1969

Users manual for cstes--finite element program,.June 1969

E. Schultchen

C. N. Kostem

USERS' MANUAL FOR CSTES FINITE ELEMENT PROGRAM

by

Erhard G. Schultchen

Celal N. Kostem

Department of Civil Engineering

Fritz Engineering Laboratory

Lehigh University

Bethlehem, Pennsylvania

June 1969

Fritz Engineering Laboratory Report 237.58

TABLE OF CONTENTS

1. INTRODUCTION

CSTES program is for the stress and deformation analysis of two dimensional elasticity problems by Finite Element Techniques. This approach discretizises the planar continuum by using macro-components, "Finite Elements". In the formulation "Constant Stress Triangle" (CST) type finite elements are used (Ref. 1). Program is based on E. L. Wilson's study on the finite element analysis of planar continuum by using CSTs (Ref. 2).

CSTES program can be used for the linear elastic analysis of two dimensional continuum subjected to gravity, body, thermal, distributed or concentrated forces and their combinations. Planar problems can be treated as plane stress or plane strain cases depending upon the input information provided by the user.

2. ANALYTIC FORMULATION

2.1 Displacement Field

Basic assumptions of finite element formulation requires the definition of displacement field. In CST type formulation displacement field is defined by

$$u(x,y) = C_1 + C_2x + C_3y$$

$$v(x,y) = C_4 + C_5x + C_6y$$

in x and y directions respectively. This assumption also presupposes the stress and strain variation within the element:

$$\epsilon_x = \frac{\partial u}{\partial x} = C_2$$

$$\epsilon_y = \frac{\partial v}{\partial y} = C_6$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = C_3 + C_5$$

Thus the strains are constant throughout the element. Consequently the stresses within an element are also constant.

Furthermore, the assumed displacement field results in the continuity between the elements after the deformation; that is initially straight lines remain straight in their displaced position.

Stepwise change of stresses and strains in the continuum from element to element imposes a restriction for the types of problems that can be analyzed with given method. Utilization of CST approach for problems with high stress or strain gradients yields poor results. This is in part due to the approximation in the use of stepwise functions, discrete representation, for non-linear curves, stress gradients. Application of CST type formulation, such as CSTES, to high stress gradient problems should not be used if accurate assessment of stress and deformation variations are required.

2.2 Basic Formulation

Linkage between the finite elements is made at the vertices of the triangles which are the nodal points. If the reactions developed between the elements are shown by $\underline{F}$, with components in x and y directions, and the corresponding nodal point displacements are shown by $\underline{\delta}$ with similar components, then the following canonical equilibrium equation can be written:

$$\underline{F} = \underline{K}\ \underline{\delta}$$

Here $\underline{K}$ is the global stiffness matrix of the problem under study.

2.3 Fundamental Matrices

In CST formulation there are three fundamental matrices.

(1) $\underline{B}$, relates elemental strains to nodal point displacements

(2) $\underline{D}$, relates elemental stresses to elemental strains, known as the Elasticity Matrix

(3) $\underline{k}^e$, relates nodal point reactions to nodal point displacements, known as the Element Stiffness Matrix

Derivation of these fundamental matrices can be found in many sources (Ref. 1 and 3). However, for user's convenience, the general form of these matrices are listed.

(1) $\underline{B}$:

$$\underline{\varepsilon} = \underline{B}\ \underline{\delta}^e$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{bmatrix}$$

where

$$b_i = y_j - y_k \qquad c_i = x_k - x_j$$

$$b_j = y_k - y_i \qquad c_j = x_i - x_k$$

$$b_k = y_i - y_j \qquad c_k = x_j - x_i$$

$$2\Delta = c_k\, b_j - c_j\, b_k$$

(2) Elasticity Matrix $\underline{D}$:

$$\underline{\sigma} = \underline{D}\ \underline{E}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

This relation holds for the plane stress formulation of isotropic materials.

(3) Element Stiffness Matrix, $\underline{k}^e$:

$$\underline{k}^e = \iint \underline{B}^T\ \underline{D}\ \underline{B}\ t\ dx\ dy = \underline{B}^T\ \underline{D}\ \underline{B}\ t\ \Delta$$

$$\underline{F}^e = \underline{k}^e\ \underline{\delta}^e$$

$$\begin{bmatrix} F_i^e \\ F_j^e \\ F_k^e \end{bmatrix} = \begin{bmatrix} \underline{k}_{ii}^e & \underline{k}_{ij}^e & \underline{k}_{ik}^e \\ \underline{k}_{ji}^e & \underline{k}_{jj}^e & \underline{k}_{jk}^e \\ \underline{k}_{ki}^e & \underline{k}_{kj}^e & \underline{k}_{kk}^e \end{bmatrix} \begin{bmatrix} \delta_i^e \\ \delta_j^e \\ \delta_k^e \end{bmatrix}$$

where $\underline{k}^e_{rs}$ are 2 x 2 submatrices of the following form.

$$\begin{bmatrix} F^e_{xr} \\ F^e_{yr} \end{bmatrix} = \begin{bmatrix} k^e_{xxrs} & k^e_{xyrs} \\ k^e_{yxrs} & k^e_{yyrs} \end{bmatrix} \begin{bmatrix} \delta^e_{xs} \\ \delta^e_{ys} \end{bmatrix}$$

Because of their simplicity the $\underline{B}$ and $\underline{D}$ matrices are directly generated in the computer program while the element stiffness matrix $\underline{k}^e$ is calculated numerically.

2.4 Global Stiffness Matrix

One of the major problems in a Finite Element program is the assemblage of the global stiffness matrix $\underline{K}$ of the structure.

$$\underline{F} = \underline{K}\ \underline{\delta}$$

$$\begin{bmatrix} F_1 \\ \cdots \\ F_r \\ \cdots \\ F_s \\ \cdots \\ F_n \end{bmatrix} = \begin{bmatrix} \underline{k}_{11}\cdots\underline{k}_{1r}\cdots\underline{k}_{1s}\cdots\underline{k}_{1n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \underline{k}_{r1}\cdots\underline{k}_{rr}\cdots\underline{k}_{rs}\cdots\underline{k}_{rn} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \underline{k}_{s1}\cdots\underline{k}_{sr}\cdots\underline{k}_{ss}\cdots\underline{k}_{sn} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \underline{k}_{n1}\cdots\underline{k}_{nr}\cdots\underline{k}_{ns}\cdots\underline{k}_{nn} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \cdots \\ \delta_r \\ \cdots \\ \delta_s \\ \cdots \\ \delta_n \end{bmatrix}$$

$\underline{k}_{rs}$ are 2 x 2 submatrices which can be obtained by the following summation:

$$\underline{k}_{rs} = \sum_{e=1}^{m} \underline{k}_{rs}^{e} .$$

n = total number of nodal points

m = total number of elements

Obviously $\underline{k}_{rs}$ is only then different from zero, if $r = s$ or r and s are adjacent nodal points. The restriction of the CSTES program that a nodal point may not have more than 8 adjacent nodal points thus is equivalent to the statement that every row of the global stiffness matrix does not have more than 8 non-zero off-diagonal terms.

The following arrays are used for the assemblage and storage of the $\underline{K}$ matrix:

NST (350,9)	"Bookkeeping" matrix, containing Col. 1: the nodal point number Col. 2-9: The number of adjacent nodal points
FXX (350,10) FXY (350,10) FYX (350,10) FYY (350,10)	arrays containing the elements of the (2 x 2) $\underline{k}$ submatrices Col. 1: Elements of the main diagonal sub-matrices $\underline{k}_{rr}$ Col. 2-9: Elements of the non-zero off-diagonal submatrices $\underline{k}_{rs}$

Column 10 contains the elements of the inverted main diagonal submatrices (f_{rr} = nodal point flexibility matrix) which are required for the iteration procedure.

In the assemblage procedure the proper storage location in the "bookkeeping" matrix for every nodal point of every element is determined by means of indirect addressing. Subsequently the elements of the element stiffness matrix are added to the global stiffness matrix.

It should be noted that an over-relaxation approach (for solving the system of simultaneous equations) is used in this program. The storage of the rather large global stiffness matrix can therefore be done in a very efficient way and problems concerning the "band width" of this matrix will not come into the picture.

2.5 Solution Procedure

Solution of the canonical equilibrium equation system can be solved by using Gauss-Seidel type iterative procedure. Equilibrium equations can be re-written for the ease of this iterative approach:

$$\underline{\delta}_n^{(j+1)} = \underline{k}_{nn}^{-1} \left[\underline{F}_n - \sum_{i=1}^{n-1} \underline{K}_{ni} \, \underline{\delta}_i^{(j+1)} - \sum_{i=n+1}^{N} \underline{K}_{ni} \, \underline{\delta}_i^{(j)} \right]$$

Superscript -j denotes the iteration counter. Since the stiffness matrix $\underline{K}$ is positive definite, the iterative approach will always converge. By introducing the over-relaxation factor - β, the expression can be modified,

$$\underline{\delta}_n^{(j+1)} = \underline{\delta}_n^{(j)} + \beta\, \underline{k}_{nn}^{-1} \left[\underline{F}_n - \sum_{i=1}^{n-1} \underline{\delta}_i^{(j+1)} - \sum_{i=n}^{N} \underline{K}_{ni}\, \underline{\delta}_i^{(j)} \right]$$

Iteration can be terminated

(1) If specified maximum number of iterations are reached, or

(2) if the iteration scheme converges within specified tolerance limit.

Convergence of the problem is checked by summing the absolute values of unbalanced forces at each iteration cycle and comparing the sums with the sum from the first cycle. The formula used for each cycle is:

$$c^{(j)} = \sum_{i=1}^{N} \left[\left| f_{xi}\, \Delta\, \delta_{xi}^{(j)} \right| + \left| f_{yi}\, \Delta\, \delta_{yi}^{(j)} \right| \right]$$

Expressions in absolute value operators corresponds to the unbalanced forces in x and y directions due to the displacement increments $\Delta\ \delta_{xi}$ and $\Delta\ \delta_{yi}$. Here superscript - (j) is the iteration counter, (N) number of nodal points and subscript - (i) denotes the nodal points.

2.6 Extension of CSTES

The CSTES program is flexible enough to be modified for different applications, such as non-linear materials. Possible non-standard methodologies are listed in References 3, 4, and 5.

3. PROGRAM DESCRIPTION

3.1 Major Program Blocks

To provide a better understanding of the program and also to explain the functions of program blocks for future modifications, if required, the breakdown of the program is included.

Card	Function
9-15	dimension statements
18	over-relaxation factor (DATA statement)
23	input of control variables
24-31	print-out of control variables
32-33	input of element data
34-35	input of nodal point data
36-38	print-out of element data
39-41	print-out of nodal point data
42-48	initializing arrays for global stiffness and "bookkeeping" matrix
53-133	Do-loop (stiffness matrix)
54-56	indirect addressing
57-62	generating elements of B-matrix
63-65	calculating triangle area
66-68,75	calculating common factors
69-74	modification of nodal point loads

Card	Function
76-81	initializing $\underline{B}$, $\underline{D}$ and $\underline{k}^e$ matrices
85-90	generating $\underline{B}$-matrix
91-93	generating $\underline{D}$-matrix
94-109	calculating element stiffness matrix (matrix multiplication)
113-132	assembling element stiffness matrix in global stiffness matrix
137-142	calculating total number of adjacent nodal points for every point
143-149	calculating nodal point flexibility matrix
156-174	Do-loop (boundary conditions)
157	input of boundary conditions data
158	print-out of boundary conditions data
159-173	modification of nodal point flexibility
175-177	initializing iteration counters
181-198	iteration procedure
199-205	checking iteration counters
210	print-out of unbalanced force
211-212	print-out of nodal point displacements
214-246	Do-loop (element stresses)
215-217	indirect addressing
218-223	regenerating elements of $\underline{B}$-matrix
224-233	calculating element stresses
234-244	calculating principal stresses and direction

Card	Function
245	print-out of element stresses
251-293	Do-loop (nodal point stresses)
252-253	initializing variables
254-277	averaging procedure
255-265	determining the element numbers adjacent to every nodal point
266-276	calculating weighting coefficients
278-280	calculating nodal point stresses
281-291	calculating principal stresses and direction
292	print-out nodal point stresses
294-295	checking iteration counters
296-298	error messages
302-332	output Format statements
333-336	input Format statements

3.2 Input Information

For each problem the following data deck must be generated.

A. Control Card (6I4, 6X, E10.3)

Cols.	Input
1-4	Total number of elements
5-8	Total number of nodal points
9-12	Total number of restrained boundary points

Cols.	Input
13-16	Cycle interval for the print of the force unbalance
17-20	Cycle interval for the print of the displacements and stresses
21-24	Maximum number of cycles
31-40	Convergence limit for unbalanced forces

B. Element Data - 1 Card per element (3I4, 8X, 5F10.0)

Cols.	Input
1-4	Nodal point number-i
5-8	Nodal point number-j
9-12	Nodal point number-k
21-30	Modulus of elasticity
31-40	Density of element
41-50	Poisson's ratio
51-60	Coefficient of thermal expansion
61-70	Temperature change within the element

C. Nodal Point Data - 1 Card per nodal point (6F10.0)

Cols.	Input
1-10	X - Ordinate
11-20	Y - Ordinate
21-30	X - Load

Cols.	Input
31-40	Y - Load
41-50	X - Displacement (*)
51-60	Y - Displacement (*)

Note: (*) On free nodal points these are initial guesses, on restrained nodal points these are specified displacements.

D. Boundary Point Data - 1 Card per restrained boundary point (I4, 3X, I1, 2X, F15.0)

Cols.	Input
1-4	Number of restrained boundary points
8	0 if nodal point is fixed in both directions
	1 if nodal point is fixed in x - direction
	2 if nodal point is free to move along a line of slope S
11-25	Slope S (type 2 boundary point only)

Note: Cards of type B and C have to be placed in their natural sequence. To avoid errors the element or nodal point number should be punched in columns 71-80

3.3 Output Information

The following information is printed by the program.

A. Input Data

1. Control variables
2. Element data
3. Nodal point data
4. Boundary point data

B. Force unbalance

(cycle interval INUNB)

C. Displacements and Stresses

(cycle interval INSTR)

1. Cycle number, force unbalance
2. Nodal point displacements
3. Element stresses

   (element numbers, σ_x, σ_y, τ_{xy}, σ_1, σ_2, direction of σ_1)
4. Nodal point stresses

   (nodal point numbers, σ_x, σ_y, τ_{xy}, σ_1, σ_2, direction of σ_1)

3.4 Limitations and Assumptions

The listed assumptions and remarks should always be considered in the preparation of input and in the interpretation of output.

1. Limitations

   (a) Maximum number of elements: 600

   (b) Maximum number of nodal points: 350

   (c) Maximum number of nodal points adjacent to a certain point: 8

   (d) Elements and nodal points have to be numbered in the natural sequence (Data cards have to be placed in the same order).

2. Assumptions

   (a) Over-relaxation factor: 1.84

   (b) Body forces: gravity is assumed to act in the negative direction of Y .

3. Remarks

   (a) Initial guesses for displacements

   For boundary condition type 2 (nodal point is forced to move along a line of slope S): Initial guesses of displacements perpendicular to the prescribed slope will be treated as imposed displacements.

   (b) Nodal point stresses

   The nodal point stresses are calculated as weighted average of stresses in all elements adjacent to a particular nodal point. The method used for this average procedure is described in Ref. 1. For nodal points at the interface of different types of materials the nodal point stresses loose their meaning, since the computations are based on stress rather than strain values.

(c) Direction of principal stresses

The calculated value is the angle between the positive x - axis and the direction of the maximum positive (or minimum negative) principal stress. In case that both values $(\sigma_x - \sigma_y)$ and τ_{xy} are very small, this angle becomes meaningless.

(d) Required field length

CM - $100\ 000_8$

By using CM - $140\ 000_8$ the program can handle up to 560 nodal points and 1120 elements.

(e) Plane strain problems

With the following modified values for E, ν, and α plane strain problems can be transformed into plane stress problems:

$$E^* = E \frac{(1-\nu)^2}{(1-2\nu)} = \frac{1}{(1-\nu^{*2})}$$

$$\nu^* = \frac{\nu}{1-\nu}$$

$$\alpha^* = (1+\nu)\alpha$$

3.5 Nomenclature

Nomenclature should be referred to in program modifications:

A. Arrays Related to Elements

NPI (600)	Nodal point numbers i, j, k
NPJ (600)	(1st level indirect addressing)
NPK (600)	
E (600)	modulus of elasticity
DE (600)	(a) density
	(b) common factor
P (600)	Poisson's ratio
AL (600)	(a) coefficient of thermal expansion
	(b) element stress in x direction
DT (600)	(a) temperature change
	(b) element stress in y direction
TXY (600)	Element shear stress

B. Arrays Related to Nodal Points

XORD (350)	X - coordinate
YORD (350)	Y - coordinate
XLOAD (350)	Nodal point load in x direction
YLOAD (350)	Nodal point load in y direction
DX (350)	Nodal point displacement in x direction
DY (350)	Nodal point displacement in y direction
FXX (350,10)	Elements of global stiffness matrix
FXY (350,10)	Col. 1: elements of main diagonal submatrices

FYX (350,10)	Col. 2-9: elements of off-diagonal submatrices
FYY (350,10)	Col. 10: elements of nodal point flexibility matrices
NST (350,9)	"Bookkeeping" matrix Col. 1: (a) number of nodal points (b) total number of adjacent nodal points Col. 2-9: numbers of adjacent nodal points

C. Other Arrays

B (6,6)	$\underline{B}$ - matrix
D (6,6)	$\underline{D}$ - matrix
S (6,6)	$\underline{k}^e$ - matrix
LM (3)	Scratch vector (2nd level indirect addressing)

D. Variables

REL	Over-relaxation factor
NELEM	Total number of elements
NNOPO	Total number of nodal points
NREPO	Total number of restrained boundary points
INUNB	Cycle interval for the print-out of unbalanced force
INSTR	Cycle interval for the print-out of displacements and stresses

MAXCY	Maximum number of cycles
TOLER	Convergence limit for unbalanced force
BI, BJ, BK	Elements of B - matrix
CI, CJ, CK	
AREA	Area (signed value)
ARE	Area (absolute value)
BODY, THE, FAC	Common factors
NBP	Number of restrained boundary points
NTYPE	Type of boundary condition
SLOPE	Slope of line along a restrained boundary point is allowed to move
NCYCLE	Iteration counter
IUB	Cycle number for the print-out of unbalanced force
IST	Cycle number for the print-out of displacements and stresses
DDX, DDY	Displacement (in x and y direction) during iteration cycle
SUM	Unbalanced force
EPX	
EPY	Element strains
GAM	
X, Y, XY	Stresses
XMAX, XMIN	Principal stresses
FI	Direction of maximum principal stress
SRX, SRY, R	Weighting coefficients for nodal point stress averaging procedure

4. REFERENCES

1. Zienkiewicz, O. C.
"The Finite Element Method in Structural and Continuum Mechanics", McGraw-Hill Publishing Company, Ltd., London, 1967

2. Wilson, E. L.
"Finite Element Analysis of Two-Dimensional Structures", Ph.D. Dissertation, University of California, Berkeley, 1963

3. Connor, J., et. al.
"Finite Element Methods in Structural Mechanics", Summer Institute Lecture Notes, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1968

4. Felippa, C. A.
"Refined Finite Element Analysis of Linear and Non-Linear Two-Dimensional Structures", SESM 66-22, University of California, Berkley, 1966

5. Kostem, C. N.
"A Bibliography on Discretized Systems in Structural Mechanics", Fritz Engineering Laboratory Report 237.52, 1969

5. PROGRAM LISTING

```
      PROGRAM CST(INPUT,TAPE1=INPUT,OUTPUT,TAPE2=OUTPUT)
C***********************************************************************
C*
C*        FINITE ELEMENT ANALYSIS OF PLANAR CONTINUUM
C*        THIS IS THE MODIFIED VERSION OF THE PROGRAM LISTED
C*        IN E.L.WILSON S DISSERTATION, BERKELEY-1963
C*
C***********************************************************************
      DIMENSION NPI(600),NPJ(600),NPK(600),E(600),DE(600),P(600)
      DIMENSION AL(600),DT(600),TXY(600)
      DIMENSION XORD(350),YORD(350),XLOAD(350),YLOAD(350)
      DIMENSION DX(350),DY(350)
      DIMENSION FXX(350,10),FXY(350,10),FYX(350,10),FYY(350,10)
      DIMENSION NST(350,9)
      DIMENSION B(6,6),D(6,6),S(6,6),LM(3)
      IN=1
      IO=2
      DATA REL/1.84/
C*    * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C*    INPUT DATA AND LISTING
C*    * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      WRITE(IO,300)
      READ(IN,400) NELEM,NNOPO,NREPO,INUNB,INSTR,MAXCY,TOLER
      WRITE(IO,301) NELEM
      WRITE(IO,302) NNOPO
      WRITE(IO,303) NREPO
      WRITE(IO,304) INUNB
      WRITE(IO,305) INSTR
      WRITE(IO,306) MAXCY
      WRITE(IO,307) TOLER
      WRITE(IO,308) REL
      READ(IN,401) (NPI(M),NPJ(M),NPK(M),E(M),DE(M),P(M),AL(M),DT(M),M=1
     1,NELEM)
      READ(IN,402) (XORD(N),YORD(N),XLOAD(N),YLOAD(N),DX(N),DY(N),N=1,NN
     1OPO)
      WRITE(IO,309)
      WRITE(IO,310) (M,NPI(M),NPJ(M),NPK(M),E(M),DE(M),P(M),AL(M),DT(M),
     1M=1,NELEM)
      WRITE(IO,311)
      WRITE(IO,312) (N,XORD(N),YORD(N),XLOAD(N),YLOAD(N),DX(N),DY(N),N=1
     1,NNOPO)
      DO 101 N=1,NNOPO
      DO 100 MM=1,9
      FXX(N,MM)=FXY(N,MM)=FYX(N,MM)=FYY(N,MM)=0.0
      NST(N,MM)=0
  100 CONTINUE
      NST(N,1)=N
  101 CONTINUE
C*    * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C*    MODIFICATION OF NODAL POINT LOADS AND ELEMENT DIMENSIONS
C*    GENERATION OF THERMIC LOADS
C*    * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      DO 108 M=1,NELEM
      I=NPI(M)
      J=NPJ(M)
      K=NPK(M)
      BI=YORD(J)-YORD(K)
```

```
      BJ=YORD(K)-YORD(I)                                                CSTES 58
      BK=YORD(I)-YORD(J)                                                CSTES 59
      CI=XORD(K)-XORD(J)                                                CSTES 60
      CJ=XORD(I)-XORD(K)                                                CSTES 61
      CK=XORD(J)-XORD(I)                                                CSTES 62
      AREA=(CK*BJ-CJ*BK)/2.0                                            CSTES 63
      IF(AREA.EQ.0.0) GO TO 135                                         CSTES 64
      ARE=ABS(AREA)                                                     CSTES 65
      BODY=ARE*DE(M)/3.0                                                CSTES 66
      DE(M)=E(M)*AL(M)*DT(M)/(1.0-P(M))                                 CSTES 67
      THE=ARE*DE(M)/(2.*AREA)                                           CSTES 68
      XLOAD(I)=XLOAD(I)+THE*BI                                          CSTES 69
      XLOAD(J)=XLOAD(J)+THE*BJ                                          CSTES 70
      XLOAD(K)=XLOAD(K)+THE*BK                                          CSTES 71
      YLOAD(I)=YLOAD(I)+THE*CI-BODY                                     CSTES 72
      YLOAD(J)=YLOAD(J)+THE*CJ-BODY                                     CSTES 73
      YLOAD(K)=YLOAD(K)+THE*CK-BODY                                     CSTES 74
      FAC=E(M)/(4.0*ARE*(1.-P(M)*P(M)))                                 CSTES 75
      DO 102 MA=1,6                                                     CSTES 76
      DO 102 MB=1,6                                                     CSTES 77
      B(MA,MB)=0.0                                                      CSTES 78
      D(MA,MB)=0.0                                                      CSTES 79
      S(MA,MB)=0.0                                                      CSTES 80
  102 CONTINUE                                                          CSTES 81
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES 82
*     FORMATION OF ELEMENT STIFFNESS MATRICES                           CSTES 83
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES 84
      B(1,1)=B(3,2)=BI                                                  CSTES 85
      B(1,3)=B(3,4)=BJ                                                  CSTES 86
      B(1,5)=B(3,6)=BK                                                  CSTES 87
      B(2,2)=B(3,1)=CI                                                  CSTES 88
      B(2,4)=B(3,3)=CJ                                                  CSTES 89
      B(2,6)=B(3,5)=CK                                                  CSTES 90
      D(1,1)=D(2,2)=FAC                                                 CSTES 91
      D(2,1)=D(1,2)=FAC*P(M)                                            CSTES 92
      D(3,3)=FAC*(1.0-P(M))/2.0                                         CSTES 93
      DO 103 J=1,6                                                      CSTES 94
      DO 103 I=1,3                                                      CSTES 95
      S(I,J)=0.0                                                        CSTES 96
      DO 103 K=1,3                                                      CSTES 97
      S(I,J)=S(I,J)+D(I,K)*B(K,J)                                       CSTES 98
  103 CONTINUE                                                          CSTES 99
      DO 104 I=1,3                                                      CSTES100
      DO 104 J=1,6                                                      CSTES101
      D(J,I)=S(I,J)                                                     CSTES102
  104 CONTINUE                                                          CSTES103
      DO 105 J=1,6                                                      CSTES104
      DO 105 I=1,6                                                      CSTES105
      S(I,J)=0.0                                                        CSTES106
      DO 105 K=1,3                                                      CSTES107
      S(I,J)=S(I,J)+D(I,K)*B(K,J)                                       CSTES108
  105 CONTINUE                                                          CSTES109
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES110
*     FORMATION OF NODAL POINT STIFFNESS MATRICES                       CSTES111
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES112
      LM(1)=NPI(M)                                                      CSTES113
      LM(2)=NPJ(M)                                                      CSTES114
      LM(3)=NPK(M)                                                      CSTES115
```

```
      DO 108 L=1,3                                                      CSTES116
      DO 108 K=1,3                                                      CSTES117
      LX=LM(L)                                                          CSTES118
      MX=0                                                              CSTES119
  106 MX=MX+1                                                           CSTES120
      IF(MX.GE.10) GO TO 136                                            CSTES121
      IF((NST(LX,MX)-LM(K)).EQ.0) GO TO 107                             CSTES122
      IF(NST(LX,MX).NE.0) GO TO 106                                     CSTES123
  107 NST(LX,MX)=LM(K)                                                  CSTES124
      LB=2*L                                                            CSTES125
      LA=LB-1                                                           CSTES126
      KB=2*K                                                            CSTES127
      KA=KB-1                                                           CSTES128
      FXX(LX,MX)=FXX(LX,MX)+S(LA,KA)                                    CSTES129
      FXY(LX,MX)=FXY(LX,MX)+S(LA,KB)                                    CSTES130
      FYX(LX,MX)=FYX(LX,MX)+S(LB,KA)                                    CSTES131
      FYY(LX,MX)=FYY(LX,MX)+S(LB,KB)                                    CSTES132
  108 CONTINUE                                                          CSTES133
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES134
*     INVERSION OF NODAL POINT STIFFNESS MATRICES,(FLEXIBILITY MATRIX)  CSTES135
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES136
      DO 110 N=1,NNOPO                                                  CSTES137
      NX=10                                                             CSTES138
  109 NX=NX-1                                                           CSTES139
      IF(NST(N,NX).EQ.0) GO TO 109                                      CSTES140
      NST(N,1)=NX                                                       CSTES141
  110 CONTINUE                                                          CSTES142
      DO 111 N=1,NNOPO                                                  CSTES143
      DET=FXX(N,1)*FYY(N,1)-FXY(N,1)*FYX(N,1)                           CSTES144
      FXX(N,10)=FYY(N,1)/DET                                            CSTES145
      FXY(N,10)=-FXY(N,1)/DET                                           CSTES146
      FYX(N,10)=-FYX(N,1)/DET                                           CSTES147
      FYY(N,10)=FXX(N,1)/DET                                            CSTES148
  111 CONTINUE                                                          CSTES149
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES150
*     MODIFICATION OF BOUNDARY POINT FLEXIBILITY MATRICES               CSTES151
*     ACCORDING TO THE PRESCRIBED BOUNDARY CONDITIONS                   CSTES152
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES153
      WRITE(IO,313)                                                     CSTES154
      WRITE(IO,314)                                                     CSTES155
      DO 116 L=1,NREPO                                                  CSTES156
      READ(IN,403) NBP,NTYPE,SLOPE                                      CSTES157
      WRITE(IO,325) NBP,NTYPE,SLOPE                                     CSTES158
      IF(NTYPE-1) 114,113,112                                           CSTES159
  112 DET=(FXX(NBP,10)*SLOPE-FXY(NBP,10))/(FYX(NBP,10)*SLOPE-FYY(NBP,10)CSTES160
     1)                                                                 CSTES161
      COF=1.-DET*SLOPE                                                  CSTES162
      FXX(NBP,10)=(FXX(NBP,10)-DET*FYX(NBP,10))/COF                     CSTES163
      FXY(NBP,10)=(FXY(NBP,10)-DET*FYY(NBP,10))/COF                     CSTES164
      FYX(NBP,10)=FXX(NBP,10)*SLOPE                                     CSTES165
      FYY(NBP,10)=FXY(NBP,10)*SLOPE                                     CSTES166
      GO TO 116                                                         CSTES167
  113 FYY(NBP,10)=FYY(NBP,10)-FYX(NBP,10)*FXY(NBP,10)/FXX(NBP,10)       CSTES168
      GO TO 115                                                         CSTES169
  114 FYY(NBP,10)=0.0                                                   CSTES170
  115 FXX(NBP,10)=0.0                                                   CSTES171
      FXY(NBP,10)=0.0                                                   CSTES172
      FYX(NBP,10)=0.0                                                   CSTES173
```

```
  116 CONTINUE                                                          CSTES174
      NCYCLE=0                                                          CSTES175
      IUB=INUNB                                                         CSTES176
      IST=INSTR                                                         CSTES177
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES178
*     ITERATION ON NODAL POINT DISPLACEMENTS                            CSTES179
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES180
  117 WRITE(IO,326)                                                     CSTES181
  118 SUM=0.0                                                           CSTES182
      DO 120 N=1,NNOPO                                                  CSTES183
      NUM=NST(N,1)                                                      CSTES184
      FX=XLOAD(N)                                                       CSTES185
      FY=YLOAD(N)                                                       CSTES186
      DO 119 L=2,NUM                                                    CSTES187
      M=NST(N,L)                                                        CSTES188
      FX=FX-FXX(N,L)*DX(M)-FXY(N,L)*DY(M)                               CSTES189
      FY=FY-FYX(N,L)*DX(M)-FYY(N,L)*DY(M)                               CSTES190
  119 CONTINUE                                                          CSTES191
      DDX=FXX(N,10)*FX+FXY(N,10)*FY-DX(N)                               CSTES192
      DDY=FYX(N,10)*FX+FYY(N,10)*FY-DY(N)                               CSTES193
      DX(N)=DX(N)+REL*DDX                                               CSTES194
      DY(N)=DY(N)+REL*DDY                                               CSTES195
      SUM=SUM+ABS(FXX(N,1)*DDX+FXY(N,1)*DDY)+ABS(FYX(N,1)*DDX+FYY(N,1)*DCSTES196
     1DY)                                                               CSTES197
  120 CONTINUE                                                          CSTES198
      NCYCLE=NCYCLE+1                                                   CSTES199
      IF(NCYCLE.GE.IUB) 121,122                                         CSTES200
  121 IUB=IUB+INUNB                                                     CSTES201
      WRITE(IO,315) NCYCLE,SUM                                          CSTES202
  122 IF(SUM.LE.TOLER) GO TO 123                                        CSTES203
      IF(NCYCLE.GE.MAXCY) GO TO 123                                     CSTES204
      IF(NCYCLE.LT.IST) GO TO 118                                       CSTES205
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES206
*     PRINT OF DISPLACEMENTS AND STRESSES                               CSTES207
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * CSTES208
      IST=IST+INSTR                                                     CSTES209
  123 WRITE(IO,316) NCYCLE,SUM                                          CSTES210
      WRITE(IO,317)                                                     CSTES211
      WRITE(IO,318) (N,DX(N),DY(N),N=1,NNOPO)                           CSTES212
      WRITE(IO,319)                                                     CSTES213
      DO 127 M=1,NELEM                                                  CSTES214
      I=NPI(M)                                                          CSTES215
      J=NPJ(M)                                                          CSTES216
      K=NPK(M)                                                          CSTES217
      BI=YORD(J)-YORD(K)                                                CSTES218
      BJ=YORD(K)-YORD(I)                                                CSTES219
      BK=YORD(I)-YORD(J)                                                CSTES220
      CI=XORD(K)-XORD(J)                                                CSTES221
      CJ=XORD(I)-XORD(K)                                                CSTES222
      CK=XORD(J)-XORD(I)                                                CSTES223
      EPX=BI*DX(I)+BJ*DX(J)+BK*DX(K)                                    CSTES224
      EPY=CI*DY(I)+CJ*DY(J)+CK*DY(K)                                    CSTES225
      GAM=CI*DX(I)+CJ*DX(J)+CK*DX(K)+BI*DY(I)+BJ*DY(J)+BK*DY(K)         CSTES226
      FAC=E(M)/((1.0-P(M)*P(M))*(CK*BJ-CJ*BK))                          CSTES227
      X=FAC*(EPX+P(M)*EPY)-DE(M)                                        CSTES228
      Y=FAC*(EPY+P(M)*EPX)-DE(M)                                        CSTES229
      XY=FAC*GAM*(1.0-P(M))/2.0                                         CSTES230
      AL(M)=X                                                           CSTES231
```

```
      DT(M)=Y                                                           CSTES232
      TXY(M)=XY                                                         CSTES233
      SIG=(X+Y)/2.0                                                     CSTES234
      RIG=X-Y                                                           CSTES235
      R=SQRT(RIG*RIG/4.0+XY*XY)                                         CSTES236
      XMAX=SIG+R                                                        CSTES237
      XMIN=SIG-R                                                        CSTES238
      T=SIGN(1.0,XY)                                                    CSTES239
      IF(RIG.EQ.0.0) 124,125                                            CSTES240
  124 FI=T*45.0                                                         CSTES241
      GO TO 126                                                         CSTES242
  125 FI=28.647890*ATAN(2.0*XY/RIG)                                     CSTES243
      IF(RIG.LT.0.0) FI=T*90.0+FI                                       CSTES244
  126 WRITE(IO,320) M,X,Y,XY,XMAX,XMIN,FI                               CSTES245
  127 CONTINUE                                                          CSTES246
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *  CSTES247
*     NODAL POINT STRESSES                                              CSTES248
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *  CSTES249
      WRITE(IO,321)                                                     CSTES250
      DO 134 N=1,NNOPO                                                  CSTES251
      X=Y=XY=0.0                                                        CSTES252
      SRX=SRY=R=0.0                                                     CSTES253
      DO 130 M=1,NELEM                                                  CSTES254
      I=NPI(M)                                                          CSTES255
      J=NPJ(M)                                                          CSTES256
      K=NPK(M)                                                          CSTES257
      IF(N.EQ.I) GO TO 129                                              CSTES258
      IF(N.EQ.J) GO TO 128                                              CSTES259
      IF(N.NE.K) GO TO 130                                              CSTES260
      I=NPK(M)                                                          CSTES261
      K=NPI(M)                                                          CSTES262
      GO TO 129                                                         CSTES263
  128 I=NPJ(M)                                                          CSTES264
      J=NPI(M)                                                          CSTES265
  129 AC=ABS(XORD(J)-XORD(I))+ABS(XORD(K)-XORD(I))                      CSTES266
      BC=ABS(YORD(J)-YORD(I))+ABS(YORD(K)-YORD(I))                      CSTES267
      AB=AC+BC                                                          CSTES268
      RX=AC/AB                                                          CSTES269
      SRX=SRX+RX                                                        CSTES270
      X=X+AL(M)*RX                                                      CSTES271
      RY=BC/AB                                                          CSTES272
      SRY=SRY+RY                                                        CSTES273
      Y=Y+DT(M)*RY                                                      CSTES274
      R=R+1.0                                                           CSTES275
      XY=XY+TXY(M)                                                      CSTES276
  130 CONTINUE                                                          CSTES277
      X=X/SRX                                                           CSTES278
      Y=Y/SRY                                                           CSTES279
      XY=XY/R                                                           CSTES280
      SIG=(X+Y)/2.0                                                     CSTES281
      RIG=X-Y                                                           CSTES282
      TR=SQRT(RIG*RIG/4.0+XY*XY)                                        CSTES283
      XMAX=SIG+TR                                                       CSTES284
      XMIN=SIG-TR                                                       CSTES285
      T=SIGN(1.0,XY)                                                    CSTES286
      IF(RIG.EQ.0.0) 131,132                                            CSTES287
  131 FI=T*45.0                                                         CSTES288
      GO TO 133                                                         CSTES289
```

```
  132 FI=28.647890*ATAN(2.0*XY/RIG)                                    CSTES290
      IF(RIG.LT.0.0) FI=T*90.0+FI                                      CSTES291
  133 WRITE(IO,322) N,X,Y,XY,XMAX,XMIN,FI                              CSTES292
  134 CONTINUE                                                         CSTES293
      IF(SUM.LE.TOLER) GO TO 500                                       CSTES294
      IF(NCYCLE.LT.MAXCY) 117,500                                      CSTES295
  135 WRITE(IO,323) M                                                  CSTES296
      GO TO 500                                                        CSTES297
  136 WRITE(IO,324) LX                                                 CSTES298
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *  CSTES299
*     FORMAT STATEMENTS                                                CSTES300
*     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *  CSTES301
  300 FORMAT(1H1,//,* INPUT  DATA*,///)                                CSTES302
  301 FORMAT(* NUMBER OF ELEMENTS              =*,I5,/)                CSTES303
  302 FORMAT(* NUMBER OF NODAL POINTS          =*,I5,/)                CSTES304
  303 FORMAT(* NUMBER OF BOUNDARY POINTS       =*,I5,/)                CSTES305
  304 FORMAT(* CYCLE PRINT INTERVAL            =*,I5,/)                CSTES306
  305 FORMAT(* OUTPUT INTERVAL OF RESULTS      =*,I5,/)                CSTES307
  306 FORMAT(* CYCLE LIMIT                     =*,I5,/)                CSTES308
  307 FORMAT(* TOLERANCE LIMIT                 =*,E13.3,/)             CSTES309
  308 FORMAT(* OVERRELAXATION FACTOR           =*,F13.3,/)             CSTES310
  309 FORMAT(////,* ELEMENT*,4X,*I*,7X,*J*,7X,*K*,6X,*E-MODULUS*,5X,*DENCSTES311
     1SITY*,5X,*POISSON*,8X,*ALPHA*,8X,*DELTA T*,//)                   CSTES312
  310 FORMAT(1X,4(I4,4X),E11.3,2F12.4,F16.8,F12.3)                     CSTES313
  311 FORMAT(////,* POINT*,8X,*X-ORD*,10X,*Y-ORD*,9X,*X-LOAD*,9X,*Y-LOADCSTES314
     1*,10X,*X-DISP*,11X,*Y-DISP*,//)                                  CSTES315
  312 FORMAT(I4,1X,4F15.5,2F17.8)                                      CSTES316
  313 FORMAT(1H1,//,* BOUNDARY  CONDITIONS*,//)                        CSTES317
  314 FORMAT(* POINT*,5X,*TYPE*,6X,*SLOPE*,//)                         CSTES318
  315 FORMAT(3X,I5,7X,E15.8)                                           CSTES319
  316 FORMAT(1H1,//,* CYCLE :*,I6,5X,*FORCE UNBALANCE :*,3X,E15.8,///) CSTES320
  317 FORMAT(////,* POINT*,9X,*X-DISPLACEMENT*,6X,*Y-DISPLACEMENT*,//) CSTES321
  318 FORMAT(I4,11X,E14.7,6X,E14.7)                                    CSTES322
  319 FORMAT(///,8X,*ELEMENT*,9X,*X-STRESS*,12X,*Y-STRESS*,11X,*XY-STRESCSTES323
     1S*,10X,*MAX-STRESS*,5X,*MIN-STRESS*,6X,*DIRECTION*,//)           CSTES324
  320 FORMAT(I12,3F20.4,5X,3F15.2)                                     CSTES325
  321 FORMAT(///,8X,* POINT *,9X,*X-STRESS*,12X,*Y-STRESS*,11X,*XY-STRESCSTES326
     1S*,10X,*MAX-STRESS*,5X,*MIN-STRESS*,6X,*DIRECTION*,//)           CSTES327
  322 FORMAT(I12,3F20.4,5X,3F15.2)                                     CSTES328
  323 FORMAT(1H1,10X,*ZERO AREA                ELEMENT NUMBER :*,I5)   CSTES329
  324 FORMAT(1H1,10X,*MORE THAN 8 NODAL POINTS ADJACENT TO POINT :*,I5) CSTES330
  325 FORMAT(I4,9X,I1,F13.5)                                           CSTES331
  326 FORMAT(1H1,//,4X,*CYCLE*,6X,*FORCE UNBALANCE*,/)                 CSTES332
  400 FORMAT(6I4,6X,E10.3)                                             CSTES333
  401 FORMAT(3I4,8X,5F10.0)                                            CSTES334
  402 FORMAT(6F10.0)                                                   CSTES335
  403 FORMAT(I4,3X,I1,2X,F15.0)                                        CSTES336
  500 CALL EXIT                                                        CSTES337
      END                                                              CSTES338
```

6. ACKNOWLEDGMENTS

Thanks and appreciation are due to Mrs. K. Michele Kostem for her help in the preparation of the report and to Mrs. Shirley Labert for typing and proofreading the manuscript.