

1951

# Interaction curves for circular column sections, 1951

R. L. Ketter

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INTERACTION CURVES FOR CIRCULAR COLUMN SECTIONS

by

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for

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Bethlehem, Pennsylvania

July 9, 1951

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## INTRODUCTION

This report is divided into three parts. The first presents the derivation of equations for predicting failure of a column, fixed at one end with moment and axial thrust applied at the other, where initial yield is used as a criterion of failure. Also included is a derivation of the collapse solution using simple plastic theory where  $L/r \rightarrow 0$ .

The second part presents sample calculations necessary for plotting the interaction curves for a 1/2 inch rod,  $L/r = 120$ .

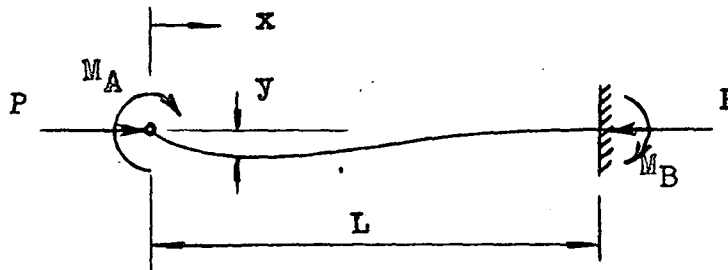
The third part contains a complete set of curves for bar sizes ranging from 3/8 " to 7/8" in diameter, varying by sixteenths of an inch for slenderness ratios of 80, 90, 100, 110, and 120.

The modulus of elasticity has been assumed equal to 29,500,000 psi with an elastic limit of 36,000 psi. All calculations are on file at the Fritz Engineering Laboratory, Lehigh University, Bethlehem, Pennsylvania.

I - DERIVATION

INITIAL YIELD INTERACTION CURVE:\*

Consider the column with axial load and moment as shown below:



The general equation of the deflection curve\*\* for such loading is:

$$y = - \frac{M_B}{P} \left[ \frac{\sin kx}{\sin kL} - \frac{x}{L} \right] + \frac{M_A}{P} \left[ \frac{\sin k(L-x)}{\sin kL} - \frac{(L-x)}{L} \right] \text{-----(1)}$$

If this expression is differentiated with respect to x an equation is obtained for the slope at any point along the deflected beam. By letting  $x = L$  and setting the resulting slope at point  $B = 0$ , we can solve for  $M_B$  in terms of  $M_A$ .

$$\theta_B = y' = - \frac{M_B L}{3EI} \psi + \frac{M_A L}{6EI} \phi$$

where;

$$\psi = \frac{3}{u} \left[ \frac{1}{\sin 2u} - \frac{1}{2u} \right]$$

$$\phi = \frac{3}{2u} \left[ \frac{1}{2u} - \frac{1}{\tan 2u} \right]$$

$$u = \frac{kL}{2}$$

(Values of  $\psi$  and  $\phi$  are tabulated in Timoshenko's "Theory of Elastic Stability", pages 499-505).

\* Based on Progress Report L, WELDED CONTINUOUS FRAMES AND THEIR COMPONENTS, "Interaction Curves for Columns", by Robert L. Ketter and Lynn S. Beedle.

\*\* Timoshenko, S., "Theory of Elastic Stability", McGraw Hill, New York, 1936, page 12.

Since  $\theta_B = 0$ ,

$$\frac{M_B L}{3EI} \psi = \frac{M_A L}{6EI} \phi$$

Therefore,

$$M_B = \frac{\phi}{2\psi} M_A$$

Substituting this value in the general equation for deflection, (Eq. 1), an equation for a fixed - end condition is obtained.

$$y = -\frac{M_A}{P} \frac{\phi}{2\psi} \left[ \frac{\sin kx}{\sin kL} - \frac{x}{L} \right] + \frac{M_A}{P} \left[ \frac{\sin k(L-x)}{\sin kL} - \frac{(L-x)}{L} \right]$$

$$y = \frac{M_A}{P} \left[ \frac{\sin k(L-x)}{\sin kL} - \frac{(L-x)}{L} - \frac{\phi}{2\psi} \left( \frac{\sin kx}{\sin kL} - \frac{x}{L} \right) \right] \text{-----}(2)$$

Differentiating Equation (2) twice with respect to x, the curvature of the member is obtained as a function of distance along the member.

$$y' = \frac{M_A}{P} \left[ \frac{-k}{\sin kL} (\cos k(L-x)) + \frac{1}{L} - \frac{\phi}{2\psi} \left( \frac{k \cos kx}{\sin kL} - \frac{1}{L} \right) \right]$$

$$y'' = \frac{M_A}{P} \left[ \frac{+k^2}{\sin kL} (-\sin k(L-x)) - \frac{\phi}{2\psi} \frac{-k^2 \sin kx}{\sin kL} \right]$$

$$y'' = \frac{M_A}{P} \frac{k^2}{\sin kL} \left[ -\sin k(L-x) + \frac{\phi}{2\psi} \sin kx \right]$$

Since the curvature = -M/EI,

$$M = -EI \cdot \frac{M_A}{P} \cdot \frac{P}{EI} \cdot \frac{1}{\sin kL} \left[ -\sin k(L-x) + \frac{\phi}{2\psi} \sin kx \right]$$

$$M = \frac{M_A}{\sin kL} \left[ \sin k(L-x) - \frac{\phi}{2\psi} \sin kx \right] \text{-----}(3)$$

To find the point of maximum moment in order that initial yield can be investigated for this section, the expression for moment, Equation (3), is differentiated with respect to x and set equal to zero. The solution of the resulting equation will

determine the distance from the point of applied moment to the section of maximum moment in terms of the axial load on the column.

$$M' = -\frac{M_A k}{\sin kL} \left[ \cos k(L-x) + \frac{\phi}{2\psi} \cos kx \right] = 0$$

Therefore,

$$\cos k(L-x) = -\frac{\phi}{2\psi} \cos kx$$

or

$$x^* = \frac{\tan^{-1} \left[ \frac{-\frac{\phi}{2\psi} - \cos kL}{\sin kL} \right]}{k} \text{-----(4)}$$

To find the axial load, P, that will be required to maintain a moment gradient of zero at the end of the column, (the maximum load that can be applied and still have the maximum moment occur at the end of the column), set  $x = 0$  in equation (4). Then,

$$\frac{\phi}{2\psi} = -\cos kL$$

Solution occurs when  $2u = 2.33$ , where  $2u = kL = L \sqrt{\frac{P}{EI}}$   
Therefore,

$$P = \frac{(2.33)^2 EI}{L^2}$$

Substituting the value of  $L^2$  as determined from Euler buckling load for a condition of one end fixed, the other pinned,  $P_{E1}$ ;

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\* Included in this report is a curve for determining x.  
See page 22 .

$$P = \frac{(2.33)^2 E I}{\pi^2 E I} = 0.27 P_E,$$

$$(0.7)^2 P_E,$$

Therefore, as long as P is less than or equal to (0.27 P<sub>E</sub>), the maximum moment will occur at the end of the column.

To plot the interaction curve, consider the basic equation for yielding at a section subjected to both bending moment and axial load.

$$\sigma_y = \frac{P}{A} + \frac{M_{max}}{S} \text{-----}(5)$$

Where M<sub>max</sub>, for our particular case, would be determined from Equation (3) substituting in values of x determined from Equation (4) for selected values of P.

Applying the above equation to members of circular cross-section,

$$\sigma_y = \frac{P}{\frac{\pi d^2}{4}} + \frac{M_{max}}{\frac{\pi d^3}{32}}$$

or

$$M_{max} = \frac{\sigma_y \pi d^3 - 4 P d}{32}$$

Where M<sub>max</sub> is determined by equation (3). Therefore,

$$M_A = \frac{[\sigma_y \pi d^3 - 4 P d] \sin kL}{32 [\sin k(L-x) - \delta/2 \psi \sin kx]}$$

Using the recommended value of  $\sigma_y = 36,000$  psi,

$$M_A = \frac{[9\pi d^3 - Pd] \sin kL}{8 [\sin k(L-x) - \delta/2 \sin kx]} \text{-----}(6)$$

Substituting value of d for the various columns under consideration in Equation (6) results in the following initial yield interaction curve equations

<u>d</u>	<u>Interaction Curve Equation</u>
3/8"	$M_A = 0.0469(3.9757-P)(Q)$
7/16"	$M_A = 0.0547(5.4119-P)(Q)$
1/2"	$M_A = 0.0625(7.0686-P)(Q)$
9/16"	$M_A = 0.0703(8.9463-P)(Q)$
5/8"	$M_A = 0.0781(11.045-P)(Q)$
11/16"	$M_A = 0.0859(13.364-P)(Q)$
3/4"	$M_A = 0.0938(15.905-P)(Q)$
13/16"	$M_A = 0.1016(18.666-P)(Q)$
7/8"	$M_A = 0.1094(21.648-P)(Q)$

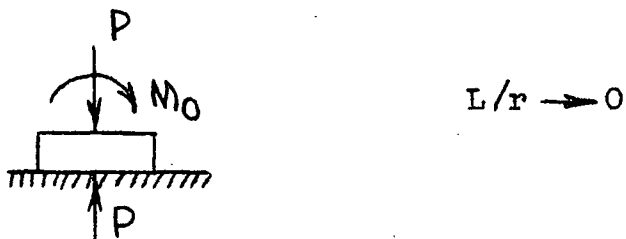
$$Q = \frac{\sin kL}{\sin k(L-x) - \frac{\phi}{2\psi} \sin kx}$$

Note: Terms are defined on page 2.



"SIMPLE PLASTIC THEORY" COLLAPSE Interaction Curves\* for  $L/r = 0$

Consider a column with axial load and end moment applied as shown below:



For the limiting case where the axial load,  $P$ , is equal to zero, (a beam problem), the ultimate moment that can be applied to the column, using the simple plastic theory, is

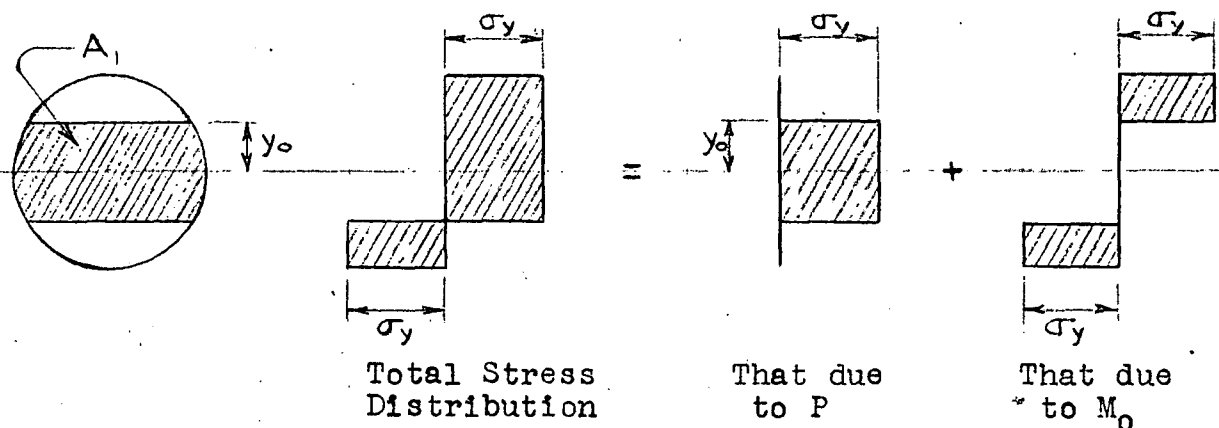
$$M_p = \sigma_y Z$$

where  $Z$  is defined as the statical moment of the section.

For the other limiting case, that where the applied end moment,  $M_0$ , is equal to zero, the maximum axial load the column can carry is

$$P_p = P_y = \sigma_y A$$

For the case where axial load and end moment both exist, (investigating only the case of complete plasticity), a stress distribution similar to that shown below would occur:



\* Based on Progress Report L, WELDED CONTINUOUS FRAMES AND THEIR COMPONENTS, "Interaction Curves for Columns", by Robert L. Ketter and Lynn S. Beedle.

The cross-sectional area,  $A_1$ , supporting the axial load,  $P$ , is shown as the shaded portion in the above figure. The remaining area determines the moment,  $M_0$ .

$$A_1 = 2 \left[ y_0 \sqrt{(d/2)^2 - (y_0)^2} + (d/2)^2 \sin^{-1} \left( \frac{y_0}{d/2} \right) \right]$$

Since  $P = \sigma_y A_1$

$$P = 2 \sigma_y \left[ y_0 \sqrt{(d/2)^2 - (y_0)^2} + (d/2)^2 \sin^{-1} \left( \frac{y_0}{d/2} \right) \right] \text{-----(7)}$$

The supporting cross-sectional moment is

$$M_0 = 4 \sigma_y \int_{y_0}^{\frac{d}{2}} y \sqrt{(d/2)^2 - y^2} dy$$

Or,

$$M_0 = + 4/3 \sigma_y \left[ (d/2)^2 - (y_0)^2 \right]^{3/2} \text{-----(8)}$$

By selecting various values of  $y_0$  and substituting these in equations (7) and (8), expressions for  $P$  and  $M_0$  are determined in terms of the diameter,  $d$ , of the column. The following were used:

$y_0$	$P$	$M_0$
$d/16$	$4.488d^2$	$5.866d^3$
$d/8$	$8.905d^2$	$5.448d^3$
$d/4$	$17.219d^2$	$3.893d^3$
$3d/8$	$24.197d^2$	$1.738d^3$
$7d/16$	$26.804d^2$	$0.682d^3$

II. SAMPLE SOLUTION

Using a 1/2" rod,  $L/r = 120$ , for illustration.

SECTIONAL PROPERTIES:

d	=	0.50 in.	L	=	15.00 in.
A	=	0.1963 in.	$P_y$	=	7.0686 kips
$I_x$	=	0.003068 in. <sup>4</sup>	$M_y$	=	0.4417 in. kips
$r_x$	=	0.1250 in.	$M_p$	=	0.7499 in. kips
S	=	0.01227 in. <sup>3</sup>	$P_{E'}$	=	8.100 kips
Z	=	0.02083 in. <sup>3</sup>	$0.27P_{E'}$	=	2.187 kips

INITIAL YIELD FAILURE: Using the Equation shown on page 6.

$$M_o = 0.0625 (7.0686 - P) \frac{\sin kL}{\sin k(L-x) - \phi/2\psi(\sin kx)}$$

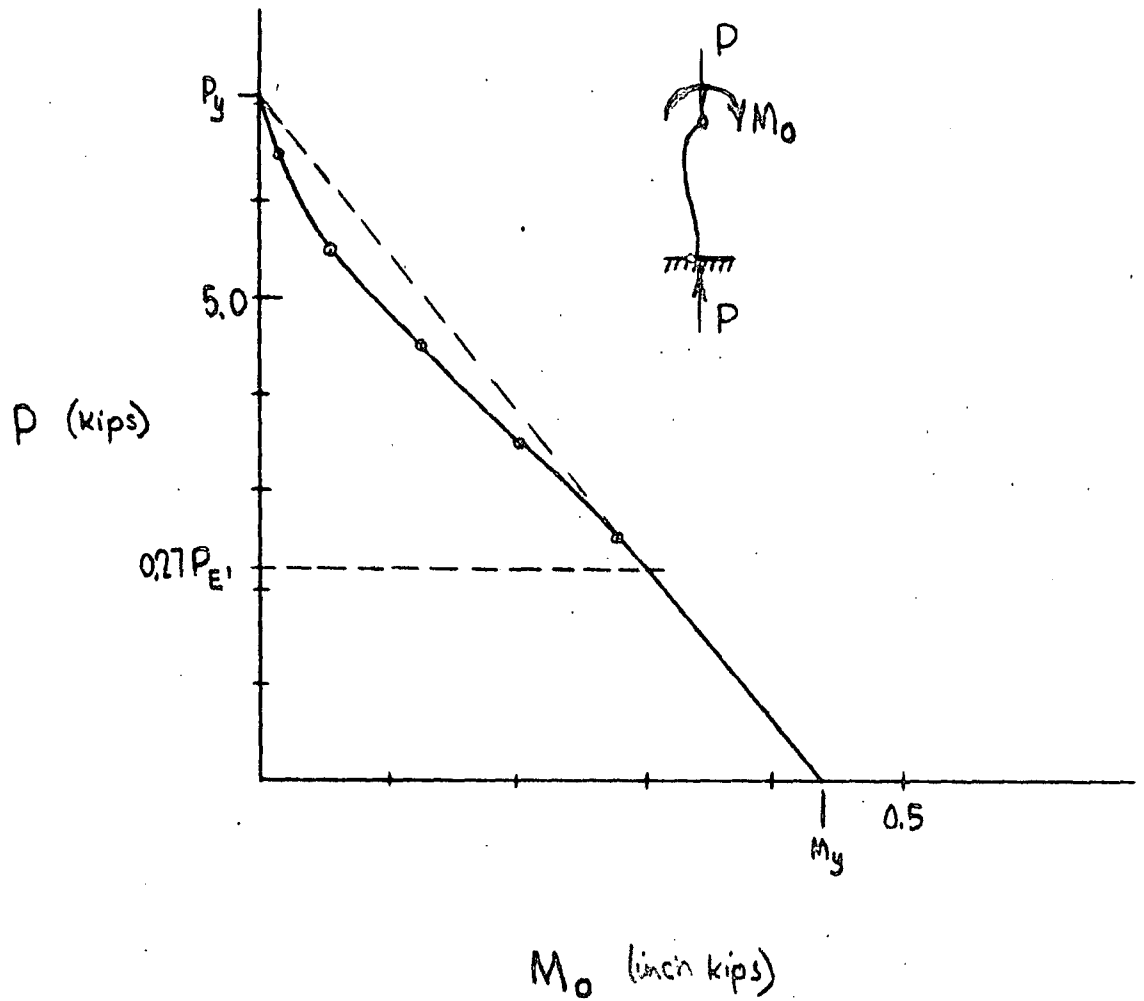
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
P	P/EI	$\sqrt{(2)} = k$	$kL = 2u$	$x/L$	x	L-x	$k(L-x)$
2.5	0.02762	0.1662	2.493	0.0460	0.690	14.310	2.378
3.5	0.03867	0.1966	2.949	0.1465	2.198	12.802	2.517
4.5	0.04972	0.2230	3.345	0.2130	3.195	11.805	2.633
5.5	0.06077	0.2465	3.698	0.2605	3.908	11.092	2.734
6.5	0.07182	0.2680	4.020	0.2985	4.478	10.522	2.820

(9)	(10)	(11)	(12)	(13)	(14)
$kx$	$\sin k(L-x)$	$\sin kx$	$\phi$	$\psi$	$2\psi$
0.1147	0.69036	0.11445	3.0190	2.0704	4.1408
0.4321	0.58347	0.41877	9.9422	5.5628	11.1256
0.7125	0.48556	0.65372	-9.9835	-4.3639	-8.7278
0.9633	0.39493	0.82107	-3.5136	-1.0863	-2.1726
1.2001	0.31457	0.93204	-2.3147	-0.4353	-0.8706

(15)	(16)	(17)	(18)	(19)	(20)
$\phi/2\psi$	$\phi/2\psi(\sin kx)$	(10) - (16)	$\sin kL$	(18)/(17)	$7.0686 - P$
0.7291	0.0834	0.6070	0.60279	0.9931	4.5686
0.8936	0.3742	0.2093	0.18984	0.9070	3.5686
1.1439	0.7478	-0.2622	-0.20357	0.7764	2.5686
1.6172	1.3278	-0.9329	-0.52949	0.5676	1.5686
2.6587	2.4780	-2.1634	-0.77074	0.4563	0.5686

(21)	(22)
$0.0625 \times (20)$	$(21) \times (19) = M_o$
0.2855	0.2835
0.2230	0.2023
0.1605	0.1246
0.0980	0.0556
0.0355	0.0162

Plotting these points give the following curve.

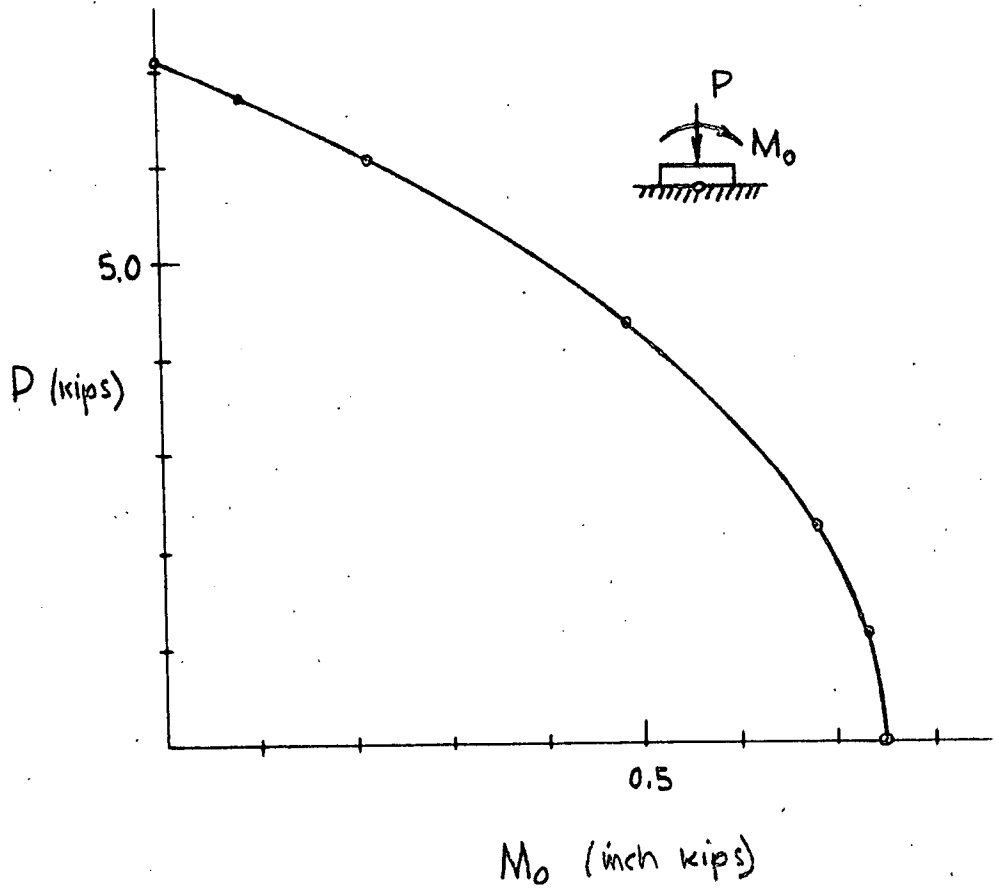


SIMPLE PLASTIC THEORY COLLAPSE ( $L/r=0$ ):

(Using equations shown on page 8.)

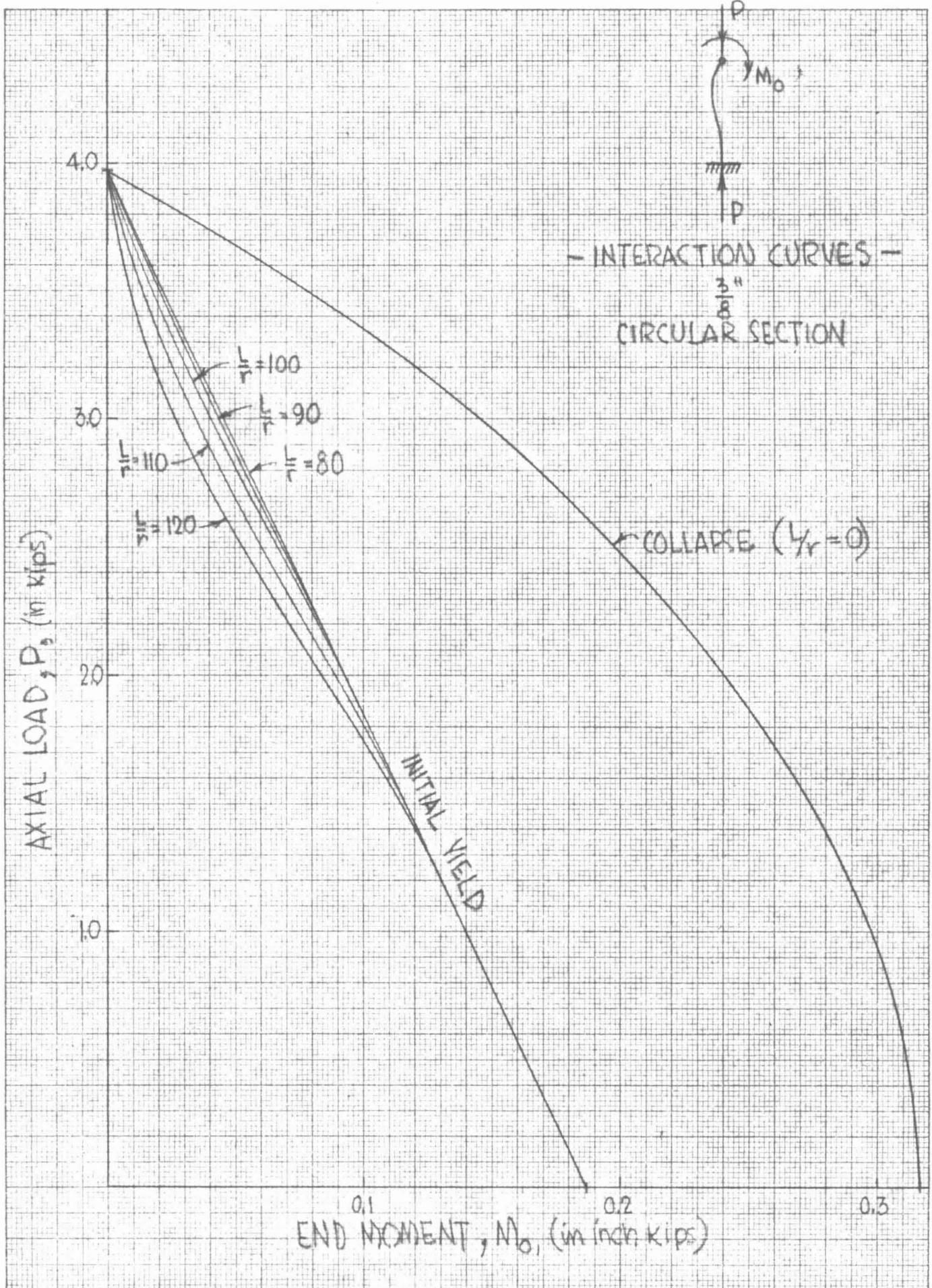
$y_0$	P	M
$d/16$	1.122	0.733
$d/8$	2.226	0.681
$d/4$	4.305	0.487
$3d/8$	6.049	0.217
$7d/16$	6.701	0.085

Plotting these points give the following curve.



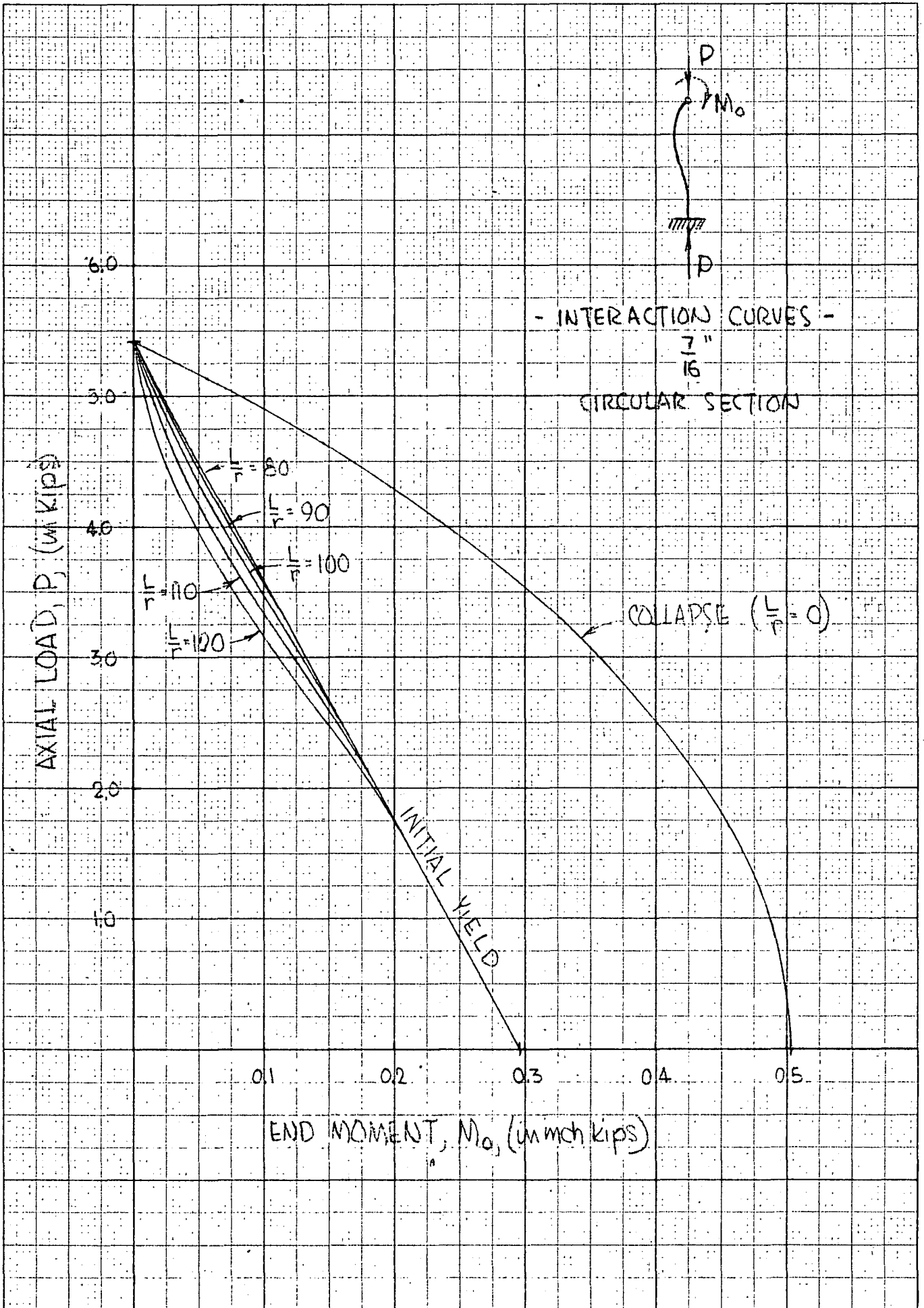
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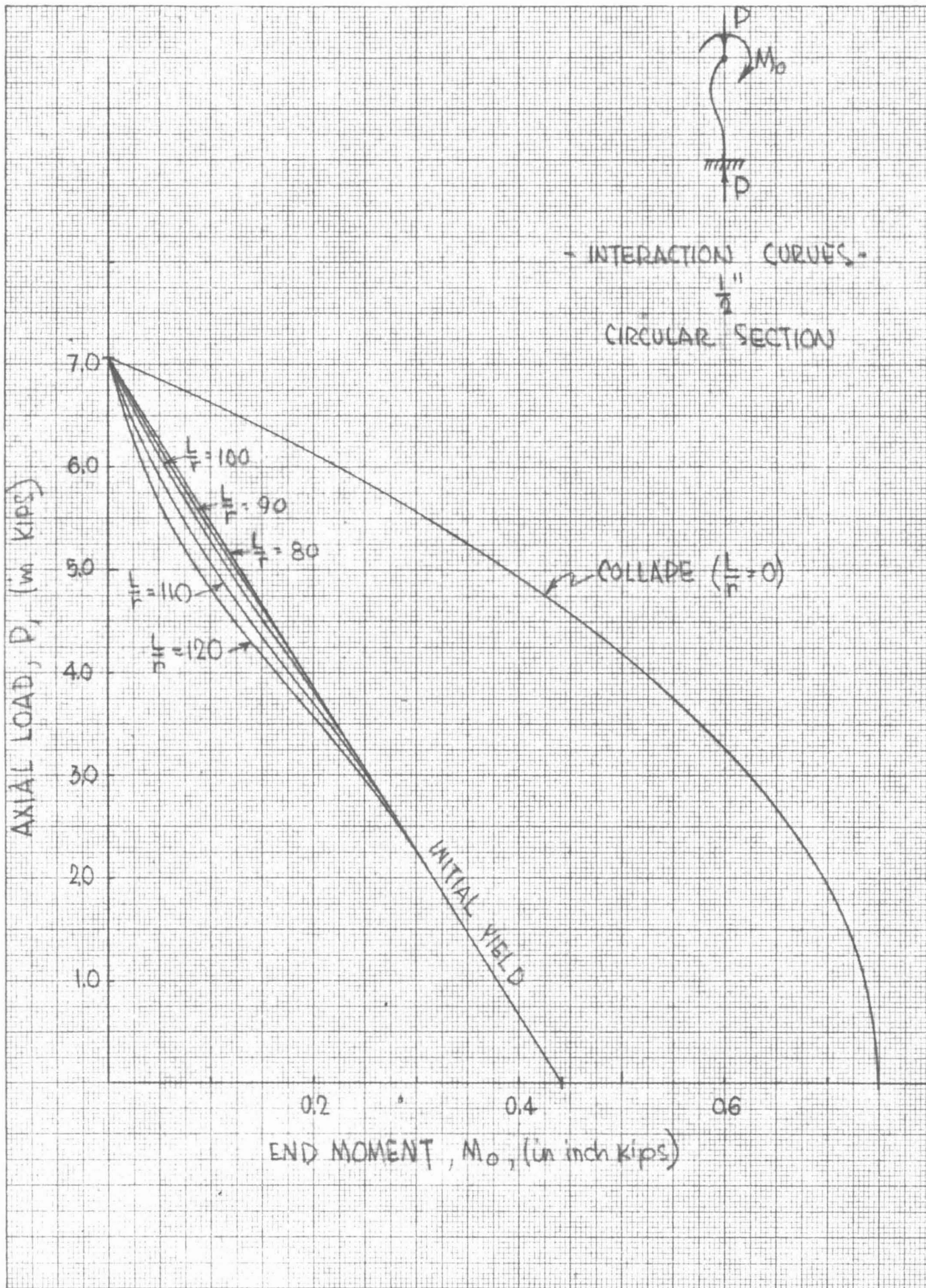
FIG. 1



NO. 340R-20 DIETZGEN GRAPH PAPER 20 X 20 PER INCH EUGENE DIETZGEN CO. MADE IN U.S.A.

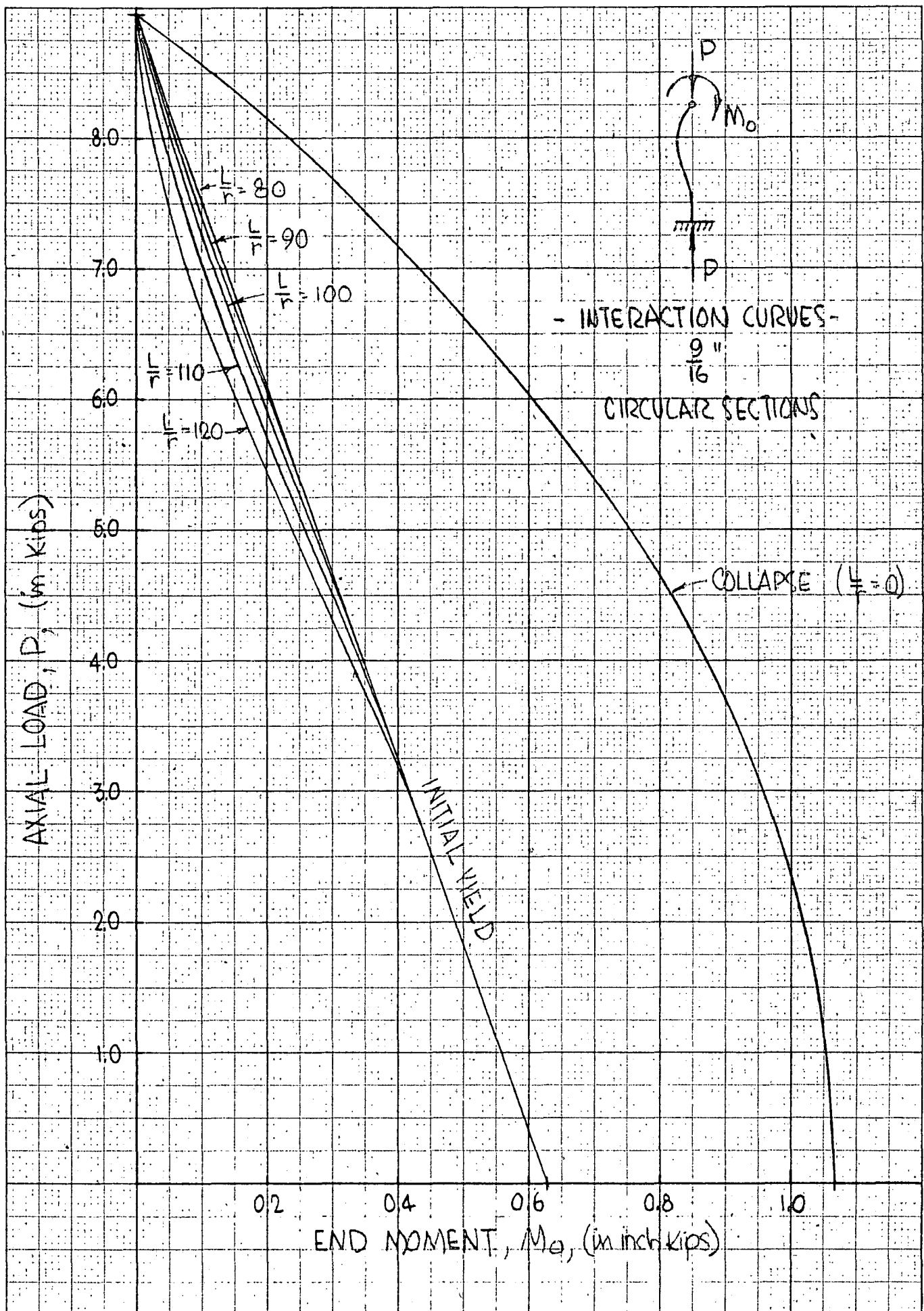
FIG. 2





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FIG. 3



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FIG. 4

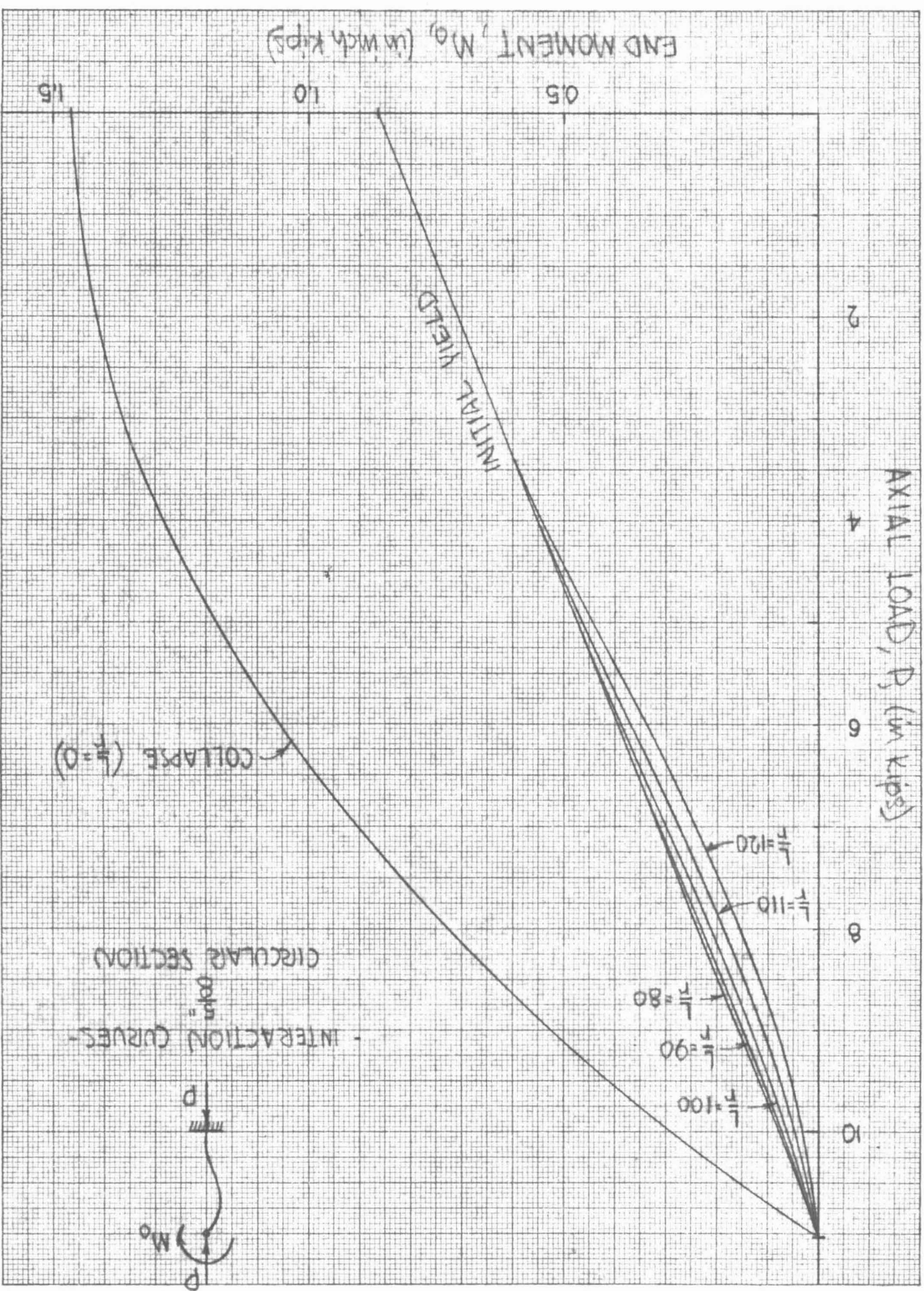
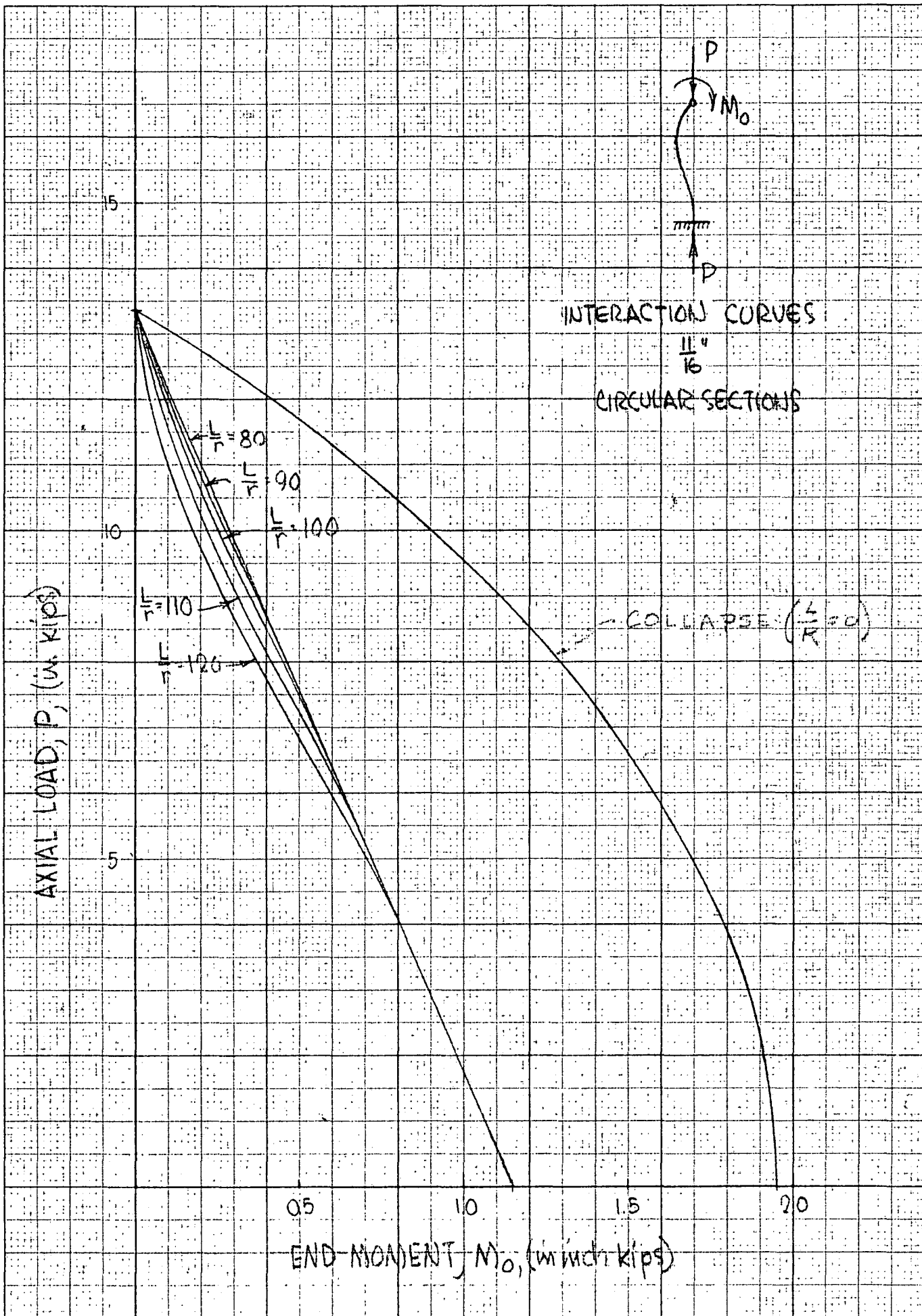


FIG. 5

8/5" R.L.K.



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20 X 20 PER INCH

FIG. 6

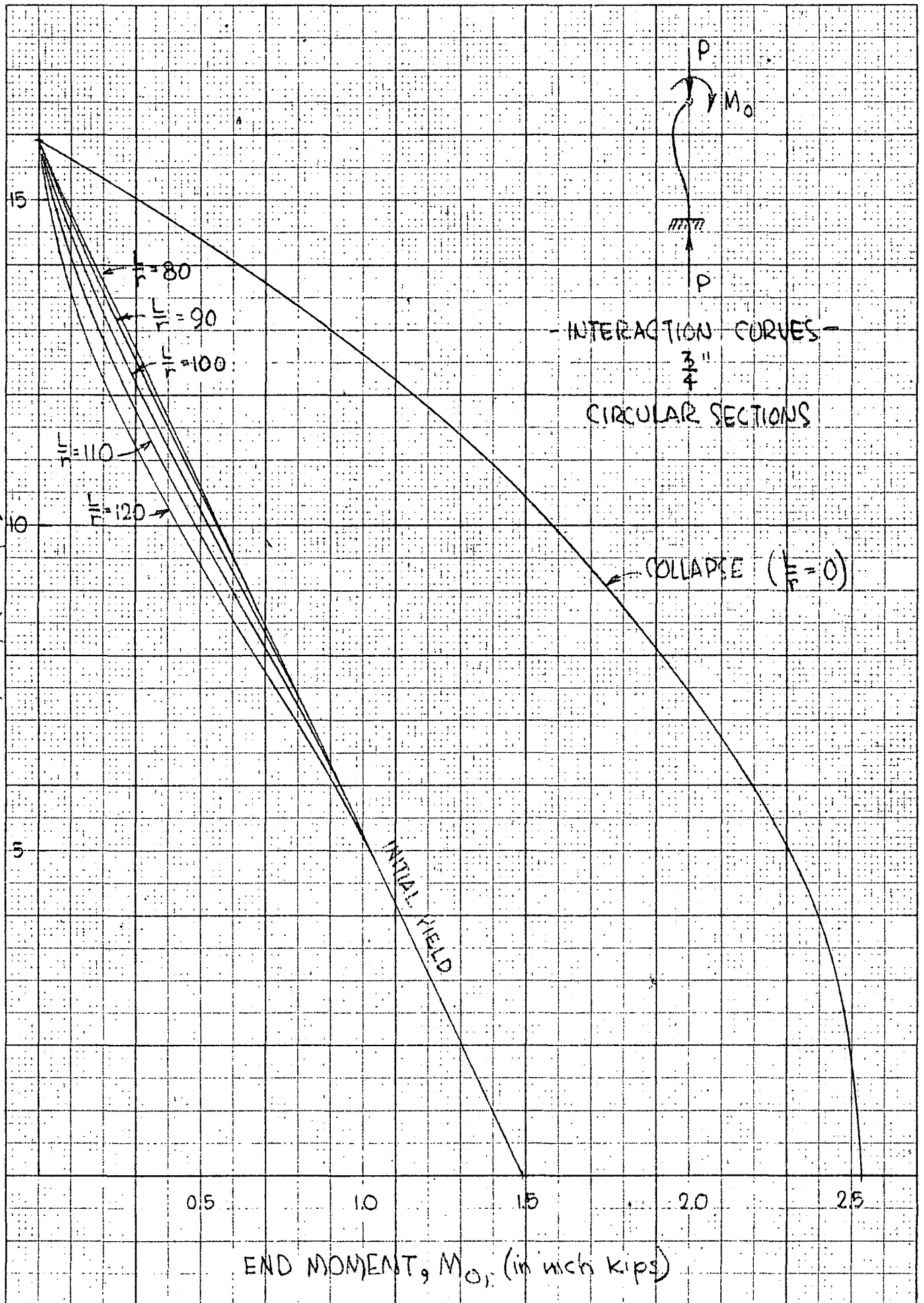
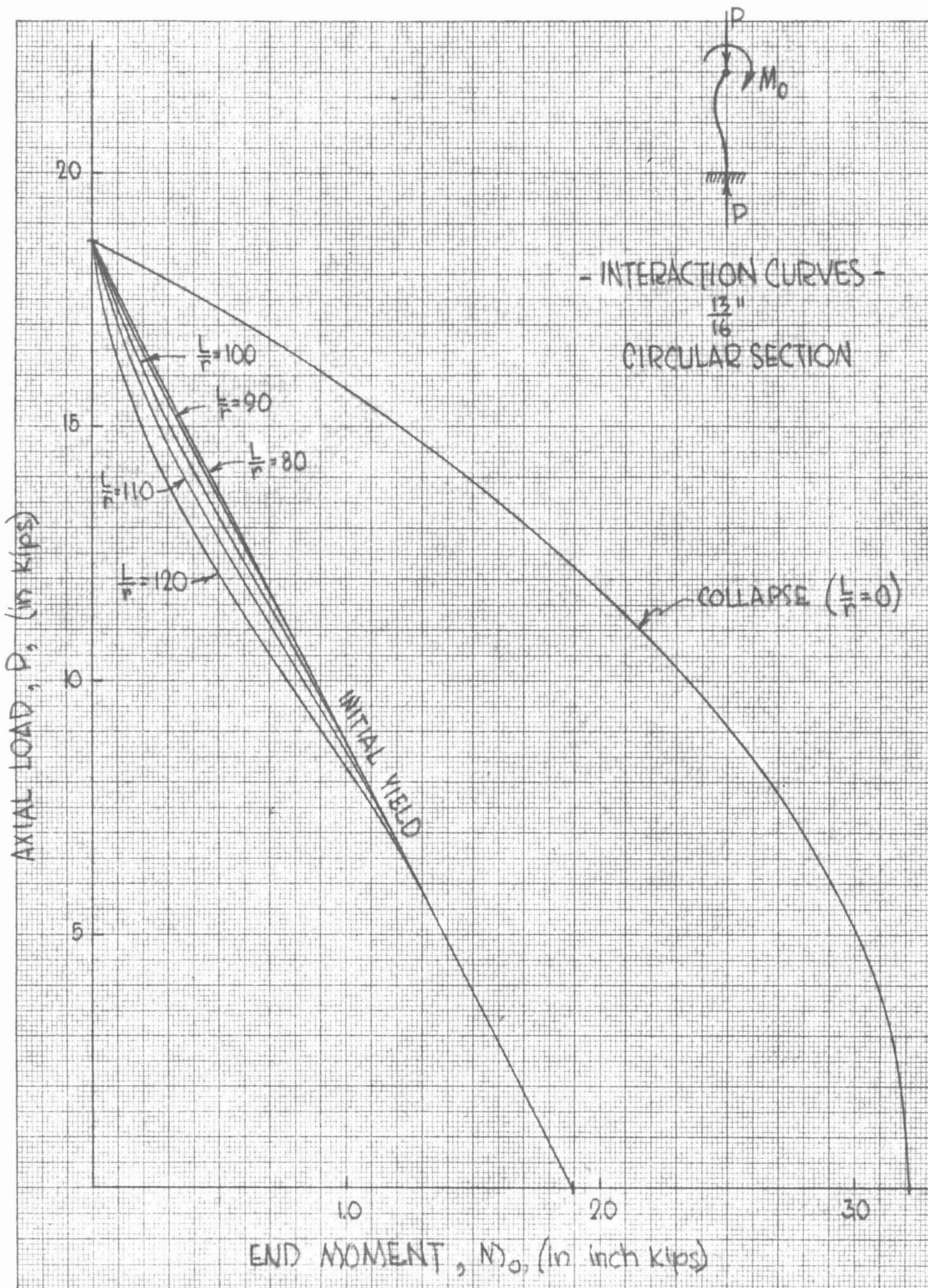


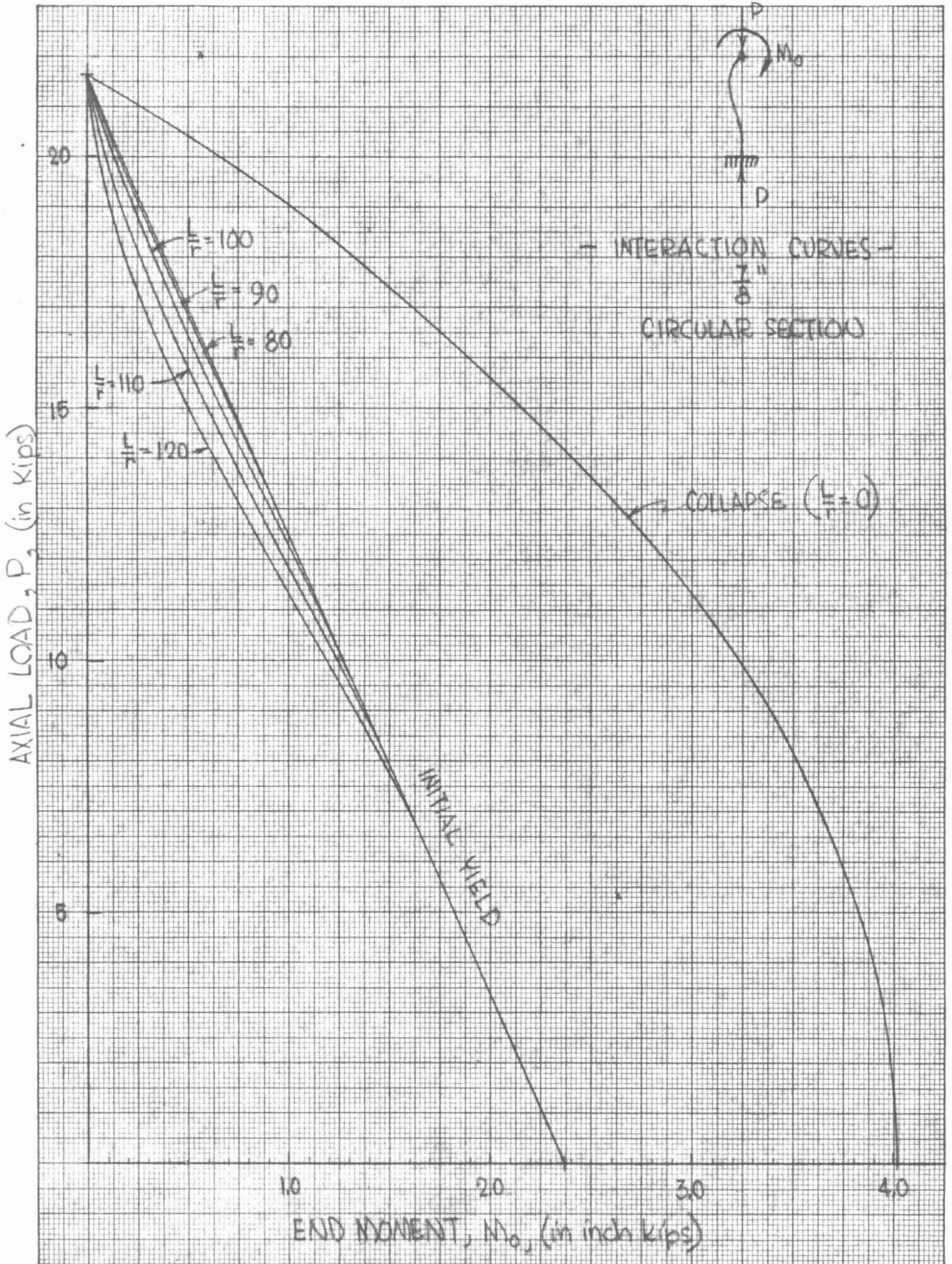
FIG. 7

NO. 240R-20 DIETZEN GRAPH PAPER  
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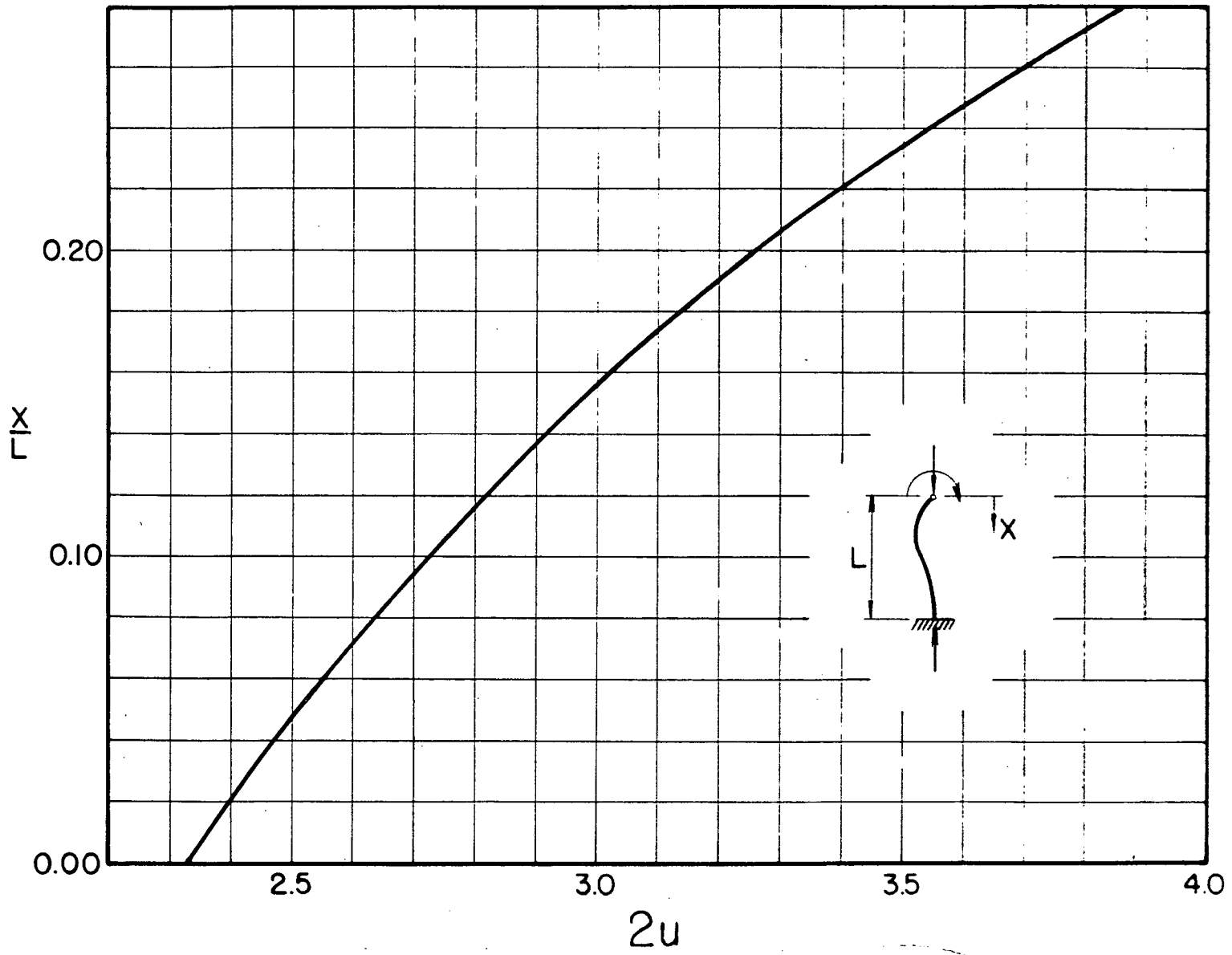
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FIG. 8



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FIG. 9



Distance to Section of Maximum Moment