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INFLUENCE SURFACES FOR MOMENTS IN SLABS
CONTINUOUS OVER FLEXIBLE CROSS BEAMS

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I. INTRODUCTION

In two recent articles [1],[2]* influence surfaces for bending moments of continuous slabs have been presented. In reference (1) Hoeland developed solutions for a slab continuous over rigid and flexible cross beams in the form of infinite series.** However, as the influence functions exhibit singular behavior at the influence point proper (i.e. point for which the influence functions are determined), such solutions are divergent at this point and slowly convergent in its neighborhood. For a discussion of the singularity and for exact numerical computations in the immediate vicinity, solutions in finite form are required.

Such solutions have been presented in reference (2) for the case of rigid cross beams. In this paper solutions in finite form for cases of flexible cross beams are developed. The singular behavior of the influence functions for the support moments over the cross beams is discussed. Finally, numerical solutions are presented in graphical form.

Whereas in references (1) and (2) solutions have been obtained by using the differential equation of a transversely loaded plate, the approach using the integral equation is presented in this paper. Apart from a different formulation of

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* Refers to List of References.
**An error in the assumption of the boundary conditions expressed by the equation for q_y on page 127 of reference (1) will be pointed out shortly.

the problem this approach has the advantage of avoiding boundary conditions along the cross beam. This, in turn, shows that the boundary condition for the shearing forces along the cross beam in reference (1), equation for q_y, p. 127, are in error. Instead of using the expression for the boundary shear V_x containing the contribution of the twisting moment, e.g.

$$V_x = \left(Q_x - \frac{\partial M_{xy}}{\partial x} \right) = -D \left[\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2} \right]$$

the expression for the shearing force Q_x, e.g.

$$Q_x = -D \left(\frac{\partial^3 W}{\partial x^3} + \frac{\partial^3 W}{\partial x \partial y^2} \right)$$

should have been used. As the twisting moments M_{xy} are continuous over the support beam, they should not appear in the boundary condition.

II. INFLUENCE SURFACE FOR DEFLECTION OF PLATE STRIP CONTINUOUS OVER FLEXIBLE CROSS BEAM

An infinite plate strip with simply supported parallel edges is considered (Fig.1). At y=0 the plate is continuous over an elastic cross beam with a constant bending stiffness EI. The coordinates of a point on the cross beam are taken as (z,0) --z being the x-coordinate -- in order to distinguish this point from a general point (x,y). The deflection w of the plate at a given point (u,v), referred to as the influence point, due to a concentrated load P at point (x,y) can be expressed by the following integral equation:

$$W(u,v;x,y) = P\,G(u,v;x,y) - \int_0^a EI\, \frac{\partial^4 W(z,0;x,y)}{\partial z^4}\, G(u,v;z,0)\, dz \qquad (1)$$

The function $G(u,v;x,y)$ is the Green's function for the deflection of point (u,v) of an infinite plate strip with simply supported edges. It is given by the following series:

$$G(u,v;x,y) = \frac{a^2}{2\pi^3 D} \sum_{n=1}^{\infty} \frac{1}{n^3} \phi_n (u,v;x,y) \qquad (2)$$

where
$$\phi_n(u,v;x,y) = \left[1 \mp \frac{n\pi}{a}(v-y) \right] e^{\pm \frac{n\pi}{a}(v-y)} \sin \frac{n\pi u}{a} \sin \frac{n\pi x}{a}$$

with the upper sign for $v \leq y$.

and the lower sign for $v \geq y$.

The first term under the intergral sign in equation (1), $EI \frac{\partial^4 w(z,o;x,y)}{\partial z^4}$ expresses the distributed reaction of the cross beam acting on the plate. When multiplied by Green's function $G(u,v;z,0)$ and integrated over the length of the cross beam the integral constitutes the influence of this beam on the deflection at point (u,v).

To simplify the following derivations, dimensionless co-ordinates are introduced[2], viz:

$$\left. \begin{array}{l} \dfrac{\pi u}{a} = \alpha \;, \quad \dfrac{\pi v}{a} = \beta \;, \\[2ex] \dfrac{\pi x}{a} = \xi \;, \quad \dfrac{\pi y}{a} = \eta \;, \quad \dfrac{\pi z}{a} = \zeta \end{array} \right\} \qquad (3)$$

Assuming the deflection surface w in the form

$$W(\alpha,\beta;\xi,\eta) = \phi(\alpha,\beta;\xi,\eta) + P G(\alpha,\beta;\xi,\eta) \qquad (4)$$

the function ϕ is determined by substituting equation (4) into equation (1):

$$\phi(\alpha,\beta;\xi,\eta) = - \frac{\pi^3 EI}{a^3} \int_0^{\pi} \frac{\partial^4 W(\zeta,0;\xi,\eta)}{\partial \zeta^4} G(\alpha,\beta,\zeta,0) d\zeta \qquad (5)$$

Since ϕ is a continuous function with respect to α and β, it can be developed into eigen-functions associated with the Green's function G as follows:

$$\phi(\alpha,\beta;\xi,\eta) = \sum_{n=1}^{\infty} a_n(\xi,\eta)\, \varphi_n(\alpha,\beta)$$

$$\varphi_n(\alpha,\beta) = (1+n\beta)\, e^{-n\beta} \sin n\alpha \tag{6}$$

for positive values of β. Substituting into equation (5) and replacing G by equation (2) gives:

$$\sum_{n=1}^{\infty} a_n(\xi,\eta)\, \varphi_n(\alpha,\beta) = -\frac{\pi^2 EI}{a^3} \int_0^{\pi} \frac{\partial^4 W(\zeta,0)}{\partial \zeta^4} \left\{ \frac{a^2}{2\pi^3 D} \sum_{m=1}^{\infty} \frac{1}{m^3} \psi_m(\alpha,\beta;\zeta,0) \right\} d\zeta \tag{7}$$

Multiplying both sides by $\sin n\alpha$ and integrating with respect to α from 0 to π, the orthogonality relations simplify equation (7) considerably. Taking into account that

$$\psi_m(\alpha,\beta;\zeta,0) = \varphi_m(\alpha,\beta) \sin m\zeta$$

the following expression is obtained:

$$\frac{\pi}{2} a_n(\xi,\eta)(1+n\beta)e^{-n\beta} = -\frac{\pi}{2} \cdot \frac{EI}{2n^3 a D}(1+n\beta)e^{-n\beta} \int_0^{\pi} \frac{\partial^4 W(\zeta,0)}{\partial \zeta^4} \sin n\zeta\, d\zeta \tag{8}$$

With the substitution

$$\frac{\partial^4 W(\zeta,0)}{\partial \zeta^4} = \frac{\partial^4}{\partial \zeta^4} \left[\phi(\zeta,0;\xi,\eta) + P G(\zeta,0;\xi,\eta) \right]$$

$$= \sum_{m=1}^{\infty} \left[m^4 a_m(\xi,\eta) \sin m\zeta + \frac{Pa^2}{2\pi^3 D} m\, \psi_m(\zeta,0;\xi,\eta) \right]$$

the function a_n can be determined. Again the orthogonality relations are used.

Introducing the parameter

$$\rho = \frac{4aD}{\pi EI} \tag{9}$$

$$a_n = -\frac{Pa^2}{2n^3\pi^3 D} \cdot \frac{n}{n+\rho}(1+n\eta)e^{-n\eta}\sin n\xi \tag{10}*$$

The non-dimensional parameter ρ depends on the ratio of the bending stiffness of the plate, $D = \frac{E\,h^3}{12(1-\nu^2)}$ to the bending stiffness of the cross beam EI. Substituting the pertinent values into equation (4) with P=1 yields the influence function for deflection:

$$W(\alpha,\beta;\xi,\eta) = \frac{a^2}{2\pi^3 D}\sum_{n=1}^{\infty}\left[\frac{1}{n^3}(1\mp n(\beta-\eta))e^{\pm n(\beta-\eta)}\right.$$
$$\left.-\frac{1}{n^2(n+\rho)}(1+n\eta)(1+n\beta)e^{-n(\beta+\eta)}\right]\sin n\alpha \sin n\xi \tag{11}*$$

upper sign $\eta \gtreqless \beta$

lower sign $\eta \lesseqgtr \beta$

The first term within the parenthesis represents the influence surface for the deflection of point (u,v) of a simply supported plate strip without cross beam. The second term expresses the influence of this beam. If the cross beam is infinitely rigid, viz. EI$\longrightarrow\infty$ and $\rho \longrightarrow 0$, the coefficient of the second term reduces to:

$$\lim_{\rho\to 0}\frac{1}{n^2(n+\rho)} = \frac{1}{n^3}$$

On the other hand, in the absence of a cross beam, EI$\longrightarrow 0$ and $\rho \longrightarrow \infty$ such that

$$\lim_{\rho\to\infty}\frac{1}{n^2(n+\rho)} = 0$$

and the second term will disappear.

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*If $\eta < 0$ the sign preceding η should be changed in equation (10) and in the second series of equation (11).

III. <u>INFLUENCE SURFACES FOR BENDING MOMENTS:</u>

The expressions for the influence functions of the bending moments M_x and M_y are obtained by differentiating equation (11) with respect to α and β as follows (see for example[3], p. 260):

M_x-influence surface:

$$m_x(\alpha,\beta) = -\frac{\pi^2 D}{a^2}\left(\frac{\partial^2 W}{\partial \alpha^2} + \nu \frac{\partial^2 W}{\partial \beta^2}\right)$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\left\{(1+\nu) \mp (1-\nu)n(\beta-\eta)\right\}e^{\pm n(\beta-\eta)}\right. \tag{12}*$$

$$\left. - \frac{1}{n+p}(1+n\eta)\left\{(1+\nu)+(1-\nu)n\beta\right\}e^{-n(\beta+\eta)}\right]\sin n\alpha \sin n\xi$$

M_y-influence surface:

$$m_y(\alpha,\beta) = -\frac{\pi^2 D}{a^2}\left(\nu\frac{\partial^2 W}{\partial \alpha^2} + \frac{\partial^2 W}{\partial \beta^2}\right)$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}\left\{(1+\nu) \mp (1-\nu)n(\beta-\eta)\right\}e^{\pm n(\beta-\eta)}\right. \tag{13}*$$

$$\left. - \frac{1}{n+p}(1+n\eta)\left\{(1+\nu)+(1-\nu)n\beta\right\}e^{-n(\beta+\eta)}\right]\sin n\alpha \sin n\xi$$

The influence function for the beam moment is proportional to the curvature of the cross beam, hence:

Influence Surface for Moment of Cross Beam:

$$m_b(\alpha,0) = -\frac{\pi^2 EI}{a^2}\frac{\partial^2 W(\alpha,0)}{\partial \alpha^2}$$

$$= \frac{2a}{\pi^2 p}\sum_{n=1}^{\infty}\left\{\frac{1}{n} - \frac{1}{n+p}(1-p\eta)\right\}e^{-n\eta}\sin n\alpha \sin n\xi \tag{14}*$$

For the particular case of the influence functions for the support moments over the cross beam, i.e., $\beta = 0$, the expressions for m_x and m_y reduce to:

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* If $\eta < 0$ the sign preceding η should be changed in the second series of equations (12) and (13) and in equation (14).

$$m_x(\alpha,0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[(1+\nu)\left\{ \frac{1}{n} - \frac{1}{n+\rho}(1-\rho\eta) \right\} - 2\nu\eta \right] e^{-n\eta} \sin n\alpha \sin n\xi \qquad (15)$$

$$m_y(\alpha,0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[(1+\nu)\left\{ \frac{1}{n} - \frac{1}{n+\rho}(1-\rho\eta) \right\} - 2\eta \right] e^{-n\eta} \sin n\alpha \sin n\xi \qquad (16)$$

The series with the terms $2\nu\eta$ and 2η in the above equations for m_x and m_y respectively, are the expressions for the influence functions in case of a rigid cross beam.

If $\rho = 0$ the above equations reduce to:

$$m_x(\alpha,0) = -\frac{\nu\eta}{\pi} \sum_{n=1}^{\infty} e^{-n\eta} \sin n\alpha \sin n\xi \qquad (17)$$

$$m_y(\alpha,0) = -\frac{\eta}{\pi} \sum_{n=1}^{\infty} e^{-n\eta} \sin n\alpha \sin n\xi \qquad (18)$$

$$m_b(\alpha,0) = 0 \qquad (19)$$

Equation (18) checks with a previously obtained result, equation (12) of reference (2). The value of m_x is ν times the value of m_y as the curvature along the cross beam disappears. Finally, the expression for m_b reduces to zero as the cross beam will not deflect.

The other extreme case is obtained if the bending rigidity of the cross beam disappears, $\rho = \infty$, or

$$m_x(\alpha,0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[(1+\nu)+(1-\nu)n\eta \right] e^{-n\eta} \sin n\alpha \sin n\xi \qquad (20)$$

$$m_y(\alpha,0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[(1+\nu)-(1-\nu)n\eta \right] e^{-n\eta} \sin n\alpha \sin n\xi \qquad (21)$$

corresponding to already known results (see for example[2], equation (6)).

In general, it appears to be impossible to sum the series of equation (12) to (16) into finite expressions. However, at least for the two specific values of $\rho = 1/2$ and $\rho = 1$ such a summation is possible. Fortunately they correspond to practical values as will be shown presently. The details of the summation and also the finite expressions for the general case, i.e., $m_x(\alpha,\beta)$ and $m_y(\alpha,\beta)$, are given in the Appendix.

Considering the support moments for point (α,0) and the specific value $\rho = 1$, equation (14) to (16) take the following form:

$$m_x(\alpha,0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[(1+\nu)\left\{ \frac{1}{n} - \frac{1}{n+1}(1-\eta) \right\} - 2\nu\eta \right] e^{-n\eta} \sin n\alpha \sin n\xi \tag{22}$$

$$m_y(\alpha,0) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[(1+\nu)\left\{ \frac{1}{n} - \frac{1}{n+1}(1-\eta) \right\} - 2\eta \right] e^{-n\eta} \sin n\alpha \sin n\xi \tag{23}$$

$$m_b(\alpha,0) = \frac{2a}{\pi^2\rho} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}(1-\eta) \right) e^{-n\eta} \sin n\alpha \sin n\xi \tag{24}$$

The finite expressions for point ($\frac{\pi}{2}$, 0) -- i.e., influence point over the cross beam at half span -- and the ratio $\rho = 1$ are:

Case: $\alpha = \frac{\pi}{2}$; $\beta = 0$; $\rho = 1$

$$m_x\left(\frac{\pi}{2},0\right) = \frac{1}{8\pi}\left[(1+\nu)\left\{ (1-(\eta-1)e^{\eta}\sin\xi) \log(\cosh\eta+\sin\xi) - (1+(\eta-1)e^{\eta}\sin\xi) \log(\cosh\eta-\sin\xi) \right.\right.$$

$$\left. - 2(\eta-1)e^{\eta}(\log 2 - \eta)\sin\xi + 2(\eta-1)e^{\eta}\cos\xi \left(\tan^{-1}\left(\frac{\cos\xi}{e^{\eta}-\sin\xi}\right) - \tan^{-1}\left(\frac{\cos\xi}{e^{\eta}+\sin\xi}\right) \right) \right\} \tag{25}$$

$$\left. - 2\nu\eta\sinh\eta \left(\frac{1}{\cosh\eta - \sin\xi} - \frac{1}{\cosh\eta + \sin\xi} \right) \right]$$

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$$m_y(\tfrac{\pi}{2},0) = -\frac{1}{8\pi}\Big[(1+\nu)\{1-(\eta-1)e^\eta\sin\xi\}\log(\cosh\eta+\sin\xi) - (1+(\eta-1)e^\eta\sin\xi)\times$$

$$\log(\cosh\eta-\sin\xi) - 2(\eta-1)e^\eta(\log 2-\eta)\sin\xi + 2(\eta-1)e^\eta\cos\xi\times \qquad (26)$$

$$\left(\tan^{-1}\left(\frac{\cos\xi}{e^\eta-\sin\xi}\right)-\tan^{-1}\left(\frac{\cos\xi}{e^\eta+\sin\xi}\right)\right)\Big\} - 2\eta\sinh\eta\left(\frac{1}{\cosh\eta-\sin\xi}-\frac{1}{\cosh\eta+\sin\xi}\right)\Big]$$

$$m_b(\tfrac{\pi}{2},0) = \frac{a}{2\pi^2}\Big[(1-(\eta-1)e^\eta\sin\xi)\log(\cosh\eta+\sin\xi)-(1+(\eta-1)e^\eta\sin\xi)\log(\cosh\eta-\sin\xi)$$

$$-2(\eta-1)e^\eta(\log 2-\eta)\sin\xi+2(\eta-1)e^\eta\cos\xi\left(\tan^{-1}\left(\frac{\cos\xi}{e^\eta-\sin\xi}\right)-\tan^{-1}\left(\frac{\cos\xi}{e^\eta+\sin\xi}\right)\right)\Big]$$

$$(27)$$

Similarly, expressions for the case of $\rho = 1/2$ can be derived, leading finally to the following equations:

Case: $\alpha = \tfrac{\pi}{2}$; $\beta = 0$; $\rho = 1/2$

$$m_x(\tfrac{\pi}{2},0)=\frac{1}{8\pi}\Big[(1+\nu)\Big\{\log\frac{\cosh\eta+\sin\xi}{\cosh\eta-\sin\xi}+(\tfrac{\eta}{2}-1)e^{\frac{\eta}{2}}\Big(\cos(\tfrac{\xi}{2}-\tfrac{\pi}{2})\log\frac{\cosh\frac{\eta}{2}+\cos(\frac{\xi}{2}-\frac{\pi}{4})}{\cosh\frac{\eta}{2}-\cos(\frac{\xi}{2}-\frac{\pi}{4})}$$

$$-\cos(\tfrac{\xi}{2}+\tfrac{\pi}{4})\log\frac{\cosh\frac{\eta}{2}+\cos(\frac{\xi}{2}+\frac{\pi}{4})}{\cosh\frac{\eta}{2}-\cos(\frac{\xi}{2}+\frac{\pi}{4})}-2\sin(\tfrac{\xi}{2}+\tfrac{\pi}{4})\Big(\tan^{-1}\Big(\frac{\sin(\frac{\xi}{2}+\frac{\pi}{4})}{\sinh\frac{\eta}{2}}\Big)-2\sin(\tfrac{\xi}{2}-\tfrac{\pi}{4})\times$$

$$\tan^{-1}\Big(\frac{\sin(\frac{\xi}{2}+\frac{\pi}{4})}{\sinh\frac{\eta}{2}}\Big)\Big)\Big\}-2\nu\eta\sinh\eta\Big(\frac{1}{\cosh\eta-\sin\xi}-\frac{1}{\cosh\eta+\sin\xi}\Big)\Big] \qquad (28)$$

$$m_y(\tfrac{\pi}{2},0)=\frac{1}{8\pi}\Big[(1+\nu)\Big\{\log\frac{\cosh\eta+\sin\xi}{\cosh\eta-\sin\xi}+(\tfrac{\eta}{2}-1)e^{\frac{\eta}{2}}\Big(\cos(\tfrac{\xi}{2}-\tfrac{\pi}{4})\log\frac{\cosh\frac{\eta}{2}+\cos(\frac{\xi}{2}-\frac{\pi}{4})}{\cosh\frac{\eta}{2}-\cos(\frac{\xi}{2}-\frac{\pi}{4})}$$

$$-\cos(\tfrac{\xi}{2}+\tfrac{\pi}{4})\log\frac{\cosh\frac{\eta}{2}+\cos(\frac{\xi}{2}+\frac{\pi}{4})}{\cosh\frac{\eta}{2}-\cos(\frac{\xi}{2}+\frac{\pi}{4})}-2\sin(\tfrac{\xi}{2}+\tfrac{\pi}{4})\tan^{-1}\Big(\frac{\sin(\frac{\xi}{2}+\frac{\pi}{4})}{\sinh\frac{\eta}{2}}\Big)$$

$$-2\sin(\tfrac{\xi}{2}-\tfrac{\pi}{4})\tan^{-1}\Big(\frac{\sin(\frac{\xi}{2}-\frac{\pi}{4})}{\sinh\frac{\eta}{2}}\Big)\Big)\Big\}-2\eta\sinh\eta\Big(\frac{1}{\cosh\eta-\sin\xi}-\frac{1}{\cosh\eta+\sin\xi}\Big)\Big] \qquad (29)$$

$$m_b\left(\tfrac{\pi}{2},0\right) = \frac{4a}{\pi^2}\left[\ \log\frac{\cosh\eta+\sin\xi}{\cosh\eta-\sin\xi} + \left(\tfrac{\eta}{2}-1\right)e^{\tfrac{\eta}{2}}\left(\cos\left(\tfrac{\xi}{2}-\tfrac{\pi}{4}\right)\log\frac{\cosh\tfrac{\eta}{2}+\cos\left(\tfrac{\xi}{2}-\tfrac{\pi}{4}\right)}{\cosh\tfrac{\eta}{2}-\cos\left(\tfrac{\xi}{2}-\tfrac{\pi}{4}\right)}\right.\right.$$

$$\left.-\cos\left(\tfrac{\xi}{2}+\tfrac{\pi}{4}\right)\log\frac{\cosh\tfrac{\eta}{2}+\cos\left(\tfrac{\xi}{2}+\tfrac{\pi}{4}\right)}{\cosh\tfrac{\eta}{2}-\cos\left(\tfrac{\xi}{2}+\tfrac{\pi}{4}\right)} - 2\sin\left(\tfrac{\xi}{2}-\tfrac{\pi}{4}\right)\tan^{-1}\left(\frac{\sin\left(\tfrac{\xi}{2}-\tfrac{\pi}{4}\right)}{\sinh\tfrac{\eta}{2}}\right)\right.$$

$$\left.\left.-2\sin\left(\tfrac{\xi}{2}+\tfrac{\pi}{4}\right)\tan^{-1}\left(\frac{\sin\left(\tfrac{\xi}{2}+\tfrac{\pi}{4}\right)}{\sinh\tfrac{\eta}{2}}\right)\right)\right] \tag{30}$$

IV. DISCUSSION OF THE SINGULARITY OF THE SUPPORT MOMENTS:

Whereas it is impossible to express the influence functions in finite form except for specific values of ρ , a general discussion of the functional behavior in the immediate vicinity of the influence point $(\tfrac{\pi}{2}, 0)$ can be given. In equation (15) and (16) the last term of the series for m_x and m_y can be summed[2]:

$$\sum_{n=1}^{\infty} e^{-n\eta}\sin n\alpha \sin n\xi = \frac{1}{4}\sinh\eta\left(\frac{1}{\cosh\eta-\cos(\alpha-\xi)} - \frac{1}{\cosh\eta-\cos(\alpha+\xi)}\right)$$

Considering points in the immediate neighborhood of $(\tfrac{\pi}{2},0)$ only, or

$$\xi = \tfrac{\pi}{2}+\varepsilon \qquad , \qquad \eta = \delta$$

$$\varepsilon \doteq 0 \ , \qquad \delta \doteq 0$$

and neglecting higher order terms, the above equations reduce to:

$$m_x \ \sim\ -\frac{1}{2\pi}\left[\frac{\nu\delta^2}{\varepsilon^2+\delta^2} - (1+\nu)J(\rho)\right] \tag{31}$$

$$m_y \ \sim\ -\frac{1}{2\pi}\left[\frac{\delta^2}{\varepsilon^2+\delta^2} - (1+\nu)J(\rho)\right] \tag{32}$$

$$m_b \ \sim\ \frac{2a}{\pi^2\rho}J(\rho) \tag{33}$$

$$J(\rho)=\sum_{n=1,3,5\cdots}^{\infty}\left(\frac{1}{n}-\frac{1}{n+\rho}\right)$$

Introducing polar coordinates θ and $r = \sqrt{\varepsilon^2+\delta^2}$ as shown in

Fig. 2 it follows that

$$\frac{\delta^2}{\varepsilon^2 + \delta^2} = \sin^2\theta$$

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} m_x\left(\frac{\pi}{2}, 0; \frac{\pi}{2} + \varepsilon, \delta\right) = -\frac{1}{2\pi}\left[\nu \sin^2\theta - (1+\nu)J(\rho)\right] \tag{34}$$

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} m_y\left(\frac{\pi}{2}, 0; \frac{\pi}{2} + \varepsilon, \delta\right) = -\frac{1}{2\pi}\left[\sin^2\theta - (1+\nu)J(\rho)\right] \tag{35}$$

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} m_b\left(\frac{\pi}{2}, 0; \frac{\pi}{2} + \varepsilon, \delta\right) = \frac{2a}{\pi^2\rho}J(\rho) \tag{36}$$

The function $J(\rho)$ can be computed as follows:

$$J(\rho) = \sum_{n=1,3,5\cdots}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\rho}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\rho}\right)\left(1 - (-1)^n\right)$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\rho}\right) - \sum_{n=1}^{\infty}\frac{(-1)^n}{n} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n+\rho}\right] \tag{37}$$

However, the theory of Gamma functions furnishes the following relationships (for example,[4] p. 458):

$$\frac{d}{d\rho}\log\Gamma(\rho) + \gamma = \sum_{\kappa=0}^{\infty}\left(\frac{1}{1+\kappa} - \frac{1}{\rho+\kappa}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\rho}\right) - \frac{1}{\rho}$$

$$\therefore \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\rho}\right) = \frac{\Gamma'(\rho)}{\Gamma(\rho)} + \gamma + \frac{1}{\rho} \tag{38}$$

Making use of the relations (for example,[5] p. 28(1)):

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\log 2$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n+\rho} = \int_0^1\frac{x^{\rho-1}}{1+x}dx - \frac{1}{\rho} = \frac{1}{2}\left[\frac{\Gamma'\left(\frac{\rho+1}{2}\right)}{\Gamma\left(\frac{\rho+1}{2}\right)} - \frac{\Gamma'\left(\frac{\rho}{2}\right)}{\Gamma\left(\frac{\rho}{2}\right)}\right] - \frac{1}{\rho} \tag{39}$$

$J(\rho)$ becomes:

$$J(\rho) = \frac{1}{2}\left[\frac{\Gamma'(\rho)}{\Gamma(\rho)} + \gamma + \log 2 + \frac{1}{2}\left\{\frac{\Gamma'(\frac{\rho+1}{2})}{\Gamma(\frac{\rho+1}{2})} - \frac{\Gamma'(\frac{\rho}{2})}{\Gamma(\frac{\rho}{2})}\right\}\right]$$

$$= \frac{1}{2}\left[\bar{\Psi}(\rho) + \gamma + \log 2 + \frac{1}{2}\left\{\bar{\Psi}(\frac{\rho+1}{2}) - \bar{\Psi}(\frac{\rho}{2})\right\}\right]^{*} \tag{39}$$

where $\quad \bar{\Psi}(\rho) = \frac{\Gamma'(\rho)}{\Gamma(\rho)} = \int_0^\infty (e^{\alpha} - \frac{1}{(1+\alpha)^\rho})\frac{d\alpha}{\alpha}$,

Since $\bar{\Psi}(\rho)$ satisfies the following two relations:

$$\bar{\Psi}(\frac{\ell}{n}) + \gamma = -\frac{\pi}{2}\cot\frac{\ell\pi}{n} + 2\sum_{\nu=1}^{|\frac{n+1}{2}-1|}\left\{\cos(\frac{2\ell\nu\pi}{n})\log\sin(\frac{\nu\pi}{n})\right\} - \log(2n)$$

$$\bar{\Psi}(\rho) + \gamma = \sum_{n=1}^{\rho-1}\frac{1}{n} \quad (\rho = 1,2,3,\cdots) \qquad \begin{array}{l}(n=2,3,4\cdots \quad \ell=1,2,\cdots(n-1))\end{array}$$

$J(\rho)$ can be readily computed. Values are shown graphically in Fig. 3. It is a positive function which increases monotonically from zero to infinity with increasing ρ.

The limit of m_x, m_y, m_b, can now be discussed. For simplicity the assumption $\nu = 0$ is made, such that:

$$\lim_{\substack{\varepsilon\to 0 \\ \delta\to 0}} m_x(\frac{\pi}{2},0;\frac{\pi}{2}+\varepsilon,\delta) = \frac{1}{2\pi}J(\rho) \tag{40}$$

$$\lim_{\substack{\varepsilon\to 0 \\ \delta\to 0}} m_y(\frac{\pi}{2},0;\frac{\pi}{2}+\varepsilon,\delta) = -\frac{1}{2\pi}\left[\sin^2\theta - J(\rho)\right] \tag{41}$$

$$\lim_{\substack{\varepsilon\to 0 \\ \delta\to 0}} m_b(\frac{\pi}{2},0;\frac{\pi}{2}+\varepsilon,\delta) = \frac{2a}{\pi^2\rho}J(\rho) \tag{42}$$

As only the limit of m_y depends upon the angle of incidence θ, the discussion will be restricted to this case. Inspection of Fig. 3 and equation (41) shows that for $J(\rho) > 1$, no angle of incidence exists for which m_y will be zero. However for $J(\rho) \leqq 1$, m_y becomes zero for specific values $\theta = \theta_0$. Five cases are considered:

---

* $\quad \gamma = 0.5772156649\ldots$ (Euler's Constant)

(I) $\rho = 0$:

$$J(\rho) = 0 \qquad\qquad m_y(\tfrac{\pi}{2}, 0; \tfrac{\pi}{2}, 0) = -\frac{1}{2\pi} \sin^2\theta$$

$$\theta_0 = 0 \quad \text{Rad.}$$

(II) $\rho = 1/2$:

$$J(\tfrac{1}{2}) = \tfrac{1}{2}(\tfrac{\pi}{2} - \log 2) = 0.4388$$

$$m_y(\tfrac{\pi}{2}, 0; \tfrac{\pi}{2}, 0) = -\frac{1}{2\pi}(\sin^2\theta - 0.4388)$$

and solving the equation: $\sin^2\theta - 0.4388 = 0$

$$\theta_0 = 0.7243 \quad \text{Rad.}$$

(III) $\rho = 1$:

$$J(1) = 0.69315$$

$$m_y(\tfrac{\pi}{2}, 0; \tfrac{\pi}{2}, 0) = -\frac{1}{2\pi}(\sin^2\theta - 0.69315)$$

$$\theta_0 = 0.9818 \quad \text{Rad}$$

(IV) $\rho = 2$:

$$J(2) = 1$$

$$m_y(\tfrac{\pi}{2}, 0; \tfrac{\pi}{2}, 0) = -\frac{1}{2\pi}(\sin^2\theta - 1)$$

$$\theta_0 = \frac{\pi}{2} \quad \text{Rad}$$

(V) $\rho > 2$

$$J(\rho) > 1, \text{ no } \theta_0 \text{ exists}$$

and since $\qquad \lim\limits_{\rho \to \infty} J(\rho) = \infty$

$$\lim\limits_{\rho \to \infty} m_y(\tfrac{\pi}{2}, 0; \tfrac{\pi}{2}, 0) = +\infty$$

This limiting case corresponds to an infinite strip with no cross beam.

The results are plotted graphically in Fig. 4 (a) to (e). It can be seen that the angle of incidence θ_0 for the zero line increases successively from 0 to $\frac{\pi}{2}$ corresponding to a change in ρ from zero to two. For values of $\rho > 2$ no zero line passes through the influence point.

Knowing the behavior of m_y at the influence point $(\frac{\pi}{2}, 0)$ and the tangents to the contour lines meeting at this point, the general shapes of the m_y influence surfaces can be easily visualized. Fig.5 (a) to (e) illustrate the general appearance schematically. Cases (a) and (e) are the known cases for an infinitely rigid cross beam, $\rho = 0$, and no cross beam, $\rho = \infty$, respectively. Between the two cases a steady transition takes place.

V. EXTENSION TO THE CASE OF MULTIPLE CROSS BEAMS:

The method of the integral equation used in Chapter 2 for the case of a single cross beam can be extended to the case of a plate strip supported by a series of cross beams as indicated in Fig. 6. If in addition to the bending resistance, consideration is given to the warping and the torsional resistance of the cross beams, the influence function for the deflection w of point (α, β) takes the following form:

$$W(\alpha, \beta; \xi, \eta) = G_1(\alpha, \beta; \xi, \eta) - \sum_{i=1}^{K} \frac{\pi^3}{a^3} \int_0^\pi EI_i \frac{\partial^4 W(\zeta_i, \gamma_i; \xi, \eta)}{\partial \zeta_i^4} G_1(\alpha, \beta; \zeta_i, \gamma_i) d\zeta_i$$

$$+ \sum_{i=1}^{K} \frac{\pi^4}{a^4} \int_0^\pi \left\{ EI_{\omega i} \frac{\partial^5 W(\zeta_i, \gamma_i; \xi, \eta)}{\partial \zeta_i^4 \partial \gamma_i} - \frac{a^2}{\pi^2} GK_T \frac{\partial^3 W(\zeta_i, \gamma_i; \xi, \eta)}{\partial \zeta_i^2 \partial \gamma_i} \right\} G_2(\alpha, \beta; \zeta_i, \gamma_i) d\zeta_i$$

$$(43)$$

where ζ_i, γ_i = dimensionless coordinates of i-th cross beam

  EI_i = bending stiffness of i-th cross beam

  $EI_{\omega i}$ = warping rigidity of i-th cross beam

  GK_T = torsional rigidity (St. Venant's Torsion) of

              i-th cross beam

$G_1(\alpha, \beta; \xi, \eta)$ = Green's function for deflection of point (α, β)

due to concentrated transverse load at point (ξ, η)

$G_2(\alpha, \beta; \xi, \eta)$ = Green's function for deflection of point (α, β)

due to a concentrated moment at point (ξ, η)

acting about an axis parallel to the ξ -axis.

The function G_1 is identical with equation (2), or in non-dimensional form:

$$G_1(\alpha, \beta; \xi, \eta) = \frac{a^2}{2\pi^3 D} \sum_{n=1}^{\infty} \frac{1}{n^3} \psi_n(\alpha, \beta; \xi, \eta) \tag{44}$$

with

$$\psi_n(\alpha, \beta; \xi, \eta) = (1 \mp n(\beta - \eta)) e^{\pm n(\beta - \eta)} \sin n\alpha \sin n\xi$$

G_2 is the limiting case of the difference between $PG_1(\alpha, \beta; \xi, \eta + \delta)$ and $PG_1(\alpha, \beta; \xi, \eta)$ as $P\delta$ approaches unity and δ itself approaches zero, or:

$$G_2(\alpha, \beta; \xi, \eta) = \lim_{\substack{P\delta \to 1 \\ \delta \to 0}} \left[PG_1(\alpha, \beta; \xi, \eta + \delta) - PG_1(\alpha, \beta; \xi, \eta) \right]$$

$$= -\frac{(\beta - \eta)}{2\pi D} \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm n(\beta - \eta)} \sin n\alpha \sin n\beta \tag{45}$$

The solution follows the pattern outlined in Chapter 2, the only difficulty being the extent of arithmetical operations.

VI. TWO-SPAN CONTINUOUS SLAB WITH FLEXIBLE CROSS BEAMS:

The slab as shown in Fig.7 is simply supported along its edges $\xi = 0$ and $\xi = \pi$. The two end beams at $\eta = \pm \gamma$ have the same constant bending rigidity EI as the cross beam at $\eta = 0$. As an example the influence surface for the support moment M_y at point $(\alpha, 0)$ will be computed. Referring to [2] the m_y-function is taken in two parts:

$$m_y = m_{yo} + m_{y1} \tag{46}$$

where m_{yo} is the solution of the infinite plate strip with an elastic cross beam at $\eta = 0$ as given by equation (16) or (27) and m_{y1} is a solution of the homogeneous plate equation:

$$m_{y_1} = \sum_{n=1}^{\infty} (a_n \sinh n\eta + b_n \cosh n\eta + c_n n\eta \sinh n\eta + d_n n\eta \cosh n\eta)$$

$$\times \quad \sin n\alpha \sin n\xi \tag{47}$$

The sum $(m_{yo} + m_{y1})$ must fulfill all boundary conditions. Considering symmetry with respect to $\eta = 0$ these conditions are:

At $\eta = 0$:

(a) $\quad \dfrac{\partial m_y}{\partial \eta} = 0 \tag{48}$

However as $\quad \dfrac{\partial m_{yo}}{\partial \eta} = 0$ the condition becomes:

$$\dfrac{\partial m_{y1}}{\partial \eta} = 0 \tag{49}$$

(b) $\quad -\dfrac{\pi^3 D}{a^3} \left(\dfrac{\partial^3 m_y}{\partial \eta^3} + \dfrac{\partial^3 m_y}{\partial \xi \partial \eta} \right) = \dfrac{\pi^4 EI}{2a^4} \dfrac{\partial^4 m_y}{\partial \xi^4} \tag{50}$

Equation (50) expresses the condition that the "shearing force"[*] at ᐩ0 is equal to half the reaction of the cross beam. However as m_{yo} fulfills this condition separately the equation reduces to:

$$-\dfrac{\pi^3 D}{a^3} \left(\dfrac{\partial^3 m_{y1}}{\partial \eta^3} + \dfrac{\partial^3 m_{y1}}{\partial \xi \partial \eta} \right) = \dfrac{\pi^4 EI}{2a^4} \dfrac{\partial^4 m_{y1}}{\partial \xi^4} \tag{51}$$

----

[*]As m_y must fulfill the same boundary conditions as the deflection w such terminology seems appropriate.

At $\eta = \dfrac{\pi c}{a} = \gamma$:

(c) $\qquad \dfrac{\partial^2 m_y}{\partial \eta^2} = 0$ $\qquad\qquad\qquad\qquad$ (52)

(d) $\qquad -\dfrac{\pi^3 D}{a^3}\left(\dfrac{\partial^3 m_y}{\partial \eta^3} + (2-\nu)\dfrac{\partial^3 m_y}{\partial \xi^2 \partial \eta} \right) = -\dfrac{\pi^4 EI}{a^4}\dfrac{\partial^4 m_y}{\partial \xi^4}$ $\qquad$ (53)

The first condition assumes the edge free of bending moments M_y whereas the second condition postulates the identity between the plate boundary shear and the loading of the cross beam. Introducing the expressions (16) and (47) for m_{yo} and m_{y1} into the boundary conditions (49), (51), (52) and (53) furnishes a system of equations for the determination of the constants a_n to d_n. Explicit expressions for these constants in general form are too complex. However, their actual computation for a given example does not offer any particular difficulty.

VII. NUMERICAL EXAMPLES:

The foregoing derivations were used for computation of two examples:

(a) Plate Strip Continuous Over Flexible Cross Beam

Choosing the influence point $(\dfrac{\pi}{2},0)$ and a stiffness ratio $\rho = \dfrac{4aD}{\pi EI} = 1$, equation (14), (15) and (16) were used to compute the influence surfaces for the beam moment m_b and the bending moments m_x and m_y. Furthermore, the assumption of Poisson's ratio $\nu = 0$ makes the expressions for m_b and m_x identical except for a constant multiplier. The results are plotted in two graphs (9) and (10). It should be noted that the plotted values correspond to 8π times the influence values for m_x and m_y and

$2\,\pi^2/a$ times for m_b. For cases other than $\nu = 0$, the graphs (9) and (10) are still applicable. Due to the fact that ν does not enter the boundary conditions, the influence surface can be taken in the following form:

$$m_y = (m_y)_{\nu=0} + \nu(m_x)_{\nu=0}$$

and similarly for m_x. However for other cases where ν influences the boundary conditions (e.g., free edges, elastically supported edges, etc.) such a procedure is not rigorously applicable and its accuracy must be investigated from case to case.

A remark is indicated concerning the choice of $\rho = 1$. A cross section through the cross beam shown in Fig. 8 is considered. The dimensions are given in terms of the span length a. The interaction between the beam and the slab is usually taken into account by determining an effective width of the slab acting as a flange of the beam (see e.g.,[3] p. 119). However, for the present qualitative consideration, it is sufficiently accurate to assume that the action of the slab forces the neutral axis of the beam to coincide with the middle plane of the slab n-n. Computing the moment of inertia I of the rectangular beam a/10 x a/25 with respect to n-n the value of ρ is determined, $\rho = 0.98$. A more rigorous investigation would lead to a neutral axis slightly below n-n and hence to a smaller I and a somewhat higher value of ρ. Nevertheless the example shows that the case $\rho = 1$ will correspond in practice to a rather flexible cross beam, the usual cases being limited between $0 < \rho < 1$.

(b) Two-Span Continuous Slab With Flexible Cross Beams:

The plates were chosen as two square plates, $\gamma = \pi$ simply supported along the edges $\xi = 0$ and $\xi = \pi$ (Fig. 7). Taking the ratio $\rho = 1$, equal bending stiffeness EI for all three cross beams was assumed. Only the influence surface for $m_y (-\frac{\pi}{2}, 0)$ was computed. It should be pointed out that the calculations were done for $\nu = 0$. Using the method outlined in Chapter 6 the computed constants a_n to d_n of equation (47) for the first and third term of the series are given in the following table:

a_1	5.151×10^{-3}	a_3	7.674×10^{-10}
b_1	5.151×10^{-3}	b_3	2.558×10^{-10}
c_1	2.830×10^{-3}	c_3	7.538×10^{-10}
d_1	5.151×10^{-3}	d_3	7.674×10^{-10}

In calculating the m_{y1} values, only the first term of the series was considered.

VIII. ACKNOWLEDGEMENTS

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APPENDIX

(a) Summation Formulae:

For summation of equations (12) to (14) use of the following formulae was made:

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \cos nx = -\frac{1}{2} \log(1 - 2r\cos x + r^2) \tag{A}$$

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \sin nx = \tan^{-1}\left(\frac{r\sin x}{1 - r\cos x}\right) \tag{B}$$

$$\sum_{n=1,3,5\cdots}^{\infty} \frac{r^n}{n} \cos nx = \frac{1}{4} \log \frac{1 + 2r\cos x + r^2}{1 - 2r\cos x + r^2} \tag{C}$$

$$\sum_{n=1,3,5\cdots}^{\infty} \frac{r^n}{n} \sin nx = \tan^{-1}\left(\frac{2r\sin x}{1 - r^2}\right) \tag{D}$$

$$\sum_{n=1}^{\infty} r^n \cos nx = \left(\frac{1 - r\cos x}{1 - 2r\cos x + r^2} - 1\right) = \frac{1}{2}\left(\frac{1 - r^2}{1 - 2r\cos x + r^2} - 1\right) \tag{E}$$

$$\sum_{n=1}^{\infty} r^n \sin nx = \frac{r\sin x}{1 - 2r\cos x + r^2} \tag{F}$$

$$\sum_{n=1}^{\infty} n r^n \cos nx = \frac{r\{(1 + r^2)\cos x - 2r\}}{(1 - 2r\cos x + r^2)^2} \tag{G}$$

for values of $|r| < 1$. Equations (A),(B),(D) may be found in reference (6), p.190. The derivation of equation (E) is given in reference (2), p.496, Appendix. Equations (C),(F),(G) were developed similarly.

(b) Summation of Equations (12) to (14):

With the aid of the formulae (A) to (G) the series of equations (12) to (14) for the specific value $\varrho = 1$ are summed. The results are given without repeating the arithmetical operations.

For $\varrho = 1$:

$$8\pi\, m_x(\alpha,\beta;\xi,\eta) = (1+\nu)\,\log\frac{\cosh(\beta-\eta)-\cos(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} + (1-\nu)(\beta-\eta)\sinh(\beta-\eta)\left(\frac{1}{\cosh(\beta-\eta)-\cos(\alpha-\xi)}\right.$$

$$\left.-\frac{1}{\cosh(\beta-\eta)-\cos(\alpha+\xi)}\right) + (1+\nu)\left\{\mp\eta\,\sinh(\beta\pm\eta)\left(\frac{1}{\cosh(\beta\pm\eta)-\cos(\alpha-\xi)}-\right.\right.$$

$$\left.\frac{1}{\cosh(\beta\pm\eta)-\cos(\alpha+\xi)}\right) + (\pm\eta-1)e^{\beta\pm\eta}\left[\cos(\alpha+\xi)\log\big(\cosh(\beta\pm\eta)-\cos(\alpha+\xi)\big)-\right.$$

$$\cos(\alpha-\xi)\log\big(\cosh(\beta\pm\eta)-\cos(\alpha-\xi)\big) - 2(\log 2\mp\eta)\sin\alpha\sin\xi -$$

$$\left.\left.2\sin(\alpha+\xi)\tan^{-1}\left(\frac{\sin(\alpha+\xi)}{e^{\beta\pm\eta}-\cos(\alpha+\xi)}\right) + 2\sin(\alpha-\xi)\tan^{-1}\left(\frac{\sin(\alpha-\xi)}{e^{\beta\pm\eta}-\cos(\alpha-\xi)}\right)\right]\right\} \div$$

$$(1-\nu)\beta\left\{(1\mp\eta)e^{\beta\pm\eta}\left[\cos(\alpha+\xi)\log\big(\cosh(\beta\pm\eta)-\cos(\alpha+\xi)\big) - \cos(\alpha-\xi)\times\right.\right.$$

$$\log\big(\cosh(\beta\pm\eta)-\cos(\alpha-\xi)\big) - 2(\log 2\mp\eta)\sin\alpha\sin\xi - 2\sin(\alpha+\xi)\times$$

$$\left.\tan^{-1}\left(\frac{\sin(\alpha+\xi)}{e^{\beta\pm\eta}-\cos(\alpha+\xi)}\right) + 2\sin(\alpha-\xi)\tan^{-1}\left(\frac{\sin(\alpha-\xi)}{e^{\beta\pm\eta}-\cos(\alpha-\xi)}\right)\right] \pm\eta\left(\frac{\cosh(\beta\pm\eta)\cos(\alpha-\xi)-1}{\big(\cosh(\beta\pm\eta)-\cos(\alpha-\xi)\big)^2}\right.$$

$$\left.-\frac{\cosh(\beta\pm\eta)\cos(\alpha+\xi)-1}{\big(\cosh(\beta\pm\eta)-\cos(\alpha+\xi)\big)^2}\right) + (1\mp\eta)\sinh(\beta\pm\eta)\left(\frac{1}{\cosh(\beta\pm\eta)-\cos(\alpha-\xi)}-\right.$$

$$\left.\left.\frac{1}{\cosh(\beta\pm\eta)-\cos(\alpha+\xi)}\right)\right\} \tag{H}$$

$$8\pi\, m_g(\alpha,\beta;\xi,\eta) = (1+\nu)\log\frac{\cosh(\beta-\eta)-\cos(\alpha+\xi)}{\cosh(\beta-\eta)-\cos(\alpha-\xi)} - (1-\nu)(\beta-\eta)\sinh(\beta-\eta)\left(\frac{1}{\cosh(\beta-\eta)-\cos(\alpha-\xi)}\right.$$

$$\left. - \frac{1}{\cosh(\beta-\eta)-\cos(\alpha+\xi)}\right) + (1+\nu)\left\{\mp\eta\sinh(\beta\pm\eta)\left(\frac{1}{\cosh(\beta\pm\eta)-\cos(\alpha\mp\xi)} - \frac{1}{\cosh(\beta\pm\eta)-\cos(\alpha+\xi)}\right)\right.$$

$$+ (\pm\eta-1)e^{\beta\pm\eta}\left[\cos(\alpha+\xi)\log(\cosh(\beta\pm\eta)-\cos(\alpha-\xi)) - \cos(\alpha-\xi)\log(\cosh(\beta\pm\eta)-\cos(\alpha+\xi))\right.$$

$$\left. - 2(\log 2 \mp \eta)\sin\alpha\sin\xi - 2\sin(\alpha+\xi)\tan^{-1}\left(\frac{\sin(\alpha+\xi)}{e^{\beta\pm\eta}-\cos(\alpha+\xi)}\right) + 2\sin(\alpha-\xi)\times\right.$$

$$\left. \tan^{-1}\left(\frac{\sin(\alpha-\xi)}{e^{\beta\pm\eta}-\cos(\alpha-\xi)}\right)\right] + (1-\nu)\beta\left\{(1\mp\eta)e^{\beta\pm\eta}\left[\cos(\alpha+\xi)\log(\cosh(\beta\pm\eta)-\cos(\alpha+\xi))\right.\right.$$

$$\left. -\cos(\alpha-\xi)\log(\cosh(\beta\pm\eta)-\cos(\alpha-\xi)) - 2(\log 2\mp\eta)\sin\alpha\sin\xi - 2\sin(\alpha+\xi)\times\right.$$

$$\left.\tan^{-1}\left(\frac{\sin(\alpha+\xi)}{e^{\beta\pm\eta}-\cos(\alpha+\xi)}\right) + 2\sin(\alpha-\xi)\tan^{-1}\left(\frac{\sin(\alpha-\xi)}{e^{\beta\pm\eta}-\cos(\alpha-\xi)}\right)\right] \pm\eta\left(\frac{\cosh(\beta\pm\eta)\cos(\alpha-\xi)-1}{(\cosh(\beta\pm\eta)-\cos(\alpha-\xi))^2}\right.$$

$$\left.\left. - \frac{\cosh(\beta\pm\eta)\cos(\alpha+\xi)-1}{(\cosh(\beta\pm\eta)-\cos(\alpha+\xi))^2}\right) + (1\mp\eta)\sinh(\beta\pm\eta)\left(\frac{1}{\cosh(\beta\pm\eta)-\cos(\alpha-\xi)} - \frac{1}{\cosh(\beta\pm\eta)-\cos(\alpha+\xi)}\right)\right\}$$

$$\text{(I)}$$

$$\frac{\pi^2}{2a}\, m_b(\alpha,0;\xi,\eta) = \log\frac{\cosh\eta-\cos(\alpha+\xi)}{\cosh\eta-\cos(\alpha-\xi)} + (\pm\eta-1)e^{\pm\eta}\left\{\cos(\alpha+\xi)\log(\cosh\eta-\cos(\alpha+\xi))\right.$$

$$\left. -\cos(\alpha-\xi)\log(\cosh\eta-\cos(\alpha-\xi)) - 2(\log 2\mp\eta)\sin\alpha\sin\xi - 2\sin(\alpha+\xi)\times\right.$$

$$\left.\tan^{-1}\left(\frac{\sin(\alpha+\xi)}{e^{\pm\eta}-\cos(\alpha+\xi)}\right) + 2\sin(\alpha-\xi)\tan^{-1}\left(\frac{\sin(\alpha-\xi)}{e^{\pm\eta}-\cos(\alpha-\xi)}\right)\right\}\qquad\text{(K)}$$

It may be pointed out that the above expressions hold anywhere on the plate strip, Fig.1. The equations for the support moments of point $(-\frac{\pi}{2}, 0)$ follow either by introducing $\alpha = \frac{\pi}{2}$, $\beta = 0$ into the above equation (H) to (K) or by summing equations (22) to (24), expressed in series form, directly with the use of the formulae (A) to (G). Both procedures will lead to equations (25) to (27).

For the case of $\rho = 1/2$ summation is also possible. The results for the support moments are given by equations (28) to (30). However, as indicated before, no summation for a general value of ρ is possible.

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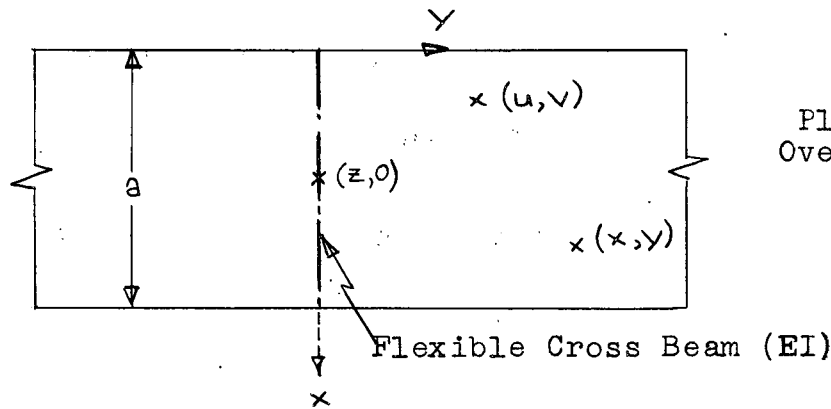
Figure 1
Plate Strip Continuous
Over Flexible Cross Beam

Flexible Cross Beam (EI)

Non-dimensional Coordinates: $\frac{\pi u}{a} = \alpha$; $\frac{\pi v}{a} = \beta$

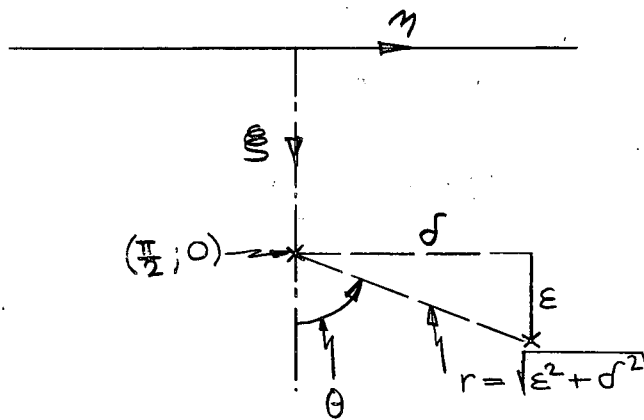$\frac{\pi x}{a} = \xi$; $\frac{\pi y}{a} = \eta$; $\frac{\pi z}{a} = \zeta$

Figure 2
Polar Coordinates Around
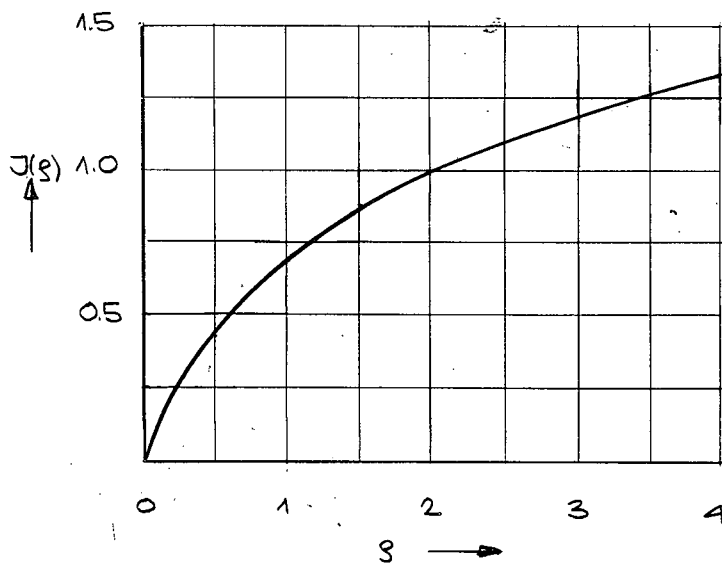Influence Point $(\frac{\pi}{2}, 0)$

$r = \sqrt{\varepsilon^2 + \delta^2}$
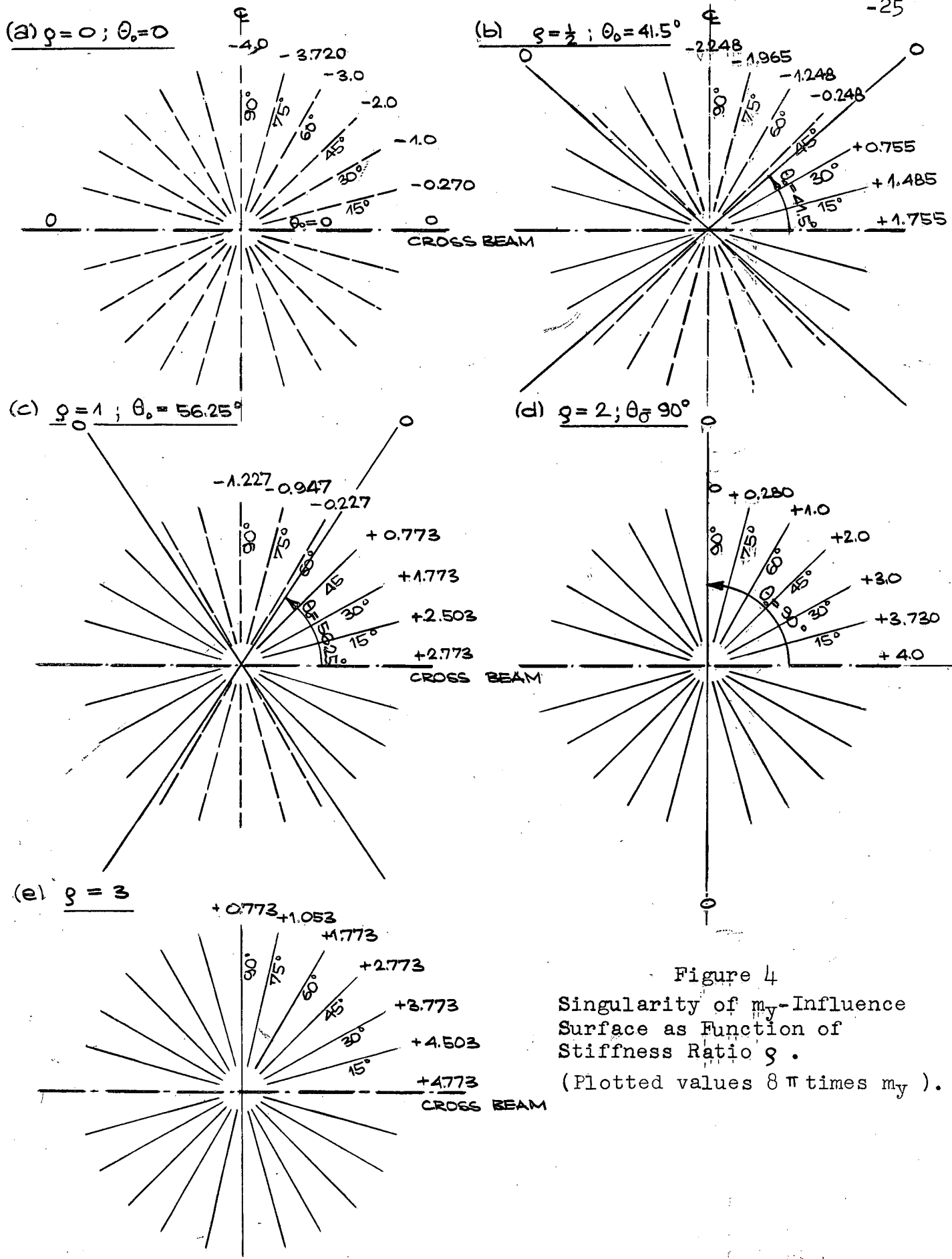

Figure 3
Function $J(\varrho)$

Figure 4

Singularity of m_y-Influence
Surface as Function of
Stiffness Ratio ϱ.

(Plotted values 8π times m_y).

(a) $\varrho = 0$

(b) $\varrho = 1$

(c) $\varrho = 2$

(d) $\varrho > 2$

(e) $\varrho \to \infty$

CROSS BEAM
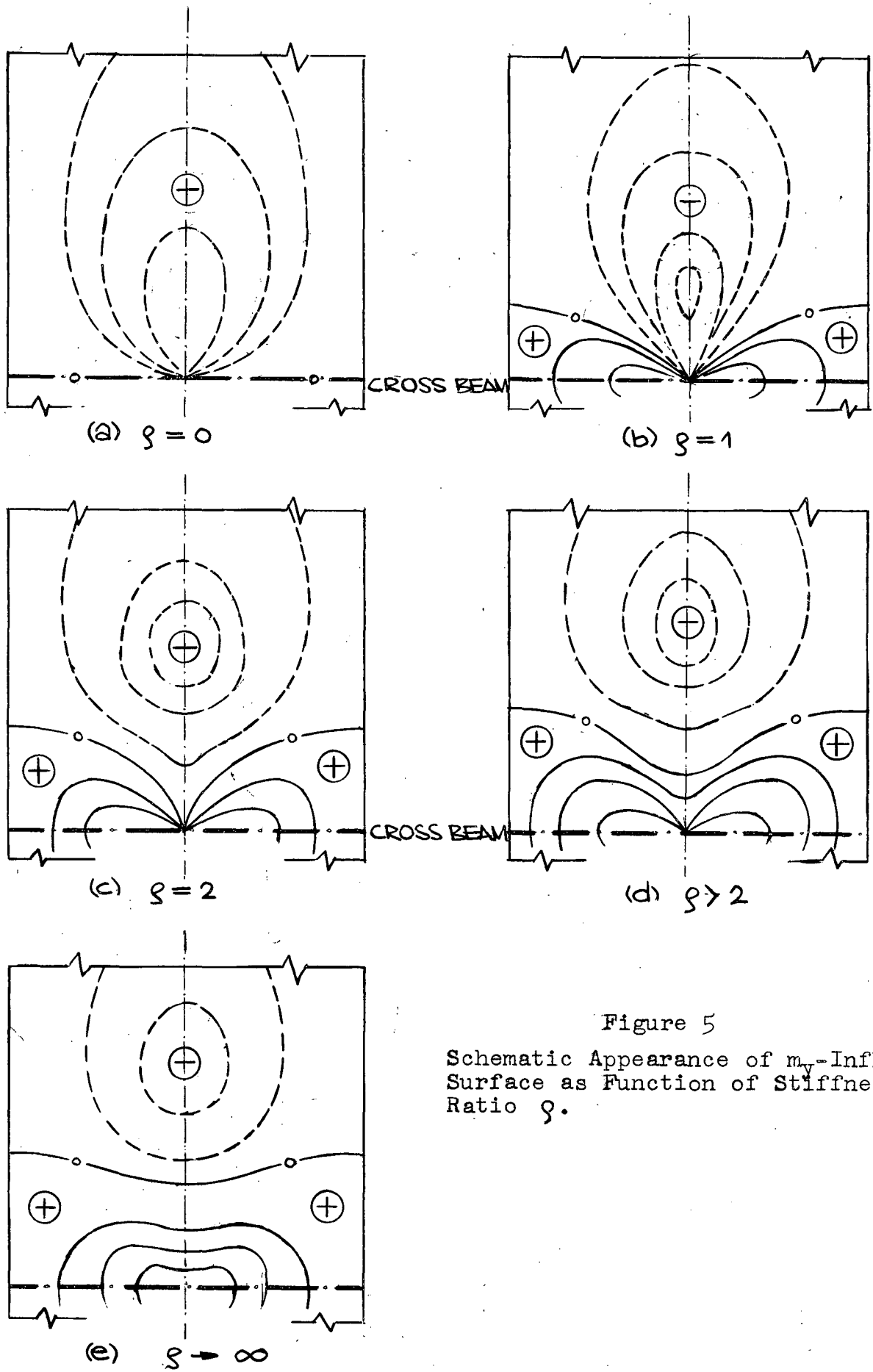
CROSS BEAM

Figure 5

Schematic Appearance of m_y-Influence Surface as Function of Stiffness Ratio ϱ.
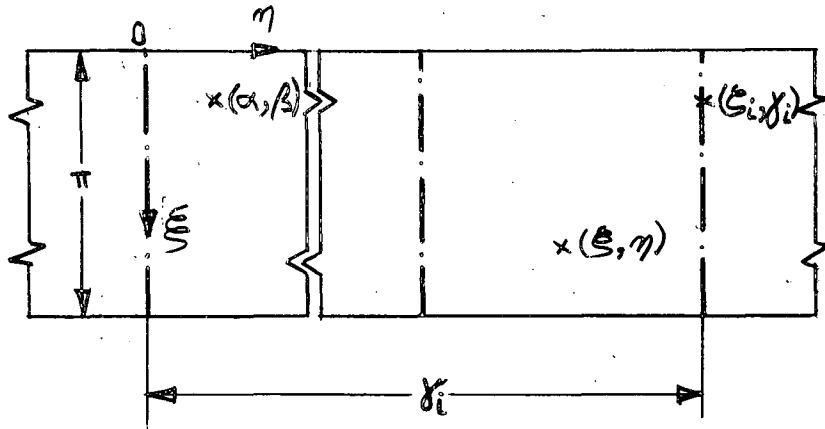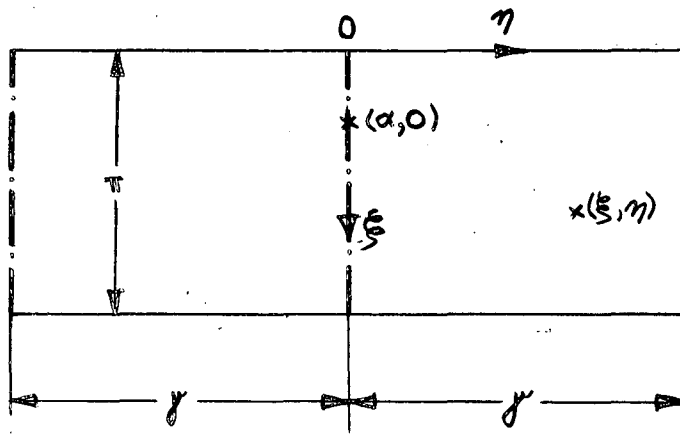
Figure 6
Plate Strip With
Multiple Cross Beams
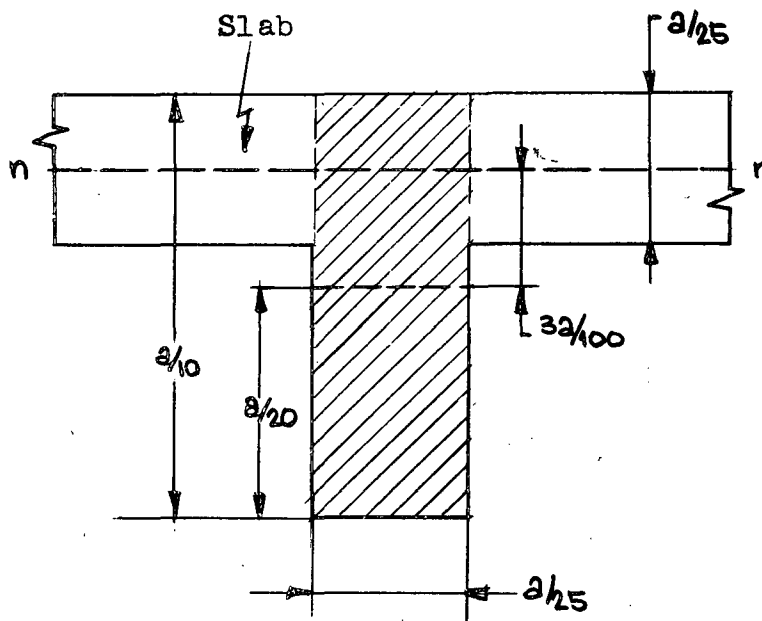

Figure 7
Two-Span Continuous
Slab With Flexible
Cross Beams

Figure 8
Section Through
Cross Beam

Span Length $= a$
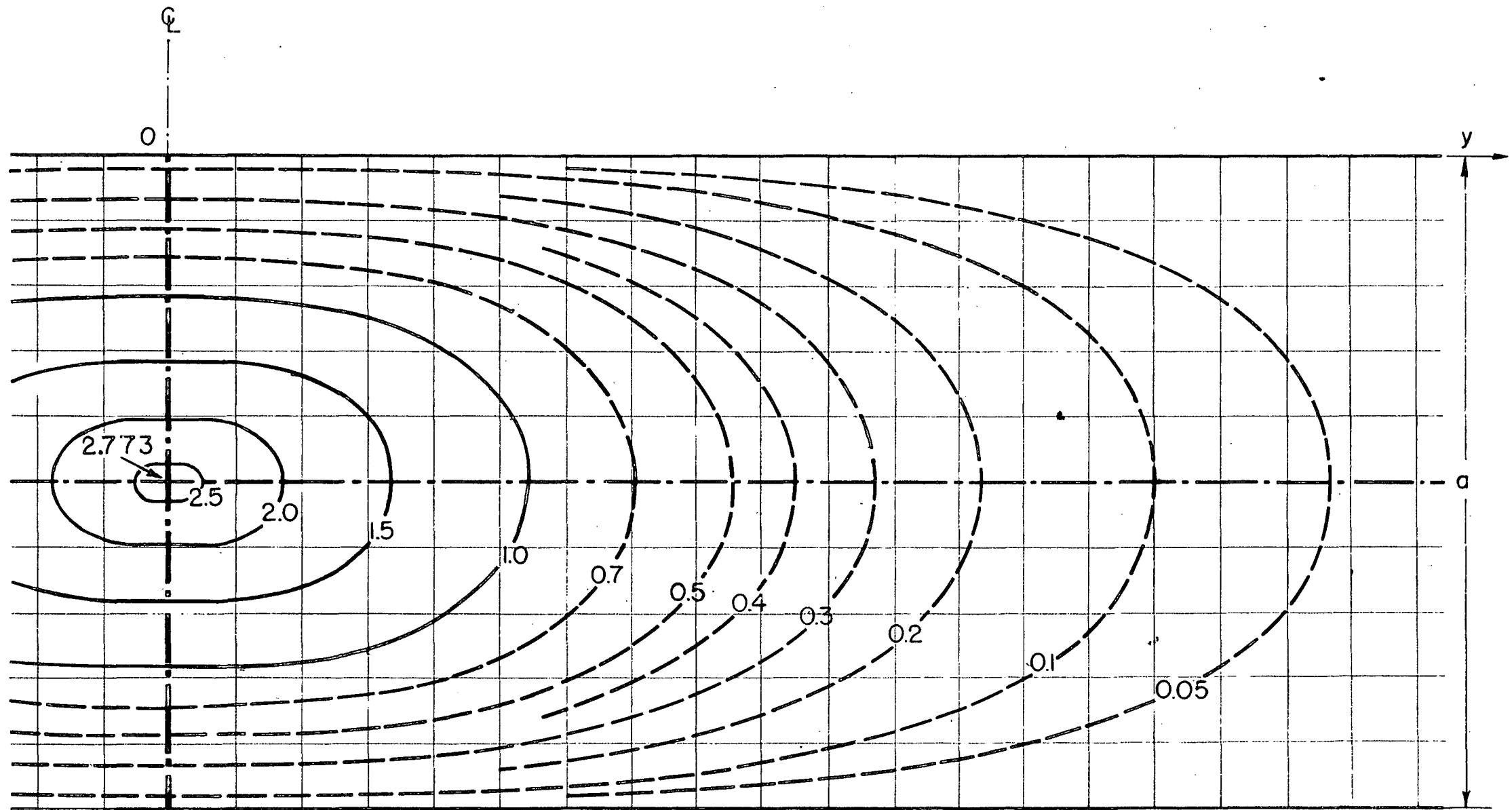Stiffness Ratio $\rho = \dfrac{4aD}{\pi EI} = 0.98$

Plate Strip with Flexible Cross Beam   Figure 9
($g = 4aD / \pi EI = 1$)

m_x — Influence Surface for Plate Moment over Cross Beam (8π times)

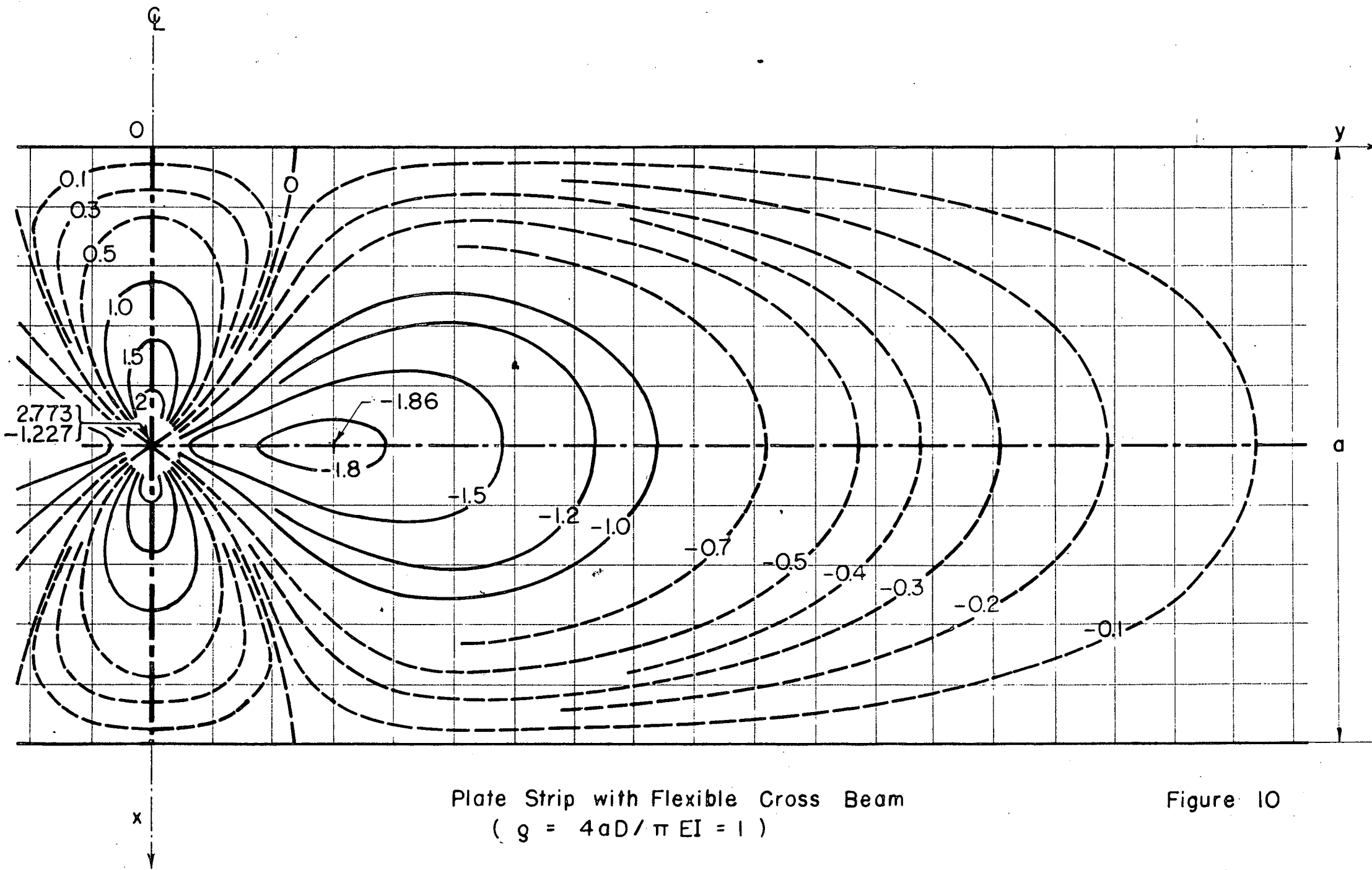m_b — Influence Surface for Bending Moment of Cross Beam ($2\pi^2 /a$ times)

Plate Strip with Flexible Cross Beam

Figure 10

$(\; g \; = \; 4aD / \pi \, EI \; = \; 1 \;)$

m_y — Influence Surface for Plate Moment over Cross Beam (8π times)
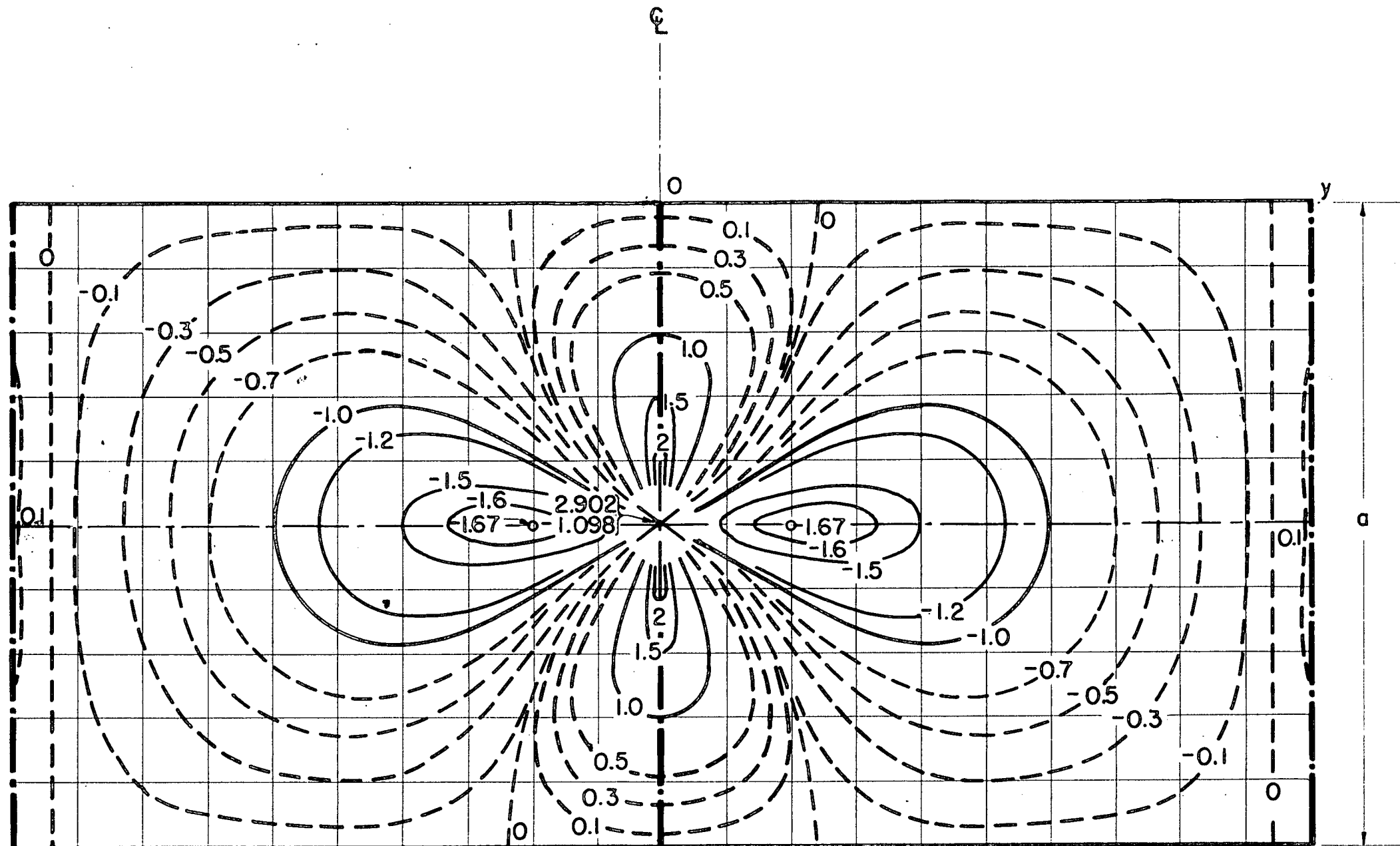
Figure 11

Two Span Continuous Slab with Flexible Cross Beams

$$\left(\varrho = \frac{4aD}{\pi EI} = 1 \right)$$

m_y — Influence Surface for Plate Moment over Middle Cross Beam (8π times)