\bar{H}_K)\big[\psi_j - \psi_i\big] \quad --------- (11) \end{aligned}$$

Axial shortening of the beam

$$\Delta_B = \frac{1}{2}\int_0^L \left[\frac{d\omega(x)}{d(x)}\right]^2 dx$$

$$= \frac{1}{2}\int_0^L \left(\pi a_K \cos\frac{\pi x}{L} + \frac{(\psi_i+\psi_j)}{2}\cos\frac{2\pi x}{L} + \frac{(\psi_i-\psi_j-2\pi a_K)}{2}\cos\frac{3\pi x}{L}\right)^2 dx$$

or

$$\Delta_B = \frac{L}{8}\left[\psi_i^2 + \psi_j^2 + 2\pi a_K\psi_j - 2\pi a_K\psi_i + 4\pi^2 a_K^2\right] \quad ------- (12)$$

Potential energy of the thrust in the beam

$$V = -H\Delta_B = \frac{-HL}{\psi 8}\left(\psi_i^2 + \psi_j^2 + 2\pi a_K \psi_j - 2\pi a_K \psi_i + 4\pi^2 a_K^2\right) \quad \text{--------(13)}$$

Potential energy of uniform load ω,

$$V_\omega = -\omega \int_0^L \omega(x)\,dx = -\frac{\omega L^2}{8\pi^2}\left(\psi_i - \psi_j + 16\pi a_K\right) \quad \text{------(14)}$$

Potential energy of external load,

$$V_B = V_\omega + V_H \quad \text{---------------------(15)}$$

Total Potential energy of the beam,

$$U_B + V_B = U_B + V_\omega + V_H \quad \text{----------------(16)}$$

4. POTENTIAL ENERGY OF A COLUMN

Imposing the boundary conditions at column ends, the deflection curve can be expressed as the following equation. Notations are shown in Fig. 4. Let column height be $2l$,

$$\omega(x) = \frac{l}{4}\Big[\psi_i - \psi_j + 4\delta_K - (\psi_i + \psi_j - 4\delta_K)\frac{x}{l} - (\psi_i - \psi_j)\left(\frac{x}{l}\right)^2 + (\psi_i + \psi_j - 2\delta_K)\left(\frac{x}{l}\right)^3\Big] \quad \text{--------(17)}$$

Slope equation:

$$\frac{d(\omega x)}{d(x)} = -\frac{\psi_i}{4} - \frac{3}{8}\psi_j + \frac{3}{2}\delta_K - \frac{(\psi_i - \psi_j)}{2}\left(\frac{x}{l}\right) + \left(\frac{3}{4}\psi_i + \frac{3}{4}\psi_j - \frac{3}{2}\delta_K\right)\left(\frac{x}{l}\right)^2 \quad \text{--------(18)}$$

Curvature equation:

$$\phi = \frac{d^2\omega(x)}{dx^2} = \frac{1}{l}\left[-\frac{\psi_i}{2} + \frac{\psi_j}{2} + \left(\frac{3}{2}\psi_i + \frac{3}{2}\psi_j - 3\delta_K\right)\frac{x}{l}\right] \quad \text{--------(19)}$$

Substitute eq. (19) in eq. (2), and integrate through the length of column, we get the bending strain energy of the column,

$$U_c = \frac{2M_y f_1(\bar{P})}{\phi_y^2 h^2}\Big[-5\psi_i^3 + 5\psi_j^3 - 3\psi_i^2\psi_j + 3\psi_i\psi_j^2 - 18\psi_j^2\delta_K + 18\psi_i^2\delta_K + 18\psi_j\delta_K^2 - 18\psi_i\delta_K^2\Big] + \frac{4M_y}{\phi_y h} f_2(\bar{P})\Big[\psi_i^2 + \psi_j^2 + \psi_i\psi_j - 3\psi_i\delta_K - 3\psi_j\delta_K + 3\delta_K^2\Big] + M_y f_3(\bar{P})\left[\psi_j - \psi_i\right] \quad \text{--------(20)}$$

The axial shortening of the column,

$$\Delta c = 1/2 \int_{-\ell}^{\ell} \left[\frac{d(\omega x)}{dx}\right]^2 dx \qquad (21)$$

Let h = 2 ℓ,

$$\Delta c = -\frac{h}{15}\left[\psi_i^2 + \psi_j^2 + 9\rho_K^2 - \frac{3}{2}\psi_i\rho_K - \frac{3}{2}\psi_j\rho_K - \frac{\psi_i\psi_j}{2}\right] \qquad (22)$$

The potential energy of the axial load on the column

$$V_c = -\frac{Ph}{30}\left[2\psi_i^2 + 2\psi_j^2 + 18\rho_K^2 - \psi_i\psi_j - 3\psi_i\rho_K - 3\psi_j\rho_K\right] \qquad (23)$$

Total potential energy of the column is

$$U_c + V_c \qquad (24)$$

5. TOTAL POTENTIAL ENERGY OF A FRAME

Total potential energy of a frame is the sum of the potential energy of beams and columns of which the frame is composed.

Therefore total potential energy of the system is,

$$\sum(U+V) = \sum(U_c+V_c) + \sum(U_B+V_B) \qquad (25)$$

6. EQUILIBRIUM CONFIGURATIONS

The first derivative of total potential energy of the structure with respect to each deformation parameter should vanish identically, if the structure is in the state of equilibrium under the given loading condition. The equilibrium conditions ~~are~~ is

$$\frac{\partial \sum(U+V)}{\partial(\psi, \rho, a)} = 0 \qquad (26)$$

Equation (26) is a set of simultaneous equations with joint rotation ψ's, horizontal deflection ρ's and beam deflection parameters a's as the variables. Solution of the equation is an equilibrium configuration of the frame under the given loading condition.

7. STABILITY CONDITIONS

The second derivative of total potential energy of the structure with respect to deformation parameters should be positive for any mode of deformation, if the structure is stable.

Therefore the structure is stable, if

$$\frac{\partial^2 \Sigma (U+V)}{\partial (\psi, \delta, a)^2} > 0 \qquad \text{- - - - - - - - - -} \quad (27)$$

When

$$\frac{\partial^2 \Sigma (U+V)}{\partial (\psi, \delta, a)^2} < 0 \qquad \text{- - - - - - - - - -} \quad (28)$$

the structure, under the given loading condition, is unstable.

To determine the magnitude of the load at which stable equilibrium changes to unstable equilibrium. The stability condition can be established by

$$\frac{\partial^2 \Sigma (U+V)}{\partial (\psi, \delta, a)^2} = 0 \qquad \text{- - - - - - - - - -} \quad (29)$$

A set of equations (29) represent the conditions of limiting stability corresponding to each instability mode. A minimum load obtained from equation (29) should be the critical load of the structure which is expected to fail in that particular instability mode.

Therefore instability mode is a failure mechanism taking into account the effects of thrust and deformation of the structure.

7. SCOPE AND LIMITATION OF THE THEORY

The theory is particularly designed to solve frame instability problem in elastic and inelastic range. However, it can equally be applied to analyze other structures. Partically no assumption, except the effect of strain reversal, has been introduced in the derivation of beam and column potential energy. Therefore generality of the method can be summarized as follows

a) There is no limitation on the geometry of the structure.

b) There is no limitation on slenderness ratio of columns and stiffness of beams. They could be any odd number.

c) The range of axial load on columns could be any odd number, from $\frac{P}{P_y} = 0$ to $\frac{P}{P_y} = 1.0$

d) The method is applicable to unsymmetrical structure and loading condition. For the solution of the sturcture under combined horizontal and vertical loads, the extra labor introduced is practically negligible.

e) For high strength steel or aluminum alloy, only the energy coefficients $f_1(\bar{P}), f_2(\bar{P}), f_3(\bar{P})$ --- need be changed.

f) The method is equally good for any end condition-- pin-ended, partial fixed or fix-ended. No extra labor or manipulation is required.

g) A limitation of the method by the capacity of computer can be shown as follows

Let s = number of story

b = number of bay

n = Total maximum unknown variables of the frame of s-story and b-bay.

then $n = (2s+1)\ (b+1)$ - - - - - - - - - - - - - (29)

Therefore the computer should be able to evaluate nxn square matrix for solution of n simultaneous equations.

8. ILLUSTRATIVE EXAMPLE

Theoretical prediction of test frame W-1 can best be served as an illustrative example as well as verification of the theory.

The following data are taken from Fritz Lab. Report No. 276.9.

$L = 87.6''$ $\qquad M_y = 40.5^{k''}$

$h = 43.8''$ $\qquad \Phi_y = 1.023 \times 10^{-3}$

$d = 2.625''$ $\qquad P_y = 44.26^k$

$E = 31{,}800$ ksi

The axial thrust on the beam is $H = \frac{R}{12}$, where R is the axial load on a column. In non-dimensional form

$$\bar{H} = \frac{H}{P_y}$$
$$\bar{R} = \frac{R}{P_y}$$

Total energy of the frame is listed in Fig. 5. For equilibrium configuration, the second degree $\bar{u}' - \bar{\Phi}$ curves are used for solution of ψ_s's, δ_1 and α_1 as shown in Fig. 6.

The solution of the simultaneous equations for $\bar{R} = 0.25$ is shown in Fig. 7.

Substitute
$$\psi_0 = 0.011990$$
$$\psi_1 = 0.037448$$
$$\psi_2 = 0.011988$$
$$\psi_3 = -0.037449$$
$$\delta_1 = 0$$
$$\alpha_1 = 0.003389$$

in sidesway buckling condition

$$\frac{\partial^2 \Sigma (u+v)}{\partial \delta_1^2} = \frac{72 m_y f_1(\bar{R})}{\Phi_y^2 h^2}\left[\psi_1 - \psi_3 - \psi_0 + \psi_2\right] + \frac{48 m_y f_2(\bar{R})}{\Phi_y h} - \frac{72}{30}\bar{R} p_y h$$

$$= \frac{72 \times 40.5\,(-0.163)}{(1.023)^2 \times 10^{-6} \times 43.8^2}(0.098875) + \frac{48 \times 40.5 \times (0.620)}{1.023 \times 10^{-3} \times 43.9} - \frac{72}{30} \times 0.25 \times 44.26 \times 43.8$$

$$= 2237 > 0 \qquad \text{- - - - (30)}$$

<u>**stable!**</u>

For $\bar{R} = 0.27$, repeat the same calculation in Fig. 8 and substitute the results in the buckling condition

$$\frac{\partial^2 \Sigma (u+v)}{\partial \delta_1^2} = -1100 < 0 \qquad \text{- - - - (31)}$$

<u>**unstable!**</u>

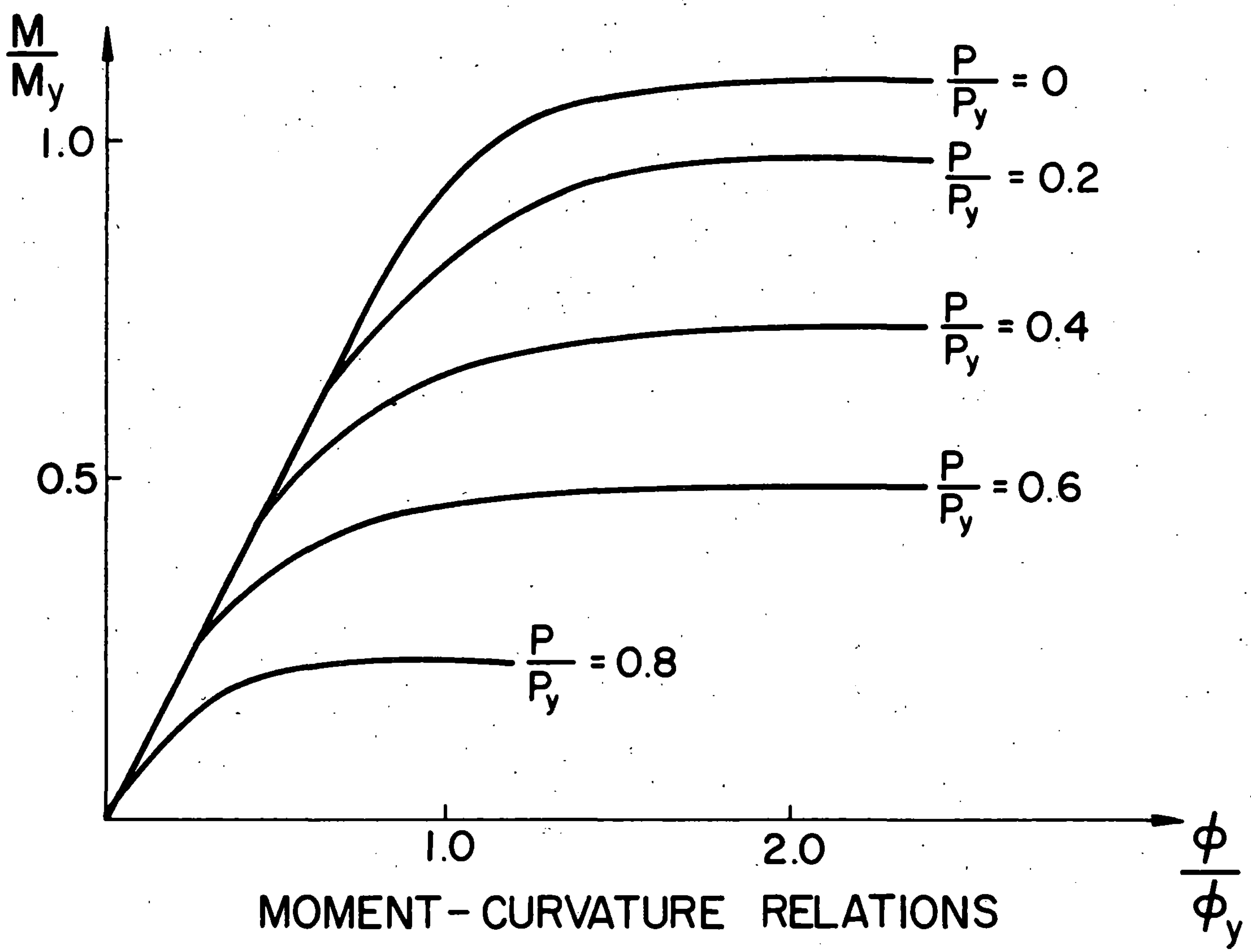
From eq. (30) and eq. (31), the buckling load of the frame is

$$\bar{R} = 0.25 + 0.0134 = 0.2634$$

or
$$R = P_y\,\bar{R} = 44.26 \times 0.2634 = 11.65 \text{ kips}$$

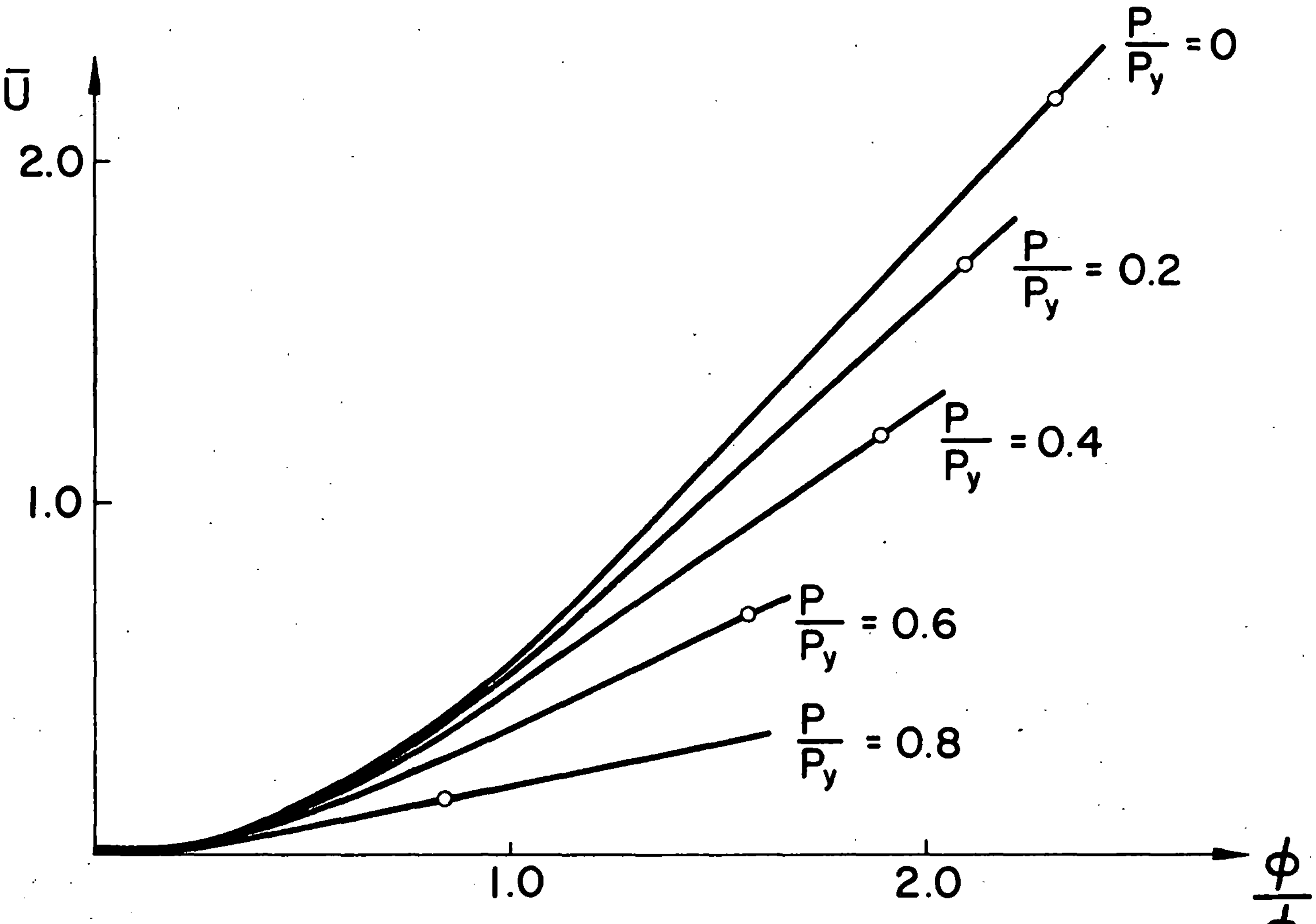
Compared with $\bar{P}_{ult} = 11.17$ kips from test results,

$$\text{Percent error} = \frac{(11.65 - 11.17) \times 100}{11.65} = 4\%.$$



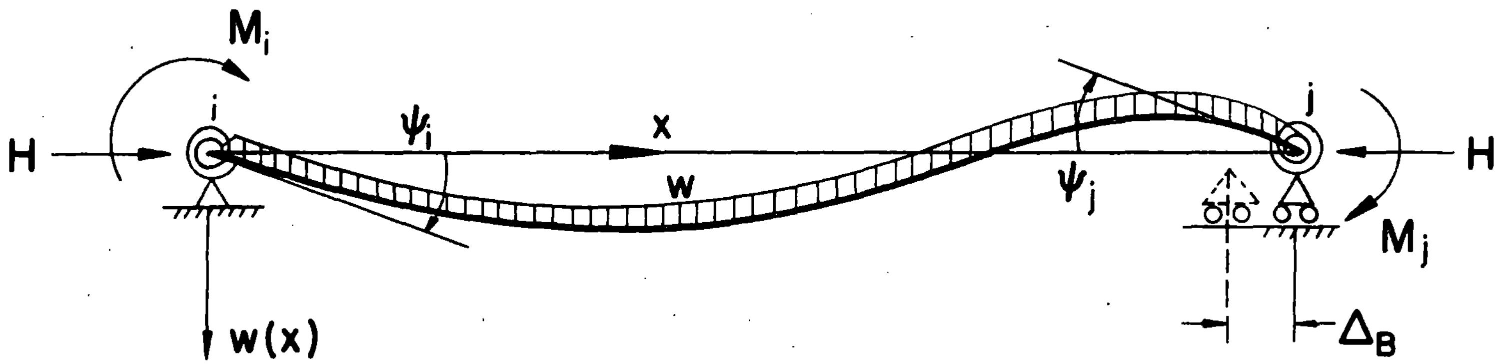
MOMENT - CURVATURE RELATIONS

Fig. 1



STRAIN-ENERGY CURVATURE RELATIONS

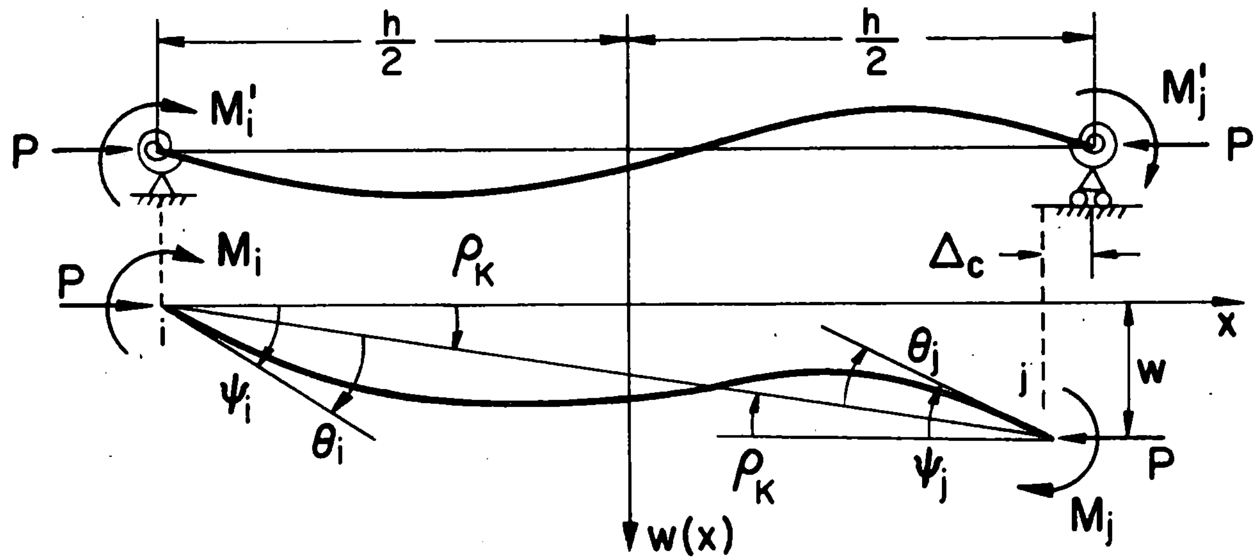
Fig. 2.2



BEAM

Deflection Curve	$w(x)$
Bending Strain Energy	U_B
Potential Energy of Uniform Load	V_W
Potential Energy of Thrust	V_H

Fig. 3



<u>COLUMN</u>

Deflection Curve $w(x)$

Bending Strain Energy U_c

Potential Energy of Thrust V_c

Fig. 4.4

TOTAL POTENTIAL ENERGY OF SINGLE-STORY FRAME ; $\Sigma(U+V)$

Column

U_c (0-1)

$$\frac{2M_y f_1(\bar{R})}{\Phi_y^2 h^2}\left\{5\psi_1^3-5\psi_0^3-3\psi_0^2\psi_1+3\psi_0\psi_1^2-18\psi_1^2\rho+18\psi_0^2\rho+18\psi_1\rho^2-18\psi_0\rho^2\right\}$$
$$+\frac{4M_y f_2(\bar{R})}{\Phi_y h}\left\{\psi_0^2+\psi_1^2+\psi_0\psi_1-3\psi_0\rho-3\psi_1\rho+3\rho^2\right\}+M_y f_3(\bar{R})\left\{\psi_1-\psi_0\right\}$$

U_c (3-2)

$$\frac{2M_y f_1(\bar{R})}{\Phi_y^2 h^2}\left\{5\psi_2^3-5\psi_3^3-3\psi_3^2\psi_2+3\psi_3\psi_2^2-18\psi_2^2\rho+18\psi_3^2\rho+18\psi_2\rho^2-18\psi_3\rho^2\right\}$$
$$+\frac{4M_y f_2(\bar{R})}{\Phi_y h}\left\{\psi_3^2+\psi_2^2+\psi_3\psi_2-3\psi_3\rho-3\psi_2\rho+3\rho^2\right\}+M_y f_3(\bar{R})\left\{\psi_2-\psi_3\right\}$$

V_c (0-1)

$$-\frac{\bar{R}P_y h}{30}\left\{2\psi_0^2+2\psi_1^2+18\rho^2-3\psi_0\rho-3\psi_1\rho-\psi_0\psi_1\right\}$$

V_c (3-2)

$$-\frac{\bar{R}P_y h}{30}\left\{2\psi_2^2+2\psi_3^2+18\rho^2-3\psi_2\rho-3\psi_3\rho-\psi_2\psi_3\right\}$$

Beam

U_B (1-3)

$$\frac{\pi^2 M_y}{\Phi_y^2 L^2} f_1(\bar{H}_K)\frac{1}{210}\left\{-1035\psi_1^3+1035\psi_3^3-4096\pi^3 a_1^3+225\psi_1^2\psi_3\right.$$
$$+1200\pi\psi_3^2 a_1+2304\pi^2 a_1^2\psi_1-225\psi_1\psi_3^2-2304\pi^2 a_1^2\psi_3$$
$$\left.+1200\pi\psi_1^2 a_1+672\pi a_1\psi_1\psi_3\right\}+\frac{\pi^2 M_y}{8\Phi_y L} f_2(\bar{H}_K)\left\{13\psi_1^2\right.$$
$$\left.+13\psi_3^2+40\pi^2 a_1^2-10\psi_1\psi_3+36\pi\psi_3 a_1-36\pi\psi_1 a_1\right\}$$
$$+M_y f_3(\bar{H}_K)\left\{\psi_3-\psi_1\right\}$$

V_B (1-3)

$$-\frac{\bar{R}P_y L}{96}\left\{\psi_1^2+\psi_3^2+4\pi^2 a_1^2+2\psi_3\pi a_1-2\pi\psi_1 a_1\right\}$$
$$-\frac{2}{27}\frac{\bar{R}P_y L}{\pi^2}\left\{\psi_1-\psi_3+16\pi a_1\right\}$$

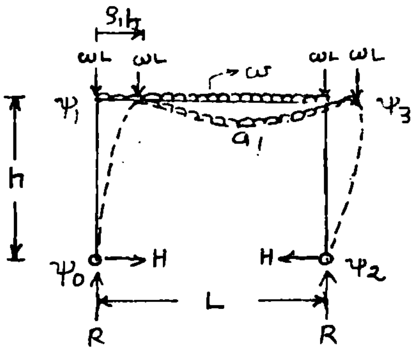


Fig. 5

EQUILIBRIUM CONDITION OF SINGLE-STORY FRAME ; $\frac{\partial \Sigma (U+V)}{\partial (\psi, \rho, a)} = 0$

W.r.t	ψ_0	ψ_1	ψ_2	ψ_3	ρ	a_1	constant
ψ_0	$+\frac{8M_y}{\Phi_y h} f_2'(\bar{R}) - \frac{2}{15}\bar{R} P_y h$	$+\frac{4M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{30} P_y h$			$-\frac{12M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{10} P_y h$		$+M_y f_3'(\bar{R})$
ψ_1	$+\frac{4M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{30} P_y h$	$+\frac{8M_y}{\Phi_y h} f_2'(\bar{R}) - \frac{2}{15}\bar{R} P_y h + \frac{26\pi^2 M_y}{8\Phi_y L} f_2'(\bar{H}_K) - \frac{\bar{R}}{48} P_y L$		$-\frac{10}{8}\frac{\pi^2 M_y}{\Phi_y L} f_2'(\bar{H}_K)$	$-\frac{12M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{10} P_y h$	$-\frac{36}{8}\frac{\pi^3 M_y}{\Phi_y L} f_2'(\bar{H}_K) + \frac{\bar{R} P_y L \pi}{48}$	$-M_y f_3'(\bar{R}) + M_y f_3'(\bar{H}_K) + \frac{2}{27}\frac{\bar{R} P_y L}{\pi^2}$
ψ_2			$+\frac{8M_y}{\Phi_y h} f_2'(\bar{R}) - \frac{2}{15}\bar{R} P_y h$	$+\frac{4M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{30} P_y h$	$-\frac{12M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{10} P_y h$		$-M_y f_3'(\bar{R})$
ψ_3		$-\frac{10}{8}\frac{\pi^2 M_y}{\Phi_y L} f_2'(\bar{H}_K)$	$+\frac{4M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{30} P_y h$	$+\frac{8M_y}{\Phi_y h} f_2'(\bar{R}) - \frac{2}{15}\bar{R} P_y h + \frac{26\pi^2 M_y}{8\Phi_y L} f_2'(\bar{H}_K) - \frac{\bar{R}}{48} P_y L$	$-\frac{12M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{10} P_y h$	$+\frac{36}{8}\frac{\pi^3 M_y}{\Phi_y L} f_2'(\bar{H}_K) - \frac{\bar{R}}{48} P_y L \pi$	$+M_y f_3'(\bar{R}) - M_y f_3'(\bar{H}_K) - \frac{2}{27}\frac{\bar{R} P_y L}{\pi^2}$
ρ	$-\frac{12M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{10} P_y h$	$-\frac{12M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{10} P_y h$	$-\frac{12M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{10} P_y h$	$-\frac{12M_y}{\Phi_y h} f_2'(\bar{R}) + \frac{\bar{R}}{10} P_y h$	$+\frac{48M_y}{\Phi_y h} f_2'(\bar{R}) - \frac{12}{5}\bar{R} P_y h$		
a_1		$-\frac{36}{8}\frac{\pi^3 M_y}{\Phi_y L} f_2'(\bar{H}_K) + \frac{\bar{R}}{48} P_y L \pi$		$+\frac{36}{8}\frac{\pi^3 M_y}{\Phi_y L} f_2'(\bar{H}_K) - \frac{\bar{R}}{48} P_y L \pi$		$+10\frac{\pi^4 M_y}{\Phi_y L} f_2'(\bar{H}_K) - \frac{\bar{R}}{12} P_y L \pi^2$	$+\frac{32}{27}\frac{\bar{R} P_y L}{\pi}$

FIG. 6

SOLUTION OF EQUILIBRIUM CONFIGURATION BY CROUT REDUCTION METHOD ; $\bar{R}=0.25$

STEPS		a_1	ρ	ψ_0	ψ_2	ψ_1	ψ_3	CONSTANT	Σ	CHECK
	(1)	10.8005	0	0	0	−1.5546	1.5546	0.0366	10.8371	10.8371
	(2)	0	7.268	−2.058	−2.058	−2.058	−2.058	0	−0.964	−0.964
	(3)	0	−2.058	1.340	0	0.718	0	0.01082	0.01082	0.01082
	(4)	0	−2.058	0	1.340	0	0.718	−0.01082	−0.01082	−0.01082
	(5)	−15.546	−2.058	0.718	0	4.910	−1.380	0.00666	−13.3493	−13.3493
	(6)	15.546	−2.058	0	0.718	−1.380	4.910	−0.00666	17.7293	17.7293
(1) ÷ 10.8005	(7)	1	0	0	0	−0.14394	0.14394	0.003389	1.003389	1.003389
(2) − (7)(0)	(8)	0	7.268	−2.058	−2.058	−2.058	−2.058	0	−0.964	−0.964
(3) − (7)(0)	(9)	0	−2.058	1.340	0	0.718	0	0.01082	0.01082	0.01082
(4) − (7)(0)	(10)	0	−2.058	0	1.340	0	0.718	−0.01082	−0.01082	−0.01082
(5) + (7)(15.546)	(11)	0	−2.058	0.718	0	2.6723	0.8577	0.059345	2.249345	2.249385
(6) − (7)(15.546)	(12)	0	−2.058	0	0.718	0.8577	2.6723	−0.059345	2.13065	2.13062
(8) ÷ 7.268	(13)		1	−0.28316	−0.28316	−0.28316	−0.28316	0	−0.13264	−0.13264
(9) + (13)(2.058)	(14)		0	0.75726	−0.58274	0.13526	−0.58274	0.01082	−0.26214	−0.262145
(10) + (13)(2.058)	(15)		0	−0.58274	0.75726	−0.58274	0.13526	−0.01082	−0.28378	−0.283785
(11) + (13)(2.058)	(16)		0	0.13526	−0.58274	2.08956	0.27496	0.059345	1.9764	1.97642
(12) + (13)(2.058)	(17)		0	−0.58274	0.13526	0.27496	2.08956	−0.059345	1.8576	1.85765
(14) ÷ 0.75726	(18)			1	−0.76954	0.17862	−0.76954	0.014288	−0.34617	−0.346175
(15) + (18)(0.58274)	(19)			0	0.30882	−0.47865	−0.31318	−0.002494	−0.48550	−0.485515
(16) − (18)(0.13526)	(20)			0	−0.47865	2.06540	0.37905	0.0574124	2.02321	2.02324
(17) + (18)(0.58274)	(21)			0	−0.31318	0.37905	1.64112	−0.051019	1.65597	1.65592
(19) ÷ 0.30882	(22)				1	−1.549932	−1.014118	−0.0080759	−1.572126	−1.57216
(20) + (22)(0.47865)	(23)				0	1.32353	−0.10635	0.0535469	1.27073	1.27073
(21) + (22)(0.31318)	(24)				0	−0.10635	1.32352	−0.053548	1.1636	1.1636
(23) ÷ (1.32353)	(25)					1	−0.08035	0.04045	0.9601	0.9601
(24) + (25)(0.10635)	(26)					0	1.31498	−0.049247	1.2657	1.2657
	ψ_3						1	−0.037449		
	ψ_1					1		+0.037448		
	ψ_2				1			+0.011988		
	ψ_0			1				−0.011990		
	ρ		1					0		
	a_1	1						+0.003389		

Fig. 7

SOLUTION OF EQUILIBRIUM CONFIGURATION BY CROUT REDUCTION METHOD ; $\bar{R}=0.27$

STEPS		ψ_0	ψ_1	ψ_2	ψ_3	ρ	a_1	CONSTANT	Σ	CHECK
	(1)	1.293	0.699	0	0	−1.992	0	0.0109	0.0109	0.0109
	(2)	0.699	4.844	0	−1.374	−1.992	−15.463	0.0072	−13.2788	−13.2788
	(3)	0	0	1.293	0.699	−1.992	0	−0.0109	−0.0109	−0.0109
	(4)	0	−1.374	0.699	4.844	−1.992	+15.463	−0.0072	17.6328	17.6328
	(5)	−1.992	−1.992	−1.992	−1.992	6.925	0	0	−1.043	−1.043
	(6)	0	−15.463	0	15.463	0	107.519	0.395	107.914	107.914
(1) ÷ 1.293	(7)	1	0.5406	0	0	−1.5406	0	0.0084	0.0084	0.0084
(2) − (7)(0.699)	(8)	0	4.4662	0	−1.374	−0.9152	−15.463	0.0014	−13.2846	−13.2846
(3) − (7)(0)	(9)	0	0	1.293	0.699	−1.992	0	−0.0109	−0.0109	−0.0109
(4) − (7)(0)	(10)	0	−1.374	0.699	4.844	−1.992	15.463	−0.0072	17.6328	17.6328
(5) − (7)(−1.992)	(11)	0	−0.9152	−1.992	−1.992	3.8562	0	0.0167	−1.0263	−1.0263
(6) − (7)(0)	(12)	0	−15.463	0	15.463	0	107.519	0.395	107.914	107.914
(8) ÷ 4.462	(13)		1	0	−0.3076	−0.2049	−3.4622	0.0003	−2.9744	−2.9744
(9) − (13)(0)	(14)		0	1.293	0.699	−1.992	0	−0.0109	−0.0109	−0.0109
(10) − (13)(−1.374)	(15)		0	0.699	4.4214	−2.2735	10.706	−0.0068	13.5461	13.5461
(11) − (13)(−0.9152)	(16)		0	−1.992	−2.2735	3.6687	−3.1686	0.0169	−3.7485	−3.7485
(12) − (13)(−15.463)	(17)		0	0	10.7066	−3.1683	53.983	0.3996	61.9209	61.9209
(14) ÷ 1.293	(18)			1	0.5406	−1.5406	0	−0.0084	−0.0084	−0.0084
(15) − (18)(0.699)	(19)			0	4.0436	−1.1967	10.706	−0.001	13.5519	13.5519
(16) − (18)(−1.992)	(20)			0	−1.1967	0.5999	−3.1686	0.0002	−3.7652	−3.7652
(17) − (18)(0)	(21)			0	10.7066	−3.1683	53.983	0.3996	61.9209	61.9209
(19) ÷ 4.0436	(22)				1	−0.2959	2.6476	−0.0002	3.3515	3.3515
(20) − (22)(−1.1967)	(23)				0	0.2458	−0.0003	0	0.2455	0.2455
(21) − (22)(10.7066)	(24)				0	−0.0003	+25.6363	0.4017	26.0377	26.0377
(23) ÷ 0.2458	(25)					1	−0.0012	0	0.9998	0.9998
(24) − (25)(−0.0003)	(26)					0	25.6363	0.4017	26.0380	26.0380
a_1							1	0.0156		
ρ						1		0.00001872		
ψ_3					1			−0.0415		
ψ_2				1				0.014		
ψ_1			1					0.0416		
ψ_0		1						−0.014		

Fig. 8