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Welded Continuous Frames and Their Components

PROGRESS REPORT 11

A VIRTUAL DISPLACEMENT METHOD FOR DETERMINING
STABILITY OF BEAM COLUMNS ABOVE
THE ELASTIC LIMIT

by

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I. SYNOPSIS

This paper, which discusses the behavior of pin-ended, steel, wide-flange beam-columns, is the development of a virtual displacement method of solution to the inelastic instability problem. The method is equally applicable to the solution of elastic or inelastic beam-column problems but its greatest application seems to be in the inelastic field. Using the proposed method of solution, the critical value of loading for members subjected to concentric, eccentric and/or lateral loads, can be determined in a fraction of the time required by existing solutions.

II. INTRODUCTION

The problem under consideration is the determination of the maximum loading a beam-column type member can carry. The importance of the problem is evident, since if it is possible to predict the "collapse" loading of a member, it is also possible to evaluate for a member with a given loading the true factor of safety.

A large number of variables enter in the solution of problems of this type, and to reduce these in the development that will follow, it will be assumed that:

a. The mode of failure is that of bending in the plane of the applied moments. Furthermore, this plane is coincident with a principal axis of the section. (Lateral-torsional and/or local buckling are not considered.)

b. The material under study possesses stress-strain properties as shown in Fig. 1.

c. Members are originally straight, free from accidental eccentricities, and of uniform cross-section along their lengths.

To illustrate the phenomena being studied, consider the typical load-deflection curves of Figs. (2) and (3).

For the axially loaded member, Fig. 2 (elastic behavior), axial thrust, P, can be increased from zero to a certain critical value, P_e, with the member remaining in the straight equilibrium position. However, on application of the slightest additional increment of thrust above the critical value, lateral

deformation occurs. Thus at the instant of bending there exist
two adjacent positions of equilibrium of the same load; that
is to say, the member is indifferent with regard to either a
straight or a deflected configuration. This condition, known
as bifurcation of the equilibrium position, is called a "con-
dition of instability".

For the eccentrically loaded column problem of Fig. 3
a different type of behavior is observed. First of all, the
member starts to deflect on the slightest application of thrust.
Elastically this deformation increases with the maximum strength
approaching the Euler load. However, as deflections increase
with load, the outermost compressive fiber stress at the center
of the column increases as the product of P and y_0. Because of
this rapidly increasing stress situation, there is reached a
load at which yielding of the member takes place, (e.g., point
"a" in Fig. 3). As the load is further increased, yielding
progresses across and along the member, thereby reducing its
resistance to further loading. Finally, there is reached a point
at which an increase in load is impossible, the member progres-
sively deflecting as load remains constant. At this instant,
the member is indifferent with regard to two possible adjacent
equilibrium configurations and it becomes unstable.

Herein then, "instability" is considered as that pheno-
menon where under a given constant load there exist two possible
adjacent equilibrium positions.

To determine this condition of instability, several
basic methods of solution are available. Three of these are as
follows:

a. If the force required to displace the loaded
column into a different, deflected position equals
zero, then the member is said to be indifferent with
regard to which of the positions it will assume, and
a condition of instability exists.

b. If the total work associated with a loaded column
remains the same as the member is deflected, that is,
the rate of change with respect to deflection equals
zero, indifference again has taken place.

c. If it is possible to determine the axial load -
lateral deflection relation, the member will become
unstable when

$$\frac{\partial P}{\partial y} = 0$$

since for that value of axial load there exist two
adjacent equilibrium positions.

III. EXISTING SOLUTIONS

In the main, previous solutions of the inelastic in-
stability, beam-column problem have been based on the criterion
of instability that involves the determination of the maximum
point of a load-deflection curve. In general each of those solu-
tions has been based on the work of Karman, who, starting from
an assumed strain distribution pattern, (and thus a stress dis-
tribution pattern), calculated the corresponding moment, thrust
and curvature values which would have produced such a stress
configuration. Since the three equations defining these variables,
M, P and $\emptyset$, contain five unknowns (see Fig. 4), one expression
of the form

$$M_{int} = f(\emptyset, P) \tag{1}$$

can be obtained by eliminating the two cross-section variables,
(α and σ_a), which define the generalized yield pattern.

Consider now a typical beam-column (Fig. 5). Since
equilibrium must be satisfied at all times, i.e.

$$M_{ext} = M_{int} \tag{2}$$

and since the equation for external moment, M_{ext}, at any point
along the column can be expressed as a function of P, y, lateral
loads, etc., i.e.

$$M_{ext} = g \ (P, \ y, \ lateral \ loads, \ etc.),$$

solution to equation (1) will be of the form

$$P = h \ (\emptyset, \ y, \ lateral \ loads, \ etc.) \tag{3}$$

At this point the various methods of solutions differ. Karman and Chwalla [3] used numerical or graphical integration to solve equation (3) for direct values of P versus y. Jezek [7], Westergaard and Osgood [16] and others assumed a known deflection curve such that $\emptyset$ could be obtained directly in terms of y by differentiation. Both of these methods of solution have as their objective the determination of a means of describing thrust as a function of deflection,

$$P = f(y), \tag{4}$$

for which, according to criterion (c), instability occurs when

$$\frac{\partial P}{\partial y} = 0$$

(See Fig. 3.)

The difficulties involved in either of these methods of solution arise from the fact that for any conventional type of cross-section other than a simple rectangle or circle, it is impossible to write a continuous function for Eq. (1); therefore curves or graphs must first be devised to supply the M-P-$\emptyset$ information corresponding to various stress conditions [9]. Proceeding from these plots it is necessary to integrate suitable $\emptyset$ values and obtain the actual load-deflection curve.

To circumvent this difficulty, Jezek [7] introduced in his approximate solution a "shape factor"* the purpose of which was to relate the relative stiffness of the section in question to that of the rectangle. This type of solution lends itself readily to the formulation of design equations* which are comparatively easy to solve.

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* Page 45, Ref. (1).

One notably different type of solution to the eccentric instability problem was presented by Ros [11], [12], who reasoned that if the rate of increase in effective, internal, resistance eccentricity, M/P, is the same as the rate of increase in the external applied centerline eccentricity, then instability will occur. Using this condition he obtained a graphical solution for the rectangular section problem by expressing the internal eccentricity as a function of the sum of the outer fiber strains (a measure of curvature).

It has been shown that the presence of residual stress is of definite influence on the strength of columns [6], [9]. This makes solution by the above methods even more complicated.

To overcome the difficulties encountered in each of these solutions, the method presented in the next section of this paper was developed. The mechanics are similar to those presented by Ros [12] for the rectangular section.

IV. THE PROPOSED VIRTUAL

DISPLACEMENT METHOD

OF SOLUTION

Bending instability can be considered as a breakdown
in the resistance the member offers to further bending. Since
bending is synonymous with moment application, it should be
possible, then, to determine for a given loading condition if
the section offers sufficient resistance to further bending by
displacing the loaded specimen a virtual amount and noting the
change in external applied moment at the most critical section
and the internal moment of resistance offered at that same sec-
tion. If the increase in external moment is greater than that
which the internal stress pattern can supply, the member fails
since resisting moment cannot "keep up" with applied moments.
When these two ratios of increase are the same a condition of
indifference exists. The necessary condition for instability
can then be stated:

> When external moment at the most critical section
> increases at a rate equal to the rate of increase
> in resistance moment supplied by the internal stress
> system at that same section, the member becomes un-
> stable.

Writing in the form of an equation

$$\left.\frac{\Delta M}{\Delta \phi}\right|_{ext} = \left.\frac{\Delta M}{\Delta \phi}\right|_{int} \tag{5}$$

where $\Delta M/\Delta \phi$ is the rate of change in moment with respect to
curvature, ϕ.

For illustration consider the axially loaded column shown in Fig. 2 assumed, for ease of solution, to buckle in the elastic stress range. As the thrust, P, is increased the potential of the external system to subject the various cross-sections along the member to bending moment in addition to applied thrust increase as the thrust, i.e.

$$\Delta M_{ext} = P\Delta y \qquad (6)$$

Whereas, internally (as shown in Fig. 6) the resistance to this possible external moment is constant at a value defined by the equation

$$\Delta M_{int} = EI\Delta \phi \qquad (7)$$

ΔM_{ext} and ΔM_{int} represent changes in moment when going from the straight (original) to the deflected position (a virtual displacement away).

Assuming the deflection curve is of the form

$$\Delta y = \Delta y_0 \quad \cos \frac{\pi x}{L},$$

then

$$\Delta \phi_0 = \frac{\partial^2}{\partial x^2} (\Delta y_0 \quad \cos \frac{\pi x}{L}) = \Delta y_0 \frac{\pi^2}{L^2}$$

or

$$\Delta y_0 = \Delta \phi_0 \frac{L^2}{\pi^2}, \qquad (8)$$

where the zero subscript denotes conditions at the center of the column (the most critically deformed section). Substituting Eq. 8 in Eq. 6,

$$\Delta M_o \Big|_{ext} = P \frac{L^2}{\pi^2} \Delta \phi_o$$

Writing this expression and Eq. (7) as the change in moment divided by the virtual curvature to obtain the rate of change in moment with respect to ϕ_o gives

$$\frac{\Delta M_o}{\Delta \phi_o} \Big|_{ext} = P \frac{L^2}{\pi^2}$$

$$\frac{\Delta M_o}{\Delta \phi_o} \Big|_{int} = EI$$

(9)

Each of Eqs. 9 can be plotted as function of P and $(\Delta M / \Delta \phi)$, (see Fig. 7). There is reached a point at which the rate of external potentiality of moment application $(\Delta M / \Delta \phi \big|_{ext})$ exactly equals the rate of internal resistance to lateral deformation $(\Delta M / \Delta \phi \big|_{int})$. At that point the member can assume either the straight or the deflected position, and therefore a condition of instability exists. Above that critical value of load, P_{cr}, it is impossible for the internal change in resistance moment to "keep up" with the potential change in externally applied moments and therefore the member collapses. To evaluate the magnitude of the load corresponding to this critical condition, use the instability condition of Eq. 5,

$$\frac{\Delta M_o}{\Delta \phi_o} \Big|_{ext} = \frac{\Delta M_o}{\Delta \phi_o} \Big|_{int}$$

or

$$P_{cr} \frac{L^2}{\pi^2} = EI \, ,$$

which when solved for P_{cr} gives

$$P_{cr} = \frac{\pi^2 E I}{L^2} \ ,$$

the Euler buckling load for a straight, axially loaded, elastic column.

Had the thrust been applied eccentrically and/or a lateral load superimposed thereby introducing bending from the start, the following would have resulted. In the original deflected equilibrium position (see Fig. 8), prior to subjecting the member to the virtual disturbance,

$$M_0 \Big|_{ext} = P \, (e + y_0) + \frac{RL}{4}$$

whereas, after the $\triangle y_0$ change

$$M_0' \Big|_{ext} = P \, (e + y_0 + \triangle y_0) + \frac{RL}{4}$$

The increase in external moment caused by this disturbance is equal to the difference between the final and original moment values, i.e.

$$\triangle M_0 \Big|_{ext} = M_0' \Big|_{ext} - M_0 \Big|_{ext}$$

or

$$\triangle M_0 \Big|_{ext} = P \, (\triangle y_0)$$

It should be noted that this is the same as for the axially

loaded column problem shown previously[*].

Internally, again considering elastic conditions for simplification, (see Fig. 9), resistance to bending is constant at a value

$$\frac{\Delta M}{\Delta \phi}\bigg|_{int} = EI$$

Solution to this problem then is the same as for the pure axial load case, instability occurring at the Euler buckling load. However from the discussion of Fig. 3 and from the bending resistance plot of Fig. 10 it is obvious that this condition can never be reached and satisfy the assumption of an elastic internal stress situation. To solve the problem then, it is first necessary to determine internal bending stiffness above the elastic limit.

Bending Stiffness Above the Elastic Limit

As was discussed in Section III (Fig. 4) the moment-curvature relation above the elastic limit is a function of axial thrust as well as cross-sectional dimensions and mechanical properties of the material. Thus to be able to use the resulting $M-\phi$ curves for integration it is necessary to first determine compatible moment and thrust quantities corresponding to given ϕ values. This can become a major problem for anything other than simple shapes; however, by modifying the problem under study

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[*] Generalizing, it can be shown that if the member is pin-ended, regardless of lateral loads, inelastic or elastic action, etc.,

$$\Delta M\bigg|_{ext} = P(\Delta y).$$

For the case of restrained members, a factor taking into account the end condition will need to be added to the right hand side of the equation.

to one where axial thrust is held constant and moments increased
to collapse, a relatively simple, semigraphical solution is
available [9] whereby compatible M, P, $\emptyset$ values can be determined
for any type of cross-section having one axis of symmetry.

Two sets of curves resulting from that previous study
are here shown as Fig. 11 and Fig. 12 and are the M-$\emptyset$ plots (at
constant P-values) for the 8WF31 (strong axis bending) and for
the rectangular section respectively. These plots have been
made non-dimensional by dividing the moment value by M_y, the
moment corresponding to initial yield in the absence of axial
thrust, and the curvature value by $\emptyset_y$, the curvature correspond-
ing to M_y.

To have these results in a form suitable for direct
use in Eq. 5 it is necessary to further develop them by obtain-
ing tangent values at various points along each curve. These
tangent values will be of the form

$$\frac{\Delta M}{\Delta \emptyset} \cdot \frac{\emptyset_y}{M_y}$$

which can be considered as a measure of the desired internal
stiffness term, $\Delta M/\Delta \emptyset \big|_{int}$, in Eq. 5. The tangent curves obtained
from Figs. 11 and 12 are shown in Figs. 13 and 14. In each case
the tangent values are plotted as a function of the two coordinates
M/M_y and $\emptyset/\emptyset_y$.

If a cosine deflection curve is assumed and the exter-
nal rate of change substituted in

$$\frac{\Delta M}{\Delta \emptyset} \cdot \frac{\emptyset_y}{M_y}\bigg|_{ext} = \frac{\Delta M}{\Delta \emptyset} \cdot \frac{\emptyset_y}{M_y}\bigg|_{int}$$

then,

$$\frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y}\bigg|_{int} = \frac{P_{cr}}{P_e} , \qquad\qquad (10)$$

since

$$\frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y}\bigg|_{ext} = \frac{PL^2}{\pi^2} \cdot \frac{\phi_y}{M_y} = \frac{PL^2}{\pi^2} \cdot \frac{1}{EI} = \frac{P}{P_e}$$

In these equations, P_e is the Euler buckling load for the member of length L.

V. APPLICATIONS

For any member subjected to both end thrust and bending moments, the total moment at any section is composed of two parts: one is due to the axial thrust times the deflection; the other is independent of deflection (e.g. end moments, lateral loads, etc.). That part of the centerline moment which is independent of deflection is noted throughout the remainder of this discussion as "m_0".

If it is assumed that the moment producing loads are symmetrically placed on the structure and that the deflection curve can be approximated by the equation

$$y = y_0 \cos \frac{\pi x}{L} \qquad\qquad (11)$$

the external rate of increase in centerline moment with respect to curvature will be given by the equation

$$\left.\frac{\Delta M}{\Delta \emptyset}\right|_{ext} = P \frac{L^2}{\pi^2}$$

However, to have this in the non-dimensional form suitable for use in Figs. 13 and 14, it is necessary to multiply both sides of the equation by 1/EI. This then, as was shown in the last section, gives

$$\left.\frac{\Delta M}{\Delta \emptyset} \cdot \frac{\emptyset_y}{M_y}\right|_{int} = \frac{P}{P_e} = \left(\frac{P}{P_y}\right) \left(\frac{\sigma_y}{\pi^2 E}\right) \left(\frac{L}{r}\right)^2 \qquad (12)$$

where P_e is the Euler buckling load for the pin-ended member.

Since a cosine deflection curve (Eq. 11) was assumed,
the centerline deflection can be expressed as

$$y_0 = \left(\frac{\phi_0}{\phi_y}\right) \frac{L^2}{\pi^2} \phi_y \qquad\qquad (11a)$$

Substituting Eq. (11a) in the following expression for m_0

$$\frac{m_0}{M_y} = \frac{M_0}{M_y} - \frac{P y_0}{M_y}, \qquad\qquad (13)$$

$$\frac{m_0}{M_y} = \frac{M_0}{M_y} - \left(\frac{P}{P_e}\right)\left(\frac{\phi_0}{\phi_y}\right) \qquad\qquad (14)$$

It should here be noted in the derivation of Eq. 14
it was not necessary to consider the explicit form of the moment
producing loads; only the axial thrust, an assumed deflection
curve and the length of the member was used. Eq. (14) then is
the solution to the general inelastic instability problem for
members, which due to applied loads, deform symmetrically ac-
cording to Eq. (11).

To illustrate the procedure, consider the following
numerical example:

Given an 8WF31 section, $L/r = 80$, subjected to a
constant axial thrust value of $P/P_y = 0.4$; determine
the critical values of various "moment producing,
loading components". (See Fig. 15.)

For this problem, according to Eq. (12), $\Delta M/\Delta\phi \cdot \phi_y/M_y\big|_{int}$
$= (0.346)$, which from Figs. 13 and 14 give the critical centerline

moment* and corresponding curvature* values as

$\frac{\Delta M}{\Delta \emptyset} \cdot \frac{\emptyset_y}{M_y}$		Strong Axis Bending	Weak Axis Bending (Rectangle)
0.346	$\frac{M_0}{M_y}$	0.624	0.904
	$\frac{\emptyset_0}{\emptyset_y}$	0.653	1.105

Substitution of these quantities in Eq. 14 gives the following values for m_0/M_y

	Strong Axis Bending	Weak Axis Bending (Rectangle)
$(\frac{m_0}{M_y})_{cr}$	0.398	0.522

Since in Fig. 15 the moment at the center of the column exclusive of thrust times deflection is equal to

$$m_0 = (M_e + \frac{RL}{4}),$$

the critical value of the moment producing loads will be given by the following expressions:

$$(\frac{m_0}{M_y})_{cr} = \frac{(M_e + RL/4)}{M_y}_{cr} = 0.398 \text{ (for strong axis bending)}$$
$$= 0.522 \text{ (for weak axis bending)}$$

- -

* These values have been circled in Figs. 13 and 14.

Had other loading conditions been imposed, a similar procedure would have been used. The expressions for a few typical problems are shown in Table I.

In each of these problems, the quantity of primary importance is m_0/M_y. To facilitate solution of other problems where length and P/P_y values are different from this one, values of m_0/M_y have been obtained for both strong axis and weak axis bending in a range of slenderness from $0 \leq L/r \leq 120$. The plots resulting from these calculations are shown in Figs. 16 and 17. They are applicable only to those cases of load which produce single curvature deformation that can be approximated by a "full" cosine curve.

Figs. 16 and 17 can be used to solve three basically different types of beam-column problems.

1. Given a constant axial thrust, determine the maximum lateral load and/or end moment the member can carry.

2. Given the lateral loads and/or end moments, determine the maximum axial thrust to which the member can be subjected.

3. Given an eccentricity, determine the maximum axial thrust.

The first two are solvable directly from the figures. For the third case, however, it is first necessary to "lay-off" a line which described the relationship between m_0 and P.* The point of intersection of this "e-line" with the given slenderness ratio will determine the critical axial thrust.

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* This line would correspond to the equation $m_0 = Pe$ or
$$(m_0/M_y) = (P/P_y)(ec/r^2).$$

VI. D I S C U S S I O N   A N D   S U M M A R Y

In the foregoing presentation a statical method was
used for determining conditions of indifference. This method
implies that an equilibrium position is known and it is desired
to determine if that position is stable or unstable. Solution
was achieved by comparing the rate of change in external moment
to the rate of change in internal resistance moment for a given
virtual displacement of the loaded specimen. If the internal
moment increased at a rate greater than the external moment, the
specimen was stable. If the rate of change in external was the
larger, the member was unstable. For the case where the rates of
change were equal, a condition of indifference (incipient in-
stability) existed.

a. Influence of Assumed Deflected Configuration

For ease of solution, it was assumed throughout this
report that the over-all behavior of the member could be pre-
dicted with sufficient accuracy by assuming a deflection curve,
the magnitude of which would be governed by the strain distri-
bution pattern at the most highly deformed cross-section. This
simplification eliminated the need of minimizing integrals and
resulted in a direct determination of an end moment, axial thrust
and critical length corresponding to a given stress distribution
pattern. However, when using such a procedure, a certain amount
of error is introduced, the member essentially being forced into
an unnatural configuration. To determine the magnitude of this
error, the following three comparisons will be made.

a. Comparison of strength results obtained for several different curves.

b. Comparison of the deflection and strength results of each of these curves with individual solutions obtained by numerical integration, and

c. Comparison with previous published solutions; namely, Chwalla's[1], Jezek's[1] and Timoshenko's.[14]

In the first comparison, the three types of configuration shown in Fig. 18 are considered. The first of these, that of a "full-cosine" curve, was used in the development of the general method of solution (Section III) and also in the preceding section on "Applications". The second is a parabola, and the third is a "partial cosine" curve of the form suggested by Westergaard and Osgood [16]. In the first two cases the moment boundary condition is violated.

Since each of these assumed curves involves one parameter, y_0, the previously derived instability condition,

$$\left.\frac{\Delta M}{\Delta \emptyset}\right|_{ext} = \left.\frac{\Delta M}{\Delta \emptyset}\right|_{int} ,$$

can be directly applied. For the first two cases,

a. "Full Cosine" curve

$$\left.\frac{\Delta M}{\Delta \emptyset}\right|_{int} = P \frac{L^2}{\pi^2} , \quad \text{and} \tag{15}$$

b. "Parabolic" curve

$$\left.\frac{\Delta M}{\Delta \emptyset}\right|_{int} = P \frac{L^2}{8} \tag{16}$$

For the "Partial Cosine" curve (assumption "c"), it is necessary to further develop $\Delta M/\Delta \phi|_{ext}$ so that the ends of the "L" length are held at a given deflection, e, rather than maintaining a hypothetical "l"-length member. (See Fig. 19.) If such a procedure is carried out a complex equation containing functions of Δy_0 and Δl and cosines thereof is obtained which cannot be solved explicitly for Δy_0 in terms of $\Delta \phi_0$. However, by considering only the first two terms of a cosine series, the expression can be reduced and results in the following equation:

$$\left.\frac{\Delta M}{\Delta \phi}\right|_{int} = P \frac{L^2}{\pi^2} \tag{17}$$

Even though this is the same external rate of moment increase as found for case "a", the solutions for members "a" and "c" are different, the M_e values being determined by different equations.

   a. "Full Cosine" curve

$$M_e = M_0 - P \phi_0 \frac{L^2}{\pi^2} \tag{18}$$

   b. "Partial Cosine" curve

$$M_e = M_0 \cos \frac{\pi}{2} \frac{L}{l} \tag{19}$$

l is determined by considering the full value of M_0 and ϕ_0 at the center section, i.e.

$$l^2 = \frac{M_0}{\phi_0} \cdot \frac{\pi^2}{P} \tag{20}$$

$$\left(\frac{l}{r}\right)^2 = \frac{M_0/M_y}{(P/P_y)\,(\sigma_y/\pi^2 E)} \tag{20a}$$

The comparison between these solutions is illustrated
in Figs. 20 and 21. Here is presented a series of interaction
curves for various length ratios for the 8WF31 member bent about
both its strong and weak axes. The "full cosine" deflection
curve gave the highest strength values. The "parabola" gave
the lowest prediction.

A recent paper by John Clark [4] discusses, in part,
this same question. Therein he not only investigates the appli-
cability of the "partial cosine" curve but also a much more com-
plicated three parameter deflection equation. Comparison with
test results is also shown in that report. In general, however,
his results indicate that the increase in accuracy obtained by
using the more complicated deflection expression does not warrant
the added work required to obtain a solution.

The second means of comparison, numerical integration,
was used as a check in a previous report [9] and one figure based
on that work is here included to indicate the observed trends.
(See Fig. 22.) There is difference between the load deflection
curves obtained by the various methods; however, the ultimate
carrying capacities of the members are in comparatively good
agreement for these examples.

The third comparison is that between previous published
solutions and the curves of ultimate strength as predicted by the
"partial cosine" deflection curves presented earlier in this dis-
cussion. Since previous information regarding WF section behavior
is lacking, comparison is made only for the rectangular section;
furthermore, only for two slenderness ratios.

Chwalla's solution is based on a numerical integration procedure and should therefore be considered as being "exact". His derivation is based on (a) a typical German steel with a stress-strain curve having a proportional limit of 27 ksi, and a yield stress level of 34 ksi. Jezek's is an approximation based on (a) a "full cosine" deflection curve and (b) a stress-strain curve of the idealized elastic type (see Fig. 1). Timoshenko's solution also considers (a) the "full cosine" curve but (b) the stress-strain curve has a proportional limit 30 ksi, a yield level of 36 ksi and an ultimate strength of 56 ksi.

Fig. 23 shows the comparison obtained between each of these previous solutions and the "partial cosine" solution of this report. All curves have been adjusted to a yield stress level of 40 ksi. Length adjustment was according to the equation

$$\left(\frac{L}{r}\right)_{adj} = \left(\frac{L}{r}\right) \sqrt{\frac{\sigma_y}{40.0}}$$

In general, Chwalla's solution gives the lowest strength values; Jezek's gives the highest. The partial cosine solution is between these two limits, as is Timoshenko's solution.

b. Elastically Restrained Members

In actual structures, columns are usually restrained at their ends rather than being pin-ended, and a recent paper[2] discusses such a condition. In that study a method was developed for reducing the actual restrained system to one of an equivalent, pin-ended, eccentrically loaded column. Having determined by that

method the equivalent member for a restrained column problem,
the solution and curves presented in this paper directly apply.

c. Residual Stresses

     To include the variable of residual stress, it is
necessary to develop a complementary set of curves for the de-
sired pattern and magnitude of residual stress. This develop-
ment would start from a given M-$\emptyset$ curve [9], which included
the effect of the residual stress pattern, and proceed along
the same lines previously described.

d. Load-Deflection History

     It should be pointed out that no special considera-
tion has been given to the manner in which a condition of in-
stability is approached; that is, the load-deflection history
of the member in question. The presented curves and numerical
work assume that an axial load, smaller than that required for
failure, is first applied and that bending is then increased
from zero to its maximum value. Under this condition there
will be no unloading of previously yielded fibers. The result-
ing critical loads then will be comparable to the reduced modu-
lus load for axially loaded members.

     As was clearly demonstrated, first by Shanley [13]
and later by Duberg and Wilder [5] and others, if an axially
loaded column is considered as an imperfect compression member
with the imperfection approaching zero, the column will start
to bend at the tangent modulus load. Increased thrust beyond
this value will be accompanied by an increasing rate of lateral
deflection, the member reaching its maximum strength at some
point prior to attainment of the reduced modulus load.

Since one limit of the generalized beam-column prob-
lem is the axially loaded column, there will be a certain amount
of error involved in the method of solution developed in this
report. While it is felt that for the assumed stress strain
curve of Fig. 1, the resulting error is quite small, further
extension of this work to materials having curved stress-strain
relations (either as a basic material property or as a result
of residual stress, etc.) should be accompanied by suitable
check calculations to indicate the amount of error involved in
this type of a procedure. The error will be primarily a func-
tion of the initial eccentricity. Other important factors would
be the type of cross-section and the stress-strain relation.

A recent NACA report [17] provided one source of in-
formation for these check calculations. Therein load-deflection
curves are presented (for an idealized WF section column) for
various initial eccentricities and stress-strain curves.

Since a majority of the practical beam-column prob-
lems encountered in the analysis of Civil Engineering type
structures contain moment producing loads greater than that
which could be attributed to accidental imperfections, this
"tangent modulus influence" will more than likely not be a real
problem.

It is interesting to note that the early numerical
work on collapse strength of rectangular section, beam-columns
carried out by Chwalla was based on a typical "German Steel"
type stress-strain relation (linear up to the proportional
limit, parabolic from the proportional limit to the yield stress
level and then horizontal to the point at which strain hardening

commences). Recently [2] this work has been corrected and it
was found that adjustment was needed only when the ratio of the
centerline deflection at maximum load to the depth of the sec-
tion was less than 0.1, a comparatively small deflection value.

e. Local and Lateral Instability

Neither local nor lateral types of instability were
considered in the development presented in this paper. The
considered condition of instability in each case has been that
of excessive bending in the plane of the applied moment. How-
ever, when bending is forced about the major axis of a section
(that condition most frequently encountered in practice); when
there is appreciable difference in bending stiffness between the
two principal axes of the section; and when there is little if
any lateral support along the member, the mode of failure will
more than likely be that due to a combined bending and twisting
action [8]. In practice, however, this will not necessarily be
the case since members will be frequently restrained by bracing,
walls, floors, etc. Where sufficient lateral restraint is pro-
vided, the problem of bending plus twist is nonexistent, and
collapse can be determined by the methods previously described.

Another entirely different type of problem can develop
when the cross-section is composed of connected, relatively thin,
projecting elements. For those types of members, the projecting
elements may individually or collectively deform in certain local
regions and thereby change the original cross-sectional properties
of the member.

Assuming the validity of any of the currently avail-
able inelastic local buckling theories, the maximum strain, ϵ_{max},

that a flange can sustain and remain in the straight position can be predicted. From this can be determined the critical value of $\epsilon_{max}/\epsilon_y$. Using the procedure outlined in Ref. 9 and noting that

$$\frac{\epsilon_{max}}{\epsilon_y} = 1 + 2\alpha \left(\frac{\phi}{\phi_y}\right)$$

(see Fig. 24), the corresponding critical value of ϕ_0/ϕ_y for available, rolled, wide-flange shapes can be determined and is found to be comparatively large ($\phi_0/\phi_y \gtrsim 5.0$). From Fig. 13 it can be seen that such a condition will be realized only at extremely low values of thrust, ($P_{cr}/P_e \lesssim 0.05$).

S U M M A R Y

This report presents a virtual displacement solution to the inelastic instability problem, the method being equally adaptable to elastic, inelastic, concentrically loaded, eccentrically loaded and/or laterally loaded columns.

The essence of the solution is that of applying a virtual curvature change to the loaded member and equating the rate of change in external moment at the most highly stressed section to the internal stiffness resulting from that curvature change. For ease of solution, deflection curves of one parameter were assumed causing the system to reduce to collocation at one point. It was shown that the ultimate strength of such members is relatively unaffected by choice of deflection curve, the "full cosine" giving the highest strength prediction and the parabola giving the lowest prediction. (Figs. 20 and 21.) The "partial cosine" solution compared favorably with previously available solutions. (Fig. 23.)

To facilitate solution of problems other than those discussed herein, curves have been presented which ensure rapid determination of critical loading values (Figs. 20 and 21) for any symmetrically deformed pin-ended member.

VII. A C K N O W L E D G M E N T S

VIII. N O M E N C L A T U R E

A	Area of cross-section
b	Width of cross-section
d	Depth of cross-section
e	End eccentricity of axial load application
E	Young's modulus of elasticity
I	Moment of inertia of total cross-section about the principle axis of section under consideration
L	Length of member (distance between pins)
l	Length of hypothetical member (used in "Partial Cosine" deflection curve assumption)
Δl	Change in length of hypothetical member due to a virtual change in centerline curvature
M	Bending moment
M_{int}	Internal bending moment due to stress distribution across section
M_{ext}	External bending moment due to lateral loads, end moment, initial curvature, and axial thrust times deflection
M_y	Bending moment at which yield point is reached in flexure
M_o	Bending moment at the centerline section of the member
m_o	That part of the centerline moment which is independent of deflection
M_e	End applied bending moment
ΔM_o	Change in value of centerline moment due to a virtual change in curvature at that same section
P	Axial thrust
P_y	Axial thrust corresponding to yield stress level over entire section
P_e	Euler buckling load
P_{cr}	Maximum thrust a member can carry

R	Concentrated lateral load
t	Flange thickness (wide-flange section)
u	Deflection parallel to web in undeformed position
v	Deflection perpendicular to web in undeformed position
y	Deflection in plane of moment application
y_o	Centerline deflection
Δy_o	Change in centerline deflection due to a virtual change in centerline curvature
α	Ratio of the depth of yield penetration in compression to total depth of section
ϵ	Unit strain
ϵ_y	Unit strain corresponding to initial yield
$\epsilon_{max.}$	Maximum average unit strain on compression flange of wide-flange section (strong axis bending)
σ	Unit normal stress
σ_y	Lower yield point stress
σ_a	Variable normal stress on tension side of cross-section (positive when in tension)
ϕ	Curvature
ϕ_o	Centerline curvature
$\Delta\phi_o$	Virtual change in centerline curvature

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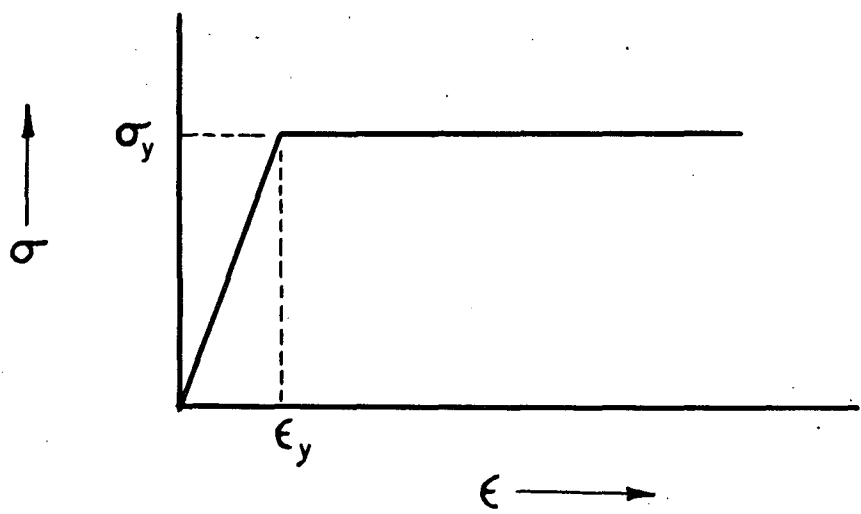
FIG. I   IDEALIZED STRESS-STRAIN RELATION

σ_y

σ

ϵ_y

ϵ

FIG. 2   TYPICAL LOAD-DEFLECTION RELATION FOR AXIALLY LOADED COLUMNS

1.0

$\dfrac{P}{P_e}$

0.5

Typical Elastic Behavior

Typical Inelastic Behavior

y_0

P

L

P

$\left[\dfrac{L^2}{r^2} < \dfrac{\pi^2 E}{\sigma_y}\right]$

$y_0 \longrightarrow$

1.0

$\dfrac{P}{P_e}$

P_{cr}/P_e

0.5

Elastic Behavior

"a"

Typical Inelastic Behavior

e   P

y_0

L

e   P

5.0

10.0

$\dfrac{y_0}{e} \longrightarrow$
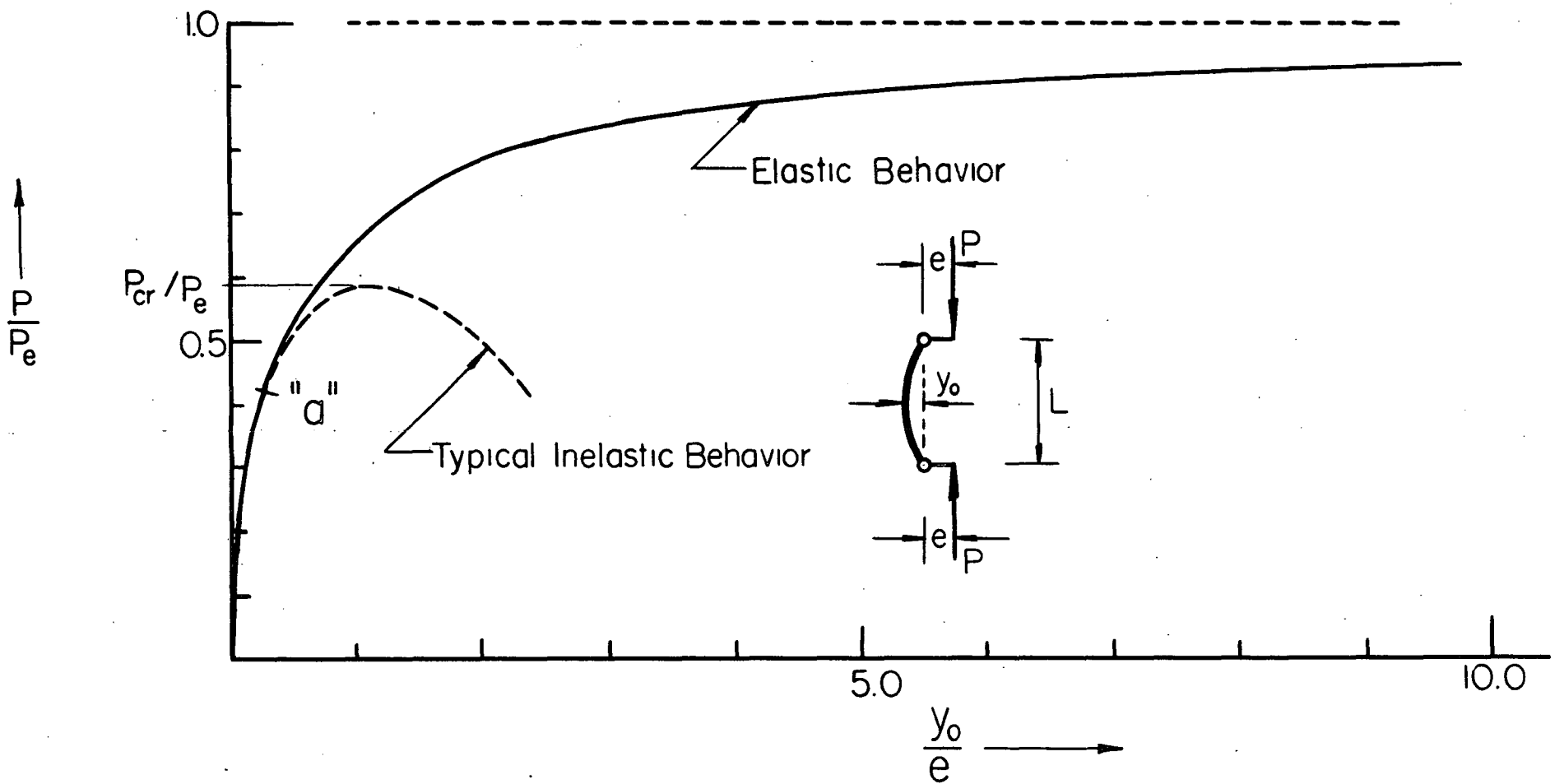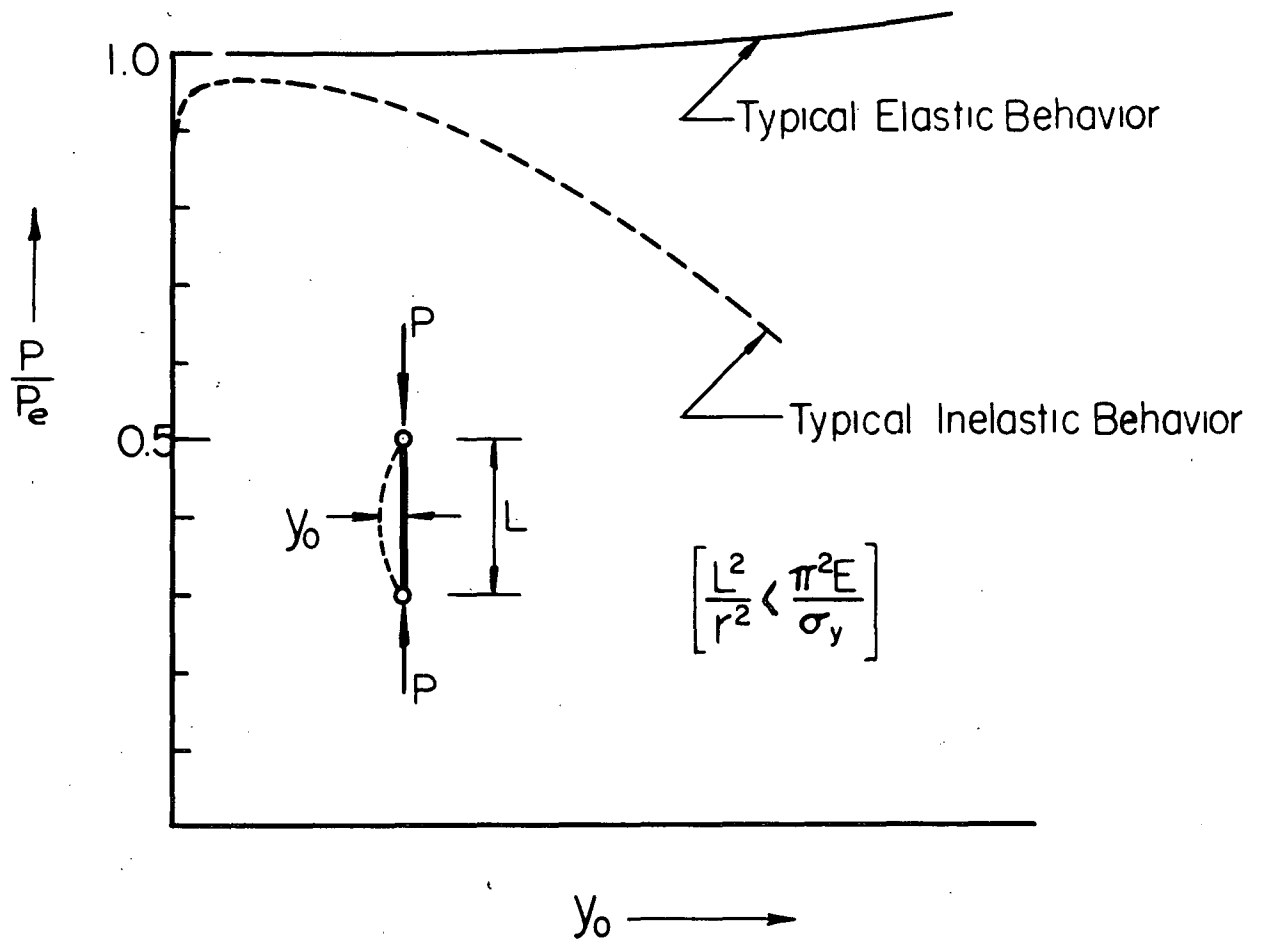
FIG. 3   LOAD-DEFLECTION RELATION FOR ECCENTRICALLY LOADED COLUMNS[9]

Cross-Section   Stress Distribution
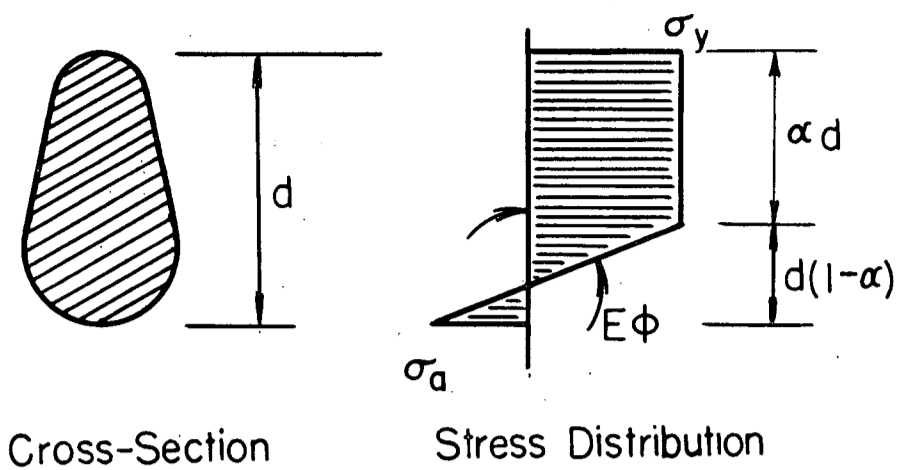
$$M = \int_A \sigma\, y\, dA$$
$$P = \int_A \sigma\, dA$$
$$E\phi = \frac{d\sigma}{dy}\Big|_{neutral\ axis}$$

FIG. 4 BASIC INTERNAL STRESS-
LOAD RELATIONS
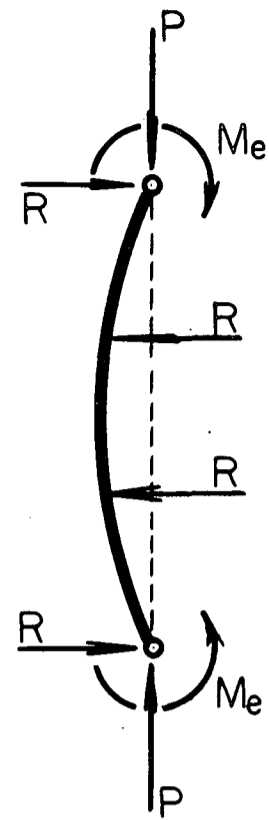
FIG. 5 TYPICAL BEAM-COLUMN
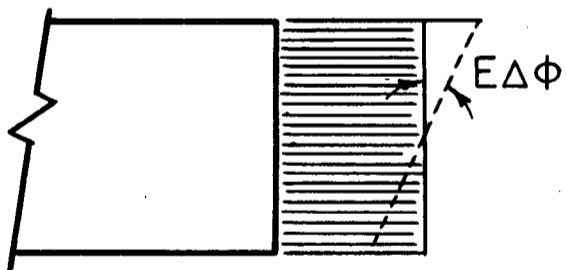
$$\Delta M_{int.} = \int_A \sigma\, y\, dA = EI\Delta\phi$$

FIG. 6 CHANGE IN STRESS
DISTRIBUTION AND RESISTING
MOMENT DUE TO A VIRTUAL
CHANGE IN CURVATURE
(ELASTIC-AXIAL LOAD)

A Measure of the
Resistance to Bending

$$\frac{\Delta M}{\Delta\phi}\Big|_{ext.} = \frac{PL^2}{\pi^2}$$

$$\frac{\Delta M}{\Delta\phi}\Big|_{int.} = EI$$

$$\frac{\Delta M}{\Delta\phi} \longrightarrow$$
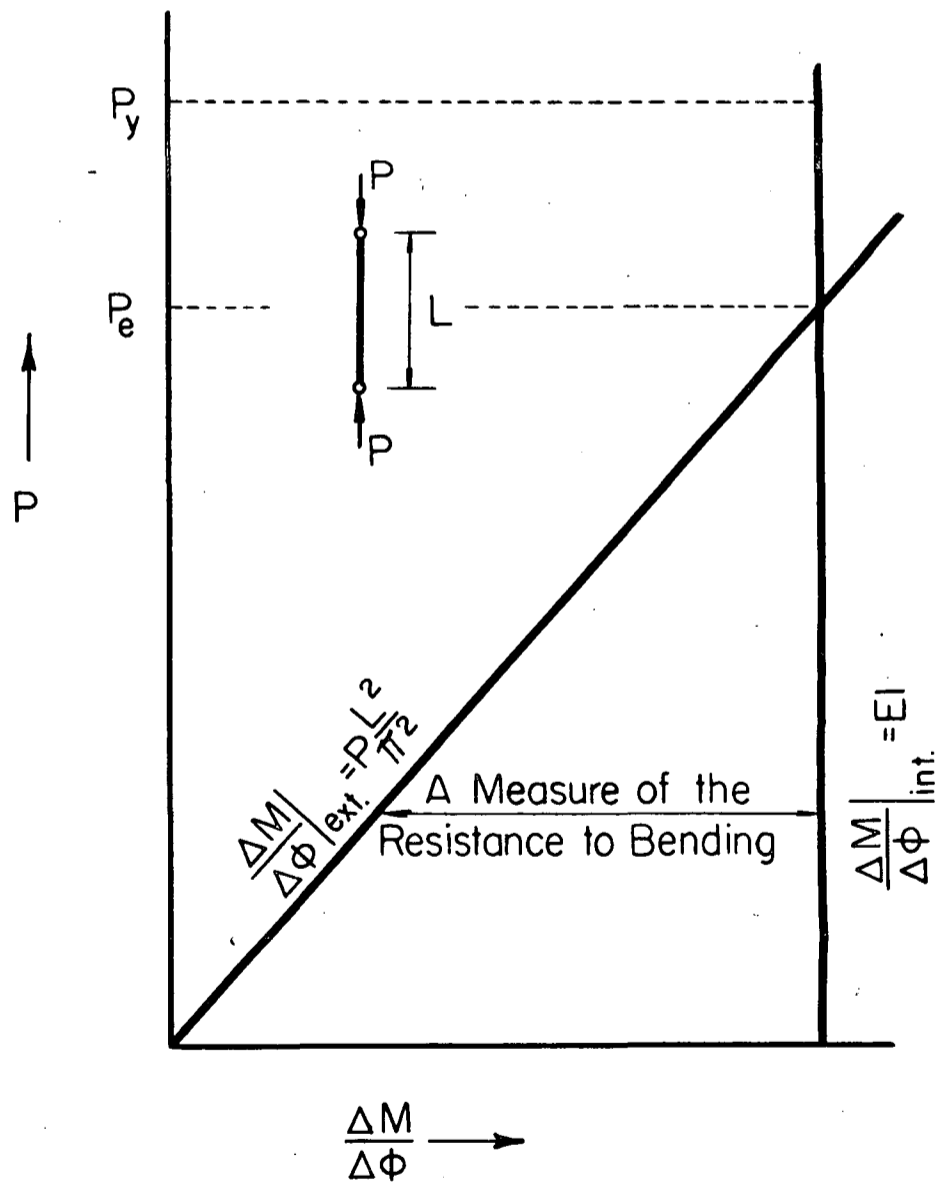
FIG. 7 TYPICAL BENDING RESISTANCE
RELATION FOR AXIALLY LOADED,
ELASTIC COLUMNS
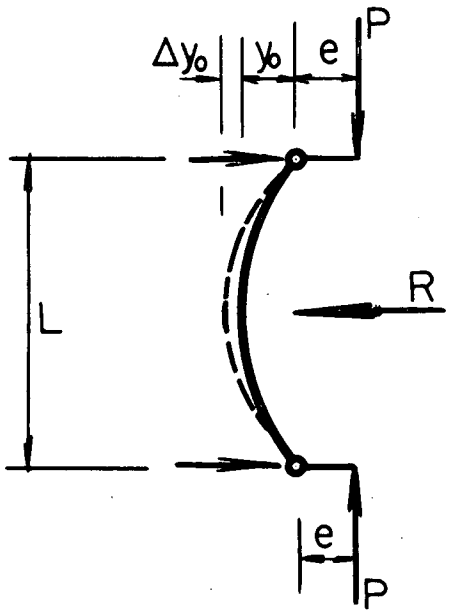
FIG. 8 TYPICAL BEAM-COLUMN
SUBJECTED TO VIRTUAL
DISPLACEMENT
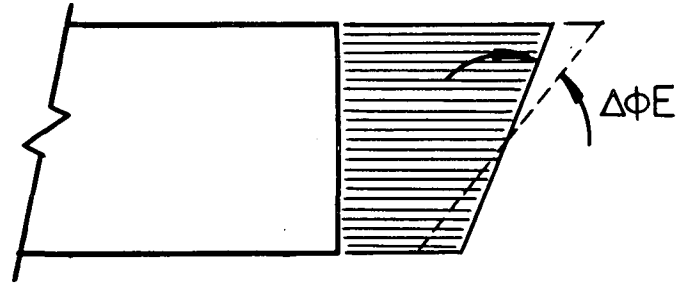
$$\Delta M_{int} = \int_A \sigma \, y \, dA = EI \, \Delta \Phi$$

FIG. 9 CHANGE IN STRESS DISTRIBUTION
AND RESISTING MOMENT DUE TO A
VIRTUAL CHANGE IN CURVATURE
(ELASTIC-ECCENTRIC LOAD)
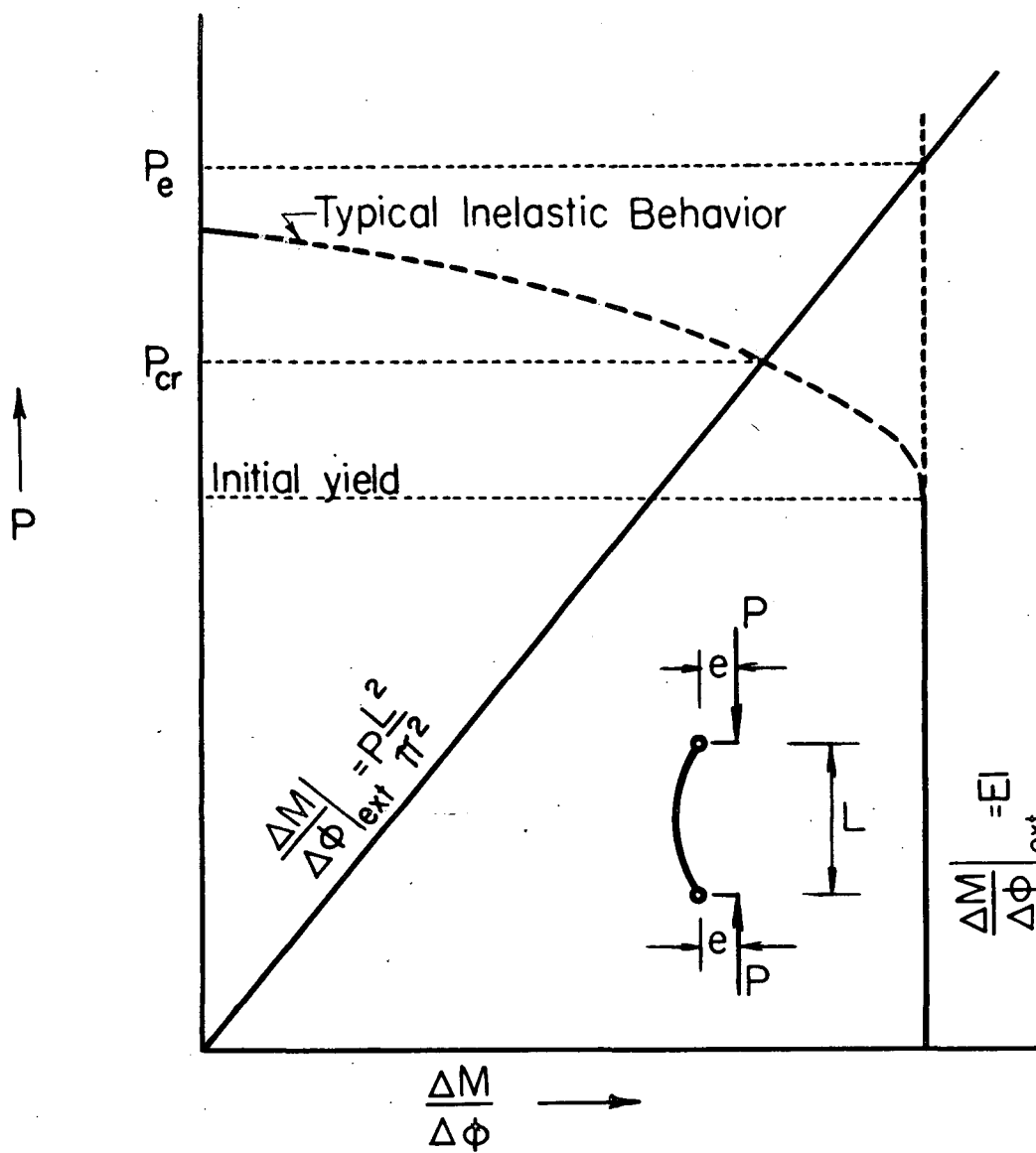
FIG. 10 TYPICAL BENDING RESISTANCE RELATION FOR
ECCENTRICALLY LOADED COLUMNS
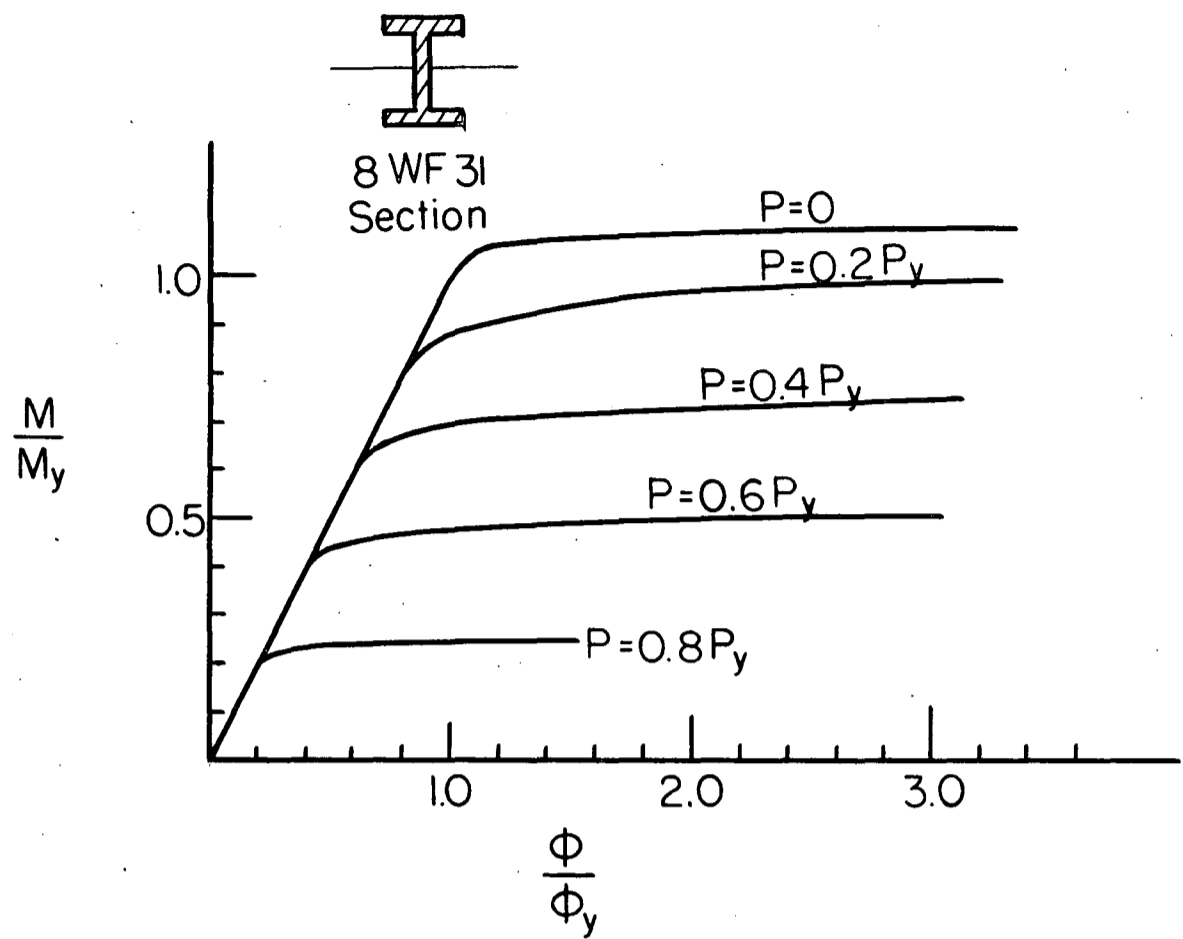
FIG. II MOMENT-CURVATURE RELATIONS FOR THE
8 WF 31 SECTION, STRONG AXIS BENDING [9]
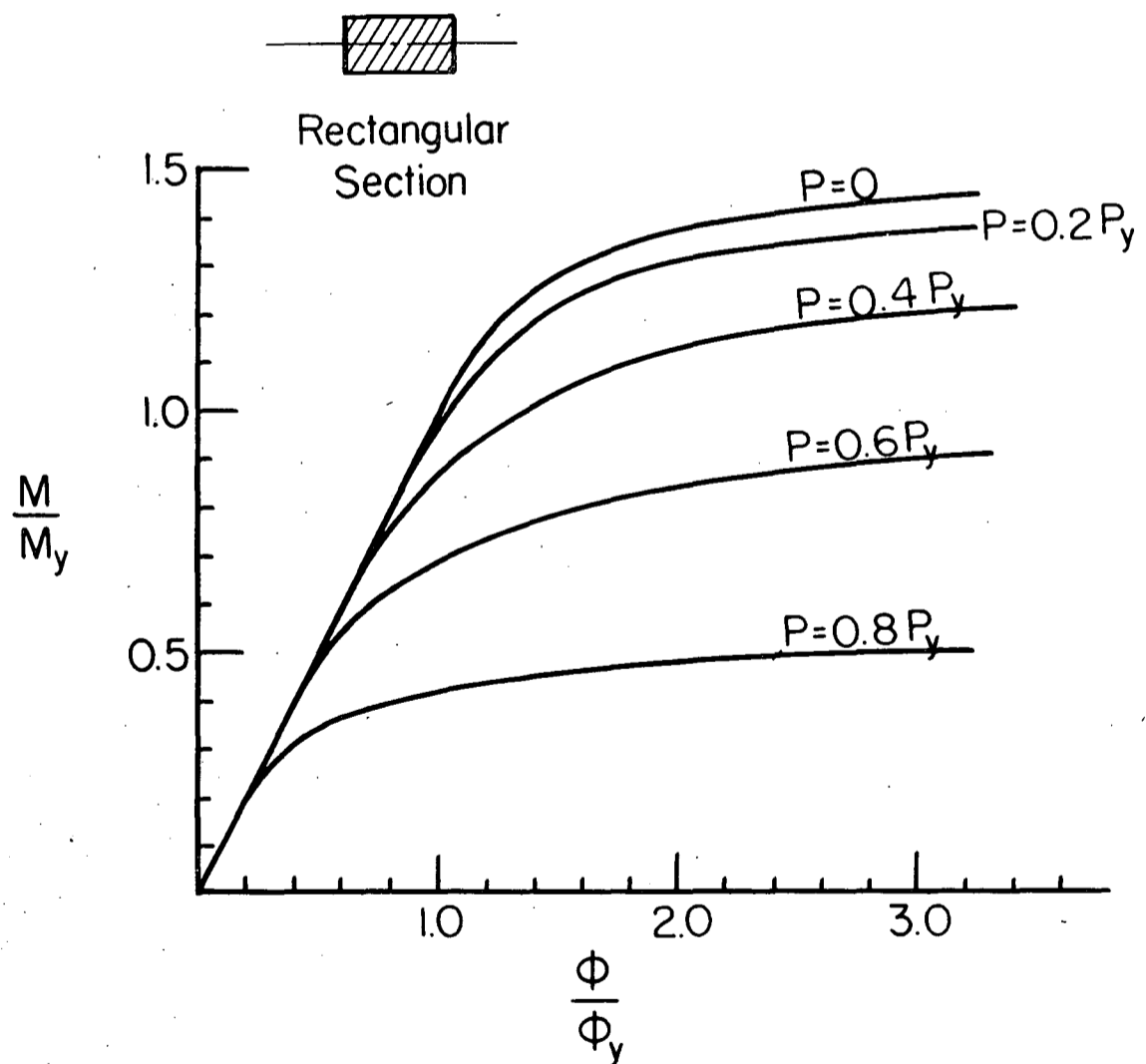
FIG. 12 MOMENT-CURVATURE RELATIONS FOR
THE RECTANGULAR SECTION [9]
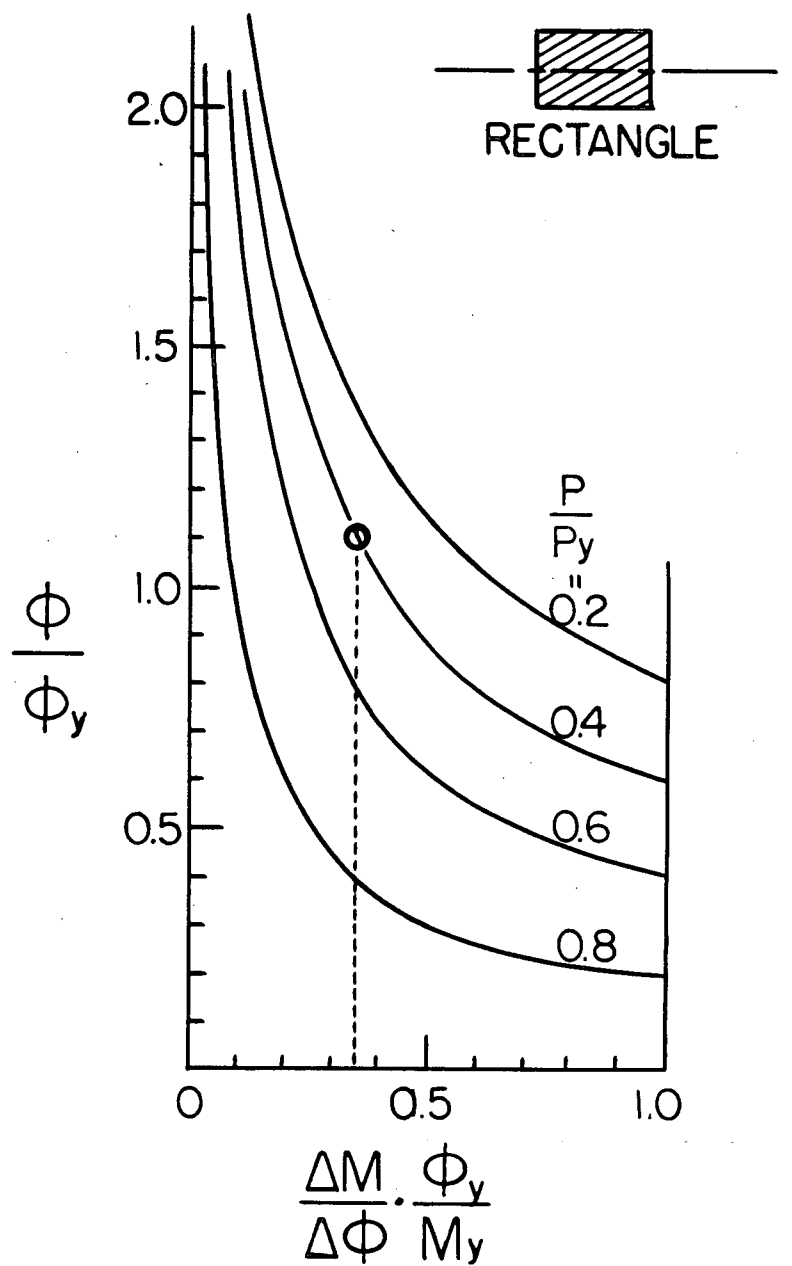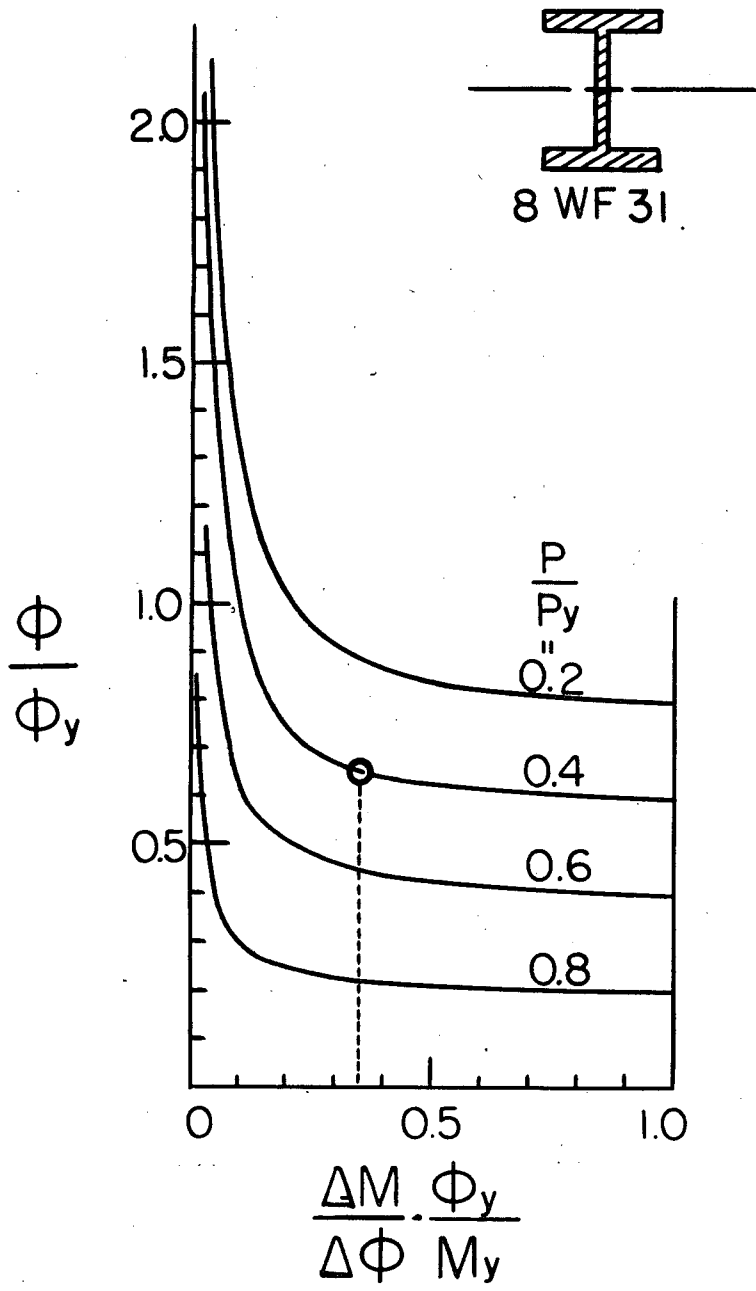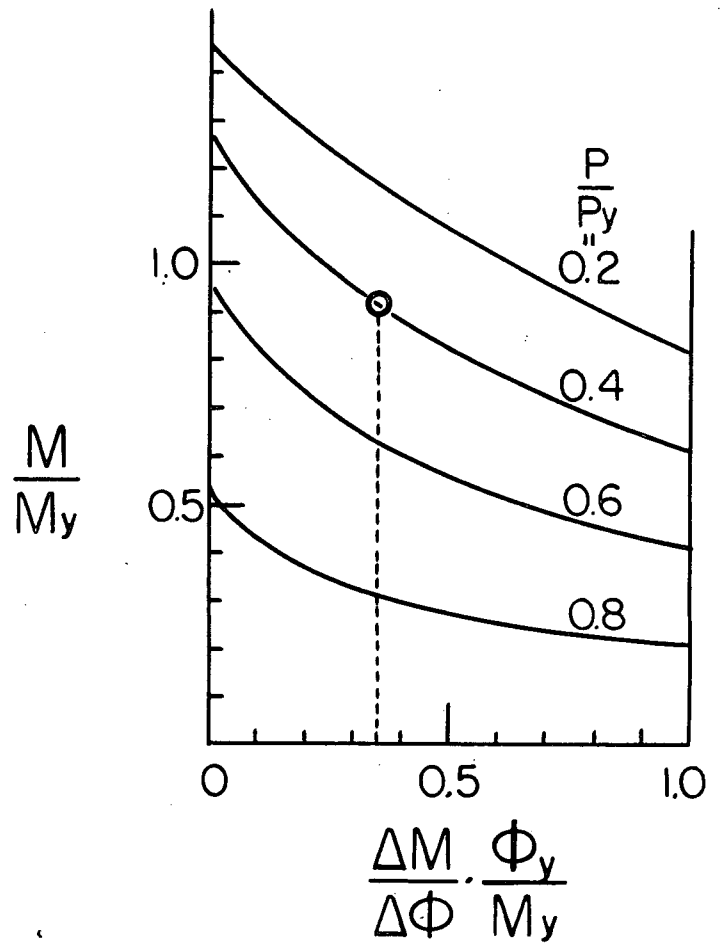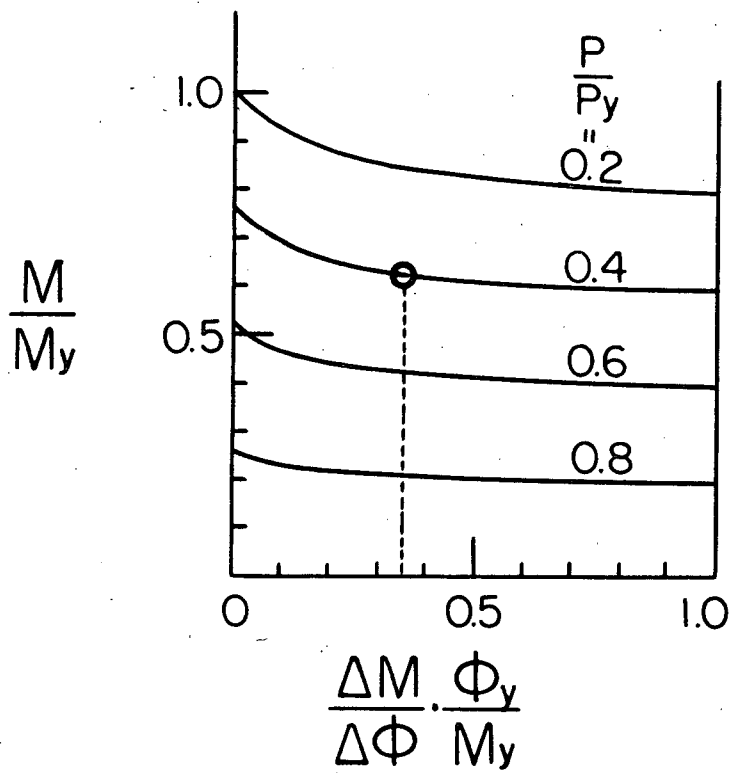
FIG. 13

FIG. 14

INTERNAL RATE OF CHANGE OF MOMENT VERSUS
MOMENT AND CURVATURE FOR CONSTANT AXIAL
THRUST VALUES
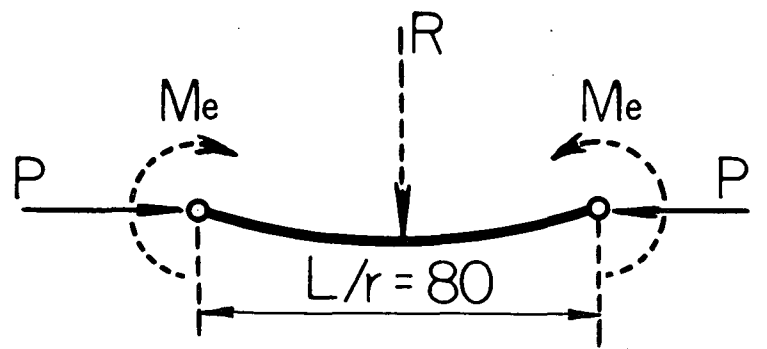
FIG. 15   TYPICAL BEAM-COLUMN
(used in example problem)
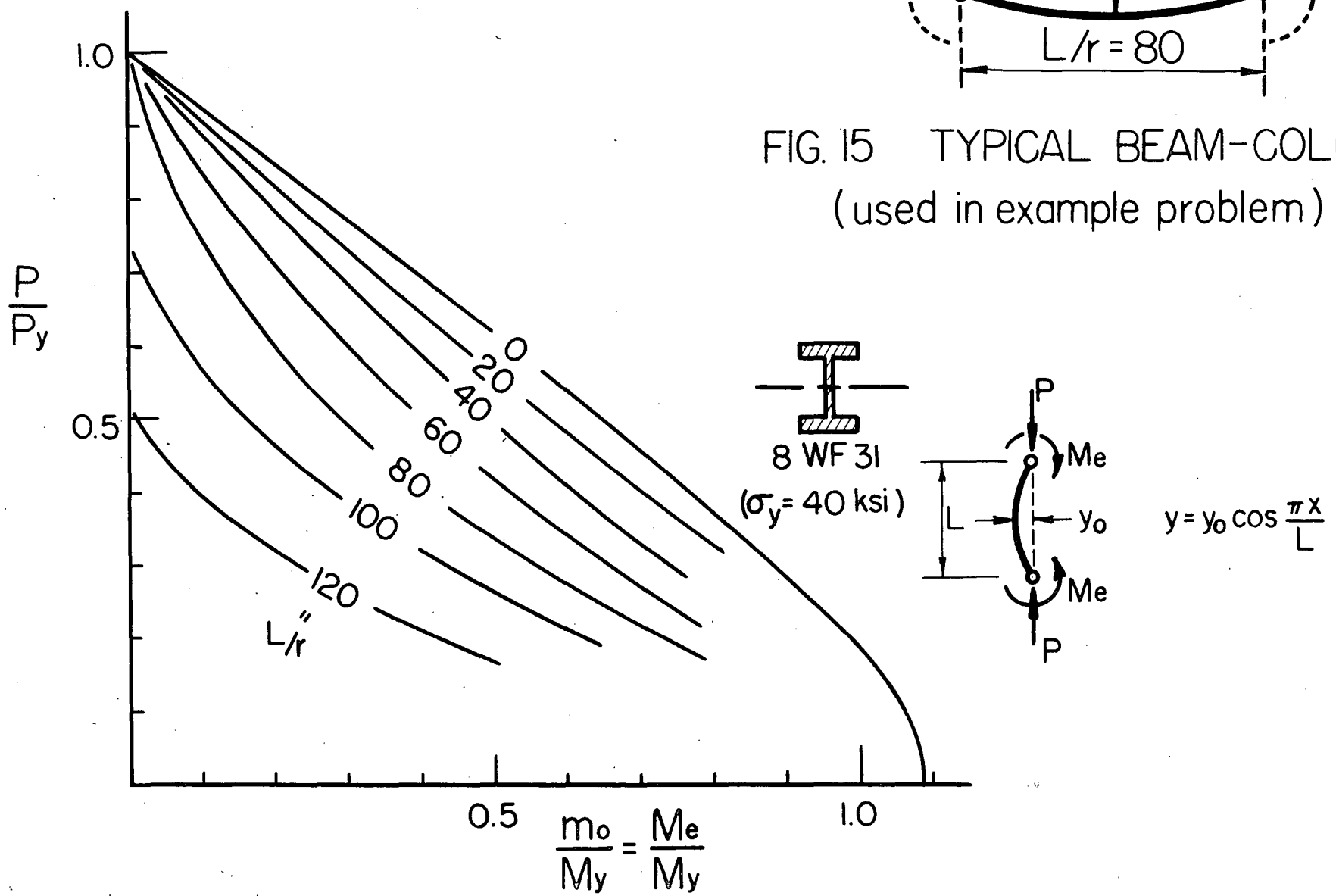
FIG. 16   INTERACTION CURVES FOR 8 WF 31 SECTION
(STRONG AXIS BENDING)
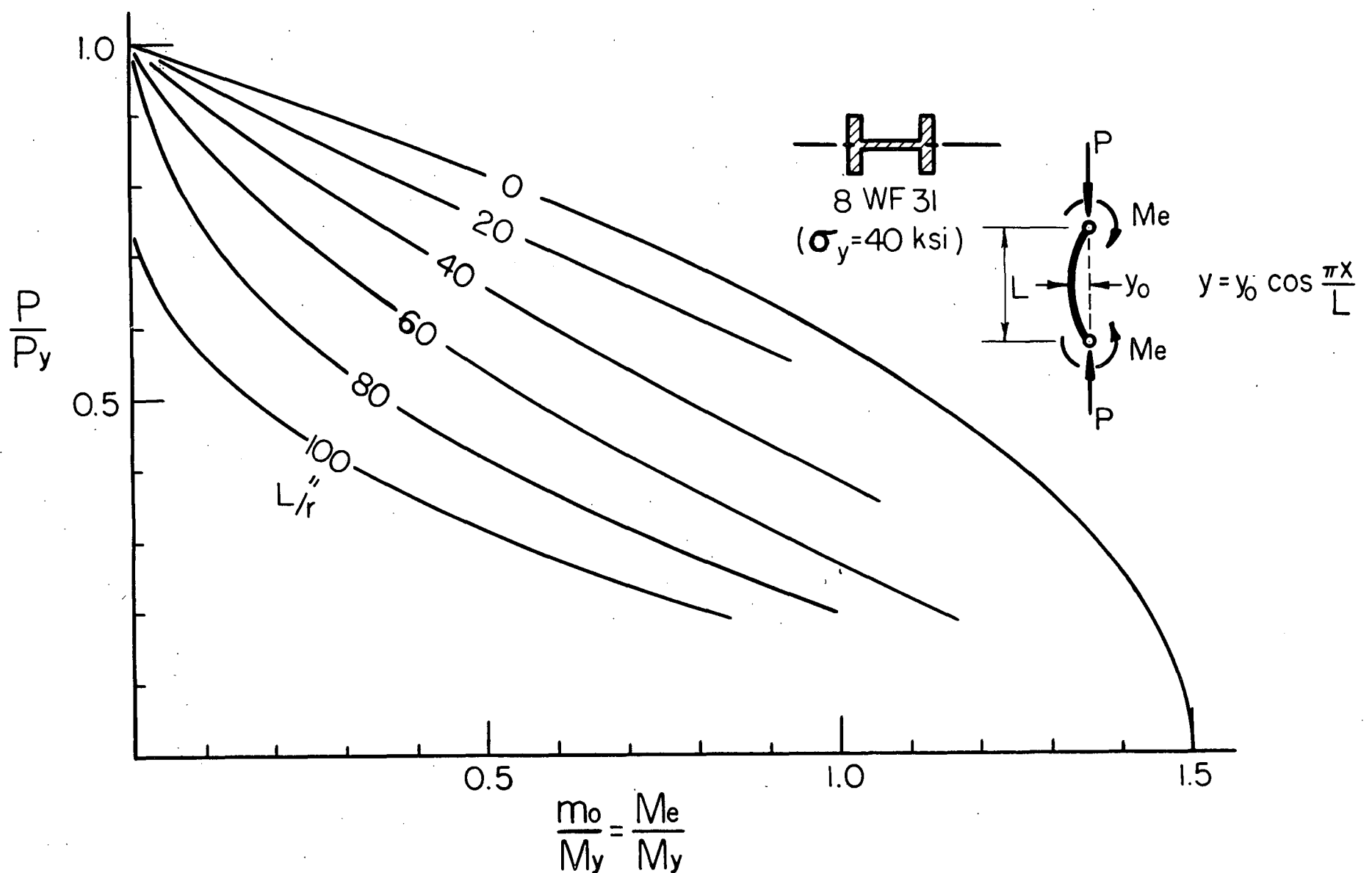
FIG. 17   INTERACTION CURVES FOR 8 WF 31 SECTION
(WEAK AXIS BENDING)

$y = y_0 \cos \pi x/L$ $y = y_0[1 - 4x^2/L^2]$ $y = y_0 \cos \pi x/l$
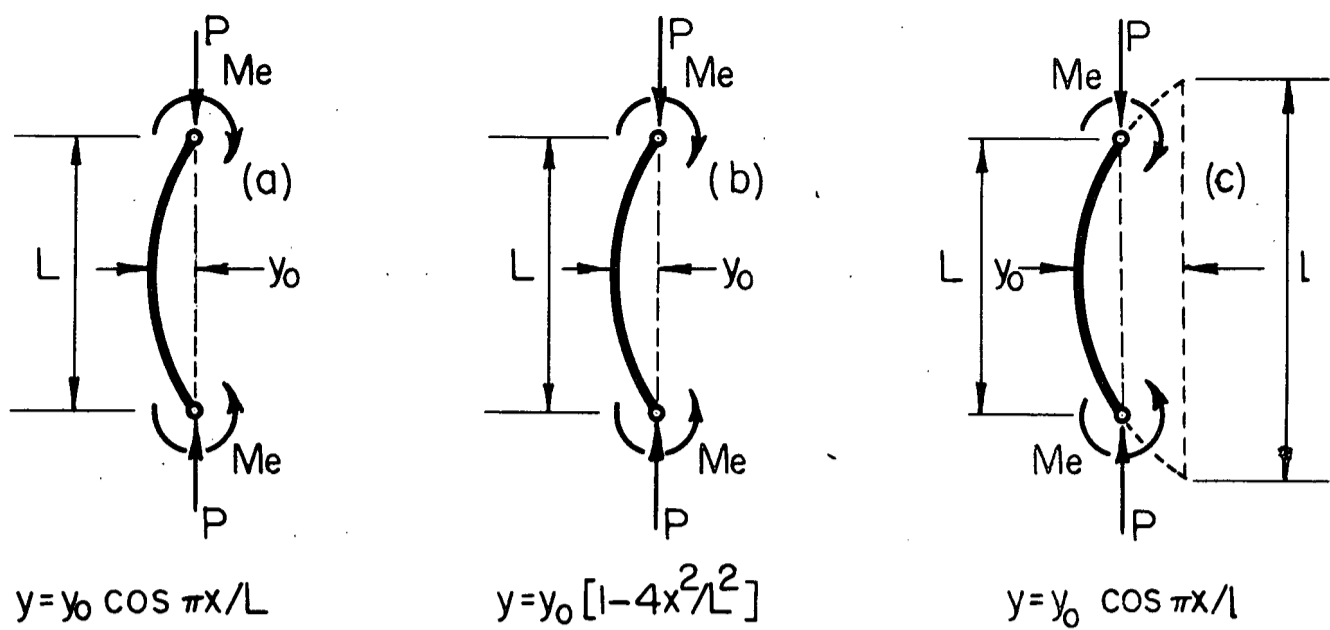
FIG. 18 ASSUMED DEFLECTION CURVES
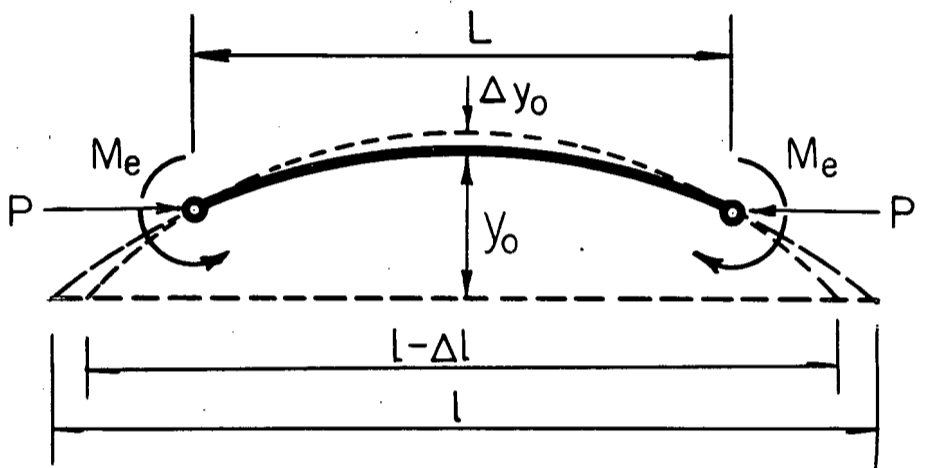
FIG. 19 VIRTUAL DEFLECTION FOR "PARTIAL COSINE" ASSUMPTION

-------- ASSUMED "FULL COSINE" DEFLECTION CURVE

————— ASSUMED "PARTIAL COSINE" DEFLECTION CURVE

—·—·— ASSUMED "PARABOLIC" DEFLECTION CURVE

8 WF 31

$(\sigma_y = 40 \text{ ksi})$

$$\frac{m_0}{M_y} = \frac{M_e}{M_y}$$
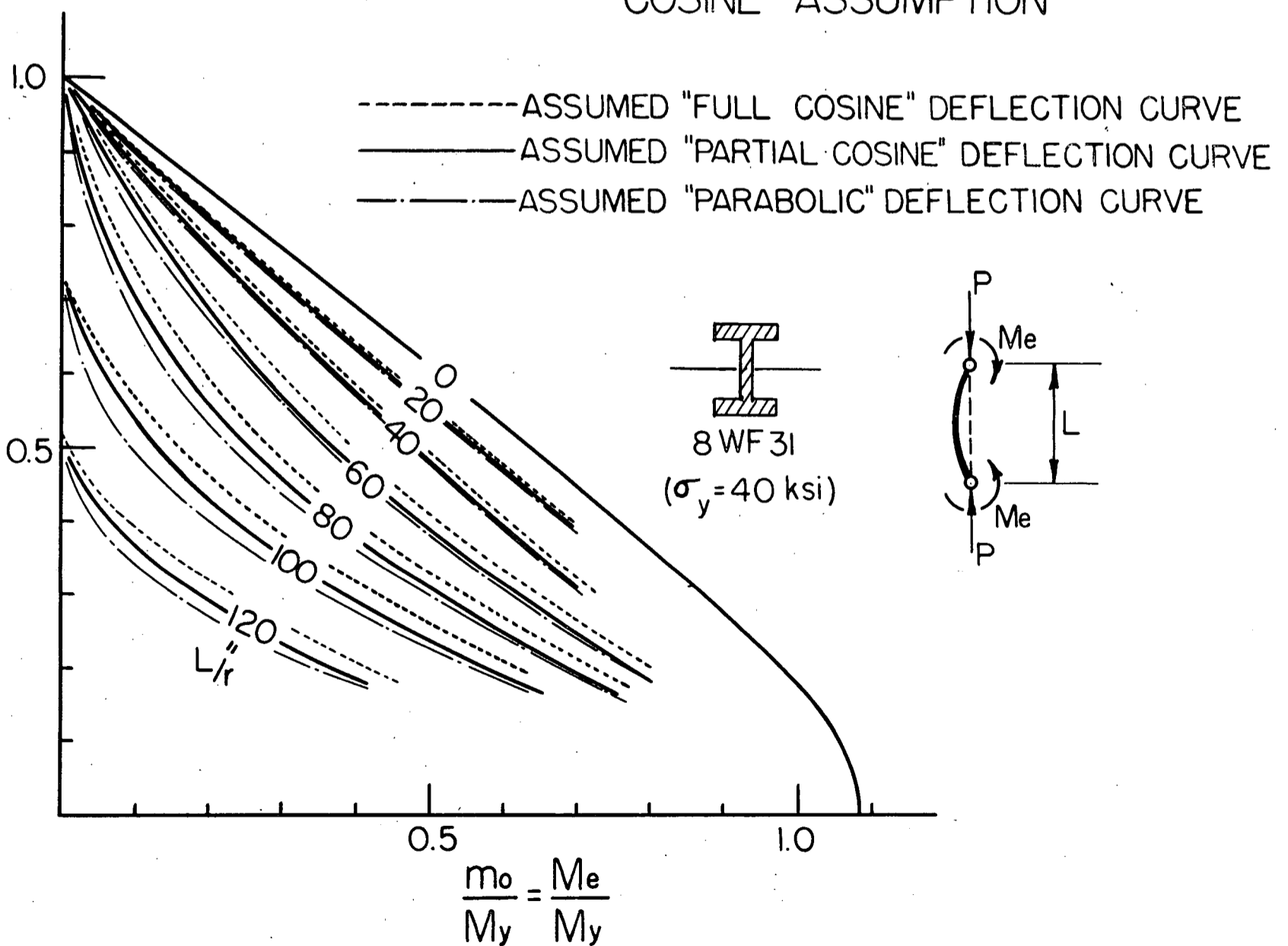
FIG. 20 INTERACTION CURVES SHOWING INFLUENCE OF
DEFLECTION CURVE ASSUMPTION
(STRONG AXIS BENDING)

FIG. 21 INTERACTION CURVES SHOWING INFLUENCE OF DEFLECTION CURVE ASSUMPTION (WEAK AXIS)

ASSUMED "FULL COSINE" CURVE
ASSUMED "PARTIAL COSINE" CURVE
ASSUMED "PARABOLIC" CURVE

$\dfrac{P}{P_y}$

$\dfrac{M_e}{M_y} = \dfrac{m_o}{M_y}$

8 WF 31
($\sigma_y = 40$ ksi)

FIG. 22 LOAD-DEFLECTION CURVES SHOWING INFLUENCE OF DEFLECTION CURVE ASSUMPTION (WEAK AXIS BENDING)

"FULL COSINE"
"PARTIAL COSINE"
"PARABOLA"
NUMERICAL INTEGRATION

$P = 0.12 P_y$
M_e
y_o
$L/r = 41$
M_e
$P = 0.12 P_y$

$P = 0.4 P_y$
M_e
y_o
$L/r = 80$
M_e
$P = 0.4 P_y$

"PARABOLA"
"PARTIAL COSINE"
"FULL COSINE"
NUMERICAL INTEGRATION

$\dfrac{M_e}{M_y}$

y_o (in inches)

FIG. 23 INTERACTION CURVES SHOWING COMPARISON
WITH PREVIOUS SOLUTIONS

* Lengths have been adjusted to account for differences in yield level.
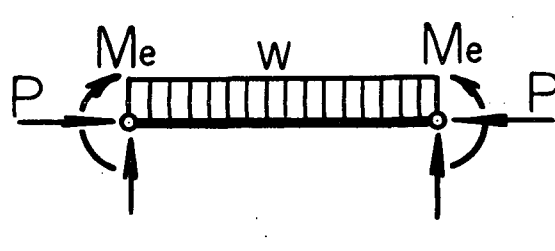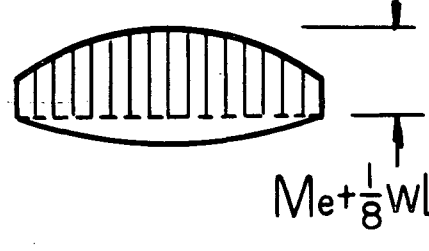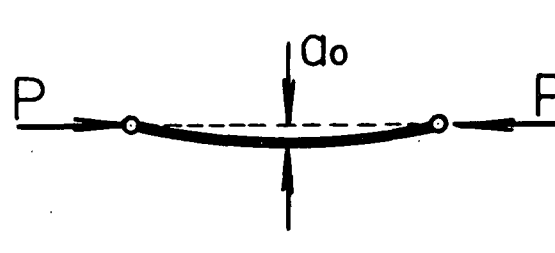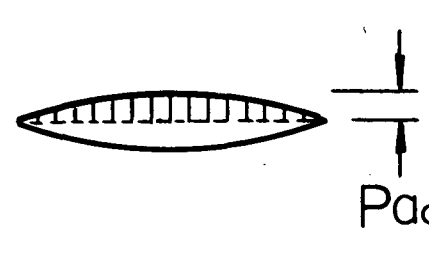Curves correspond to $\sigma_y = 40$ ksi
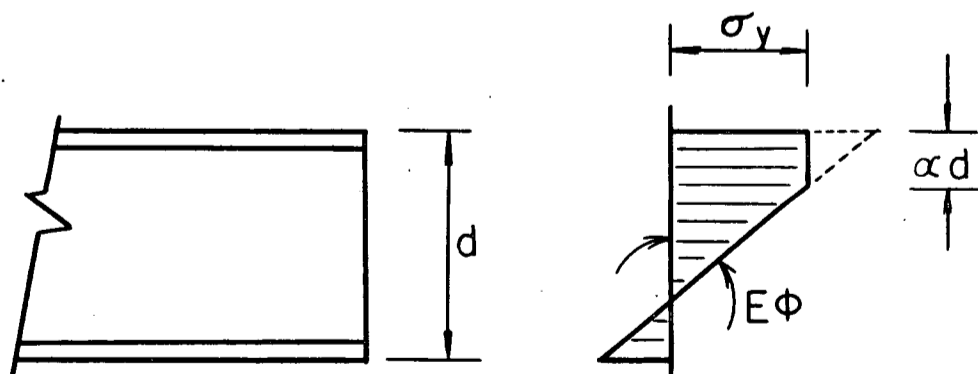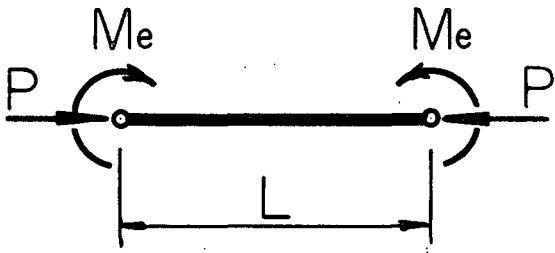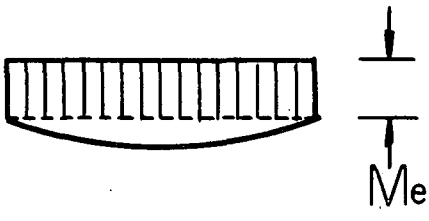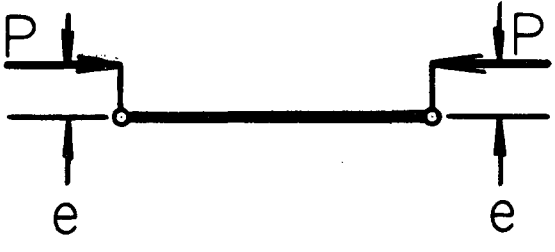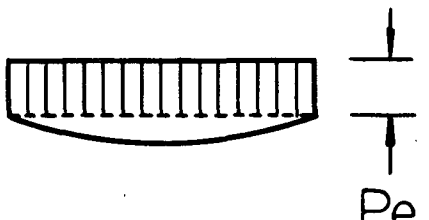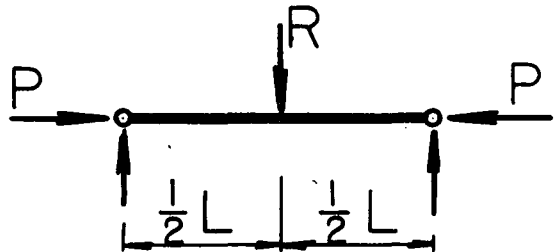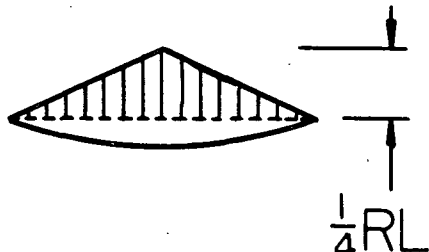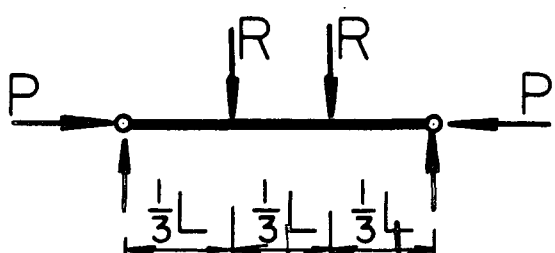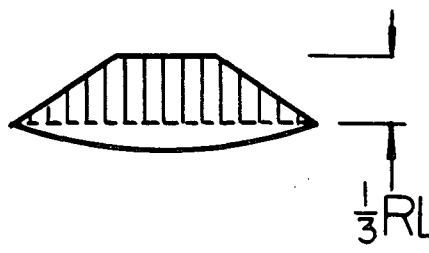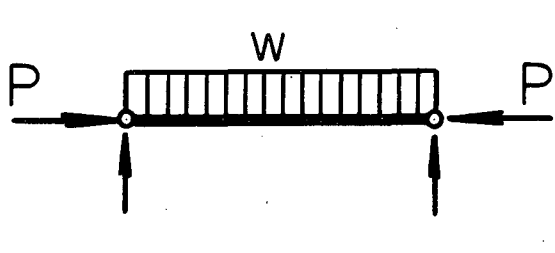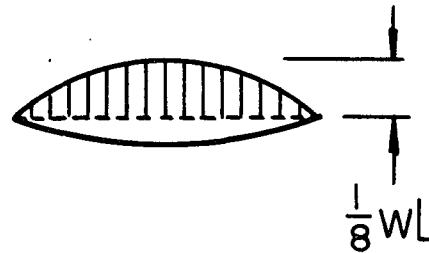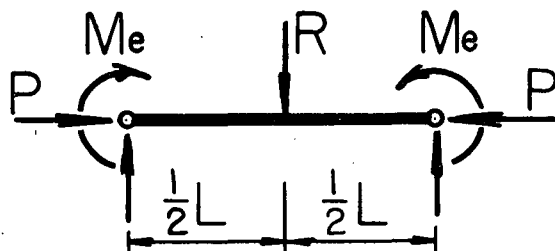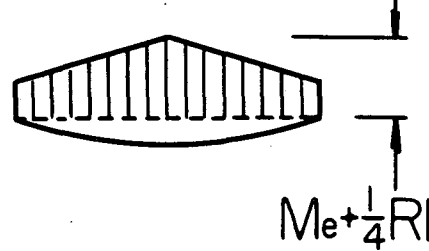
FIG. 24 STRESS DISTRIBUTION PATTERN WHEN
STRAINS EXCEED THE ELASTIC LIMIT

TABLE I

LOADING CONDITION	MOMENT DIAGRAM	DETERMINATION OF UNKNOWN
P — M_e ... M_e — P, span L	M_e	$\left(\dfrac{m_o}{M_y}\right)_{cr} = \left(\dfrac{M_e}{M_y}\right)_{cr}$
P ... P, eccentricity e	Pe	$\left(\dfrac{m_o}{M_y}\right)_{cr} = \dfrac{P(e)_{cr}}{M_y}$ *
P — R — P, $\frac{1}{2}L \mid \frac{1}{2}L$	$\frac{1}{4}RL$	$\left(\dfrac{m_o}{M_y}\right)_{cr} = R_{cr}\left[\dfrac{L}{4M_y}\right]$
P — R R — P, $\frac{1}{3}L \mid \frac{1}{3}L \mid \frac{1}{3}L$	$\frac{1}{3}RL$	$\left(\dfrac{m_o}{M_y}\right)_{cr} = R_{cr}\left[\dfrac{L}{3M_y}\right]$
P — w — P	$\frac{1}{8}wL^2$	$\left(\dfrac{m_o}{M_y}\right)_{cr} = w_{cr}\left[\dfrac{L^2}{8M_y}\right]$
P — M_e R M_e — P, $\frac{1}{2}L \mid \frac{1}{2}L$	$M_e + \frac{1}{4}RL$	$\left(\dfrac{m_o}{M_y}\right)_{cr} = \dfrac{(M_e + \frac{1}{4}RL)_{cr}}{M_y}$
P — M_e w M_e — P	$M_e + \frac{1}{8}wL^2$	$\left(\dfrac{m_o}{M_y}\right)_{cr} = \dfrac{(M_e + \frac{1}{4}wL^2)_{cr}}{M_y}$
P — a_o — P	Pa_o	$\left(\dfrac{m_o}{M_y}\right)_{cr} = \dfrac{Pa_o}{M_y} = \left[\left(\dfrac{P}{P_y}\right)\dfrac{c}{r^2}(a_o)_{cr}\right]$

$$*\left(\frac{ec}{r^2}\right) = \left[\frac{m_o/M_y}{P/P_y}\right]$$