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BY

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Paper No. 2152

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ANALYSIS OF BUILDING FRAMES WITH SEMI-RIGID CONNECTIONS

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WITH DISCUSSION BY MESSRS. MAURICE P. VAN BUREN, WAYNE W. SMITH, LEONARD P. ZICK, JR., AND CONRAD C. WAN, S. D. LASH, DEAN F. PETERSON, JR., R. W. STEWART, JAROSLAV J. POLIVKA, AND BRUCE JOHNSTON) AND EDWARD H. MOUNT.

SYNOPSIS

Methods applicable to the analysis of building frames with semi-rigid riveted or welded connections between the beams and columns are presented in this paper. The methods of analysis are too complex for ordinary design use, but the writers have presented simple design procedures, based on these methods of analysis, elsewhere (2) (3),[3] and have made them expeditious by the use of charts and diagrams. Such design methods effect permissible economy in the required beam sizes, made possible by considering the partial restraint afforded by standard or near standard connections, particularly riveted or welded connections of the top and seat angle type.

This paper also presents test results of a welded building frame that corroborate the methods of analysis. A study of the effect of neglecting the width of members in the analysis is presented. ("Member width" is used in this paper to indicate column width or beam depth, as the frame is viewed in elevation.) The essential features of the methods of analysis have been presented elsewhere (1) (13) (14), and it is the intention of the writers to modify them slightly so as to simplify the technique by conforming in every way to the usual slope-deflection and moment-distribution procedures.

INTRODUCTION

The design of the steel frames that form the skeleton of multiple-story steel buildings is usually based upon certain simplifying assumptions, chief of which are: (a) For the purpose of beam design the beam-column connections

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are assumed to be pin connections, or simple supports; (b) columns are designed without attempting to evaluate the moments introduced by frame action; and (c) the beam-column connections are assumed to be rigid in calculating stresses due to lateral or wind loads.

These assumptions have afforded a means of rapid design calculation. Riveted building frames constructed on the basis of these design assumptions have proved to be safe and reliable, but there remains the possibility of achieving greater economy through the use of more nearly correct design assumptions. According to the British investigations (1), an average saving in the weight of beams of as much as 20% may be expected by taking advantage of the partial end restraint of riveted beam-column connections. Welded construction, with its inherent continuity, also makes similar saving in weight possible.

The basis for the application of more accurate design methods to frames with semi-rigid connections has already been laid in Great Britain (1) and in work of a parallel nature in the United States (2) (3) (4). The comparison of analytical and experimental results presented in this paper was made possible by the construction and test of a welded building frame with three bays and two stories. The beam-column connections used in this frame were of a semi-rigid type previously studied at the Fritz Engineering Laboratory (2) (5) in connection with research programs sponsored by the American Welding Society. Similar tests on riveted building frames have been made in Great Britain (1).

The slope-deflection and moment-distribution methods have as a common purpose the determination of the bending moments at the ends of the individual members of a statically indeterminate frame. Both methods in their usual form are based on the assumption that the deformations of the frames are caused entirely by bending of the members and that the relation between bending moment and distortion is given by the beam formula:

$$\frac{M}{E\,I} = \frac{d^2y}{dx^2} \dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

The derivation of the beam theory and the assumptions on which it is based may be found in any text on the strength of materials.

In 1915 the slope-deflection method was applied in the United States (by Wilbur M. Wilson and George A. Maney, Members, Am. Soc. C. E.) to the analysis of wind stresses in tall buildings (6). A more complete treatment followed in 1918 (by Professor Wilson, with F. E. Richart and Camillo Weiss, Members, Am. Soc. C. E.) (7), and in 1931 a modification (8) was introduced by L. T. Evans, Assoc. M. Am. Soc. C. E., to take care of members with varying moments of inertia. The method of moment distribution was first presented in mimeographed form by Hardy Cross, M. Am. Soc. C. E., in 1926 (9) (10) (12). Innumerable variations and short cuts have been applied to the moment-distribution method and, although some of these have merit, the original method remains the outstanding development in recent times for rapid and effective analysis of continuous frames. The application of both the slope-deflection and moment-distribution methods to the analysis of frames with semi-rigid connections was made by John F. Baker, Assoc. M.

Am. Soc. C. E. (13). The width of the members and the semi-rigid nature of the connections are both taken into account, as it is found that the neglect of member width gives rise to considerable error, particularly in the case of analyses of frames with semi-rigid connections.

ANALYSIS OF FRAMES WITH SEMI-RIGID JOINTS

It will be assumed that the reader is already familiar with the usual application of the slope-deflection and moment-distribution methods, for which references are readily available (12) (15) (16) (17). The methods herein presented are identical in mode of application to the usual simple form, with the exception that special coefficients must be used in the slope-deflection equations and for the carry-over and distribution factors in the moment-distribution method. The following method is identical with that previously presented by Professor Baker (13) (14) when the width of member is neglected. When the width of member is considered, the following method differs in two respects from that of Professor Baker: (a) The interior of the joint between connections is assumed infinitely rigid, whereas Professor Baker assumes it to have the same stiffness as the beam; and (b) hypothetical moments are computed at the joint centers by the usual slope-deflection and moment-distribution procedures, whereas in Professor Baker's method separate expressions are given for the moment and shear at the connection, or column face, and are dealt with separately.

The following assumptions apply both to the slope-deflection and to the moment-distribution relations as herein presented: (a) Members are of uniform cross section between their end connections; (b) the semi-rigid connection at the end of a member behaves elastically as defined by the connection constant γ; and (c) the interior of the joint between connections is assumed to be infinitely rigid, although free to rotate as a rigid body.

The notation used is shown subsequently in Figs. 5 and 6. The hypothetical moments and shears at the joint centers, when width of column is considered, are designated by the bar above the letter M or V, thus; $\bar{M}$ and $\bar{V}$.

The "Semi-Rigid" Connection.—The semi-rigid connection may be thought of as a locally weakened section between the end of the beam and the face of the column to which the connection is made. The effect on analysis is the inverse of the effect produced by end haunches or added cover plates. The typical test behavior of a riveted or welded connection of this type is shown in Fig. 1, which presents the plot of the relationship between moment transmitted through the connection and the angle change between the joint center and the end of the beam. In the design range the relationship is assumed linear and the inverse slope is termed the connection constant, γ, thus:

$$\gamma = \frac{\phi}{M} = \frac{\text{Angle change}}{\text{Moment}} \dots\dots\dots\dots\dots\dots(2)$$

The connection constant γ may be defined as "angle change for unit moment" and may be determined experimentally by testing typical joints. The vertical

line through the origin in Fig. 1 would indicate the behavior of a perfectly fixed connection with $\gamma = 0$, and the horizontal line would represent the behavior of a frictionless pin connection, in which case $\gamma = \infty$. Fig. 2(a)
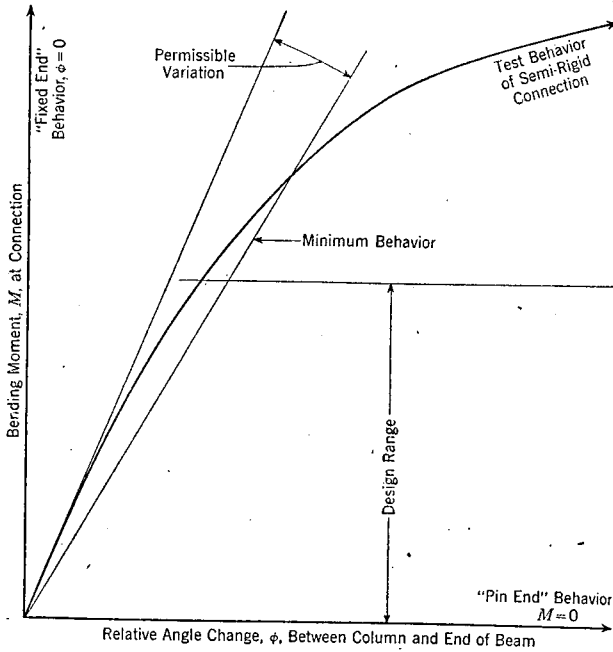
FIG. 1.—TYPICAL M-ϕ RELATION IN THE TEST OF A WELDED OR RIVETED BEAM-COLUMN CONNECTION

shows the test setup for determining the connection constant at an interior joint of a frame with beam-to-column-flange connections, and Fig. 2(b) shows a similar setup to test the connection between a beam and an exterior column
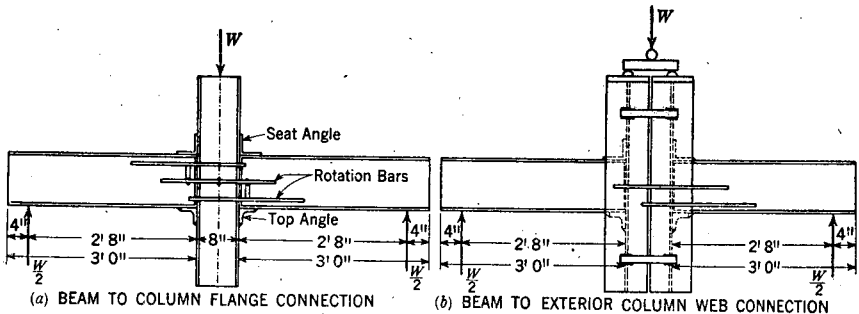
FIG. 2.—TEST ARRANGEMENT TO DETERMINE CONNECTION CONSTANTS

web. These connections correspond to those used on two frames tested by the writers. The relative rotation between the ends of the beam and the center of the column at the joint were measured with a 20-in. level bar which is shown in Fig. 3 in position for measurement of angle changes of the actual

test frame. Fig. 4 is a graph within the test-design range of measured angle change plotted against moment in typical joint tests corresponding to the actual test frame.
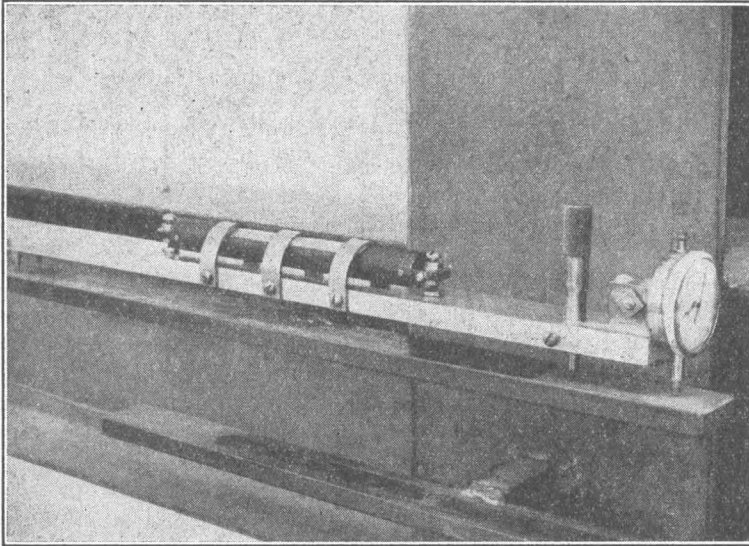
Fig. 3.—Level Bar Used to Measure Rotations

The Slope-Deflection Equations.—For any individual member of a frame, the relation between its end moments, the angle changes at each end, and the angle change of the member as a whole may be expressed by a pair of slope-deflection equations. For the usually assumed case of uniform beam cross

Fig. 4.—Test Results in Design Range for Typical Beam-Column Connections

section, and with rigid end connections, these equations are written:

$$M_{AB} = 2 E K (2 \theta_A + \theta_B - 3 R) - M_{RAB} \ldots \ldots \ldots (3a)$$

and

$$M_{BA} = 2 E K (2 \theta_B + \theta_A - 3 R) + M_{RBA} \ldots \ldots \ldots (3b)$$

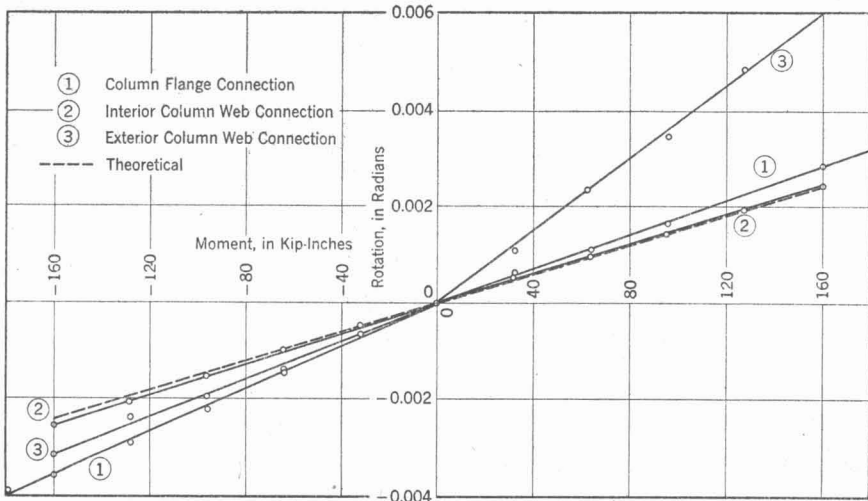In Eqs. 3, $K = \dfrac{I}{l}$, in which $I =$ the moment of inertia and $l =$ the length of the member (distance between joint connections). The angle changes at ends A and B are θ_A and θ_B; and $R = \dfrac{\Delta}{l} =$ the angle change of the entire member, Δ being the relative lateral displacements of ends A and B. The fixed-end moments for the lateral loads alone are M_{RAB} and M_{RBA}. The slope-deflection equations may be derived directly from Eq. 1 or by an application of the moment-area principles.

. Slope-deflection equations such as Eqs. 3 may be derived by similar methods for members which frame with semi-rigid connections. Fig. 5 shows the notation used and the geometry of the general deflected curve of any such member. Note especially Fig. 5(b), which shows the hypothetical moments at the center of the joint. The slope-deflection equations corresponding to Eqs. 3 for the hypothetical moments $\bar{M}_{AB}$ and $\bar{M}_{BA}$ at the joint centers for any member $A B$, as shown in Fig. 5, may be written:

$$\bar{M}_{AB} = \frac{1}{1 + 2\alpha + 2\beta + 3\alpha\beta} [2 E K(C_{AA} \theta_A + C_{AB} \theta_B - C_{AC} R)$$
$$- F_{AA} M_{RAB} - F_{AB} M_{RBA}] - V_{AB}' b_{AB} \ldots \ldots \ldots (4a)$$

and

$$\bar{M}_{BA} = \frac{1}{1 + 2\alpha + 2\beta + 3\alpha\beta} [2 E K(C_{BB} \theta_B + C_{BA} \theta_A - C_{BC} R)$$
$$+ F_{BB} M_{RBA} + F_{BA} M_{RAB}] + V_{BA}' b_{BA} \ldots \ldots \ldots (4b)$$

Except for the fact that new coefficients replace the even integer coefficients in Eqs. 3, the application of Eqs. 4 to any particular problem is identical with the usual slope-deflection procedure. (Note that $R = \dfrac{\Delta}{l}$, not $\dfrac{\Delta}{L}$, in Eqs. 4.)

(a) SEMI-RIGID CONNECTIONS AND
FINITE WIDTH OF MEMBER

(b) DEFORMATION OF MEMBER WITH SEMI-RIGID
CONNECTIONS AND FINITE JOINT WIDTH
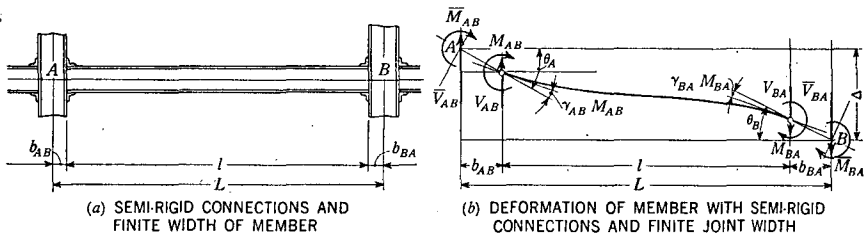
FIG. 5.—DEFORMATION OF MEMBER CONSIDERING SEMI-RIGID CONNECTIONS AND FINITE JOINT WIDTH

In Eqs. 4 the new constants C_{AA}, C_{AB}, C_{AC}, C_{BB}, C_{BA}, F_{AA}, F_{AB}, F_{BA}, and F_{BB} depend on the dimensions of the members and on the value of the joint constant. Factor K again is given by $\dfrac{I}{l}$, and it should be noted that l is the length between connections rather than the length between joint centers.

The connection constants γ_A and γ_B for the connections at the two ends of the beam are introduced into new constants α and β by the relations (1):

$$\alpha = 2 E K \gamma_A \dots\dots\dots\dots\dots\dots\dots(5a)$$

and

$$\beta = 2 E K \gamma_B \dots\dots\dots\dots\dots\dots\dots(5b)$$

The fixed-end moments for a member with fixed, rigidly connected, ends of span length l are M_{RAB} and M_{RBA}; and V_{AB}' and V_{BA}' are the shears or reactions at the ends of a member with freely supported ends and span length l.

The constants C_{AA}, C_{AB}, C_{AC}, F_{AA}, and F_{AB} in Eq. 4a are given in Table 1 for four different cases. All four cases are for unsymmetrical conditions, the first and the third considering semi-rigid connections, and the second and fourth considering rigid connections. In cases I and II, a finite width of member is considered; and, in cases III and IV, width is ignored. The values in case IV are those commonly assumed—that is, frames with rigid joints and with width of member neglected. In Eq. 4b the constants C_{BB}, C_{BA}, C_{BC}, F_{BB}, and F_{BA} for $\bar{M}_{BA}$ are obtained from C_{AA}, C_{AB}, C_{AC}, F_{AA}, and F_{AB}, respectively, by interchanging α and β and the subscripts A and B. It is noted that C_{BA} is equal to C_{AB}.

Eqs. 4 with the preceding coefficients are for moments at the joint centers and therefore are used with exactly the same equilibrium conditions as in the simpler form (2); namely,

For joint equilibrium,

$$\Sigma \bar{M} = 0 \dots\dots\dots\dots\dots\dots\dots\dots(6a)$$

and, for story equilibrium,

$$\Sigma \bar{M} + V h = 0 \dots\dots\dots\dots\dots\dots\dots(6b)$$

In Eq. 6b, h is the story height between neutral axes of two beams.

With Eqs. 4 it is now possible to determine the hypothetical end moments $\bar{M}_{AB}$ and $\bar{M}_{BA}$ at the joint centers. Moments and shears are assumed as positive when they act on the ends of the beam with a clockwise sense, or act on the joint with a counterclockwise sense. The hypothetical shears in the joint are constant and equal to the actual shear at the connection. The shears $\bar{V}_{AB}$ and $\bar{V}_{BA}$ may be calculated from the moments $\bar{M}_{AB}$ and $\bar{M}_{BA}$ by the following:

$$V_{AB} = \bar{V}_{AB} = - \left(\frac{\bar{M}_{AB} + \bar{M}_{BA}}{L} \right) + \bar{V}_{AB}' \dots\dots\dots\dots(7a)$$

and

$$V_{BA} = \bar{V}_{BA} = - \left(\frac{\bar{M}_{AB} + \bar{M}_{BA}}{L} \right) - \bar{V}_{BA}' \dots\dots\dots\dots(7b)$$

in which $\bar{V}_{AB}'$ and $\bar{V}_{BA}'$ are the end shears in a member having span length L with simply or freely supported ends. The moments at the connections, M_{AB} and M_{BA}, may now be calculated from the following:

$$M_{AB} = \bar{M}_{AB} + \bar{V}_{AB} b_{AB} \dots\dots\dots\dots\dots\dots(8a)$$

and

$$M_{BA} = \bar{M}_{BA} + \bar{V}_{BA} b_{BA} \dots\dots\dots\dots\dots\dots(8b)$$

TABLE 1.—Slope-Deflection and Moment-Distribution Constants—Nonsymmetrical Cases

Constant	Case I—Nonsymmetrical, semi-rigid connections, finite width of members	Case II—Nonsymmetrical, rigid connections, finite widths	Case III—Nonsymmetrical, semi-rigid connections, zero width	Case IV—Rigid connections, 0 width
		(a) Slope Deflection		
C_{AA}	$2 + 3\beta + 6(1+\beta)\frac{b_{AB}}{l} + 3(2+\alpha+\beta)\frac{b_{AB}^2}{l^2}$	$2 + \frac{6\,b_{AB}}{l} + \frac{6\,b_{AB}^2}{l^2}$	$2 + 3\beta$	2
$C_{AB} = C_{BA}$	$1 + 3(1+\alpha)\frac{b_{AB}}{l} + 3(1+\beta)\frac{b_{BA}}{l} + 3(2+\alpha+\beta)\frac{b_{AB}b_{BA}}{l^2}$	$1 + \frac{3\,b_{AB}}{l} + \frac{3\,b_{BA}}{l} + \frac{6\,b_{AB}b_{BA}}{l^2}$	1	1
C_{BB}	$2 + 3\alpha + 6(1+\alpha)\frac{b_{BA}}{l} + 3(2+\alpha+\beta)\frac{b_{BA}^2}{l^2}$	$2 + \frac{6\,b_{BA}}{l} + \frac{6\,b_{BA}^2}{l^2}$	$2 + 3\alpha$	2
C_{AC}	$3(1+\beta) + 3(2+\alpha+\beta)\frac{b_{AB}}{l}$	$3 + \frac{6\,b_{AB}}{l}$	$3(1+\beta)$	3
C_{BC}	$3(1+\alpha) + 3(2+\alpha+\beta)\frac{b_{BA}}{l}$	$3 + \frac{6\,b_{BA}}{l}$	$3(1+\alpha)$	3
F_{AA}	$1 + 2\beta + (1-\alpha+2\beta)\frac{b_{AB}}{l}$	$1 + \frac{b_{AB}}{l}$	$1 + 2\beta$	1
F_{AB}	$\beta + (\beta-2\alpha-1)\frac{b_{AB}}{l}$	$-\frac{b_{AB}}{l}$	β	0
F_{BB}	$1 + 2\alpha + (1-\beta+2\alpha)\frac{b_{BA}}{l}$	$1 + \frac{b_{BA}}{l}$	$1 + 2\alpha$	1
F_{BA}	$\alpha + (\alpha-2\beta-1)\frac{b_{BA}}{l}$	$-\frac{b_{BA}}{l}$	α	0
		(b) Moment Distribution		
$\bar{M}_{SAB}$ semi-fixed end moment at joint center	$\dfrac{\left\{\left[1+2\beta+\frac{b_{AB}}{l}(1+2\beta-\alpha)\right]M_{RAB} + \left[\beta-\frac{b_{AB}}{l}(1+2\alpha-\beta)\right]M_{RBA}\right\}}{1+2\alpha+2\beta+3\alpha\beta} + V_{AB}'b_{AB}$	$\left(1+\frac{b_{AB}}{l}\right)M_{RAB} - \frac{b_{AB}}{l}M_{RBA} + V_{AB}'b_{AB}$	$\dfrac{[(1+2\beta)\,M_{RAB} + \beta\,M_{RBA}]}{1+2\alpha+2\beta+3\alpha\beta}$	M_{RAB}
r_{AB} carry-over factor between joint centers	$\dfrac{1+3(1+\alpha)\frac{b_{AB}}{l} + 3(1+\beta)\frac{b_{BA}}{l} + 3(2+\alpha+\beta)\frac{b_{AB}b_{BA}}{l^2}}{2+3\beta+6(1+\beta)\frac{b_{AB}}{l} + 3(2+\alpha+\beta)\frac{b_{AB}^2}{l^2}}$	$\dfrac{1+\frac{3\,b_{AB}}{l} + \frac{3\,b_{BA}}{l} + \frac{6\,b_{AB}b_{BA}}{l^2}}{2 + \frac{6\,b_{AB}}{l} + \frac{6\,b_{AB}^2}{l^2}}$	$\dfrac{1}{2+3\beta}$	$\dfrac{1}{2}$
$\bar{S}_{MAB}$ end rotation stiffness	$\dfrac{2EK\left[2+3\beta+6(1+\beta)\frac{b_{AB}}{l} + 3(2+\alpha+\beta)\frac{b_{AB}^2}{l^2}\right]}{1+2\alpha+2\beta+3\alpha\beta}$	$2EK\left(2 + \frac{6\,b_{AB}}{l} + \frac{6\,b_{AB}^2}{l^2}\right)$	$\dfrac{2EK(2+3\beta)}{1+2\alpha+2\beta+3\alpha\beta}$	$4EK$
$\bar{S}_{VAB}$ sidesway stiffness	$\dfrac{6EK}{l^2}\left(\dfrac{2+\alpha+\beta}{1+2\alpha+2\beta+3\alpha\beta}\right)$	$\dfrac{12EK}{l^2}$	$\dfrac{6EK}{l^2}\left(\dfrac{2+\alpha+\beta}{1+2\alpha+2\beta+3\alpha\beta}\right)$	$\dfrac{12EK}{l^2}$
$\bar{M}_{VAB}$ sidesway end moment at A	$-\dfrac{6EK}{l}\left[\dfrac{1+\beta+\frac{b_{AB}}{l}(2+\alpha+\beta)}{1+2\alpha+2\beta+3\alpha\beta}\right]$	$-\dfrac{6EK}{l}\left(1+\frac{2\,b_{AB}}{l}\right)$	$-\dfrac{6EK}{l}\left(\dfrac{1+\beta}{1+2\alpha+2\beta+3\alpha\beta}\right)$	$-\dfrac{6EK}{l}$

The slope-deflection equations are simplified when symmetrical conditions of loading and structure exist with respect to any particular member. In such a case $\alpha = \beta$. Furthermore, $b_{AB} = b_{BA} = b$; $M_{RAB} = M_{RBA} = M_R$; $\bar{V}_{AB}' = \bar{V}_{BA}' = V_{AB}' = V_{BA}' = V'$; and the slope-deflection equations corresponding to case I of Table 1 may be reduced to:

$$\bar{M}_{AB} = \frac{2\,E\,K}{1+3\,\alpha}\left[\left(\frac{2+3\,\alpha}{1+\alpha}+\frac{6\,b}{l}+\frac{6\,b^2}{l^2}\right)\theta_A + \left(\frac{1}{1+\alpha}+\frac{6\,b}{l}+\frac{6\,b^2}{l^2}\right)\theta_B\right.$$
$$\left.-\left(3+\frac{6\,b}{l}\right)R\right]-\left(\frac{M_R}{1+\alpha}+V'\,b\right)\ldots\ldots\ldots\ldots(9a)$$

and

$$\bar{M}_{BA} = \frac{2\,E\,K}{1+3\,\alpha}\left[\left(\frac{2+3\,\alpha}{1+\alpha}+\frac{6\,b}{l}+\frac{6\,b^2}{l^2}\right)\theta_B + \left(\frac{1}{1+\alpha}+\frac{6\,b}{l}+\frac{6\,b^2}{l^2}\right)\theta_A\right.$$
$$\left.-\left(3+\frac{6\,b}{l}\right)R\right]+\left(\frac{M_R}{1+\alpha}+V'\,b\right)\ldots\ldots\ldots\ldots(9b)$$

For case II of Table 1, considering the semi-rigidity of the joints but neglecting the width of the members, the equations for symmetrical conditions may be reduced to a simple form by letting $b=0$ in Eqs. 9; thus:

$$\bar{M}_{AB} = M_{AB} = \frac{2\,E\,K}{1+3\,\alpha}\left[\left(\frac{2+3\,\alpha}{1+\alpha}\right)\theta_A\right.$$
$$\left.+\left(\frac{1}{1+\alpha}\right)\theta_B - 3\,R\right]-\frac{M_R'}{1+\alpha}\ldots\ldots\ldots\ldots(10a)$$

and

$$\bar{M}_{BA} = M_{BA} = \frac{2\,E\,K}{1+3\,\alpha}\left[\left(\frac{2+3\,\alpha}{1+\alpha}\right)\theta_B\right.$$
$$\left.+\left(\frac{1}{1+\alpha}\right)\theta_A - 3\,R\right]+\frac{M_R}{1+\alpha}\ldots\ldots\ldots\ldots(10b)$$

A similar simplification may be made for case III of Table 1 by letting $\alpha = 0$ in Eqs. 9, in which case the following equations result:

$$\bar{M}_{AB} = 2\,E\,K\left[\left(2+\frac{6\,b}{l}+\frac{6\,b^2}{l^2}\right)\theta_A + \left(1+\frac{6\,b}{l}+\frac{6\,b^2}{l^2}\right)\theta_B\right.$$
$$\left.-\left(3+\frac{6\,b}{l}\right)R\right]-(M_R+V_b')\ldots\ldots\ldots\ldots(11a)$$

and

$$\bar{M}_{BA} = 2\,E\,K\left[\left(2+\frac{6\,b}{l}+\frac{6\,b^2}{l^2}\right)\theta_B + \left(1+\frac{6\,b}{l}+\frac{6\,b^2}{l^2}\right)\theta_A\right.$$
$$\left.-\left(3+\frac{6\,b}{l}\right)R\right]+(M_R+V_b')\ldots\ldots\ldots\ldots(11b)$$

Moment Distribution Applied to Frames with Semi-Rigid Connections.—The moment-distribution method serves identically the same purpose as the slope-deflection method; that is, the moments at the ends of the individual members

of a frame or continuous beam are determined. It will be assumed, as in the case of the slope-deflection method, that the reader is familiar with the usual moment distribution procedure. The procedure as applied to frames with semi-rigid connections is identical with the usual method, although there are differences in the numerical value of the carry-over, stiffness, and other factors.

The factors used in the moment-distribution procedure may be derived from the slope-deflection equations or by use of the column analogy (11), as will be described herein. Fig. 6 shows the deformation conditions for determining
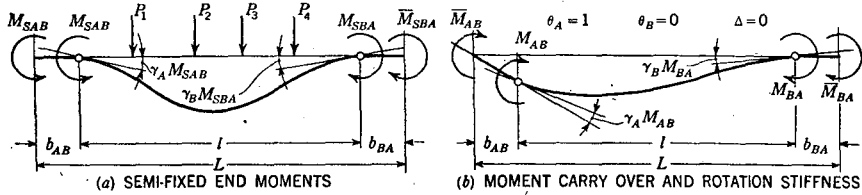

FIG. 6.—DEFORMATION CONDITIONS USED IN MOMENT DISTRIBUTION FOR MEMBERS WITH SEMI-RIGID CONNECTIONS AND FINITE JOINT WIDTH

semi-rigid end moments and the carry-over factor for moment distribution. Table 1 gives the "semi-rigid" end moments, "carry-over" factor, rotation-stiffness factors, sidesway-stiffness factors, and sidesway end moments for the nonsymmetrical cases I, II, III, and IV. Table 2 presents the same factors for the symmetrical cases I, II, and III.

TABLE 2.—MOMENT DISTRIBUTION FOR SYMMETRICAL CASES
$$\alpha = \beta, \ b_{AB} = b_{BA} = b, \ \text{AND} \ M_{RA} = M_{RB} = M_R$$

Constant	Symmetrical case I, semi-rigid connections, finite width of members	Symmetrical case II, rigid connections, finite member width	Symmetrical case III, semi-rigid connections, zero member width
$\bar{M}_{SAB}$ semi-fixed-end moment at joint center	$\dfrac{M_R}{1 + \alpha} + V' b$	$M_R + V' b$	$\dfrac{M_R}{1 + \alpha}$
$\bar{r}_{AB}$ carry-over factor between joint centers	$\dfrac{1 + 6(1+\alpha)\frac{b}{l} + 6(1+\alpha)\frac{b^2}{l^2}}{2 + 3\alpha + 6(1+\alpha)\frac{b}{l} + 6(1+\alpha)\frac{b^2}{l^2}}$	$\dfrac{1 + \frac{6b}{l} + \frac{6b^2}{l^2}}{2 + \frac{6b}{l} + \frac{6b^2}{l^2}}$	$\dfrac{1}{2 + 3\alpha}$
$\bar{S}_{MAB}$ end rotation stiffness	$\dfrac{2EK}{1+3\alpha}\left(\dfrac{2+3\alpha}{1+\alpha} + \dfrac{6b}{l} + \dfrac{6b^2}{l^2}\right)$	$2EK\left(2 + \dfrac{6b}{l} + \dfrac{6b^2}{l^2}\right)$	$2EK\left(\dfrac{2+3\alpha}{1+4\alpha+3\alpha^2}\right)$
$\bar{S}_{VAB}$ sidesway stiffness	$\dfrac{12EK}{l^2(1+3\alpha)}$	$\dfrac{12EK}{l^2}$	$\dfrac{12EK}{l^2(1+3\alpha)}$
$\bar{M}_{VAB}$ sidesway end moment at A	$-\dfrac{6EK}{l}\left(\dfrac{1+\frac{2b}{l}}{1+3\alpha}\right)$	$-\dfrac{6EK}{l}\left(1 + \dfrac{2b}{l}\right)$	$-\dfrac{6EK}{l(1+3\alpha)}$

All of the moment-distribution factors relate to hypothetical moments at the centers of the joints, and the general procedure is therefore identical with that used in the simple case. After the hypothetical moments at the joint centers are obtained, the moments and shears at the connections result as before from Eqs. 7 and 8.

In applying the moment-distribution method to frame problems, sidesway may be induced either by lateral loads or by unsymmetrical conditions of loading or frame arrangement. The simplest form of sidesway problem is one involving only lateral loads applied to a one-story frame. The lateral load is distributed to the columns in proportion to their "sidesway stiffness" S_V, and semi-fixed or fixed-end moments are distributed to the ends of each column in proportion to the $\bar{M}_V$ moments for unit sidesway. The next step is the usual moment distribution, but at the conclusion the summation of column shears will not account for the total lateral force. All of the end moments are then multiplied by a constant ratio sufficient to bring the shears into equilibrium with the external lateral force. If the structure is more than one story in height, the procedure is progressively more complicated. Shears are distributed in any one story in proportion to their lateral or sidesway stiffness; but in special cases in which two-story sections adjoin open auditoriums or halls, the combined rigidities of the two stories of columns "in series" must be determined. The stiffness of a two-story group of columns "in series" is given by:

$$S_{VABC} = \frac{S_{VAB}\, S_{VBC}}{S_{VAB} + S_{VBC}} \ldots\ldots\ldots\ldots\ldots\ldots (12a)$$

in which S_{VABC} = sidesway stiffness of two columns or groups of columns "in series." The combined rigidity of three tiers of columns "in series" is given by:

$$S_{VABCD} = \frac{S_{VAB}\, S_{VBC}\, S_{VCD}}{S_{VAB}\, S_{VBC} + S_{VBC}\, S_{VCD} + S_{VCD}\, S_{VAB}} \ldots\ldots (12b)$$

The analysis of two-story and three-story problems of the foregoing type is taken up in texts (12) (16) (17) and the procedure for frames with semi-rigid joints follows exactly the same course. In applying these analytical methods to the development of design methods for multi-storied buildings, under the action of vertical loads alone, it is reasonable to neglect the effect of sidesway.

Certain short cuts for special conditions may be used, provided that their use in simpler forms of moment distribution is already familiar. For example, if the end B of member $A\,B$ is pin-connected, or freely supported, β becomes equal to ∞. The "semi-fixed" end moment at A then becomes:

$$\bar{M}_{SAB} = \left(\frac{2 + \dfrac{2\, b_{AB}}{l}}{2 + 3\,\alpha}\right) M_{RAB} + V_{A}'\, b_{AB} \ldots\ldots\ldots\ldots (13)$$

In Eq. 13 M_{RAB} is the fixed-end moment at A due to lateral loads in a beam freely supported at B. The rotation stiffness, or distribution factor, for end A of member $A\,B$ with B freely supported is:

$$S_{MAB} = \frac{2\,E\,K}{2 + 3\,\alpha}\left(3 + \frac{6\, b_{AB}}{l} + \frac{3\, b_{AB}^2}{l^2}\right) \ldots\ldots\ldots\ldots (14a)$$

The sidesway stiffness factor for the same case, with one end freely supported,

is as follows:

$$\bar{S}_{VAB} = \frac{6\,E\,K}{l^2\,(2 + 3\,\alpha)}\dots\dots\dots\dots(14b)$$

Cases II and III in Table 2 may be obtained from Eqs. 13, 14a, and 14b by letting α and b_{AB}, respectively, be equal to zero.

Another type of special case occurs when the entire frame and loading upon it are symmetrical. If the center line of the frame is on line with a column, there will be no rotation of the column joints and the center line of the center column may be assumed equivalent to a fixed wall. If there are an odd number of panels, the center line of the frame will cut through the center of the beams in the center bay. The rotation of the two ends of each beam in the center panel will be equal in magnitude and opposite in sign. From this condition it follows that the modified moment stiffness or "distribution factor" may be taken as:

$$\bar{S}_{MAB} = \frac{2\,E\,K}{1 + \alpha}\dots\dots\dots\dots(14c)$$

for the ends of beams in the center panel. No carry-over is used in the center panel when the modified stiffness factor is used.

Application of the Column Analogy to Members with Semi-Rigid Connections. —It will be assumed that the reader is familiar with the application of the column analogy, originally developed by Professor Cross (11), to the determination of moment-distribution factors for beams with variable moments of inertia. The width of the "analogous column" is equal to $\frac{1}{E\,I}$, and the area of any elemental length ds of the analogous column is therefore equal to $\frac{ds}{E\,I}$. From the fundamental relations of the bent beam,

$$\frac{d\phi}{M} = \frac{ds}{E\,I}\dots\dots\dots\dots(15)$$

in which $d\phi$ = the angle change in any elemental length of beam. At the

(a) BEAM WITH SEMI-RIGID CONNECTIONS

Area of Column = γ at Connections

0 Width (I Assumed Infinite)
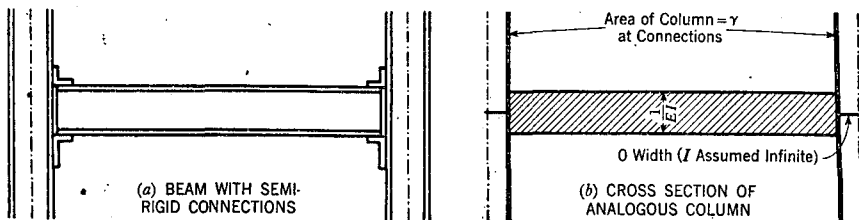
(b) CROSS SECTION OF ANALOGOUS COLUMN

FIG. 7.—APPLICATION OF COLUMN ANALOGY TO BEAMS WITH SEMI-RIGID CONNECTIONS

particular location of the semi-rigid joint, from the definition of the "connection constant,"

$$\gamma = \frac{d\phi}{M} = \frac{ds}{E\,I}\dots\dots\dots\dots(16)$$

Hence, the localized area of the analogous column at the semi-rigid connection

is equal to the connection constant γ. Professor Cross (11) has shown that the area of the analogous column at a pin connection is infinite, and in the region of a completely rigid zone it is equal to zero. The semi-rigid connection obviously is a case somewhere between these two extremes, and the column analogy may readily be used to obtain the moment-distribution factors for a member so connected. Fig. 7 illustrates a cross section through the analogous column of a member with semi-rigid end connections.

Analysis by Slope-Deflection Method

An illustrative example will be presented in detail to demonstrate the application of both the slope-deflection and moment-distribution methods to the analysis of a frame with semi-rigid joints, taking into account the width of the members.

The frame shown in Fig. 8 corresponds to one of the frames actually tested (see Fig. 9), and the connection constant used in the analysis was obtained experimentally from tests of a sample joint. All of the connections were identically alike, and each beam, therefore, was individually symmetrical. The results of the connection test gave an experimental value of $\gamma = 0.01775 \times 10^{-3}$ in inch-kip units. The stiffness of the frame members was measured by bending tests preliminary to fabrication of the frame, and the quantity $E I$ was thus found to be $3,550 \times 10^3$ and $3,321 \times 10^3$ for the beams and columns, respectively, in inch-kip units. The net length l of the beams between con-nections was 168 in. $-$ 8 in. $=$ 160 in. The columns are continuous, and the beam connections were of the welded seat and top angle type. An approximate correction for column length may be shown to be one third of the beam depth at each end that frames with a beam (see heading "Effect of Width of Member Upon Analysis"). Hence, for the second-story columns, $l = 120 - 6.67 = 113.33$ and, for the first-story columns, $l = 120 - 3.33 = 116.67$. This correction could well be omitted with but little error.

The constant α for the beams was

$$\alpha = \beta = \frac{2\,E\,I\,\gamma}{l} = \frac{2 \times 3,550 \times 10^3 \times 0.01775 \times 10^{-3}}{160} = 0.7877.$$

The fixed-end moment for the loading shown is 221.0 in-kips.

Because of the individual symmetry of the beams, the slope-deflection equations in the form of Eqs. 9 were applicable. The typical equation for any beam is written by substituting the values of α, $E K$, b, l, etc., in Eqs. 9, which for any loaded beam results in the following:

$$\bar{M}_{AB} = \frac{2 \times 3,550}{160(1 + 3 \times 0.7877)}\left\{ \left[\left(\frac{2 + 3 \times 0.7877}{1 + 0.7877}\right) + \frac{6 \times 4}{160} + \frac{6 \times 4^2}{160^2}\right]\theta_A \right.$$
$$\left. + \cdots + \left[\left(\frac{1}{1 + 0.7877}\right) + \frac{6 \times 4}{160} + \frac{6 \times 4^2}{160^2}\right]\theta_B \right\} - \left(\frac{221.0}{1.7877} + 6.5 \times 4\right) \ldots (17)$$

The right-hand side of this and the following equations has been divided by 1,000 to give more convenient values of θ. The moment $\bar{M}_{AB} = 34.233\,\theta_A$
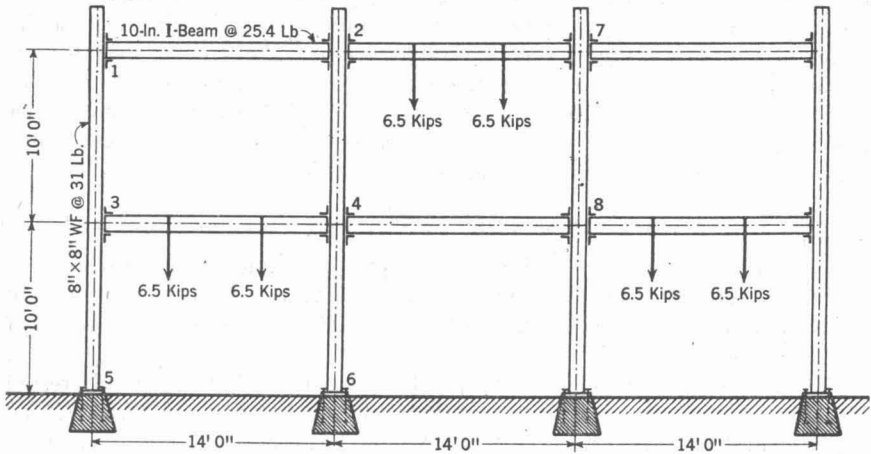
FIG. 8.—TEST FRAME AND ILLUSTRATIVE ANALYSIS (LOAD SPACING: 4 FT 5 IN., 5 FT 2 IN., AND 4 FT 5 IN., EQUALS 14 FT 0 IN.)
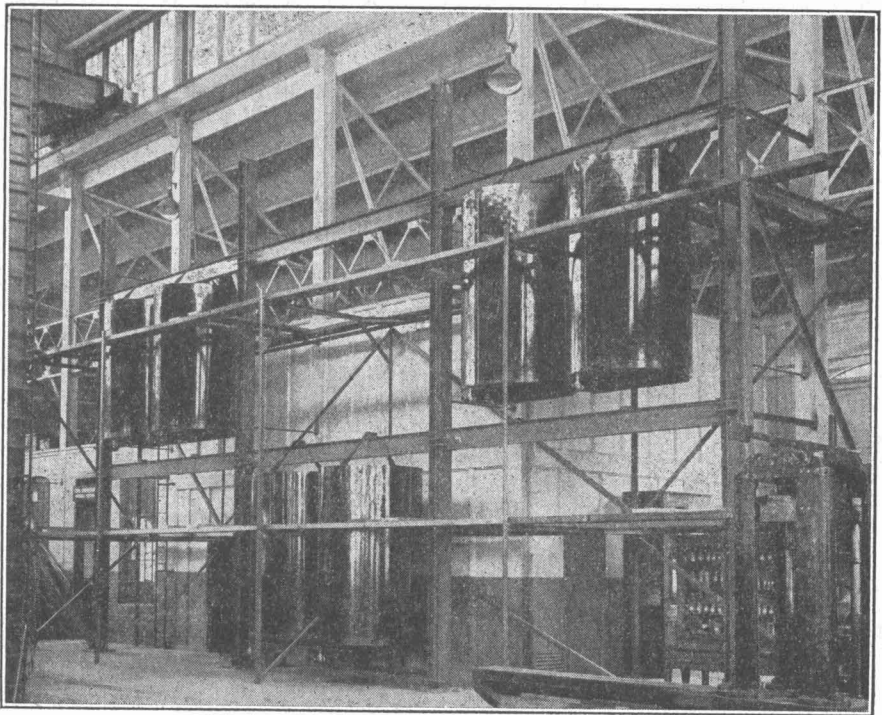

FIG. 9.—VIEW OF TEST FRAME

$+ 9.411\ \theta_B - 149.62$; and similarly:

$$\bar{M}_{BA} = 34.233\ \theta_B + 9.411\ \theta_A + 149.62 \ldots\ldots\ldots\ldots\ldots (18)$$

Equations of this type are written for all of the beams in a symmetrical half of the frame, as follows:

$$
\left.
\begin{aligned}
\bar{M}_{12} &= 34.233\ \theta_1 + 9.411\ \theta_2 \\
\bar{M}_{21} &= 34.233\ \theta_2 + 9.411\ \theta_1 \\
\bar{M}_{27} &= 34.233\ \theta_2 + 9.411\ \theta_7 - 149.62 \quad (-\theta_2 = +\theta_1) \\
&= 24.822\ \theta_2 - 149.62 \\
\bar{M}_{34} &= 34.233\ \theta_3 + 9.411\ \theta_4 - 149.62 \\
\bar{M}_{43} &= 34.233\ \theta_4 + 9.411\ \theta_3 + 149.62
\end{aligned}
\right\} \ldots (19)
$$

and

$$\bar{M}_{48} = 24.822\ \theta_4 \qquad\qquad (-\theta_4 = +\theta_8)$$

Similar equations for the column moments are written by making the proper substitutions in slope-deflection Eqs. 11, as follows:

$$
\left.
\begin{aligned}
\bar{M}_{13} &= 127.865\ \theta_1 + 69.257\ \theta_3 \\
\bar{M}_{24} &= 127.865\ \theta_2 + 69.257\ \theta_4 \\
\bar{M}_{31} &= 127.865\ \theta_3 + 69.257\ \theta_1 \\
\bar{M}_{42} &= 127.865\ \theta_4 + 69.257\ \theta_2 \\
\bar{M}_{35} &= 123.910\ \theta_3 \qquad\qquad (\theta_5 = 0) \\
\bar{M}_{46} &= 123.910\ \theta_4 \qquad\qquad (\theta_6 = 0) \\
\bar{M}_{53} &= 61.810\ \theta_3 \qquad\qquad (\theta_5 = 0)
\end{aligned}
\right\} \ldots\ldots\ldots\ldots (20)
$$

and

$$\bar{M}_{64} = 61.810\ \theta_4 \qquad\qquad (\theta_6 = 0)$$

The sidesway is obviously zero because of symmetry, and the only unknowns are the four angle changes θ_1, θ_2, θ_3, and θ_4. The necessary and sufficient conditions for the solution are obtained by applying the joint equilibrium equation $\Sigma M = 0$ to the four joints 1, 2, 3, and 4:

$$
\left.
\begin{aligned}
\bar{M}_{12} + \bar{M}_{13} &= 0 \\
\bar{M}_{21} + \bar{M}_{24} + \bar{M}_{27} &= 0 \\
\bar{M}_{31} + \bar{M}_{34} + \bar{M}_{35} &= 0
\end{aligned}
\right\} \ldots\ldots\ldots\ldots\ldots (21)
$$

and

$$\bar{M}_{43} + \bar{M}_{42} + \bar{M}_{48} + \bar{M}_{46} = 0$$

Rewriting these equations in terms of the unknown θ's:

$$
\left.
\begin{aligned}
+ 162.098\ \theta_1 + \quad 9.411\ \theta_2 + \quad 69.257\ \theta_3 \qquad\qquad\qquad &= \qquad 0 \\
9.411\ \theta_1 + 186.920\ \theta_2 \qquad\qquad\qquad + \quad 69.257\ \theta_4 &= + 149.620 \\
+ \quad 69.257\ \theta_1 \qquad\qquad + 286.008\ \theta_3 + \quad 9.411\ \theta_4 &= + 149.620 \\
+ \quad 69.257\ \theta_2 + \quad 9.411\ \theta_3 + 310.830\ \theta_4 &= - 149.620
\end{aligned}
\right\} \ldots (22)
$$

The solution of these four simultaneous equations may be made by systematic elimination of unknowns (15) (16) or by a method of successive approximations (15). The following solution was obtained by the first method: $\theta_1 = - 0.33174$;

$\theta_2 = +1.09276$; $\theta_3 = +0.62795$; and $\theta_4 = -0.74385$. These values of θ are 1,000 times the actual values but will give the correct moments when substituted in the moment equations which previously had been divided by 1,000. The actual moments at the connections may be found from the hypothetical joint center moments by computing the shears with Eqs. 7 and the connection moments with Eqs. 8. An alternate method would be to construct, graphically, the simple beam moment diagram for the full lengths L upon the joint moment base line. The connection moments then could be scaled off as the ordinate to the moment diagram at the face of the connecting member. Table 3 gives the results by the analytical method.

TABLE 3.—COMPUTATION BY SLOPE DEFLECTION

Location, joint and member	Joint moment $\overline{M}$ from slope-deflection equation	$\overline{V}$ by Eqs. 7	Connection moment M by Eqs. 8
1–2	− 1.07	−0.198	− 1.86
2–1	+ 34.29	−0.198	+ 33.50
2–7	−122.49	+6.500	− 96.49
3–4	−135.12	+6.531	−109.00
4–3	+130.06	−6.469	+104.18
4–8	− 18.46	0	− 18.46
1–3	+ 1.07	−0.515	− 0.65
3–1	+ 57.31	−0.515	+ 55.59
3–5	+ 77.79	−1.000	+ 74.46
5–3	+ 38.82	−1.000	+ 38.82
2–4	+ 88.20	−0.607	+ 86.18
4–2	− 19.44	−0.607	− 21.46
4–6	− 92.15	+1.186	− 88.20
6–4	− 45.99	+1.186	− 45.99

ANALYSIS BY MOMENT DISTRIBUTION

The factors required in the moment-distribution procedure have been presented in Tables 1 and 2. Some of the necessary computations in the following have already been made under the heading "Analysis by Slope-Deflection Method":

Semi-Fixed End Moment at Joint Center.—

$$\bar{M}_{SAB} = \frac{221.0}{1.7877} + 6.5 \times 4 = 149.62 \text{ in-kips.}$$

Carry-Over Factors Between Joint Centers.—

Beams (Both Directions).—

$$\bar{r}_{ab} = \bar{r}_{ba} = \frac{1 + 6\,(1.7877)\,\dfrac{4}{160} + 6\,(1.7877)\,\dfrac{4^2}{160^2}}{2 + 3\,(0.7877) + 6\,(1.7877)\,\dfrac{4}{160} + 6\,(1.7877)\,\dfrac{4^2}{160^2}} = 0.275.$$

Second-Story Columns (Both Directions).—

$$r = \frac{1 + 6\left(\dfrac{3.33}{113.33}\right) + 6\left(\dfrac{3.33^2}{113.33^2}\right)}{2 + 6\left(\dfrac{3.33}{113.33}\right) + 6\left(\dfrac{3.33^2}{113.33^2}\right)} = 0.542.$$

First-Story Columns (Top to Bottom) (See Table 1).—

$$r = \frac{1 + 3\left(\dfrac{3.33}{116.67}\right)}{2 + 6\left(\dfrac{3.33}{116.67}\right) + 6\left(\dfrac{3.33}{116.67}\right)^2} = 0.499.$$

End Rotation Stiffness at Joint Centers.—
    Beams (End Bay).—

$$S_M = \frac{2\,(3{,}550) \times 10^6}{160\,[1 + 3\,(0.7877)]} \left[\frac{2 + 3\,(0.7877)}{1.7877} + 6\left(\frac{4}{160}\right) + 6\left(\frac{4^2}{160^2}\right) \right]$$
$$= 34.2 \times 10^6.$$

Beams (Modified Stiffness in Center Bay Due to Symmetry Requiring Analysis of Only One Half of Frame—Eq. 14c).—

$$S_M = \frac{2\,(3{,}550) \times 10^6}{160\,(1 + 0.7877)} = 24.8 \times 10^6.$$

Second-Story Columns.—

$$S_M = \frac{2\,(3{,}321) \times 10^6}{113.33} \left[2 + 6\left(\frac{3.33}{113.33}\right) + 6\left(\frac{3.33^2}{113.33^2}\right) \right] = 127.9 \times 10^6.$$

First-Story Columns (Upper End).—

$$S_M = \frac{2\,(3{,}321) \times 10^6}{116.67} \left[2 + 6\left(\frac{3.33}{116.67}\right) + 6\left(\frac{3.33}{116.67}\right)^2 \right] = 123.9 \times 10^6.$$

Proportional factors for distributing moments to ends of members are given in Table 4. The distribution may be done either directly on a diagram

TABLE 4.—PROPORTIONAL FACTORS FOR DISTRIBUTING MOMENTS
TO ENDS OF MEMBERS

	JOINT 1			JOINT 2			JOINT 3			JOINT 4	
Member	End rotation stiffness	Distribution factor	Member	End rotation stiffness	Distribution factor	Member	End rotation stiffness	Distribution factor	Member	End rotation stiffness	Distribution factor
1–3	127.9	0.789	2–1	34.2	0.182	3–5	123.9	0.433	4–3	34.2	0.110
1–2	34.2	0.211	2–4	127.9	0.685	3–1	127.9	0.447	4–2	127.9	0.411
...	.....	.....	2–7	24.8	0.133	3–4	34.2	0.120	4–8	24.8	0.080
...	.....	.....	...	.....	.....	...	.....	.....	4–6	123.9	0.399
...	162.1	1.000	...	186.9	1.000	...	286.0	1.000	...	310.8	1.000

of the frame in the manner frequently followed or in tabular form. The solution is herein presented in tabular form (see Fig. 10), through five cycles after the initial distribution. Each cycle consists successively of: (a) The carry-over of moments from the previously distributed moments; and (b) the distribution of the new unbalanced moment at each joint to the ends of the members. The final summation of moments may be compared with the results of the solution by the slope-deflection method, and the results are seen to check with a maximum error of two in the third significant figure, or a fraction of 1%, except in the case of the smallest moment of 1.07 in-kips,

when the error is about 2%. The moments resulting from the distribution procedure are hypothetical moments at the joint center, and the actual moments at the connection may be found by the method previously described.
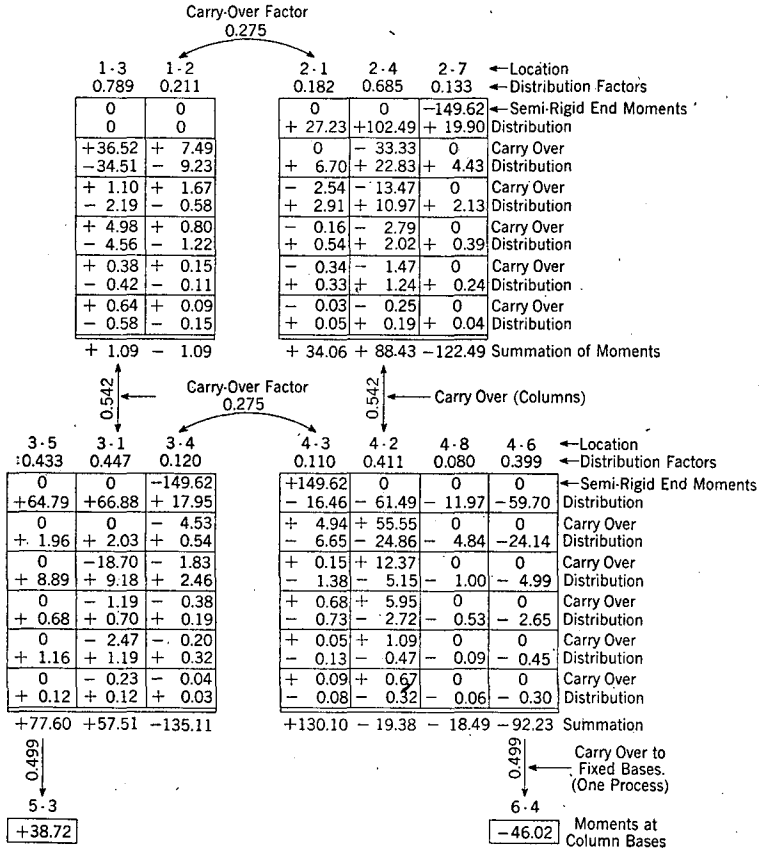
Carry-Over Factor 0.275

1-3	1-2		2-1	2-4	2-7	←Location
0.789	0.211		0.182	0.685	0.133	← Distribution Factors
0	0		0	0	−149.62	←Semi-Rigid End Moments
0	0		+ 27.23	+102.49	+ 19.90	Distribution
+36.52	+ 7.49		0	− 33.33	0	Carry Over
−34.51	− 9.23		+ 6.70	+ 22.83	+ 4.43	Distribution
+ 1.10	+ 1.67		− 2.54	− 13.47	0	Carry Over
− 2.19	− 0.58		+ 2.91	+ 10.97	+ 2.13	Distribution
+ 4.98	+ 0.80		− 0.16	− 2.79	0	Carry Over
− 4.56	− 1.22		+ 0.54	+ 2.02	+ 0.39	Distribution
+ 0.38	+ 0.15		− 0.34	− 1.47	0	Carry Over
− 0.42	− 0.11		+ 0.33	+ 1.24	+ 0.24	Distribution
+ 0.64	+ 0.09		− 0.03	− 0.25	0	Carry Over
− 0.58	− 0.15		+ 0.05	+ 0.19	+ 0.04	Distribution
+ 1.09	− 1.09		+ 34.06	+ 88.43	−122.49	Summation of Moments

0.542 ← Carry-Over Factor 0.275   0.542 ← Carry Over (Columns)

3-5	3-1	3-4		4-3	4-2	4-8	4-6	←Location
0.433	0.447	0.120		0.110	0.411	0.080	0.399	←Distribution Factors
0	0	−149.62		+149.62	0	0	0	←Semi-Rigid End Moments
+64.79	+66.88	+ 17.95		− 16.46	− 61.49	− 11.97	−59.70	Distribution
0	0	− 4.53		+ 4.94	+ 55.55	0	0	Carry Over
+. 1.96	+ 2.03	+ 0.54		− 6.65	− 24.86	− 4.84	−24.14	Distribution
0	−18.70	− 1.83		+ 0.15	+ 12.37	0	0	Carry Over
+ 8.89	+ 9.18	+ 2.46		− 1.38	− 5.15	− 1.00	− 4.99	Distribution
0	− 1.19	− 0.38		+ 0.68	+ 5.95	0	0	Carry Over
+ 0.68	+ 0.70	+ 0.19		− 0.73	− 2.72	− 0.53	− 2.65	Distribution
0	− 2.47	−. 0.20		+ 0.05	+ 1.09	0	0	Carry Over
+ 1.16	+ 1.19	+ 0.32		− 0.13	− 0.47	− 0.09	− 0.45	Distribution
0	− 0.23	− 0.04		+ 0.09	+ 0.67	0	0	Carry Over
+ 0.12	+ 0.12	+ 0.03		− 0.08	− 0.32	− 0.06	− 0.30	Distribution
+77.60	+57.51	−135.11		+130.10	− 19.38	− 18.49	−92.23	Summation

0.499 ← Carry Over to Fixed Bases. (One Process)   0.499

5-3	6-4	
+38.72	−46.02	Moments at Column Bases

Fig. 10.—Solution by Moment Distribution

Sidesway Induced by Unsymmetrical Vertical Loads

The writers have analyzed the frame shown in Fig. 11 by both slope deflection and moment distribution in order to study the effect of neglecting sidesway as induced by unsymmetrical vertical loads. Space does not permit the details of the analysis, which follows usual procedures, however. The dimensions of the frame and size of members are shown in Fig. 11 and the beam a b is assumed to carry a uniformly distributed load of one kip per foot. As in the previous case, it was assumed that the columns were fixed at the base, but the beam-column connections were assumed to have "50% rigidity," which corresponds to $\alpha = 1$. The results are presented in Table 5.

Although no general conclusions should be drawn from this single case, it is seen that in Table 5 sidesway could have been neglected without great
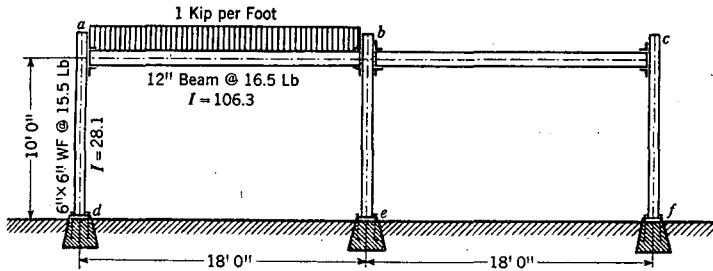
FIG. 11.—UNSYMMETRICALLY LOADED FRAME

error in the end moments. Sidesway due to vertical loads usually will be less in frames with semi-rigid connections as compared with the same frames rigidly connected, and will be further decreased in the actual structure by walls and concrete incasement.

TABLE 5.—SIDESWAY INDUCED BY UNSYMMETRICAL VERTICAL LOADS

Joint and member	BY MOMENT DISTRIBUTION			By slope deflection: Moment at joint center by slope-deflection method	CONNECTION MOMENTS		
	Moments at joint centers, neglecting sidesway	Sidesway moment to balance shear	Moment at joint center, corrected for sidesway		b, in in.	Shear at end of member by Eqs. 7	Moment at face of connecting member
a d	+119.02	− 6.74	+112.28	+112.32	4	−1.34	+106.96
d a	+ 59.32	−10.53	+ 48.79	+ 48.80	0	−1.34	+ 48.80
a b	−119.02	+ 6.74	−112.28	−112.32	3	+8.59	− 86.52
b a	+141.71	+ 5.10	+146.81	+146.85	3	−8.91	+120.12
b e	− 88.61	−10.20	− 98.81	− 98.88	4	+1.29	− 93.72
e b	− 44.16	−12.25	− 56.41	− 56.42	0	+1.29	− 51.26
b c	− 53.10	+ 5.10	− 48.00	− 47.97	3	+0.23	− 47.28
c b	− 7.63	+ 6.74	− 0.89	− 0.89	3	+0.23	− 0.20
c f	+ 7.63	− 6.74	+ 0.89	+ 0.89	4	+0.05	+. 1.09
f c	+ 3.80	−10.53	− 6.73	− 6.72	0	+0.05	− 6.52

EFFECT OF WIDTH OF MEMBER UPON ANALYSIS

In the analysis of frames, the length of each member is often assumed to be equal to the distance center-to-center of joints. The moments thus computed at the joint centers will usually be higher than the actual moment at the connection at the end of the member. This method of computation is usually on the safe side in determining end-connection moments but generally will be on the unsafe side in determining the maximum positive moment near the center of the beam.

An approximate correction is sometimes made for the effect of member width. The end moments computed in the foregoing manner are used to construct the moment diagram. The actual end-connection moment to be used in design is then taken as the ordinate to the moment diagram at the face of the column or connecting member. This method usually gives values of end-connection moments that are too low.

The error by either of the foregoing methods becomes greater as the ratio between the width of the joint and the length of the member increases. The

average errors are also greater for frames with semi-rigid connections than for frames with rigid connections.

In order to develop criteria to determine when, and when not, to consider member width in analysis, the behavior of the frame shown in Fig. 12 was studied for various ratios of joint width to member length. The load was assumed to act uniformly on beams 1–2 and 3–4, and the analyses were made for four different ratios of joint width to member length—namely, $\frac{1}{15}$, $\frac{1}{12}$, $\frac{1}{9}$, and $\frac{1}{6}$. Analyses also were made neglecting member width entirely, and the method of arbitrary correction previously outlined was tried also. The
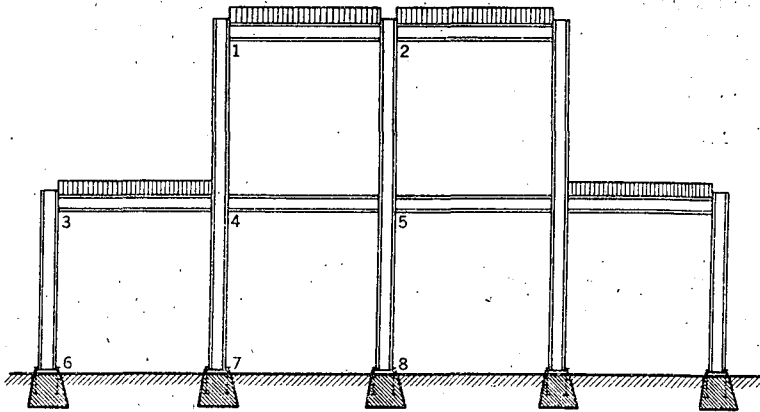
Fig. 12

analyses were made both for a frame with rigid connections and for a frame with continuous columns but with semi-rigid beam-to-column connections. All of the analyses were made by the method of moment distribution.

Special note should be made of the length of the columns in relation to beam depth. The methods herein presented to take account of width of joint are based on the assumption that the interior of the joint may be considered infinitely rigid in comparison with the bending stiffness of the member. In the case of a beam framing into a column, this assumption seems reasonable, particularly if the column runs through the joint without a splice. In the case of the continuous column, however, the connection moments are introduced by concentrated lateral forces acting at the top and bottom of the beam in the type of connection shown in Fig. 2(b). In such a case it may be shown that nearly correct results may be obtained for the moments in the column by assuming a length correction for the column of one third the beam depth at each end instead of one half the beam depth. This correction was made in the analyses under consideration and was found to give good results in actual frame tests.

The results of these studies are shown in Fig. 13 for the frames with rigid and semi-rigid connections, respectively. The solid lines give the percentage of error of moments determined with a neglect of joint width as compared with

corresponding moments correctly computed at the face of the joint. The broken lines give the percentage of error resulting from the arbitrary correction for joint width by neglecting it in the analysis but using the ordinate of the moment diagram at the face of the joint.

It is noted in Fig. 13 that the maximum percentage of error occurs in the case of the large end moments in the loaded beams at joints 1 and 4. It also
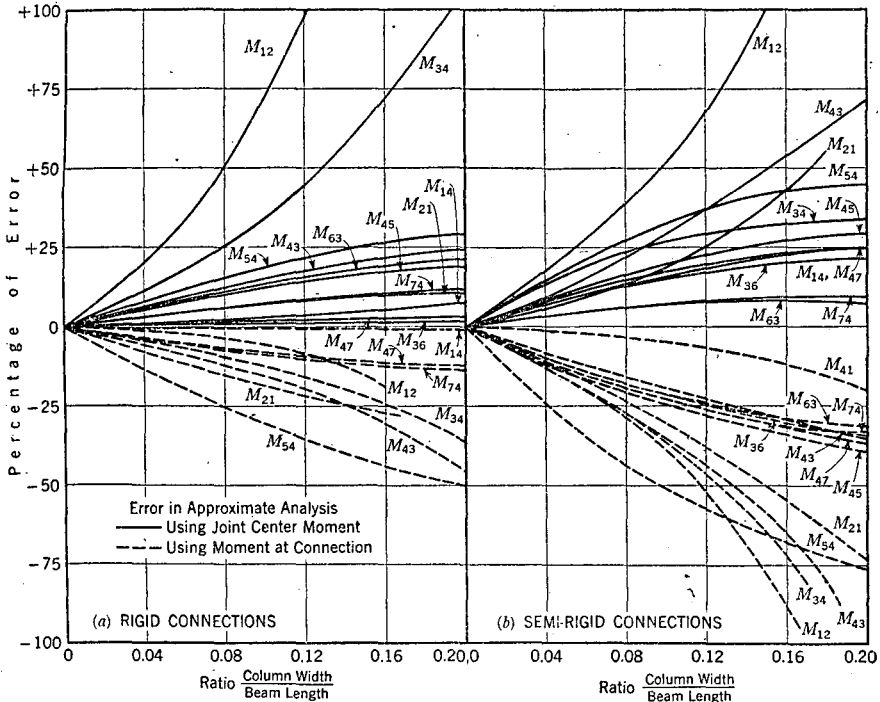

FIG. 13.—PERCENTAGE OF ERROR IN CONNECTION MOMENT, NEGLECTING MEMBER WIDTH

may be seen that the errors are usually larger in the frame with semi-rigid connections than in the frame with rigid connections. The errors are appreciable even for low ratios of joint width to beam length. In the case of a one-to-twenty ratio, for example, the error may be as high as 20%, with the average error about 5% for the rigid frame and as high as 25%, and with an average error close to 10% for the frame with semi-rigid connections. As the ratio of joint width to beam length increases, the errors become increasingly larger.

A fairly close approximation for the moment at the connection is obtained by neglecting joint width in the analysis and using, as the connection moment, the moment halfway between the connection and joint center.

COMPARISON BETWEEN THEORETICAL ANALYSES AND TEST RESULTS

In order to compare the results of analyses with the actual behavior of building frames, two full-size, all-welded, model building frames were con-
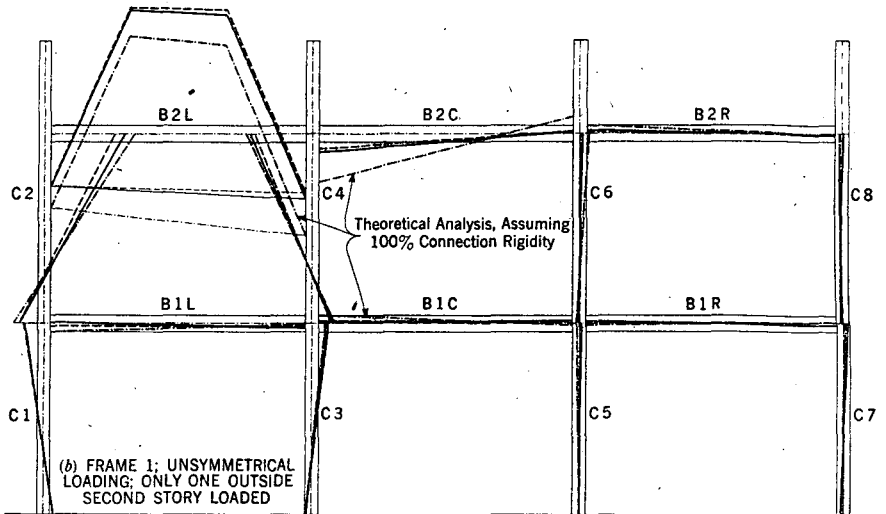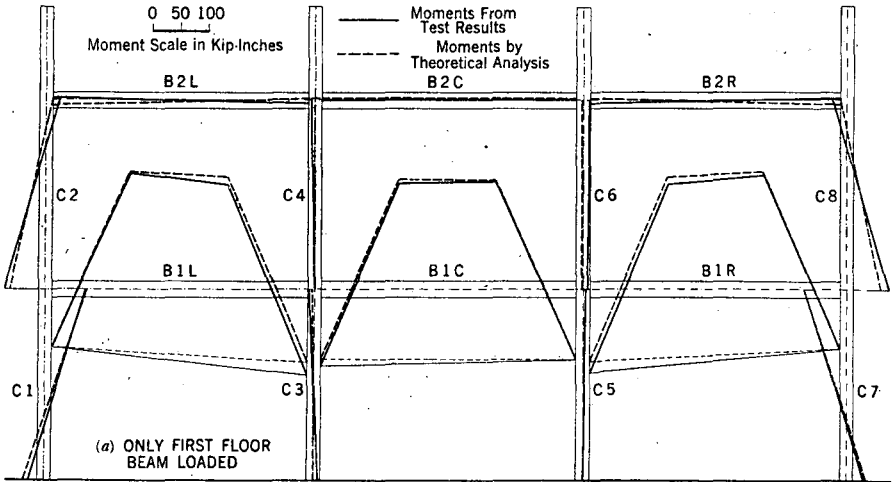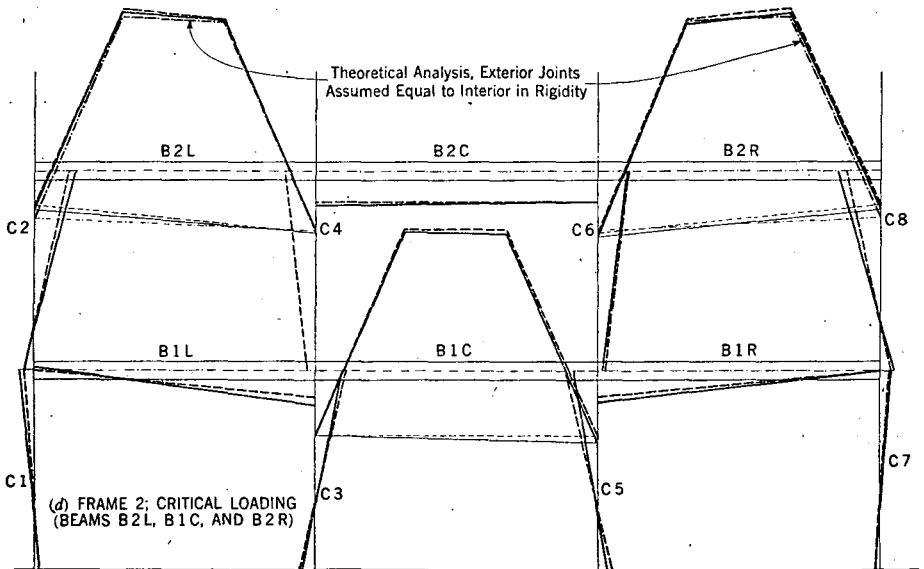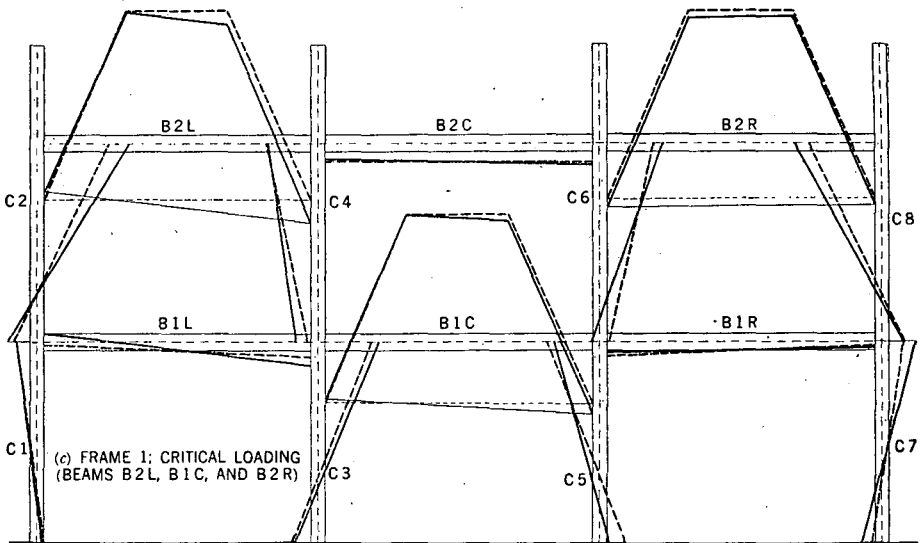
0 50 100
Moment Scale in Kip-Inches

Moments From Test Results

Moments by Theoretical Analysis

B2L   B2C   B2R

C2   C4   C6   C8

B1L   B1C   B1R

C1   C3   C5   C7

(a) ONLY FIRST FLOOR BEAM LOADED

B2L   B2C   B2R

C2   C4   C6   C8

Theoretical Analysis, Assuming 100% Connection Rigidity

B1L   B1C   B1R

C1   C3   C5   C7

(b) FRAME 1; UNSYMMETRICAL LOADING; ONLY ONE OUTSIDE SECOND STORY LOADED

FIG. 14.—COMPARISON OF COMPUTED AND

(c) FRAME 1; CRITICAL LOADING
(BEAMS B2L, B1C, AND B2R)

Theoretical Analysis, Exterior Joints
Assumed Equal to Interior in Rigidity

(d) FRAME 2; CRITICAL LOADING
(BEAMS B2L, B1C, AND B2R)

OBSERVED MOMENTS, FOR CRITICAL CONDITIONS

structed. Details of these tests have already been presented in another paper by the writers (2).

Frame No. 1 was made with beam-to-column flange connections, whereas frame No. 2 had beam-to-column web connections. The general dimensions and size of members of frame No. 1 are shown in Fig. 8 in connection with the illustrative example (see heading "Analysis by Slope-Deflection Method"), and a photograph of the same frame is shown in Fig. 9. The beam-to-column connection used in these frames consisted of welded seat and top angles, the details and semi-rigid properties of which have been described elsewhere (2). Vertical loads were applied to the frames by means of water tanks, which are shown in Fig. 9 in one of the loading positions. Each frame was braced laterally near each joint by means of flexible ties welded between columns of the frame and columns of the laboratory. These ties had reduced sections near each end that allowed the frame full freedom to bend or move laterally in its own plane but that prevented movement out of its own plane.

The computation of the moments developed during tests of the frames was made by measuring the rotation at the ends of each beam and at the joint centers by means of the 20-in. level bar which was illustrated in Fig. 3. Then the moments at the end of each beam and column could be calculated by the slope-deflection equations (see Eqs. 3).

The connection constants for typical joints in the frame were determined by means of the setup shown in Fig. 2(b). The experimentally determined values of these connection constants, as determined by Fig. 4, have been used in the theoretical analyses. The method of moment distribution was used and a typical analysis, taking account of the width of member, has been presented in the illustrative example.

Fig. 14 shows both the computed and experimentally determined moments for several of the critical conditions of load that were applied to the two frames. Fig. 14(a) shows moment diagrams for frame No. 1 with only first-floor beam loaded. Fig. 14(b) is for frame No. 1 with unsymmetrical loading in which only one outside second-story beam was loaded. Sidesway was neglected in the analysis but the agreement between analysis and experimental result is excellent. A comparison is made in this case with an analysis assuming completely rigid joints. The actual test results agree well with the analysis for semi-rigid joints but are widely divergent from the analysis for rigid points. It should be noted that the moments "taper out" much more rapidly in a frame with semi-rigid connections than in one with rigid joints. Fig. 14(c) is for a critical condition of loading. In applying the test load for this case, the order of loading was purposely unbalanced but the moments by test are in fairly good agreement with the theoretical analysis. Fig. 14(d) is for frame No. 2, with beam-to-column web connections, and is for the same critical loading condition as Fig. 14(c). The outside column connections in frame No. 2 has less rigidity than the inside, and this was taken into account in the analysis. The analysis based on the assumption that the outside joints are as rigid as

the interior joints is also given, and it is seen that the test results usually fall between the two different analyses.

In general, the test results agree well with the methods of analysis which have been presented. The results also show that the test of a single joint to determine the connection constant gives a satisfactory measure of the behavior of the same type of joint used in an actual frame.

THE DESIGN OF FRAMES FOR PARTIAL RIGIDITY

The methods of analysis which have been presented in this paper obviously are not directly applicable to design. Any method of statically indeterminate analysis requires an assumed structure as a preliminary to design. To assume a building design, and then to analyze such a highly redundant structure by the methods that have been presented, would be an impractical design procedure, warranted only for very special problems.

For routine building design, a suitable method must be direct and simple in application. Such design methods have been presented by the writers in conjunction with a particular type of all-welded, beam-to-column connection (2), and in a more general article covering the application to any semi-rigidly connected structure (3). The British Steel Structures Research Committee (1) has developed design procedures for frames with semi-rigid riveted connections. In a letter dated December 26, 1940, S. D. Lash, secretary of the Subcommittee on Steel Construction, National Building Code, National Research Council of Canada, stated that simplifications in the original design method have been made in Great Britain and that similar steps are in progress in Canada.

The design procedure for the beams in welded building frames with semi-rigid connections that has been developed by the writers (2) may be outlined as follows:

1. The beams are designed by the usual procedure of computing the required section modulus for maximum simple beam moment.

2. The section modulus for maximum simple beam moment is multiplied by a reduction factor that depends on the distribution of load and relative stiffness of the simple beam and adjacent column sections. This reduction factor is obtained from a graph or simple formula and is based on the most critical combination of load possible.

3. The final beam selection is determined by the reduced section modulus found by step 2.

Although the method was developed for designing beams with welded connections, it is applicable to any frame having connections with the desired semi-rigid properties.

In computing the reduction factor, the stiffening effect of adjacent beams was neglected, and the same formula applies to exterior and interior bays. By this procedure, with end connections designed for 50% end restraint, an

average saving in the weight of beams of between 15% and 20% was found possible. If greater refinement and complexity are introduced into the design procedure, the average saving in weight of beams might be raised to more than 20%.

CONCLUSION

The methods presented and corroborated by test in this paper represent a refinement in the analysis of building frames. It may be questioned whether such refinement is warranted. The concrete encasement of beams, columns, walls, and partitions, and the uncertainties of applied load, all represent indeterminate quantities which, undoubtedly, may have as great, or greater, effect upon frame behavior as does the semi-rigidity of the bare steel connection. Nevertheless, discounting these uncertainties as assets that cannot be counted upon definitely, there remains the certain dependable bare connection end-restraint. This influence can be determined and applied to the development of improved and more economical methods of design.

ACKNOWLEDGMENT

APPENDIX

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DISCUSSION

MAURICE P. VAN BUREN,[4] ASSOC. M. AM. SOC. C. E. (by letter).—A basis for effecting valuable economies in certain types of structures is afforded by this interesting paper. However, one important correction to the analysis should be made. In comparing the moments at the face and at the center of a joint, consideration must be given to the reaction as a load on the structure. It does not appear that this has been included in the examples given. It can be provided for easily by the addition of a term to both Eq. 4a and Eq. 4b. At the point of inflection of the column, the reaction is uniformly distributed, and will cause a moment about the joint center of $V g b$, in which $g b$ is the distance from the joint center to the centroid of the half column. This moment is of opposite sense to the moments $V b$, and the last terms of Eqs. 4a and 4b will then read: $- V_{AB}' (1 - g) b_{AB}$ and $V_{BA}' (1 - g) b_{BA}$, respectively. The significance of this effect is apparent in the case of the flanged column illustrated in Fig. 5(a), for which g approaches 1, and the term involving V practically vanishes. The authors' comments on the extent to which final results will be affected is awaited with interest.

WAYNE W. SMITH,[5] LEONARD P. ZICK, JR.,[6] JUNIORS, AM. SOC. C. E., AND CONRAD C. WAN,[7] ESQ. (by letter).—As the authors have shown, there are reasonable possibilities for a more economical design of frame construction by considering the semi-rigidity of connections. As they have stated, their method, in practice, is too cumbersome to be used directly. Their suggested procedure using simple beam moments and reduction constants thus appears to be the most practical method of design. However, it should be emphasized that every continuous structure or frame is an individual problem and should be dealt with as such. Graphs, charts, and simple formulas may be used to simplify certain designs, but a thorough knowledge of their limitations must guide their use.

According to the authors' test results, the maximum moment developed was about 274 kip-in. (Fig. 14(d)). For the 10-in., 25.4-lb, I-beam used, this meant that a stress of 11 kips per sq in. was developed, a stress considerably below the elastic limit of steel. The use of the connection constant γ seems quite proper within the "design range," but as seen in Fig. 15 the constant changes rapidly beyond this range. This design range is not clearly specified in the paper. Making the logical assumption that the design range is one half the elastic limit, there is an "overload range" for which γ is no longer constant. Failure should not occur until the elastic limit is reached, which limit is the farther boundary of this overload range. Realizing that the maximum moment does not necessarily occur at the joint, the question arises as to how important is the discrepancy due to assuming γ constant when the member is stressed

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nearly to the elastic limit. To ignore this point may reduce the factor of safety for the member.

It seems that the authors' suggestion of neglecting the sidesway moments due to vertical unsymmetrical loadings is contradictory to the degree of refinement in their analysis. As shown in Table 5, the sidesway moments range from 6.74 to −12.25 kip-in., which, in the latter case, amounts to a decrease of about 29%.

In the comparison by the method of moment distribution of a rigid structure and a semi-rigid structure, it is very interesting to notice the changes in the carry-over factors and the end rotation stiffness (distribution factor).

The solutions in Table 6 are for the test frame shown in Fig. 8 assuming rigid connections and neglecting joint width. The authors' results are taken from Fig. 10. As shown, the carry-over factor for the beams was decreased from the usual value of 0.5 to 0.275. Since the case of the semi-rigid beam is the opposite of the haunched beam, the results are as expected. The changes in the carry-
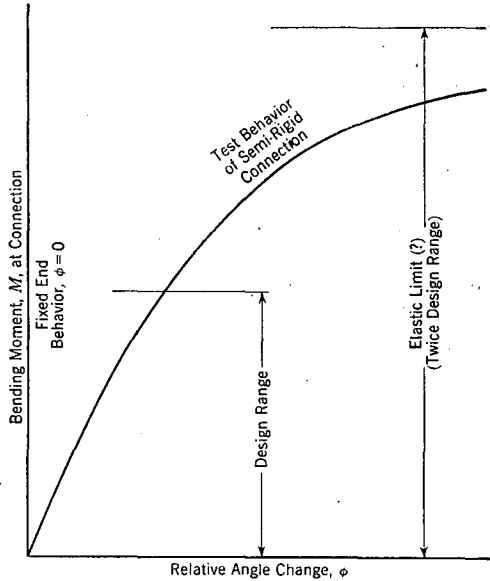
FIG. 15

TABLE 6.—COMPARISON OF MOMENTS AND STIFFNESS FACTORS FOR RIGID AND SEMI-RIGID CONDITIONS

Member	JOINT MOMENT		CARRY-OVER FACTOR		DISTRIBUTION FACTOR	
	Authors	100% fixed	Authors	100% fixed	Authors	100% fixed
1-2	−1.09	7	0.275	0.500	0.211	0.443
1-3	1.09	−7	0.542	0.500	0.789	0.557
2-1	34.06	93	0.275	0.500	0.182	0.362
2-4	88.43	91	0.542	0.500	0.685	0.456
2-7	−122.49	−184	.....	.....	0.133	0.182
3-1	57.51	83	0.542	0.500	0.447	0.358
3-4	−135.11	−187	0.275	0.500	0.120	0.284
3-5	77.60	104	0.449	0.500	0.433	0.358
4-2	−19.38	−40	0.542	0.500	0.411	0.313
4-3	130.10	198	0.275	0.500	0.110	0.249
4-6	−92.23	−113	0.449	0.500	0.399	0.313
4-8	−18.49	−45	.....	.....	0.080	0.125
5-3	38.72	52	0.449	0.500	.....	.....
6-4	−46.02	−56	0.449	0.500	.....	.....

over factors for the columns are due to the consideration of the joint width only. The results given in Table 6 for the analysis with joints 100% rigid is merely additional information. The authors have already shown the large

disagreement considering 100% rigid connections (see heading "Comparison Between Theoretical Analysis and Test Results").

The authors are to be commended for the close agreement between the theoretical and experimental results that they obtained. Also, in their consideration of the joint width, their results showed that for large ratios of the joint width to the member length some adjustment should be made.

S. D. LASH,[8] ESQ. (by letter).—It is gratifying to find that attention is being directed toward problems connected with the design of beams in steel building frames. The fact that considerable economies are possible without change in construction procedure makes it the more surprising that this subject has been comparatively neglected by practising engineers.

The authors do not claim that the methods presented by them are suitable for the practical design of beams in steel building frames. However, since the methods of analysis are presented as a basis for simpler design procedures, it appears legitimate to consider the paper from the point of view of design rather than analysis.

The calculation of end moments in beams with semi-rigid connections differs from most other design calculations, inasmuch as it is necessary to rely upon the results of laboratory tests for the properties of the connections. Many investigators have assumed that the relation between applied moment and angular deformation for any semi-rigid connection is approximately linear, since this assumption is the obvious way of introducing modifications of regular design procedures. The laboratory investigations referred to by the authors have shown, however, that such an assumption is incorrect for most types of semi-rigid connections. The relation between moment and angular deformation is not linear, and the behavior of a connection on first loading differs considerably from its behavior on subsequent reloadings. In an experiment on a steel frame, if the loads are applied, removed, and reapplied, measurements of strains being taken on the reloading, these strains will depend upon the reloading curves for the connections, but the total strains will be those corresponding to the initial loading curves.

Since moment-angle curves for semi-rigid connections are not linear, it would appear to be necessary to introduce some form of "limit design" when determining allowable restraining moments. Thus, for example, if a factor of safety, or load factor, of n is desired, the allowable restraining moment at working load should be $\frac{1}{n}$ times the computed restraining moment at n times the working load.

It should also be pointed out that the "connection constant γ," referred to by the authors, depends not only upon the type of connection, but also, in the case of flange-angle connections, at least, upon the depth of the beam. The constant as determined experimentally cannot be used for beams of a depth other than that used in the original tests unless suitable corrections are made.

In order to make the foregoing points more specific, Fig. 16 has been prepared using information previously published by the Steel Structures Research

Committee in Great Britain.[9] Curve (1) represents the moment-angle relationship for 4-in. by 4-in. by $\frac{1}{2}$-in. flange-angle connections on a beam of 12-in. depth. Actually, this curve defines a lower limit obtained by combining the curves resulting from tests on seven riveted specimens submitted by different fabricators.
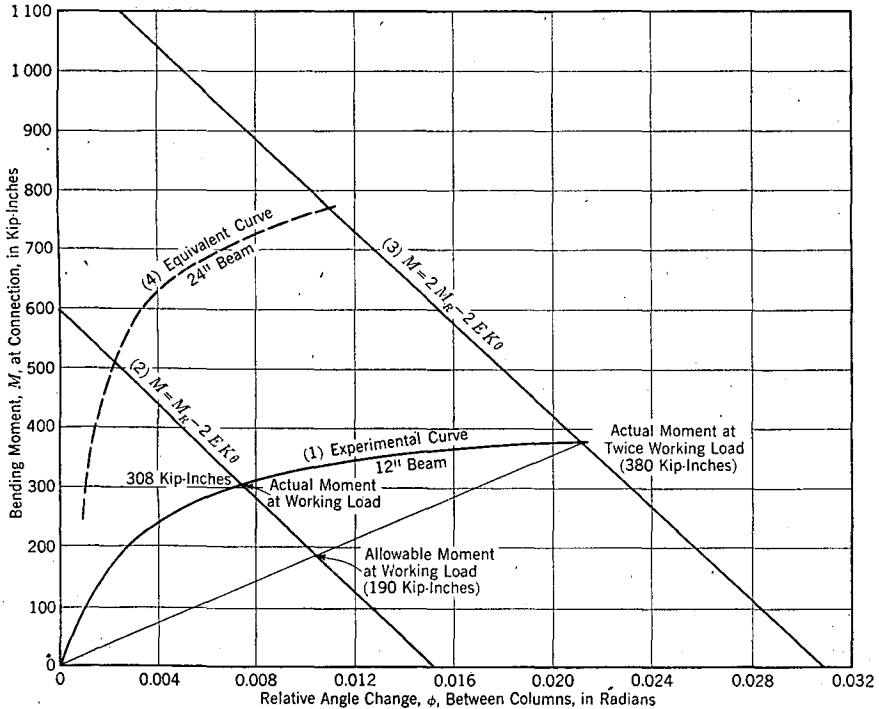

FIG. 16

For simplicity it will be assumed that a beam is attached to completely rigid columns. In such a case for symmetrical loading, the end moment may be conveniently determined by solving the equations graphically:

$$M - f(\phi) \dots\dots\dots\dots\dots\dots\dots\dots (23)$$

representing the curve of the connection; and

$$M = M_R - 2\,E\,K\,\phi \dots\dots\dots\dots\dots\dots (24)$$

representing the slope-deflection equation.

In Fig. 16 two lines, (2) and (3), are drawn corresponding to slope-deflection equations for two different magnitudes of load applied to a 12-in. beam 20 ft long. Line (2) corresponds approximately to a working load and line (3) to a load twice as great. In the first instance the end moment is 308 kip-in. and in the second, 380 kip-in. Thus, for a load factor of two, the allowable moment

[9] Final Report, Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, p. 282.

at working load should be 190 as indicated by line (5) instead of 308 kip-in. This diagram will also serve to indicate that the working range of the curve for a connection is actually quite extensive. For case shown it will extend at working loads from angular deformations of about 0.002 radians up to 0.12 radians and, if overload is being considered, up to twice these values. For this reason it is desirable to take experimental readings over a somewhat greater range than has usually been done in the past.

Fig. 16 also shows, approximately, the effect of depth of beam upon the moment-angle relationship obtained. The moment-angle curve (1) shown is based on tests using 12-in. beams. If 24-in. beams had been used in these tests, it is probable that the curve would have been similar to that shown by curve (4). If it is desired to present the properties of connections without relation to depth of beam, a convenient way of doing so is to transform the ordinates and abscissas from M and ϕ to $\dfrac{M}{D}$ and ϕD, respectively. The ratio $\dfrac{M}{D}$ may be thought of as the pull on the connection and ϕD as its linear deformation. The application of this method to web connections has not been investigated.

In frames having semi-rigid beam connections, the most reasonable approach to the problem of making allowance for width of members appears to be to assume that the columns are continuous members to which the beams are attached. After all, this is what actually occurs. It then appears logical to assume the effective length of a column as the length measured between the neutral axes of the beams, and the effective length of a beam as the length measured between the faces of the columns.

In the case of flange-angle connections particularly, this method leads to an overestimate of column moments, since the loads are transmitted to the columns by the connection angles as concentrated loads and not as moments at the neutral axes of the beams. No appreciable error will be introduced by assuming the maximum moment in the column to be the moment at the estimated level of the point of contraflexure of the flange angle connected to it. Although this correction may be worth making from the point of view of the design of the column, it is doubtful if it is worth making when considering the distribution of moments throughout a framework, since an appreciable rotation of the end of the beam may result from local shear deformation of the column in the vicinity of the connection. Results published elsewhere[10] have shown that this deformation may increase the total rotation of the column at the level of the beam by 30% or 40%.

The National Research Council of Canada has recently published Part 3 (Engineering Requirements) of the National Building Code (a model code for the use of Canadian municipalities). In this Code an attempt has been made to give some weight to the considerations mentioned by the authors at the beginning of their paper. In particular, it is required that bending moments in columns other than those supporting a symmetrical arrangement of beams of approximately equal span shall be investigated, and the stresses resulting from them shall be provided for. With this mandatory requirement is a

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[10] Final Report, Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, p. 357.

permissive requirement stating that "where a beam is restrained at either end due allowance may be made for such restraint." The Code does not restrict the designer to any particular method for estimating moments in columns or restraining moments in beams, but acceptable methods of doing these things are given as Appendixes. A description of the method given for computing allowable end moments was published in 1941.[11]

DEAN F. PETERSON, JR.,[12] ASSOC. M. AM. SOC. C. E. (by letter).—The methods presented by the authors have made an excellent rational start toward a better basic understanding of what actually occurs in a loaded frame in which the members are connected in the customary way. Expressing the laboratory-determined "stiffness" of the connection in terms of the "stiffness" of the
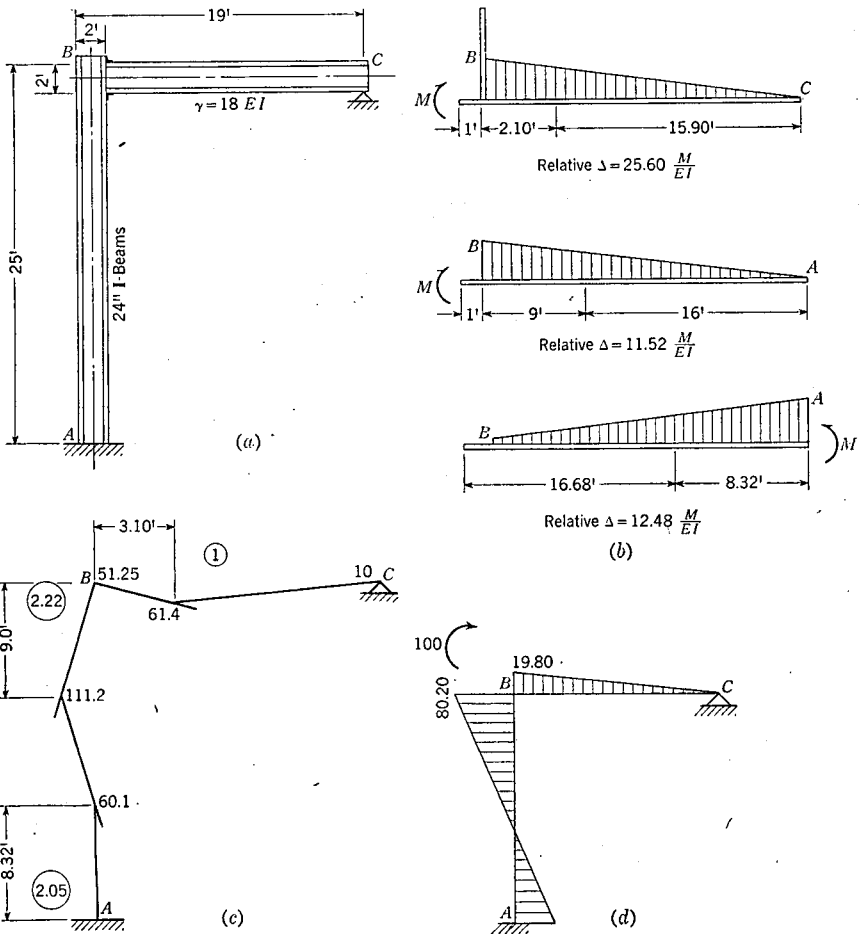

FIG. 17

[11] "The Design of Beams in Steel Frame Buildings," N. R. C. 992, National Research Council of Canada, 1941.

[12] Asst. Engr., Sanderson Porter, Pine Bluff, Ark.

connecting member "opens the door" for the application of the "moment-distribution" and "slope-deflection" methods. The writer believes, however, that the most direct and simple method for solving such an elastically complex structure as a building frame with semi-rigid connections is the "flexure factor" method.[13] Using this method it is unnecessary to solve either simultaneous equations or to make the rather involved adjustments of the "moment-distribution" method. Since no mention is made of the application of the "flexure factor" method, either in the text or in the references given, it may add to the completeness of the discussion to call attention to it.

To illustrate the application of this method, the writer has chosen the simple frame of Fig. 17(a). The "stiffness" of any member is the moment required at one end, with the other end freely supported, to produce an angle change of unity in the direction of the end tangents. In Fig. 17(b) the proportionate stiffnesses of the members are determined from the $\frac{M}{E\,I}$ -diagrams. Fig. 17(c) is the traverse of the elastic curve due to an unbalanced moment of 100 at Point B and Fig. 17(d) is the resulting moment diagram. For a vertical load of 10 kips at a point 9 ft from Point B the "fixed-end" moment at Point B, by the "moment-area" method or by Eq. 13, is 13.86 kip-ft and $M_{BC} = M_{BA} = (0.8020)(13.86) = 11.15$ kip-ft. The writer was able to check this value by means of Eqs. 4.

The writer doubts that the effect of sidesway should be generally ignored. If sidesway were allowed in his example M_{BA} would be decreased from $+ 11.15$ kip-ft to $+ 6.54$ kip-ft, and M_{AB} would change from $+ 5.55$ kip-ft to $- 6.54$ kip-ft. What would be true, however, of such buildings as powerhouses or low warehouses would probably not be so true of a multi-story office building.

The authors are to be complimented on an excellent research project. Perhaps the time will come when the designer will be able to predict to a reasonable extent what the value of γ will be for a certain connection, or, better still, be able to design a connection for a predetermined value of α.

R. W. STEWART,[14] M. AM. SOC. C. E. (by letter).—Considerable effort has been made by the authors to overcome the complexities which result when the effect of semi-rigid connections is incorporated into the computation of the bending moments in a steel frame.

Each of the methods presented has defects which render its use difficult. This may account for the authors' statement that the methods of analysis are too complex for ordinary design use, but can be made expeditious by the use of charts and diagrams in connection with simpler design procedures. It may also account for the fact that the general case, which would occur if an unsymmetrical beam having variable moment of inertia were used between the riveted joints, was not included in the scope of the paper.

A specific list of the defects in the authors' solutions is as follows:

(1) The constants used in end-moment distribution, which are based on the

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[13] "Relative Flexure Factors for Analyzing Continuous Structures," by Ralph W. Stewart, *Transactions*, Am. Soc. C. E., Vol. 104 (1939), p. 521.

[14] Engr. of Bridge and Structural Design, City of Los Angeles, Los Angeles, Calif.

moment required to produce unit rotation at one end of a beam when the other end is fixed, require long series of algebraic terms to express their values if a member is affected by asymmetry or other special conditions like the yielding joints treated by the authors. A glance at the right-hand column and then at the left column (Case I) of Table 1 will disclose how the expressions for moment-distribution constants expand when conditions other than the simplest are introduced. If these constants were further encumbered by unsymmetrical tapering members between the riveted joints, the computation of α and β would become complex, making the use of the formula very tedious and difficult.

(2) The slope deflection equations (4a and 4b) are based on $K = \frac{I}{l}$-values which are applicable only to members of uniform section between joints. The introduction of unsymmetrical tapering members would add substantially to the difficulty of computing the necessary constants.

(3) The use of the authors' methods independently of a set of charts pertaining thereto would require having at hand, for reference, formulas that are too complicated to remember.

(4) The authors' methods offer no easy facility for detecting errors which may occur in computations.

The following solution of Fig. 11 entirely eliminates defect 3 attributed to the authors' methods, as any one familiar with this method of attack would not use any references to solve any of the authors' problems, except a steel handbook giving the moments of inertia of rolled beams. It greatly alleviates defect 4 in that an automatic check involving a single setting of a slide rule will verify a large portion of the solution. It alleviates to a considerable degree defects 1 and 2 by using "illustrated" member constants which are simpler than moment-distribution constants and which can be altered without difficulty to include the general case of unsymmetrical members between joints. Fig. 18(a) represents an $\frac{M}{I}$-diagram due to a moment at one end of a beam shown in Fig. 11, the other end being hinged. It requires no explanation; Fig. 18(b) is the appurtenant graph of the tangents to the elastic curves in the beam, known as a traverse of the elastic curves, in which Δ represents the curvature between joints. To evaluate ϕ, which represents the angle of yield at a joint, Eqs. 2 and 5 yield

$$\alpha = 2 E K \frac{\phi}{M} \dots\dots\dots\dots\dots\dots\dots(25)$$

For the flexure of a beam of constant section, hinged at one end, $\Delta = \frac{M}{2 E K}$. For $\alpha = 1$ (which is the condition for Fig. 11), by solving each of these equations for M, it is found that $\phi = \Delta$ for the case in which Δ is the curvature represented by a triangular $\frac{M}{I}$-diagram. For Fig. 18(b), ϕ will equal the area of the $\frac{M}{I}$-diagram under the dotted line in Fig. 18(a). With this explanation and

the traverse principle[15] that in traverse triangles the lengths of the sides (considered as their horizontal projections since altitudes are negligible) are proportional to the opposite angles, Fig. 18(b) can be readily sketched and evaluated.   The difference between Figs. 18(a) and 18(b) is of interest.   Fig.
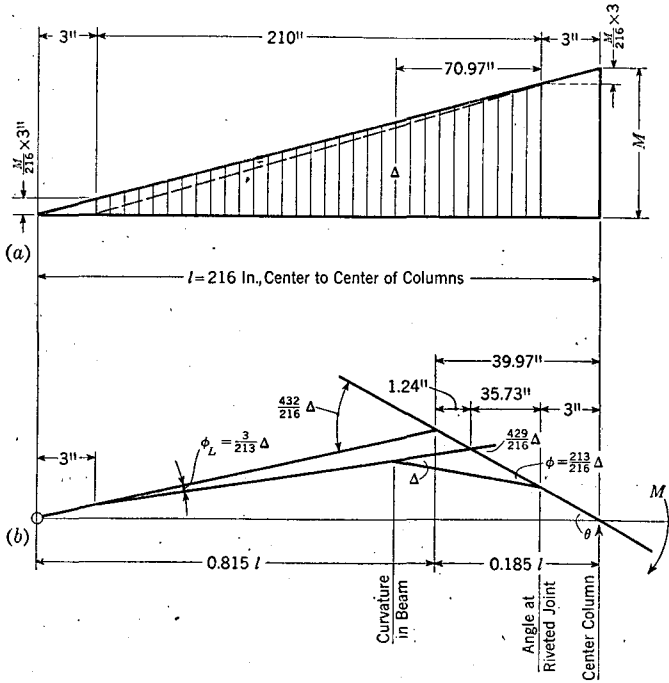

Fɪɢ. 18

18(a) is a moment-area diagram that cannot show joint yield angles or joint rotation angles.   Fig. 18(b) is a traverse diagram that shows all the elements of the flexure.   The moment-distribution constants in Table 1 are necessarily based on flexure due to the existence of moments at both ends of a beam which is why they become more complicated for unsymmetrical conditions than Fig. 18(b) which deals with a moment at one end only.   The stiffness factor computed from Fig. 18(b) is first computed as the moment necessary to give the value of unity to the angle shown as $\frac{432}{216} \Delta$, which represents the over-all curvature in the beam.   Top and bottom stiffnesses for the columns are computed by a procedure that is similar but is simplified because there are no ϕ-angles in the columns.   After-all stiffness factors are computed, they are multiplied by a factor which will reduce the deck stiffness to unity in order to simplify subsequent arithmetical work.

The computation for Fig. 19(a) is a much faster procedure than the un-initiated would suspect.   When the traverse computation arrives at the top of a column both column moments are obtained by one setting of the slide

rule as they bear the same ratio to the moments written on column *cf* that the angle at the top of the column bears to the angle at the top of column *cf*.

Fig. 19(*b*) is nearly all copied from Fig. 19(*a*). The computations for Fig. 19(*b*) take less than one minute. As soon as they are complete a single slide-
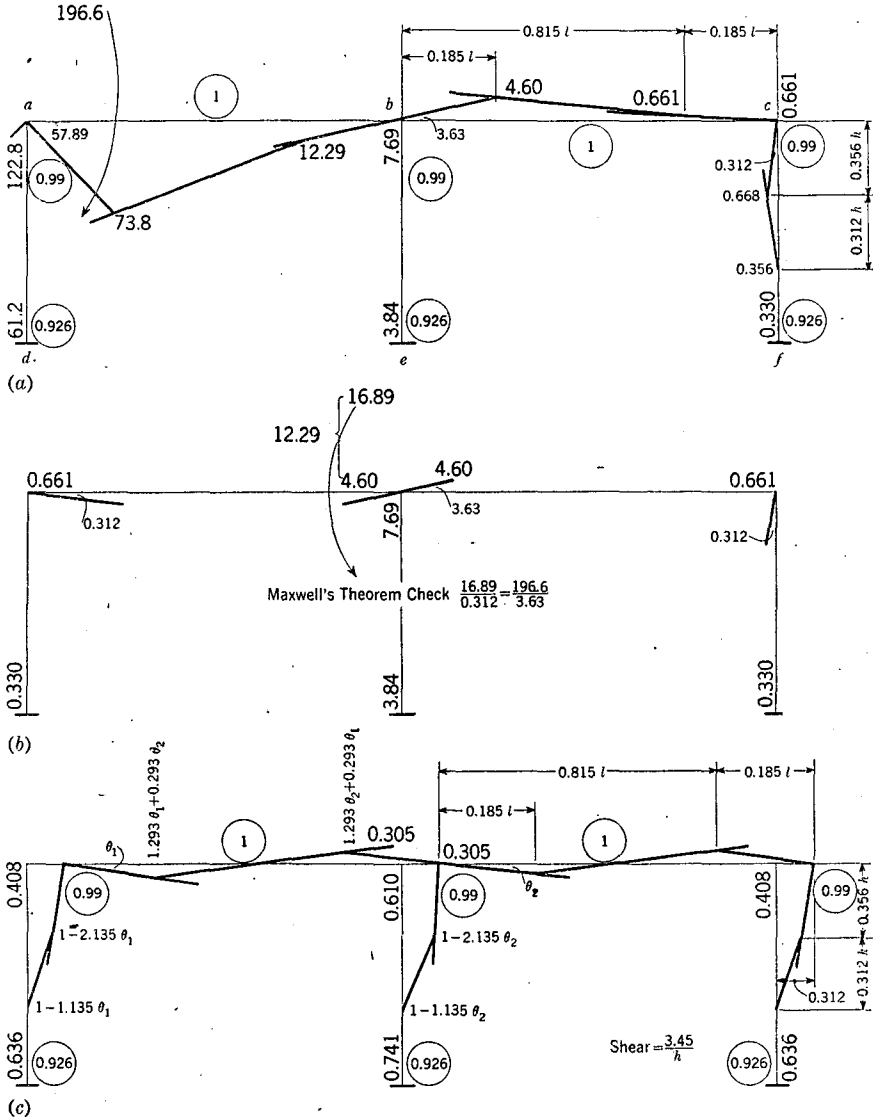

Fig. 19.

rule setting will disclose whether either Fig. 19(*a*) or 19(*b*) contains an error.

There is more than one way to compute sidesway moments using the traverse method. Fig. 19(*c*) represents an elastic curve traverse converted into an amplified form of slope deflection which can be applied to the two-story

structures treated in the authors' paper. The key to this method is to express the values of the angles which govern the stiffnesses in terms of the θ-angles and then proceed as in slope deflection.

Table 7 shows the moments computed from Fig. 2. The check with the authors' moments is exact.

TABLE 7.—MOMENTS COMPUTED FROM FIG. 19

Description	d	a	ba	be	e	bc	c	f	Shear
$\frac{179.38}{196.6}$ × Fig. 19(a)	55.8	112.0	11.2	7.0	3.5	4.2	0.6	0.3	158.2 ÷ h
$\frac{179.38}{16.89}$ × Fig. 19(b)	3.5	7.0	130.5	81.6	40.7	48.9	7.0	3.5	101.3 ÷ h
Moments without sidesway . . .	59.3	119.0	141.7	88.6	44.2	53.1	7.6	3.8	56.9 ÷ h (check)
Johnston-Mount.	59.3	119.0	141.7	88.6	44.2	53.1	7.6	3.8	. . . .
Sidesway Correction:									
$\frac{56.9}{3.45}$ × Fig. 19(c)	10.5	6.7	5.1	10.2	12.2	5.1	6.7	10.5	56.9 ÷ h
Moments with sidesway	48.8	112.3	146.8	98.8	56.4	48.0	0.9	6.7	Zero
Johnston-Mount.	48.8	112.3	146.8	98.8	56.4	48.0	0.9	6.7	. . . .

To summarize: The methods demonstrated by the authors to establish a basis for the design of structures with semi-rigid joints required the use of derived constants and derived formulas which are inherently very complex.

The use of the basic constants of flexure[16] in an orderly manner assisted by a pictorial graph of the flexure will eliminate the necessity of making reference either to special slope deflection equations or to formula for constants in an independent solution of problems of this class. ("Independent" solution means one not dependent on a set of charts and diagrams prepared by some one else. In court testimony in a building failure case the solution should be independent.)

It will also eliminate most of the anxiety regarding the possibility of errors.

JAROSLAV J. POLIVKA,[17] M. AM. SOC. C. E. (by letter).—A complete analysis of building frames with semi-rigid connections is presented in this interesting and valuable paper. The authors have corroborated their analytical methods by tests. Both methods applied (slope deflection and moment distribution) have the same disadvantage—namely, that the computations must be made separately for each type of loading, and the exact and refined method, demonstrated by the authors, becomes complex and tedious, even for such a simple case as a two-story building with three equal bays (Fig. 8). The authors agree that the method is not applicable to ordinary design. Three types of loading (Fig. 14) require the determination of sixteen unknown slopes and this number increases to forty eight for an unsymmetrical four-story building.

The writer has attempted to find the answer to three questions relating to the analysis of building frames with semi-rigid connections:

(1) How are the results affected by certain assumptions that would simplify the analysis considerably?

[16] *Transactions*, Am. Soc. C. E., Vol. 102 (1937), pp. 41–44.

[17] Research Associate, Univ. of California, and Cons. Engr., Berkeley, Calif.

(2) Is it possible to determine certain constants of a given building frame that would be applicable for any type of loading?

(3) What are the short cuts of a general exact method permitting sufficient accuracy in the analysis?

Using the method of elastic weights,[18] two characteristic points for each structural member can be determined. The characteristic points are constants of the building frame and represent sufficient data to analyze the frame for any
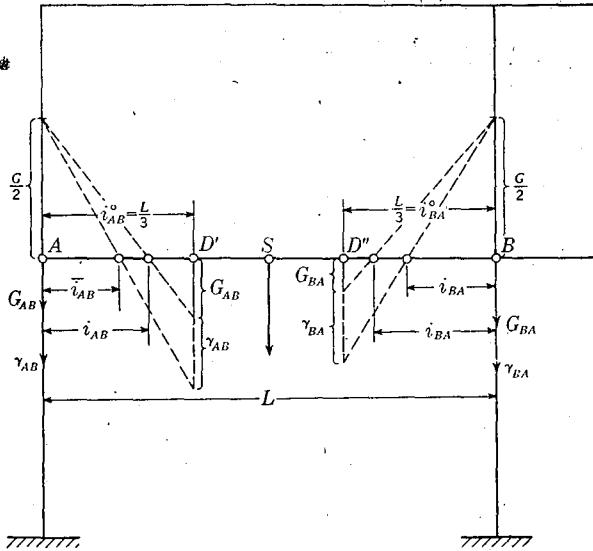

Fig. 20

type of loading, either vertical or horizontal. The characteristic points are centers of simultaneous elastic rotations and can be determined by the following relationships (see Fig. 20):

$$i_{AB} = \frac{L\,G}{3\,(G + 2\,G_{AB})} \dots\dots\dots\dots\dots\dots (26a)$$

and

$$i_{BA} = \frac{L\,G}{3\,(G + 2\,G_{BA})} \dots\dots\dots\dots\dots\dots (26b)$$

in which, in addition to the notation of the paper, G is the elastic weight of the beam $\left(= \frac{L}{E\,I} \right)$; and G_A and G_B are the elastic weights of the supports A and B. Angular rotations at A and B due to a unit moment are G_{AB} and G_{BA}.

For semi-rigid connections Eqs. 26 become

$$i_{AB} = \frac{L\,G}{3\,[G + 2\,(G_{AB} + \gamma)]} \dots\dots\dots\dots (27a)$$

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[18] "Graphical Methods of Analyzing Statically Indeterminate Structures," mimeographed lectures by J. J. Polivka, Berkeley, Calif., 1940 and 1941.

and

$$\tilde{i}_{BA} = \frac{L\,G}{3\,[G + 2\,(G_{BA} + \gamma)]} \quad\cdots\cdots\cdots\cdots\cdots (27b)$$

In the simple case in which the width of a member is assumed equal to zero (Fig. 20), the centers of rotation D' and D'' relating to rigidly restrained supports at A and B are distant $\frac{L}{3}$ from the supports.

For a finite member width, this distance can be determined graphically, as shown in Fig. 21, or algebraically by

$$i^\circ = \frac{G\,[(l + 2\,b)^2 + 2\,b\,(l + b)] + 12\,\gamma\,b\,(l + b)}{3\,(G + 2\,\gamma)\,(l + 2\,b)} \quad\cdots\cdots\cdots (28)$$
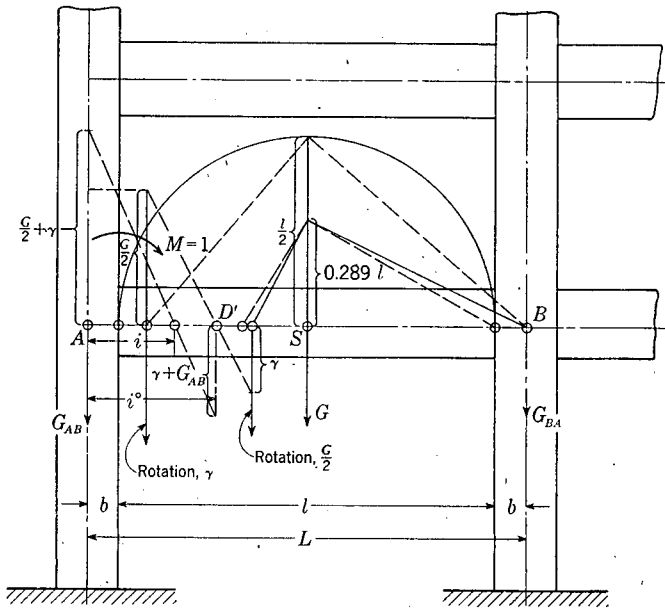

FIG. 21

Using the values in the authors' example—$G = 0.04507$, $l = 160$ in., $b = 4$ in., and $\gamma = 0.01775$—the distance $i^\circ = 36.223$ in. Since the carry-over factor is

$$r = \frac{i^\circ}{L - i^\circ} \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (29)$$

the accuracy of this method may be checked with the results obtained by the authors: $i^\circ = \frac{36.223}{131.777} = 0.27488$ (compared with 0.275 found by the authors).

In the example discussed (Fig. 8), the elastic weight of support of the beam 3-4 results in

$$G_{3-4} = \frac{G_{3-5}\,G_{3-1}}{G_{3-5} + G_{3-1}} \quad\cdots\cdots\cdots\cdots\cdots\cdots (30)$$

That is, $G_{3\text{-}4} = \dfrac{0.00878 \times 0.01008}{0.00878 \times 0.01008} = 0.00469$ and the center of resultant rota-

tion is determined by $i = i° \dfrac{G + 2\gamma}{G + 2\,(G_{3\text{-}4} + \gamma)} = 32.444$ in. Practically the

same value is obtained using the simple term $i = \dfrac{L\,G}{3_{\dot{x}}[G + 2\,(G_{3\text{-}4} + \gamma)]}$
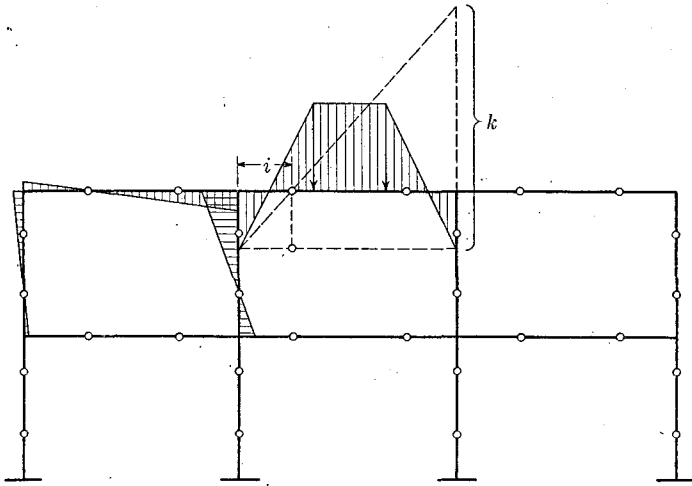

FIG. 22

$+\,b = 28.058 + 4 = 32.058$ in., the difference for this short cut being only 1.2% affecting the center moment of the beam on the side of safety. This simplification eliminates entirely the complex term in Eq. 30.

Knowing the characteristic points of each structural member determining all carry-over factors, the moment diagrams for any type of loading can readily be plotted, as shown in Fig. 22.

. BRUCE JOHNSTON,[19] Assoc. M. Am. Soc. C. E., AND EDWARD H. MOUNT,[20] EsQ.—The worth-while discussions contributed to this paper are appreciated.

Mr. van Buren assumes uniform distribution of the beam reaction in the column at a point of inflection and then concludes that the "reaction as a load on the structure" should be considered to act at the centroid of the half-column width adjacent to the beam. This would produce a moment at the joint center equal to "$V\,g\,b$, in which $g\,b$ is the distance from the joint center to the centroid of the half column." If a complete free-body diagram is drawn of the end of the beam, extended just to the column center line, it will be obvious that the moment $V\,g\,b$ is balanced by an equal and opposite moment resulting from normal stresses acting on the vertically cut section of the column web. These stresses are in equilibrium with the section of the column cut away from the imaginary free body.

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[20] Formerly Am. Welding Soc. Research Fellow at Fritz Eng. Laboratory, Lehigh Univ., Bethlehem, Pa.

It should be emphasized that the "hypothetical" moment at the joint center is a pure fiction, created simply for convenience of analysis, as a result of which moments are balanced at the joint center without separate consideration of the shears. The actual distribution of stresses acting in and around the joint is very complex. Exact stress analysis is out of the question and the general behavior of the connection in so far as it affects frame behavior is of more importance than a complete knowledge of stresses.

In a "rigid" frame joint, the reaction stress introduced into a column by a beam (apart from the moment producing stresses) would be uniformly distributed over the column cross section a short distance below the connection; but, in the case of a semirigid joint, the reaction of the beam could cause a slight tilting of the column and would shift the resultant of the column stress (again assuming no rotation of the end of the beam— that is, no frame action). The reaction would not shift all the way from the center to the column face, but this amount of shift makes a convenient assumption and happens to coincide with Mr. van Buren's suggestion. Table 8, Col. 2, presents the results of an analysis of the frame as loaded in Fig. 8, giving the results of the writers' original analysis taken from Table 3. Col. 3 is for the analysis with column reactions assumed at the connection face (as suggested by Mr. van Buren for flanged columns), and Col. 4 gives moments actually determined by test. The results are not very consistent, but may be compared on the basis of the following percentage of summed differences:

TABLE 8.—COMPARISON OF MOMENTS

Location, joint and member	Connection moment, by Table 3	Connection moments, revised assumptions	Average moments, by test
(1)	(2)	(3)	(4)
1-2	−1.86	−1.36	−1.9
2-1	+33.50	+27.83	+41.9
2-7	−96.49	−101.10	−103.5
3-4	−109.00	−111.57	−102.9
4-3	+104.18	+107.50	+117.0
4-8	−18.46	−15.28	−21.3
1-3	+0.65	+0.68	+1.6
3-1	+55.59	+46.79	+61.5
3-5	+74.46	+62.76	+70.9
5-3	+38.82	+31.97	+28.7
2-4	+86.18	+72.47	+89.1
4-2	−21.46	−16.74	−32.8
4-6	−88.20	−74.69	−88.5
6-4	−45.99	−38.05	−37.2

$$\frac{\Sigma(\text{Difference between Col. 4 and Col. 3 or Col. 2})}{\Sigma \text{ Col. 4}} \times 100$$

This percentage of error in the summed differences gives most weight to the moments of largest magnitude. The percentage of the summed differences is 9.95% for the writer's original analysis (Col. 2), and 15.05% for the assumption of column reaction at column face (Col. 3).

Although the analysis assuming column reaction at column face appears to be definitely less in agreement with test results, it is perhaps more important that the difference between the two is less than the percentage of error in the summed differences by the writers' original analysis. It should be added that the percentage of error in the summed differences of a "rigid frame analysis" is 58.61% (see analysis in discussion by Messrs. Smith, Zick, and Wan). Obviously, no analysis, however complicated, can ever account for variations in fabrication, and all of the other factors that affect the behavior of the actual

bare frame, to say nothing of what finally happens in the actual building when cross framing, fireproofing, walls, and partitions are added. Nevertheless, the behavior of the bare frame represents a dependable minimum on which to base a more rational design procedure than that commonly used at present.

Messrs. Smith, Zick, and Wan, in their discussion, state: "It seems that the authors' suggestion of neglecting the sidesway moments due to vertical unsymmetrical loadings is contradictory to the degree of refinement in their analysis." The writers' statement was not intended to apply to analysis but rather to design, and particularly to design of sections of multi-story tier buildings where enough duplication and regularity make semi-rigid design a practicable possibility. In such cases local tendencies toward sidesway will be absorbed largely by adjacent parts of the building frame. For example, the agreement between test results and theoretical analysis for the unsymmetrical loading in Fig. 14(b) should be noted. Sidesway was neglected in this analysis. Hence, refined methods of analysis may be used to show that sidesway in some cases may be neglected in simplified and necessarily less refined procedures for design. In cases where sidesway affects analysis to an appreciable degree, as in the illustrative example, Fig. 11, there probably would not be enough duplication of structure to make actual design for semi-rigid behavior an economical procedure.

Mr. Lash provides a stimulating discussion on the subject of design. Messrs. Smith, Zick, and Wan also discussed the design aspects of the non-linear relation between moment and angle change in a semi-rigid connection. The general subject of design is one that requires much more study before complete, simple, and practical rules may be specified for all load conditions, types of structure, and connection. The writers avoided this subject, their primary object being to set up methods of analysis that might be used as a basis from which to build rational procedures of design. One of the writers, in collaboration with Robert A. Hechtman, Jun. Am. Soc. C. E., has presented elsewhere a simple approach to the design problem (3).

Mr. Hechtman, in collaboration with one of the writers, recently completed a program of tests on forty-seven different riveted beam-column connections with beam depths ranging from 12 in. to 24 in. These tests have been sponsored at Lehigh University by the American Institute of Steel Construction. Preliminary studies of these test results indicate that the procedure of estimating the behavior of a 24-in. beam connection from test results obtained with a 12-in. beam connection is misleading, if not entirely unsatisfactory. This procedure was used in the reports of the Steel Structures Research Committee of Great Britain, as illustrated by Mr. Lash in Fig. 16. The M-ϕ relationship of a top and seat angle beam connection is influenced by size and arrangement of rivets, thickness of seat and top angles, thickness of beam and column flanges, and other factors, many of which cannot be similar in the 12-in. and 24-in. beam connection because of practical design requirements.

The beam lines for symmetrical loads, as shown by Mr. Lash in Fig. 16, originally suggested as a criterion of design requirement by Cyril Batho,[21]

[21] "Investigations on Beam and Stanchion Connections," by C. Batho and H. C. Rowan, Second Rept., Steel Structures Research Committee, Dept. of Scientific and Industrial Research of Great Britain, H. M. Stationery Office, London, pp. 61–137.

have been used to advantage in reporting recent tests of welded beam connections.[22]   These tests and later tests now in progress indicate the possibility of contriving welded semi-rigid connections that will develop nearly linear behavior up to a moment which gives an adequate factor of safety.

It should be noted that the "beam line" shown by Mr. Lash in Fig. 16 is for "constant beam load." If the semi-rigid connections are actually considered in design, the load capacity of the beam will increase as the connection stiffness increases. If the columns are assumed not to rotate, and the semi-
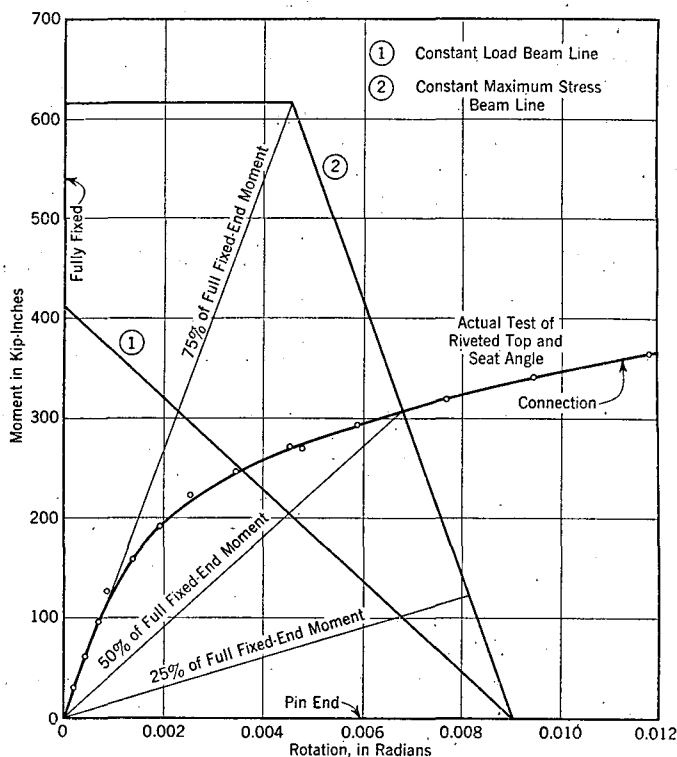

FIG. 23.—DESIGN REQUIREMENTS FOR 12 WF 25 BEAM·

rigid connections are counted upon to their full capacity, the design will be for "constant maximum beam stress," and the design requirements are then more severe than for "constant beam load."

The beam-line equation for "constant beam load" has been noted by Mr. Lash in Eq. 24 ($M_R = M + 2 E K \phi$).   Eq. 24 is plotted as curve (1) in Fig. 23 for a 12 WF 25 beam (12-in., 25-lb wide flange I-beam) for $\frac{l}{d} = 20$.   For the condition of constant maximum beam stress, M_R is not a constant.   If the maximum design stress of 20 kips per sq in. is developed, the following equa-

tion gives M_R for the uniformly loaded beam:

$$M_R = \tfrac{2}{3}(20\,S + M)\dots\dots\dots\dots\dots\dots\dots(31)$$

in which S is the section modulus of beam. Substituting Eq. 31 in Eq. 24, the beam-line equation for "constant maximum beam stress" is obtained:

$$M + 6\,E\,K\,\phi = 40\,S\dots\dots\dots\dots\dots\dots\dots(32)$$

Eq. 32 is plotted as curve (2) in Fig. 23. The beam line, curve (2), is cut off by a horizontal line corresponding to joint rigidities equal to or greater than 75%, in which case the end moment governs the design. The experimental curve in Fig. 23 represents results of a typical connection test of a riveted seat and top angle connection, using a 12 WF 25 beam, as tested at the Fritz Laboratory.

In the foregoing, the columns are assumed not to bend or rotate at the joint. When column bending due to unbalanced beam loads is allowed for, the design loads will be less than assumed in Eq. 32. In such cases the design requirements usually will be somewhere between beam curves (1) and (2), Fig. 23. The beam line for uniform load and "constant maximum stress," therefore, seems to be a good criterion of connection performance for actual semi-rigid design.

When column bending is allowed for, there may be some question as to whether or not it is necessary to penalize the design load by the straight-line construction shown in Fig. 16, drawn from the "twice working load" beam line, because the design requirements will be modified by column rotation. However, in order to maintain a safe structural load factor, some reduction in design value must be made if the M-ϕ curve is decidedly nonlinear up to a beam line constructed on the basis of (working load) $\times$ (factor of safety). Mr. Lash has suggested a factor of safety of 2, in Fig. 16, and this is a conservative value.

The discussions by Messrs. Peterson and Stewart both claim advantages for the "flexure factor" method of analysis as compared with moment distribution or slope deflection. In Table 7, Mr. Stewart[13] shows an exact check by the flexure factor method with the writers' illustrative example, Fig. 11.

Any designer who is actively engaged in analyzing continuous structures will be well repaid by a thorough study of the flexure factor method, as is attested by the discussions of Mr. Stewart's paper[13] by a number of designing engineers who have used it in their work. The method is particularly advantageous if variable load conditions are to be considered on any one structure.

In spite of the advantages of the flexure factor method as an analytical tool, the ultimate application to semi-rigid design probably will be made only in cases of multi-story buildings with regular repetition of bays, or in the case of a large factory with similar repetition of design in roof construction. In the case of the multi-story building, the design probably will be made by means of standardized connections and simplified design rules, and the designer will not make a complete analysis at all, either by the writers' method or by Mr. Stewart's method. It appears to the writers that the slope-deflection method will be the most suitable in establishing general design formulas for typical and limiting conditions, thereby leading to simplified design rules or charts.

Mr. Polivka has shown, in his well-written discussion, how readily the graphical method of "characteristic points" lends itself to the semi-rigid analysis problem.   His discussion, like Mr. Stewart's, is a valuable addition to what might be termed a symposium of different methods of semi-rigid frame analysis. Perhaps it would be better to say "techniques" than "methods," since all are merely special ways of determining terminal moments in framed members, with the fundamental basis in each case being expressed by

$$\frac{M}{E\,I} = \frac{d^2y}{dx^2} \qquad\qquad\qquad\qquad\qquad (33)$$

which is the well-known equation of the bent beam.

In conclusion, the writers wish to emphasize that the important problem is not analysis, but design.   More rational methods of analysis are significant only to the extent that they result in more rational methods of design, thereby reducing the "factor of ignorance" and effecting greater economy in the use of materials.