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OF

PRESTRESSED CONCRETE BEAMS

PART III: ANALYSIS OF STRESSES IN BEAMS
            SUBJECTED TO REPEATED
            FLEXURAL LOADINGS

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SYNOPSIS

A theoretical analysis is made of the behavior of prestressed concrete members subjected to repeated flexural loadings. Equations are derived for the stresses and deformations in the steel reinforcement and in the concrete in beams subjected to moments varying in value between zero and the static ultimate.

INTRODUCTION

Before the strand fatigue test data presented in Part II[1] of this series can be used to estimate beam fatigue life, a method of

analysis must be developed for predicting as accurately as possible the stresses and deformations in a member subjected to repeated loadings. Of prime importance is the relation between steel stress and applied moment in any given load cycle, since the transformation of the loading history of a beam into the corresponding stress history for the tension reinforcement is an essential step in determining beam fatigue life.

In this analytic study, an analysis is first made of the response of prestressed concrete members of rectangular section with the steel reinforcement placed in one horizontal layer. The more complicated cases of beams with the reinforcement distributed between several levels and beams with I sections are then treated briefly.

Loading is considered in two stages; zero moment to M_{on}, and M_{on} to static ultimate moment, where M_{on} is the moment at which cracks begin to open. In the first loading stage, increments of strain in the steel and concrete are relatively small and linear stress-strain relations are assumed for both materials. Furthermore, any previously formed flexural cracks are closed by the internal prestressing force, and so even the cracked regions are assumed to behave elastically provided the stresses remain compressive, i.e., provided M_{on} is not exceeded.

Conditions in the second stage of loading are considerably more complicated. The analysis of beam behavior is based on a consideration of the following:

    (a) Stress-strain relations for concrete and steel,

    (b) An assumed pattern of deformation in the beam in

        the region of flexural cracking,

    (c) Equilibrium of internal forces.

Tests on concrete cylinders were conducted to determine the effect on the stress-strain relation of a prior history of fatigue loading. The concrete stress-strain data obtained from the cylinder tests provide a basis for the choice of an equation for the concrete stress-strain relation. An idealized pattern of beam deformation is assumed which describes, approximately, the concrete deformations observed in the beam tests. Finally, a compatibility factor ψ, which measures the degree of bond breakdown between strand and concrete in the beam near a flexural crack, is evaluated empirically from the deformation measurements made on the beams during the fatigue tests.

NOTATION

A_c area of concrete section

A_s cross sectional area of longitudinal tension steel

b width of rectangular beam, width of top flange of I beam

d effective depth of beam

e distance from center of gravity of A_s to center of gravity of A_c

E_{cn} modulus of elasticity of concrete in the n-th load cycle

E non-dimensionalized concrete strain; $E = \dfrac{\epsilon_c}{\epsilon_u}$

F non-dimensionalized concrete stress; $F = \dfrac{f_c}{f'_c}$

F_n prestressing force in beam during the n-th load cycle

F_{se} prestressing force in test beam just prior to first load cycle

f_c concrete stress

f'_c ultimate cylinder strength for the concrete

f'_r modulus of rupture of the concrete

f_{sl} total steel stress at moment $M_1 > M_{on}$

h full depth of concrete section

I moment of inertia of steel-concrete transformed section about centroidal axis

I_c moment of inertia of concrete area about its centroidal axis

k dimensionless factor defining depth to neutral axis at a cracked section

k_1 dimensionless factor relating average concrete compressive stress at beam failure to $k_3 f'_c$

k_2 dimensionless factor defining location of compressive force in concrete compressive stress block

k_3 dimensionless factor relating concrete strength in beam and cylinder

ℓ_b distance from crack over which full or partial bond breakdown occurs

ℓ_c crack spacing

m $\dfrac{E_s}{E_c}$

M applied moment

M_{on} moment in n-th load cycle at which cracks begin to open

$\bar{x}$ distance from the center of gravity of the steel area to the centroidal axis

α dimensionless quantity defining the shape of the concrete stress-strain relation; $\alpha = E_{cn} \dfrac{\epsilon_u}{f'_c}$

β, γ dimensionless parameters defining the shape of the unloading portion of the concrete stress-strain curve

ϵ_c concrete strain

$\Delta\epsilon'_c$ inelastic strain in a concrete cylinder test

ϵ_s steel strain

ϵ_{c1} concrete strain in top fiber of the beam

ϵ_{s1} total steel strain at the cracked section at moment $M_1 > M_{on}$

ϵ_{cF} elastic strain in concrete at the steel level due to pre-stressing force F_n

ϵ_{sF} steel strain due to prestressing force F_n

ϵ_u concrete strain in cylinder at f'_c

ψ compatibility factor

INITIAL LOADING STAGE, CRACKS NOT OPENED

In the first loading stage, linear relations are assumed between stress and strain for both concrete and steel, and strains at different levels in the beam are assumed to vary linearly with depth. Considering the steel-concrete composite section of a rectangular beam, we determine the position of the centroidal axis as

$$\overline{x} = \frac{A_c e}{A_c + (m-1)A_s}$$

where e is the distance from the center of gravity of the concrete area A_c to the center of gravity of the steel area A_s, $\overline{x}$ is the distance from the center of gravity of the steel area to the centroidal axis, and m is the modular ratio. The moment of inertia of the steel-concrete section with respect to the centroidal axis is

$$I = A_c \left[\frac{h^2}{12} + (e-\overline{x})^2 + (m-1) \frac{A_s}{bh} \overline{x}^2 \right]$$

With tensile stresses positive, the top and bottom concrete fiber stresses and the steel stress induced by moment M are, respectively

$$f_c^t = -\frac{M}{I} \left[\frac{h}{2} + e - \overline{x} \right]$$

$$f_c^b = +\frac{M}{I} \left[\frac{h}{2} - e + \overline{x} \right]$$

$$f_s = +m \frac{M}{I} \overline{x}$$

If the prestressing force in the steel prior to the application of the n-th load cycle is F_n, the corresponding stresses in the unloaded beam are

$$f_{cF}^{t} = -F_n \left[\frac{1}{A_c} - \frac{he}{2I_c} \right]$$

$$f_{cF}^{b} = -F_n \left[\frac{1}{A_c} + \frac{he}{2I_c} \right]$$

$$f_{sF} = + \frac{F_n}{A_s}$$

where I_c is the moment of inertia of the rectangular concrete section about its center of gravity. The total stresses at moment M in the n-th cycle are therefore

$$f_{cn}^{t} = -F_n \left[\frac{1}{A_c} - \frac{he}{2I_c} \right] - \frac{M}{I} \left[\frac{h}{2} + e - \overline{x} \right] \qquad (3.1)$$

$$f_{cn}^{b} = -F_n \left[\frac{1}{A_c} + \frac{he}{2I_c} \right] + \frac{M}{I} \left[\frac{h}{2} - e + \overline{x} \right] \qquad (3.2)$$

$$f_{sn} = \frac{F_n}{A_s} + m \frac{M}{I} \overline{x} \qquad (3.3)$$

For the first cycle of loading, i.e. when n = 1, the value of F_{se} will either be known by measurement or estimated in the design calculations. Cracking will take place in this initial load cycle when f_{c1}^{b} becomes equal to the modulus of rupture of the concrete, i.e. when

$$-F_{se} \left[\frac{1}{A_c} + \frac{he}{2I_c} \right] + \frac{M}{I} \left[\frac{h}{2} - e + \overline{x} \right] = f_r'$$

Thus, the value of M_{on} in the first load cycle is

$$M_{o1} = I \frac{f_t' - f_{cF}^{b}}{\frac{1}{2} h - e + \overline{x}} \qquad (3.4)$$

In subsequent load cycles cracks will begin to open when the value of f_{cn}^b is zero; thus, for $n > 1$,

$$M_{on} = I \frac{-f_{cF}^b}{\frac{1}{2} h - e + \overline{x}}$$

(3.5)

The prestressing force in the steel prior to the application of the n-th load cycle, F_n, will vary slightly during the lifetime of the member. Appropriate values must be chosen in each particular instance on the basis of an analysis for creep and shrinkage losses and an estimation of other possible effects.

SECOND LOADING STAGE, CRACKS OPEN

Concrete Stress-Strain Relation

The loading portion of the concrete stress-strain curve obtained from cylinder tests can be represented with fair approximation by the cubic parabola

$$F = \alpha E + (3-2\alpha) E^2 + (\alpha - 2) E^3$$

(3.6)

in which stress and strain are expressed by the non-dimensional terms

$$F = \frac{f_c}{f'_c}$$

and

$$E = \frac{\epsilon_c}{\epsilon_u}$$

Besides fulfilling the requirements that $F = 1$ when $E = 1$, $\frac{dF}{dE} = 0$ when

E = 1, and F = 0 when E = 0, Eq. 3.6 also contains an open parameter which represents the tangent modulus of elasticity at zero load for the beginning of the n-th loading cycle,

$$\alpha = E_{cn} \frac{\epsilon_u}{f'_c}$$

A limitation must however be placed on the value of α to ensure that the curve increases monotonically in the range $0 \leq E \leq 1$. If the initial slope is too steep, the curve reaches a maximum for $E < 1$ and then becomes a minimum at $E = 1$. It is therefore stipulated that

$$\frac{d^2F}{dE^2} \leq 0 \quad \text{at} \quad E = 1.0$$

which leads to the result

$$\alpha \leq 3$$

It will be noted that Eq. 3.6 exhibits concave-up curvature in the lower load range when

$$\frac{d^2F}{dE^2} < 0 \quad \text{at} \quad E = 0$$

i.e. when

$$\alpha < 1.5$$

Equation 3.6 has been plotted for values of α varying between 0.5 and 3.0 in Fig. 1.

Concrete cylinders were tested to determine stress-strain relations for the concrete and to observe the effect upon the stress-strain relation of a prior history of fatigue loading. Strain measurements were made with two six inch SR4 A-9 electric resistance strain

gages placed 180 degrees apart on the side of the 6 x 12-in. cylinder. In tests involving large numbers of load applications, a Whittemore deformeter was used with aluminum targets cemented to the side of the cylinders to check the strain gage readings against drift. The first load cycle was applied statically to allow strain readings to be made with a static strain indicator. The predetermined number of load cycles was then applied at a rate of 500 cycles per minute, and finally the specimen was tested statically to failure with strain readings being taken at regular intervals up to the ultimate load.

Two different load cycles were used for pre-loading the 6 x 12-in. cylinders. Each cycle had a minimum load level of 20 kips, while maximum load levels were 100 and 130 kips. Concrete fatigue tests currently being conducted at Fritz Engineering Laboratory indicate for this strength concrete that the smaller load cycle, 20 to 100 kips, may be regarded for all practical purposes as an understress, i.e. to have an infinite fatigue life. Fatigue tests to failure on three cylinders indicated an average fatigue life of 300,000 cycles for the 20 to 130 kip load cycle. Tests were conducted with pre-loadings of 0, 20, 30,000 and 100,000 cycles. An additional test was conducted with one million pre-loadings of the smaller cycle. Each test was replicated at least three times.

Typical strain readings from the final loading cycle to failure are plotted non-dimensionally in Figs. 2 through 7. The results of the initial static test were used to determine the value of the tangent modulus of elasticity at the commencement of the test, E_{co}. The amount

of inelastic strain in the cylinder due to the repeated loadings, $\Delta\epsilon_c'$, was measured prior to the final static test. Ultimate values of stress and strain measured during the static test to failure, f_c' and ϵ_u, together with the modulus of elasticity, E_{cn}, were obtained during the final test. Mean values of these quantities are shown in the figures.

Equation 3.6 has been plotted on Figs. 2 through 7 for the α values providing best fit. The stress-strain data for the specimens without prior loading, Fig. 2 and very light prior loading, Fig. 3, follow quite well the cubic parabola with an α value lying between 1.5 and 2.0. When the fatigue loading has been more intense there is a distinct tendency for the stress-strain relation to assume a concave-up region in the lower load range. In Fig. 5, which shows test points for specimens which had been subjected to approximately one third of the number of load cycles required to cause fatigue failure, the concave-up shape is clearly seen. Even in such cases a cubic parabola provides a reasonable approximation to the stress-strain curve. A more complicated equation is certainly not justified when account is taken of the considerable variation which is observed between replications of the same test, even when differences in values of maximum stresses and strains have been removed by non-dimensionalizing.

The maximum value of α used for the correlation of test data for high strength concrete is 2.0 which is well within the limiting value of 3.0. It appears that values of α less than 3.0 will be adequate for practically all types of concrete. It should be noted that when the experimental curve has an initial concave-up section, the best fit equation will not necessarily be obtained by substituting for α the

observed initial slope, but by choosing α to provide a good fit at all load levels.

Equation 3.6 will be used to represent the stress-strain relation for concrete subjected to axial loading. Before the stress-strain relation obtained from axially loaded test pieces is applied to the concrete in the compressive stress block of a flexural member, two further effects must be considered.

In previous studies[3,4] it has been found that the general stress-strain characteristics in axial compression are applicable to conditions involving a compressive stress gradient provided account is taken of the unloading portion of the stress-strain relation at high loads. The unloading curve may be represented by a second order parabola, as shown in Fig. 8 which is continuous with the loading curve at $E = 1$, and descends to the point $F = \beta$, $E = 1 + \delta$, i.e.

$$F = 1 - \frac{\beta}{\delta^2} (E-1)^2 \tag{3.7}$$

The second effect which must be considered is the variation in concrete strength between beam and cylinder. In static ultimate strength theory, the strength of the concrete in the beam is usually written as $k_3 f'_c$, where f'_c is the cylinder strength. In most ultimate strength studies, only the product of the factor k_3 and another factor, k_1, is evaluated, such that $k_1 k_3 f'_c$ is the average stress in the concrete compressive stress block at failure. Reliable information on the relation between concrete strengths in beam and cylinder has not yet been published. In ultimate strength theory for reinforced concrete columns and beams a value for k_3 of 0.85 is commonly used to account

for size effect, poorer concrete compaction in the column or beam, etc. This value will also be adopted here because of the lack of more reliable information. For convenience, the terms F and α, which henceforth will be used in connection with the concrete in the beam, will be taken to be

$$F = \frac{f_c}{k_3 f'_c} \tag{3.8}$$

and

$$\alpha = E_{cn} \frac{\epsilon_u}{k_3 f'_c} \tag{3.9}$$

The area under the curve represented by Eq. 3.6 in the range $0 \leq E \leq E_1$, for $E_1 \leq 1.0$, is

$$A = \int_0^{E_1} F\,dE$$

i.e.,

$$A = \frac{\alpha}{2} E_1^2 + \frac{3-2\alpha}{3} E_1^3 + \frac{\alpha-2}{4} E_1^4 \tag{3.10}$$

With the interval between E_1 and the center of gravity of the area defined as $k_2 E_1$,

$$k_2 E_1 = E_1 - \frac{\int_0^{E_1} FE\,dE}{\int_0^{E_1} F\,dE}$$

so that

$$k_2 = \frac{\alpha + (1.5-\alpha)E_1 + (0.3\alpha-0.6)E_1^2}{3\alpha + (6-4\alpha)E_1 + (1.5\alpha-3)E_1^2} \tag{3.11}$$

Deformations in the Beam

In the idealized case of perfect bonding between steel and concrete, Bernoulli's linear strain hypothesis leads to the following compatibility equation

$$\epsilon_{s1} = \epsilon_{sF} + \epsilon_{cF} + \frac{1-k}{k}\epsilon_{c1} \qquad (3.12)$$

where k is the ratio of depth of compression block to d, ϵ_{s1} and ϵ_{c1} are the strains in the steel reinforcement and the concrete top fiber respectively, and ϵ_{sF} and ϵ_{cF} are the strains in the steel and in the concrete at the steel level due to the prestressing force F_n. Equation 3.12 quantitatively describes a condition of uniform bending in which the tensile deformations consist of infinitesimal hair cracks at infinitesimal spacings; in actual fact, however, cracks at finite width form at finite spacings and a slip of the steel relative to the concrete occurs for some length on either side of the crack. To represent quantitatively the deformations in the beam for moments greater than M_{on}, a condition is assumed in which evenly spaced vertical tension cracks break the beam into a series of blocks. Tensile deformations below the neutral axis are concentrated in the crack, while compressive deformations exist as strain in the concrete above the crack. The deformation of the beam is thus pictured as a series of slight kinks which exist at the cracked sections, each kink consisting of a rotation of two adjacent blocks about the neutral axis as shown in Fig. 9. Since infinite strains cannot exist in the steel reinforcement, full or partial breakdown of bond must exist over some finite length on either side of each crack. To take account of this situation a com-

patibility factor, ψ , may be introduced into Eq. 3.12 as follows

$$\epsilon_{s1} = \epsilon_{sF} + \epsilon_{cF} + \frac{1-k}{k} \epsilon_{c1} \psi \qquad (3.13)$$

This type of compatibility factor has been used for ultimate strength analysis for both flexural and shear failures[5,6]. For a stress analysis at loads less than the ultimate, a theoretical expression for the dimensionless term ψ has been derived based on several reasonable but approximate assumptions[7].

In general, theoretical evaluation of ψ is not feasible, because of the lack of basic information on steel-concrete bond properties and the scattersome nature of the phenomenon. Values must therefore be determined empirically from beam tests. Data obtained during the beam fatigue tests described in Part I[2] have been used to obtain the empirical ψ values contained in Fig. 10. A very marked change is seen to occur in ψ during the life of a member. The value, at first usually in excess of unity, falls by as much as 50 percent during the early load cycles, then tends to become reasonably constant. Considerable scatter is also seen to exist in ψ, even when obtained from beams of almost identical properties.

If the results of the initial loading sequences are disregarded, the mean value of ψ for all six beams is 0.93. Before reliable figures can be recommended a much greater number of beam tests will have to be conducted, but for the purposes of the present study the simple value $\psi = 1.0$ will be taken as characteristic for the test beams after the initial load sequence has been applied. It is emphasized, however, that values considerably different from unity are usually to be expected.

The effect on fatigue life of variations in ψ will be discussed in Part IV of this series of papers.

Equilibrium

The distribution of internal stresses at a cracked section during a particular load cycle will depend upon the magnitude of previous loadings. When the loading under consideration is considerably larger than all previous ones, a field of tension stresses will exist in the concrete immediately below the neutral axis of stress. In some circumstances, especially at lower loads, the concrete tensile stresses are an important consideration in the equilibrium of the section. When, however, a previous loading has been greater than the one under consideration, cracks will have extended above the present level of the neutral axis and there will be no concrete tensile stresses at the section. In the present analysis it is assumed that overloadings are evenly distributed through the life of the member and concrete tensile stresses will be ignored in the equilibrium considerations.

Assuming a linear distribution of strain above the crack, one obtains for the total compressive force,

$$C = b \int_{o}^{kd} f_c \, d_y \qquad (3.14)$$

With

$$F = \frac{f_c}{k_3 f'_c}$$

and

$$E = \frac{\epsilon_c}{\epsilon_u}$$

Eq. 3.14 may be written as

$$C = \frac{bd\ k_3\ f'_c\ k}{E_1} \int_o^{E_1} F dE \qquad (3.15)$$

where E_1 is the extreme fiber value of E.

Equations 3.10 and 3.15 together yield

$$C = bd\ k_3\ f'_c\ k \left[\frac{\alpha}{2} E_1 + \frac{3-2\alpha}{3} E_1^2 + \frac{\alpha-2}{4} E_1^3 \right] \qquad (3.16)$$

provided $E_1 \leq 1$. It should be noted that the range of values $1.0 \leq E_1 \leq 1 + \gamma$ corresponds to over-loadings approaching static ultimate load, and will not be met with in practical situations. The stress corresponding to such high loads can however be investigated by considering also the un-loading portion of the stress-strain relation represented by Eq. 3.7. Horizontal forces may now be equated to yield

$$\frac{f_{s1}\ A_s}{bd\ k_3\ f'_c} = k \left[\frac{\alpha}{2} E_1 + \frac{3-2\alpha}{3} E_1^2 + \frac{\alpha-2}{4} E_1^3 \right] \qquad (3.17)$$

and equating internal and external moments, we obtain

$$M_1 = f_{s1}\ A_s\ d\ (1 - k_2 k) \qquad (3.18)$$

where k_2 is given in terms of E_1 in Eq. 3.11.

Stresses in a Rectangular Section

The equations derived for the cracked rectangular section are now summarized.

$$\frac{f_{s1} A_s}{bd \ k_3 \ f'_c} = k \left[\frac{\alpha}{2} E_1 + \frac{3-2\alpha}{3} E_1^2 + \frac{\alpha-2}{4} E_1^3 \right] \tag{3.17}$$

$$M_1 = f_{s1} A_s d (1 - k_2 k) \tag{3.18}$$

$$\epsilon_{s1} = \epsilon_{sF} + \epsilon_{cF} + \frac{1 - k}{k} \epsilon_{c1} \psi \tag{3.13}$$

$$k_2 = \frac{\alpha + (1.5-\alpha) E_1 + (0.3\alpha-0.6) E_1^2}{3\alpha + (6-4\alpha) E_1 + (1.5\alpha-3) E_1^2} \tag{3.11}$$

These equations, together with the steel stress-strain relation, may be used to evaluate, for a moment M, in the n-th load cycle, the unknowns f_{s1}, ϵ_{s1}, k, k_2, and E_1. It will be noted that the value of F_n for the n-th load cycle must be known or estimated since it is used to determine ϵ_{sF} and ϵ_{cF}. In general, the calculations for the stress-moment relation will be simplified if values of f_{s1} and ϵ_{s1} are first chosen and substituted in Eqs. 3.17 and 3.13 to obtain k, and hence, through Eq. 3.11, k_2. The corresponding values of M_1 may be obtained by substitution of values in Eq. 3.18. The process may then be repeated to obtain different points on the M_1 - f_{s1} curve.

When a large number of stress-moment calculations are to be made, it may be convenient to plot Eqs. 3.17 and 3.13 in the form of an intercept chart on coordinates of E_1 and k, for appropriate values of α. Equation 3.13 may be divided throughout by ϵ_u and rearranged to the form

$$\frac{1}{\psi \ \epsilon_u} \left[\epsilon_{s1} - \epsilon_{sF} - \epsilon_{cF} \right] = E_1 \ \frac{1 - k}{k} \tag{3.13a}$$

Equations 3.17 and 3.13a now contain the terms E_1 and k in the right hand sides, while their left hand sides are functions of either f_{s1} or

ϵ_{s1}. Each equation may be used to plot a family of curves on the E_1-k coordinates. The resulting intercept chart provides values directly for k and E_1 without a trial and error procedure. It is convenient also to plot k_2 against E_1 using Eq. 3.11 so that k_2 can be evaluated without calculation.

Steel at Several Different Levels in the Beam

The above equations were derived for a rectangular section in which all of the steel reinforcement is at a depth d below the top surface. When the steel lies at z different levels, there will be z different values of stress and strain for the steel. Letting the depths of the different steel layers be d_1, d_2,.....d_z, the corresponding steel stresses be f_{s1}, f_{s2},.....f_{sz}, and the depth to the neutral axis be a, one obtains for the two equilibrium equations,

$$M = f_{s1} A_{s1} (d_1 - k_2 a)$$

$$+ f_{s2} A_{s2} (d_2 - k_2 a)$$

$$................$$

$$................$$

$$+ f_{sz} A_{sz} (d_z - k_2 a) \tag{3.19}$$

and

$$\frac{\alpha}{2} E_1 + \frac{3-2\alpha}{3} E_1^2 + \frac{\alpha-2}{4} E_1^3 = \frac{1}{ba\ k_3\ f'_c} \begin{bmatrix} f_{s1}\ A_{s1} \\ +f_{s2}\ A_{s2} \\ .... \\ .... \\ +f_{sz}\ A_{sz} \end{bmatrix} \tag{3.20}$$

If it is further assumed that the bond parameter does not vary with the steel level, the following z compatibility equations are obtained,

$$\epsilon_{s1} = \epsilon_{sF1} + \epsilon_{cF1} + (\frac{d_1}{a} - 1)\, \epsilon_{c1}\, \psi$$

$$\epsilon_{s2} = \epsilon_{sF2} + \epsilon_{cF2} + (\frac{d_2}{a} - 1)\, \epsilon_{c1}\, \psi$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\epsilon_{sz} = \epsilon_{sFz} + \epsilon_{cFz} + (\frac{d_z}{a} - 1)\, \epsilon_{c1}\, \psi$$

These equations, together with Eqs. 3.19, 3.20, 3.11, and the stress-strain relation for the steel, must be used to obtain, by trial and error procedure, the z different values of steel stress.

Beams with I Shaped Sections

In the case of beams of I section, loading must be considered in three stages; zero moment to M_{on}, M_{on} to M_t, and M_t to static ultimate moment, where M_{on} is the moment at which cracks begin to open and M_t is the moment at which the depth to the neutral axis is equal to the depth of the top flange.

Linear stress-strain relations may be assumed for the first loading stage and a simple elastic analysis of the section, similar to the analysis for the initial loading stage for the rectnagular cross section discussed earlier, provides values for steel stress and cracking moment. In the third loading stage the neutral axis will lie in the top flange and the steel stresses may be determined from an analysis of the

beam assuming a rectangular section with the width equal to the width of the upper flange.

In the second loading stage, $M_{on} < M_1 < M_t$, the beam is cracked and the neutral axis lies in the web. Letting the value of the dimensionless strain term E in the top concrete fiber be E_1, and assuming a linear distribution of compressive strains, one obtains for the value of E at the bottom level of the top flange

$$E = \frac{kd - t}{kd} E_1$$

where t is the flange thickness. The compressive force is therefore

$$C = b' \int_0^{kd} f_c \, dy \quad + \quad (b-b') \int_{(kd-t)}^{kd} f_c \, dy$$

i.e.

$$C = \frac{k_3 f'_c d}{E_1} \; k \left[b' \int_0^{E_1} F dE \; + \; (b-b') \int_{(1-\frac{t}{kd})E_1}^{E_1} F dE \right] \tag{3.21}$$

The compressive force is located at a distance $k_2 kd$ from the top fiber, where

$$k_2 = \frac{\displaystyle\int_0^{E_1} EF dE + (1 - \frac{b'}{b}) \int_{(1-\frac{t}{kd})E_1}^{E_1} EF dE}{\displaystyle\int_0^{E_1} F dE + (1 - \frac{b'}{b}) \int_{(1-\frac{t}{kd})E_1}^{E_1} F dE} \tag{3.22}$$

Equations for horizontal equilibrium and moment equilibrium are

$$f_{s1} A_s = k_3 f_c' d k \left[\frac{b'}{E_1} \int_o^{E_1} FdE + \frac{(b-b')}{E_1} \int_{(1-\frac{t}{kd})E_1}^{E_1} FdE \right] \qquad (3.23)$$

and

$$M_1 = f_{s1} A_s d (1 - k_2 k).$$

An analysis of the deformations in the beam yields the compatibility condition represented by Eq. 3.13. It should be noted that values of the bond parameter ψ may well vary considerably from those for rectangular sections.

The integrals appearing in Eqs. 3.22 and 3.23 may be evaluated without difficulty in terms of E_1 and k, and then Eqs. 3.13, 3.18, 3.22 and 3.23, together with the steel stress-strain relation, may be used to solve for steel stress by a trial and error procedure. The main difference between these calculations and those for a rectangular section is that k_2 is now a function of k as well as E_1. It is convenient to begin the calculations by assuming a steel stress f_{s1}, and make trial values of k until the correct value of ϵ_{s1} is given by Eq. 3.13. The moment corresponding to f_{s1} is then obtained by substitution in Eq. 3.18.

CONCLUDING REMARKS

Assumption of a deformation condition with finite spacing of vertical tension cracks has led to the use of a compatibility factor ψ

in the compatibility equation. Analysis of the beam test data has shown that the value of ψ varies considerably during the fatigue life of a member, usually decreasing by up to 50 percent during the early load cycles, but becoming fairly steady for the major portion of the fatigue life. Before reliable quantitative values can be quoted for the compatibility factor, more extensive beam test data will be required.

The method presented in this paper may be used to determine the steel and concrete stresses in prestressed concrete members subjected to bending moments varying in value between zero and static ultimate. A numerical example of the use of the equations will be included in the fourth and final paper of this series as part of a sample calculation of the probable fatigue life of a beam of rectangular section.

ACKNOWLEDGEMENTS

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$$F = \alpha E + (3 - 2\alpha)E^2 + (\alpha - 2)E^3$$
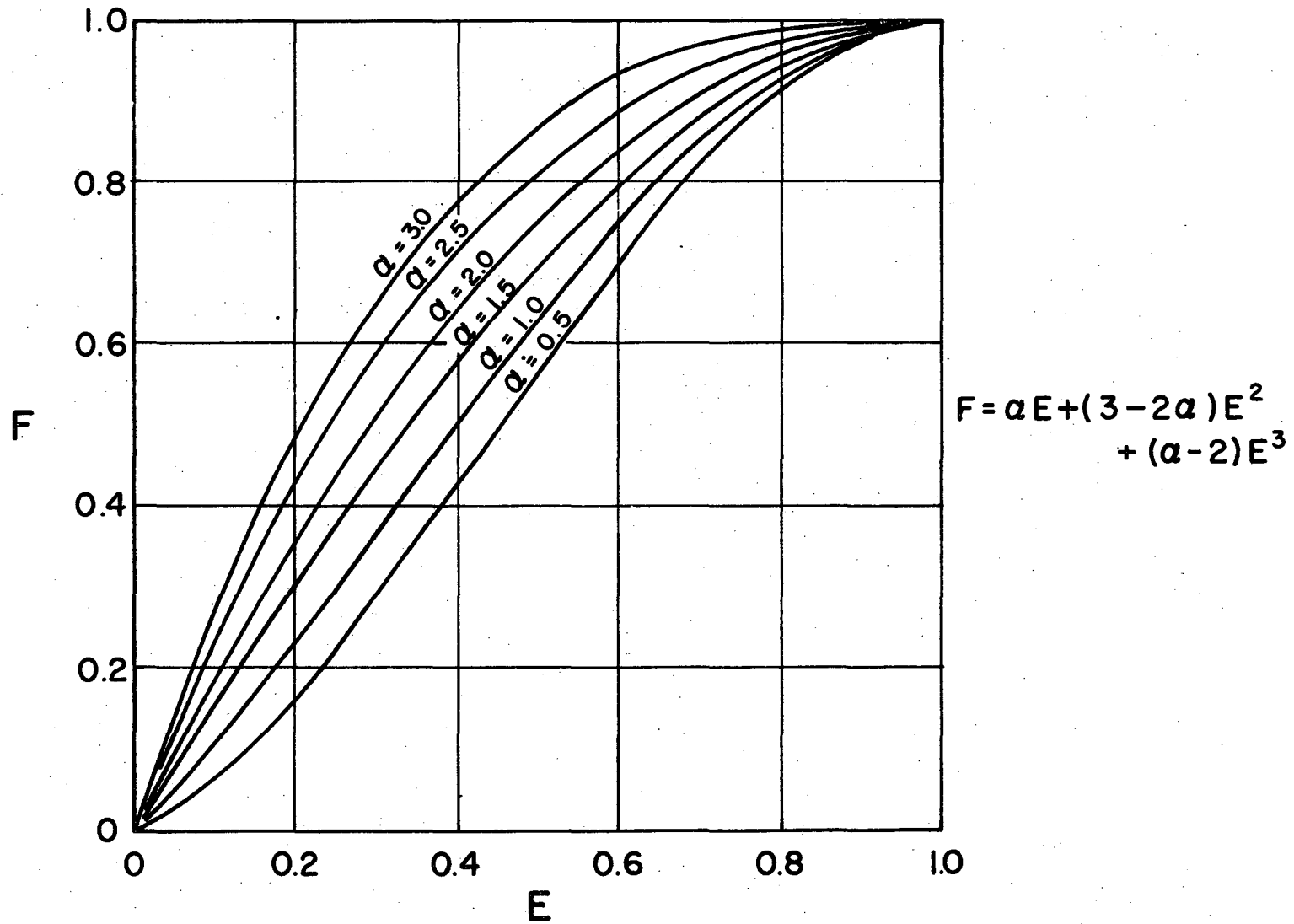
FIG. I — CUBIC PARABOLA FOR VARIOUS VALUES OF α
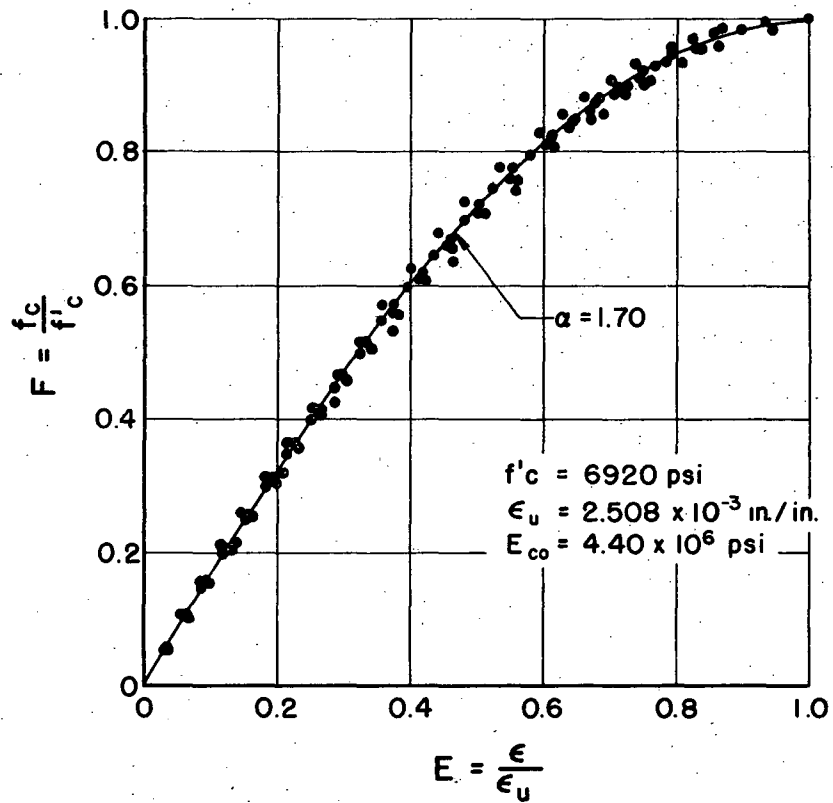
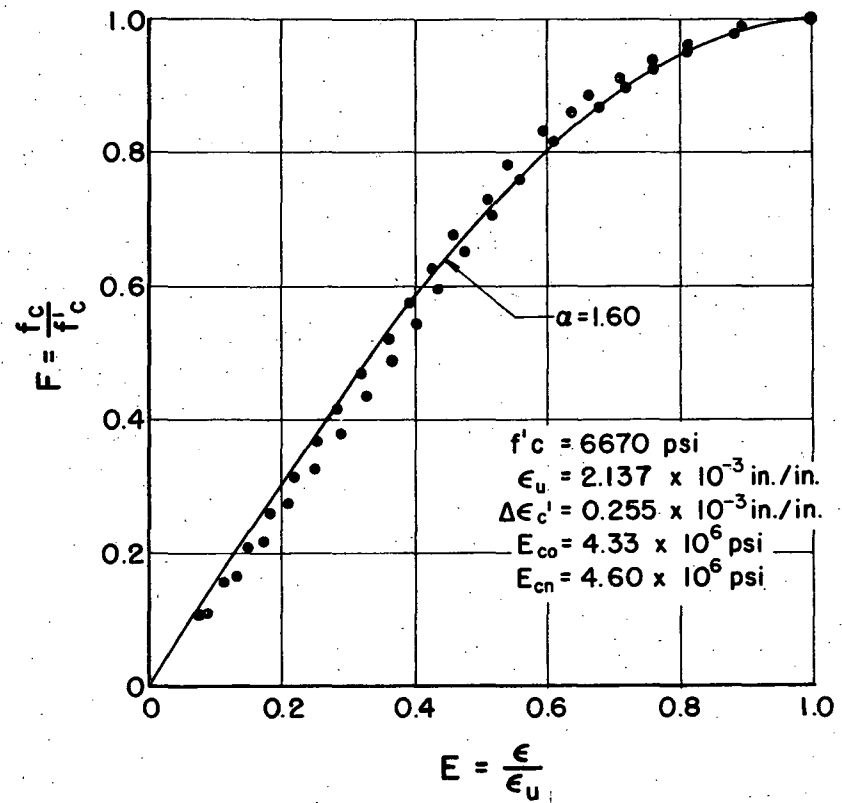FIG. 2   —CONCRETE STRESS-STRAIN RELATION
PRE-LOADING: NONE

FIG. 3   —CONCRETE STRESS-STRAIN RELATION
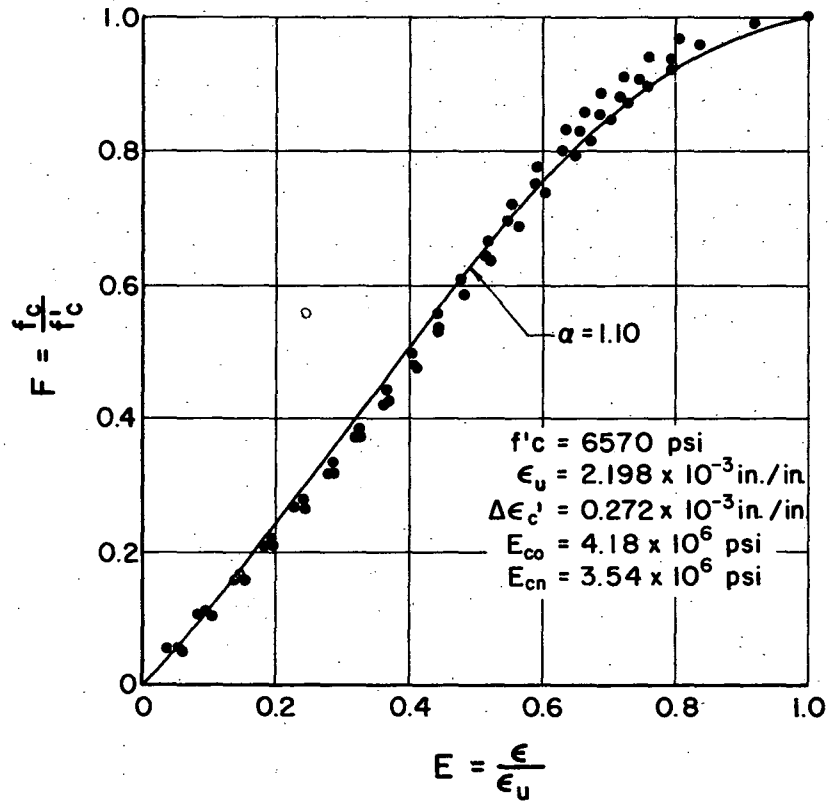PRE-LOADING: 20 CYCLES , 20-130 KIPS

FIG. 4   -CONCRETE STRESS-STRAIN RELATION
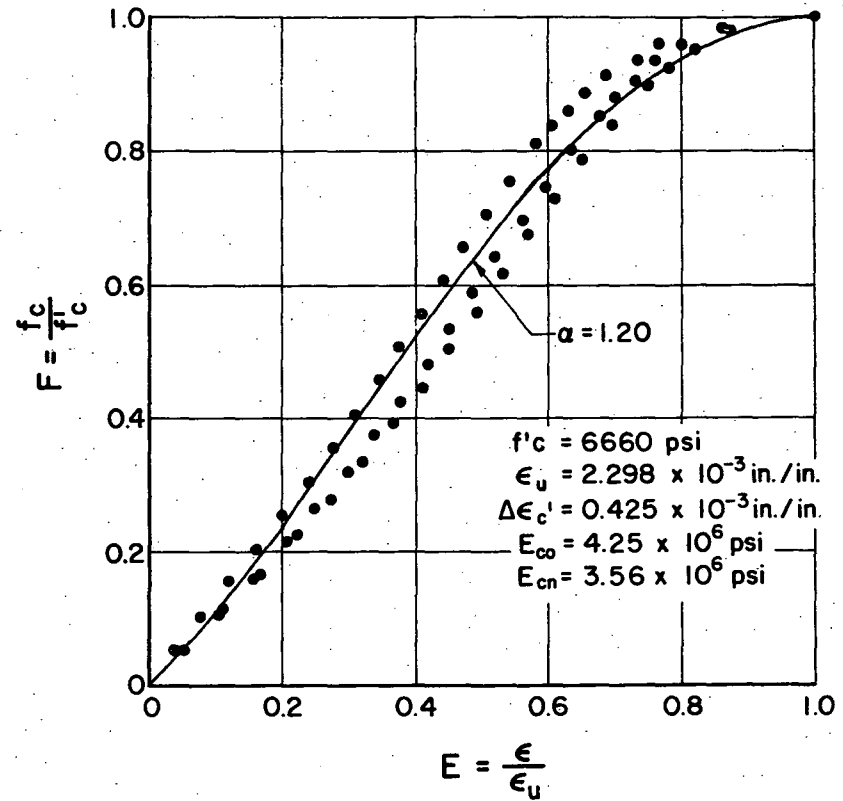PRE-LOADING : 30,000 CYCLES, 20-130 KIPS

FIG. 5   -CONCRETE STRESS-STRAIN RELATION
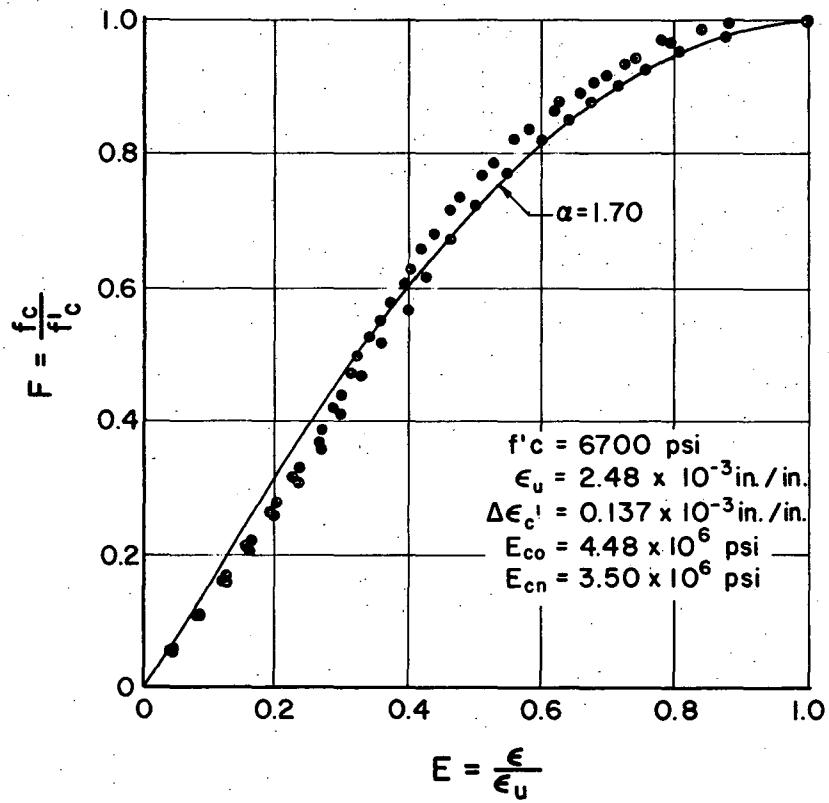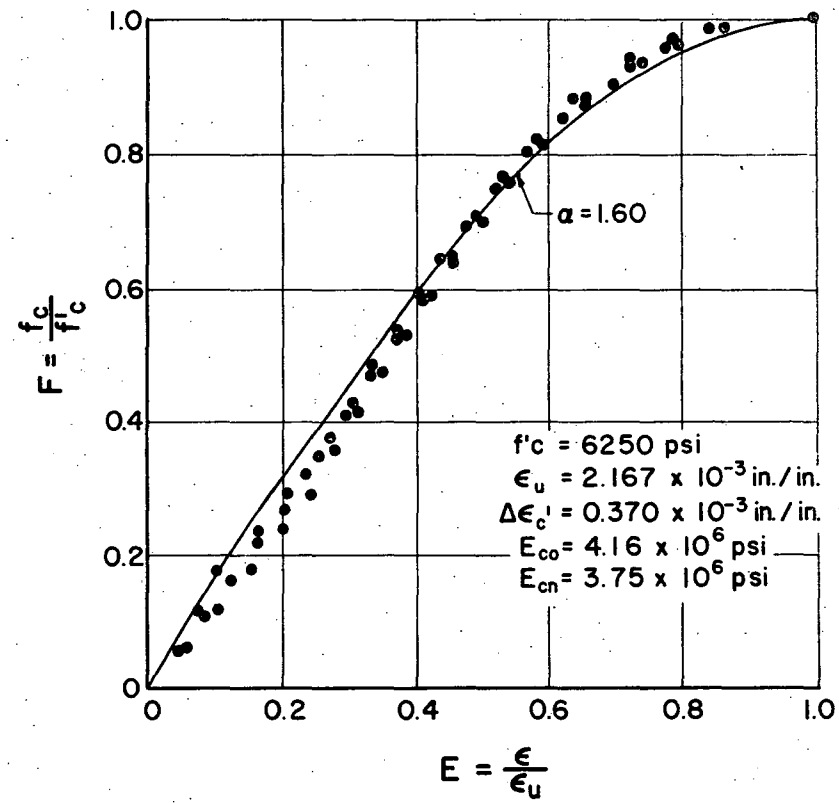PRE-LOADING: 100,000 CYCLES, 20-130 KIPS

FIG. 6   -CONCRETE STRESS-STRAIN RELATION
PRE-LOADING: 100,000 CYCLES, 20-100 KIPS

FIG. 7   -CONCRETE STRESS-STRAIN RELATION
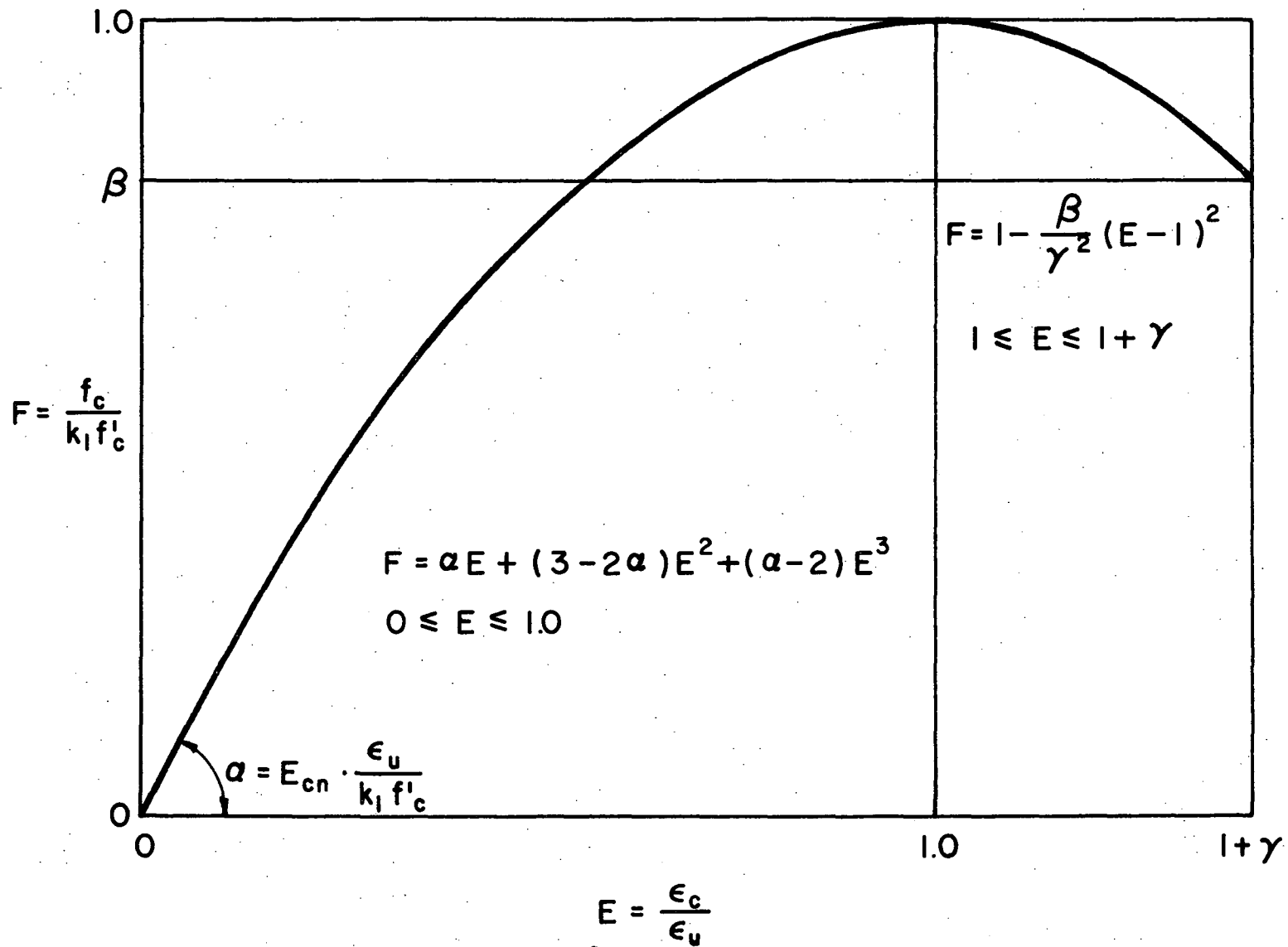PRE-LOADING: 1,000,000 CYCLES, 20-100 KIPS

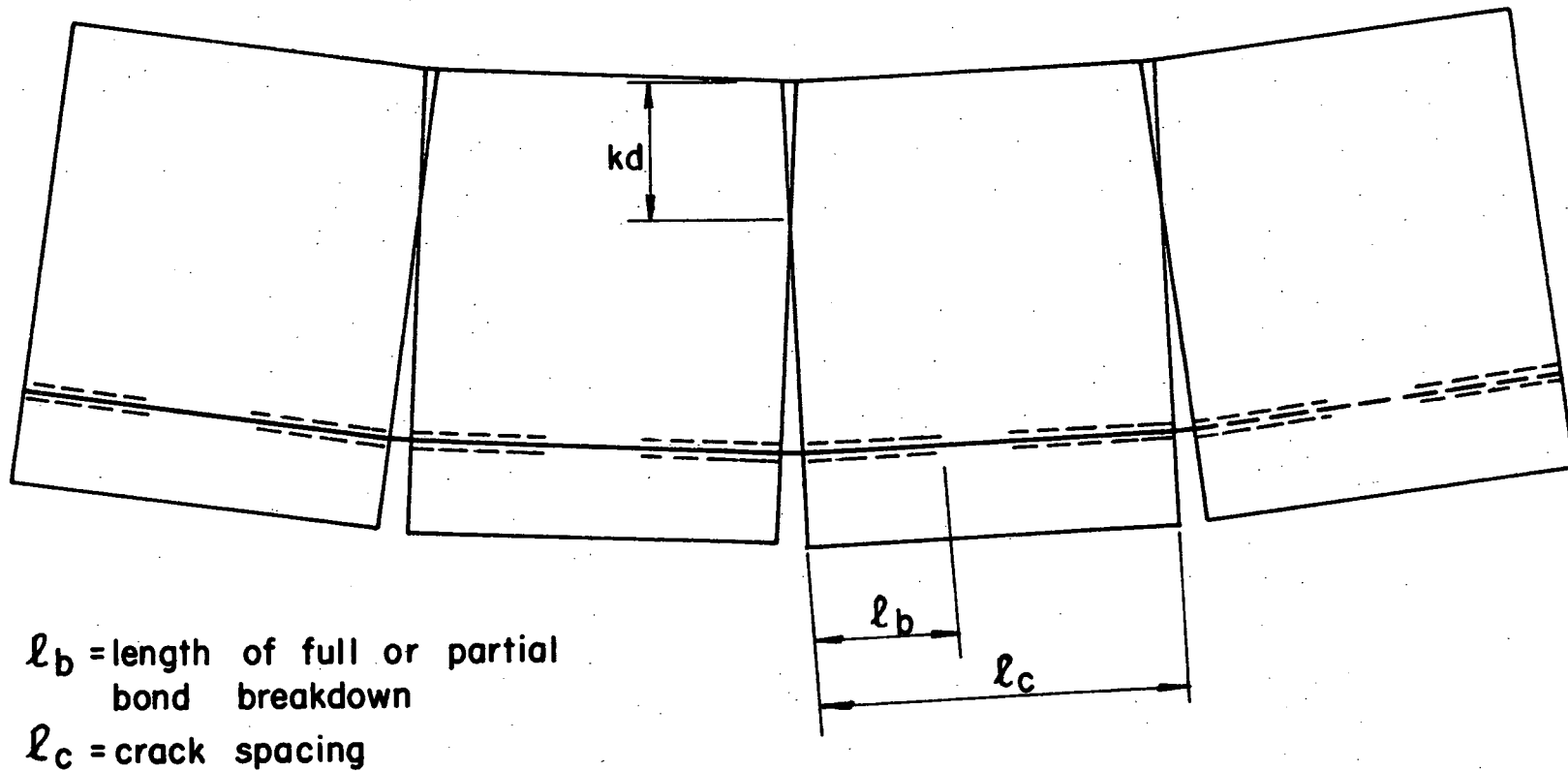FIG. 8   COMPLETE   STRESS - STRAIN   RELATION   FOR   CONCRETE

ℓ_b = length of full or partial
bond breakdown

ℓ_c = crack spacing
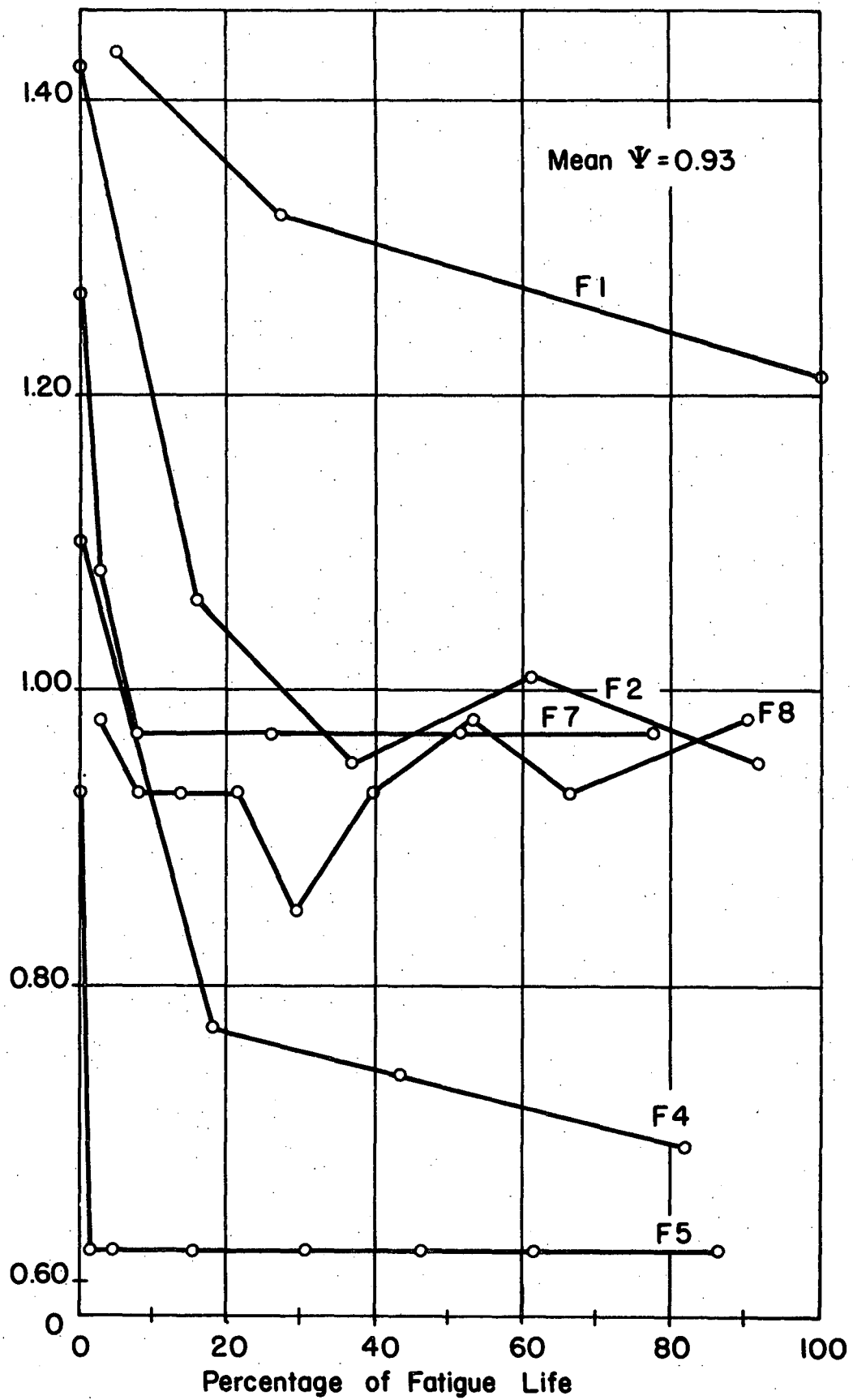
FIG. 9 – IDEALIZED DEFORMATION CONDITION AT CRACKED SECTION

FIG. 10-EXPERIMENTAL VALUES OF BOND PARAMETER, Ψ