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INFLUENCE SURFACES OF ORTHOTROPIC PLATES

by

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TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

SYNOPSIS

Modern developments of reinforced concrete structures have presented many problems in the field of theory of elasticity. Especially in the case of plate and shell structures, theoretical investigations based on the theory of elasticity have become indispensable for a safe and economical design. The application of plate theory, that is, influence surfaces of plates has been taking more and more important roles in the design of bridge floor slabs.

In this dissertation, the extension of the theory of influence surfaces to orthotropic plates are made, the approach being based on the mathematical concept of "Green's Function" for the deflection of a plate.

Solutions for the moments of semi-infinite strips as well as infinite strips with various boundary conditions are derived mostly in closed form.

Such a solution in closed form will render numerical computations much easier than series solutions as presented by Pucher and other investigators. A general discussion of the singularities of the surfaces are presented with several numerical examples.

CHAPTER I

Introduction

1.1 The Importance of Influence Surfaces in the Design of Bridge Floors

The use of influence lines for the design of bridges subjected to live loads has become a standard practice, even to the extent that no further method is accepted. The influence lines allow to determine the maximum moment, shearing force, axial load, etc. for a given section in a bridge member under live loads.

A logical extension of this method to the design of bridge slabs is the development of influence surfaces (two-dimensional influence lines). They allow the determination of the maximum moment (and shearing force, twisting moment, etc. if desired) at a given point of the slab subjected to concentrated wheel loads. The proper detailing of the slab can readily be handled, once the extreme moment values are known.

In this chapter, the fundamental equation of an orthotropic plate will be introduced first. Then the engineering concept of influence surfaces will be described. Finally, some important theorems as well as properties of influence surfaces will be listed without proof.

1.2 Bending of Orthotropic Plates (for example, (1) p.188)

It is assumed that the material of the plate has three planes of symmetry with respect to its elastic properties.

Taking these planes as the coordinate planes, the relations between the stress and strain components for a case of plane stress in the xy-plane can be represented by the following equations: (Fig.(1-1))

$$\left.\begin{aligned}\sigma_x &= E_x' \epsilon_x + E'' \epsilon_y \\ \sigma_y &= E_y' \epsilon_y + E'' \epsilon_x \\ \tau_{xy} &= G \gamma_{xy}\end{aligned}\right\} \tag{1.1}$$

It is seen that in the case of plane stress, four constant E_x', E_y'', E" and G are needed to characterize the elastic properties of a material.

Considering the bending of a plate made of such a material, it is assumed that linear elements perpendicular to the middle plane (xy-plane) of the plate before bending remain straight and normal to the deflection surface of the plate after bending. Hence, the usual expressions for the components of strain can be used((1)p.34).

$$\epsilon_x = -z\frac{\partial^2 W}{\partial x^2}, \quad \epsilon_y = -z\frac{\partial^2 W}{\partial y^2} \quad \gamma_{xy} = -2z\frac{\partial^2 W}{\partial x \partial y} \tag{1.2}$$

The corresponding stress components, are

$$\left.\begin{aligned}\sigma_x &= -z\left(E_x'\frac{\partial^2 W}{\partial x^2} + E''\frac{\partial^2 W}{\partial y^2}\right) \\ \sigma_y &= -z\left(E_y'\frac{\partial^2 W}{\partial y^2} + E''\frac{\partial^2 W}{\partial x^2}\right) \\ \tau_{xy} &= -2Gz\frac{\partial^2 W}{\partial x \partial y}\end{aligned}\right\} \tag{1.3}$$

With these expressions for the stress components the bending and twisting moments are

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z \, dz = -\left(D_x\frac{\partial^2 W}{\partial x^2} + D_1\frac{\partial^2 W}{\partial y^2}\right) \tag{1.4}$$

$$\left.\begin{aligned} M_y &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z\, dz = -\left(D_y \frac{\partial^2 W}{\partial y^2} + D_1 \frac{\partial^2 W}{\partial x^2} \right) \\ M_{xy} &= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} z\, dz = 2 D_{xy} \frac{\partial^2 W}{\partial x \partial y} = -M_{yx} \end{aligned}\right\} \qquad (1.4)$$

in which

$$D_x = \frac{E_x' h^3}{12}, \quad D_y = \frac{E_y' h^3}{12}, \quad D_1 = \frac{E'' h^3}{12} \quad D_{xy} = \frac{G h^3}{12} \qquad (1.5)$$

Substituting expressions (1.4) into the equations of equilibrium for a differential element in x, y and z directions. (Fig. 1-2)

$$\begin{aligned} \frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x &= 0 \\ \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y &= 0 \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q &= 0 \end{aligned} \qquad (1.6)$$

the equation for an orthotropic plate is obtained

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = q \qquad (1.7)$$

where

$$H = D_1 + 2 D_{xy}$$

In the particular case of isotropy,

$$E_x' = E_y' = \frac{E}{1-\nu^2} \qquad E'' = \frac{\nu E}{1-\nu^2} \qquad G = \frac{E}{2(1+\nu)}$$

Hence

$$D_x = D_y = \frac{E h^3}{12(1-\nu^2)} = D$$

$$H = D_1 + 2 D_{xy} = \frac{h^3}{12}\left(\frac{\nu E}{1-\nu^2} + \frac{E}{1+\nu} \right) = \frac{E h^3}{12(1-\nu^2)} = D$$

Therefore equation (1.7) reduces to the ordinary plate equation:

$$D \Delta \Delta W = q \qquad (1.8)$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

In addition to equation (1.5) and equation (1.7), the expressions for the shearing force Q_x, Q_y and the boundary shear V_x, V_y are are collected here:

$$\left.\begin{aligned} Q_x &= \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_x}{\partial y} = -\frac{\partial}{\partial x}\left(D_x \frac{\partial^2 W}{\partial x^2} + H \frac{\partial^2 W}{\partial y^2}\right) \\ Q_y &= \frac{\partial M_y}{\partial y} - \frac{\partial M_x}{\partial x} = -\frac{\partial}{\partial y}\left(H \frac{\partial^2 W}{\partial x^2} + D_y \frac{\partial^2 W}{\partial y^2}\right) \\ V_x &= Q_x - \frac{\partial M_{xy}}{\partial y} = -\frac{\partial}{\partial x}\left(D_x \frac{\partial^2 W}{\partial x^2} + (2H - D_1) \frac{\partial^2 W}{\partial y^2}\right) \\ V_y &= Q_y - \frac{\partial M_{yx}}{\partial x} = -\frac{\partial}{\partial y}\left((2H - D_1) \frac{\partial^2 W}{\partial x \partial y} + D_y \frac{\partial^2 W}{\partial y^2}\right) \end{aligned}\right\} \quad (1.9)$$

W_{xy}

1.3 Engineering Concept of Influence Function for the Deflection of a Plate

Consider a plate of any shape with prescribed boundary conditions subjected to a concentrated load P=1 acting at the point (x,y). (Fig. 1-3) The deflection W(u,v;x,y) of a point (u,v) is called the Green's function (influence function) for the deflection of the given plate.

The influence function W(u,v;x,y) depends on the four variables u,v and x,y. For the graphical presentation of the function a two-dimensional contour line system will be employed. For instance, if (u,v) is fixed ((u,v) being the influence point), the function depends upon x and y, therefore W(u,v;x,y) will form a surface. This surface, W(x,y), will be called influence surface for the deflection of point (u,v). On the other hand, if x,y is fixed ((x,y) being the loading point) the function, W(u,v) represents another surface, which is the deflection surface of the plate under a concentrated load P=1 at (x,y). The theory of influence surfaces is based on the ordinary theory of plate. Therefore, following assumptions made in section (1.2) apply:

1. The plate thickness h is assumed to be constant and small compared to other dimensions.
2. The material is orthotropic and follows Hooke's law.
3. The deflection of plates is small against the thickness h.

1.4 Some Important Theorems and Properties of Influence Functions

It is not the purpose of this section to introduce the general theory of influence surfaces developed by A. Pucher. However, several fundamental theorems and properties of influence surfaces will be pointed out.

(a) The influence function for the deflection of a plate W(u,v;x,y) consists of two functions, that is,

$$W(u,v;x,y) = W_0(u,v;x,y) + W_1(u,v;x,y)$$

where $W_0(u,v;x,y)$ is the particular solution of the differental equation:

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = q(x,y)$$

and $W_1(u,v;x,y)$ is the homogeneous solution of the above equation whose constants are determined such that W(u,v;x,y) will fulfill the prescribed boundary conditions, $W_0(w,v;x,y)$ contains the singular solution corresponding to $r^2 \log \frac{r}{r_0}$ in case of isotropic plates[(2)p. 261]. The corresponding solution for orthotropic plates has been derived by Mossakowski[(11)].

It is this part which plays the important role for the singular behavior of influence surfaces as will be shown later.

(b) The influence function F(u,v;x,y) for any effect in a plate (such as bending moment, twisting moment, shearing force, etc.) at a given point (u,v) is obtained by differentiating the influence function for the deflection, W(u,v;x,y), with respect to u and v.

Following are the formulae for the derivation of such influence functions:

Bending Moments

$$M_x(u,v;x,y) = -\left(D_x \frac{\partial^2 W}{\partial u^2} + D_1 \frac{\partial^2 W}{\partial v^2}\right)$$

$$M_y(u,v;x,y) = -\left(D_1 \frac{\partial^2 W}{\partial u^2} + D_y \frac{\partial^2 W}{\partial v^2}\right)$$

Twisting Moments

$$M_{xy}(u,v;x,y) = 2\, D_{xy} \frac{\partial^2 W}{\partial u \partial v} = -M_{yx}(u,v;x,y)$$

Shearing Forces

$$Q_x(u,v;x,y) = -\frac{\partial}{\partial u}\left(D_x \frac{\partial^2 W}{\partial u^2} + H \frac{\partial^2 W}{\partial v^2}\right)$$

$$Q_y(u,v;x,y) = -\frac{\partial}{\partial v}\left(H \frac{\partial^2 W}{\partial u^2} + D_y \frac{\partial^2 W}{\partial v^2}\right)$$

Boundary Shear

$$V_x(u,v;x,y) = -\frac{\partial}{\partial u}\left[D_x \frac{\partial^2 W}{\partial u^2} + (2H - D_1)\frac{\partial^2 W}{\partial v^2}\right]$$

$$V_y(u,v;x,y) = -\frac{\partial}{\partial v}\left[(2H - D_1)\frac{\partial^2 W}{\partial u^2} + D_y \frac{\partial^2 W}{\partial v^2}\right]$$

The function F(u,v;x,y), can be used in two different ways. If the point (u,v) is fixed (this point (u,v) will be called from now on influence point), the function will represent the influence surface for the particular effect (for example, bending moment, etc.) with respect to the influence point (u,v) and will be written f(x,y).

On the other hand if the point (x,y), the loading point, is fixed the function determines the distribution of the effect over the plate due to the load P acting at (x,y). For example, in case of $M_x(u,v;x,y)$ it represents the M_x-moment surface due to a concentrated load P=1. It will be written as F(u,v).

(c) From section (b) it can be concluded that the influence function F(u,v;x,y) for any effect in a plate is a solution of

$$D_x \frac{\partial^4 F}{\partial x^4} + 2H \frac{\partial^4 F}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 F}{\partial y^4} = 0$$

with a singularity at the influence point (u,v). The function F(u,v;x,y) fulfills the same prescribed boundary condition as W(u,v;x,y). In references(5)(6) some cases were solved directly for moments using this principle instead of deriving W(u,v;x,y). However, in this dissertation, W(u,v;x,y) is always thought first and M_x, M_y are obtained through differentions. This is done for the following two reasons.

(i) Once, W(u,v;x,y) is determined, any other influence function is obtained quickly by simple differentiation.

(ii) W(u,v;x,y) can be successfully applied to solve other important problems such as eigen value problems of plates (Vibration, buckling), dynamical behavior of plates due to impulsive loading, etc.

(d) Magnitude of particular effect in a plate under arbitrary loading:

The magnitude is given by the following expression

$$F = \sum_i P_i f(u,v;x_i,y_i) + \int p(s) f(u,v;x(s),y(s))\,ds$$

$$+ \iint p(x,y) f(u,v;x,y)\,dx\,dy$$

where

P_i: concentrated loads acting at (x,y)

p(s): line load distributed along some line *load density*

p(x,y): distributed load over some area.

With the use of influence surface diagrams[4] this computation can be done graphically and numerically.

(e) Influence surfaces are generally controlled by following four conditions:

(i) location of the influence point (u,v)

(ii) shape of plate boundaries

(iii) boundary conditions

(iv) material properties of plates: that is, the two parameters : $\lambda = \frac{H}{D_y}$, $\mu = \sqrt{\frac{D_x}{D_y}}$ *values?*

(f) All influence functions f(u,v;x,y) have singularities at the influence point (u,v) with the exception of the one for deflection. Values of M_x, M_y for interior points of plates, edge moments along free edge become infinitely large at the influence point (u,v). Though other influence functions show singular behaviors at the influence point (u,v), the corresponding values stay finite. In the vicinity of the influence point (u,v), the signular part of the solution F_0(u,v;x,y) becomes predominant.

(g) In order to clarify the adopted definitions and notation they are summarized in the following table:

(i) For the influence function W(u,v;x,y) of the deflection (u,v) and (x,y) are completely interchangeable (Maxwell's Law). However, for the influence function of any effect F(u,v;x,y) obtained through differentiation from $W(u,v;x,y)$, such a reciprocity does not apply in general.

(ii)

Notation	Definition	Coordinates of Influence point (u,v)	Loading Point (x,y)
$F(u,v;x,y)$	Influence function for any effect in a plate at a given point (u,v) to a unit concentrated load P=1 at (x,y)	Variable	Variable
$F(u,v)$	The distribution of any effect over the plate due to the unit load P=1 acting at (x,y) example--$M_x(u,v)$, Moment surface for Bending Moment M_x	Variable	Fixed
$f(x,y)$	Influence surface for any effect with respect to the influence point Q(u,v) example--$m_x(x,y)$, influence surface for bending moment M_x at point u,v.	Fixed	Variable
F	The magnitude of any effect at (u,v) due to specific loads.	Fixed	Fixed

1.5 Application of the Theory of Orthotropic Plates to Actual Bridge Floor Systems

There are quite a few specific cases to which the theory of orthotropic plate is applicable: two-way reinforced concrete slabs, stiffened plates, corrugated plates, gridwork systems, plywood plates, etc. are typical examples of orthotropic plates. In order to study the behavior of such plates, applying the theory of misprint $\sqrt{\frac{D_x}{D_y}}$

orthotropic plates, elastic constants, D_x, D_y, H must be determined either by experiment or on the basis of theoretical consideration.

As stated in (1.4,e) the shape of an influence surface of an orthotropic plate is controlled by the two ratios of the elastic constants: $\lambda = \frac{H}{D_y}$, $\mu = \sqrt{\frac{H}{D_y}}$ It is very important to study the methods to determine these constants. Since Huber's work on reinforced concrete slabs a great number of investigation have been carried out on this particular problem. However it may be premature to say that accurate methods for the determination of λ and μ have been established. It is a problem beyond the scope of this dissertation. However, in order to get a picture on the variation of λ and μ as encountered in practice, numerical data on actual bridge floor systems have been collected and represented in Fig(1-4) (See also Chapter XI, References.[(19)--(26)])

These data were obtained either by theoretical analysis or by direct tests. The domain of λ - μ diagram is bisected by the $\lambda = \mu$ line, and most of the points (λ, μ) are located in the domain $\lambda < \mu$, with several points ((2),(3),(4),(14)) are very close to μ -axis.

Along the μ- axis, λ =0, or, in other words, H=0. This is the case for gridwork system for which the torsional rigidity of the floor may be negligible. On the other hand, along λ -axis μ =0, that ie, D_x=0 this is the case of articulated plates[(26)]. In general, for actual orthotropic plates, λ and μ values can be limited.

$$0 \leqq \lambda \leqq \lambda_0$$

$$0 \leqq \mu \leqq \mu_0$$

where λ_0, μ_0 present some maximum upper limits*. The other limit $\lambda = \mu = 0$ is practically less important, because the structure is effectively reduced to a system of beams side by side without connection ($D_x = H = 0$).

It is also interesting that the case $\lambda < \mu$ is quite common as far as bridge floor systems are concerned. However, it is the more complicated case for practical computation of influence surfaces as will be seen later.

1.6 Historical Review of Investigation on Influence Surfaces

Since theory of influence surfaces is essentially the theory of Green's functions associated with the linear fourth order partial differential plate equation the problem is closely related to the bending of plates in the theory of elasticity. The first solution of the problem of bending of a simply supported rectangular plate with the use of double trigonometric series is due to Navier in 1820. This famous solution in case of a single concentrated load P is actually the Green function for this particular plate in double series form of eigen functions.[(1), p.117]

In discussing problems of bending of rectangular plates with two opposite edges simply supported M. Levy suggested the single series solution in 1899. Thus, the Green's function of this problem has become possible to be expressed in a single series form (Levy's solution)[(1),p.125]

Almost, at the same time, J.H. Michell has derived the Green's function for a circular plate whose boundary is clamped, using the method of inversion in 1901.[7]

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*For the numerical discussion of the singularities of influence surfaces in Chapter VII $\lambda_0 = \mu_0 = 10$ is assumed and twelve values of λ and μ are considered.

However, the first attempt to compute influence surfaces for the stresses in slabs was probably made by Westergaard[8]. Realizing the reciprocity between the bending moment at point (u,v) due to a load at (x,y) and vice-versa in the case of a simply supported plate strip he obtained a moment influence surface.

Subsequent investigators[9],[10] followed the same line of reasoning by basing the influence surfaces on Maxwell's reciprocity theorem. However, this theorem on the reciprocity of deflections, if applied to moments holds for a limited number of cases only (that is, simply supported plate strip, simply supported rectangular plates, etc.).

Pucher has developed the general theory of influence surfaces in 1938[5] and he furnished a great number of important results in form of contour line diagrams.[4] But his work and that of work made by other investigators is confined to the case of isotropic plates.

The extension of the theory of influence surfaces to the case of orthotropic plates is presented in this dissertation.

Incidentally, a recent literature review disclosed that such work has been started independently in Poland by Nowacki, Mossakowski and others since 1950[11],[12],[13]. It should be pointed out that some minor results developed in this dissertation have been already derived by these investigators, employing methods similar to the ones in this dissertation.

CHAPTER II

Practical Application of Influence Surfaces

The practical application of influence surfaces will be discussed shortly in this chapter. Since the influence surfaces are generally presented in the form of contour line diagrams, it is important to know how to use these surfaces in order to get accurate results. Furthermore the consistency between theory and experiments will be discussed.

2.1 Application of Influence Surfaces to Actual Problems

As pointed out in (1.4,d) already, the determination of any effect (bending moment, shearing force, etc) at a given point due to an arbitrary load, requires only the computation of simple area or volume integrals by making use of influence surfaces. (similar to influence lines).

(i) for a distributed load p(x,y)

$$F = \iint p(x,y)\, f(u,v;x,y)\, dx\, dy$$

(ii) for a line load p(s)

$$F = \int p(s)\, f(u,v;x(s),y(s))\, ds$$

(iii) for several concentrated loads $P_i(x,y)$

$$F = \sum_i P_i(x_i,y_i)\, f(u,v;x_i,y_i)$$

In actual computation, (for case (i)) surfaces are sectioned by horizontal or vertical planes and for each section, the area is computed using a planemeter or applying Simpson's Rule. The volume can be computed by repeating Simpson's Rule on the areas.

At the influence point the value of the influence function very often grows to infinity. In numerical computations the volume in the immediate neighborhood of this singular point is usually neglected. In order to justify this practice the following example is given:

Consider the singular part of $m_x(u,v)$ in the vicinity of the influence point (u,v). (Fig. 2-1) Since the singular part of m_x is predominant around this point the volume of neglected portion of the surface ΔV is essentially governed by this singular part and can hence be computed as follows.

In the case of an isotropic plate the singular part is:

$$(m_x)_o = -\frac{1}{8\pi}\left(2\log\frac{r}{r_o} + 2\cos^2\varphi + 1\right)$$

assuming $(m_x)_o = \chi$

$$\log\frac{r}{r_o} = -\frac{1}{2}(8\pi\chi + 2\cos^2\varphi + 1)$$

$$\text{or} \quad r(\chi,\varphi) = r_o\, e^{-\frac{8\pi\chi+1}{2}}\, e^{-\cos^2\varphi}$$

This is the equation of a section $(m_x)_o = \chi$ of the surface. The area of the section follows to:

$$A(\chi) = \frac{1}{2}\int_0^{2\pi} r^2 d\varphi = \frac{1}{2} r_o^2 e^{-(8\pi\chi+1)}\int_0^{2\pi} e^{-2\cos^2\varphi} d\varphi$$

$$= \frac{1}{2}\times 2.926\, r_o^2\, e^{-(8\pi\chi+1)}\,{}^{*}$$

Therefore the volume $V(\chi)$ of the surface above plane χ is obtained:

$$V(\chi) = \int_\chi^\infty F(\chi)\,d\chi = \frac{r_o^2}{2e}\cdot 2.926\int_\chi^\infty e^{-8\pi\chi} d\chi = \frac{r_o^2}{2e}\cdot\frac{2.926}{8\pi} e^{-8\pi\chi}$$

$$\text{or} \quad V(\chi) = 0.02146\, r_o^2 e^{-8\pi\chi}$$

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*by numerical integration

$$\int_0^{2\pi} e^{-2\cos^2\varphi} d\varphi = 2.926$$

Using $V(x)$, ΔV is easily estimated

$$\Delta V = V\left(\frac{7}{8\pi}\right) = 0.02146\, r_0^2 e^{-7} = 1.957\, r_0^2 \times 10^{-5}$$

such that it can be usually neglected in the computation of M_x. In case of orthotropic plates, magnitude of ΔV will change depending upon λ and μ, however it is still of order 10^{-5}.

Since influence surfaces have singularities at the influence point, careful consideration must be paid to the computation in the vicinity of that point.

Further details concerning practical computation will be found in Pucher's book.(4) **Careful computation yields always very** accurate results (max. error = 5%).

2.2 Consistency Between Theory and Experiments

Since the theory of influence surfaces is based on the ordinary theory of plates, results obtained are certainly correct within the limitation of the theory of elasticity. Therefore it can be expected that corresponding results are much superior than present semi-empirical formulae given in specifications such as AASHO. Theory of plates subjected to concentrated loads and hence the theory of influence surfaces has been checked experimentally. Especially Dutch investigators have recently carried out a very successful experimental study of slabs subjected to concentrated loads.(14)

The experiments were conducted on a steel model to obtain information about the stress-strain distribution in slabs, subjected to concentrated loads.

(i) Investigation of influence of the size of the loading surface (the concentration of the load) on the bending moments in the slab.

The load was in succession transmitted by a ball (which gave a contract area with a diameter of about 0.45 cm) and by circular distribution pads with diameters D of 1.6 cm, 3.6 cm, 5.4 cm and 7.6 cm. The ratios e/a (radius of distributor pad/span) were respectively 0.0024, 0.0087, 0.0195, 0.0293 and 0.0411. For these measurements investigations on the influence of various intermediate layers such as, 3 mm cardboard and rubber with various thicknesses were also made.

(ii) Investigation of the stress-distribution in the slab as a function of the boundary conditions and the locations of the load. (Fig. 2-3)

Summarizing the test results, the following conclusions were drawn:

(a) Outside the immediate neighborhood of the load there was a good agreement between the experiments and the elementary theory of plates.
(for concentration e/a=0.0024 to e/a=0.0411 no noticeable influence was found outside an area with a radius of 5 cm (about $\frac{1}{18}$ of the span) around the center of gravity of the load)

(b) For the bending moments under the load, the correction presented by Westergaard[8] was in good agreement with the experiments. (Fig. 2-4 and 2-5).

As will be seen later, influence functions for any effect except the deflection exhibit singular behavior in the neighborhood of the influence point. This is due to the assumption of an idealized concentrated load. Actually, this ideal concentration of load cannot be realized.

Instead, a small portion of the plate just under the load must be subjected to rather high compressive pressure because of highly localized loads.

Therefore it is impossible to apply the ordinary plate theory in the vicinity of the applied loads. Nadai,[3] Woinowsky-Krieger,[1] Westergaard,[8] and other, have investigated the stress distribtuion directly under the loads (theory of thick plates). Nevertheless, such a disturbance has such localized effects that the accuracy of the theory is practically not affected (by St. Venant's Principle), because, the volume of influence surfaces above the certain limiting values is usually negligible as stated before.

CHAPTER III

Deflections, Moments And Influence Functions For The Infinite Plate Strip With Simply Supported Parallel Edges

3.1 Method of Solution

In order to obtain the solution, the usual approach solving directly differential equation will be employed. Although the deflection surface is obtained in a series form, bending moments twisting moment, shearing forces can be expressed in closed form as will be seen later. The expressions consist of a singular part due to the particular solution of the generalized Biharmonic equation and a regular part due to homogeneous solution of

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = 0$$

3.2 Formation of the Problem and Derivation of the Solution

Consider an infinite plate strip with simply supported parallel edges (Fig. 3-1).

The problem consists of deriving the deflection surface and hence the influence function for deflections (Green's function) of this infinite plate strip. The deflection surface must satisfy the following differential equation

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = 0 \qquad (3.1)$$

except at the point where the concentrated load P=1 is applied.

The corresponding boundary conditions are as follows:

$$x = 0 \quad : \quad W = 0 \qquad M_x = -\left(D_x \frac{\partial^2 W}{\partial x^2} + D_1 \frac{\partial^2 W}{\partial y^2}\right) = 0 \qquad \text{(i)}$$

$$x = a \quad : \quad W = 0 \qquad M_x = 0 \quad \text{or} \quad \frac{\partial^2 W}{\partial x^2} = 0 \qquad \text{(ii)}$$

$$y \longrightarrow \pm\infty \quad : \quad W \longrightarrow 0 \qquad (3.2)$$

And $\quad y = 0 \quad : \quad \dfrac{\partial W}{\partial y} = 0 \qquad Q_y(d,0) = -\dfrac{P}{2}, \; P = 1 \quad \text{(iii)}$ $\qquad y > 0?$

$$\lim_{b \to 0+} \int_{d-\epsilon}^{d+\epsilon} Q_y(x,b)\, dx = -P/2$$

Condition (iii) assures that the deflection surface is symmetrical with respect to the x-axis and the shearing force Q_y disappears except at the loading point (d,0).

Assuming the deflection surface

$$W(x,y) = \sum_{n=1}^{\infty} Y_n(y) \sin \frac{n\pi x}{a} \qquad (3.3)$$

and substituting equation (3.3) into equation (3.1), gives the following expression for the n th-term

$$D_y Y_n'''' - 2H\left(\frac{n\pi}{a}\right)^2 Y_n'' + D_x\left(\frac{n\pi}{a}\right)^4 Y_n = 0 \qquad (n = 1, 2, 3, \cdots) \qquad (3.4)$$

Taking $Y_n(y) = e^{\lambda_n y}$ and substituting it into equation (3.4):

$$D_y \lambda_n^4 - 2H\left(\frac{n\pi}{a}\right)^2 \lambda_n^2 + D_x\left(\frac{n\pi}{a}\right)^4 = 0$$

The roots of the corresponding characteristic equation are:

$$\lambda_n = \pm\left(\frac{n\pi}{a}\right)\sqrt{\frac{H}{D_y} \pm \sqrt{\left(\frac{H}{D_y}\right)^2 - \frac{D_x}{D_y}}} \qquad (3.5)$$

The following three cases must be considered

(1) $H^2 - D_x D_y > 0$ $\qquad$ (2) $H^2 - D_x D_y = 0$ $\qquad$ (3) $H^2 - D_x D_y < 0$ $\qquad$ (3.6)

In the first case all the roots of equation (3.5) are real. However, in the second case, the characteristic equation has two double roots, and the function Y_n has the same form as in the case of an isotropic plate. In the third case, the roots of the characteristic equation are imaginary, and Y_n are expressed by trigonometric functions.

For the time being, the first case is considered. All the roots of the characteristic equation (3.6) are real. Considering the part of the plate with positive y and observing that the deflection w and its derivatives must vanish at large distances from the load (Boundary condition (3.2,ii)), only the negative roots can be retained.

Using the notation

$$\left.\begin{aligned} K_1 &= \sqrt{\frac{H}{D_y} + \sqrt{\left(\frac{H}{D_y}\right)^2 - \frac{D_x}{D_y}}} = \sqrt{\lambda + \sqrt{\lambda^2 - \mu^2}} \\ K_2 &= \sqrt{\frac{H}{D_y} - \sqrt{\left(\frac{H}{D_y}\right)^2 - \frac{D_x}{D_y}}} = \sqrt{\lambda - \sqrt{\lambda^2 - \mu^2}} \end{aligned}\right\} \qquad (3.7)$$

where

$$\lambda = \frac{H}{D_y}, \quad \mu^2 = \frac{D_x}{D_y} \quad \text{and} \quad \lambda^2 - \mu^2 > 0$$

The integral of equation (3.4) becomes

$$Y_n(y) = A_n e^{-\frac{n\pi K_1 y}{a}} + B_n e^{-\frac{n\pi K_2 y}{a}}$$

and expression (3.3) can be represented as follows:

$$W(x,y) = \sum_{n=1}^{\infty} \left(A_n e^{-\frac{n\pi K_1 y}{a}} + B_n e^{-\frac{n\pi K_2 y}{a}} \right) \sin \frac{n\pi x}{a} \qquad (y>0) \quad (3.8)$$

Since it is easily seen that the boundary condition (i),(ii) of (3.2) are satisfied already, the coefficients A_n and B_n must be determined by (3.2,iii).

From $\left(\frac{\partial W}{\partial y}\right)_{y=0} = 0$

$$A_n K_1 + B_n K_2 = 0$$

The other condition $(Q_y)_{y=d} = -\frac{1}{2}$ can be written as follows

$$-\frac{\partial}{\partial y}\left(D_y \frac{\partial^2 W}{\partial y^2} + H \frac{\partial^2 W}{\partial x^2} \right) = -\frac{1}{2}$$

Expanding the term of external load P=1 into a Fourier Sine series, that ie,

$$P = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a}$$

where $b_n = \frac{2P}{n\pi a} \sin \frac{n\pi d}{a}$

and substituting for w its expression (3.8) and using (3.7)

$$A_n K_2 + B_n K_1 = \frac{b_n}{2\left(\frac{n\pi}{a}\right)^3 \sqrt{D_x D_y}}$$

$$\text{or} \quad A_n K_2 + B_n K_1 = \frac{a^3}{2 n^3 \pi^3 \sqrt{D_x D_y}} \sin \frac{n\pi d}{a}$$

Thus A_n, B_n are determined,

$$\left.\begin{aligned} A_n &= \frac{K_2 a^2}{2 n^4 \pi^4 \mu D_y \sqrt{\lambda^2 - \mu^2}} \\ B_n &= -\frac{K_1 a^2}{2 n^4 \pi^4 \mu D_y \sqrt{\lambda^2 - \mu^2}} \end{aligned}\right\} \qquad (3.9)$$

combining equation (3.8) and equation (3.9) the solution w(x,y) becomes finally

$$W(x,y) = \frac{a^2}{2\pi^3 \mu D_y \sqrt{\lambda^2 - \mu^2}} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(K_2 e^{-\frac{n\pi K_1 y}{a}} - K_1 e^{-\frac{n\pi K_2 y}{a}} \right) \sin \frac{n\pi d}{a} \sin \frac{n\pi x}{a} \qquad (3.10)$$

$$(\lambda > \mu)$$

Differentiating the solution for w(x,y) in equation (3.10) the bending moments $M_x(x,y)$, $M_y(x,y)$, the twisting moment $M_{xy}(x,y)$ and shearing forces $Q_x(x,y)$, $Q_y(x,y)$ are easily derived.

$$M_x = -\left(D_x \frac{\partial^2 W}{\partial x^2} + D_1 \frac{\partial^2 W}{\partial y^2} \right)$$

$$= \frac{1}{2\pi\mu D_y\sqrt{\lambda^2-\mu^2}} \sum_{n=1}^{\infty} \frac{1}{n} \left[D_x \left(K_1 e^{-\frac{n\pi K_2 y}{a}} - K_2 e^{-\frac{n\pi K_1 y}{a}} \right) - D_1 K_1 K_2 \left(K_1 e^{-\frac{n\pi K_1 y}{a}} - K_2 e^{-\frac{n\pi K_2 y}{a}} \right) \right] \sin\frac{n\pi d}{a} \sin\frac{n\pi x}{a}$$

$$= \frac{1}{2\pi\sqrt{\lambda^2-\mu^2}} \sum_{n=1}^{\infty} \frac{1}{n} \left[\left(K_1\mu - K_2\left(\frac{D_1}{D_y}\right)\right) e^{-\frac{n\pi K_2 y}{a}} - \left(K_2\mu - K_1\left(\frac{D_1}{D_y}\right)\right) e^{-\frac{n\pi K_1 y}{a}} \right] \times \sin\frac{n\pi d}{a} \sin\frac{n\pi x}{a}$$

This series solution can be expressed in closed form by making use of the summation formulae listed in the Appendix.

$$M_x = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}} \left[\left(K_1\mu - K_2\left(\frac{D_1}{D_y}\right)\right) \log \frac{\cosh\frac{\pi K_2 y}{a} - \cos\frac{\pi}{a}(x+d)}{\cosh\frac{\pi K_2 y}{a} - \cos\frac{\pi}{a}(x-d)} - \left(K_2\mu - K_1\left(\frac{D_1}{D_y}\right)\right) \log \frac{\cosh\frac{\pi K_1 y}{a} - \cos\frac{\pi}{a}(x+d)}{\cosh\frac{\pi K_1 y}{a} - \cos\frac{\pi}{a}(x-d)} \right]$$

Similarly

$$M_y = -\left(D_1 \frac{\partial^2 W}{\partial x^2} + D_y \frac{\partial^2 W}{\partial y^2} \right)$$

$$= \frac{1}{2\pi\mu D_y\sqrt{\lambda^2-\mu^2}} \sum_{n=1}^{\infty} \frac{1}{n} \left[D_1 \left(K_1 e^{-\frac{n\pi K_2 y}{a}} - K_2 e^{-\frac{n\pi K_1 y}{a}} \right) - D_y K_1 K_2 \left(K_2 e^{-\frac{n\pi K_2 y}{a}} - K_1 e^{-\frac{n\pi K_1 y}{a}} \right) \right] \sin\frac{n\pi d}{a} \sin\frac{n\pi x}{a}$$

$$= \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}} \left[\left(\frac{K_1 D_1}{\mu D_y} - K_2 \right) \log \frac{\cosh\frac{\pi K_2 y}{a} - \cos\frac{\pi}{a}(x+d)}{\cosh\frac{\pi K_2 y}{a} - \cos\frac{\pi}{a}(x-d)} - \left(\frac{K_2 D_1}{\mu D_y} - K_1 \right) \log \frac{\cosh\frac{\pi K_1 y}{a} - \cos\frac{\pi}{a}(x+d)}{\cosh\frac{\pi K_1 y}{a} - \cos\frac{\pi}{a}(x-d)} \right]$$

$$M_{xy} = 2 D_{xy} \left(\frac{\partial^2 W}{\partial x \partial y} \right)$$

$$= \frac{D_{xy}}{\pi D_y\sqrt{\lambda^2-\mu^2}} \sum_{n=1}^{\infty} \frac{1}{n} \left(e^{-\frac{n\pi K_2 y}{a}} - e^{-\frac{n\pi K_1 y}{a}} \right) \sin\frac{n\pi d}{a} \sin\frac{n\pi x}{a}$$

$$= \frac{D_{xy}}{2\pi D_y\sqrt{\lambda^2-\mu^2}}\Big[\tan^{-1}\Big(\frac{\sin\frac{\pi}{a}(x+d)}{e^{\frac{\pi K_2 y}{a}}-\cos\frac{\pi}{a}(x+d)}\Big)-\tan^{-1}\Big(\frac{\sin\frac{\pi}{a}(x-d)}{e^{\frac{\pi K_2 y}{a}}-\cos\frac{\pi}{a}(x-d)}\Big)$$

$$-\tan^{-1}\Big(\frac{\sin\frac{\pi}{a}(x+d)}{e^{\frac{\pi K_1 y}{a}}-\cos\frac{\pi}{a}(x+d)}\Big)+\tan^{-1}\Big(\frac{\sin\frac{\pi}{a}(x-d)}{e^{\frac{\pi K_1 y}{a}}-\cos\frac{\pi}{a}(x-d)}\Big)\Big]$$

$$Q_x = -\frac{\partial}{\partial x}\Big(D_x\frac{\partial^2 W}{\partial x^2}+H\frac{\partial^2 W}{\partial y^2}\Big)$$

$$= \frac{1}{2a\sqrt{\lambda^2-\mu^2}}\sum_{n=1}^{\infty}\Big[(\mu K_1-\lambda K_2)\,e^{-\frac{n\pi K_2 y}{a}}-(\mu K_2-\lambda K_1)\,e^{-\frac{n\pi K_1 y}{a}}\Big]\sin\frac{n\pi d}{a}\sin\frac{n\pi x}{a}$$

$$= \frac{1}{8a\sqrt{\lambda^2-\mu^2}}\Big[(\mu K_1-\lambda K_2)\Big\{\frac{\sin\frac{\pi}{a}(x+d)}{\cosh\frac{\pi K_2 y}{a}-\cos\frac{\pi}{a}(x+d)}-\frac{\sin\frac{\pi}{a}(x-d)}{\cosh\frac{\pi K_2 y}{a}-\cos\frac{\pi}{a}(x-d)}\Big\}$$

$$-(\mu K_2-\lambda K_1)\Big\{\frac{\sin\frac{\pi}{a}(x+d)}{\cosh\frac{\pi K_1 y}{a}-\cos\frac{\pi}{a}(x+d)}-\frac{\sin\frac{\pi}{a}(x-d)}{\cosh\frac{\pi K_1 y}{a}-\cos\frac{\pi}{a}(x-d)}\Big\}\Big]$$

$$Q_y = -\frac{\partial}{\partial y}\Big(H\frac{\partial^2 W}{\partial x^2}+D_y\frac{\partial^2 W}{\partial y^2}\Big)$$

$$= \frac{1}{2a\sqrt{\lambda^2-\mu^2}}\sum_{n=1}^{\infty}\Big[\lambda\Big(e^{-\frac{n\pi K_2 y}{a}}-e^{-\frac{n\pi K_1 y}{a}}\Big)-\Big(K_2^2 e^{-\frac{n\pi K_2 y}{a}}-K_1^2 e^{-\frac{n\pi K_1 y}{a}}\Big)\Big]$$

$$\times\sin\frac{n\pi d}{a}\sin\frac{n\pi x}{a}$$

$$= \frac{1}{8a}\Big[\Big\{\frac{\sinh\frac{\pi K_2 y}{a}}{\cosh\frac{\pi K_2 y}{a}-\cos\frac{\pi}{a}(x+d)}-\frac{\sinh\frac{\pi K_2 y}{a}}{\cosh\frac{\pi K_2 y}{a}-\cos\frac{\pi}{a}(x-d)}\Big\}$$

$$-\Big\{\frac{\sinh\frac{\pi K_1 y}{a}}{\cosh\frac{\pi K_1 y}{a}-\cos\frac{\pi}{a}(x+d)}-\frac{\sinh\frac{\pi K_1 y}{a}}{\cosh\frac{\pi K_1 y}{a}-\cos\frac{\pi}{a}(x-d)}\Big\}\Big]$$

Turning to case (3) $H^2 - D_x D_y < 0$, or $\lambda < \mu$ the following abbreviations are introduced:

$$\left.\begin{aligned} K_3 &= \sqrt{\frac{\sqrt{D_x D_y} + H}{2 D_y}} = \sqrt{\frac{1}{2}(\mu + \lambda)} \\ K_4 &= \sqrt{\frac{\sqrt{D_x D_y} - H}{2 D_y}} = \sqrt{\frac{1}{2}(\mu - \lambda)} \end{aligned}\right\} \qquad (3.11)$$

Observing the following relations.

$$K_1 = K_3 + i K_4 \,, \qquad K_2 = K_3 - i K_4$$

the solution W(x,y) can be easily derived.

$$W(x,y) = \frac{a^2}{\pi^3 \mu D_y \sqrt{\mu^2 - \lambda^2}} \sum_{n=1}^{\infty} \frac{e^{-\frac{n\pi K_3 y}{a}}}{n^3} \left(K_4 \cos\frac{n\pi K_4 y}{a} + K_3 \sin\frac{n\pi K_4 y}{a} \right) \times \sin\frac{n\pi d}{a} \sin\frac{n\pi x}{a} \qquad (3.12)$$

For case (2) $H^2 - D_x D_y = 0$, or $\lambda = \mu$, λ approaches the μ in (3.12). Taking the limit, the solution W(x,y) becomes:

$$W(x,y) = \frac{a^2}{2\pi^3 D_y \sqrt{\lambda^3}} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(1 + \frac{n\pi\sqrt{\lambda} y}{a} \right) e^{-\frac{n\pi\sqrt{\lambda} y}{a}} \times \sin\frac{n\pi d}{a} \sin\frac{n\pi x}{a} \qquad (3.13)$$

Likewise, closed form expression for M_x, M_y, M_{xy}, Q_x and Q_y can be derived for both cases $\lambda < \mu$ and $\lambda = \mu$.

So far the point where the load P=1 is applied has been located on the x-axis. However, it is quite simple to derive the expressions for the general case. Assuming that the load P=1 is applied at (x,y) and the influence point is (u,v) then y is replaced by $\pm(v-y)$ (upper sign for $v \leq y$, lower sign for $v \geq y$)* $(x \pm d)$ is replaced by $u \pm x$. (Fig.3-2)

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*Hereafter this rule should be applied to any double sign, unless otherwise noted.

Furthermore, for simplicity, non-dimensional coordinates are introduced:

$$\frac{\pi x}{a} = \xi , \quad \frac{\pi y}{a} = \eta \; ; \quad \frac{\pi u}{a} = \alpha , \quad \frac{\pi v}{a} = \beta$$

Using the above notation, several important functions are defined in Table I.

Referring to these functions general expressions for the influence functions of an infinite strip are obtained.

(I) Deflection $W(\alpha,\beta,\xi,\eta)$

(i) $\lambda > \mu$

$$\frac{a^2}{2\pi^3 \mu D_y \sqrt{\lambda^2-\mu^2}} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(K_1 e^{\pm n K_2(\beta-\eta)} - K_2 e^{\pm n K_1(\beta-\eta)} \right) \sin n\alpha \sin n\xi$$

(ii) $\lambda < \mu$

$$\frac{a^2}{\pi^3 \mu D_y \sqrt{\mu^2-\lambda^2}} \sum_{n=1}^{\infty} \frac{e^{\pm n K_3(\beta-\eta)}}{n^3} \left(K_4 \cos n K_4(\beta-\eta) \mp K_3 \sin n K_4(\beta-\eta) \right) \sin n\alpha \sin n\xi \qquad (3.14)$$

(iii) $\lambda = \mu$

$$\frac{a^2}{2\pi^3 D_y \sqrt{\lambda^3}} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(1 \mp n\sqrt{\lambda}(\beta-\eta) \right) e^{\pm n\sqrt{\lambda}(\beta-\eta)} \sin n\alpha \sin n\xi$$

(II) Bending Moments $M_x(\alpha,\beta;\xi,\eta)$ $M_y(\alpha,\beta;\xi,\eta)$

(i) $\lambda > \mu$

$$\begin{cases} M_x = \dfrac{1}{8\pi\sqrt{\lambda^2-\mu^2}} \left[\left(K_1\mu - K_2\left(\frac{D_1}{D_y}\right)\right) R_2 - \left(K_2\mu - K_1\left(\frac{D_1}{D_y}\right)\right) R_1 \right] \\[2ex] M_y = \dfrac{1}{8\pi\sqrt{\lambda^2-\mu^2}} \left[\left(\frac{K_1 D_1}{\mu D_y} - K_2\right) R_2 - \left(\frac{K_2 D_1}{\mu D_y} - K_1\right) R_1 \right] \end{cases}$$

(ii) $\lambda < \mu$

$$\begin{cases} M_x = \dfrac{1}{8\pi\sqrt{\mu^2-\lambda^2}} \left[K_4\left(\mu + \frac{D_1}{D_y}\right) R_3 + 2K_3\left(\mu - \frac{D_1}{D_y}\right) R_4 \right] \\[2ex] M_y = \dfrac{1}{8\pi\sqrt{\mu^2-\lambda^2}} \left[K_4\left(\frac{D_1}{\mu D_y} + 1\right) R_3 + 2K_3\left(\frac{D_1}{\mu D_y} - 1\right) R_4 \right] \end{cases} \qquad (3.15)$$

(iii) $\lambda = \mu$

$$M_x = \frac{1}{8\pi}\left[\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\left(\frac{D_1}{D_y}\right)\right) R_5 \mp \left(\lambda - \frac{D_1}{D_y}\right)(\beta-\eta)\, S_1 \right]$$

$$M_y = \frac{1}{8\pi}\left[\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda^3}}\left(\frac{D_1}{D_y}\right)\right) R_5 \mp \left(\frac{1}{\lambda}\left(\frac{D_1}{D_y}\right)-1\right)(\beta-\eta)\, S_1 \right]$$

(III) Twisting Moment $M_{xy}(\alpha,\beta,\xi,\eta)$

(i) $\lambda > \mu$

$$= \frac{\pm D_{xy}}{2\pi D_y\sqrt{\lambda^2-\mu^2}} \left(R_7 - R_6 \right)$$

(ii) $\lambda < \mu$ (3.16)

$$= \frac{\pm D_{xy}}{4\pi D_y\sqrt{\mu^2-\lambda^2}}\, R_8$$

(iii) $\lambda = \mu$

$$= \frac{\pm(\beta-\eta)D_{xy}}{4\pi D_y \lambda}\, S_2$$

(IV) Shearing Forces $Q_x(\alpha,\beta;\xi,\eta)$, $Q_y(\alpha,\beta;\xi,\eta)$

(i) $\lambda > \mu$

$$Q_x = \frac{1}{8a\sqrt{\lambda^2-\mu^2}}\left[(K_1\mu - K_2\lambda)\, S_4 - (K_2\mu - K_1\lambda)\, S_3 \right]$$

$$Q_y = \frac{1}{8a}\left(S_5 + S_6 \right)$$

(ii) $\lambda < \mu$

$$Q_x = \frac{1}{8a}\left(K_3\, S_7 + K_4\, S_8 \right)$$

(3.17)

$$Q_y = \frac{1}{8a}\, S_9$$

(iii) $\lambda = \mu$

$$Q_x = \frac{\sqrt{\lambda}}{4a}\, S_2$$

$$Q_y = \frac{1}{4a}\, S_1$$

In the case of isotropic plate ($D_x=D_y=H=D$)

$$\lambda = \mu = 1 \qquad \frac{D_1}{D_y} = \nu \qquad \frac{D_{xy}}{D_y} = \frac{1-\nu}{2}$$

and the above expressions reduce as follows:

$$W = \frac{a^2}{2\pi^3 D}\sum_{n=1}^{\infty} \frac{1}{n^3}\left(1 \mp n(\beta-\eta)\right)e^{\pm n(\beta-\eta)} \sin n\alpha \sin n\xi$$

$$\left.\begin{matrix} M_x \\ M_y \end{matrix}\right\} = \frac{1}{8\pi}\left[(1+\nu)\log\frac{\cosh(\beta-\eta)-\cos(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} \pm (1-\nu)(\beta-\eta)\times \right.$$

$$\left.\left(\frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)}\right)\right]$$

upper sign for M_x
lower sign for M_y (3.18)

$$M_{xy} = \frac{\pm(1-\nu)(\beta-\eta)}{8\pi}\left[\frac{\sin(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)} - \frac{\sin(\alpha-\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)}\right]$$

$$Q_x = \frac{1}{4a}\left[\frac{\sin(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)} - \frac{\sin(\alpha-\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)}\right]$$

$$Q_y = \frac{\mp 1}{4a}\left[\frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)}\right]$$

CHAPTER IV

Influence Surfaces For The Semi-Infinite Plate Strips With Simply-Supported Parallel Edges

4.1 General Method to Obtain the Solutions

In Chapter III, the solution for the infinite plate strip was obtained. It will constitute the particular solution $W_0(\alpha,\beta;\xi,\eta)$ for solutions of semi-infinite plate strips or rectangular plates.

Taking the solution $W(\alpha,\beta;\xi,\eta)= W_0(\alpha,\beta;\xi,\eta)+ W_1(\alpha,\beta;\xi,\eta)$ with W_0 as the particular solution and W_1 as a general integral of the homogeneous plate equation, the sum must satisfy all the boundary conditions. The homogeneous solution for a plate strip is generally expressed as follows: $(\beta \geqq 0)$

$$W_1 = \begin{cases} \dfrac{a^2}{2\pi^3\mu D_y\sqrt{\lambda^2-\mu^2}}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}\left(A_n e^{-nK_2\beta}+B_n e^{-nK_1\beta}\right)\sin n\alpha & (\lambda>\mu) \\ \dfrac{a^2}{\pi^3\mu D_y\sqrt{\mu^2-\lambda^2}}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}\left(A_n\cos nK_4\beta+B_n\sin nK_4\beta\right)e^{-nK_3\beta}\sin n\alpha & (\lambda<\mu) \\ \dfrac{a^2}{2\pi^3 D_y\sqrt{\lambda^3}}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}\left(A_n+B_n(n\sqrt{\lambda}\beta)\right)e^{-n\sqrt{\lambda}\beta}\sin n\alpha & (\lambda=\mu) \end{cases} \tag{4.1}$$

Since the particular solution and homogeneous solutions satisfy the boundary conditions imposed on the parallel edges:

$$\alpha=0 \qquad W=0 \quad , \quad \frac{\partial^2 W}{\partial\alpha^2}=0$$

$$\alpha=\pi \qquad W=0 \quad , \quad \frac{\partial^2 W}{\partial\alpha^2}=0$$

the boundary condition of the third edge, that is, $\beta=0$ will determine the unknown constants A_n, B_n of the homogeneous solution (4.1)

In this dissertation 3 different cases are considered, that is, (a) simply supported (b) clamped (c) free edge. (Fig. 4-1)

4.2 Influence Functions for the Simply-Supported Strip

(i) The particular solution $W_o(\alpha,\beta;\xi,\eta)$ is rewritten here

$$W_o = \begin{cases} \dfrac{a^2}{2\pi^3\mu D_y\sqrt{\lambda^2-\mu^2}}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}\left(K_2 e^{\pm nK_1(\beta-\eta)} - K_1 e^{\pm nK_2(\beta-\eta)}\right)\sin n\alpha \sin n\xi & (\lambda>\mu) \\ \dfrac{a^2}{\pi^3\mu D_y\sqrt{\mu^2-\lambda^2}}\displaystyle\sum_{n=1}^{\infty}\frac{e^{\pm nK_3(\beta-\eta)}}{n^3}\left(K_4 \cos nK_4(\beta-\eta) \mp K_3 \sin nK_4(\beta-\eta)\right)\sin n\alpha \sin n\xi & (\lambda<\mu) \\ \dfrac{a^2}{2\pi^3 D_y\sqrt{\lambda^3}}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1 \mp n\sqrt{\lambda}(\beta-\eta)\right)e^{\pm n\sqrt{\lambda}(\beta-\eta)}\sin n\alpha \sin n\xi & (\lambda=\mu) \end{cases}$$

Assuming the solution $W(\alpha,\beta;\xi,\eta) = W_o(\alpha,\beta;\xi,\eta) + W_1(\alpha,\beta;\xi,\eta)$ and applying the boundary condition along the α axis:

$$\beta=0 \;:\; W=0 \;,\; \frac{\partial^2 W}{\partial\beta^2}=0 \qquad \text{(Fig. 4-1)}$$

Thus A_n, B_n are determined.

$$A_n = \begin{cases} -K_1 e^{-nK_2\eta}\sin n\xi & (\lambda>\mu) \\ -e^{-nK_3\eta}\left(K_3 \sin nK_4\eta + K_4 \cos nK_4\eta\right)\sin n\xi & (\lambda<\mu) \\ -(1+n\sqrt{\lambda}\eta)e^{-n\sqrt{\lambda}\eta}\sin n\xi & (\lambda=\mu) \end{cases} \tag{4.2}$$

$$B_n = \begin{cases} +K_2 e^{-nK_1\eta}\sin n\xi & (\lambda>\mu) \\ -e^{-nK_3\eta}\left(K_3 \cos nK_4\eta + K_4 \sin nK_4\eta\right)\sin n\xi & (\lambda<\mu) \\ -e^{-n\sqrt{\lambda}\eta}\sin n\xi & (\lambda=\mu) \end{cases}$$

Substituting equation (4.2) into equation (4.1), the general solution for the deflection are derived.

(I) Influence Functions for the deflection $W(\alpha,\beta;\xi,\eta)$

(i) $\lambda>\mu$

$$= \frac{a^2}{2\pi^3\mu D_y\sqrt{\lambda^2-\mu^2}}\sum_{n=1}^{\infty}\frac{1}{n^3}\Big[K_1 e^{\pm nK_2(\beta-\eta)} - K_2 e^{\pm nK_1(\beta-\eta)} - K_1 e^{-nK_2(\beta+\eta)} + K_2 e^{-nK_1(\beta+\eta)}\Big]\sin n\alpha \sin n\xi$$

(ii) $\lambda<\mu$

$$= \frac{a^2}{\pi^3\mu D_y\sqrt{\mu^2-\lambda^2}}\sum_{n=1}^{\infty}\frac{1}{n^3}\Big[e^{\pm nK_3(\beta-\eta)}\left(K_4 \cos nK_4(\beta-\eta) \mp K_3 \sin nK_4(\beta-\eta)\right) \tag{4.3}$$

$$- e^{-nK_3(\beta+\eta)}(K_4 \cos nK_4(\beta+\eta) + K_3 \sin nK_4(\beta+\eta))\,] \sin n\alpha \sin n\xi$$

(iii) $\lambda = \mu$

$$= \frac{a^2}{2\pi^3 D_y \sqrt{\lambda^3}} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[\{1 \mp n\sqrt{\lambda}(\beta-\eta)\} e^{\pm n\sqrt{\lambda}(\beta-\eta)} - \{1 + n\sqrt{\lambda}(\beta+\eta)\} e^{-n\sqrt{\lambda}(\beta+\eta)} \right] \times \sin n\alpha \sin n\xi \qquad (4.3)$$

(II) Influence Functions for the Bending Moments $M_x(\alpha,\beta;\xi,\eta)$, $M_y(\alpha,\beta;\xi,\eta)$

Bending moments, twisting moments, etc. can be derived by differentiating equation (4.3) and summing up the series solution into closed form expression as explained in Chapter III.

Here only the final results are summarized without showing the intermediate mathematical operations .

(i) $\lambda > \mu$

$$M_x = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}} \left[(K_1\mu - K_2(\tfrac{D_1}{D_y}))(R_2 - \bar{R}_2) - (K_2\mu - K_1(\tfrac{D_1}{D_y}))(R_1 - \bar{R}_1) \right]$$

$$M_y = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}} \left[(\frac{K_1 D_1}{\mu D_y} - K_2)(R_2 - \bar{R}_2) - (\frac{K_2 D_1}{\mu D_y} - K_1)(R_1 - \bar{R}_1) \right]$$

(ii) $\lambda < \mu$ (4.4)

$$M_x = \frac{1}{8\pi\sqrt{\mu^2-\lambda^2}} \left[K_4(\mu + \tfrac{D_1}{D_y})(R_3 - \bar{R}_3) + 2K_3(\mu - \tfrac{D_1}{D_y})(R_4 - \bar{R}_4) \right]$$

$$M_y = \frac{1}{8\pi\sqrt{\mu^2-\lambda^2}} \left[K_4(\frac{D_1}{\mu D_y} + 1)(R_3 - \bar{R}_3) + 2K_3(\frac{D_1}{\mu D_y} - 1)(R_4 - \bar{R}_4) \right]$$

(iii) $\lambda = \mu$

$$M_x = \frac{1}{8\pi\sqrt{\lambda}}\left[\left(\lambda + \frac{D_1}{D_y}\right)\left(R_5 - \bar{R}_5\right) - \left(\lambda - \frac{D_1}{D_y}\right)\left\{ \pm\sqrt{\lambda}(\beta-\eta)\, S_1 + \sqrt{\lambda}(\beta+\eta)\, \bar{S}_1 \right\} \right]$$

$$M_y = \frac{1}{8\pi\sqrt{\lambda^3}}\left[\left(\frac{D_1}{D_y} + \lambda\right)\left(R_5 - \bar{R}_5\right) + \left(\lambda - \frac{D_1}{D_y}\right)\left\{ \pm\sqrt{\lambda}(\beta-\eta)\, S_1 + \sqrt{\lambda}(\beta+\eta)\, \bar{S}_1 \right\} \right] \qquad (4.4)$$

(III) Influence Function for the Twisting Moments $M_{xy}(\alpha,\beta;\xi,\eta)$

(i) $\lambda > \mu$

$$= \frac{D_{xy}}{2\pi D_y\sqrt{\lambda^2-\mu^2}}\left[\pm\left(R_7 - R_6\right) + \left(\bar{R}_7 - \bar{R}_6\right) \right]$$

(ii) $\lambda < \mu$ (4.5)

$$= \frac{D_{xy}}{2\pi D_y\sqrt{\mu^2-\lambda^2}}\left(\pm R_8 + \bar{R}_8 \right)$$

(iii) $\lambda = \mu$

$$= \frac{D_{xy}}{4\pi\lambda D_y}\left[\mp(\beta-\eta)\, S_2 + (\beta+\eta)\, \bar{S}_2 \right]$$

IV. Influence Surface for Corner Reaction $r(\xi,\eta)$

In order to prevent the uplifting of the plate at the corners (for example, origin $\alpha=\beta=0$) concentrated corner reaction must exist acting downward. According to geometrical consideration and observing that the angle of the corner is equal to $\frac{\pi}{2}$ so that $M_{xy}=-M_{yx}$, it is concluded that

$$\gamma(\xi,\eta) = M_{xy}(0,0;\xi,\eta) - M_{yx}(0,0;\xi,\eta) = 2M_{xy}(0,0;\xi,\eta)$$

Therefore the corresponding influence surfaces are easily derived.

(i) $\lambda > \mu$

$$= \frac{4D_{xy}}{\pi D_y\sqrt{\lambda^2-\mu^2}}\left[\tan^{-1}\left(\frac{\sin\xi}{e^{K_2\eta}-\cos\xi}\right) - \tan^{-1}\left(\frac{\sin\xi}{e^{K_1\eta}-\cos\xi}\right)\right]$$

(ii) $\lambda < \mu$ (4.6)

$$= \frac{2D_{xy}}{\pi D_y\sqrt{\mu^2-\lambda^2}}\log\frac{\cosh K_3\eta - \cos(\xi - K_4\eta)}{\cosh K_3\eta - \cos(\xi + K_4\eta)}$$

(iii) $\lambda = \mu$

$$= \frac{2D_{xy}}{\pi\lambda D_y}\left(\frac{\sin\xi}{\cosh\sqrt{\lambda}\,\eta - \cos\xi}\right)$$

For the case of an isotropic plate, $\lambda=\mu=1$, the expressions simplify considerably:

$$W = \frac{a^2}{2\pi^3 D}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[\left\{1 \mp n(\beta-\eta)\right\}e^{\pm(\beta-\eta)} - \left\{1+n(\beta+\eta)\right\}e^{-n(\beta+\eta)}\right]\sin n\alpha\,\sin n\xi$$

$$\left.\begin{matrix} M_x \\ M_y \end{matrix}\right\} = \frac{1}{8\pi}\left[(1+\nu)\log\frac{\{\cosh(\beta-\eta)-\cos(\alpha+\xi)\}\{\cosh(\beta+\eta)-\cos(\alpha+\xi)\}}{\{\cosh(\beta-\eta)-\cos(\alpha-\xi)\}\{\cosh(\beta+\eta)-\cos(\alpha-\xi)\}}\right.$$

$$\pm(1-\nu)(\beta-\eta)\left\{\frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)}\right\} \quad (4.7)$$

$$\left.-(1-\nu)(\beta+\eta)\left\{\frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)} - \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha+\xi)}\right\}\right]$$

upper sign for M_x

lower sign for M_y

$$M_{xy} = \frac{(1-\nu)}{8\pi}\left[\pm(\beta-\eta)\left\{ \frac{\sin(\alpha-\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} - \frac{\sin(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)} \right\}\right.$$

$$\left. -(\beta+\eta)\left\{ \frac{\sin(\alpha-\xi)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)} - \frac{\sin(\alpha+\xi)}{\cosh(\beta+\eta)-\cos(\alpha+\xi)} \right\} \right] \qquad (4.7)$$

$$\gamma = \frac{(1-\nu)}{2\pi}\cdot\frac{\eta\sin\xi}{\cosh\eta-\cos\xi}$$

4.3 Influence Functions for the Clamped Edge

The corresponding boundary conditions are (Fig. 4-1)

$$\beta = 0 \quad : \quad W = 0 \quad , \quad \frac{\partial W}{\partial \beta} = 0$$

The general solutions can be derived by determining the two constants A_n, B_n.

(I) Influence Function for the Deflection $W(\alpha,\beta;\xi,\eta)$

(i) $\lambda > \mu$

$$= \frac{a^2}{2\pi^3\mu D_y\sqrt{\lambda^2-\mu^2}}\sum_{n=1}^{\infty}\frac{1}{n^3}\Big[K_1 e^{\pm nK_2(\beta-\eta)} - K_2 e^{\pm nK_1(\beta-\eta)} + \frac{K_1(K_1+K_2)}{K_2-K_1}e^{-nK_1(\beta+\eta)}$$

$$- \frac{2K_1K_2}{K_2-K_1}e^{-n(K_1\eta+K_2\beta)} - \frac{2K_1K_2}{K_2-K_1}e^{-n(K_2\eta+K_1\beta)} + \frac{K_2(K_1+K_2)}{K_2-K_1}e^{-nK_2(\beta+\eta)}\Big]$$

$$\times \sin n\alpha \sin n\xi$$

(ii) $\lambda < \mu$ (4.8)

$$= \frac{a^2}{\pi^3\mu D_y\sqrt{\mu^2-\lambda^2}}\sum_{n=1}^{\infty}\frac{1}{n^3}\Big[e^{\pm nK_3(\beta-\eta)}(K_4\cos nK_4(\beta-\eta) \mp K_3\sin nK_4(\beta-\eta))$$

$$+ \frac{2K_3}{K_4}e^{-nK_3(\beta+\eta)}(K_3\cos nK_4(\beta+\eta) - K_4\sin nK_4(\beta+\eta)) +$$

$$\frac{2(K_3^2+K_4^2)}{K_4}e^{-nK_3(\beta+\eta)}\cos nK_4(\beta-\eta)\Big]\sin n\alpha \sin n\xi$$

(iii) $\lambda = \mu$

$$= \frac{a^2}{2\pi^3 D_y\sqrt{\lambda^3}}\sum_{n=1}^{\infty}\frac{1}{n^3}\Big[(1 \mp n\sqrt{\lambda}(\beta-\eta))e^{\pm n\sqrt{\lambda}(\beta-\eta)} - (1+n\sqrt{\lambda}\eta)e^{-n\sqrt{\lambda}(\beta+\eta)}$$

$$-n\sqrt{\lambda}\beta(1+2n\sqrt{\lambda}\eta)e^{-n\sqrt{\lambda}(\beta+\eta)}\Big]\sin n\alpha \sin n\xi$$

Only the influence surfaces for bending moments $M_x(\alpha,\beta;\xi,\eta)$ $M_y(\alpha,\beta;\xi,\eta)$ will be derived in this case. The corner reactions disappear as one of the edges is clamped.

(II) Influence Function for the Bending Moments $M_x(\alpha,\beta;\xi,\eta)$ and $M_y(\alpha,\beta;\xi,\eta)$

(i) $\lambda > \mu$

$$M_x = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\Big[(K_1\mu - \frac{K_2 D_1}{D_y}) R_2 - (K_2\mu - \frac{K_1 D_1}{D_y}) R_1 + \frac{(K_1\mu - \frac{K_2 D_1}{D_y})}{K_2 - K_1}\Big\{ (K_1+K_2)\bar{R}_2 - 2K_2 R_{10}\Big\} - \frac{(K_2\mu - \frac{K_1 D_1}{D_y})}{K_2-K_1}\Big\{ 2K_1 R_{11} - (K_1+K_2)\bar{R}_1 \Big\}\Big]$$

$$M_y = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\Big[(\frac{K_1 D_1}{\mu D_y} - K_2) R_2 - (\frac{K_2 D_1}{\mu D_y} - K_1) R_1 + \frac{(\frac{K_1 D_1}{\mu D_y} - K_2)}{K_2 - K_1}\Big\{ (K_1+K_2)\bar{R}_2 - 2K_2 R_{10}\Big\} - \frac{(\frac{K_2 D_1}{\mu D_y} - K_1)}{K_2-K_1}\Big\{ 2K_1 R_{11} - (K_1+K_2)\bar{R}_1 \Big\}\Big]$$

(ii) $\lambda < \mu$

$$M_x = \frac{1}{8\pi\sqrt{\mu^2-\lambda^2}}\Big[K_4(\mu + \frac{D_1}{D_y}) R_3 - 2K_3(\mu - \frac{D_1}{D_y}) R_4 + \frac{K_3^2}{K_4}(\mu - \frac{D_1}{D_y})\bar{R}_3 + 2K_3(\mu + \frac{D_1}{D_y})\bar{R}_4 + \frac{1}{K_4}(\mu^2 - \frac{\lambda D_1}{D_y}) R_{12} - 4K_1(\frac{D_1}{D_y}) R_{13} \Big] \tag{4.9}$$

$$M_y = \frac{1}{8\pi\sqrt{\mu^2-\lambda^2}}\Big[K_4(\frac{D_1}{\mu D_y} + 1) R_3 - 2K_3(\frac{D_1}{\mu D_y} - 1) R_4 + \frac{K_3^2}{K_4}(\frac{D_1}{\mu D_y} - 1)\bar{R}_3 + 2K_3(\frac{D_1}{\mu D_y} + 1)\bar{R}_4 + \frac{1}{K_4}(\frac{D_1}{D_y} - \lambda) R_{12} - 4K_1 R_{13} \Big]$$

(iii) $\lambda = \mu$

$$M_x = \frac{1}{8\pi\sqrt{\lambda}}\Big[(\lambda + \frac{D_1}{D_y}) R_5 \mp (\lambda - \frac{D_1}{D_y})\sqrt{\lambda}(\beta-\eta) S_1 - (\lambda + \frac{D_1}{D_y})\bar{S}_1 + \Big\{ (\lambda + 3(\frac{D_1}{D_y}))\eta + (\lambda - \frac{D_1}{D_y})\beta \Big\}\bar{S}_1 - 2\lambda\beta\eta(\lambda - \frac{D_1}{D_y}) T_1 \Big]$$

$$M_y = \frac{1}{8\pi\sqrt{\lambda^3}}\Big[(\frac{D_1}{D_y} + \lambda)(R_5 - \bar{R}_5) \mp (\frac{D_1}{D_y} - \lambda)\sqrt{\lambda}(\beta-\eta) S_1 + \Big\{ (\frac{D_1}{D_y} - \lambda)\beta + (\frac{D_1}{D_y} + 3\lambda)\eta \Big\}\bar{S}_1 - 2\lambda\beta\eta(\frac{D_1}{D_y} - \lambda) T_1 \Big]$$

(iv) $\lambda = \mu = 1$ (isotropic)

$$W = \frac{a^2}{2\pi^3 D}\sum_{n=1}^{\infty}\frac{1}{n^3}\Big[\,(1\mp n(\beta-\eta))e^{\pm n(\beta-\eta)} - (1+n\eta)e^{-n(\beta+\eta)} - n\beta(1+2n\eta)e^{-n(\beta+\eta)}\Big]\sin n\alpha \sin n\xi$$

$$M_x = \frac{1}{8\pi}\Big[(1+\nu)\log\frac{\cosh(\beta-\eta)-\cos(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} - (1-\nu)(\beta-\eta)\Big\{\frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)}\Big\} - (1+\nu)\log\frac{\cosh(\beta+\eta)-\cos(\alpha+\xi)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)}$$
$$-((1+3\nu)\eta+(1-\nu)\beta)\Big\{\frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha+\xi)} - \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)}\Big\}$$
$$-2(1-\nu)\beta\eta\Big\{\frac{\cosh(\beta+\eta)\cos(\alpha-\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha-\xi))^2} - \frac{\cosh(\beta+\eta)\cos(\alpha+\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha+\xi))^2}\Big\}\Big]$$

$$M_y = \frac{1}{8\pi}\Big[(1+\nu)\log\frac{\cosh(\beta-\eta)-\cos(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} + (1-\nu)(\beta-\eta)\Big\{\frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)}\Big\} - (1+\nu)\log\frac{\cosh(\beta+\eta)-\cos(\alpha+\xi)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)}$$
$$-((1+3\nu)\eta-(1-\nu)\beta)\Big\{\frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)}\Big\}$$
$$+2(1-\nu)\beta\eta\Big\{\frac{\cosh(\beta+\eta)\cos(\alpha-\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha-\xi))^2} - \frac{\cosh(\beta+\eta)\cos(\alpha+\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha+\xi))^2}\Big\}\Big]$$

(4.10)

4.4 Influence Functions for the Free Edge

The boundary conditionsfor this case are (Fig. 4-1)

$$\beta = 0 \quad : \qquad M_y = -\left(D_1 \frac{\partial^2 W}{\partial x^2} + D_y \frac{\partial^2 W}{\partial \beta^2} \right) = 0$$

$$\bar{V}_y = -\left((H + 2D_{xy}) \frac{\partial^3 W}{\partial \alpha^2 \partial \beta} + D_y \frac{\partial^3 W}{\partial \beta^3} \right) = 0$$

The general solution for $W(\alpha,\beta;\xi,\eta)$ is obtained as follows.

(I) Influence Functions for the deflection $W(\alpha,\beta;\xi,\eta)$

(i) $\lambda > \mu$

$$= \frac{a^2}{2\pi^3 \mu D_y \sqrt{\lambda^2 - \mu^2}} \sum_{n=1}^{\infty} \frac{1}{n^3} \Big[K_1 e^{\pm n K_2(\beta-\eta)} - K_2 e^{\pm n K_1(\beta-\eta)} + \frac{K_1(K_1+K_2)M}{(K_2-K_1)L} e^{-nK_2(\beta+\eta)}$$

$$- \frac{2K_1K_2N}{(K_2-K_1)L} e^{-n(K_1\eta + K_2\beta)} - \frac{2K_1K_2N}{(K_2-K_1)L} e^{-n(K_1\beta+K_2\eta)} + \frac{K_2(K_1+K_2)M}{(K_2-K_1)L} e^{-nK_1(\beta+\eta)} \Big]$$

$$\times \sin n\alpha \sin n\xi$$

(ii) $\lambda < \mu$

$$= \frac{a^2}{\pi^3 \mu D_y \sqrt{\mu^2 - \lambda^2}} \sum_{n=1}^{\infty} \frac{1}{n^3} \Big[e^{\pm nK_3(\beta-\eta)} \big(K_4 \cos nK_4(\beta-\eta) \mp K_3 \sin nK_4(\beta-\eta) \big) \tag{4.11}$$

$$+ \frac{K_3 M}{K_4 L} e^{-nK_3(\beta+\eta)} \big(K_3 \cos nK_4(\beta+\eta) - K_4 \sin nK_4(\beta+\eta) \big) - \frac{N(K_3^2 + K_4^2)}{K_4 L} e^{-nK_3(\beta+\eta)} \times$$

$$\cos nK_4(\beta-\eta) \Big] \sin n\alpha \sin n\xi$$

(iii) $\lambda = \mu$

$$= \frac{a^2}{2\pi^3 D_y \sqrt{\lambda^3}} \sum_{n=1}^{\infty} \frac{1}{n^3} \Big[(1 \mp n\sqrt{\lambda}(\beta-\eta)) e^{\pm n\sqrt{\lambda}(\beta-\eta)} + \frac{(2H^2 - 2D_{xy}H + D_{xy}^2) + D_{xy}^2 (n\sqrt{\lambda}\eta)}{D_{xy}(2H - D_{xy})} e^{-n\sqrt{\lambda}(\beta+\eta)}$$

$$+ \frac{D_{xy}(1 + 2n\sqrt{\lambda}\eta)(n\sqrt{\lambda}\beta)}{2H - D_{xy}} e^{-n\sqrt{\lambda}(\beta+\eta)} \Big] \sin n\alpha \sin n\xi$$

where

$$L = 4D_{xy}\sqrt{D_x D_y} - D_1^2 + D_x D_y$$

$$M = -4D_{xy}\sqrt{D_x D_y} - D_1^2 + D_x D_y$$

$$N = 4D_1 D_{xy} + D_1^2 - D_x D_y$$

(II) Influence Functions for Bending Moment $M_x(\alpha,\beta;\xi,\eta)$, $M_y(\alpha,\beta;\xi,\eta)$

(i) $\lambda > \mu$

$$M_x = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\Big[(K_1\mu - \frac{K_2 D_1}{D_y})\Big\{R_2 + \frac{M(K_1+K_2)}{L(K_2-K_1)}\bar{R}_2 - \frac{2K_2N}{L(K_2-K_1)}R_{10}\Big\} - (K_2\mu - \frac{K_1D_1}{D_y})\Big\{R_1 - \frac{M(K_1+K_2)}{L(K_2-K_1)}\bar{R}_1 + \frac{2K_1N}{L(K_2-K_1)}R_{11}\Big\}\Big]$$

$$M_y = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\Big[(\frac{D_1}{K_1D_y} - K_2)\Big\{R_2 + \frac{M(K_1+K_2)}{L(K_2-K_1)}\bar{R}_2 - \frac{2K_2N}{L(K_2-K_1)}R_{10}\Big\} - (\frac{D_1}{K_1D_y} - K_1)\Big\{R_1 - \frac{M(K_1+K_2)}{L(K_2-K_1)}\bar{R}_1 + \frac{2K_1N}{L(K_2-K_1)}R_{11}\Big\}\Big]$$

(ii) $\lambda < \mu$

$$M_x = \frac{1}{8\pi\sqrt{\mu^2-\lambda^2}}\Big[K_4(\mu + \frac{D_1}{D_y})R_3 - 2K_3(\mu - \frac{D_1}{D_y})R_4 + \frac{K_3^2M}{K_4L}(\mu - \frac{D_1}{D_y})\bar{R}_3 + \frac{2K_3M}{L}(\mu + \frac{D_1}{D_y})\bar{R}_4 - \frac{N}{K_4L}(\mu^2 - \lambda(\frac{D_1}{D_y}))R_{12} - \frac{4K_3D_1N}{D_yL}R_{13}\Big]$$

$$M_y = \frac{1}{8\pi\sqrt{\mu^2-\lambda^2}}\Big[K_4(\frac{D_1}{\mu D_y} + 1)R_3 - 2K_3(\frac{D_1}{\mu D_y} - 1)R_4 + \frac{K_3^2M}{K_4L}(\frac{D_1}{\mu D_y} - 1)\bar{R}_3 + \frac{2K_3M}{L}(\frac{D_1}{\mu D_y} + 1)\bar{R}_4 + \frac{N}{K_4L}(\lambda - \frac{D_1}{D_y})R_{12} - \frac{4K_3N}{L}R_{13}\Big]$$

(iii) $\lambda = \mu$

$$M_x = \frac{1}{8\pi\sqrt{\lambda}}\Big[(\lambda + \frac{D_1}{D_y})R_5 \pm (\lambda - \frac{D_1}{D_y})\sqrt{\lambda}(\beta-\eta)S_1 + \frac{(2H^2 - 2D_{xy}H + D_{xy}^2)(\lambda - \frac{D_1}{D_y}) + 2(\frac{D_1}{D_y})D_{xy}^2}{D_{xy}(2H - D_{xy})}\bar{R}_5 + \frac{\sqrt{\lambda}D_{xy}}{2H - D_{xy}}((\lambda + \frac{3D_1}{D_y})\eta - (\lambda - \frac{D_1}{D_y})\beta)\bar{S}_1 + \frac{2D_{xy}(\lambda - \frac{D_1}{D_y})}{2H - D_{xy}}\lambda\beta\eta T_1\Big]$$

$$M_y = \frac{1}{8\pi\sqrt{\lambda^3}}\Big[(\frac{D_1}{D_y} + \lambda)R_5 \pm (\frac{D_1}{D_y} - \lambda)\sqrt{\lambda}(\beta-\eta)S_1 + \frac{(2H^2 - 2D_{xy}H + D_{xy}^2)(\frac{D_1}{D_y} - \lambda) + 2\lambda D_{xy}^2}{D_{xy}(2H - D_{xy})}R_5 - \frac{\sqrt{\lambda}D_{xy}}{2H - D_{xy}}((\frac{D_1}{D_y} + \lambda)\eta + (\frac{D_1}{D_y} - \lambda)\beta)\bar{S}_1 + \frac{2D_{xy}(\frac{D_1}{D_y} - \lambda)}{2H - D_{xy}}\lambda\beta\eta T_1\Big]$$

(4.12)

(iv) $\lambda = \mu = 1$ (isotropic)

$$W = \frac{a^2}{2\pi^3 D}\sum_{n=1}^{\infty}\frac{1}{n^3}\Big[\,(1 \mp n(\beta-\eta))e^{\pm n(\beta-\eta)} + \Big\{\frac{5+2\nu+\nu^2}{(3+\nu)(1-\nu)} + \frac{1-\nu}{3+\nu}\times (n(\beta+\eta)+2n^2\beta\eta)\Big\}e^{-n(\beta+\eta)}\Big]\sin n\alpha \sin n\xi$$

$$M_x = \frac{1}{8\pi}\Big[\,(1+\nu)\log\frac{\cosh(\beta-\eta)-\cos(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} + (1-\nu)(\beta-\eta)\Big\{\frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)}\Big\} + \frac{(5+\nu)(1-\nu)}{3+\nu}\times$$
$$\log\frac{\cosh(\beta+\eta)-\cos(\alpha+\xi)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)} - \frac{1-\nu}{3+\nu}((1+3\nu)\eta+(1-\nu)\beta)\Big\{\frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)} - \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha+\xi)}\Big\} + \frac{2(1-\nu)^2\beta\eta}{3+\nu}\times$$
$$\Big\{\frac{\cosh(\beta+\eta)\cos(\alpha-\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha-\xi))^2} - \frac{\cosh(\beta+\eta)\cos(\alpha+\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha+\xi))^2}\Big\}\Big] \tag{4.13}$$

$$M_y = \frac{1}{8\pi}\Big[\,(1+\nu)\log\frac{\cosh(\beta-\eta)-\cos(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} - (1-\nu)(\beta-\eta)\Big\{\frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)}\Big\} + \frac{(5+\nu)(1-\nu)}{3+\nu}\times$$
$$\log\frac{\cosh(\beta+\eta)-\cos(\alpha+\xi)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)} - \frac{1-\nu}{3+\nu}((1+3\nu)\eta+(1-\nu)\beta)\Big\{\frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)} - \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha+\xi)}\Big\} - \frac{2(1-\nu)^2\beta\eta}{3+\nu}\times$$
$$\Big\{\frac{\cosh(\beta+\eta)\cos(\alpha-\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha-\xi))^2} - \frac{\cosh(\beta+\eta)\cos(\alpha+\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha+\xi))^2}\Big\}\Big]$$

(4.13)

From the practical point of view, the most general case is the case where the third edge is elastically supported. The corresponding boundary conditions of the third edge are:

u=0

$$M_y = EI_\omega \frac{\partial^5 W}{\partial u^4 \partial v} - GK_T \frac{\partial^3 W}{\partial u^2 \partial v}$$

or

$$D_1 \frac{\partial^2 W}{\partial u^2} + D_y \frac{\partial^2 W}{\partial v^2} = GK_t \frac{\partial^3 W}{\partial u^2 \partial v} - EI_\omega \frac{\partial^5 W}{\partial u^4 \partial v}$$

$$V_y = -EI \frac{\partial^4 W}{\partial u^4}$$

or

$$\frac{\partial}{\partial v}\left((2H - D_1)\frac{\partial^2 W}{\partial u \partial v} + D_y \frac{\partial^2 W}{\partial v^2}\right) = -EI \frac{\partial^4 W}{\partial u^4}$$

where

EI: Bending stiffness of the edge beam.

GK_t: St. Venant's torsional rigidity of the edge beam.

EI_W: Warping rigidity of the edge beam.

The solution can be obtained in the same way as illustrated before, though it may be very complicated.

Three cases treated in this chapter are actually the special cases of this particular problem.

CHAPTER V

Influence Function for a Rectangular Plate With Simply Supported Edges

5.1 Method of Solution

The influence surface for the deflection of a rectangular plate with simply supported edges will be derived in double Fourier series form (Navier's Solution) and thereafter it will be converted into a single series form (Levy's solution).

It turns out to be a simpler way to find the solution than the ordinary method illustrated in Chapter IV.

As far as the influence surfaces for bending moments M_x and M_y are concerned, influence functions can be expressed in terms of Jacobi's elliptic functions in this particular case.

Making the length of one side, say b, infinitely large, the solutions for semi-infinite as well as infinite plate strip will be derived again with the aid of Fourier's integrals.

5.2 Navier's Solution for a Rectangular Plate with Simply Supported Edges

Consider a rectangular plate whose sides are a and b respectively (Fig. 5.1). The concentrated load P=1 acting at (x,y) can be expressed in the following double Fourier Series:

$$P(u,v) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi u}{a} \sin\frac{n\pi v}{b} \qquad (5.1)$$

where

$$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b P(u,v) \sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b}\, du\, dv$$

$$= \frac{4}{ab}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Assuming the solution W(u,v;x,y)

$$W(u,v;x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} \sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b} \qquad (5.2)$$

It is easily seen that all boundary conditions are satisfied by equation (5.2). Substituting equation (5.2) into the original partial differential equation, b_{mn} can be determined.

$$b_{mn} = \frac{\frac{4}{ab}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\left(\frac{m\pi}{a}\right)^4 D_x + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 H + \left(\frac{n\pi}{b}\right)^4 D_y} \tag{5.3}$$

Therefore the solution can be written as follows:

$$W = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varphi_{mn}(x,y)\,\varphi_{mn}(u,v)}{\lambda_{mn}^2} \tag{5.4}$$

where

$$\varphi_{mn}(u,v) = \frac{2}{\sqrt{ab}}\sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b}$$

$$\lambda_{mn}^2 = D_x\left(\frac{m\pi}{a}\right)^4 + 2H\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_y\left(\frac{n\pi}{b}\right)^4$$

This is the solution for rectangular or orthotropic plate corresponding to Navier's solution for an isotropic plate.

5.3 Transformation of Navier's Solution into Levy's Solution

Navier's solution can be transformed into Levy's solution with the aid of the following summation formulae (See (2) p.198 Appendix)

$$\sum_{n=1}^{\infty}\frac{\cos nx}{(n^2+K^2)^2} = -\frac{1}{2K^4} + \frac{\pi^2}{4K^2}\cdot\frac{\cosh Kx}{\sinh^2 K\pi} + \frac{\pi}{4K^3}\cdot\frac{\cosh K(\pi-x)}{\sinh K\pi} + \frac{\pi x}{4K^2}\cdot\frac{\sinh K(\pi-x)}{\sinh K\pi} \quad (0 \leqq x \leqq \pi) \tag{5.5}$$

$$\sum_{n=1}^{\infty}\frac{\cos nx}{(n^2+K^2)(n^2+K'^2)} = \frac{1}{K^2-K'^2}\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2+K^2} - \frac{\cos nx}{n^2+K'^2}\right)$$

$$= \frac{1}{K^2-K'^2}\left[\frac{\pi}{2K}\cdot\frac{\cosh K(x-\pi)}{\sinh K\pi} - \frac{1}{2K^2} - \frac{\pi}{2K'}\cdot\frac{\cosh K'(x-\pi)}{\sinh K'\pi} + \frac{1}{2K'^2}\right] \quad (0 \leqq x \leqq \pi)$$

Taking the case of $\lambda > \mu$, the transformation will be illustrated briefly. From equation (5.4):

$$W = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\left(\frac{m\pi}{a}\right)^4 D_x + 2H\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4 D_y}$$

$$= \frac{2}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi u}{a}\sin\frac{m\pi x}{a}}{\left(\frac{\pi}{b}\right)^4 D_y\left[n^4 + 2\lambda\left(\frac{mb}{a}\right)^2 n^2 + \mu^2\left(\frac{mb}{a}\right)^4\right]}\left\{\cos\frac{n\pi}{b}(y-v) - \cos\frac{n\pi}{b}(y+v)\right\} \qquad (5.6)$$

$$= \frac{2}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi u}{a}\sin\frac{n\pi x}{a}}{\left(\frac{\pi}{b}\right)^4 D_y\left(n^2 + \left(\frac{mbK_1}{a}\right)^2\right)\left(n^2 + \left(\frac{mbK_2}{a}\right)^2\right)}\left\{\cos\frac{n\pi}{b}(y-v) - \cos\frac{n\pi}{b}(y+v)\right\}$$

where K_1, K_2 are the constants defined in Chapter III.

Applying the second formula of equation (5.5) to the series of y in equation (5.6):

$$W = \frac{1}{ab\left(\frac{\pi}{b}\right)^4 D_y}\sum_{m=1}^{\infty}\frac{\sin\frac{m\pi u}{a}\sin\frac{m\pi x}{a}}{\frac{m^2b^2}{a^2}\left(K_1^2 - K_2^2\right)}\left\{\frac{\pi}{\left(\frac{mbK_2}{a}\right)}\cdot\frac{\cosh\frac{mbK_2}{a}\left(\frac{\pi}{b}(y-v)-\pi\right)}{\sinh\frac{m\pi bK_2}{a}}\right.$$

$$-\frac{\pi}{\left(\frac{mbK_1}{a}\right)}\cdot\frac{\cosh\frac{mbK_1}{a}\left(\frac{\pi}{b}(y-v)-\pi\right)}{\sinh\frac{m\pi bK_1}{a}} - \frac{\pi}{\left(\frac{mbK_2}{a}\right)}\cdot\frac{\cosh\frac{mbK_2}{a}\left(\frac{\pi}{b}(y+v)-\pi\right)}{\sinh\frac{m\pi bK_2}{a}}$$

$$\left.+\frac{\pi}{\left(\frac{mbK_1}{a}\right)}\cdot\frac{\cosh\frac{mbK_1}{a}\left(\frac{\pi}{b}(y+v)-\pi\right)}{\sinh\frac{m\pi bK_1}{a}}\right\} \qquad (y \geq v)$$

This solution for an orthotropic rectangular plate corresponds to Levy's solution for an isotropic plate. Without repeating the mathematical operation, the results obtained are summarized as follows. Again non-dimensional coordinates as defined in Chapter IV are employed with another new parameter (Fig. 5-2).

$$\gamma = \frac{\pi b}{a}$$

(I) Influence Functions for the Deflection $W(\alpha,\beta;\xi,\eta)$

(i) $\lambda > \mu$

$$= \frac{a^2}{2\pi^3\mu D_y\sqrt{\lambda^2-\mu^2}}\sum_{n=1}^{\infty}\frac{1}{n^3}\left\{\frac{\cosh nK_2(\beta-\eta\mp\gamma)}{K_2\sinh nK_2\gamma}-\frac{\cosh nK_1(\beta-\eta\mp\gamma)}{K_1\sinh nK_1\gamma}\right\}\sin n\alpha\sin n\xi$$

(ii) $\lambda < \mu$

$$= \frac{a^2}{\pi^3\mu D_y\sqrt{\mu^2-\lambda^2}}\sum_{n=1}^{\infty}\frac{\sin n\alpha\sin n\xi}{n^3(\sinh^2(nK_3\gamma)+\sin^2(nK_4\gamma))}\Big[\Big\{\cosh nK_3(\beta-\eta\mp\gamma)\cos nK_4(\beta-\eta\mp\gamma)$$

$$-\cos nK_3(\beta+\eta-\gamma)\cos nK_4(\beta+\eta-\gamma)\Big\}(K_4\sinh nK_3\gamma\cos nK_4\gamma+$$

$$K_3\cosh nK_3\gamma\sinh nK_4\gamma)-\Big\{\sinh nK_3(\beta-\eta\mp\gamma)\sinh nK_4(\beta-\eta\mp\gamma)$$

$$-\sinh nK_3(\beta+\eta-\gamma)\sin nK_4(\beta+\eta-\gamma)\Big\}(K_3\sinh nK_3\gamma\cos nK_4\gamma-K_3\cosh nK_3\gamma\sin nK_4\gamma)\Big]$$

(iii) $\lambda = \mu$ (5.7)

$$= \frac{a^2}{\pi^3 D_y\sqrt{\lambda^3}}\sum_{n=1}^{\infty}\frac{1}{n^3}\Big[\frac{\sinh n\sqrt{\lambda}(\gamma-\beta)}{\sinh n\sqrt{\lambda}\gamma}(\sinh n\sqrt{\lambda}\eta-n\sqrt{\lambda}\eta\cosh n\sqrt{\lambda}\eta)+$$

$$\frac{\sinh n\sqrt{\lambda}\eta}{\sinh n\sqrt{\lambda}\gamma}\Big(n\sqrt{\lambda}\beta\cosh n\sqrt{\lambda}(\gamma-\beta)-\frac{n\sqrt{\lambda}\gamma\sinh n\sqrt{\lambda}\beta}{\sinh n\sqrt{\lambda}\gamma}\Big]\sin n\alpha\sin n\xi$$

$(\beta \geqq \eta)$

$$= \frac{a^2}{\pi^3 D_y\sqrt{\lambda^3}}\sum_{n=1}^{\infty}\frac{1}{n^3}\Big[\frac{\sinh n\sqrt{\lambda}\beta}{\sinh n\sqrt{\lambda}\gamma}(\sinh n\sqrt{\lambda}(\gamma-\eta)-n\sqrt{\lambda}(\gamma-\eta)\cosh n\sqrt{\lambda}(\beta-\eta))$$

$$+\frac{\sinh n\sqrt{\lambda}(\gamma-\eta)}{\sinh n\sqrt{\lambda}\gamma}\Big(n\sqrt{\lambda}(\gamma-\beta)\cosh n\sqrt{\lambda}\beta-\frac{n\sqrt{\lambda}\gamma\sinh n\sqrt{\lambda}(\gamma-\beta)}{\sinh n\sqrt{\lambda}\gamma}\Big]\sin n\alpha\sin n\xi$$

$(\beta \leqq \eta)$

5.4 Representation of M_x, M_y -Influence Functions by Jacobi's ϑ-Functions

Differentiating the solution equation (5.7) with respect to α, β twice, the influence function for the bending moments can be obtained in single series form.

However, the theory of elliptic function shows that such series can be expressed in terms of Jacobi's ϑ-functions.[18] The illustration will be made here only in case of $\lambda > \mu$.

Assuming $\beta \geqq \eta$ and carrying out the differentiation of $W(\alpha,\beta;\xi,\eta)$ with respect to α, β and forming $M_x(\alpha,\beta;\xi,\eta)$

$$\begin{aligned} M_x &= -\left(D_x \frac{\partial^2 W}{\partial \alpha^2} + D_1 \frac{\partial^2 W}{\partial \beta^2} \right) \\ &= \frac{1}{4\pi\sqrt{\lambda^2-\mu^2}} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \left(\frac{\mu^2}{K_2} - \frac{K_1 D_1}{D_y} \right) \left(\frac{\cosh nK_2(\beta-\eta-\gamma) - \cosh nK_2(\beta+\eta-\gamma)}{\sinh nK_2\gamma} \right) \right. \\ &\quad \left. - \left(\frac{1}{K_1} - \frac{K_1 D_1}{D_y} \right) \left(\frac{\cosh nK_1(\beta-\eta-\gamma) - \cosh nK_1(\beta+\eta-\gamma)}{\sinh nK_1\gamma} \right) \right\} \left(\cos n(\alpha-\xi) - \cos n(\alpha+\xi) \right) \end{aligned} \tag{5.8}$$

Using the relation:

$$\cosh x = \cos ix$$

new complex variables are introduced:

$$u + iK_1 v = a\,\zeta_0 \qquad x + iK_1 y = a\,\zeta \tag{5.9}$$

$$u + iK_2 v = a\,\zeta_0' \qquad x + iK_2 y = a\,\zeta'$$

so that

$$\alpha + iK_1\beta = \pi\,\zeta_0 \qquad \xi + iK_1\eta = \pi\,\zeta$$

$$\alpha + iK_2\beta = \pi\,\zeta_0' \qquad \xi + iK_2\eta = \pi\,\zeta'$$

Using the new notation equation (5.9), equation (5.8) is rewritten:

$$M_x = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\sum_{n=1}^{\infty}\frac{1}{n}\Big[\frac{\left(\frac{\mu^2}{K_2}-\frac{K_2D_1}{D_y}\right)}{\sinh nK_2\gamma}\Big\{\cos n\pi\left(\bar{\zeta}'-\bar{\zeta}_0'+\frac{iK_2\gamma}{\pi}\right)+\cos n\pi\left(\zeta'-\zeta_0'-\frac{iK_2\gamma}{\pi}\right)$$
$$-\cos n\pi\left(\bar{\zeta}'-\zeta_0'+\frac{iK_2\gamma}{\pi}\right)-\cos n\pi\left(\zeta'-\bar{\zeta}_0'-\frac{iK_2\gamma}{\pi}\right)-\cos n\pi\left(\bar{\zeta}'+\zeta_0'+\frac{iK_2\gamma}{\pi}\right)$$
$$-\cos n\pi\left(\zeta'+\bar{\zeta}_0'-\frac{iK_2\gamma}{\pi}\right)+\cos n\pi\left(\bar{\zeta}'+\bar{\zeta}_0'+\frac{iK_2\gamma}{\pi}\right)+\cos n\pi\left(\zeta'+\zeta_0'-\frac{iK_2\gamma}{\pi}\right)\Big\}$$
$$-\frac{\left(\frac{\mu^2}{K_1}-\frac{K_1D_1}{D_y}\right)}{\sinh nK_1\gamma}\Big\{\cos n\pi\left(\bar{\zeta}-\bar{\zeta}_0+\frac{iK_1\gamma}{\pi}\right)+\cos n\pi\left(\zeta-\zeta_0-\frac{iK_1\gamma}{\pi}\right)$$
$$-\cos n\pi\left(\bar{\zeta}-\zeta_0+\frac{iK_1\gamma}{\pi}\right)-\cos n\pi\left(\zeta-\bar{\zeta}_0-\frac{iK_1\gamma}{\pi}\right)-\cos n\pi\left(\bar{\zeta}+\zeta_0+\frac{iK_1\gamma}{\pi}\right)$$
$$-\cos n\pi\left(\zeta+\bar{\zeta}_0-\frac{iK_1\gamma}{\pi}\right)+\cos n\pi\left(\bar{\zeta}+\bar{\zeta}_0+\frac{iK_1\gamma}{\pi}\right)+\cos n\pi\left(\zeta+\zeta_0-\frac{iK_1\gamma}{\pi}\right)\Big\}\Big]$$

The theory of elliptic function furnsihes the following mathematical relations:

$$\log \vartheta_0(z) = \sum_{n=1}^{\infty}\log(1-q^{2n}) - 2\sum_{n=1}^{\infty}\frac{q^n\cos 2n\pi z}{n(1-q^{2n})} \qquad (5.10)$$

and

$$\vartheta_0\left(z\pm\tfrac{1}{2}\tau\right) = \pm i\, q^{-\frac{1}{4}} e^{\mp i\pi z}\vartheta_1(z)$$

where

$$q = e^{i\pi\tau} \qquad \tau = \text{the period of } \vartheta_0(z)$$

Using equation (5.10)

$$\sum_{n=1}^{\infty}\frac{\cos n\pi\left(\bar{\zeta}'-\bar{\zeta}_0'+\frac{iK_2\gamma}{\pi}\right)}{n\sinh nK_2\gamma} = \sum_{n=1}^{\infty}\frac{2q^{2n}}{n(1-q^{2n})}\cos n\pi\left(\bar{\zeta}'-\bar{\zeta}_0'+\frac{iK_2\gamma}{\pi}\right)$$
$$= \frac{1}{2}\sum_{n=1}^{\infty}\log(1-q^{2n}) - \frac{1}{2}\log\vartheta_0\left(\frac{\bar{\zeta}'-\bar{\zeta}_0'+\tau'}{2}\right) \qquad (5.11)$$

where

$$\tau' = \frac{iK_2\gamma}{\pi} \qquad q = e^{i\pi\tau'} = e^{-K_2\gamma}$$

Performing the mathematical operation indicated in (5.11)

$$M_x = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}} \mathrm{Re}\Big[\Big(\frac{\mu^2}{K_2} - \frac{K_2 D_1}{D_y}\Big)\Big\{ -\log\vartheta_0\Big(\frac{\bar{\zeta}'-\bar{\zeta}_0'+\tau'}{2}\Big) - \log\vartheta_0\Big(\frac{\zeta'-\zeta_0'-\tau'}{2}\Big)$$

$$+ \log\vartheta_0\Big(\frac{\bar{\zeta}'-\zeta_0'+\tau'}{2}\Big) + \log\vartheta_0\Big(\frac{\zeta'-\bar{\zeta}_0'-\tau'}{2}\Big) + \log\vartheta_0\Big(\frac{\bar{\zeta}'+\zeta_0'+\tau'}{2}\Big)$$

$$+ \log\vartheta_0\Big(\frac{\zeta'+\bar{\zeta}_0'-\tau'}{2}\Big) - \log\vartheta_0\Big(\frac{\bar{\zeta}'+\bar{\zeta}_0'+\tau'}{2}\Big) - \log\vartheta_0\Big(\frac{\zeta'+\zeta_0'-\tau'}{2}\Big)\Big\}$$

$$+ \Big(\frac{\mu^2}{K_1} - \frac{K_1 D_1}{D_y}\Big)\Big\{ -\log\vartheta_0\Big(\frac{\zeta-\bar{\zeta}_0-\tau}{2}\Big) + \log\vartheta_0\Big(\frac{\bar{\zeta}'+\zeta_0+\tau}{2}\Big) + \log\vartheta_0\Big(\frac{\bar{\zeta}-\bar{\zeta}_0+\tau}{2}\Big)$$

$$+ \log\vartheta_0\Big(\frac{\zeta-\zeta_0-\tau}{2}\Big) + \log\vartheta_0\Big(\frac{\bar{\zeta}+\zeta_0+\tau}{2}\Big) + \log\vartheta_0\Big(\frac{\zeta+\zeta_0-\tau}{2}\Big)$$

$$- \log\vartheta_0\Big(\frac{\bar{\zeta}+\bar{\zeta}+\tau}{2}\Big) - \log\vartheta_0\Big(\frac{\zeta+\zeta_0-\tau}{2}\Big)\Big\}\Big]$$

where $\tau = \dfrac{i K_1 \gamma}{\pi}$

$$= \frac{1}{4\pi\sqrt{\lambda^2-\mu^2}} \mathrm{Re}\Big[\Big(\frac{\mu^2}{K_2} - \frac{K_2 D_1}{D_y}\Big) \log \frac{\vartheta_1\big(\frac{\zeta'-\bar{\zeta}_0'}{2}\big)\vartheta_1\big(\frac{\zeta'+\bar{\zeta}_0'}{2}\big)}{\vartheta_1\big(\frac{\zeta'-\zeta_0'}{2}\big)\vartheta_1\big(\frac{\zeta'+\zeta_0'}{2}\big)}$$

$$- \Big(\frac{\mu^2}{K_1} - \frac{K_1 D_1}{D_y}\Big) \log \frac{\vartheta_1\big(\frac{\zeta-\bar{\zeta}_0}{2}\big)\vartheta_1\big(\frac{\zeta+\bar{\zeta}_0}{2}\big)}{\vartheta_1\big(\frac{\zeta-\zeta_0}{2}\big)\vartheta_1\big(\frac{\zeta+\zeta_0}{2}\big)} \Big]$$

In the same way, influence functions for the other cases of bending moments can be expressed in terms of Jacobi's elliptic function. The results obtained are summarized as follows:

(i) $\lambda > \mu$

$$M_x = \frac{1}{4\pi\sqrt{\lambda^2-\mu^2}} \mathrm{Re}\Big[\Big(\frac{\mu^2}{K_2} - \frac{K_2 D_1}{D_y}\Big) \log \frac{\vartheta_1\big(\frac{\zeta'-\bar{\zeta}_0'}{2}\big)\vartheta_1\big(\frac{\zeta'+\bar{\zeta}_0'}{2}\big)}{\vartheta_1\big(\frac{\zeta'-\zeta_0'}{2}\big)\vartheta_1\big(\frac{\zeta'+\zeta_0'}{2}\big)} - \Big(\frac{\mu^2}{K_1} - \frac{K_1 D_1}{D_y}\Big) \log \frac{\vartheta_1\big(\frac{\zeta-\bar{\zeta}_0}{2}\big)\vartheta_1\big(\frac{\zeta+\bar{\zeta}_0}{2}\big)}{\vartheta_1\big(\frac{\zeta-\zeta_0}{2}\big)\vartheta_1\big(\frac{\zeta+\zeta_0}{2}\big)} \Big] \qquad (5.12)$$

$$M_y = \frac{1}{4\pi\sqrt{\lambda^2-\mu^2}} \mathrm{Re}\Big[\Big(\frac{D_1}{K_2 D_y} - K_2\Big) \log \frac{\vartheta_1\big(\frac{\zeta'-\bar{\zeta}_0'}{2}\big)\vartheta_1\big(\frac{\zeta'+\bar{\zeta}_0'}{2}\big)}{\vartheta_1\big(\frac{\zeta'-\zeta_0'}{2}\big)\vartheta_1\big(\frac{\zeta'+\zeta_0'}{2}\big)}$$

$$- \Big(\frac{D_1}{K_1 D_y} - K_1\Big) \log \frac{\vartheta_1\big(\frac{\zeta-\bar{\zeta}_0}{2}\big)\vartheta_1\big(\frac{\zeta+\bar{\zeta}_0}{2}\big)}{\vartheta_1\big(\frac{\zeta-\zeta_0}{2}\big)\vartheta_1\big(\frac{\zeta+\zeta_0}{2}\big)} \Big]$$

(ii) $\lambda < \mu$

Remembering the relations: (5.12)

$$K_1 = K_3 + i K_4 \;, \quad K_2 = K_3 - i K_4$$

Expressions for M_x and M_y for the case $\lambda > \mu$ can be used.

(iii) $\lambda = \mu$

$$M_x = \frac{1}{4\pi} Re \Big[\left(\sqrt{\lambda} + \frac{D_1}{\sqrt{\lambda} D_y}\right) \log \frac{\vartheta_1\left(\frac{\omega-\omega_0}{2}\right) \vartheta_1\left(\frac{\omega+\bar{\omega}_0}{2}\right)}{\vartheta_1\left(\frac{\omega-\omega_0}{2}\right)\vartheta_1\left(\frac{\omega+\omega_0}{2}\right)}$$

$$-i\left(\lambda - \frac{D_1}{D_y}\right)\Big\{ (\beta+\eta)\left(\frac{\vartheta_1'\left(\frac{\omega-\bar{\omega}_0}{2}\right)}{\vartheta_1\left(\frac{\omega-\bar{\omega}_0}{2}\right)} - \frac{\vartheta_1'\left(\frac{\omega+\omega_0}{2}\right)}{\vartheta_1\left(\frac{\omega+\omega_0}{2}\right)}\right)$$

$$\pm (\beta-\eta)\left(\frac{\vartheta_1'\left(\frac{\omega+\bar{\omega}_0}{2}\right)}{\vartheta_1'\left(\frac{\omega+\bar{\omega}_0}{2}\right)} - \frac{\vartheta_1'\left(\frac{\omega-\omega_0}{2}\right)}{\vartheta_1\left(\frac{\omega-\omega_0}{2}\right)}\right)\Big\}\Big]$$

$$M_y = \frac{1}{4\pi} Re \Big[\left(\frac{D_1}{\sqrt{\lambda^3} D_y} + \frac{1}{\sqrt{\lambda}}\right) \log \frac{\vartheta_1\left(\frac{\omega-\omega_0}{2}\right)\vartheta_1\left(\frac{\omega+\bar{\omega}_0}{2}\right)}{\vartheta_1\left(\frac{\omega-\omega_0}{2}\right)\vartheta_1\left(\frac{\omega+\omega_0}{2}\right)} \tag{5.12}$$

$$-i\left(\frac{D_1}{\lambda D_y} - 1\right)\Big\{(\beta+\eta)\left(\frac{\vartheta_1'\left(\frac{\omega-\bar{\omega}_0}{2}\right)}{\vartheta_1\left(\frac{\omega-\bar{\omega}_0}{2}\right)} - \frac{\vartheta_1'\left(\frac{\omega+\omega_0}{2}\right)}{\vartheta_1\left(\frac{\omega+\omega_0}{2}\right)}\right)$$

$$\pm (\beta-\eta)\left(\frac{\vartheta_1'\left(\frac{\omega+\bar{\omega}_0}{2}\right)}{\vartheta_1\left(\frac{\omega+\bar{\omega}_0}{2}\right)} - \frac{\vartheta_1'\left(\frac{\omega-\omega_0}{2}\right)}{\vartheta_1\left(\frac{\omega-\omega_0}{2}\right)}\right)\Big\}\Big]$$

where $a\omega_0 = u + i\sqrt{\lambda}\, v$, $\quad a\omega = x + i\sqrt{\lambda}\, y$

$$a\bar{\omega}_0 = u - i\sqrt{\lambda}\, v \qquad a\bar{\omega} = x - i\sqrt{\lambda}\, y$$

$$q = e^{-\sqrt{\lambda}\, \gamma}$$

5.5 <u>Some Remarks on the Computation of $M_x(\alpha,\beta;\xi,\eta)$ and $M_y(\alpha,\beta;\xi,\eta)$</u>

According to the theory of elliptic functions an expansion formulae for $\vartheta_1(z)$ exists:

$$\vartheta_1(z) = 2\sum_{n=1}^{\infty}(-1)^n q^{\left(\frac{2n+1}{2}\right)^2} \sin(2n+1)\pi z = 2q^{\frac{1}{4}}(\sin\pi z - q^2 \sin 3\pi z + q^6 \sin 5\pi z - \cdots) \tag{5.13}$$

where $\pi z = \alpha + i\beta$

In order to investigate the convergence of this series, a value for γ/π is assumed, with $\gamma/\pi = b/a = 1.5$:

$$q = e^{-1.5\pi} = 0.00898 \quad \text{(isotropic)}$$

Therefore series (5.13) converges so rapidly that the sum of first two terms will give a very accurate result. So

$$\vartheta_1(z) \sim 2q^{\frac{1}{4}}[(\sin\alpha\cosh\beta + i\cos\alpha\sinh\beta) - q^2(\sin 3\alpha\cosh 3\beta + i\cos 3\alpha\sinh 3\beta)] \tag{5.14}$$

Putting

$$\vartheta_1 = \varphi_1 + i\phi_1$$

so

$$\varphi_1 \sim 2q^{\frac{1}{4}}(\sin\alpha\cosh\beta - q^2\sin 3\alpha\cosh 3\beta)$$

$$\phi_1 \sim 2q^{\frac{1}{4}}(\cos\alpha\sinh\beta - q^2\cos 3\alpha\sinh 3\beta)$$

$$|\vartheta_1(z)|^2 = \varphi_1^2 + \phi_1^2 \sim 4q^{\frac{1}{2}}(\cosh 2\beta - \cos 2\alpha)(\tfrac{1}{2} - q^2 - 2q^2\cosh 2\beta\cos 2\alpha) \tag{5.15}$$

Using this expansion formulae, very accurate results of M_x and M_y can be obtained. The influence functions for M_x or M_y of semi-infinite plate strip can be deduced from (5.12) making $b \longrightarrow \infty$

Consider M_x in case of $\lambda > \mu$.

With the aid of expansion formula (5.15)

$$Re\left[\log\frac{\vartheta_1(\frac{\zeta-\bar{\zeta}_0}{2})\vartheta_1(\frac{\zeta+\bar{\zeta}_0}{2})}{\vartheta_1(\frac{\zeta-\zeta_0}{2})\vartheta_1(\frac{\zeta+\zeta_0}{2})}\right] = \frac{1}{2}\left[\log\frac{\left|\vartheta_1(\frac{\zeta-\bar{\zeta}_0}{2})\right|^2\left|\vartheta_1(\frac{\zeta+\bar{\zeta}_0}{2})\right|^2}{\left|\vartheta_1(\frac{\zeta-\zeta_0}{2})\right|^2\left|\vartheta_1(\frac{\zeta+\zeta_0}{2})\right|^2}\right]$$

Since $\quad b \gg 0 \quad , \quad q \doteqdot 0$

$$\left|\vartheta_1\left(\frac{\zeta'-\bar{\zeta}_0'}{2}\right)\right|^2 \sim 4q^{\frac{1}{2}}(\cosh K_2(\beta+\eta) - \cos(\alpha-\xi))$$

$$\left|\vartheta_1\left(\frac{\zeta'+\bar{\zeta}_0'}{2}\right)\right|^2 \sim 4q^{\frac{1}{2}}(\cosh K_2(\beta-\eta) - \cos(\alpha+\xi))$$

$$\left|\vartheta_1\left(\frac{\zeta-\zeta_0}{2}\right)\right|^2 \sim 4q^{\frac{1}{2}}(\cosh K_2(\beta-\eta) - \cos(\alpha-\xi))$$

$$\left|\vartheta_1\left(\frac{\zeta+\zeta_0}{2}\right)\right|^2 \sim 4q^{\frac{1}{2}}(\cosh K_2(\beta+\eta) - \cos(\alpha+\xi))$$

$$\mathrm{Re}\left[\log\frac{\vartheta_1\left(\frac{\zeta'-\bar{\zeta}_0'}{2}\right)\vartheta_1\left(\frac{\zeta'+\bar{\zeta}_0'}{2}\right)}{\vartheta_1\left(\frac{\zeta-\zeta_0}{2}\right)\vartheta_1\left(\frac{\zeta+\zeta_0}{2}\right)}\right] \sim \frac{1}{2}\left(R_2-\bar{R}_2\right)$$

Similarly

$$\mathrm{Re}\left[\log\frac{\vartheta_1\left(\frac{\zeta-\bar{\zeta}_0}{2}\right)\vartheta_1\left(\frac{\zeta+\bar{\zeta}_0}{2}\right)}{\vartheta_1\left(\frac{\zeta-\zeta_0}{2}\right)\vartheta_1\left(\frac{\zeta+\zeta_0}{2}\right)}\right] \sim \frac{1}{2}\left(R_1-\bar{R}_1\right)$$

Making $b \longrightarrow \infty$ so that $q \longrightarrow 0$.

$$M_x = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\left[\left(K_1\mu-\frac{K_2D_1}{D_y}\right)\left(R_2-\bar{R}_2\right)-\left(K_2\mu-\frac{K_1D_1}{D_y}\right)\left(R_1-\bar{R}_1\right)\right]$$

This corresponds to the result obtained previously for a semi-infinite strip.

5.6 Application of Fourier Integrals for the Solution of Semi-Infinite Plate Strips.

Another simple way to find the solution for the semi-infinite plate strip is to start from the Navier's solution for a rectangular plate. As an example an isotropic plate is considered whose influence function for the deflection is

$$W(u,v;x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\varphi_{mn}(u,v)\,\varphi_{mn}(x,y)}{\lambda_{mn}^2}$$

$$= \frac{2}{abD}\sum_{m=1}^{\infty}\sin\frac{m\pi u}{a}\sin\frac{m\pi x}{a}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{b}(y-v)-\cos\frac{n\pi}{b}(y+v)}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}$$

Making $b \longrightarrow \infty$ the summation of series with respect to y is replaced by an integral

$$W(u,v;x,y) = \frac{2}{a\pi D}\sum_{m=1}^{\infty}\sin\frac{m\pi u}{a}\sin\frac{m\pi x}{a}\int_0^{\infty}\frac{\cos\frac{n\pi}{b}(y-v)-\cos\frac{n\pi}{b}(y+v)}{\left[\left(\frac{m\pi}{a}\right)^2+p^2\right]^2}dp \quad (5.16)$$

Using the relation:

$$\int_0^{\infty}\frac{\cos mx}{(x^2+a^2)^2}dx = \frac{\pi e^{-ma}}{4a^3}(1+ma) \quad (m \geq 0) \quad (5.17)$$

(5.16) can be transformed to:

$$W(u,v;x,y) = \frac{a^2}{2\pi^3 D}\sum_{m=1}^{\infty}\frac{1}{m^3}\left\{\left(1 \mp \frac{m\pi}{a}(y-v)\right)e^{\pm\frac{m\pi}{a}(y-v)} - \left(1+\frac{m\pi}{a}(y+v)\right)e^{-\frac{m\pi}{a}(y+v)}\right\}\sin\frac{m\pi u}{a}\sin\frac{m\pi x}{a}$$

This checks the results obtained in Chapter IV.

It is apparent that the first series represents the influence function of an infinite plate strip. The second series is due to an anti-symmetric load P with respect to the **x-axis** (Mirror Method). Further applications of the Fourier integral will be discussed in Chapter VII.

5. Other Boundary Value Problems of Rectangular Plates

If a rectangular plate has two parallel edges simply supported solutions in product form as illustrated in Chapter IV are applicable (5-1). However, for other conditions solutions can be obtained by superposition, taking equation (5.4) as the particular solution of the problem. Unfortunately the solution leads to an infinite number of simultaneous equations for which only approximate solutions are possible. (Fig. 5-4)

By making the length of one edge **infinitely long** in those solutions obtained so that changing the summation to an integral, solutions for semi-infinite plate strip can be derived in Fourier integral form. (Fig. 5-5).

CHAPTER VI

Influence Functions for Moments in Slab Continuous Over Flexible Cross Beams

6.1 An Infinite Plate Strip With Simply Supported Parallel Edge ~~(Fig. 6-1)~~

At y=0 the plate is continuous over an elastic cross beam with a constant bending stiffness EI. The coordinates of a point on the cross beam are taken as (z,0) --z being the x-coordinate -- in order to distinguish this point from a general point (u,v), referred to as the influence point. The deflection of a general point (u,v) due to a concentrated load P at point P(x,y) can be expressed by the following integral equation:

$$W(\alpha,\beta;\xi,\eta) = PG(\alpha,\beta;\xi,\eta) - \int_0^{\pi} EI\left(\frac{\pi}{a}\right)^3 \frac{\partial^4 W(\zeta,0;\xi,\eta)}{\partial \zeta^4} G(\alpha,\beta;\zeta,0)\,d\zeta \qquad (6.1)$$

Here again non-dimensional coordinates defined in Chapter III are introduced with a new parameter

$$\zeta = \frac{\pi z}{a}$$

The function $G(\alpha,\beta;\xi,\eta)$ is Green's function for the deflection of point (α,β) of an infinite plate strip with simply supported edges. (It is given in Chapter III, p.25).

The first term under the integral sign in equation (6.1)

$EI\left(\frac{\pi}{a}\right)^3 \frac{\partial^4 W(\zeta,0;\xi,\eta)}{\partial \zeta^4}$ expresses the distributed reaction of the cross beam acting on the plate.

When multiplied by Green's function $G(\alpha,\beta;\zeta,0)$ and integrated over the length of the cross beam the integral constitutes the influence of this beam on the deflection at point (α,β).

Assuming the deflection surface W in the form

$$W(\alpha,\beta;\xi,\eta) = \phi(\alpha,\beta;\xi,\eta) + P\,G(\alpha,\beta;\xi,\eta) \qquad (6.2)$$

The function ϕ is determined by substituting (6.2) into (6.1)

$$\phi(\alpha,\beta;\xi,\eta) = -\frac{\pi^3 EI}{a^3}\int_0^{\pi}\frac{\partial^4 W(\zeta,0;\xi,\eta)}{\partial\zeta^4}\,G(\alpha,\beta;\zeta,0)\,d\zeta \tag{6.3}$$

Since ϕ is a continuous function with respect to α and β, it can be developed into eigen-functions associated with Green's function G as follows:

$$\phi(\alpha,\beta;\xi,\eta) = \sum_{n=1}^{\infty} a_n(\xi,\eta)\,\varphi_n(\alpha,\beta) \tag{6.4}$$

Confining the discussion to the case $\lambda > \mu$:

$$\varphi_n(\alpha,\beta) = (K_1 e^{\mp nK_2\beta} - K_2 e^{\mp nK_1\beta})\sin n\alpha$$

Substituting into (6.3) and replacing G by equation (3.14) gives:

$$\sum_{n=1}^{\infty} a_n(\xi,\eta)\,\varphi_n(\alpha,\beta) = -\frac{\pi^3 EI}{a^3}\int_0^{\pi}\frac{\partial^4 W(\zeta,0)}{\partial\zeta^4}\Big\{\frac{a^2}{2\pi^3\mu D_y\sqrt{\lambda^2-\mu^2}}\sum_{m=1}^{\infty}\frac{1}{m^3}$$
$$(K_1 e^{\mp mK_2\beta} - K_2 e^{\mp mK_1\beta})\sin m\alpha\,\sin m\zeta\Big\}\,d\zeta \tag{6.5}$$

Multiplying both sides by sin $n\alpha$ and integrating with respect to α from 0 to π, the orthogonality relations simplify equation (6.5) considerably.

$$\frac{\pi}{2}a_m(\xi,\eta)(K_1 e^{\mp nK_1\beta} - K_2 e^{\mp nK_1\beta}) = -\frac{\pi}{2}\cdot\frac{EI}{2n^3 a\mu D_y\sqrt{\lambda^2-\mu^2}}$$
$$(K_1 e^{\mp nK_2\beta} - K_2 e^{\mp nK_1\beta})\int_0^{\pi}\frac{\partial^4 W(\zeta,0)}{\partial\zeta^4}\sin n\zeta\,d\zeta \tag{6.6}$$

with the substitution

$$\frac{\partial^4 W(\zeta,0)}{\partial\zeta^4} = \frac{\partial^4}{\partial\zeta^4}\Big[\phi(\zeta,0;\xi,\eta) + PG(\zeta,0;\xi,\eta)\Big]$$
$$= \sum_{m=1}^{\infty}\Big[m^4 a_m(\xi,\eta)\sin m\zeta + \frac{ma^2}{2\pi^3\mu D_y\sqrt{\lambda^2-\mu^2}}($$
$$K_1 e^{\mp mK_2\eta} - K_2 e^{\mp mK_1\eta})\sin m\xi\,\sin m\zeta\Big]$$

the function a_n can be determined. Again use of the orthogonality relations is made. Introducing the parameter

$$\frac{1}{\rho} = \frac{(K_1-K_2)\pi EI}{4a\mu D_y\sqrt{\lambda^2-\mu^2}} \tag{6.7}$$

$$a_n = -\frac{a^2}{2n^3\pi^3\mu(K_1-K_2)D_y\sqrt{\lambda^2-\mu^2}}\left(\frac{n}{n+\rho}\right)(K_1 e^{\mp nK_2\eta} - K_2 e^{\mp nK_1\eta})\sin n\xi \tag{6.8}$$

The non-dimensional parameter ρ depends on the ratio of the bending stiffness of the plate in y direction D_y and the bending stiffness of the cross beam EI as well as λ and μ. Substituting the pertinent values into (6.2) with P=1 yields the influence function for the deflection:

$$W(\alpha,\beta;\xi,\eta) = \frac{a^2}{2\pi^3\mu D_y\sqrt{\lambda^2-\mu^2}}\sum_{n=1}^{\infty}\left[\frac{1}{n^3}(K_1 e^{\pm nK_2(\beta-\eta)} - K_2 e^{\pm nK_1(\beta-\eta)})\right.$$

$$\left. - \frac{1}{n^2(n+\rho)(K_1-K_2)}(K_1 e^{-nK_2\eta} - K_2 e^{-nK_1\eta})(K_1 e^{-nK_2\beta} - K_1 e^{-nK_1\beta})\right]\sin n\alpha \sin n\xi \tag{6.9}$$

The first term within the parenthesis represents the influence surface for the deflection of point (α,β), of a simply supported plate strip without cross beam. The second term expresses the influence of this beam. If the cross beam is inifinitely rigid, that is, EI $\to\infty$ and $\rho \to 0$, the coefficient of the second term reduces to:

$$\lim_{\rho\to 0}\frac{1}{n^2(n+\rho)(K_1-K_2)} = \frac{1}{n^3(K_1-K_2)}$$

On the other hand, in the absence of a cross beam, EI $\rightarrow 0$, and $\rho \rightarrow \infty$ such that

$$\lim_{\rho \to \infty} \frac{1}{n^2(n+\rho)(K_1-K_2)} = 0$$

and the second term will disappear entirely (reduced to the case of infinite plate strip). The following results were obtained in a similar manner:

(I) Influence Surfaces for the deflection $W(\alpha,\beta;\xi,\eta)$

(i) $\lambda > \mu$

$$= \frac{a^2}{2\pi^3\mu D_y\sqrt{\lambda^2-\mu^2}} \sum_{n=1}^{\infty} \Big[\frac{1}{n^3}\left(K_1 e^{\pm nK_2(\beta-\eta)} - K_2 e^{\pm nK_1(\beta-\eta)}\right) - \frac{1}{n^2(n+\rho)(K_1-K_2)}$$

$$(K_1 e^{-nK_2\eta} - K_2 e^{-nK_1\eta})(K_1 e^{-nK_2\beta} - K_2 e^{-nK_1\beta})\Big] \sin n\alpha \sin n\xi$$

where $$\frac{1}{\rho} = \frac{\pi EI(K_1-K_2)}{4a\mu D_y\sqrt{\lambda^2-\mu^2}}$$

(ii) $\lambda < \mu$

$$= \frac{a^2}{\pi^3\mu D_y\sqrt{\mu^2-\lambda^2}} \sum_{n=1}^{\infty} \Big[\frac{e^{\pm nK_3(\beta-\eta)}}{n^3} \left(K_4 \cos nK_4(\beta-\eta) \mp K_3 \sin nK_4(\beta-\eta)\right) \quad (6.10)$$

$$- \frac{e^{-nK_3(\beta+\eta)}}{n^2(n+\rho)K_4} (K_4 \cos nK_4\beta + K_3 \sin nK_4\beta)(K_4 \cos nK_4\eta + K_3 \sin nK_4\eta) \sin n\alpha \sin n\xi$$

where $$\frac{1}{\rho} = \frac{K_4 EI}{2a\mu D_y\sqrt{\mu^2-\lambda^2}}$$

(iii) $\lambda = \mu$

$$= \frac{a^2}{2\pi^3 D_y\sqrt{\lambda^3}} \sum_{n=1}^{\infty} \Big[\frac{1}{n^3}\left(1 \mp n\sqrt{\lambda}(\beta-\eta)\right) e^{\pm n\sqrt{\lambda}(\beta-\eta)} - \frac{1}{n^2(n+\rho)}(1+n\sqrt{\lambda}\beta) \times$$

$$(1+n\sqrt{\lambda}\eta)\, e^{-n\sqrt{\lambda}(\beta+\eta)}\Big] \sin n\alpha \sin n\xi$$

where $$\frac{1}{\rho} = \frac{\pi EI}{4aD_y\sqrt{\lambda^3}}$$

(II) Influence Surfaces for Moments $M_x(\alpha,\beta;\xi,\eta)$ and $M_y(\alpha,\beta;\xi,\eta)$

(i) $\lambda > \mu$

$$M_x = \frac{1}{2\pi\sqrt{\lambda^2-\mu^2}}\sum_{n=1}^{\infty}\Big[\Big(K_1\mu - \frac{K_2 D_1}{D_y}\Big)\Big\{\frac{e^{\pm nK_2(\beta-\eta)}}{n} + \frac{K_1 e^{-nK_2(\beta+\eta)} - K_2 e^{-n(K_1\eta+K_2\beta)}}{(n+\rho)(K_2-K_1)}\Big\}$$

$$-\Big(K_2\mu - \frac{K_1 D_1}{D_y}\Big)\Big\{\frac{e^{\pm nK_1(\beta-\eta)}}{n} + \frac{K_1 e^{-n(K_1\beta+K_1\eta)} - K_2 e^{-n(K_1\beta+K_2\eta)}}{(n+\rho)(K_2-K_1)}\Big\}\Big]\sin n\alpha \sin n\xi$$

$$M_y = \frac{1}{2\pi\sqrt{\lambda^2-\mu^2}}\sum_{n=1}^{\infty}\Big[\Big(\frac{K_1 D_1}{\mu D_y} - K_2\Big)\Big\{\frac{e^{\pm nK_2(\beta-\eta)}}{n} + \frac{K_1 e^{-nK_2(\beta+\eta)} - K_2 e^{-n(K_1\eta+K_2\beta)}}{(n+\rho)(K_2-K_1)}\Big\}$$

$$-\Big(\frac{K_2 D_1}{\mu D_y} - K_1\Big)\Big\{\frac{e^{\pm nK_1(\beta-\eta)}}{n} + \frac{K_1 e^{-n(K_1\beta+K_2\eta)} - K_2 e^{-n(K_1\beta+K_2\eta)}}{(n+\rho)(K_2-K_1)}\Big\}\Big]\sin n\alpha \sin n\xi$$

(ii) $\lambda < \mu$

$$M_x = \frac{1}{\pi\sqrt{\mu^2-\lambda^2}}\sum_{n=1}^{\infty}\Big[\frac{e^{\pm nK_3(\beta-\eta)}}{n}\Big\{K_4\Big(\mu+\frac{D_1}{D_y}\Big)\cos nK_4(\beta-\eta) \mp K_3\Big(\mu-\frac{D_1}{D_y}\Big)\times$$

$$\sin nK_4(\beta-\eta)\Big\} - \frac{e^{-nK_4(\beta+\eta)}}{K_4(n+\rho)}\big(K_4\cos nK_4\eta + K_3\sin nK_4\eta\big)\Big\{ \tag{6.11}$$

$$K_4\Big(\mu+\frac{D_1}{D_y}\Big)\cos nK_4\beta + K_3\Big(\mu-\frac{D_1}{D_y}\Big)\sin nK_4\beta\Big\}\Big]\sin n\alpha \sin n\xi$$

$$M_y = \frac{1}{\pi\sqrt{\mu^2-\lambda^2}}\sum_{n=1}^{\infty}\Big[\frac{e^{\pm nK_3(\beta-\eta)}}{n}\Big\{K_4\Big(\frac{D_1}{\mu D_y}+1\Big)\cos nK_4(\beta-\eta) \mp K_3\Big(\frac{D_1}{\mu D_y}-1\Big)$$

$$\times\sin nK_4(\beta-\eta)\Big\} - \frac{e^{-nK_4(\beta+\eta)}}{K_4(n+\rho)}\big(K_4\cos nK_4\eta + K_3\sin nK_4\eta\big)\Big\{$$

$$K_4\Big(\frac{D_1}{\mu D_y}+1\Big)\cos nK_4\beta + K_3\Big(\frac{D_1}{\mu D_y}-1\Big)\sin nK_4\beta\Big\}\Big]\sin n\alpha \sin n\xi$$

(iii) $\lambda = \mu$

$$M_x = \frac{1}{2\pi\sqrt{\lambda}}\sum_{n=1}^{\infty}\Big[\frac{1}{n}\Big\{\Big(\lambda+\frac{D_1}{D_y}\Big) \mp \Big(\lambda-\frac{D_1}{D_y}\Big)n\sqrt{\lambda}(\beta-\eta)\Big\}e^{\pm n\sqrt{\lambda}(\beta-\eta)}$$

$$-\frac{1}{n+\rho}\big(1+n\sqrt{\lambda}\eta\big)\Big\{\Big(\lambda+\frac{D_1}{D_y}\Big)+\Big(\lambda-\frac{D_1}{D_y}\Big)n\sqrt{\lambda}\beta\Big\}e^{-n\sqrt{\lambda}(\beta+\eta)}\Big]\sin n\alpha \sin n\xi$$

$$M_y = \frac{1}{2\pi\sqrt{\lambda^3}}\sum_{n=1}^{\infty}\Big[\frac{1}{n}\Big\{\Big(\lambda+\frac{D_1}{D_y}\Big) \pm \Big(\lambda-\frac{D_1}{D_y}\Big)n\sqrt{\lambda}(\beta-\eta)\Big\}e^{\pm n\sqrt{\lambda}(\beta-\eta)}$$

$$-\frac{1}{n+\rho}\big(1+n\sqrt{\lambda}\eta\big)\Big\{\Big(\lambda+\frac{D_1}{D_y}\Big)-\Big(\lambda-\frac{D_1}{D_y}\Big)n\sqrt{\lambda}\beta\Big\}e^{-n\sqrt{\lambda}(\beta+\eta)}\Big]\sin n\alpha \sin n\xi$$

upper sign for $\beta \leqq \eta$

lower sign for $\beta \geqq \eta$

and if $\eta < 0$; the signs preceding η must be changed in the second series of above equations.

In general, it appears to be impossible to sum the series of the equation (6.11). However, for several specific values of ρ, such a summation can be made.(15)(16)(17) The results obtained for the case of a rigid cross beam, that is, $\rho = 0$ are tabulated as follows.

(III) M_x, M_y — Influence Surfaces for the Case of the Rigid Cross Beam (ρ =0)

(i) $\lambda > \mu$

$$M_x = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\Big[\Big(K_1\mu - \frac{K_2 D_1}{D_y}\Big)\Big\{ R_2 + \frac{1}{K_2-K_1}(K_1\bar{R}_2 - K_2 R_{10})\Big\} - \Big(K_2\mu - \frac{K_1 D_1}{D_y}\Big)\Big\{ R_1 + \frac{1}{K_2-K_1}(K_1\bar{R}_1 - K_2 R_{10})\Big\}\Big]$$

$$M_y = \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\Big[\Big(\frac{K_1 D_1}{\mu D_y} - K_2\Big)\Big\{ R_2 + \frac{1}{K_2-K_1}(K_1\bar{R}_2 - K_2 R_{10})\Big\} - \Big(\frac{K_2 D_1}{\mu D_y} - K_1\Big)\Big\{ R_1 + \frac{1}{K_2-K_1}(K_1\bar{R}_1 - K_2 R_{10})\Big\}$$

(ii) $\lambda < \mu$

$$M_x = \frac{1}{4\pi\sqrt{\mu^2-\lambda^2}}\Big[\Big\{K_4\Big(\mu+\frac{D_1}{D_y}\Big)R_3 + K_3\Big(\mu-\frac{D_1}{D_y}\Big)R_4\Big\} - \frac{1}{2}\Big\{K_4\Big(\mu+\frac{D_1}{D_y}\Big)(\bar{R}_3 + R_{12}) + K_3\Big(\mu+\frac{D_1}{D_y}\Big)(\bar{R}_4 - R_{12}) + K_3\Big(\mu-\frac{D_1}{D_y}\Big)(\bar{R}_4 + R_{13}) + \frac{K_3^2}{K_4}\Big(\mu-\frac{D_1}{D_y}\Big)(R_{12} - \bar{R}_3)\Big\}\Big] \tag{6.12}$$

$$M_y = \frac{1}{4\pi\sqrt{\mu^2-\lambda^2}}\Big[\Big\{K_4\Big(\frac{D_1}{\mu D_y}+1\Big)R_3 + K_3\Big(\frac{D_1}{\mu D_y}-1\Big)R_4\Big\} - \frac{1}{2}\Big\{K_4\Big(\frac{D_1}{\mu D_y}+1\Big)(\bar{R}_3 + R_{12}) + K_3\Big(\frac{D_1}{\mu D_y}+1\Big)(\bar{R}_4 - R_{13}) + K_3\Big(\frac{D_1}{\mu D_y}-1\Big)(\bar{R}_4 + R_{13}) + \frac{K_3^2}{K_4}\Big(\frac{D_1}{\mu D_y}-1\Big)(R_{12} - \bar{R}_3)\Big\}\Big]$$

(iii) $\lambda = \mu$

$$M_x = \frac{1}{8\pi\sqrt{\lambda}}\Big[\left(\lambda+\frac{D_1}{D_y}\right)R_5 \mp \left(\lambda-\frac{D_1}{D_y}\right)\sqrt{\lambda}(\beta-\eta)S_1 - \left(\lambda+\frac{D_1}{D_y}\right)\bar{R}_5 - \sqrt{\lambda}\,\eta\left(\lambda+\frac{D_1}{D_y}\right)\bar{S}_1 - \lambda\beta\eta\left(\lambda-\frac{D_1}{D_y}\right)T_1 - \sqrt{\lambda}\,\beta\left(\lambda-\frac{D_1}{D_y}\right)\bar{S}_1\Big]$$

$$M_y = \frac{1}{8\pi\sqrt{\lambda^3}}\Big[\left(\lambda+\frac{D_1}{D_y}\right)R_5 \pm \left(\lambda-\frac{D_1}{D_y}\right)\sqrt{\lambda}(\beta-\eta)S_1 - \left(\lambda+\frac{D_1}{D_y}\right)\bar{R}_5 - \sqrt{\lambda}\,\eta\left(\lambda+\frac{D_1}{D_y}\right)\bar{S}_1 + \lambda\beta\eta\left(\lambda-\frac{D_1}{D_y}\right)T_1 + \sqrt{\lambda}\,\beta\left(\lambda-\frac{D_1}{D_y}\right)\bar{S}_1\Big]$$

IV. $\lambda = \mu = 1$ (isotropic)

$$W = \frac{a^2}{2\pi^3 D}\sum_{n=1}^{\infty}\Big[\frac{1}{n^3}\left(1 \mp n(\beta-\eta)\right)e^{\pm n(\beta-\eta)} - \frac{1}{n^2(n+\rho)}(1+n\beta)\times(1+n\eta)\,e^{-n(\beta+\eta)}\Big]\sin n\alpha \sin n\xi$$

where $$\frac{1}{\rho} = \frac{\pi E I}{4aD}$$

$$M_x = \frac{1}{8\pi}\Big[(1+\nu)\log\frac{\cosh(\beta-\eta)-\cos(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} + (1-\nu)(\beta-\eta)\times\left(\frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)}\right) - (1+\nu)\times\log\frac{\cosh(\beta+\eta)-\cos(\alpha+\xi)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)} - \left\{(1+\nu)\eta+(1-\nu)\beta\right\}\left(\frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)} - \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha+\xi)}\right) - (1-\nu)\beta\eta\left(\frac{\cosh(\beta+\eta)\cos(\alpha-\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha-\xi))^2} - \frac{\cosh(\beta+\eta)\cos(\alpha+\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha+\xi))^2}\right)\Big]$$

$$M_y = \frac{1}{8\pi}\Big[(1+\nu)\log\frac{\cosh(\beta-\eta)-\cos(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} - (1-\nu)(\beta-\eta)\left(\frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} - \frac{\sinh(\beta-\eta)}{\cosh(\beta-\eta)-\cos(\alpha+\xi)}\right) - (1+\nu)\log\frac{\cosh(\beta+\eta)-\cos(\alpha+\xi)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)} - \left\{(1+\nu)\eta-(1-\nu)\beta\right\}\left(\frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha-\xi)} - \frac{\sinh(\beta+\eta)}{\cosh(\beta+\eta)-\cos(\alpha+\xi)}\right) + (1-\nu)\beta\eta\left(\frac{\cosh(\beta+\eta)\cos(\alpha-\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha-\xi))^2} - \frac{\cosh(\beta+\eta)\cos(\alpha+\xi)-1}{(\cosh(\beta+\eta)-\cos(\alpha+\xi))^2}\right)\Big]$$

CHAPTER VII

Application of Fourier Integrals and Complex Variables

7.1 Alternative Methods of Solution

In this chapter, methods other than the ordinary methods employed so far in Chapters' III and IV will be discussed briefly, particularly the application of Fourier integrals to the boundary value problem of a semi-infinite as well as an infinite plate strip, the application of conformal mapping to isotropic plates whose boundaries are simply supported.

Rather than solving any particular problem, brief discussion on the general approach of problems involved will be given.

7.2 Application of Fourier Integrals to Problems of Plate Strips

For simplicity, only isotropic plates will be considered.

(i) Influence function of plate strip with simply supported edges in form of Fourier Integral: Levy's solution obtained in equation (5.7) can readily be rewritten in the form of a y-sine series (Fig. 5-1):

if $u \geq x$

$$W = \sum_{n=1}^{\infty} \frac{1}{bD\left(\frac{n\pi}{b}\right)^3} \sin\frac{n\pi y}{b} \sin\frac{n\pi v}{a} \left[\frac{\sinh\frac{n\pi}{b}(a-u)}{\sinh\frac{n\pi a}{b}}\left(\sinh\frac{n\pi x}{b} - \frac{n\pi x}{b}\cosh\frac{n\pi x}{b}\right) + \frac{\sinh\frac{n\pi x}{b}}{\sinh\frac{n\pi a}{b}}\left(\frac{n\pi u}{a}\cosh\frac{n\pi}{b}(a-u) - \frac{\frac{n\pi a}{b}\sinh\frac{n\pi u}{b}}{\sinh\frac{n\pi a}{b}}\right)\right]$$

Making $b \to \infty$, the summation will turn into an integral. Introducing

$$\frac{\pi}{b} = dp, \qquad \frac{n\pi}{b} = p$$

$$W(u,v;x,y)=\begin{cases} \dfrac{1}{\pi D}\displaystyle\int_0^\infty \dfrac{1}{p^3}\Big[\dfrac{\sinh p(a-u)}{\sinh ap}(\sinh xp - xp\cosh xp) + \\ \dfrac{\sinh xp}{\sinh ap}\Big(pu\cosh p(a-u) - \dfrac{ap\sinh up}{\sinh ap}\Big)\Big]\sin yp \sin vp\, dp \\ \qquad\qquad\qquad\qquad (u \geqq x) \\ \dfrac{1}{\pi D}\displaystyle\int_0^\infty \dfrac{1}{p^3}\Big[\dfrac{\sinh up}{\sinh ap}(\sinh(a-x)p - (a-x)p \times \\ \cosh(a-x)p + \dfrac{\sinh(a-x)p}{\sinh ap}\Big((a-u)p\cosh up - \dfrac{ap\sinh(a-u)p}{\sinh ap}\Big)\Big] \times \\ \sin yp \sin vp\, dp \qquad (u \leqq x) \end{cases} \tag{7.1}$$

Equation (7.1) represents the influence function of deflection for a semi-infinite isotropic plate strip with simply supported edges in integral from. For an infinite strip, the corresponding solution is obtained by simply replacing sin yp sin vp by $\frac{1}{2}$ cos (v-y)p, because sin yp sin vp = $\frac{1}{2}$ (cos(v-y)p-cos (v+y)p), and the latter is the image of the former with respect to the x-axis (Fig. 7-1).

Next the homogeneous solution of $\Delta\Delta W=0$ will be obtained in Fourier integral form.

(ii) Homogeneous solution of $\Delta\Delta W=0$ in Fourier integral form.

It is easy to see that

$$(A\cosh px + B\sinh px + Cxp\cosh px + Dxp\sinh px)\cos(y-v)p$$

satisfies the equation $\Delta\Delta W=0$. Therefore, the general expression of the homogeneous solution can be written as

$$W_1(u,v;x,y) = \int_0^\infty A(p)\cosh xp + B(p)\sinh xp + C(p)xp\cosh xp \tag{7.2}$$

$$D(p)xp\sinh xp)\cos(y-v)p\, dp$$

where A(p), B(p), C(p), D(p), are arbitrary functions of p.

(iii) The solution for the infinite plate strip with clamped parallel edges.

Combining equation (7.1) and equation (7.2)

$$W(u,v;x,y) = W_0(u,v;x,y) + W_1(u,v;x,y)$$

with the boundary conditions

$$x = 0 \qquad W = 0 \qquad \frac{\partial W}{\partial x} = 0$$

$$x = a \qquad W = 0 \qquad \frac{\partial W}{\partial x} = 0$$

These four boundary conditions determine the functions A(p), B(p), C(p), and D(p) in equation (7.1).

For actual computation of influence functions, the theory of residues or methods of numerical integration must be employed.

(iv) Infinite Plate Resting on an Elastic Foundation:

The differential equation corresponding to this case is:

$$D \Delta\Delta W + K W = q(x,y) \tag{7.3}$$

kw is the reaction of the foundation. The coefficient k is usually expressed in pounds per square inches per inch of deflection. This quantity is generally referred to as modulus of the foundation.

The influence function for the deflection of a simply supported rectangular plate on an elastic foundation is given in reference((1), p. 252) in double Fourier series form.

$$W(u,v;x,y) = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi u}{a}\sin\frac{n\pi v}{b}}{\pi^4 D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+K^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$= \frac{1}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\cos\frac{m\pi}{a}(u-x)-\cos\frac{m\pi}{a}(u+x)\right)}{\pi^4 D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+k}\left(\cos\frac{n\pi}{b}(y-v)-\cos\frac{n\pi}{b}(y+v)\right) \tag{7.4}$$

Making a,b infinitely large, writing $\frac{\pi}{a} = dp,\ \frac{m\pi}{a} = p,\ \frac{\pi}{b} = dq\ \ \frac{n\pi}{b} = q$

the double Fourier integral form can be derived

$$W(u,v;x,y) = \frac{1}{\pi^2 D}\int_0^\infty\int_0^\infty \frac{1}{(p^2+q^2)^2+\chi^4}\Big(\cos p(u-x)\cos q(v-y)$$

$$-\cos p(u-x)\cos q(v+y) - \cos p(u+x)\cos q(v-y) + \cos p(u+x)\cos q(v+y)\Big)\,dp\,dq \tag{7.5}$$

where $\chi^4 = \frac{k}{D}$

Equation (7.5) represents the influence function of an infinite wedge plate whose opening angle is $\frac{\pi}{2}$ (Fig. 7-2). Observing that edges of the wedge are simply supported, it can be concluded that the first integral represent the influence function for this particular problem. The other three terms are nothing but the image of the first term with respect to either x-axis or y-axis.

$$\therefore\ W(u,v;x,y) = \frac{1}{\pi^2 D}\int_0^\infty\int_0^\infty \frac{\cos(x-u)p\,\cos(v-y)p}{(p^2+q^2)^2+\chi^4}\,dp\,dq \tag{7.6}$$

This is the solution for this particular case.

The deflection under the load can be easily computed

$$(W)_{\substack{x=u\\y=v}} = \frac{1}{\pi^2 D}\int_0^\infty\int_0^\infty \frac{dp\,dq}{(p^2+q^2)^2+\chi^2} = \frac{1}{\pi^2 D}\cdot\frac{\pi}{2\chi^2}\int_0^\infty \frac{dp}{\sqrt{p^4+\chi^4}}\times$$

$$\sqrt{\frac{\sqrt{p^4+\chi^4}-p^2}{2}} = \frac{\chi^2}{8K} = \frac{1}{8\sqrt{KD}} \qquad ((1)\ p.\ 255)$$

Equation (7.6) is the fundamental solution for the influence functions of the infinite plate on the elastic foundation.

The method illustrated so far in this chapter can be easily extended to the case of orthotropic plates. However, general solutions of such problems will not be treated here.

7.3 Application of Conformal Mapping

As mentioned in 7.1, if the shape of an isotropic plate is bounded by straight lines and the edges are simply supported, conformal mapping can be successfully applied to find the influence functions for M_x and M_y.

Consider the moment sum $M_x+M_y=M$ in Cartesian coordinate.

$$M = M_x + M_y = -D(1+\nu)\Delta W \tag{7.7}$$

so that $D\Delta\Delta W = -\frac{1}{1+\nu}\Delta M = g(x,y)$

Therefore, the fourth order plate equation reduces to a second order equation in M. The influence functions of the bending moment M_x, M_y can be easily obtained as shown subsequently, once M is derived. Since M_x, M_y, M_{xy} are integrals of linearly varying stresses, $\sigma_x, \sigma_y, \tau_{xy}$ over the thickness of the plate they have the same tensor character as a two dimensional stress field. M is an invariant of the system.

Assuming the edges of the plate to be straight segments and simply supported, M will disappear along the boundary:

$$M = 0 \tag{7.8}$$

Therefore (7.7) and (7.8) constitute the boundary value problem in a two dimensional moment field. Actually the influence functions for M is directly proportional to Green's function for the deflection of membrane of equal shape.

Since M satisfies Laplace's equation except at the loading point, it is possible to apply conformal mapping to find M in a given domain from Green's function for M of the unit circle. The theory of harmonic functions furnishes the Green's function $g(r,\theta;\rho,\varphi)$ for the unit circle:

$$g(r,\theta;\rho,\varphi) = \frac{1}{2}\log\frac{1-2\rho r\cos(\theta-\varphi)+\rho^2 r^2}{r^2-2\rho r\cos(\theta+\varphi)+\rho^2} \tag{7.9}$$

Observing the similarity between $g(r,\theta;\rho,\varphi)$ and $M(r,\theta;\rho,\varphi)$ the parameter relating the two effects is determined such that

$$M(r,\theta;\rho,\varphi) = \frac{(1+\nu)}{4\pi} \log \frac{1-2r\rho\cos(\theta-\varphi)+r^2\rho^2}{r^2-2r\rho\cos(\theta-\varphi)+\rho^2} \tag{7.10}$$

$M(r,\theta;\rho,\varphi)$ for the semi-circular domain with unit radius can be derived, taking the image of (7.10) with respect to the line of $\theta=0$ (Fig. 7-3).

$$M(r,\theta;\rho,\varphi) = \frac{1+\nu}{4\pi} \log \frac{(1-2\rho r\cos(\theta-\varphi)+\rho^2r^2)(r^2-2r\rho\cos(\theta+\varphi)+\rho^2)}{(r^2-2r\rho\cos(\theta-\varphi)+\rho^2)(1-2r\rho\cos(\theta+\varphi)+\rho^2r^2)} \tag{7.11}$$

Applying the conformal mapping $z = e^{i\omega}$ to equation (7.11)

$$z = re^{i\theta}, \qquad \omega = \alpha + i\beta$$

$$\therefore \quad r = e^{-\beta} \qquad \theta = \alpha$$

$$\rho = e^{-\eta} \qquad \varphi = \xi \tag{7.12}$$

Substituting (7.12) into (7.11):

$$M(\alpha,\beta;\xi,\eta) = \frac{1+\nu}{4\pi} \log \frac{\{\cosh(\beta+\eta)-\cos(\alpha-\xi)\}\{\cosh(\beta-\eta)-\cos(\alpha+\xi)\}}{\{\cosh(\beta-\eta)-\cos(\alpha-\xi)\}\{\cosh(\beta+\eta)-\cos(\alpha+\xi)\}} \tag{7.13}$$

This is the expression of M for a semi-infinite plate strip. M_x, M_y can be easily obtained using following relations:

$$M_x = \frac{1}{2}\left[M - \frac{1-\nu}{1+\nu} Y \frac{\partial M}{\partial Y} \right]$$
$$M_y = \frac{1}{2}\left[M + \frac{1-\nu}{1+\nu} Y \frac{\partial M}{\partial Y} \right] \tag{7.14}$$

where $\quad Y = (\beta \pm \eta)$

Making $\beta \to \infty$, $\eta \to \infty$, M for an infinite plate strip can also be derived

$$M = \frac{1+\nu}{4\pi} \log \frac{\cosh(\beta-\eta)-\cos(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} \tag{7.15}$$

This checks the result obtained by Nadai[(3), p. 89] using also conformal mapping but in a different way.

Further solutions of M for a rectangular plate or a wedge-shaped plate could be obtained with the aid of Schwarz-Christoffel's transformation.

CHAPTER VIII

Discussion of Singularities of Influence Surfaces

8.1 Singular Behavior of Influence Surfaces at the Influence Point

In general, influence surfaces exhibit singular behavior at the influence point, singularities are due to the singular (particular) part of the solutions. Since regular part of the solution does not show any singularity, they can be discarded as far as the discussion of the singularities are concerned.

In this chapter, a general discussion of the singularities of influence surfaces will be given, that is, singularities of the influence surfaces m_x, m_y, m_{xy}, q_x, q_y at an interior point of the plate, of the corner reaction r of a simply supported rectangular plate, of the boundary moment m_y of a clamped edge, of the boundary moment m_x of a free edge and of support moment m_x, m_y of slabs continuous over a flexible cross beam. Numerical values will be presented so that the general appearance of surfaces can be easily visualized.

8.2 Derivation of Singularities of Influence Surfaces

In the case of an isotropic plate the singular solution of plate equation $D\Delta\Delta W=q(x,y)$ is $r^2 \log r$ where r is the distance between the influence point and the loading point.

These singularities can be obtained considering the neighborhood of the influence point (α, β) only. Taking the coordinates in this neighborhood as

$$\xi = \alpha + \varepsilon$$
$$\eta = \beta + \delta$$

with $\varepsilon \doteqdot 0$ and $\delta \doteqdot 0$ the terms of the influence functions are expanded into series. Neglecting higher order terms in the singular part of the solutions and discarding the regular part entirely expressions for the singularity are obtained.

(I) Singularities of Influence Surfaces m_x, m_{xy}, q_y at the Interior Point of the Plate

Since m_y and q_x show the same singular behavior as m_x and q_y respectively, m_x and q_y will be discussed only. In order to distinguish the singular part of the influence function, suffix 0 will be used in every case.

Taking the solution given in (3.15) the vicinity of the influence point (α,β), ξ,η can be expressed as follows:

$$\xi = \alpha + \varepsilon , \quad \eta = \beta + \delta \qquad (\varepsilon \doteqdot 0, \ \delta \doteqdot 0)$$

Consider the case of $\lambda > \mu$. Since $\varepsilon \doteqdot 0, \ \delta \doteqdot 0$

$$\cosh K_2(\beta-\eta) - \cos(\alpha+\xi) \sim 1 - \cos 2\alpha$$

$$\cosh K_2(\beta-\eta) - \cos(\alpha-\xi) \sim \frac{1}{2}\left\{ (\alpha-\xi)^2 + K_2^2(\beta-\eta)^2 \right\} = \frac{1}{2}(\varepsilon^2 + K_2^2\delta^2)$$

$$\therefore \quad R_1 \sim \log \frac{1-\cos 2\alpha}{\frac{1}{2}(\varepsilon^2 + K_1^2\delta^2)} \qquad R_2 \sim \log \frac{1-\cos 2\alpha}{\frac{1}{2}(\varepsilon^2 + K_2^2\delta^2)}$$

Introducing polar coordinate

$$\varepsilon = r\cos\theta , \quad \delta = r\sin\theta , \quad r \doteqdot 0$$

and discarding the regular part of the influence function: $\log 2(1-\cos 2\alpha)$, following final result is obtained

$$(m_x)_0 \sim \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\Big[\left(K_2\mu - K_1\left(\tfrac{D_1}{D_y}\right)\right) \log r^2(\cos^2\theta + K_1^2\sin^2\theta) - \left(K_1\mu - K_2\left(\tfrac{D_1}{D_y}\right)\right) \log r^2(\cos^2\theta + K_2^2\sin^2\theta) \Big]$$

Similarly, $(m_{xy})_o$, $(q_y)_o$ can be obtained.

The results obtained are summarized as follows.

(i) $\lambda > \mu$

$$(m_x)_o \sim \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\left[\left(K_2\mu - K_1\left(\frac{D_1}{D_y}\right)\right)\log\, r^2(\cos^2\theta + K_1^2\sin^2\theta)\right.$$

$$\left. - \left(K_1\mu - K_2\left(\frac{D_1}{D_y}\right)\right)\log\, r^2(\cos^2\theta + K_2^2\sin^2\theta)\right]$$

$$(m_{xy})_o \sim \frac{D_{xy}}{2\pi D_y\sqrt{\lambda^2-\mu^2}}\left[\tan^{-1}\left(\frac{\cot\theta}{K_2}\right) - \tan^{-1}\left(\frac{\cot\theta}{K_1}\right)\right]$$

$$(q_y)_o \sim -\frac{1}{4ar}\left[\frac{K_2\sin\theta}{\cos^2\theta + K_2^2\sin^2\theta} + \frac{K_1\sin\theta}{\cos^2\theta + K_1^2\sin^2\theta}\right]$$

(ii) $\lambda < \mu$

$$(m_x)_o \sim \frac{1}{8\pi\sqrt{\mu^2-\lambda^2}}\left[2K_3\left(\mu - \frac{D_1}{D_y}\right)\tan^{-1}\left(\frac{2K_3K_4\sin^2\theta}{\cos^2\theta + \lambda\sin^2\theta}\right)\right.$$

$$\left. - K_4\left(\mu + \frac{D_1}{D_y}\right)\log\, r^4(\cos^4\theta + 2\lambda\sin^2\theta\cos^2\theta + \mu^2\sin^4\theta)\right]$$

$$(m_{xy})_o \sim \frac{D_{xy}}{4\pi D_y\sqrt{\mu^2-\lambda^2}}\log\frac{\cos^2\theta - 2K_4\sin\theta\cos\theta + \mu\sin^2\theta}{\cos^2\theta + 2K_4\sin\theta\cos\theta + \mu\sin^2\theta}$$

$$(q_y)_o \sim -\frac{1}{4ar}\left(\frac{K_3\sin\theta}{\cos^2\theta - 2K_4\sin\theta\cos\theta + \mu\sin^2\theta} + \frac{K_4\sin\theta}{\cos^2\theta + 2K_4\sin\theta\cos\theta + \mu\sin^2\theta}\right)$$

(iii) $\lambda = \mu$

$$(m_x)_o \sim \frac{1}{8\pi}\left[-\left(\sqrt{\lambda} + \frac{D_1}{\sqrt{\lambda}D_y}\right)\log\, r^2(\cos^2\theta + \lambda\sin^2\theta)\right.$$

$$\left. + \frac{2\sqrt{\lambda}\left(\lambda - \frac{D_1}{D_y}\right)\sin^2\theta}{\cos^2\theta + \lambda\sin^2\theta}\right]$$

$$(m_{xy})_o \sim \frac{D_{xy}}{2\pi\lambda D_y}\cdot\frac{\sin\theta\cos\theta}{\cos^2\theta + \lambda\sin^2\theta}$$

$$(q_y)_o \sim -\frac{1}{4ar}\left(\frac{\sqrt{\lambda}\sin\theta}{\cos^2\theta + \lambda\sin^2\theta}\right)$$

(II) Corner Reaction of Simply Supported Edge $(Y)_0$

(i) $\lambda > \mu$

$$= \frac{4D_{xy}}{\pi D_y\sqrt{\lambda^2-\mu^2}}\left[\tan^{-1}\left(\frac{\cot\theta}{K_2}\right) - \tan^{-1}\left(\frac{\cot\theta}{K_1}\right)\right]$$

(ii) $\lambda < \mu$

$$= \frac{2D_{xy}}{\pi D_y\sqrt{\mu^2-\lambda^2}} \log \frac{\cos^2\theta - 2K_3\sin\theta\cos\theta + \mu\sin^2\theta}{\cos^2\theta + 2K_3\sin\theta\cos\theta + \mu\sin^2\theta}$$

(iii) $\lambda = \mu$

$$= \frac{4D_{xy}}{\pi\lambda D_y}\cdot\frac{\sin\theta\cos\theta}{\cos^2\theta + \lambda\sin^2\theta}$$

(III) Boundary Moment of Clamped Edge $(m_x)_0$ $(m_y)_0$

(i) $\lambda > \mu$

$$(m_x)_0 \sim \frac{(K_1+K_2)D_1}{4\pi D_y\sqrt{\lambda^2-\mu^2}} \log \frac{\cos^2\theta + K_2^2\sin^2\theta}{\cos^2\theta + K_1\sin^2\theta}$$

$$(m_y)_0 \sim \frac{(K_1+K_2)}{4\pi\sqrt{\lambda^2-\mu^2}} \log \frac{\cos^2\theta + K_2^2\sin^2\theta}{\cos^2\theta + K_1\sin^2\theta}$$

(ii) $\lambda < \mu$

$$(m_x)_0 \sim -\frac{K_3 D_1}{\pi D_y\sqrt{\mu^2-\lambda^2}} \tan^{-1}\left(\frac{2K_3K_4\sin^2\theta}{\cos^2\theta + \lambda\sin^2\theta}\right)$$

$$(m_y)_0 \sim -\frac{K_3}{\pi\sqrt{\mu^2-\lambda^2}} \tan^{-1}\left(\frac{2K_3K_4\sin^2\theta}{\cos^2\theta + \lambda\sin^2\theta}\right)$$

(iii) $\lambda = \mu$

$$(m_x)_0 \sim -\frac{\sqrt{\lambda}D_1}{\pi D_y}\cdot\frac{\sin^2\theta}{\cos^2\theta + \lambda\sin^2\theta}$$

$$(m_y)_0 \sim -\frac{\sqrt{\lambda}}{\pi}\cdot\frac{\sin^2\theta}{\cos^2\theta + \lambda\sin^2\theta}$$

(IV) Boundary Moment of Free Edge $(m_x)_o$

(i) $\lambda > \mu$

$$(m_x)_o \sim \frac{1}{8\pi\sqrt{\lambda^2-\mu^2}}\left[\left\{\left(K_2\mu - \frac{K_1 D_1}{D_y}\right)\left(1+\frac{M(K_1+K_2)}{L(K_1-K_2)}\right)-\left(K_1\mu-\frac{K_2 D_1}{D_y}\right)\left(\frac{2K_2 N}{L(K_1-K_2)}\right)\right\}\right.$$

$$\times \log r^2(\cos^2\theta + K_1^2\sin^2\theta) - \left\{\left(K_1\mu - \frac{K_2 D_1}{D_y}\right)\left(1-\frac{M(K_1+K_2)}{L(K_1-K_2)}\right)+\right.$$

$$\left.\left.\left(K_2\mu - \frac{K_1 D_1}{D_y}\right)\left(\frac{2K_1 N}{L(K_1-K_2)}\right)\right\}\log r^2(\cos^2\theta + K_2^2\sin^2\theta)\right]$$

(ii) $\lambda < \mu$

$$(m_x)_o \sim \frac{1}{8\pi\sqrt{\mu^2-\lambda^2}}\left[\left\{\frac{N}{K_4 L}\left(\mu^2-\frac{\lambda D_1}{D_y}\right)-K_4\left(\mu+\frac{D_1}{D_y}\right)-\frac{K_3^2 M}{K_4 L}\left(\mu-\frac{D_1}{D_y}\right)\right\}\times\right.$$

$$\log\cdot r^4(\cos^4\theta + 2\lambda\cos^2\theta\sin^2\theta + \mu^2\sin^4\theta) + \left\{2K_3\left(\mu-\frac{D_1}{D_y}\right)+\frac{2K_3 M}{L}\left(\mu+\frac{D_1}{D_y}\right)\right.$$

$$\left.\left.+\frac{4K_3 D_1 N}{D_y L}\right\}\tan^{-1}\left(\frac{2K_3K_4\sin^2\theta}{\cos^2\theta+\lambda\sin^2\theta}\right)\right]$$

(iii) $\lambda = \mu$

$$(m_x)_o \sim \frac{1}{8\pi\sqrt{\lambda}}\left[-\left\{\frac{(2H^2-2D_{xy}H+D_{xy}^2)(\lambda-\frac{D_1}{D_y})+2(\frac{D_1}{D_y})D_{xy}^2}{D_{xy}(2H-D_{xy})}+\left(\lambda+\frac{D_1}{D_y}\right)\right\}\times\right.$$

$$\left.\log\cdot r^2(\cos^2\theta+\lambda\sin^2\theta)+\left\{\left(\lambda-\frac{D_1}{D_y}\right)+\frac{D_{xy}}{2H-D_{xy}}\left(1+\frac{3D_1}{D_y}\right)\right\}\times\frac{2\lambda\sin^2\theta}{\cos^2\theta+\lambda\sin^2\theta}\right]$$

(V) Support Moments $(m_x)_o, (m_y)_o$ of Slabs Continuous Over a Flexible Cross Beam

(i) $\lambda > \mu$

$$(m_x)_o \sim \frac{1}{2\pi\sqrt{\lambda^2-\mu^2}}\left[\frac{(K_1+K_2)D_1}{4D_y}\log\frac{\cos^2\theta+K_2^2\sin^2\theta}{\cos^2\theta+K_1^2\sin^2\theta}+(K_1-K_2)\left(\mu+\frac{D_1}{D_y}\right)J(\rho)\right]$$

$$(m_y)_o \sim \frac{1}{2\pi\sqrt{\lambda^2-\mu^2}}\left[\frac{(K_1+K_2)}{4}\log\frac{\cos^2\theta+K_2^2\sin^2\theta}{\cos^2\theta+K_1^2\sin^2\theta}+(K_1-K_2)\left(\frac{D_1}{\mu D_y}+1\right)J(\rho)\right]$$

where $$\rho = \frac{2\mu\sqrt{\lambda^2-\mu^2}}{K_1-K_2}\cdot\frac{4aD_y}{\pi EI}$$

(ii) $\lambda < \mu$

$$(m_x)_0 \sim \frac{1}{\pi\sqrt{\mu^2-\lambda^2}}\left[-\frac{K_3 D_1}{2D_y}\tan^{-1}\left(\frac{2K_3K_4\sin^2\theta}{\cos^2\theta+\lambda\sin^2\theta}\right)+K_4\left(\mu+\frac{D_1}{D_y}\right)J(\rho)\right]$$

$$(m_y)_0 \sim \frac{1}{\pi\sqrt{\mu^2-\lambda^2}}\left[-\frac{K_3}{2}\tan^{-1}\left(\frac{2K_3K_4\sin^2\theta}{\cos^2\theta+\lambda\sin^2\theta}\right)+K_4\left(\frac{D_1}{\mu D_y}+1\right)J(\rho)\right]$$

where $$\rho = \frac{\mu}{K_4}\sqrt{\mu^2-\lambda^2}\cdot\frac{4aD_y}{\pi EI}$$

(iii) $\lambda = \mu$

$$(m_x)_0 \sim \frac{1}{2\pi\sqrt{\lambda}}\left[-\frac{\lambda\left(\frac{D_1}{D_y}\right)\sin^2\theta}{\cos^2\theta+\lambda\sin^2\theta}+\left(\lambda+\frac{D_1}{D_y}\right)J(\rho)\right]$$

$$(m_y)_0 \sim \frac{1}{2\pi\sqrt{\lambda}}\left[-\frac{\lambda\sin^2\theta}{\cos^2\theta+\lambda\sin^2\theta}+\left(\frac{D_1}{\lambda D_y}+1\right)J(\rho)\right]$$

where $$\rho = 2\sqrt{\lambda^3}\cdot\frac{2aD_y}{\pi EI}$$

if $\rho=0$, $(m_x)_0, (m_y)_0$ are the support moments in case of a rigid cross beam. Furthermore, it is easily seen that $2(m_x)_0$, $2(m_y)_0$ are exactly identical with the boundary moments for a clamped edge.

The function $J(\rho)$ introduced here is defined as follows:

$$J(\rho) = \sum_{n=1,3,5\cdots}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\rho}\right)$$

$$= \frac{1}{2}\left[\Psi(\rho)+\gamma+\log 2+\frac{1}{2}\left\{\Psi\left(\frac{\rho+1}{2}\right)-\Psi\left(\frac{\rho}{2}\right)\right\}\right]^*$$

where $\Psi(\rho)$ is the Psi-function introduced by Gauss[28].

$$\Psi(\rho) = \frac{\Gamma'(\rho)}{\Gamma(\rho)} = \int_0^\infty \left(e^{\alpha}-\frac{1}{(1+\alpha)^\rho}\right)\frac{d\alpha}{\alpha}$$

---

* Derivation is given in Appendix.

and γ =0, 5772156649 ---(Euler's constant)

For practical computation of $J(\rho)$, the following two mathematical formulas are used.

$$\Psi\left(\frac{l}{n}\right) + \gamma = -\frac{\pi}{2}\cot\frac{l\pi}{n} + 2\sum_{\nu=1}^{\left|\frac{n+1}{2}\right|}\left\{\cos\left(\frac{2l\nu\pi}{n}\right)\log\sin\left(\frac{\nu\pi}{n}\right)\right\} - \log(2n)$$

$$(n = 2, 3, 4 \cdots\cdots,\ l = 1, 2, 3 \cdots (n-1))$$

$$\Psi(\rho) + \gamma = \sum_{n=1}^{\rho-1}\frac{1}{n} \qquad (\rho = 1, 2, 3, \cdots)$$

8.3 General Appearance of Singularities

In order to visualize the general appearance of singularities, the isotropic case $\lambda = \mu = 1$ is considered here.

(a) $(m_x)_o$, $(m_y)_o$, $(q_y)_o$ at the interior point of a slab.

$$(m_x)_o \sim \frac{1}{8\pi}\left[-(1+\nu)\log r^2 + 2(1-\nu)\sin^2\theta\right]$$

$$(m_{xy})_o \sim \frac{(1-\nu)}{8\pi}\sin 2\theta$$

$$(q_y)_o \sim -\frac{1}{4ar}\sin\theta$$

(b) Corner reaction $(r)_o$ of simply supported edge

$$(r)_o \sim \frac{(1-\nu)}{\pi}\sin 2\theta$$

(c) Boundary moments $(m_x)_o$, $(m_y)_o$ of clamped edge

$$(m_x)_o \sim -\frac{\nu}{\pi}\sin^2\theta$$

$$(m_y)_o \sim -\frac{1}{\pi}\sin^2\theta$$

(d) Boundary moment $(m_x)_o$ of free edge

$$(m_x)_o \sim \frac{1}{\pi(3+\nu)}\left[-\log r^2 + (1-\nu^2)\sin^2\theta\right]$$

(e) Support moments $(m_x)_o$, $(m_y)_o$ of a continuous slab

$$(m_x)_o \sim -\frac{1}{2\pi}\left[-\nu \sin^2\theta + (1+\nu) J(\rho)\right]$$

$$(m_y)_o \sim -\frac{1}{2\pi}\left[-\sin^2\theta + (1+\nu) J(\rho)\right]$$

The above equations, except (e), were already obtained by Pucher(4). Fig. 8-1 gives a graphical representation of these singularities.*

Knowing the singular behavior of the influence functions, their general appearance in specific cases can be easily drawn as shown in Figs. 8-2 and 8-3.

A three dimensional view of the $(m_x)_o$ surface at the interior point of a slab is also given in Fig. 2-2.

8.4 <u>Discussion on the Singularities of Orthotropic Plates</u>

As pointed out in Chapter II (4), the domain $0 \leqq \lambda \leqq 10$ $0 \leqq \mu \leqq 10$ is of practical importance. Therefore, numerical computations were made for several cases listed in Fig. 8-4. Generally, the influence functions take completely different mathematical expressions depending on the relation: $\lambda \gtreqless \mu$. However, results of numerical computation show that the influence surfaces will change their shape as well as their numerical values continuously according to the value of λ and μ.

The domain $\lambda < \mu$ is the case where the mathematical expressions take their most complicated form. However, it is exactly this domain where most of the data of actual bridge slabs

---

* $(m_x)_o$, for the interior point, (m_{xo}) for the free edge become infinitely large at the influence point. In computing the contours shown in Fig. 8-1, the assumption $\nu = 0$ was made. Furthermore, for the cases where the singularity tends to infinity a value of the influence function equal to zero was assumed. As all contours are similar this assumption does not influence their general shape.

fall, (Fig. 1-4), (especially for bridge slabs, cases $\lambda = 0$, $\lambda < \mu$ (points on the μ-axis) are of importance).

Results of numerical computation are collected in Figs. 8-5, 8-6, 8-7, 8-8, 8-9, 8-10, 8-11.

It is easy to understand how mountains (positive zone) and valleys (negative zone) will change their shapes, contracting or expanding depending on the value of λ and μ.

Some of the mathematical aspects of the singularities of orthotropic plates have recently been discussed by Mossakowski[11] using a Fourier Integral transform.

CHAPTER IX

Summary

In this dissertation mathematical expressions for the influence surfaces of orthotropic rectangular plates are derived. The principal results of the investigation can be divided into four parts:

(1) Cases Solved

The Green's function for the decflection of an infinite orthotropic plate strip with simply supported parallel edges is solved as a fundamental case (Chapter III). Combining this solution with the homogeneous solution for orthotropic rectangular plate and determining its coefficients such that the combination fulfills the boundary conditions at the third edge, the influence functions for the semi-infinite plate strip with simply supported parallel edges are derived in Chapter IV.

Using a solution in double Fourier series form (corresponding to Navier's solution for isotropic plate) rectangular plate with simply supported edges is treated (Chapter V). Through summation a solution in simple series form is developed. Finally, in Chapter VI, the plate strip continuous over a flexible cross beam is studied.

(2) Closed Form Solutions

In this dissertation, most solutions are carried through to a closed form by making use of several mathematical summation formulae. Thus, the discussion of the singularities of the influence functions become possible and the general appearance of influence surfaces around the singularities is made clear. Many previous solutions for isotropic plates are in series form which

converge very slowly in the vicinity of the influence point and are divergent for the point itself. They do not allow a discussion of singular points.

(3) Discussion of the Singularities

Discarding the regular part as well as higher order terms of the singular part in the vicinity of the influence point, the solutions are obtained for various cases. Assuming various values for the orthotropic parameters λ and μ a general investigation of the singular behavior of the influence surfaces is made.

(4) Practical Application

In practical application the orthotropic parameters λ and μ seem to be limited as follows:

$$0 \leqq \lambda \leqq 10$$
$$0 \leqq \mu \leqq 10$$

This square domain covers such cases as two-way reinforced concrete slabs, grid work systems, corrugated sheets, plywood plates, stiffened plates, etc. Orthotropic bridge slabs fall generally in the domain $\lambda < \mu$ and even $\lambda \doteqdot 0$ as shown in (Fig. 1-4).

Assuming twelve values of λ and μ, numerical computation of the singularities was carried out, and the results were represented in contour line diagrams.

The change of the shapes as well as numerical values of influence surfaces due to changes of λ and μ are easily visualized. Since the change of influence surfaces in shape and numerical value is continuous depending upon the change of λ and μ, an interpolation between the computed surfaces is admissible.

CHAPTER X

Appendix

1. Mathematical Formulae for the Summation of the Series of the Type $\sum_{n=1}^{\infty} \frac{r^n}{n} \cos nx$

If Z is a complex variable and $|z|<1$, the following expansion holds.

$$\frac{1}{1-Z} = 1 + Z + Z^2 + \cdots\cdots = \sum_{n=0}^{\infty} Z^n \tag{A}$$

Expressing Z in polar coordinates

$$Z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

and it conjugate

$$\bar{Z} = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$

yields

$$\frac{1}{1-Z} = \frac{1-\bar{Z}}{(1-Z)(1-\bar{Z})} = \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2} \tag{B}$$

$$\sum_{n=1}^{\infty} Z^n = \sum_{n=1}^{\infty} r^n \cos n\theta + i \sum_{n=1}^{\infty} r^n \sin n\theta \tag{C}$$

comparing equations (A), (B) and (C) the following expressions can be derived, provided $0 \leq r \leq 1$ (<)

$$\sum_{n=0}^{\infty} r^n \cos n\theta = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}$$

and

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} - 1 = \frac{1}{2}\left(\frac{1 - r^2}{1 - 2r\cos\theta + r^2} - 1\right) \tag{D}$$

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2} \tag{E}$$

Integrating equation (A)

$$-\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{z^n}{n} \tag{F}$$

$$1-z = (1-r\cos\theta) + ir\sin\theta = \sqrt{1-2r\cos\theta+r^2}\; e^{i\tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right)} \tag{G}$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{r^n}{n}\cos n\theta + i\sum_{n=1}^{\infty} \frac{r^n}{n}\sin n\theta \tag{H}$$

From (F),(G) and (H)

$$\sum_{n=1}^{\infty} \frac{r^n}{n}\cos n\theta = -\frac{1}{2}\log(1-2r\cos\theta+r^2) \tag{I}$$

$$\sum_{n=1}^{\infty} \frac{r^n}{n}\sin n\theta = \tan^{-1}\left(\frac{r\sin\theta}{1-r\cos\theta}\right) \tag{J}$$

In the same way several other formulae can be derived. Only the final expressions are given:

$$\sum_{n=1}^{\infty} \frac{r^n}{n}\cos nx = -\frac{1}{2}\log(1-2r\cos x+r^2)$$

$$\sum_{n=1}^{\infty} \frac{r^n}{n}\sin nx = \tan^{-1}\left(\frac{r\sin x}{1-r\cos x}\right)$$

$$\sum_{n=1,3,5\cdots}^{\infty} \frac{r^n}{n}\cos nx = \frac{1}{4}\log\frac{1+2r\cos x+r^2}{1-2r\cos x+r^2}$$

$$\sum_{n=1,3,5\cdots}^{\infty} \frac{r^n}{n}\sin nx = \tan^{-1}\left(\frac{2r\sin x}{1-r^2}\right)$$

$$\sum_{n=1}^{\infty} r^n\cos nx = \frac{1-r\cos x}{1-2r\cos x+r^2} - 1 = \frac{1}{2}\left(\frac{1-r^2}{1-2r\cos x+r^2} - 1\right)$$

$$\sum_{n=1}^{\infty} r^n\sin nx = \frac{r\sin x}{1-2r\cos x+r^2}$$

$$\sum_{n=1}^{\infty} n r^n\cos nx = \frac{r\{(1+r^2)\cos x - 2r\}}{(1-2r\cos x+r^2)^2}$$

(2) Mathematical Formulae for the Summation of the Series of the Type $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2+K^2}$

Expanding e^{kx} into Fourier series in the range $[0,2\pi]$

$$\frac{\pi e^{Kx}}{e^{2K\pi}-1} = \frac{1}{2K} + \sum_{n=1}^{\infty} \frac{K\cos nx}{n^2+K^2} - \sum_{n=1}^{\infty} \frac{n\sin nx}{n^2+K^2} \qquad (A)$$

changing k to -k

$$\frac{\pi e^{-Kx}}{e^{-2K\pi}-1} = -\frac{1}{2K} - \sum_{n=1}^{\infty} \frac{K\cos nx}{n^2+K^2} - \sum_{n=1}^{\infty} \frac{n\sin nx}{n^2+K^2} \qquad (B)$$

Combinding equation (A) and (B)

$$\frac{\pi e^{K(x-\pi)}}{e^{K\pi}-e^{-K\pi}} = \frac{1}{2K} + \sum_{n=1}^{\infty} \frac{K\cos nx}{n^2+K^2} - \sum_{n=1}^{\infty} \frac{n\sin nx}{n^2+K^2} \qquad (C)$$

$$\frac{\pi e^{-K(x-\pi)}}{e^{K\pi}-e^{-K\pi}} = \frac{1}{2K} + \sum_{n=1}^{\infty} \frac{K\cos nx}{n^2+K^2} + \sum_{n=1}^{\infty} \frac{n\sin nx}{n^2+K^2} \qquad (D)$$

Adding equations (C) and (D)

$$\frac{\pi\cosh K(x-\pi)}{\sinh K\pi} = \frac{1}{K} + 2K\sum_{n=1}^{\infty} \frac{\cos nx}{n^2+K^2}$$

or

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2+K^2} = \frac{\pi}{2K}\cdot\frac{\cosh K(x-\pi)}{\sinh K\pi} - \frac{1}{2K^2} \qquad (E)$$

Differentiating equation (E) with respect to k

$$-2K\sum_{n=1}^{\infty}\frac{\cos nx}{(n^2+K^2)^2} = -\frac{\pi}{2K^2}\cdot\frac{\cosh K(x-\pi)}{\sinh K\pi} + \frac{1}{4K^3}$$

$$+\frac{\pi}{2K}\cdot\frac{(x-\pi)\sinh K(x-\pi)\sinh K\pi - \pi\cosh K\pi\cosh K(x-\pi)}{\sinh^2 K\pi}$$

$$\therefore \sum_{n=1}^{\infty}\frac{\cos nx}{(n^2+K^2)^2} = -\frac{1}{2K^4} + \frac{\pi}{4K^3}\cdot\frac{\cosh K(x-\pi)}{\sinh K\pi}$$

$$+\frac{\pi^2}{4K^2}\cdot\frac{\cosh K(x-\pi)\cosh K\pi}{\sinh^2 K\pi} - \frac{\pi(x-\pi)}{4K^2}\cdot\frac{\sinh K(x-\pi)}{\sinh K\pi}$$

$$(0 \leqq x \leqq 2\pi) \qquad \text{(F)}$$

In the same way many useful summation formulae of series can be obtained.

(3) Derivation of $J(\rho)$

$$J(\rho) = \sum_{n=1,3,5\cdots}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\rho}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\rho}\right)\left(1-(-1)^n\right)$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\rho}\right) - \sum_{n=1}^{\infty}\frac{(-1)^n}{n} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n+\rho}\right]$$

However the theory of Gamma functions furnishes the following relationships((28)p, 458)).

$$\frac{d}{d\rho}\log\Gamma(\rho)+\gamma = \sum_{K=0}^{\infty}\left(\frac{1}{1+K}-\frac{1}{\rho+K}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\rho}\right)-\frac{1}{\rho}$$

$$\therefore \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\rho}\right) = \frac{\Gamma'(\rho)}{\Gamma(\rho)} + \gamma + \frac{1}{\rho}$$

Making use of the relations(28)

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\log 2$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n+\rho} = \int_0^1 \frac{x^{\rho-1}}{1+x}dx - \frac{1}{\rho} = \frac{1}{2}\left[\frac{\Gamma'(\frac{\rho+1}{2})}{\Gamma(\frac{\rho+1}{2})} - \frac{\Gamma'(\frac{\rho}{2})}{\Gamma(\frac{\rho}{2})}\right] - \frac{1}{\rho}$$

$J(\rho)$ becomes

$$J(\rho) = \frac{1}{2}\left[\frac{\Gamma'(\rho)}{\Gamma(\rho)} + \gamma + \log 2 + \frac{1}{2}\left\{\frac{\Gamma'(\frac{\rho+1}{2})}{\Gamma(\frac{\rho+1}{2})} - \frac{\Gamma'(\frac{\rho}{2})}{\Gamma(\frac{\rho}{2})}\right\}\right]$$

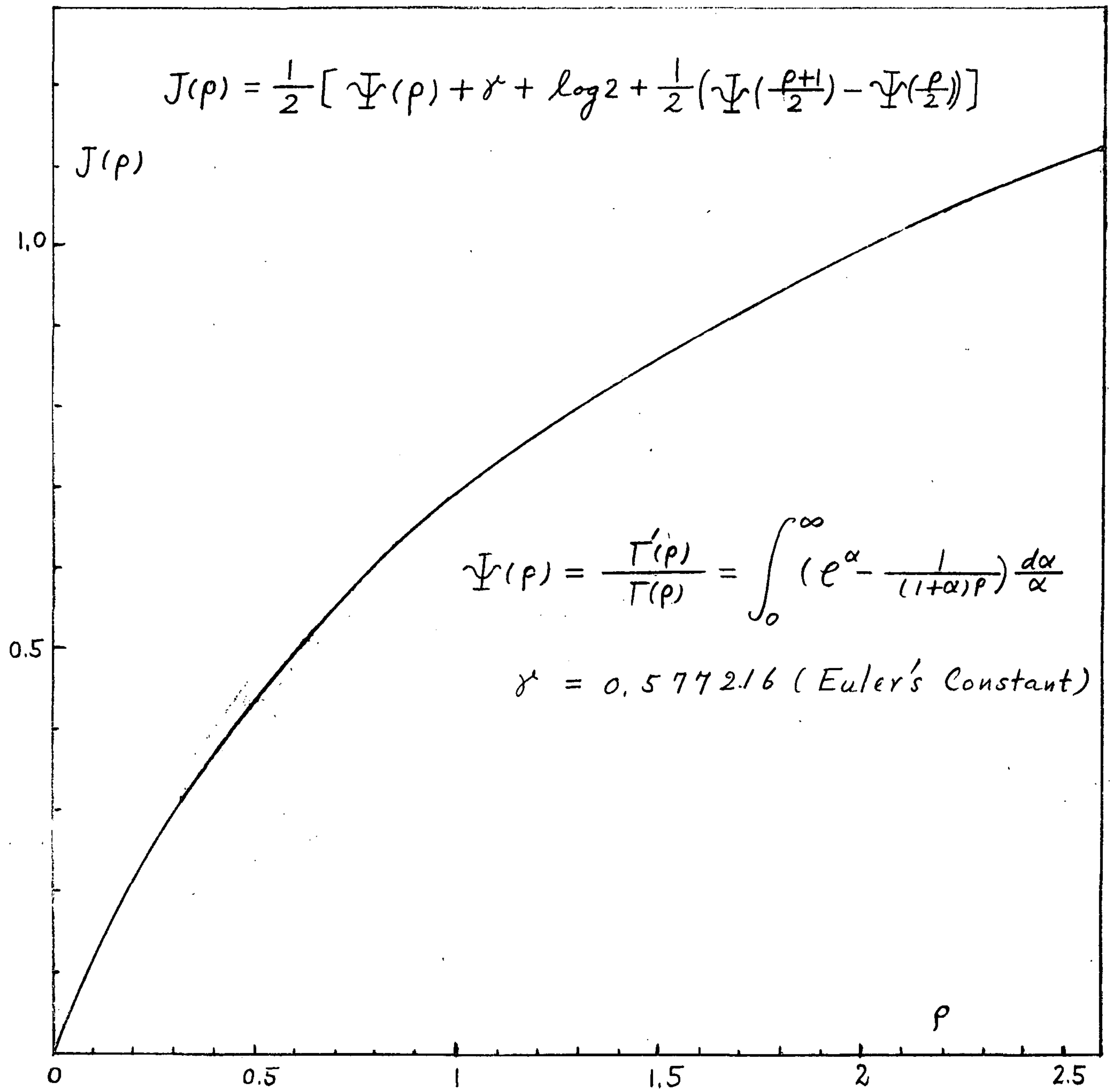
$$= \frac{1}{2}\left[\Psi(\rho) + \gamma + \log 2 + \frac{1}{2}\left\{\Psi\left(\frac{\rho+1}{2}\right) - \Psi\left(\frac{\rho}{2}\right)\right\}\right]$$

where

$$\Psi(\rho) = \frac{\Gamma'(\rho)}{\Gamma(\rho)} = \int_0^{\infty}\left(e^{\alpha} - \frac{1}{(1+\alpha)^{\rho}}\right)\frac{d\alpha}{\alpha}$$

$$\gamma = 0.577215 6649 \cdots \quad \text{(Euler's Constant)}$$

$J(\rho)$ is represented graphically in the following figure.


$J(\rho) = \frac{1}{2}\left[\Psi(\rho) + \gamma + \log 2 + \frac{1}{2}\left(\Psi\left(\frac{\rho+1}{2}\right) - \Psi\left(\frac{\rho}{2}\right)\right)\right]$
J(ρ)
1.0
0.5
$\Psi(\rho) = \frac{\Gamma'(\rho)}{\Gamma(\rho)} = \int_0^\infty \left(e^{\alpha} - \frac{1}{(1+\alpha)^{\rho}}\right)\frac{d\alpha}{\alpha}$
γ = 0, 577216 (Euler's Constant)
ρ
0
0.5
1
1.5
2
2.5

Function J(ρ)

CHAPTER XI

References

1. Timoshenko:
   Theory of Plates and Shells, (1940) McGraw-Hill Book Co., Inc., New York

2. Girkmann:
   Flächentragwerke 4aufl, (1956) Wien. Springer Verlag

3. Nadai:
   Elastische Platten (1921) Springer Verlag

4. Pucher, A.:
   Einflußfelder elastischer Platten (1951) Wien Springer Verlag

5. Pucher, A.:
   Die Momenteneinflußfelder rechteckiger Platten Deutscher Ausschuß für Eisenbeton Heft. 90 (1938)

6. Pucher, A.:
   Die Einflußfelder des Plattenstreifens mit zwei eingespannten Rändern Federhofer-Girkmann-Festschrift, s.303 (1950)

7. Michell, J.H.:
   On the Flexure of a Circular Plate Proc. London Math. Soc. 34, 223 (1902)

8. Westergaard, H.M.:
   Computation of Stresses in Bridge Slabs Due to Wheel Loads Public Roads, Vol. II (1930) p.1

9. Baron, F.M.:
   Influence Surfaces for Stresses in Slabs ASME Transactions, Vol. 63, 1941, p.A-3

10. Bittner, E.:
    Momententafeln und Einflußflächen für Kreuzweise bewehrte Eisenbeton-Platten, Springer, Vienna 1938.

11. Mossakowski, J.:
    Singular Solution in the Theory of Orthotropic Plates Arch. Mech. Stos.6 (1954) p.413-432.

12. Nowacki, W. and Mossakowski, J.:
    The Influence Surfaces of Plates Representing Annular Sectors Arch. Mech. Stos.5 (1953) p.237-272.

13. Cywinska, Z. and Mossakowski, J.:
    The Influence Surfaces of an Orthotropic Semi-Infinite Strip Arch. Mech. Stos.6 (1954) p.33-64.

14. Kist, H.J. and Bouma, A.L.:
    An Experimental Investigation of Slabs, Subjected to Concentrated Loads IABSE (1954) p.85.

15. Hoeland, G.:
Stützenmomenteneinflußfelder durchlaufender elastischer Platten mit zwei frei drehbar gelagerten Rändern
Ing. Archiv. Vol.24 (1956) p.124.

16. Thürlimann, B.:
Influence Surfaces for Support Moments of Continuous Slabs
IABSE, Publications, Vol. 16, (1956) p.485. isotropic

17. Kawai, T. and Thürlimann, B.:
Influence Surfaces for Moments in Slabs Continuous Over Flexible Cross Beams
IABSE, to be published, Vol. 17 (1957)

18. Müller, W.:
Die Momentenfläche der elastischen Platte oder Pilzdecke und die Bestimmung der Durchbiegung aus der Momenten
Ing-Archiv XXI (1953) p.63.

19. Massonnet, Ch.:
Recherches expérimentales sur les ponts à poutres multiples (Extrait des annales des travanx publics de Belgique, No.2, 1955)

20. Morice, P.B. and Little:
Load Distribution in Prestressed Concrete Bridge Systems
The Structural Engineer, March 1954, pp. 83-111.

21. Morice, P.B. and Little:
The Analysis of Right Bridge Decks Subjected to Abnormal Loading
Cement and Concrete Association, July 1956.

22. Sattler, K.:
Betrachtungen zum Berechnugverfahren von Guyon-Massonnet für frei-aufliegende Trägerroste und Erweiterung dieses Verfahrens and beliebige Systeme (Der Bauingenieur Heft 3 - 1955).

23. Cornelius, W.:
Die Berechnung der ebenen Flächentragwerke mit Hilfe der Theorie der orthogonal anisotropen Platte
Stahlbau 21,21,43,60 (1952)

24. Walter, R.:
Investigation of Multi-Beam Bridges
Lehigh University, Institute of Research, August 1956
Prestressed Concrete Bridge Members, Progress Report No.14

25. Hoppmann, W.H.,2nd.; Huffington, N.J., Jr.; and Magness, L.S.:
Orthogonally Stiffened Plates
Journal of Applied Mechanics, Vol. 23, No.3 (September 1956) p.343.

26. Roesli, A.:
Lateral Load Distribution in Multi-Beam Bridges
Prestress Concrete Bridge Members, Progress Report No. 10, Lehigh University

27. Whittaker & Watson:
Modern Analysis
Cambridge University Press, (1935)

28. Wilson:
Advanced Calculus
Ginn & Company, Boston

29. Byerly:
An Elementary Treatise On Fourier's Series and Spherical, Cylindrical and Elliposoidal Harmonics
Ginn & Company, Boston

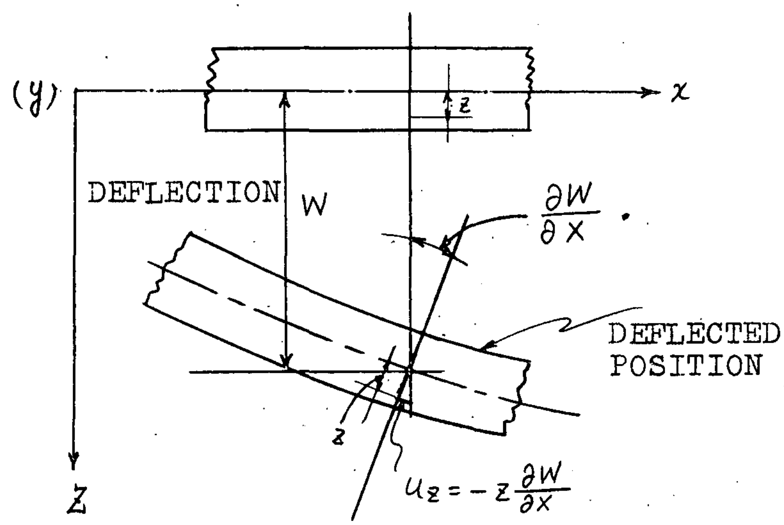
30. Bierens de-Haan, D.:
Nouvelles Tables d'Integrales Definites
P. Engels Leide, 1867

CHAPTER XII

Nomenclature

a	Width of strips
b	Length of a rectangular plate
D	Flexural rigidity of an isotropic plate
D_x, D_y	Flexural rigidity of an orthotropic plate in the x- and y-axes respectively
D_{xy}	Torsional rigidity of an orthotropic plate
D_1	Some elastic constant of an orthotropic plate
E	Modulus of elasticity in tension and compression
E'_x, E'_y, E''	Elastic constants to characterize the properties of an orthotropic material
G	Modulus of elasticity in shear
H	Torsional rigidity of an orthotropic plate, $H=D_1+2D_{xy}$
h	Thickness of a plate
I	Bending rigidity of a beam
I_w	Warping rigidity of the beam
K_t	Torsional rigidity of the beam
k_1, k_2, k_3, k_4	Some constant controlling elastic properties of an orthotropic plate (Section (3.2))
L, M, N	Some elastic constants associated with free edge boundary (Section (4.4))
M_x, M_y	Bending moment per unit length of sections of a plate perpendicular to x- and y-axes, respectively
M_{xy}	Twisting moment per unit length of section of a plate perpendicular to x-axis
m_x, m_y, m_{xy}	Influence surfaces for M_x, M_y, M_{xy}, respectively
Q_x, Q_y	Shearing forces parallel to z-axis per unit length of sections of a plate perpendicular to x- and y-axes respectively
q_x, q_y	Influence surfaces for Q_x and Q_y respectively
q	Intensity of a distributed load

R_i, S_i, T_i	Some transcendental functions defined in Table (I)
r	Influence surface for reaction of a simply supported rectangular corner (Section (4.2; IV))
(r, θ)	Polar coordinates
(u, v)	Rectangular coordinates of influence point
V_x, V_y	Boundary shears corresponding to Q_x and Q_y, respectively.
W	Deflection of a plate in z-axis
x, y, z	Rectangular coordinates
$\alpha, \beta; \xi, \eta$	Non-dimensional coordinates of the influence point (p.25)
γ	Aspect ratio of a rectangular plate (p.42)
γ_{xy}	Shearing strain component in rectangular coordinates
ϵ_x, ϵ_y	Unit elongation in x- and y-directions
σ_x, σ_y	Normal components of stress parallel to x- and y-axes
τ_{xy}	Shearing stress component in rectangular coordinates
τ, τ'	Half periods of ϑ_1-functions (p.46)
ρ	Ratio of bending rigidity of a cross beam and bending rigidity of a plate in y-direction (eq. (6.7))
λ, μ	Parameters controlling anisotropy of a plate (eq. (3.7))
ν	Poisson's ratio
$\zeta, \zeta_0; \zeta', \zeta_0'$	Non-dimensional coordinates in complex variable (eq. (5.9))



u= Displacement in the x-direction

v= Displacement in the y-direction

w= Displacement in the z-direction

DISPLACEMENTS OF THE POINT Z

$$W_z = W$$

$$U_z = -z\frac{\partial W}{\partial x}$$

$$V_z = -z\frac{\partial W}{\partial y}$$

STRAINS

$$\varepsilon_x = \frac{\partial U_z}{\partial x} = -z\frac{\partial^2 W}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial V_z}{\partial y} = -z\frac{\partial^2 W}{\partial y^2}$$

$$\gamma_{yz} = \frac{\partial U_z}{\partial y} + \frac{\partial V_z}{\partial x} = -z\frac{\partial^2 W}{\partial x \partial y}$$

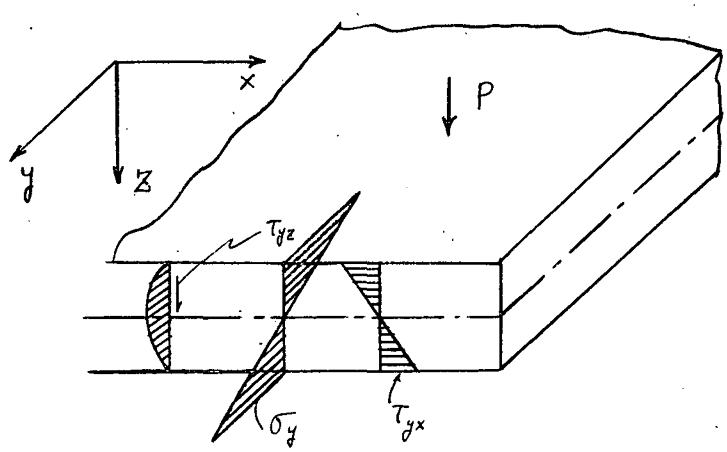


Figure 1-1 TRANSVERSELY LOADED PLATE

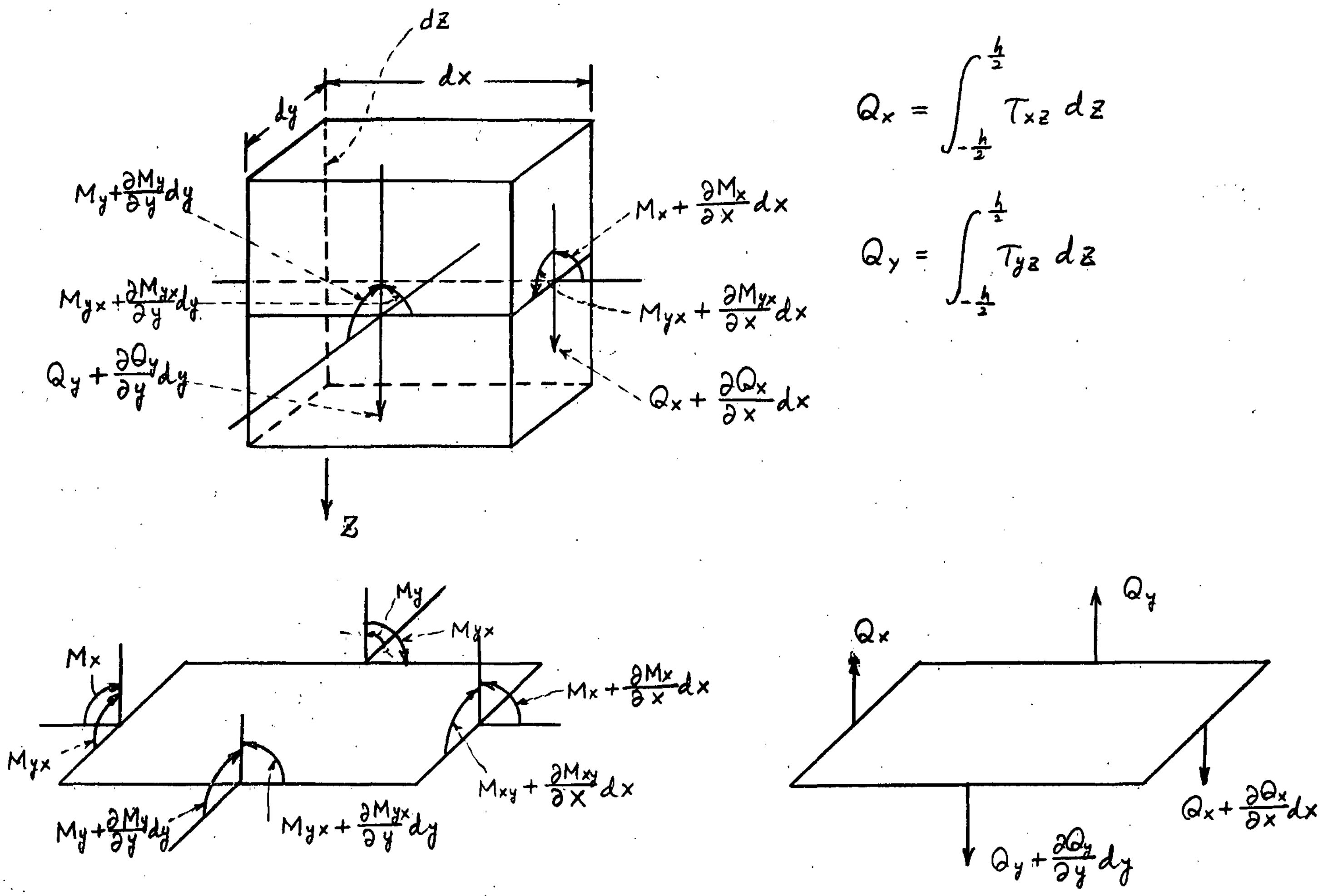
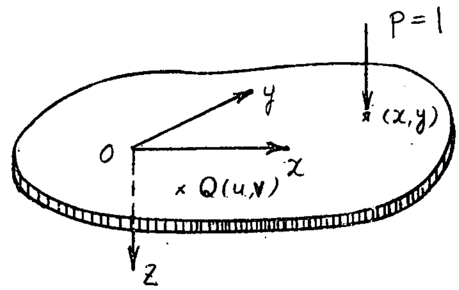


Figure 1-2 EQUILIBRIUM OF THE PLATE ELEMENT



Basic Differential Equation

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} = q(x,y)$$

q(x,y: external load acting on the plate and in this case

$$q(x,y) = \begin{cases} 0 \text{ (for any point other than } (x,y)) \\ P=1 \text{ (at } (x,y)) \end{cases}$$

with prescribed boundary conditions. (either statical or geometrical conditions)

Figure 1-3 GREEN'S FUNCTION W(u,v;x,y) FOR THE DEFLECTION OF AN ORTHOTROPIC PLATE

(1) P. B. Morice

(2) P. B. Morice

(3) Massonet (Pont de Fer à Tournai)

(4) Massonet (Pont de Rocour)

(5) Girkmann (Köln-Mülheim Hängerbricke)

(6) P. B. Morice

(7) Massonet (Pont d'Elouges)

(8) Olsen u. Reinitzhuber

(9) R. Walther

(10) Massonet (Pont de la rue du Sable à Coutrai)

(11) Girkmann (Plywood plate)

(12) Massonet (Pont de Trochinnes-Baarle)

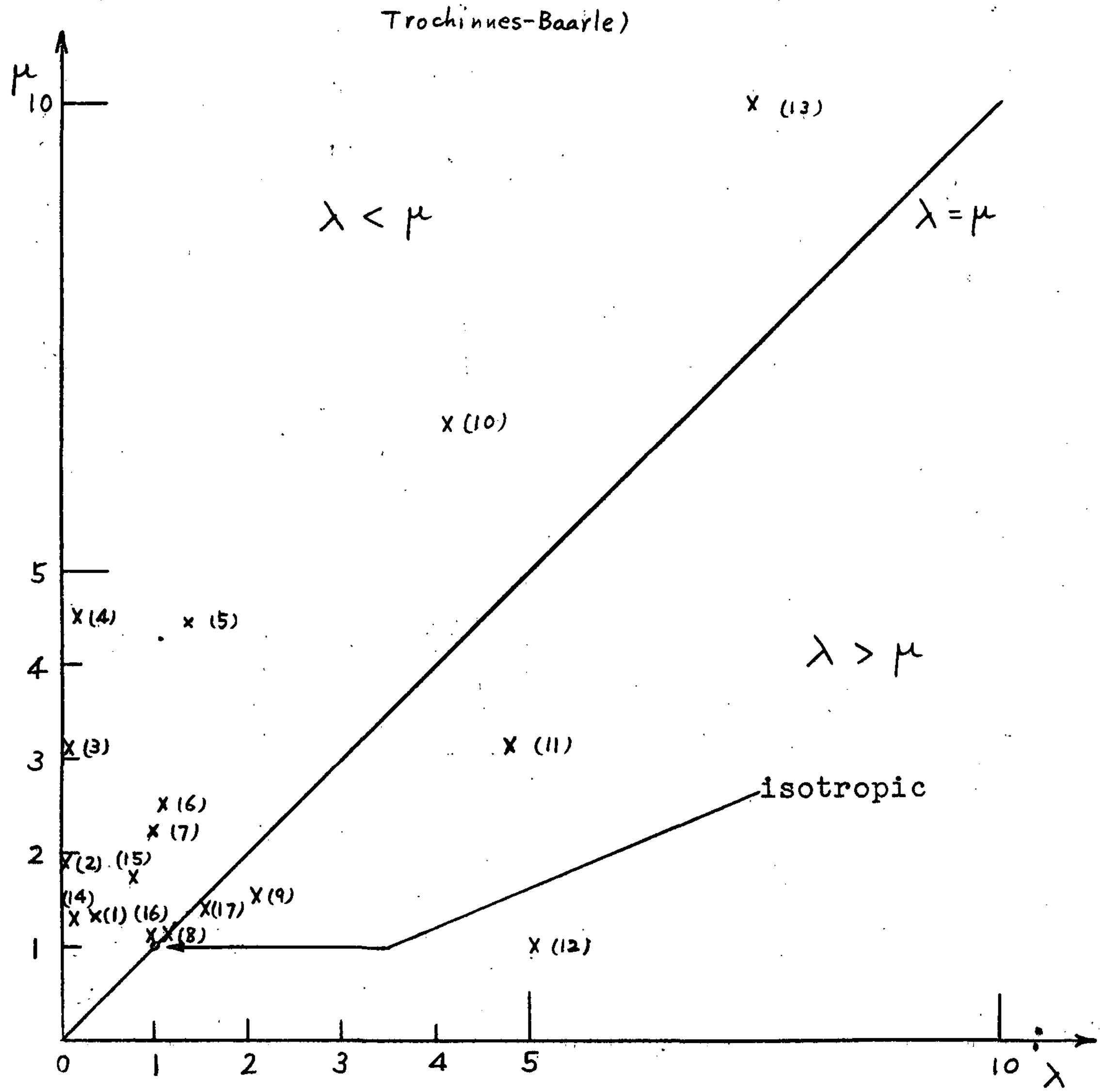
(13) R. Walther

(14) K. Sattler (model)

(15) K. Sattler (model)

(16) W. H. Hoppmann (model)

(17) W. H. Hoppmann (model)



$$\lambda = \frac{H}{D_y} \quad , \quad \mu = \sqrt{\frac{D_x}{D_y}}$$

Figure 1-4 EXAMPLES OF ORTHOTROPIC SLABS

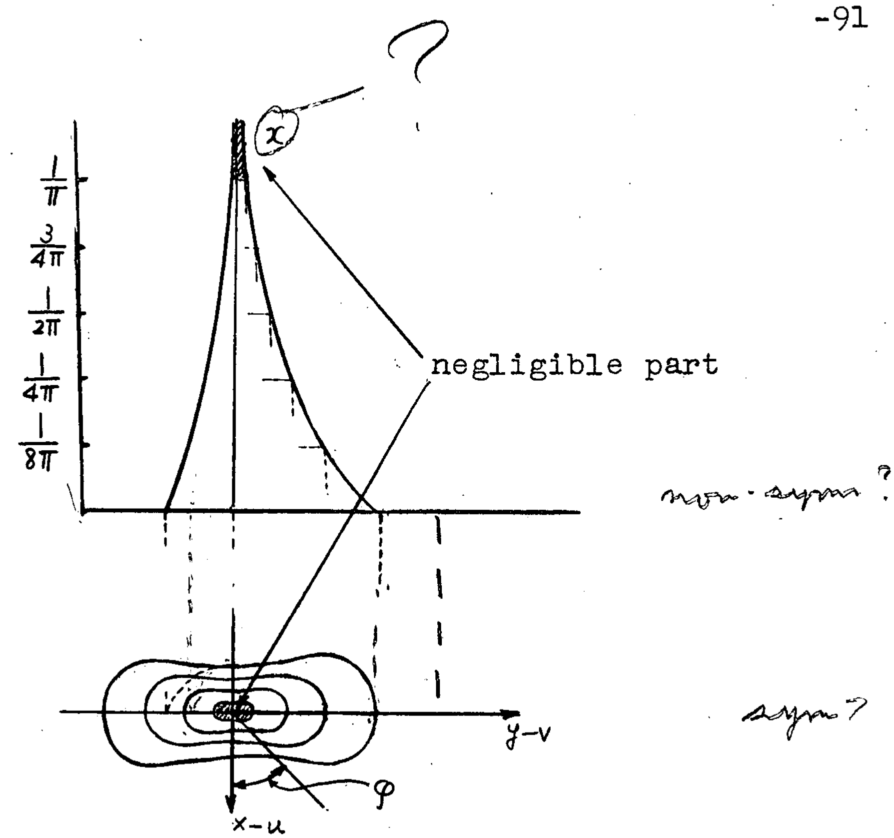


Figure 2-1 $(m_x)_o$-INFLUENCE SURFACE IN THE VICINITY OF THE INFLUENCE POINT

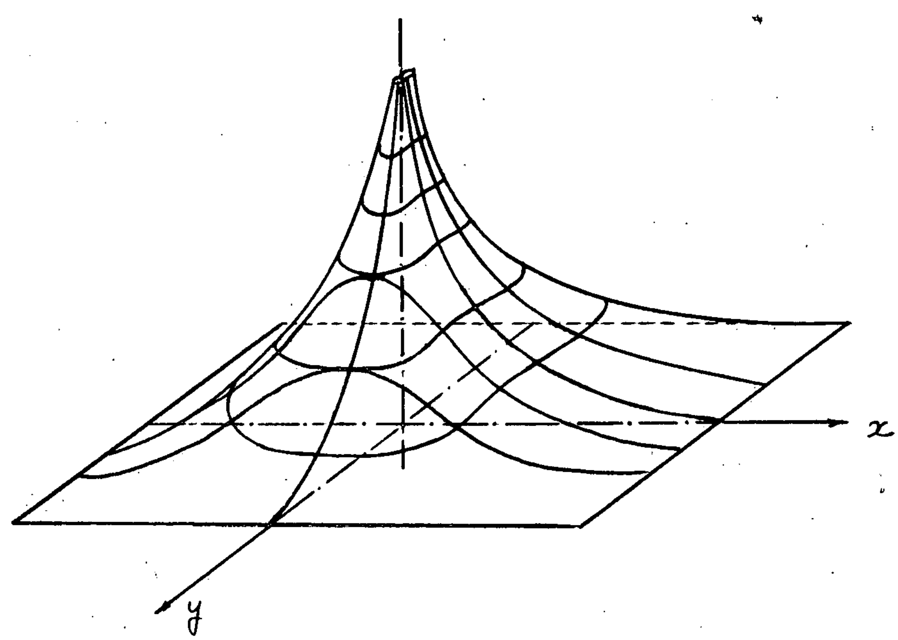


Figure 2-2 THREE-DIMENSIONAL APPEARANCE OF $m_x(u,v)$ INFLUENCE SURFACES

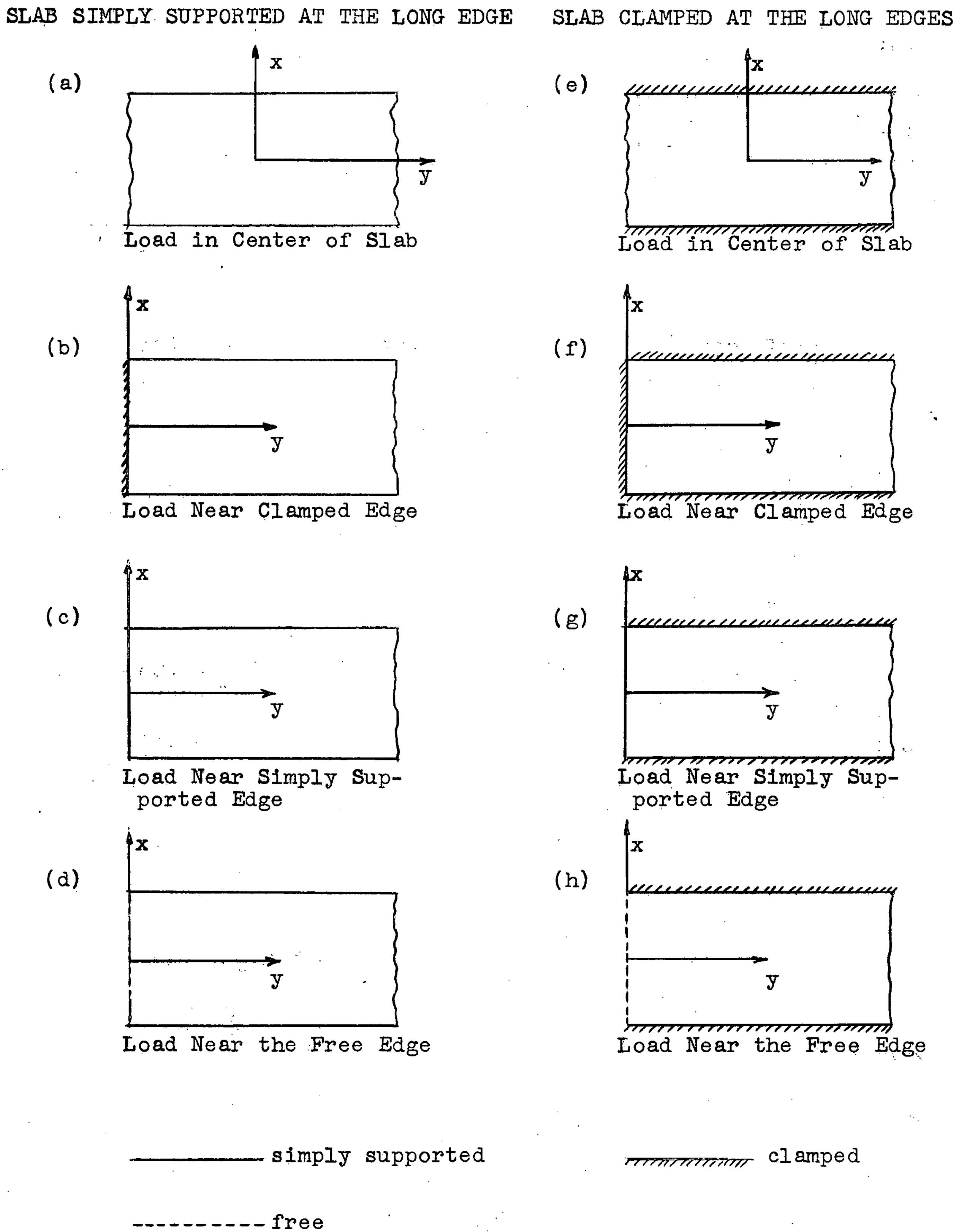
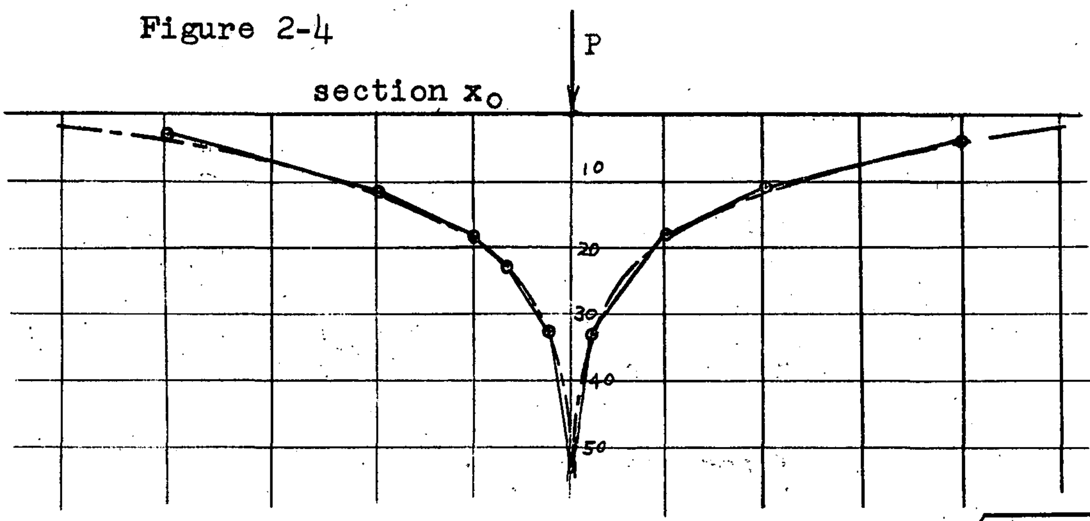


Figure 2-3 THE DIFFERENT CASES OF BOUNDARY CONDITIONS

Figure 2-4



—·—·— According to the elementary theory.

——— According to test results.

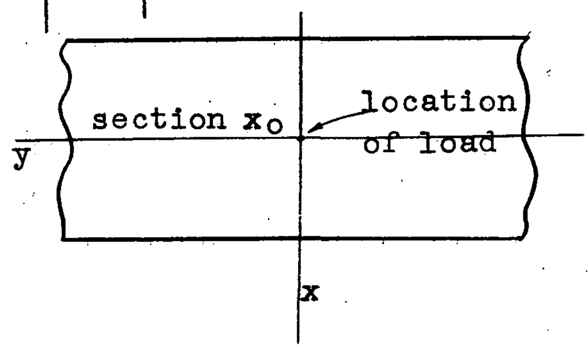
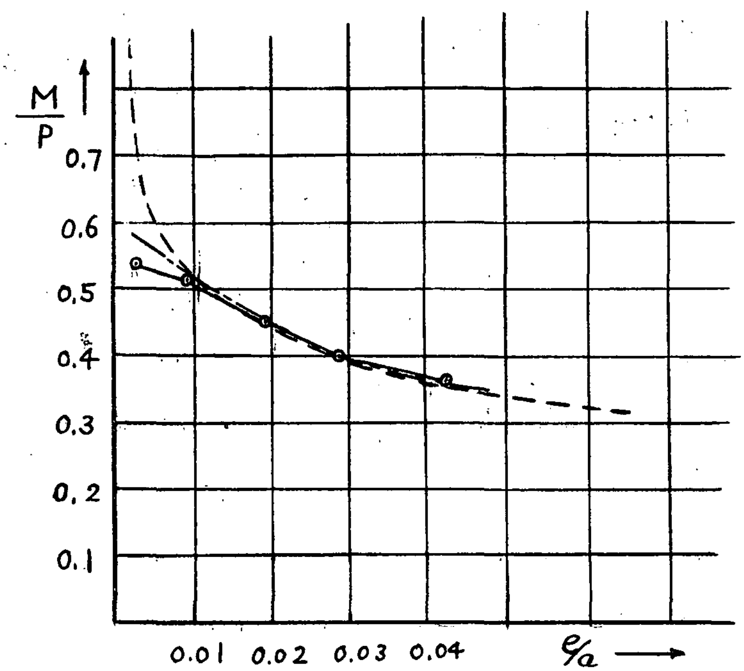


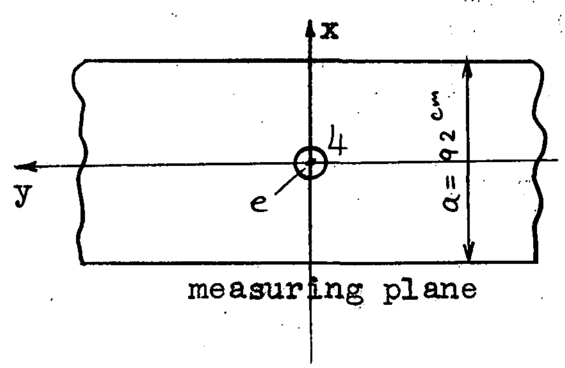
Figure 2-5



- - - - - According to elementary theory of plates (ν=0.3)

——— According to test results.

—·—·— According to Westergaard (ν=0.3)



Figures 2-4 and 2-5 CONSISTENCY BETWEEN THEORY AND EXPERIMENT

for isotropic plate (Dutch investigators)

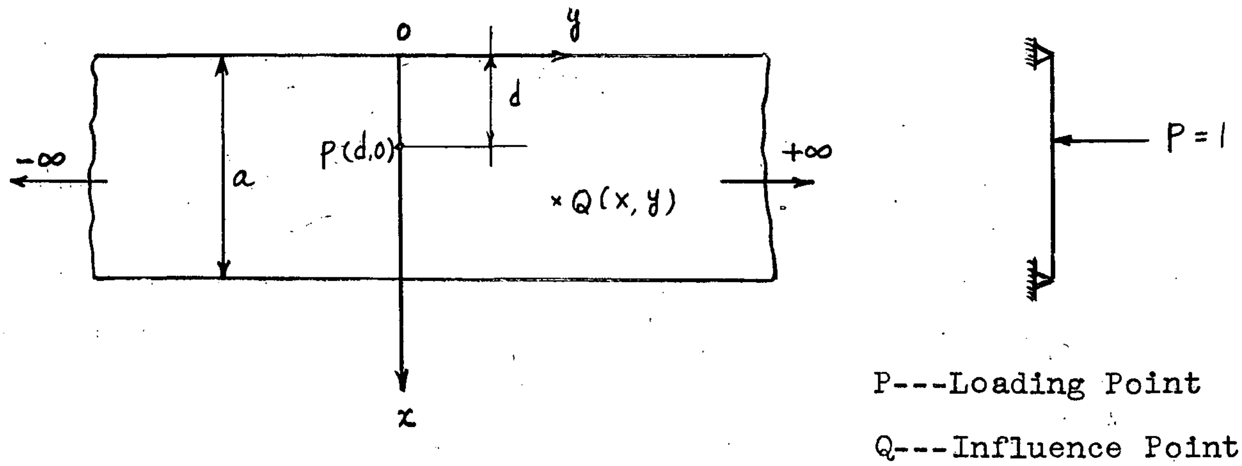
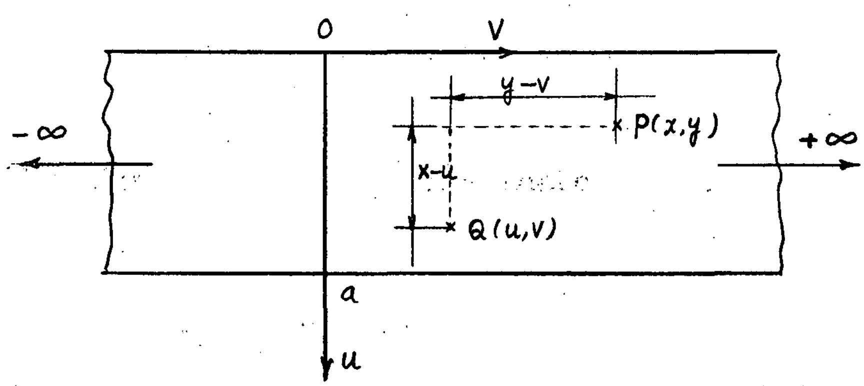
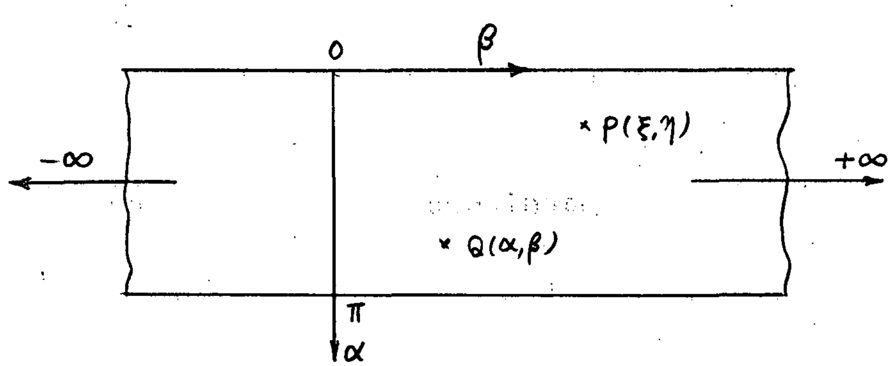


Figure 3-1 INFINITELY LONG STRIP WITH A CONCENTRATED LOAD



Non-dimensional Coordinates System



$$\alpha = \frac{\pi u}{a}$$

$$\beta = \frac{\pi v}{a}$$

$$\xi = \frac{\pi x}{a}$$

$$\eta = \frac{\pi y}{a}$$

Figure 3-2 SIMPLY SUPPORTED INFINITE STRIP

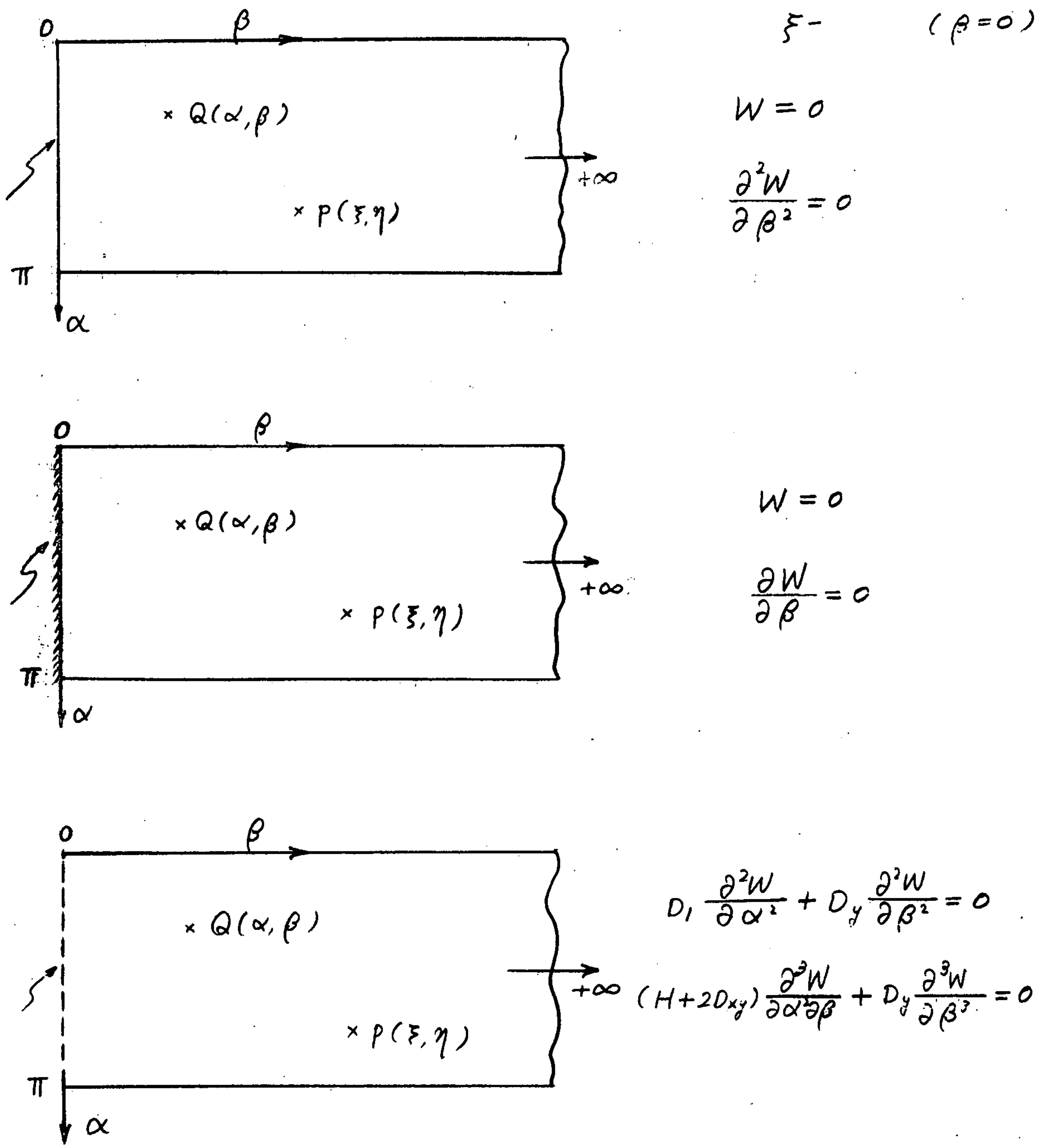
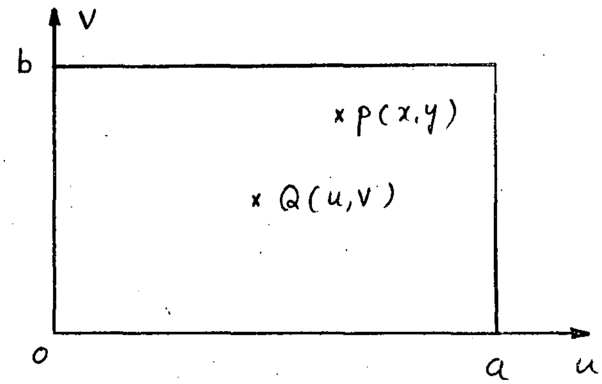


Figure 4-1 SEMI-INFINITE STRIPS



P(x,y) - Loading point
Q(u,v) - Influence point

Boundary Conditions

$x = 0$	$W = 0$,	$\frac{\partial^2 W}{\partial x^2} = 0$
$x = a$	$W = 0$	$\frac{\partial^2 W}{\partial x^2} = 0$
$y = 0$	$W = 0$	$\frac{\partial^2 W}{\partial y^2} = 0$
$y = b$	$W = 0$	$\frac{\partial^2 W}{\partial y^2} = 0$

Figure 5-1 RECTANGULAR PLATE WITH SIMPLY SUPPORTED EDGES

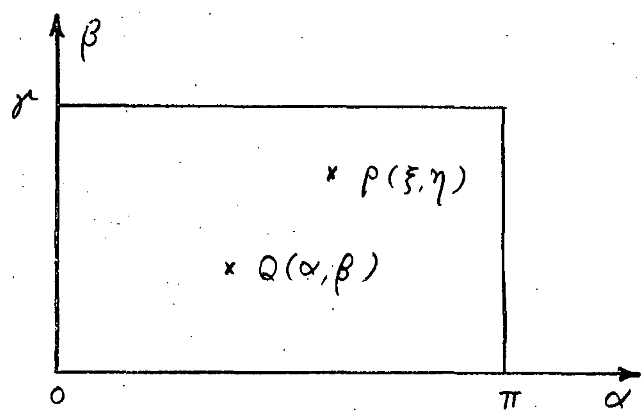
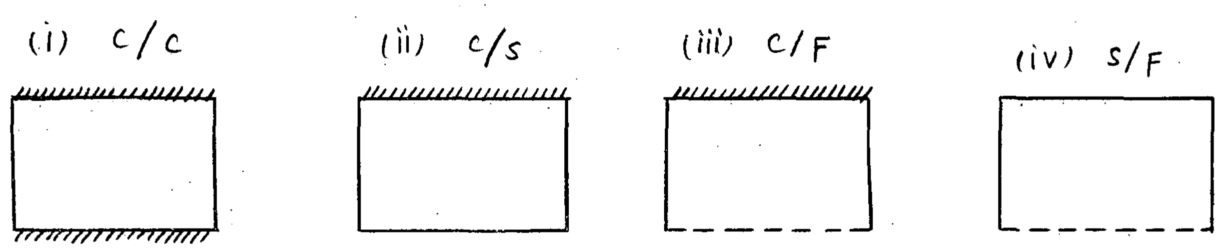


Figure 5-2 NON-DIMENSIONAL COORDINATE SYSTEM



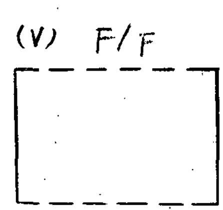


Figure 5-3 POSSIBLE BOUNDARY VALUE PROBLEMS OF A RECTANGULAR PLATE WITH TWO PARALLEL EDGES SIMPLY SUPPORTED

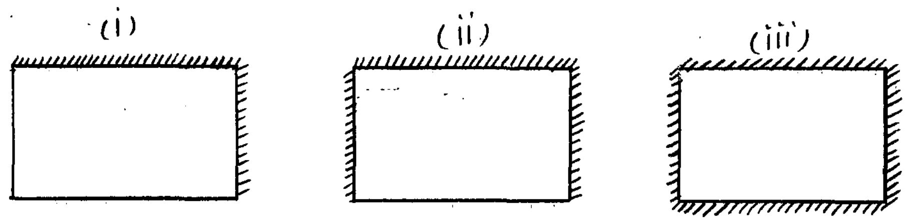


Figure 5-4 POSSIBLE BOUNDARY VALUE PROBLEMS OF A RECTANGULAR PLATE WITH ALL EDGES EITHER SIMPLY SUPPORTED OR CLAMPED

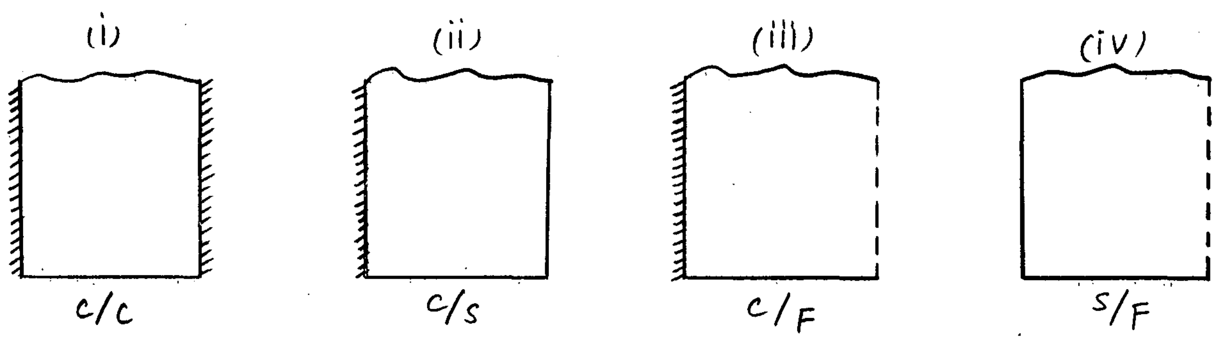
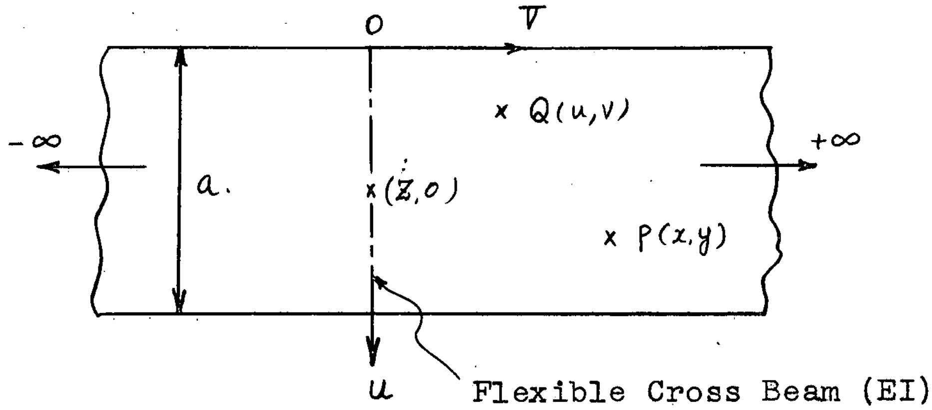


Figure 5-5 BOUNDARY VALUE PROBLEMS OF SEMI-INFINITE PLATE STRIPS WITH THE THIRD EDGE SIMPLY SUPPORTED



Non-Dimensional Coordinates

$$\frac{\pi u}{a} = \alpha, \quad \frac{\pi v}{a} = \beta, \quad \frac{\pi x}{a} = \xi, \quad \frac{\pi y}{a} = \eta, \quad \frac{\pi z}{a} = \zeta$$

Figure 6-1 PLATE STRIP CONTINUOUS OVER FLEXIBLE CROSS BEAM

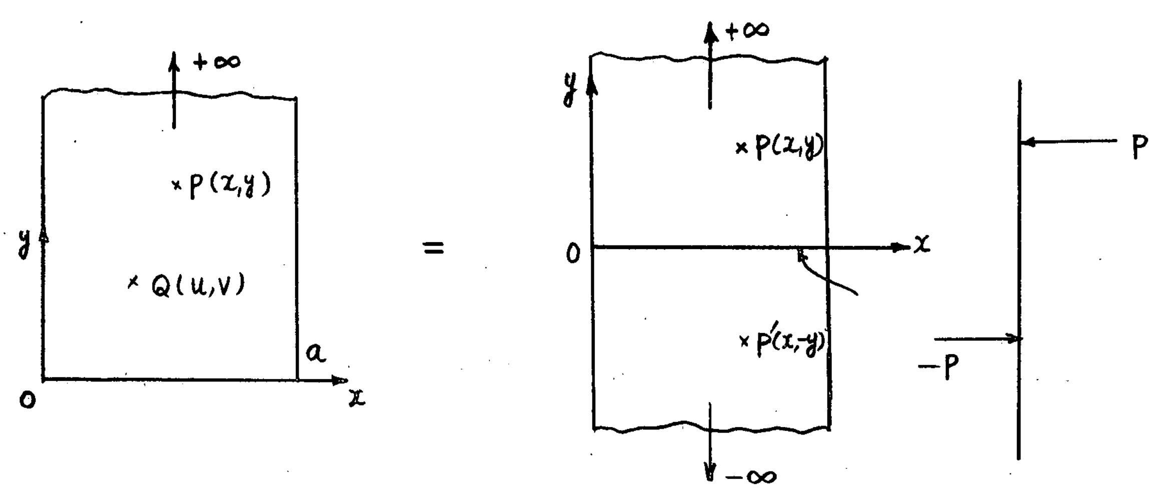


Figure 7-1 SIMPLY SUPPORTED SEMI-INFINITE STRIP

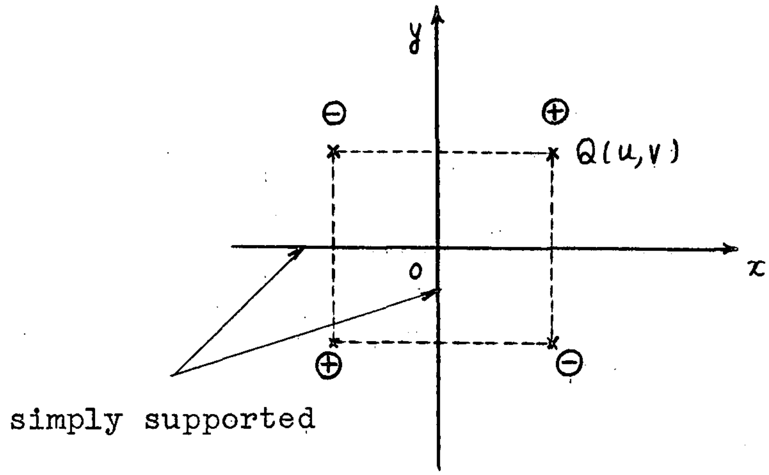


Figure 7-2 INFINITE WEDGE-SHAPED PLATE
(opening angle= $\frac{\pi}{2}$)

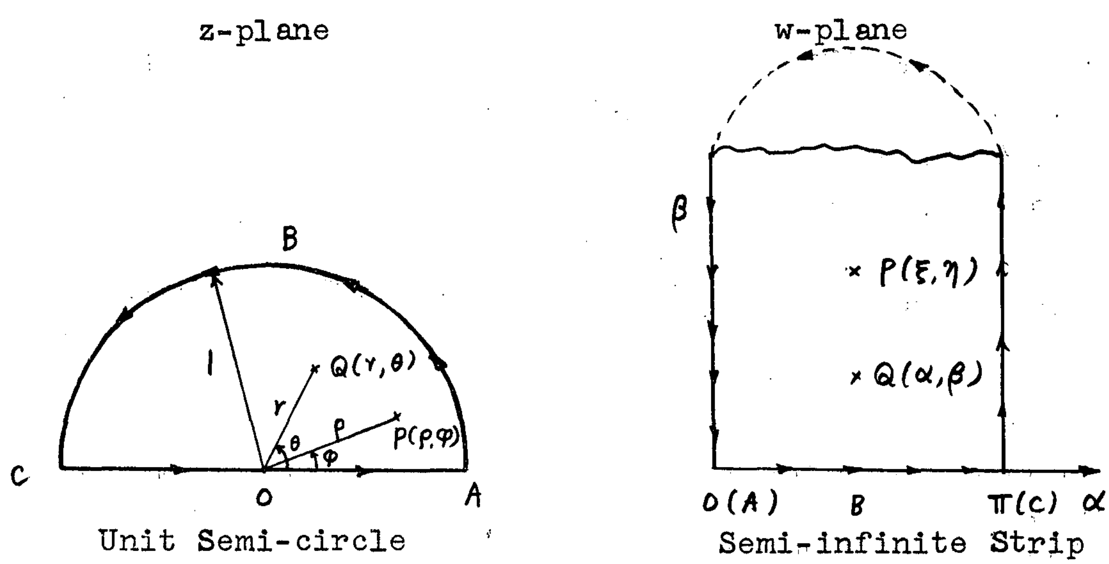


Figure 7-3 CORRESPONDENCY BETWEEN THE UNIT SEMI-CIRCLE AND SEMI-INFINITE PLATE STRIP

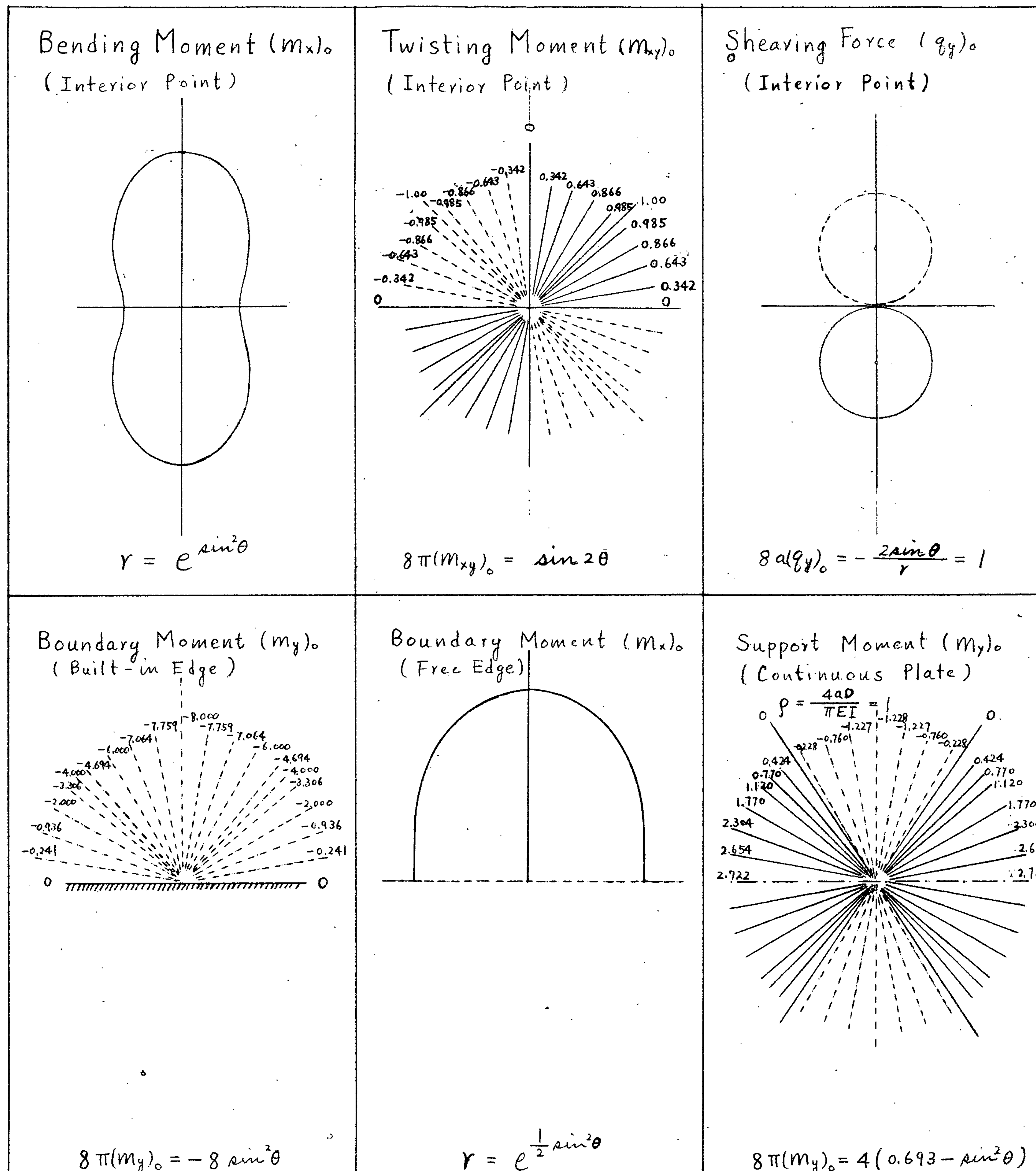


Fig 8-1

Various Singularities of Influence Surfaces

$(\lambda = \mu = 1 : \text{isotropic})$

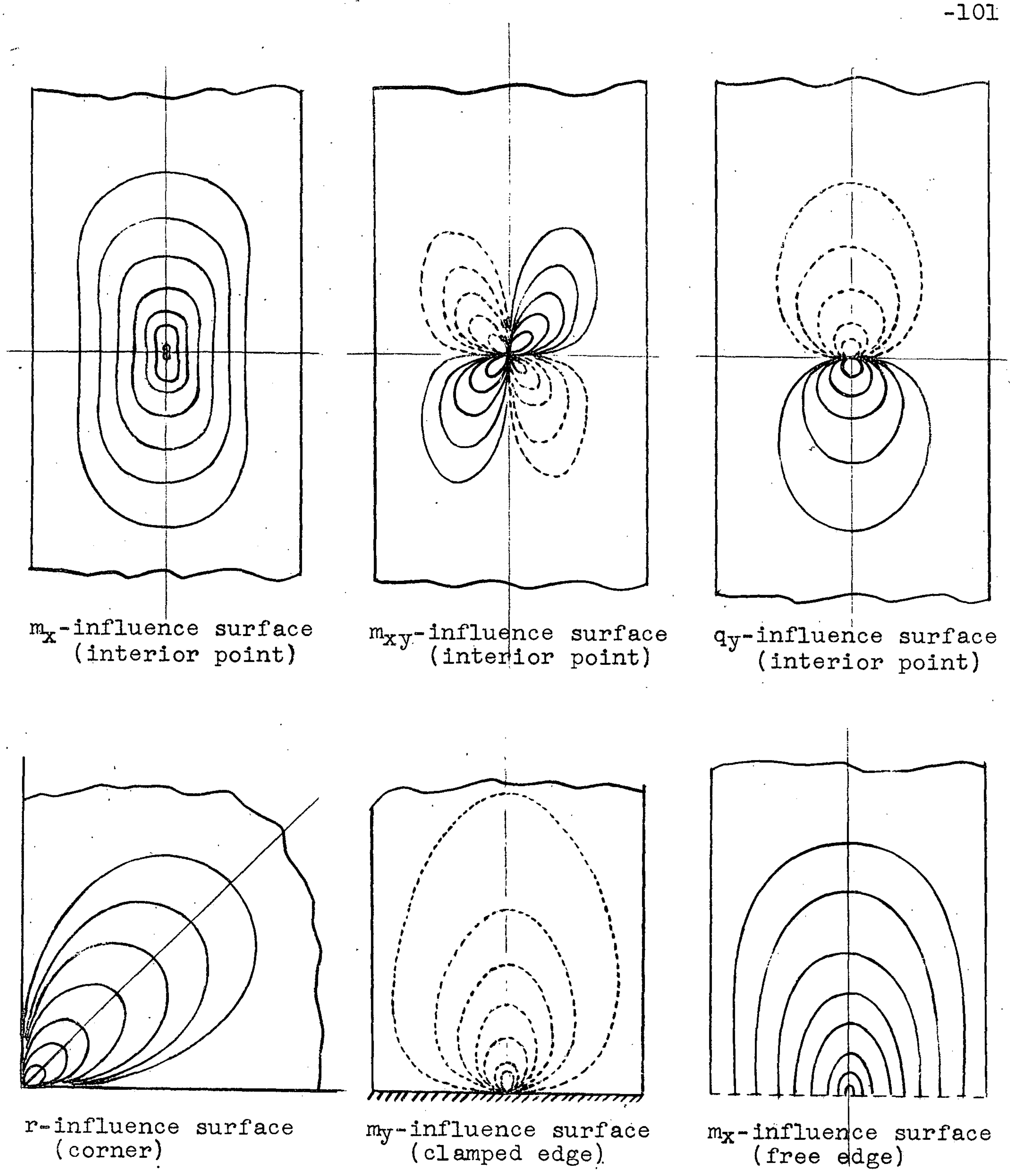


Figure 8-2 GENERAL APPEARANCE OF INFLUENCE SURFACES

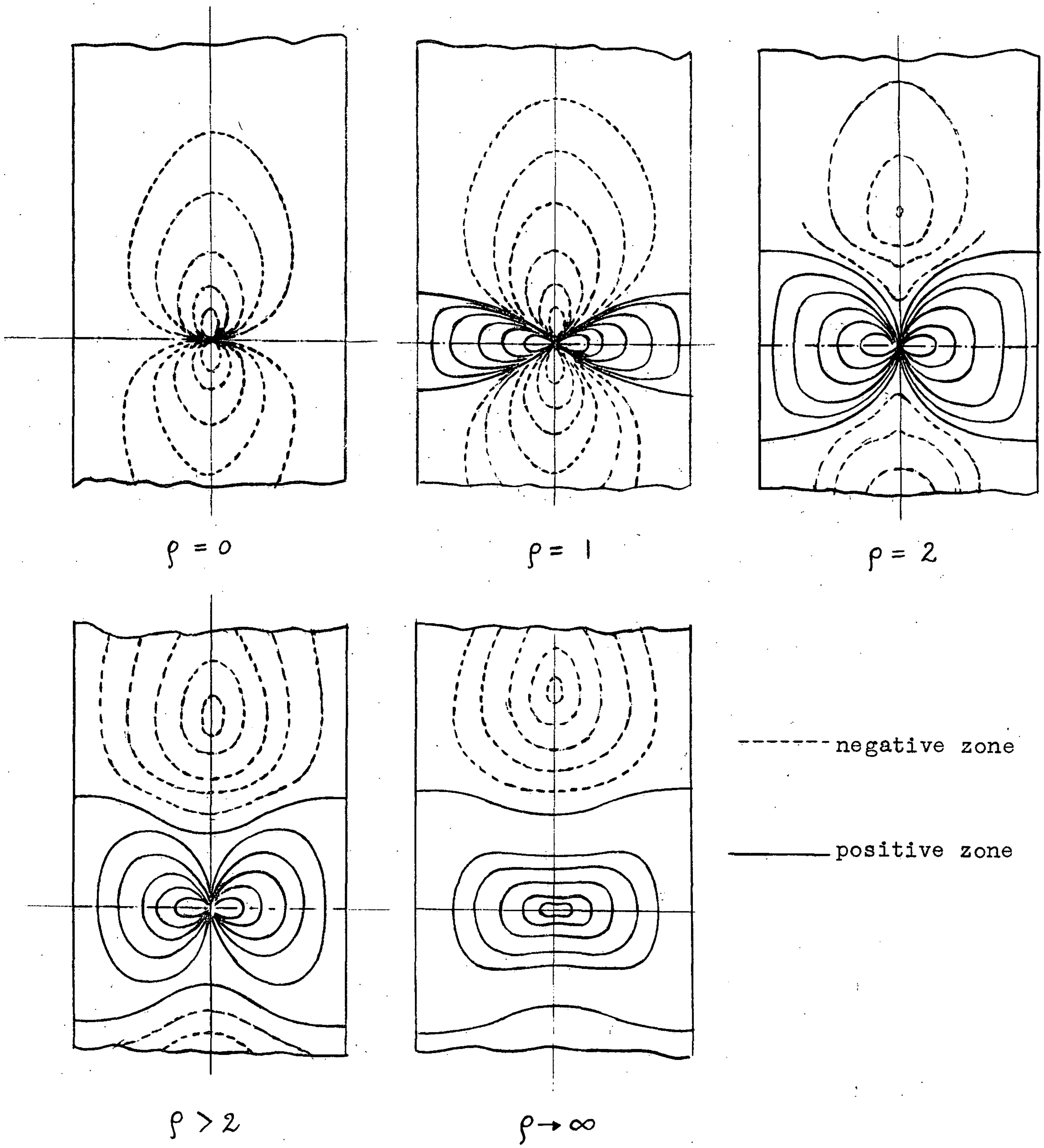
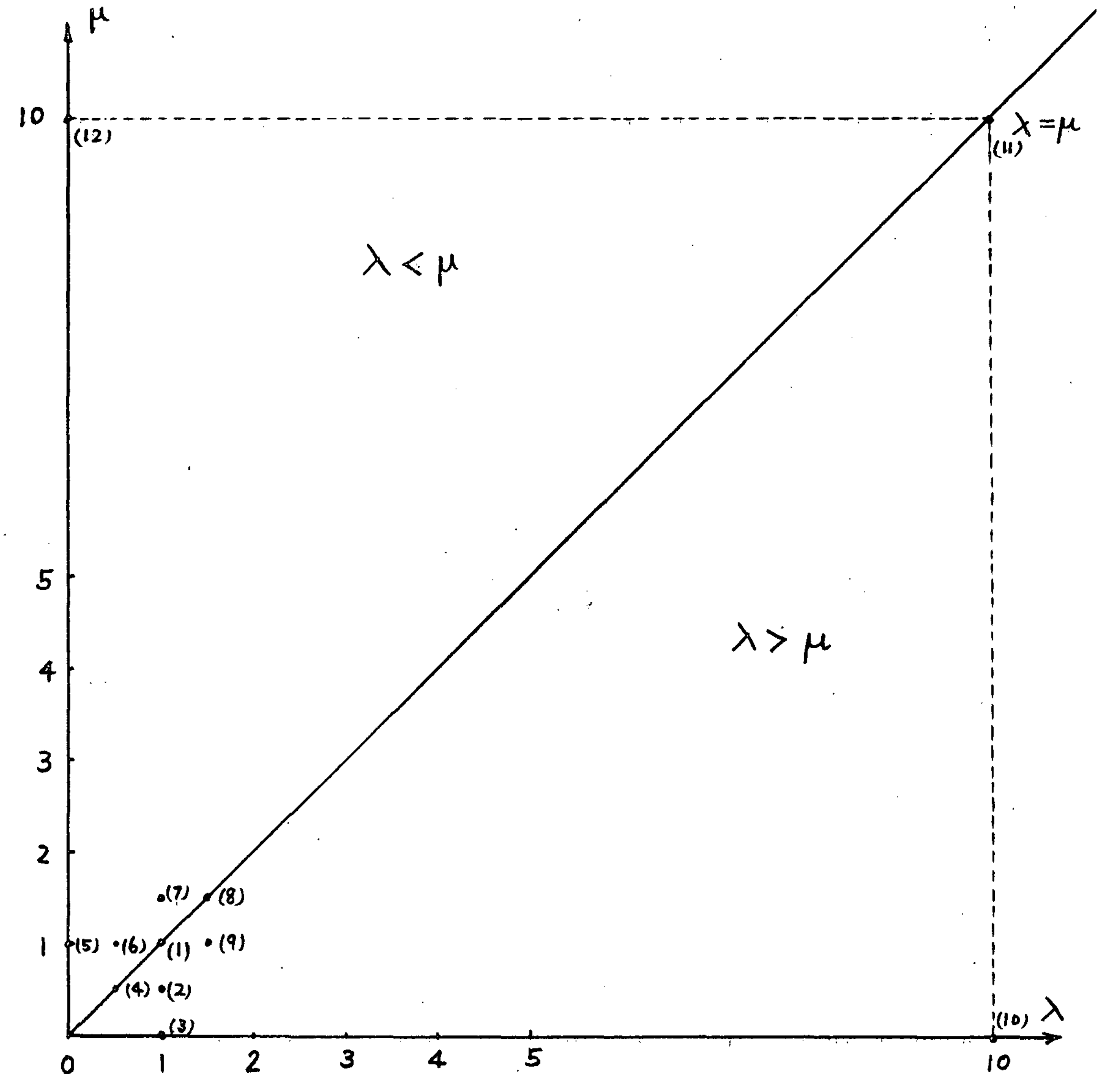


Figure 8-3 INFLUENCE SURFACE FOR SUPPORT MOMENT m_y OF INFINITE PLATE STRIP CONTINUOUS OVER ONE CROSS BEAM



μ \ λ	0	0.5	1.0	1.5	10.
10.	(12)				(11)
1.5			(7)	(8)	
1.0	(5)	(6)	(1)	(9)	
0.5		(4)	(2)		
0			(3)		(10)

Figure 8-4 SEVERAL COMPUTED CASES

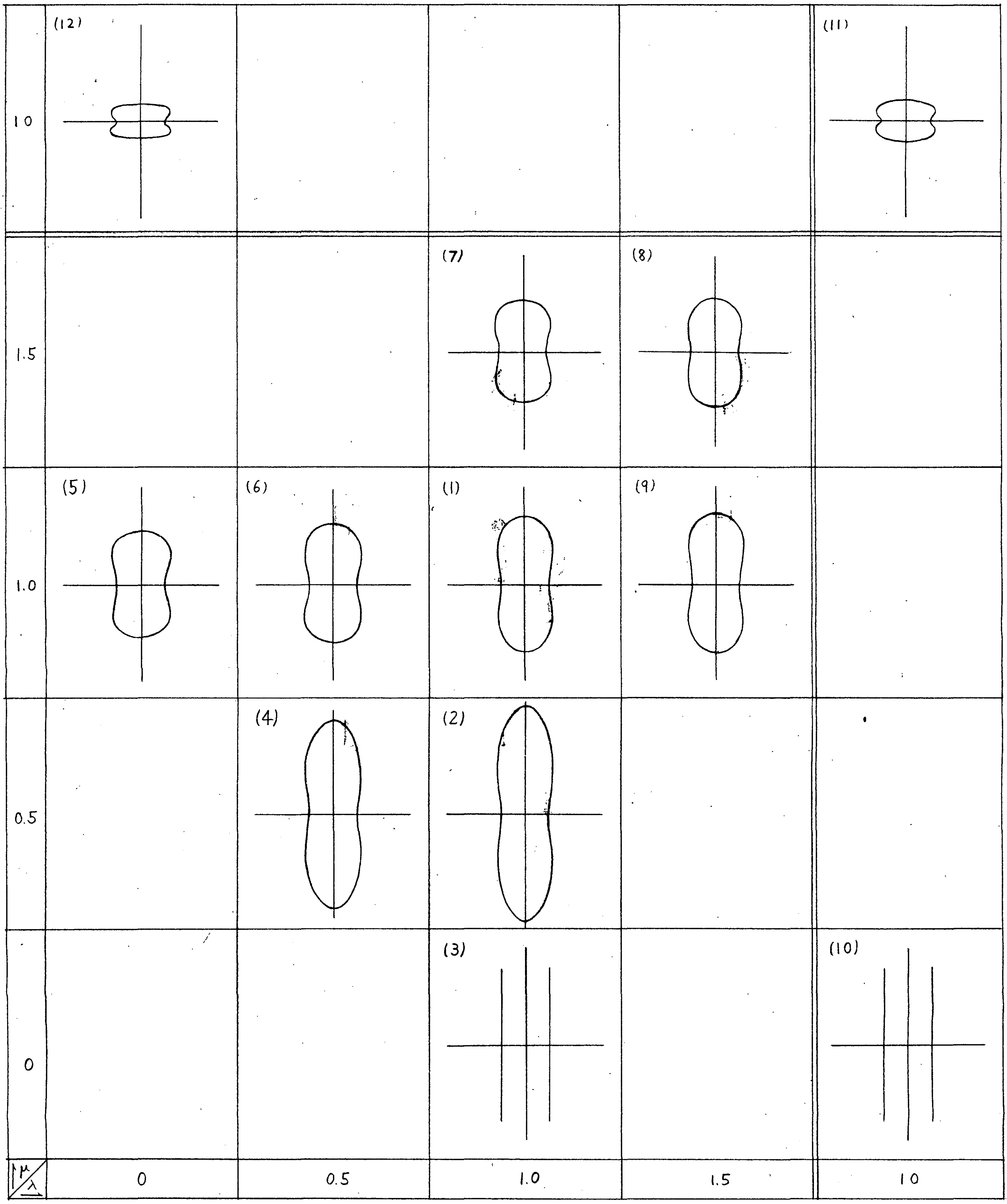


Fig 8-5 $(m_x)_o$ as a Function of λ and μ (interior point)

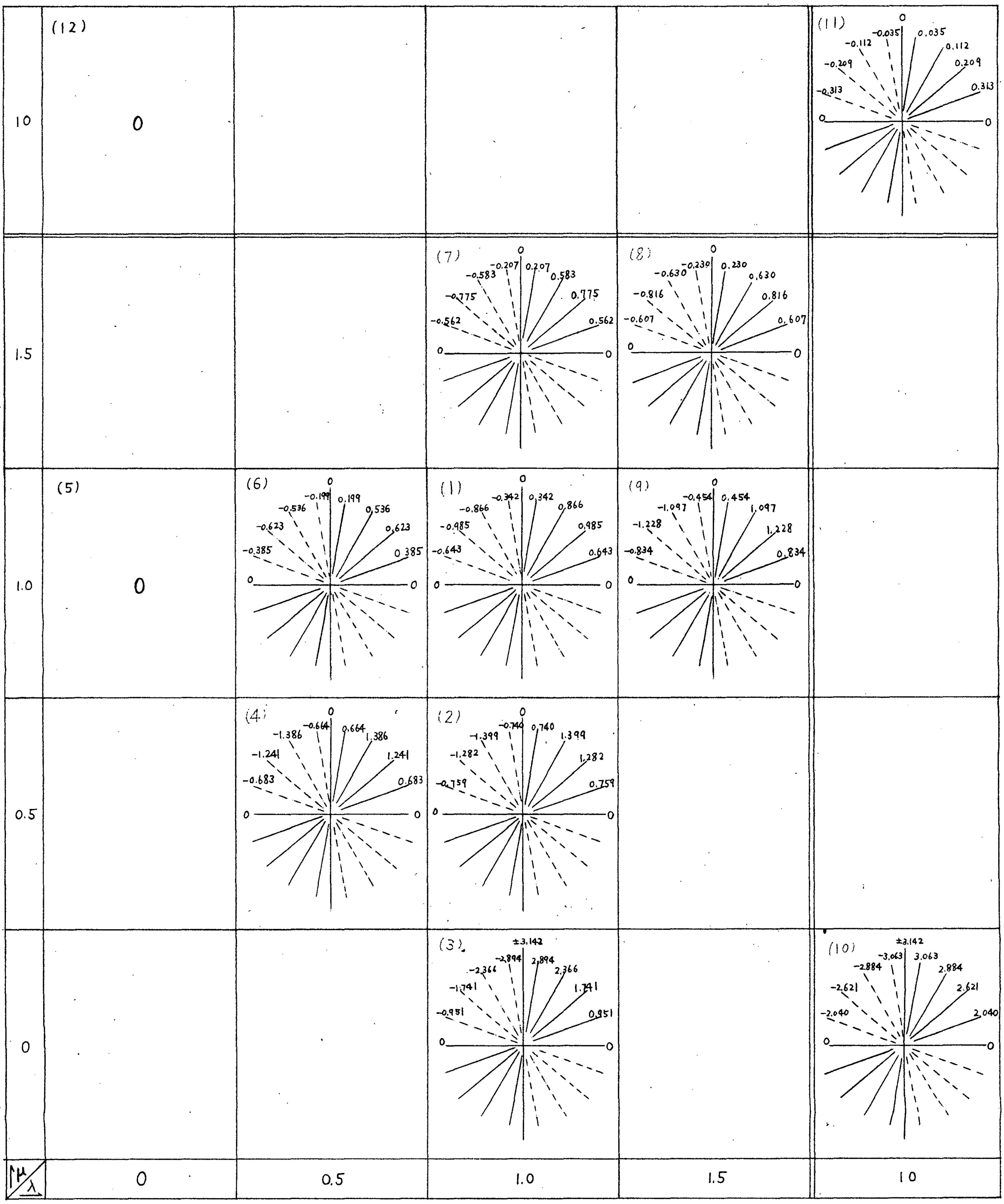


Fig 8-6 $8\pi(m_{xy})_0$ as a Function of λ and μ

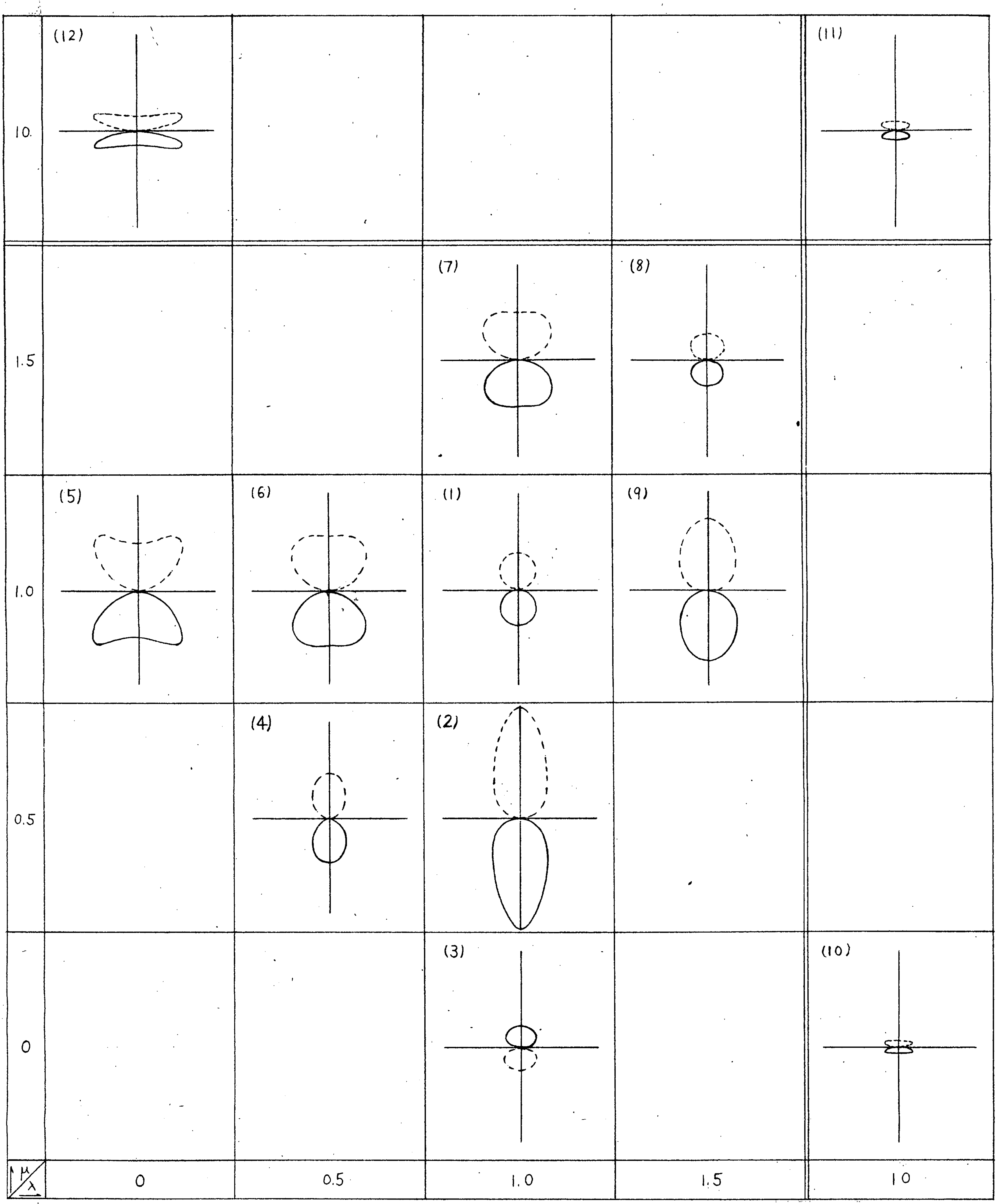


Fig 8-7 $(q_y)_o$ as a Function of λ and μ. (Interior point)

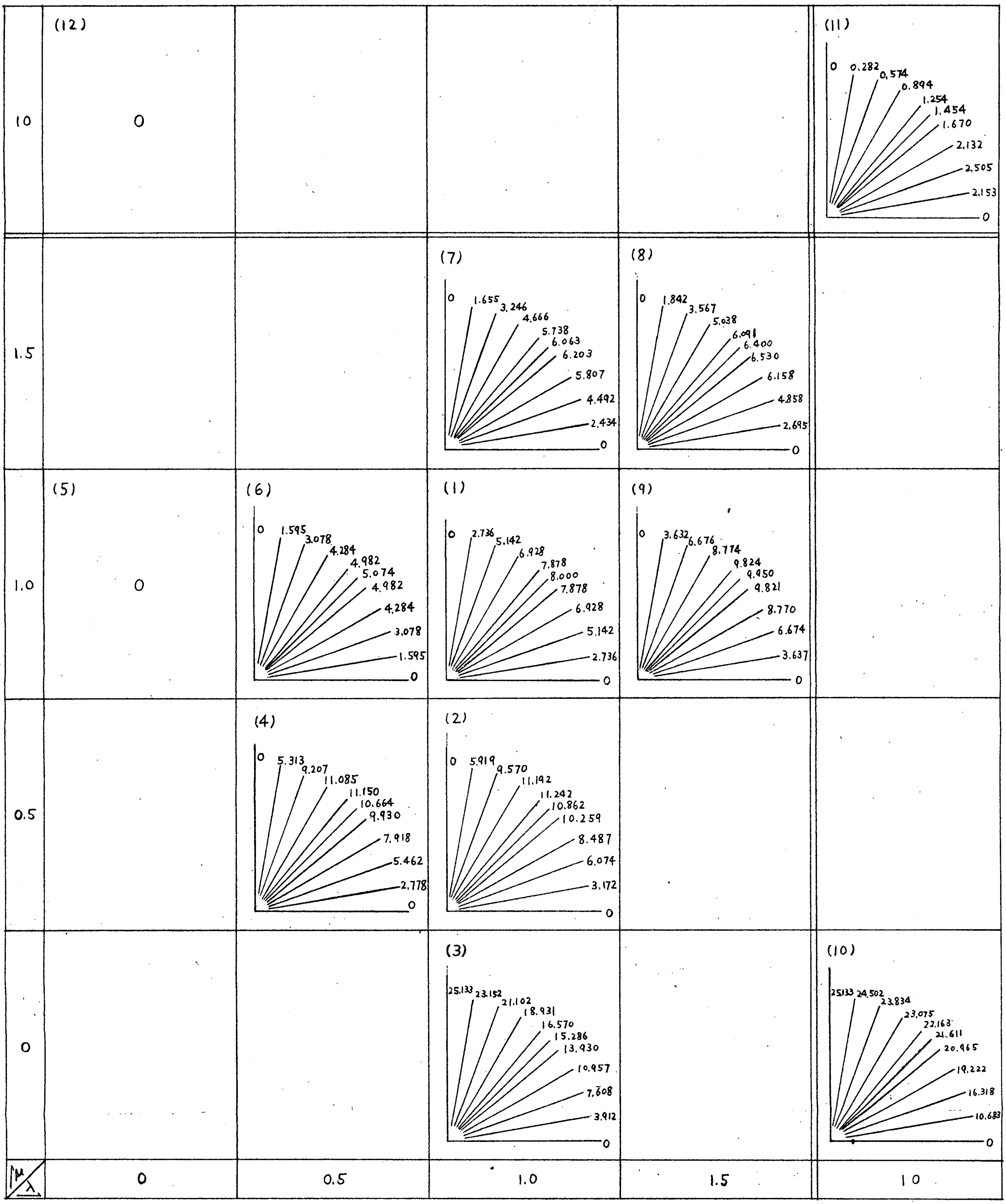


Fig 8-8 $8\pi(\gamma)_0$ as a Function of λ and μ

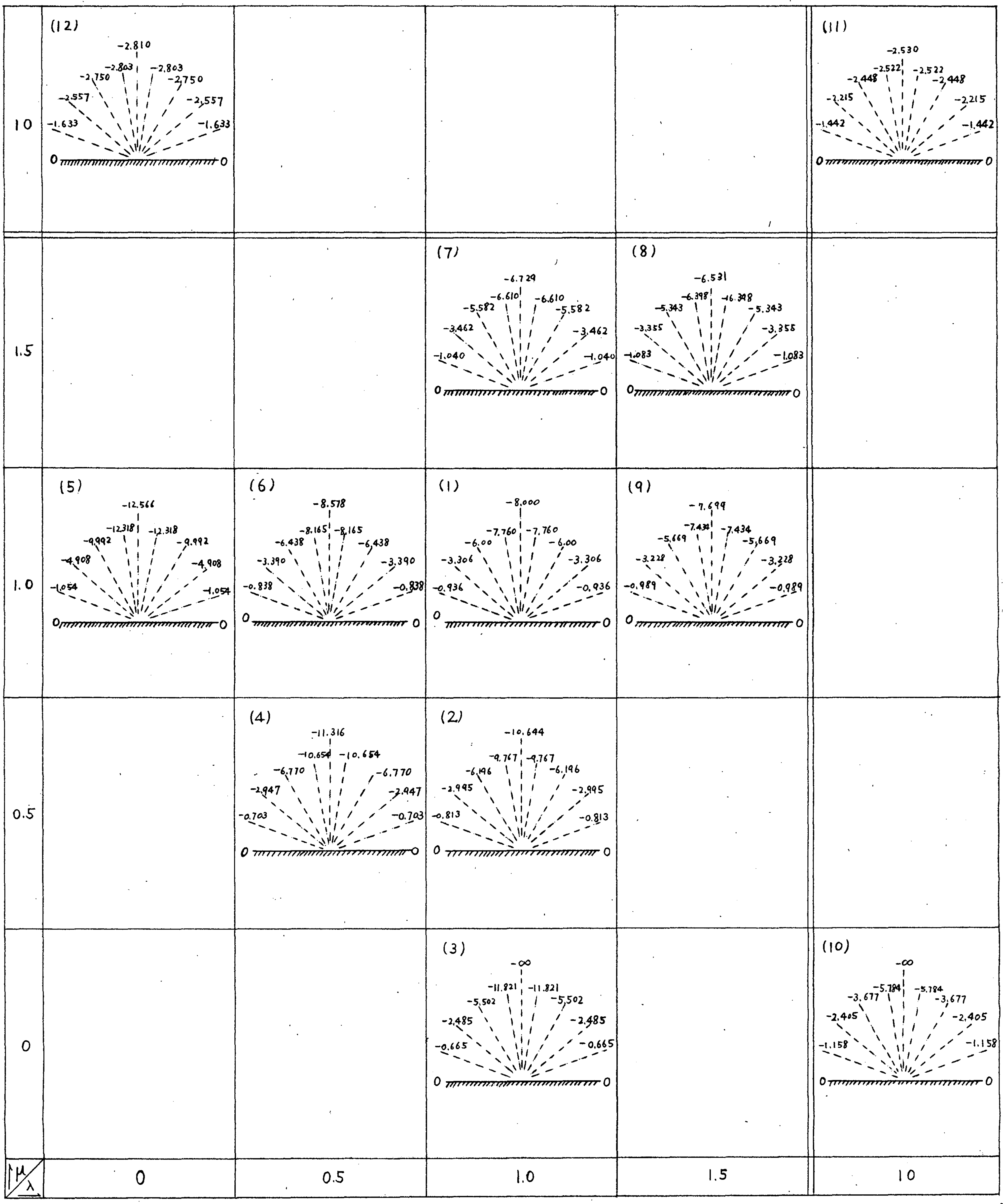


Fig 8-9 $8\pi(M_y)_0$ as a Function of λ and μ (clamped edge)

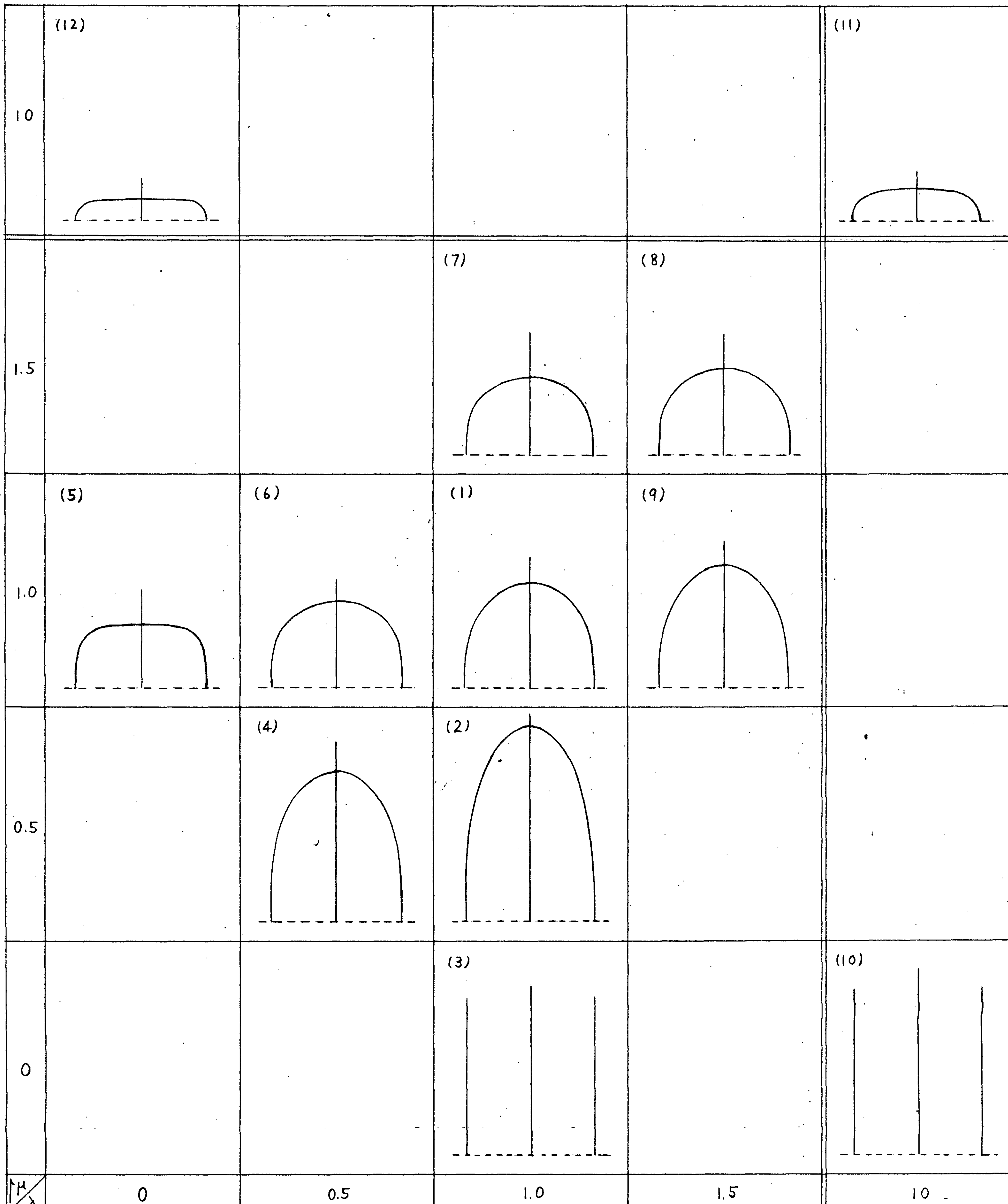


Fig 8-10 $(m_x)_0$ as a Function of λ and μ (free edge)

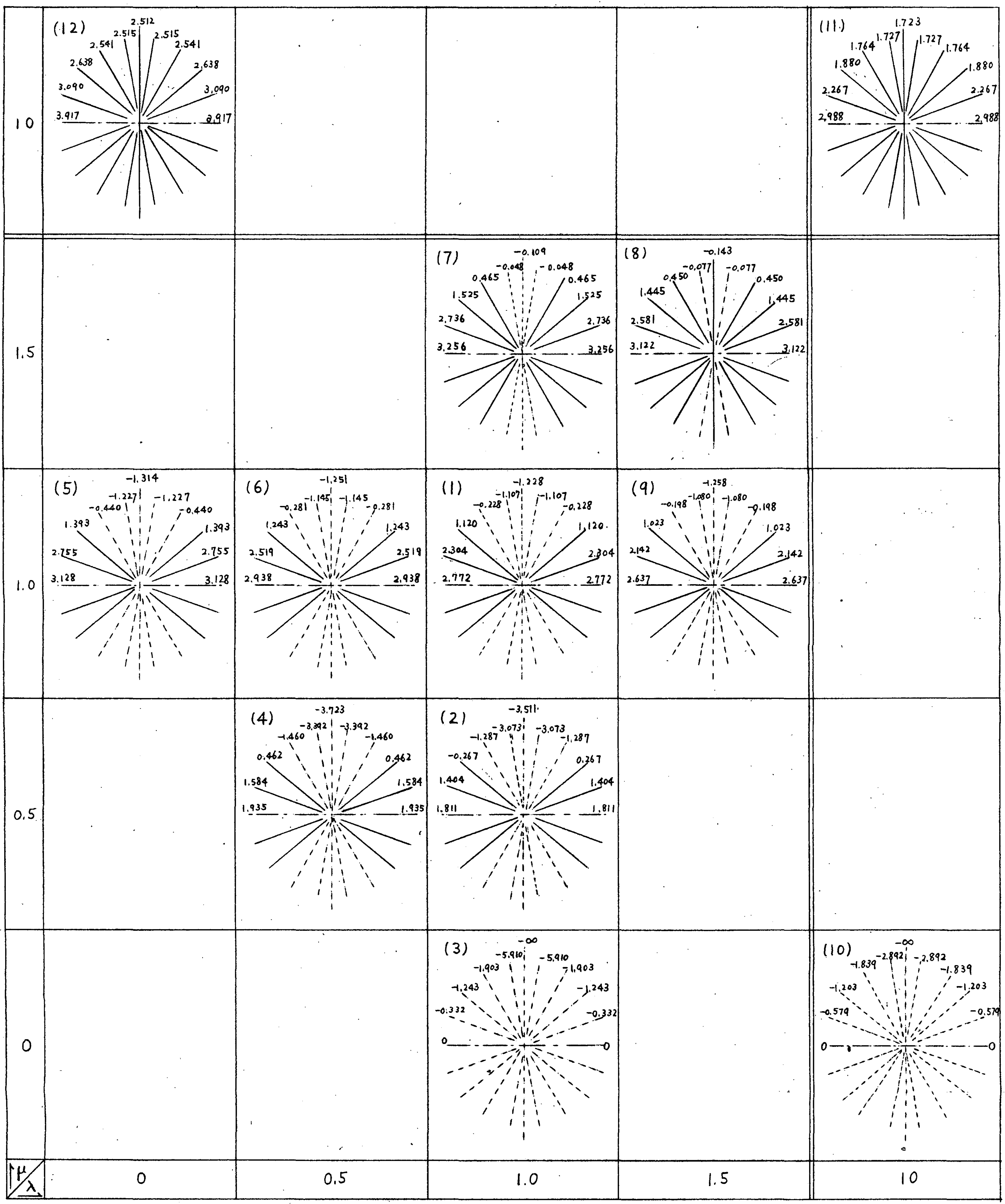


Fig 8-11 $8\pi(M_y)_0$ as a Function of λ and μ (Continuos plate)

TABLE I

DEFINITION OF SOME IMPORTANT TRANCENDENTAL FUNCTIONS

Several functions which constitute the influence functions of orthotropic plates are defined in the following table.

Following remarks should be observed for the application of this table.

(i) In order to avoid complexity, every function is written without showing four independent variables $\alpha, \beta; \xi, \eta$.

For example, $M_x = M_x(\alpha, \beta; \xi, \eta)$

$R_1 = R_1(\alpha, \beta; \xi, \eta)$, etc.

This rule should be applied to any influence functions unless otherwise noted.

(ii) The functions $\bar{R}_i$ is defined as follows:

$$\bar{R}_i = R_i(\alpha, \beta; \xi, -\eta)$$

(iii) If a function has $\pm$ sign, the following sign convention should be observed:

upper sign (+) for $\beta \leqq \eta$

lower sign (-) for $\beta \geqq \eta$

for example: if $\beta > 0$, $\eta > 0$.

$$\bar{R}_6 = R_6(\alpha, \beta; \xi, -\eta)$$

$$= \tan^{-1}\left(\frac{\sin(\alpha+\xi)}{e^{K_1(\beta+\eta)} - \cos(\alpha+\xi)}\right) - \tan^{-1}\left(\frac{\sin(\alpha-\xi)}{e^{K_1(\beta+\eta)} - \cos(\alpha-\xi)}\right)$$

(iv)

$$K_1 = \sqrt{\lambda + \sqrt{\lambda^2 - \mu^2}}, \qquad K_2 = \sqrt{\lambda - \sqrt{\lambda^2 - \mu^2}} \qquad (\lambda > \mu)$$

$$K_3 = \sqrt{\frac{\mu + \lambda}{2}}, \qquad K_4 = \sqrt{\frac{\mu - \lambda}{2}} \qquad (\lambda < \mu)$$

Functions	Series and Closed Form Expressions
R_1	$4\sum_{n=1}^{\infty}\frac{1}{n}e^{\pm nK_1(\beta-\eta)}\sin n\xi \sin n\alpha$ $=\log\frac{\cosh K_1(\beta-\eta)-\cos(\alpha+\xi)}{\cosh K_1(\beta-\eta)-\cos(\alpha-\xi)}$
R_2	$4\sum_{n=1}^{\infty}\frac{1}{n}e^{\pm nK_1(\beta-\eta)}\sin n\xi \sin n\alpha$ $=\log\frac{\cosh K_2(\beta-\eta)-\cos(\alpha+\xi)}{\cosh K_2(\beta-\eta)-\cos(\alpha-\xi)}$
R_3	$8\sum_{n=1}^{\infty}\frac{1}{n}e^{\pm nK_3(\beta-\eta)}\cos nK_4(\beta-\eta)\sin n\xi \sin n\alpha$ $=\log\frac{\{\cosh K_3(\beta-\eta)-\cos(\alpha+\xi\mp K_4(\beta-\eta))\}\{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\pm K_4(\beta-\eta))\}}{\{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\mp K_4(\beta-\eta))\}\{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\pm K_4(\beta-\eta))\}}$?
R_4	$4\sum_{n=1}^{\infty}\frac{\mp 1}{n}e^{\pm nK_3(\beta-\eta)}\sin nK_4(\beta-\eta)\sin n\xi \sin n\alpha$ $=\tan^{-1}\left(\frac{\sin(\alpha-\xi\pm K_4(\beta-\eta))}{e^{\mp K_4(\beta-\eta)}-\cos(\alpha-\xi\pm K_4(\beta-\eta))}\right)+\tan^{-1}\left(\frac{\sin(\alpha+\xi\mp K_4(\beta-\eta))}{e^{\mp K_4(\beta-\eta)}-\cos(\alpha+\xi\mp K_4(\beta-\eta))}\right)$ $-\tan^{-1}\left(\frac{\sin(\alpha-\xi\mp K_4(\beta-\eta))}{e^{\mp K_4(\beta-\eta)}-\cos(\alpha-\xi\mp K_4(\beta-\eta))}\right)-\tan^{-1}\left(\frac{\sin(\alpha+\xi\pm K_4(\beta-\eta))}{e^{\mp K_4(\beta-\eta)}-\cos(\alpha+\xi\pm K_4(\beta-\eta))}\right)$
R_5	$4\sum_{n=1}^{\infty}\frac{1}{n}e^{\pm n\sqrt{x}(\beta-\eta)}\sin n\xi \sin n\alpha$ $=\log\frac{\cosh\sqrt{x}(\beta-\eta)-\cos(\alpha+\xi)}{\cosh\sqrt{x}(\beta-\eta)-\cos(\alpha-\xi)}$
R_6	$2\sum_{n=1}^{\infty}\frac{1}{n}e^{\pm nK_1(\beta-\eta)}\sin n\xi \cos n\alpha$ $=\tan^{-1}\left(\frac{\sin(\alpha+\xi)}{e^{\mp K_1(\beta-\eta)}-\cos(\alpha+\xi)}\right)-\tan^{-1}\left(\frac{\sin(\alpha-\xi)}{e^{\mp K_1(\beta-\eta)}-\cos(\alpha-\xi)}\right)$

Functions	Series and Closed Form Expressions
R_7	$2\sum_{n=1}^{\infty}\frac{1}{n}e^{\pm nK_2(\beta-\eta)}\sin n\xi\cos n\alpha$ $=\tan^{-1}\left(\frac{\sin(\alpha+\xi)}{e^{\mp K_2(\beta-\eta)}-\cos(\alpha+\xi)}\right)-\tan^{-1}\left(\frac{\sin(\alpha-\xi)}{e^{\mp K_2(\beta-\eta)}-\cos(\alpha-\xi)}\right)$
R_8	$8\sum_{n=1}^{\infty}\frac{\mp 1}{n}e^{\pm nK_3(\beta-\eta)}\sin nK_4(\beta-\eta)\sin n\xi\cos n\alpha$ $=\log\frac{\{\cosh K_3(\beta-\eta)-\cos(\alpha+\xi\mp K_4(\beta-\eta))\}\{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\pm K_4(\beta-\eta))\}}{\{\cosh K_3(\beta-\eta)-\cos(\alpha+\xi\pm K_4(\beta-\eta))\}\{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\mp K_4(\beta-\eta))\}}$
R_9	$2\sum_{n=1}^{\infty}\frac{1}{n}e^{\pm n\sqrt{\lambda}(\beta-\eta)}\sin n\xi\cos n\alpha$ $=\tan^{-1}\left(\frac{\sin(\alpha+\xi)}{e^{\mp\sqrt{\lambda}(\beta-\eta)}-\cos(\alpha+\xi)}\right)-\tan^{-1}\left(\frac{\sin(\alpha-\xi)}{e^{\mp\sqrt{\lambda}(\beta-\eta)}-\cos(\alpha-\xi)}\right)$
R_{10}	$4\sum_{n=1}^{\infty}e^{-n(K_2\beta+K_1\eta)}\sin n\xi\sin n\alpha$ $=\log\frac{\cosh(K_2\beta+K_1\eta)-\cos(\alpha+\xi)}{\cosh(K_2\beta+K_1\eta)-\cos(\alpha-\xi)}$
R_{11}	$4\sum_{n=1}^{\infty}e^{-n(K_1\beta+K_2\eta)}\sin n\xi\sin n\alpha$ $=\log\frac{\cosh(K_1\beta+K_2\eta)-\cos(\alpha+\xi)}{\cosh(K_1\beta+K_2\eta)-\cos(\alpha-\xi)}$
R_{12}	$8\sum_{n=1}^{\infty}\frac{1}{n}e^{-nK_3(\beta+\eta)}\cos nK_4(\beta-\eta)\sin n\xi\sin n\alpha$ $=\log\frac{\{\cosh K_3(\beta+\eta)-\cos(\alpha+\xi\mp K_4(\beta-\eta))\}\{\cosh K_3(\beta+\eta)-\cos(\alpha+\xi\pm K_4(\beta-\eta))\}}{\{\cosh K_3(\beta+\eta)-\cos(\alpha-\xi\pm K_4(\beta-\eta))\}\{\cosh K_3(\beta+\eta)-\cos(\alpha+\xi\mp K_4(\beta-\eta))\}}$

Functions	Series and Closed Form Expressions
R_{13}	$4\sum_{n=1}^{\infty}\frac{\pm 1}{n}e^{-nK_3(\beta+\eta)}\sin nK_4(\beta-\eta)\sin n\alpha\sin n\xi$ $= \tan^{-1}\left(\frac{\sin(\alpha-\xi\pm K_4(\beta-\eta))}{e^{K_3(\beta+\eta)}-\cos(\alpha-\xi\pm K_4(\beta-\eta))}\right)+\tan^{-1}\left(\frac{\sin(\alpha+\xi\mp K_4(\beta-\eta))}{e^{K_3(\beta+\eta)}-\cos(\alpha+\xi\mp K_4(\beta-\eta))}\right)$ $-\tan^{-1}\left(\frac{\sin(\alpha-\xi\mp K_4(\beta-\eta))}{e^{K_3(\beta+\eta)}-\cos(\alpha-\xi\mp K_4(\beta-\eta))}\right)-\tan^{-1}\left(\frac{\sin(\alpha+\xi\pm K_4(\beta-\eta))}{e^{K_3(\beta+\eta)}-\cos(\alpha+\xi\pm K_4(\beta-\eta))}\right)$
S_1	$4\sum_{n=1}^{\infty}e^{\pm n\sqrt{\lambda}(\beta-\eta)}\sin n\xi\sin n\alpha$ $=\mp\left[\frac{\sinh\sqrt{\lambda}(\beta-\eta)}{\cosh\sqrt{\lambda}(\beta-\eta)-\cos(\alpha-\xi)}-\frac{\sinh\sqrt{\lambda}(\beta-\eta)}{\cosh\sqrt{\lambda}(\beta-\eta)-\cos(\alpha+\xi)}\right]$
S_2	$4\sum_{n=1}^{\infty}e^{\pm n\sqrt{\lambda}(\beta-\eta)}\sin n\xi\cos n\alpha$ $=\frac{\sin(\alpha+\xi)}{\cosh\sqrt{\lambda}(\beta-\eta)-\cos(\alpha+\xi)}-\frac{\sin(\alpha-\xi)}{\cosh\sqrt{\lambda}(\beta-\eta)-\cos(\alpha-\xi)}$
S_3	$4\sum_{n=1}^{\infty}e^{\pm nK_1(\beta-\eta)}\sin n\xi\cos n\alpha$ $=\frac{\sin(\alpha+\xi)}{\cosh K_1(\beta-\eta)-\cos(\alpha+\xi)}-\frac{\sin(\alpha-\xi)}{\cosh K_1(\beta-\eta)-\cos(\alpha-\xi)}$
S_4	$4\sum_{n=1}^{\infty}e^{\pm nK_2(\beta-\eta)}\sin n\xi\cos n\alpha$ $=\frac{\sin(\alpha+\xi)}{\cosh K_2(\beta-\eta)-\cos(\alpha+\xi)}-\frac{\sin(\alpha-\xi)}{\cosh K_2(\beta-\eta)-\cos(\alpha-\xi)}$
S_5	$4\sum_{n=1}^{\infty}e^{\pm nK_1(\beta-\eta)}\sin n\alpha\sin n\xi$ $=\mp\left[\frac{\sinh K_1(\beta-\eta)}{\cosh K_1(\beta-\eta)-\cos(\alpha-\xi)}-\frac{\sinh K_1(\beta-\eta)}{\cosh K_1(\beta-\eta)-\cos(\alpha+\xi)}\right]$

Functions	Series and Closed Form Expressions
S_6	$4\sum_{n=1}^{\infty} e^{\pm nK_2(\beta-\eta)} \sin n\xi \sin n\alpha$ $= \mp\left[\frac{\sinh K_2(\beta-\eta)}{\cosh K_2(\beta-\eta)-\cos(\alpha-\xi)} - \frac{\sinh K_2(\beta-\eta)}{\cosh K_2(\beta-\eta)-\cos(\alpha+\xi)}\right]$
S_7	$8\sum_{n=1}^{\infty} e^{\pm nK_3(\beta-\eta)} \cos nK_4(\beta-\eta) \sin n\xi \cos n\alpha$ $= \frac{\sin(\alpha+\xi\mp K_4(\beta-\eta))}{\cosh K_3(\beta-\eta)-\cos(\alpha+\xi\mp K_4(\beta-\eta))} - \frac{\sin(\alpha-\xi\mp K_4(\beta-\eta))}{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\mp K_4(\beta-\eta))}$ $+ \frac{\sin(\alpha+\xi\pm K_4(\beta-\eta))}{\cosh K_3(\beta-\eta)-\cos(\alpha+\xi\pm K_4(\beta-\eta))} - \frac{\sin(\alpha-\xi\pm K_4(\beta-\eta))}{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\pm K_4(\beta-\eta))}$
S_8	$\mp 8\sum_{n=1}^{\infty} e^{\pm nK_3(\beta-\eta)} \sin nK_4(\beta-\eta) \sin n\xi \cos n\alpha$ $= \frac{\sinh K_3(\beta-\eta)}{\cosh K_3(\beta-\eta)-\cos(\alpha+\xi\mp K_4(\beta-\eta))} + \frac{\sinh K_3(\beta-\eta)}{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\pm K_4(\beta-\eta))}$ $- \frac{\sinh K_3(\beta-\eta)}{\cosh K_3(\beta-\eta)-\cos(\alpha+\xi\pm K_4(\beta-\eta))} - \frac{\sinh K_3(\beta-\eta)}{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\mp K_4(\beta-\eta))}$
S_9	$8\sum_{n=1}^{\infty} e^{\pm nK_4(\beta-\eta)} \cos nK_4(\beta-\eta) \sin n\xi \sin n\alpha$ $= \pm\left[\frac{\sinh K_3(\beta-\eta)}{\cosh K_3(\beta-\eta)-\cos(\alpha+\xi\pm K_4(\beta-\eta))} + \frac{\sinh K_3(\beta-\eta)}{\cosh K_3(\beta-\eta)-\cos(\alpha+\xi\mp K_4(\beta-\eta))}\right.$ $\left.- \frac{\sinh K_3(\beta-\eta)}{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\mp K_4(\beta-\eta))} - \frac{\sinh K_3(\beta-\eta)}{\cosh K_3(\beta-\eta)-\cos(\alpha-\xi\pm K_4(\beta-\eta))}\right]$
T_1	$4\sum_{n=1}^{\infty} n\, e^{-n\sqrt{\lambda}(\beta+\eta)} \sin n\xi \sin n\alpha$ $= \frac{\cosh\sqrt{\lambda}(\beta+\eta)\cos(\alpha-\xi)-1}{(\cosh\sqrt{\lambda}(\beta+\eta)-\cos(\alpha-\xi))^2} - \frac{\cosh\sqrt{\lambda}(\beta+\eta)\cos(\alpha+\xi)-1}{(\cosh\sqrt{\lambda}(\beta+\eta)-\cos(\alpha+\xi))}$

TABLE II

VARIOUS SINGULARITIES OF INFLUENCE SURFACES AS FUNCTIONS OF λ and μ

Figures 8-5 to 8-11 are graphical representation of the equations of stress singularities given in this Table (II).

In derivation of these equations following assumptions were made:

(i) $D_1 = 0$ $H = 2D_{xy}$ $\frac{D_{xy}}{D_y} = \frac{\lambda}{2}$

(ii) For the case (G), $\frac{4aD_y}{\pi EI} = 1$

Except cases (A) and (F), limit values of the surfaces stay finite.

In cases (A) and (F), $(m_x)_o = 0$ is assumed since every contour lines are similar to each other. In case (C) $(q_y)_o = 1$ is also assumed.

(A) Bending Moment $(m_x)_o$ (interior point)

Case	λ	μ	Equation $(m_x)_o = 0$
(1)	1	1	$\log r - \sin^2\theta = 0$
(2)	1	0.5	$\log r + 0.683 \log(1 + 0.866 \sin^2\theta) - 0.183 \log(1 - 0.866 \sin^2\theta) = 0$
(3)	1	0	$r \cos\theta = 1$
(4)	0.5	0.5	$\log r + 0.5 \log(1 - 0.5 \sin^2\theta) - \frac{\sin^2\theta}{1 + \cos^2\theta} = 0$
(5)	0	1	$\log r + 0.25 \log(1 - 2 \sin^2\theta \cos^2\theta) - 0.5 \tan^{-1}(\tan^2\theta) = 0$
(6)	0.5	1	$\log r + 0.25 \log(1 - \sin^2\theta \cos^2\theta) - 0.866 \tan^{-1}\left(\frac{1.732 \sin^2\theta}{1 + \cos^2\theta}\right) = 0$
(7)	1	1.5	$\log r + 0.25 \log(1 + 1.25 \sin^4\theta) - 1.118 \tan^{-1}(1.118 \sin^2\theta) = 0$
(8)	1.5	1.5	$\log r + 0.5 \log(1 + 0.5 \sin^2\theta) - \frac{3 \sin^2\theta}{2 + \sin^2\theta} = 0$
(9)	1.5	1	$\log r + 0.809 \log(1 - 0.618 \sin^2\theta) - 0.309 \log(1 + 0.618 \sin^2\theta) = 0$
(10)	10	0	$r \cos\theta = 1$
(11)	10	10	$\log r + 0.5 \log(1 + 9 \sin^2\theta) - \frac{10 \sin^2\theta}{1 + 9 \sin^2\theta} = 0$
(12)	0	10	$\log r + 0.25 \log(\cos^4\theta + 100 \sin^4\theta) - 0.5 \tan^{-1}(10 \tan^2\theta) = 0$

(B) Twisting Moment $(m_{xy})_o$ (interior point)

Case	λ	μ	Equation : $8\pi(m_{xy})_o$
(1)	1	1	$\sin 2\theta$
(2)	1	0.5	$2.309\left[\tan^{-1}(2.732\cot\theta) - \tan^{-1}(0.732\cot\theta)\right]$
(3)	1	0	$3.142 - 2\tan^{-1}(0.707\cot\theta)$
(4)	0.5	0.5	$\frac{\sin 2\theta}{1 - 0.5\sin^2\theta}$
(5)	0	1	0
(6)	0.5	1	$0.577\log\frac{1+\sin\theta\cos\theta}{1-\sin\theta\cos\theta}$
(7)	1	1.5	$0.895\log\frac{\cos^2\theta+\sin\theta\cos\theta+1.5\sin^2\theta}{\cos^2\theta-\sin\theta\cos\theta+1.5\sin^2\theta}$
(8)	1.5	1.5	$\frac{\sin 2\theta}{1 + 0.5\sin^2\theta}$
(9)	1.5	1	$2.683\left[\tan^{-1}(1.618\cot\theta) - \tan^{-1}(0.618\cot\theta)\right]$
(10)	10	0	$3.142 - 2\tan^{-1}(0.224\cot\theta)$
(11)	10	10	$\frac{\sin 2\theta}{1 + 9\sin^2\theta}$
(12)	0	10	0

(C) Shearing Force $(q_y)_o$ (interior point)

Case	λ	μ	Equation: $8a(q_y)_o = 1$
(1)	1	1	$\gamma = -2 \sin\theta$
(2)	1	0.5	$\gamma = -2\left(\frac{0.366 \sin\theta}{1-0.866 \sin^2\theta} + \frac{1.366 \sin\theta}{1+0.866 \sin^2\theta}\right)$
(3)	1	0	$\gamma = -\frac{2.828 \sin\theta}{1+\sin^2\theta}$
(4)	0.5	0.5	$\gamma = -\frac{1.414 \sin\theta}{1-0.5 \sin^2\theta}$
(5)	0	1	$\gamma = -\frac{2.828 \sin\theta}{1-2\sin^2\theta \cos^2\theta}$
(6)	0.5	1	$\gamma = -\frac{3.464 \sin\theta}{1-\sin^2\theta \cos^2\theta}$
(7)	1	1.5	$\gamma = -2.236\left(\frac{\sin\theta}{\cos^2\theta - \sin\theta\cos\theta + 1.5\sin^2\theta} + \frac{\sin\theta}{\cos^2\theta + \sin\theta\cos\theta + 1.5\sin^2\theta}\right)$
(8)	1.5	1.5	$\gamma = -\frac{2.45 \sin\theta}{1+0.5\sin^2\theta}$
(9)	1.5	1	$\gamma = -2\left(\frac{0.618 \sin\theta}{1-0.618 \sin^2\theta} + \frac{1.618 \sin\theta}{1+1.618 \sin^2\theta}\right)$
(10)	10	0	$\gamma = -\frac{8.944 \sin\theta}{1+19\sin^2\theta}$
(11)	10	10	$\gamma = -\frac{6.324 \sin\theta}{1+9\sin^2\theta}$
(12)	0	10	$\gamma = -\frac{8.944 \sin\theta(1+9\sin^2\theta)}{\cos^4\theta + 100\sin^4\theta}$

(D) Corner Reaction $(r)_0$ of Simply Supported Rectangular Edges

Case	λ	μ	Equation : $8\pi(r)_0$
(1)	1	1	$8\sin 2\theta$
(2)	1	0.5	$18.475\left[\tan^{-1}(2.732\cot\theta)-\tan^{-1}(0.732\cot\theta)\right]$
(3)	1	0	$25.133-16\tan^{-1}(0.707\cot\theta)$
(4)	0.5	0.5	$\dfrac{8\sin 2\theta}{1-0.5\sin^2\theta}$
(5)	0	1	0
(6)	0.5	1	$4.619\log\dfrac{1+\sin\theta\cos\theta}{1-\sin\theta\cos\theta}$
(7)	1	1.5	$7.156\log\dfrac{\cos^2\theta+\sin\theta\cos\theta+1.5\sin^2\theta}{\cos^2\theta-\sin\theta\cos\theta+1.5\sin^2\theta}$
(8)	1.5	1.5	$\dfrac{8\sin 2\theta}{1+0.5\sin^2\theta}$
(9)	1.5	1	$21.467\left[\tan^{-1}(1.618\cot\theta)-\tan^{-1}(0.618\cot\theta)\right]$
(10)	10	0	$25.133-16\tan^{-1}(0.224\cot\theta)$
(11)	10	10	$\dfrac{8\sin 2\theta}{1+9\sin^2\theta}$
(12)	0	10	0

(E) Boundary Moment $(m_y)_o$ of Clamped Edge

case	λ	μ	Equation : $8\pi(m_y)_o$
(1)	1	1	$-8\sin^2\theta$
(2)	1	0.5	$4\log\frac{1-0.866\sin^2\theta}{1+0.866\sin^2\theta}$
(3)	1	0	$2.848\log\frac{\cos^2\theta}{1+\sin^2\theta}$
(4)	0.5	0.5	$-\frac{5.658\sin^2\theta}{1-0.5\sin^2\theta}$
(5)	0	1	$-8\tan^{-1}(\tan^2\theta)$
(6)	0.5	1	$-8\tan^{-1}\left(\frac{0.866\sin^2\theta}{1-0.5\sin^2\theta}\right)$
(7)	1	1.5	$-8\tan^{-1}(1.118\sin^2\theta)$
(8)	1.5	1.5	$-\frac{9.796\sin^2\theta}{1+0.5\sin^2\theta}$
(9)	1.5	1	$4\log\frac{1-0.618\sin^2\theta}{1+0.618\sin^2\theta}$
(10)	10	0	$-0.894\log\frac{\cos^2\theta}{1+19\sin^2\theta}$
(11)	10	10	$-\frac{25.3\sin^2\theta}{1+9\sin^2\theta}$
(12)	0	10	$-1.789\tan^{-1}(10\tan^2\theta)$

(F) Boundary Moment $(m_x)_o$ of Free Edge

case	λ	μ	Equation: $(m_x)_o = 0$
(1)	1	1	$\log r - 0.5 \sin^2\theta = 0$
(2)	1	0.5	$\log r - 0.039 \log(1+0.866\sin^2\theta) + 0.539 \log(1-0.866\sin^2\theta) = 0$
(3)	1	0	$r \cos\theta = 1$
(4)	0.5	0.5	$\log r + 0.5 \log(1-0.5\sin^2\theta) - \frac{0.25\sin^2\theta}{1-0.5\sin^2\theta} = 0$
(5)	0	1	$\log r + 0.25 \log(\cos^4\theta + \sin^4\theta) = 0$
(6)	0.5	1	$\log r + 0.25 \log(1-\cos^2\theta\sin^2\theta) - 0.289 \tan^{-1}\left(\frac{0.866\sin^2\theta}{1+\sin^2\theta}\right) = 0$
(7)	1	1.5	$\log r + 0.25 \log(1+1.25\sin^4\theta) - 0.447 \tan^{-1}(1.118\sin^2\theta) = 0$
(8)	1.5	1.5	$\log r + 0.5 \log(1+0.5\sin^2\theta) - \frac{0.75\sin^2\theta}{1+0.5\sin^2\theta} = 0$
(9)	1.5	1	$\log r - 0.085 \log(1+1.618\sin^2\theta) + 0.585 \log(1-0.618\sin^2\theta) = 0$
(10)	10	0	$r \cos\theta = 1$
(11)	10	10	$\log r + 0.5 \log(1+9\sin^2\theta) - \frac{5\sin^2\theta}{1+9\sin^2\theta} = 0$
(12)	0	10	$\log r + 0.25 \log(\cos^4\theta + 100\sin^4\theta) = 0$

(G) Support Moment $(m_y)_o$ of a Slab Continuous Over Flexible Cross Beam

Case	λ	μ	Equation : $8\pi (m_y)_o$
(1)	1	1	$2.772 - 4 \sin^2\theta$
(2)	1	0.5	$1.811 + 2 \log \frac{1-0.866 \sin^2\theta}{1+0.866 \sin^2\theta}$
(3)	1	0	$1.414 \log \frac{\cos^2\theta}{2-\cos^2\theta}$
(4)	0.5	0.5	$1.935 - \frac{2.829 \sin^2\theta}{1-0.5 \sin^2\theta}$
(5)	0	1	$3.128 - 2.828 \tan^{-1}(\tan^2\theta)$
(6)	0.5	1	$2.938 - 4 \tan^{-1}\left(\frac{0.866 \sin^2\theta}{1-0.5 \sin^2\theta}\right)$
(7)	1	1.5	$3.256 - 4 \tan^{-1}(1.118 \sin^2\theta)$
(8)	1.5	1.5	$3.122 - \frac{4.898 \sin^2\theta}{1+0.5 \sin^2\theta}$
(9)	1.5	1	$2.637 + 2 \log \frac{1-0.618 \sin^2\theta}{1+1.618 \sin^2\theta}$
(10)	10	0	$0.447 \log \frac{\cos^2\theta}{1+19 \sin^2\theta}$
(11)	10	10	$2.988 - \frac{12.65 \sin^2\theta}{1+9 \sin^2\theta}$
(12)	0	10	$3.917 - 0.894 \tan^{-1}(10 \tan^2\theta)$

VITA

The author was born as the second child of Kanjiro and Shizue Kawai on February 20, 1926 in Tokyo, Japan.

In April, 1949 he entered the University of Tokyo and in March 1952 was awarded the degree of B.S. in Naval Architecture. Thereafter, he continued his study in the graduate school at the University of Tokyo until August 1954.

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