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A STUDY OF STRUCTURAL SECTIONS SUBJECTED TO TORSION*

by Inge Lyse**

The general relationship between angle of twist and torsional moment in a circular section is given by: $T = J \cdot G \cdot \theta$, where T is torsional moment, J is polar moment of inertia, G is shearing modulus of elasticity, and θ is angle of twist. For non-circular sections we may similarly set: $T = K \cdot G \cdot \theta$, where K represents the resistance of the section to torsional deformation. The K-value therefore, becomes the important item for the determination of shearing stresses in structural shapes. For a rectangular section Saint Venant developed the following equation for the torsional rigidity:

$$K = b \cdot \frac{n^3}{3} - 2Vn^4 \qquad (1)$$

where: b = length of rectangular section

n = thickness or breadth of rectangular section

V = a constant depending on the ratio $\frac{b}{n}$

For the ratio b/n greater than 4 the value for V = 0.10504. Considering the soap film analogy, the expression $-2Vn^4$ represents "end loss". For any differential length, dx, along the rectangular section the rigidity is:

$$K = \frac{1}{3} n^3 \cdot dx$$

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* A detailed description of an extensive investigation of this problem has been published in a paper: STRUCTURAL BEAMS IN TORSION by Inge Lyse and Bruce G. Johnston, Proceedings, of the American Society of Civil Engineers, April 1935, pp. 369-508.

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For a trapezoidal section in which the thickness at any point is r (Fig. 1a) the rigidity for differential length is:

$$K = \frac{1}{3}\int_0^b r^3 \, dx \qquad (2)$$

Evaluating r in terms of m and n, and integrating:

$$K = \frac{b}{12}(m+n)(m^2+n^2) \qquad (3)$$

The "end loss" must be deducted from the above value, or:

$$K = \frac{b}{12}(m+n)(m^2+n^2) - V_m \cdot n^4 - V_n \cdot n^4 \qquad (4)$$

where V_m and V_n represent the constants for the two ends. Considering a section consisting of a part of a sector such as shown in Fig. 1b, the following results are obtained:

$$V_m = 0.10504 - 0.10000 \cdot S + 0.08480 \cdot S^2 - 0.06746 \cdot S^3 + 0.05153 \cdot S^4 \qquad (5)$$

and

$$V_n = 0.10504 + 0.10000 \cdot S + 0.08480 \cdot S^2 + 0.06746 \cdot S^3 + 0.05153 \cdot S^4 \qquad (6)$$

where S is the slope of the section; that is $\frac{m-n}{b}$.

A structural beam with sloping flanges may be considered made up partially of trapezoids and rectangles. The rigidity of the portion shown in Fig. 1c is:

$$K_f = \frac{b-w}{12}(m+n)(m^2+n^2) + \frac{1}{3} wm^3 - 2V_n n^4 \qquad (7)$$

The beam shown in Fig. 2 consists of two such flange sections plus the web section plus the value contributed at the junctures of these sections. The web contributes:

$$K_w = \frac{1}{3}(d-2m) \cdot w^3 \qquad (8)$$

The rigidity contributed by the juncture has generally been considered proportional to the fourth power of the largest inscribed circle; that is:

$$K_j = \alpha \cdot D^4 \qquad (9)$$

where: D = diameter of inscribed circle, and α = a factor depending upon the ratios $\frac{w}{m}$ and $\frac{r}{m}$. The values of $\underline{\alpha}$ were determined experimentally by soap film tests which gave the following results for parallel flange sides:

$$\alpha = 0.094 + 0.070\frac{r}{m} \qquad (10)$$

For flanges with sloping sides the following results were obtained. For slope of 1 on 20: $\alpha = 0.066+0.021\frac{w}{m} +0.072\frac{r}{m}$ for slope of 1 on 50: $\alpha = 0.084 + 0.007\frac{w}{m} + 0.071\frac{r}{m}$

Summarizing the various elements we obtain for parallel-sided flanges:

$$K = \frac{2}{3}bn^3 + \frac{1}{3}(d-2n) \cdot w^3 + 2\alpha D^4 - 0.42016n^4 \qquad (11)$$

and for sloping flange sections:

$$K = \frac{b-w}{6}(m+n)(m^2+n^2) + \frac{1}{3}(d-2m) \cdot w^3 + 2\alpha D - 4V_n \cdot n^4 \qquad (12)$$

The diameter of the inscribed circle is found by the following formula for the parallel-sided flange sections.

$$D = \frac{(n+r)^2 + w(r + \frac{w}{4})}{2r + n} \qquad (13)$$

and for sloping flange sections:

$$D = \frac{\left(r\cdot S\left(\sqrt{\frac{1}{S^2}+1}-1-\frac{w}{2r}\right)+z\right)^2+w\left(r+\frac{w}{4}\right)}{r\cdot S\cdot\left(\sqrt{\frac{1}{S^2}+1}-1-\frac{w}{2r}\right)+r+z} \qquad (14)$$

where z = the maximum flange depth shown in Fig. 2. The shearing stress is a function of the thickness of material and the following equations were found to give fairly accurate maximum shearing stress in the flange and in the web of structural beams:

For parallel-sided flange:

$$\tau_f = \frac{T(n+0.3r)}{K} \qquad (15)$$

For sloping flange sections:

$$\tau_f = \frac{T(m+0.3r)}{K} \qquad (16)$$

and for the web:

$$\tau_w = \frac{T\cdot(w+0.3r)}{K} \qquad (17)$$

For structural sections restrained at the ends we have the following conditions. Referring to Fig. 3 the torsional moment is readily seen to be:

$$T = T_\mu + Q\cdot h \qquad (18)$$

in which

$$T_\mu = KG\,\frac{d\psi}{dx} = \frac{2K\cdot G}{h}\cdot\frac{dy}{dx} \qquad (19)$$

where: K is the torsional constant for the section, and G is the modulus of shear. If $\frac{I_y}{2}$ denotes the larger moment of inertia of one flange, we have:

$$Q = -\frac{EI_y}{2}\cdot\frac{d^3y}{dx^3} \qquad (20)$$

Thus:

$$T = \frac{2K \cdot G}{h} \frac{dy}{dx} - \frac{EI_y}{2} \frac{d^3y}{dx^3} \tag{21}$$

Setting

$$a = \frac{h}{2} \sqrt{\frac{EI_y}{KG}}$$

and reducing, we have:

$$a^2 \frac{d^3y}{dx^3} - \frac{dy}{dx} + \frac{hT}{2K \cdot G} = 0 \tag{22}$$

The differential equation (22) has the general solution:

$$y = A \sinh \frac{x}{a} + B \cosh \frac{x}{a} + C + y_1 \tag{23}$$

where: A, B, and C are determined by the border conditions and y_1 is the solution which satisfies equation (22).

For a section fixed at both ends equation (23) becomes:

$$y = \frac{T.h.a}{2\ K \cdot G}(\cosh \frac{x}{a} \tanh \frac{\ell}{2a} - \sinh \frac{x}{a} + \frac{x}{a} - \tanh \frac{\ell}{2a}) \tag{24}$$

or for $x = \ell$:

$$y = \frac{T \cdot h \cdot a}{2\ K \cdot G} (\frac{\ell}{a} - 2 \tanh \frac{\ell}{2a}) \tag{25}$$

The bending moment in each flange is:

$$M = \frac{EI_y}{2} \cdot \frac{d^2y}{dx^2} = \frac{T \cdot a}{h} \frac{\sinh (\frac{\ell}{2a} - \frac{x}{a})}{\cosh \frac{\ell}{2a}} \tag{26}$$

the shear:

$$Q = \frac{EI_y}{2} \frac{d^3y}{dx^3} = \frac{-T}{h} \frac{\cosh (\frac{\ell}{2a} - \frac{x}{a})}{\cosh \frac{\ell}{2a}} \tag{27}$$

and the longitudinal stresses at the outer fibres of the flanges:

$$\sigma = \frac{M\cdot c}{I} = \frac{Mb}{I} = \frac{T\cdot a\cdot b}{h\cdot I_y}\,\frac{\sinh\left(\frac{\ell}{2a} - \frac{x}{a}\right)}{\cosh\frac{\ell}{2a}} \qquad (28)$$

The angular twist at the center of the section is given by:

$\theta_c = \frac{d\psi}{dx} = \frac{2}{h}\cdot\frac{dy}{dx}$. By use of equation (24) we obtain:

$$\theta_c = \frac{T}{KG}\,\frac{\cosh\frac{\ell}{2a} - 1}{\cosh\frac{\ell}{2a}} \qquad (29)$$

Due to the shortening of the beam during twisting, a correction factor must be applied when cross-sections are restrained from warping. Professor Timoshenko* has shown that a correction of: $1 + 2.95\,\frac{b^2}{\ell^2}$ should be used, that is:

$$\theta_c = \frac{T}{KG}\,\frac{\cosh\frac{\ell}{2a} - 1}{\cosh\frac{\ell}{2a}}\left(1 + 2.95\,\frac{b^2}{\ell^2}\right) \qquad (30)$$

or the torsion constant for this case is:

$$K_c = \frac{T}{\theta_c\cdot G} = K\cdot\left(\frac{\cosh\frac{\ell}{2a}}{\cosh\frac{\ell}{2a} - 1}\right)\cdot\left(\frac{1}{1+2.95\frac{b^2}{\ell^2}}\right) \qquad (31)$$

The theoretical study was substantiated by experiments on full sized structural sections. The loading rig used for these tests is shown in Fig.4. An illustration of the observed stress distribution in free-ended sections is shown in Fig.5, and in fully restrained sections in Fig.6. Fig.7 shows the relationship between computed and observed stiffnesses as well as yield points of fixed-ended beams, and Fig.8 indicates the agreement between the computed and observed shearing stresses in the beams.

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* S. Timoshenko STRENGTH OF MATERIALS, Vol. 1

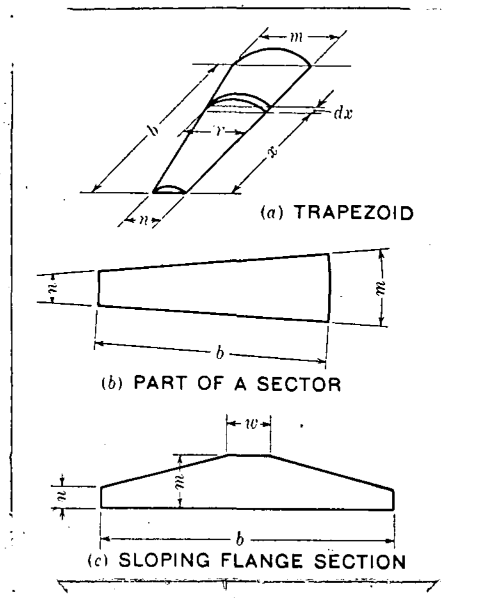

Fig. 1
Sections With
Sloping Sides

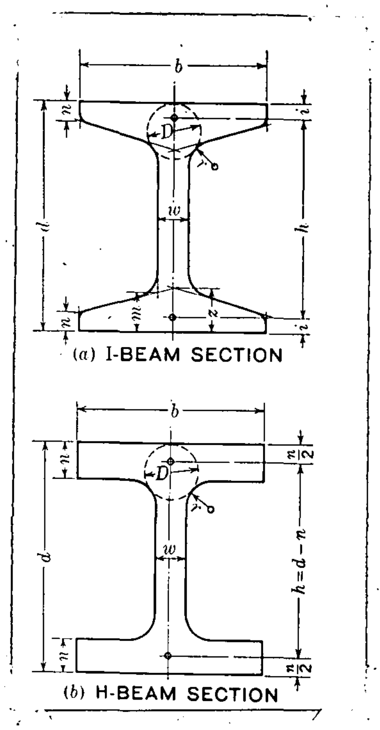

Fig. 2 - Structural
Beam Sections

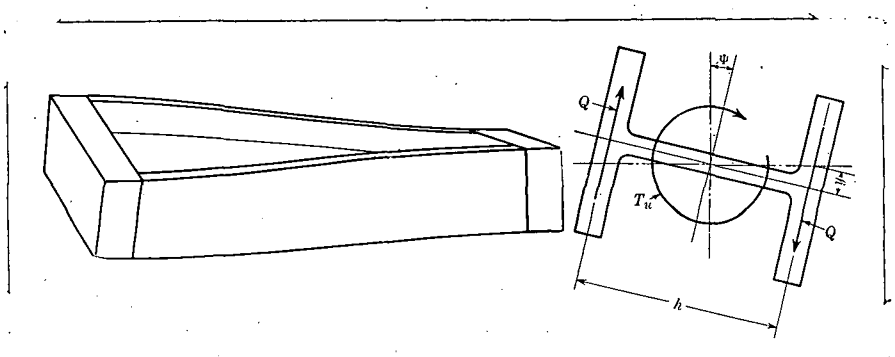

Fig. 3 - Illustration of Twisting of Beam

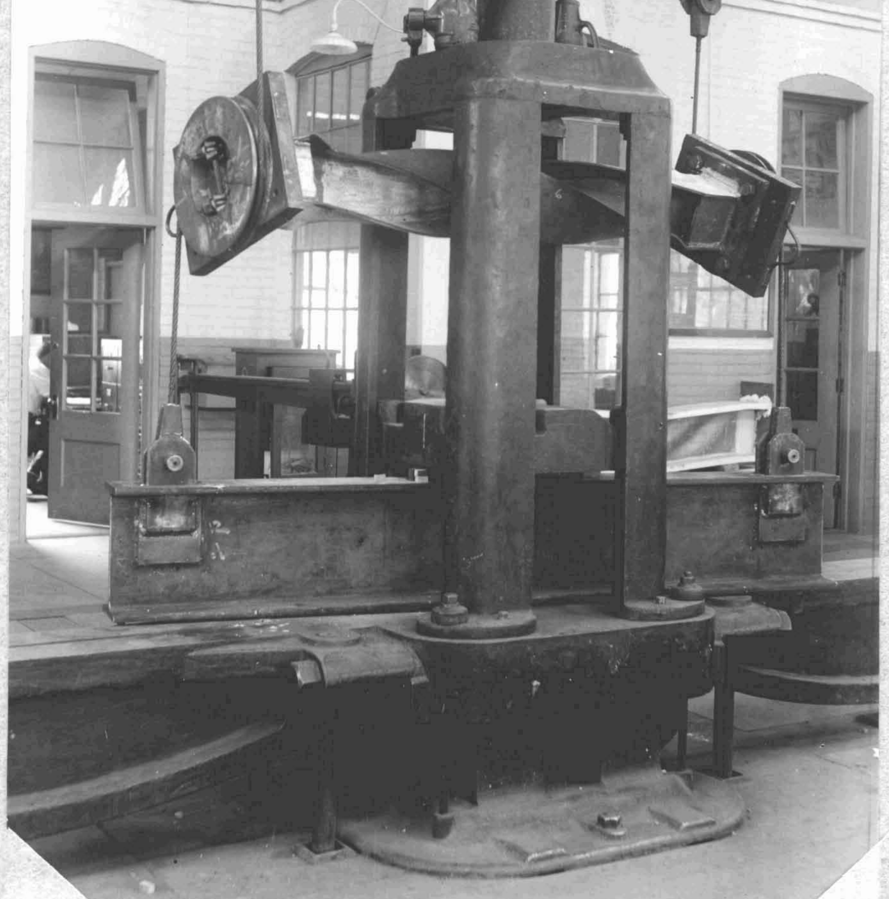

Fig. 4 - Loading Arrangement for Twisting Beams

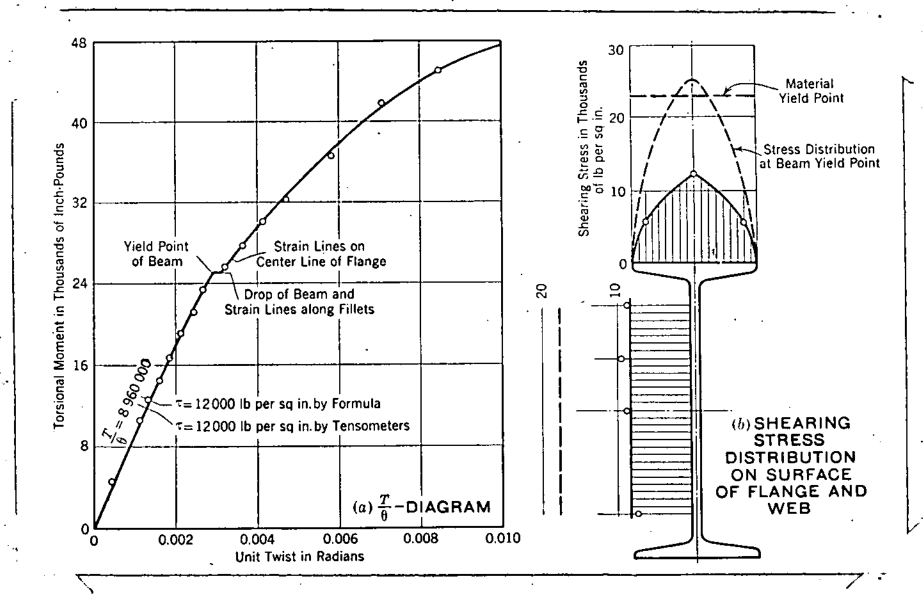

Fig. 5 - Free-Ended Torsion Test

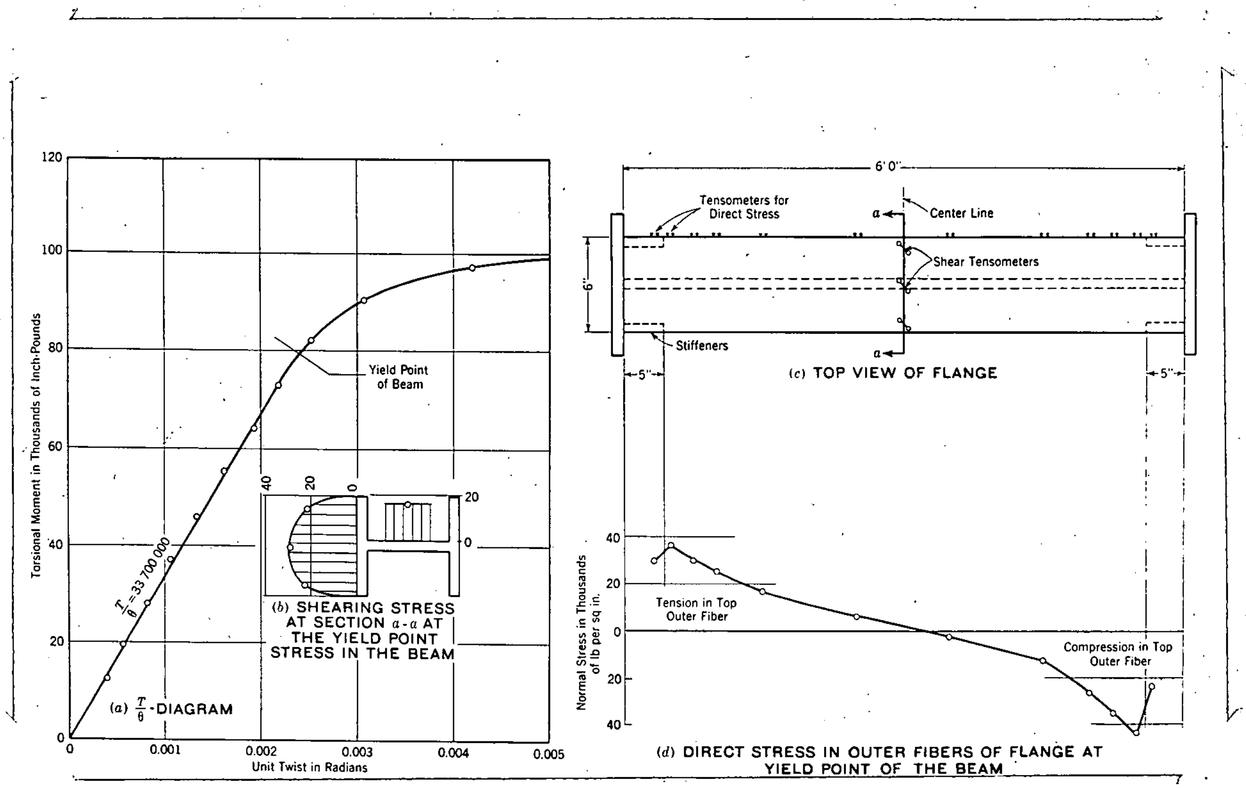

Fig. 6 - Fixed-Ended Torsion Test

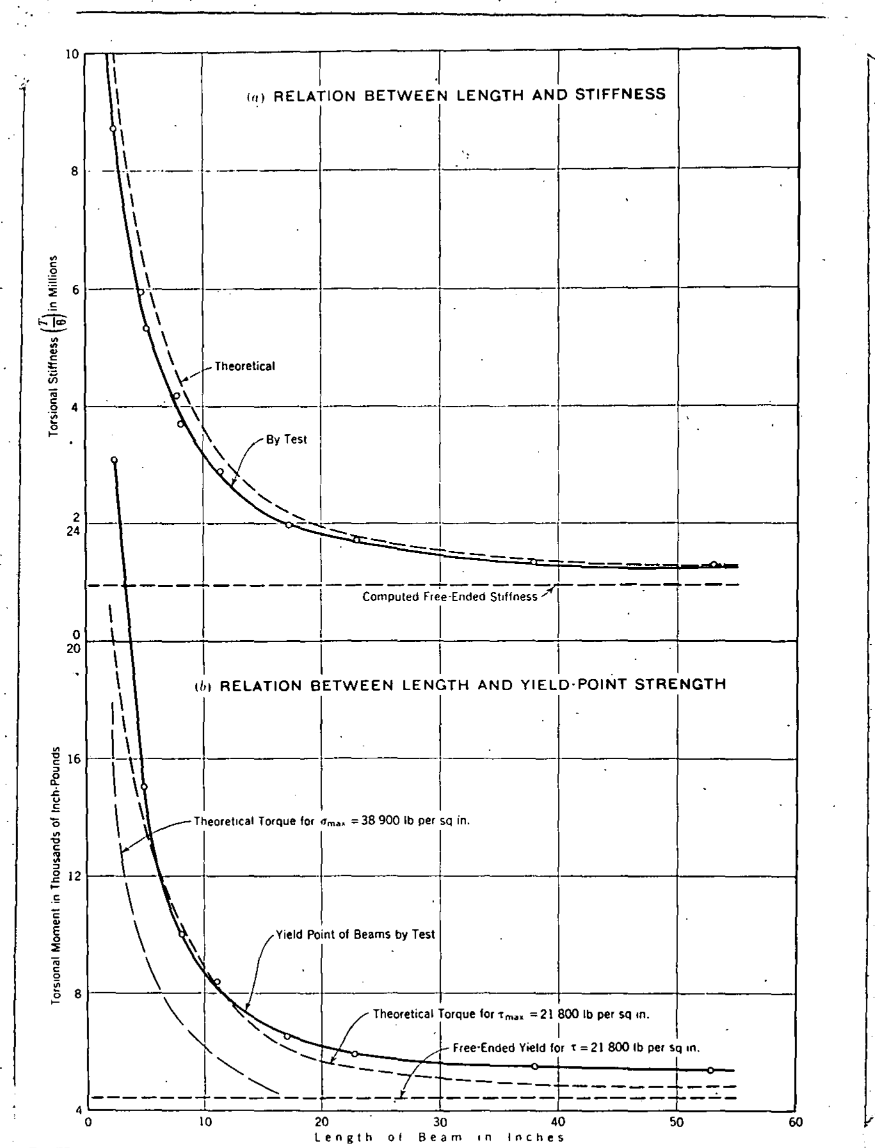

Fig. 7 - Effect of End Fixity On Torsional Rigidity of Beams

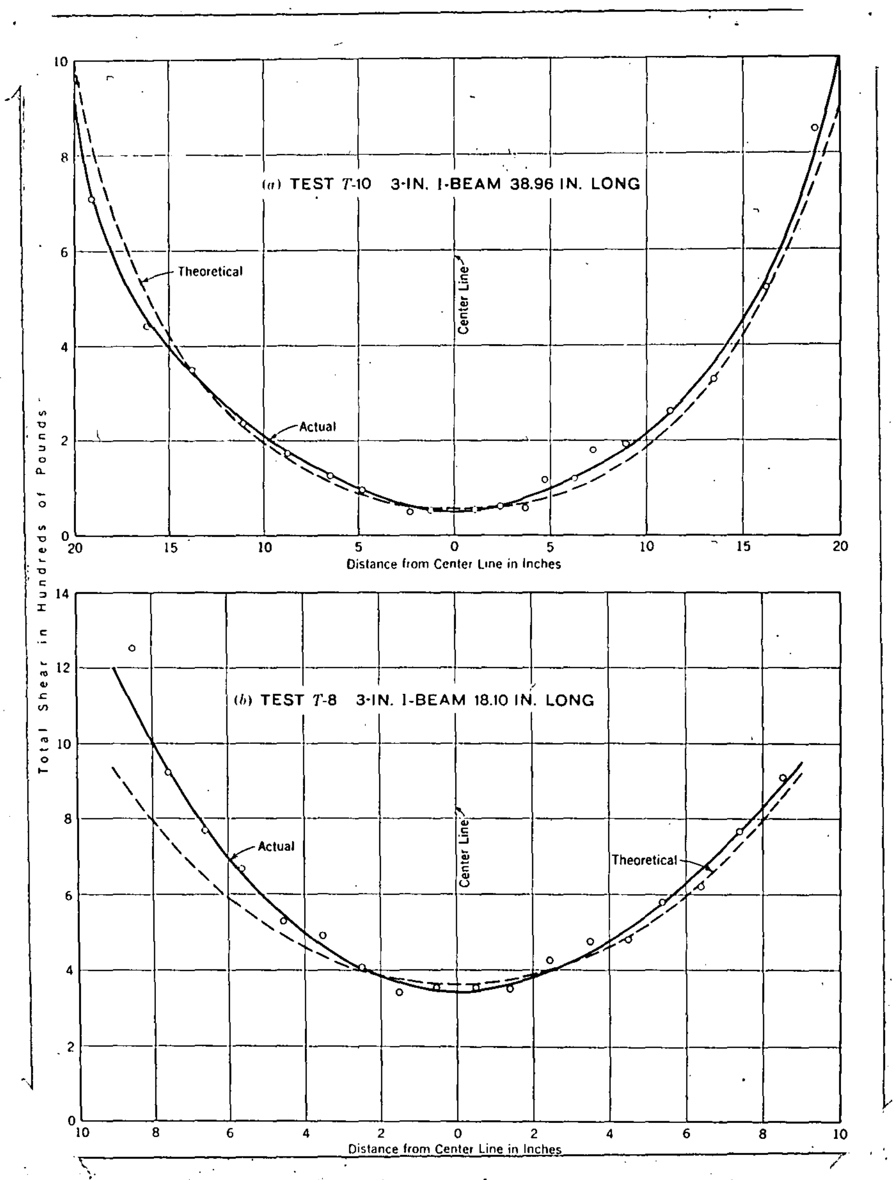

Fig. 8 - Relation Between Computed and Observed Shearing Stresses in Fixed-Ended Beams