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G. R. Irwin

B. Lingaraju

H. Tada

LEHIGH UNIVERSITY INSTITUTE OF RESEARCH

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Low-Cycle Fatigue Behavior of Joined Structures

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Fritz Engineering Laboratory

Department of Civil Engineering

Lehigh University

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1. INTRODUCTION

The concept here termed "crack opening dislocation" deserves attention for several reasons. These pertain to analytical studies of plastic strains near a tensile crack, high-stress-level crack toughness evaluations, crack growth rates during low-cycle-fatigue, and the instability tendency of a plastic strain pattern connecting the leading edge of a crack with a free boundary. The proposal for the study, "Low-Cycle Fatigue Behavior of Joined Structures", of which one phase is reported here, noted the potential applicability of the crack opening dislocation idea as an aid to the analysis and understanding of low-cycle-fatigue experiments. This report gives a brief review of the leading-edge crack-opening idea, develops the relationship of that concept to stress analysis of cracks, and discusses specific low-cycle-fatigue applications.

The initial studies of the crack opening dislocation are largely due to A. A. Wells.[1] During a working period at NRL* during the summer of 1960, Wells decided to exploit applications of the crack opening length factor, δ, defined by

$$\delta = \mathcal{G}/\sigma_Y \qquad (1)$$

where $\mathcal{G}$ is the usual crack extension force (computed with the aid of the plasticity adjustment factor, r_Y) and σ_Y is the yield strength

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*Abbreviations are defined in Section 9, Nomenclature (yield point) of the material. The length factor, δ, was initially developed during NRL discussions in the following way:

The addition of r_Y to the crack size,

where

$$r_Y = \frac{1}{2\pi} (K/\sigma_Y)^2, \tag{2}$$

gives the leading edge of the analysis model a central position within the crack border plastic zone. K is the stress intensity factor at the crack tip. The linear-elastic analysis equation for the displacements, v, normal to and close to the crack plane at a distance r from the reading edge of the analysis model crack is

$$v = \frac{2}{\pi} \frac{K}{E'} \sqrt{2\pi r} \tag{3}$$

where

$$E' = E \text{ (for plane-stress)}$$

$$E' = \frac{E}{(1 - \mu^2)} \text{ (for plane-strain)}$$

$$E = \text{Young's Modulus}$$

$$\mu = \text{Poisson's ratio}$$

If the equation for δ is written as the value of 2v when $r = r_Y$, then

$$\delta = \frac{4}{\pi} \frac{K^2}{E' \sigma_Y} \tag{4}$$

and since

$$\mathcal{G} = K^2/E' \qquad (5)$$

the value of δ is given by

$$\delta = \frac{4}{\pi}\,\mathcal{G}/\sigma_Y \qquad (6)$$

or

$$\delta \cong \mathcal{G}/\sigma_Y \qquad (7)$$

At the time of the initial 1960 discussions, it was believed that the invariance of Eq. 6 to stress state (plane-strain or plane-stress) was significant. The interpretation of $\mathcal{G}$ as the product, $\sigma_Y\delta$, suggested to Wells several plans for interpreting simple notched bar testing of steels in terms equivalent to $\mathcal{G}_c$ values for the crack toughness. At BWRA a special gauge was developed which contacted the separating surfaces near the leading edge of the crack and permitted direct observation of the crack opening dislocation. Comparisons were made to estimates of δ from plasticity analysis.[1,2] Satisfactory agreement was found both with plasticity analysis and with Eq. 7. The measurements were limited to saw-cut representations of the leading edge of a crack; the measurement precision became unsatisfactory when δ was less than about 2×10^{-3} inches. After crack motion and natural roughening of the fracture surfaces, the measurement procedure would not have been suitable. In fact no suitable method for direct measurement of δ for natural cracks in structural metals has been suggested. For such cracks, quantitative values of δ will, of necessity, require indirect methods of assessment.

Burdekin at BWRA compared direct measurements of δ with measurements of thickness contraction near the root of the saw-cut notch and concluded these were nearly equal.[3] Consideration can be given to use of measurements of thickness contraction for indirect assessment of δ. However, additional study of the analytic justification would be needed and the method would only apply to through-cracks in plates. Other methods of indirect δ measurement depend explicitly upon specific assumed elastic-plastic analysis models. A considerable simplication is possible if it can be shown that a simple proportionality between δ and the r_Y corrected value of $\mathcal{G}$ provides a satisfactory method for obtaining estimates of δ so long as the applied stress does not produce general yielding. The first portion of this report presents calculation results which support such an interpretation. The second portion of the report deals with elementary methods for estimating δ applicable for stresses high enough to produce general yielding.

2. THE STRIP-PLASTIC-ZONE ANALYSIS MODEL

The strip-plastic-zone analysis model employs an idea for removing the stress singularity at the leading edge of the linear-elastic crack which was first suggested by Barenblatt.[4] Variations of this idea were employed by Bilby[5] and by Dugdale.[6] These assumed that the effects of yielding are suitably represented if the stress (appropriate to the crack displacement mode) is relaxed to a constant (yield strength) value on a flat strip adjacent to the leading edge of the crack and co-planar with the crack. Bilby's discussion assumed an edge dislocation "pile-up" ahead of the crack which he treated in terms of a continuous distribution of dislocations in Mode II (forward shear). Dugdale used a similar analysis in Mode I (tension). This discussion will employ the same idea in Mode III (parallel shear). The mode choice is dictated by the fact that comparisons are desirable of δ calculations from the strip zone model to δ calculations by means of the Mode III elastic-plastic analysis model. A direct and exact comparison is possible only in Mode III. The immediate purpose is to calculate the influence upon the ratio of $\mathcal{G}$ to δ of two very different distributions of plastic strain. Use of Mode III analysis models is satisfactory for this purpose and provides a considerable advantage also in terms of analysis simplification. It would be of value to verify the indications from Mode III calculations by comparisons of a similar nature in Mode I. However, the possibility for new findings would be small, and the Mode I comparisons might, therefore, be deferred until analysis advancements provide methods for doing these in ways which are simpler than those currently available.

When the plastic zone is very small relative to the crack size and other dimensions, the strip-plastic-zone analysis problem close to the leading edge of the crack can be solved by use of the following (Westergaard type*) stress function

$$Z(\zeta) = \frac{2\tau_Y}{\pi} \arctan \left(\frac{b_o}{\zeta} \right)^{1/2} \quad (8)$$

where

$Z(\zeta)$ = a function of the complex variable, ζ

ζ = $x + i\,y$

τ_Y = shear yield strength

b_o = size of the "strip" yield zone

The yielded strip (of width b_o) lies to the left of the origin of the x, y coordinates. The apparent leading edge of the crack is at $x = -b_o$. The solution provided by Eq. 8 corresponds to

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*The term, Westergaard type, implies that the stress function provides Mode I and Mode II solutions for problems having the same configuration. In Mode III, the product of the displacement times the rigidity modulus is completely given by $\mathrm{Im}(\overline{Z})$.

the linear-elastic solution given by

$$Z(\zeta) = \frac{K}{\sqrt{2 \pi \zeta_1}} \tag{9}$$

where

$$\zeta_1 = \zeta + \frac{2}{3} b_o \tag{10}$$

and

$$K = \frac{2 \tau_Y}{\pi} \sqrt{2 \pi b_o} \tag{11}$$

Equation 10 infers the use of an r_Y type plasticity correction equal to $b_o/3$. It can be shown that this adjustment for the influence of plastic yielding provides a "best fit" relationship between the stresses and displacements derived from Eq. 8 and those derived from Eqs. 9 and 11.

From Eq. 11 and the fact that the problem is in Mode III,

$$\mathcal{G} = \frac{K^2}{2G} = \frac{4 \tau_Y^2 b_o}{\pi G} \tag{12}$$

In order to obtain the opening displacements to the left of the origin for the strip model (Eq. 8) one can integrate $Z\, d\zeta$ to obtain $\bar{Z}(\zeta)$. Then the crack opening, 2w, is given by

$$2w = \frac{2 \mathrm{Im} \bar{Z}(\zeta)}{G} \tag{13}$$

on $y = o$, $x < o$. The results of this operation, for $-b_o \le x \le o$ and $-x = \tau$, is

$$2w = \frac{4\ \tau_Y}{\pi G}\left\{\sqrt{b_o\ r} - (b_o - r)\ \text{arctanh}\ (\frac{r}{b_o})^{1/2}\right\}$$

where

$$o \leq r \leq b_o \qquad (14)$$

The value of δ is given by 2w at $-x = r = b_o$. Comparing the result of this to Eq. 12, one finds

$$\delta = \frac{4\ \tau_Y\ b_o}{\pi G} = \mathcal{G}/\tau_Y \qquad (15)$$

Consider next a more complete strip model in which the crack size is directly represented. The two-dimensional crack of length, $2a$, in an infinite plate with symmetrical yielded strips of size, b_o, adjacent to each leading edge is represented by the stress function

$$Z(\zeta) = \frac{2\ \tau_Y}{\pi}\ \arctan\left\{\frac{(c/a)^2 - 1}{1 - (c/\zeta)^2}\right\}^{1/2} \qquad (16)$$

where $c = a + b_o$, and the remote shear stress parallel to the leading edges of the crack, τ, is given by

$$\tau = \frac{2}{\pi}\ \tau_Y\ \text{arcsec}\ (c/a) \qquad (17)$$

The corresponding linear-elastic crack stress field is given by the stress function

$$Z(\zeta) = \frac{1}{\left\{1 - (a_1/\zeta)^2\right\}^{1/2}} \qquad (18)$$

where

$$a_1 = a + r_Y.$$

The value of r_Y is defined by

$$\frac{1}{2\pi} \left(\frac{K}{\tau_Y}\right)^2 = r_Y = \frac{1}{2} (\tau/\tau_Y)^2 \, (a + r_Y) \tag{19}$$

Thus

$$2 \, r_Y = \frac{a(\tau/\tau_Y)^2}{1 - \frac{1}{2}(\tau/\tau_Y)^2} \tag{20}$$

Computing $\mathcal{G}/\tau_Y$ as $K^2/2G \, \tau_Y$ one obtains an estimate of δ, corresponding to Eq. 15, as

$$\delta = \frac{\mathcal{G}}{\tau_Y} = \frac{\pi \, r_Y \, \tau_Y}{G} \tag{21}$$

or

$$\delta = \frac{\mathcal{G}}{\tau_Y} = \frac{\pi \, a \, \tau_Y}{G} \left(\frac{\frac{1}{2}(\tau/\tau_Y)^2}{1 - \frac{1}{2}(\tau/\tau_Y)^2} \right) \tag{22}$$

Values of δ from Eq. 22 can be compared to values of δ from the crack opening displacement at $x = \pm a$ provided by the stress function of Eq. 16. The δ values for the strip model represented by Eq. 16 can be computed from the equation

$$\delta = \frac{4}{\pi} \frac{a \, \tau_Y}{G} \log \left(\sec \frac{\pi \, \tau}{2\tau_Y}\right) \tag{23}$$

The comparison shows that the values of δ from Eq. 23 and Eq. 22 differ by less than 2 percent until τ becomes larger than 0.8 τ_Y. Even at $\tau = 0.9\ \tau_Y$ the δ predicted by the strip model is only 10 percent above the δ value from Eq. 22. As τ/τ_Y increases above 0.9, the rapid expansion of the strip-model values of δ toward infinity might be appropriate for an infinite solid with a small crack, zero strain hardening, and parallel shear loading. However, for tensile loading one would expect the two plastic zones would tend to join above and below the crack as the remote stress increased toward σ_Y. In other words, for very high tensile stress, a small crack would become enclosed within the total plastic region prior to unlimited outward spread of this region. Thus very large displacements due to plastic strain could develop remote from the crack without a proportionate increase of δ and δ would remain finite as the applied stress approached σ_Y. The strip model (in any of the three modes) and the Mode III elastic-(perfectly) plastic model, to be discussed later, cannot represent the actual behavior at high stress level expected for a small crack (in an infinite plate under tensile loading) because the plastic zones on each side of the crack cannot join.

It is concludes from the above discussion that $\delta = \mathcal{G}/\tau_Y$ provides very nearly the same δ value as does the strip plasticity model across the range of the stress ratio, τ/τ_Y, within which the tensile analogue of the Mode III "strip" analysis model is appropriate in terms of real behavior of tensile cracks.

Consider next, the strip plasticity model in application to a central crack of length, $2a$, in a finite plate of width, W. This problem can be exactly solved in Mode III using a stress function very similar to Eq. 16. One simply replaces c, a, and ζ by $\sin \frac{\pi c}{W}$, $\sin \frac{\pi a}{W}$, and $\sin \frac{\pi \zeta}{W}$. The relationship, $c = a + b_o$, holds as before where c is given in terms of a and the (τ/τ_Y) ratio by the equation

$$\tau = \frac{2}{\pi} \tau_Y \operatorname{arcsec} \left(\frac{\sin \frac{\pi c}{W}}{\sin \frac{\pi a}{W}} \right) \tag{24}$$

The Westergaard-type stress function for linear-elastic analysis of the central crack in a finite width plate, providing also an exact solution in Mode III, is as follows.

$$Z(\zeta) = \frac{\tau}{\left\{ 1 - \left(\frac{\sin \frac{\pi a_1}{W}}{\sin \frac{\pi \zeta}{W}} \right)^2 \right\}^{1/2}} \tag{25}$$

where

$$a_1 = a + r_Y$$

and

$$r_Y = \frac{1}{2\pi} \left(\frac{K}{\tau_Y}\right)^2 = \frac{1}{2\pi} \left(\frac{\tau}{\tau_Y}\right)^2 \tan \frac{\pi a_1}{W} \tag{26}$$

Equation 24 in comparison to Eq. 17 and Eq. 25 in comparison to Eq. 18 provide examples of the substitution of sine functions for length parameters. The form of Eq. 16 after modification into periodic form should be clear and need not be repeated at this point.

The introduction of free surfaces to the right and left of the central crack changes the situation with regard to potential applicability of the tensile analogue. For the same configuration under tensile loading and with a crack size, $2a$, no smaller than about 0.4W, one anticipates a tendency of the plastic yielding to stretch across from the leading edges of the crack toward the free surfaces. Even at a stress nearly large enough for general yielding, no tendency toward joining of the two plastic zones above and below the crack would be expected. Thus comparisons of δ (from the $\mathcal{G}/\tau_Y$ ratio and directly from the plasticity models) can be made up to values of τ/τ_Y equal to unity. However, for values of τ/τ_Y above 0.8, unless the value of $2a/W$ is more than 0.4, we can infer from the preceding infinite plate comparisons that the strip model would tend to over-estimate the δ values applicable to actual tensile deformation of a plate with a small central crack. The comparisons of δ values indicated above require numerical integration procedures and will be discussed later along with the corresponding comparison based upon the Mode III elastic-(perfectly) plastic analysis model.

3. THE MODE III ELASTIC-(PERFECTLY) PLASTIC MODEL

This analysis model was developed by McClintock[7] in a series of papers (with others) beginning in 1956. Until the difficulty of finding simple analysis methods for tensile yielding near a crack became evident, only limited attention was given to the Mode III elastic-plastic approach. Subsequently it was appreciated that study of general characteristics of yielding near a crack could be assisted using this approach. The relative simplicity of the model is of particular value for exploratory calculations. The relationship of this analysis model to "plasticity aspects of fracture mechanics" was discussed by McClintock and Irwin.[8] However, the attention given to the crack opening dislocation concept in that paper was inadequate. Analysis studies suitable for the δ value comparisons of interest here were developed by Rice.[9] However, Rice did not make a comparison of δ values from the Mode III elastic-plastic analysis to values from linear-elastic analysis (with an r_Y plasticity correction).

In the case of a single crack of length, $2a$, (or a periodic co-linear array of such cracks) assuming all displacements zero except those parallel to the leading edges of the cracks and assuming the remote shear stress, τ, does not produce general yielding, then the plastic zones have the following characteristics. In terms of coordinates as shown in Fig. 1, the strains γ within the plastic zone are given by:

$$\gamma = \gamma_Y(\frac{R}{r}) \qquad (27)$$

where $\gamma_Y = \tau_Y/G$, r is the distance from the apparent leading edge of the crack, and R is the value which would be reached by r if extended along the same line to the elastic-plastic boundary. At a given point (r, θ) in plastic zone, γ gives the total maximum shear strain, elastic plus plastic. The displacement gradient is maximum normal to the line of the vector, $r\,e^{i\theta}$. Thus γ acts across the plane containing the position vector.

In the limit, as the size of the plastic zone is reduced by lowering the value of τ, the elastic-plastic boundary approaches a circular shape and the strains, γ, can be calculated from the equation,

$$\gamma = \gamma_Y\left(\frac{2r_Y \cos\theta}{r}\right) = \frac{1}{r}\frac{\partial w}{\partial \theta} \qquad (28)$$

where

$$2r_Y = \frac{1}{\pi}\left(\frac{K}{\tau_Y}\right)^2.$$

Around the elastic-plastic boundary, the stresses and strains correspond exactly in size and direction to the linear-elastic values from Eq. 9, assuming the origin of ζ_1 to be at the center of the circular plastic zone. In the small plastic zone limit of the strip model, the "best fit" plasticity adjustment corresponding to r_Y was $b_o/3$ which is about 20 percent smaller than r_Y. However, in the case of Mode III elastic-plastic analysis, the customary r_Y adjustment to the leading edge of the linear analysis model is correct on a "best fit" basis. The "fit" becomes exact in the limit as the plastic zone tends toward a circular shape.

Whenever the elastic-plastic boundary is known, as well as strain directions on the boundary, the value of δ can be calculated as the integral around the elastic-plastic boundary of the gradient of the displacement (parallel to δ) along the direction of integration. In the case of the plastic zone of Eq. 28"

$$\delta = 2 \int_{\theta=0}^{\theta=\pi/2} \frac{\partial w}{\partial s}\, ds \qquad (29)$$

where w is the z-direction displacement and the element of integration path, ds, can be regarded as having the orthogonal components, dr and $rd\theta$ (at $r = R$). Because the strain gradient $\frac{\partial w}{\partial r}$ is zero, it is convenient to replace the integrand of Eq. 29 by

$$\frac{\partial w}{\partial s}\, ds = \left(\frac{\partial w}{\partial \theta}\right)_{d=R} d\theta \qquad (30)$$

$$= \gamma_Y \; (2r_Y) \; \cos\theta \; d\theta \qquad (31)$$

Completing the integration

$$\delta = \frac{2\;\tau_Y}{\pi\;G} \left(\frac{K}{\tau_Y}\right)^2 = \frac{4}{\pi}\frac{\mathcal{G}}{\tau_Y} \qquad (32)$$

this is the same result as was given by Eq. 6.

4. NUMERICAL CALCULATIONS OF δ FROM TWO PLASTICITY MODELS

Section 4.1 discusses calculations of the ratio of G to δ for a central crack in a plate of finite width, using for the δ calculations the strip-plastic zone analysis model in Mode III. This is the calculation study referred to at the close of Section 2, and it is followed in Section 4.2 by a similar study using the Mode III elastic-perfectly plastic analysis model.

4.1 The Strip-Plastic Zone Analysis Model

Use the following nondimensional notations

$$\left.\begin{array}{l} \alpha = \frac{2a}{W} ; \qquad \beta = \frac{2c}{W} ; \qquad \xi = \frac{2x}{W} \\[2ex] t = \frac{\tau_N}{\tau_Y} \quad \text{or} \quad \frac{\tau}{\tau_Y} = t(1 - \alpha) \end{array}\right\} \qquad (33)$$

(τ_N is the net section stress)

Replacing c, a, and ζ in Eqs. 16 and 17 by $\sin(\pi c/W)$, $\sin(\pi a/W)$, and $\sin(\pi\xi/W)$ respectively, and using the nondimensional quantities defined by Eq. 33, the nondimensional crack opening dislocation $\bar{\delta}_D$ is given by

$$\bar{\delta}_D = \frac{\delta_D}{(\tau_Y W/\pi G)} = \int_{\alpha}^{\beta} \varphi \, d\xi \qquad (34)$$

where

$$\varphi = \varphi(\xi;\ \alpha,\ t) = \tanh^{-1}\left[\frac{\left\{\frac{\sin(\pi\beta/2)}{\sin(\pi\xi/2)}\right\}^2 - 1}{q^2}\right]^{1/2} \qquad (35a)$$

$$\beta = \beta(\alpha,\ t) = \frac{2}{\pi}\sin^{-1}\left[\frac{\sin(\pi\alpha/2)}{\cos(\pi t(1-\alpha)/2)}\right] \qquad (35b)$$

$$q = q(\alpha,\ t) = \left\{\tan\frac{\pi}{2}\,t(1-\alpha)\right\} \qquad (35c)$$

For convenience to calculate $\bar{\delta}_D$ change the variable from ξ to φ

$$\bar{\delta}_D = \frac{4}{\pi}q^2\tan\frac{\pi}{2}\alpha\left\{\int_0^\infty F(\varphi)\,d\varphi\right. \qquad (36)$$

$$F(\varphi) = \frac{\varphi\sinh\varphi}{\left\{1 + (1+q^2)\sinh^2\varphi\right\}\left\{\left(\frac{\tan(\pi\alpha/2)}{\tan(\pi\beta/2)}\right)^2 + \sinh^2\varphi\right\}^{1/2}}$$

Since $F(\varphi) \to 0$ as $\varphi \to \infty$, separate the integral into two parts as

$$\int_0^\infty F(\varphi)d\varphi = \int_0^{\varphi_1} F(\varphi)d\varphi + \int_{\varphi_1}^\infty F(\varphi)d\varphi$$

For large $\varphi(\varphi > \varphi_1)$, expanding the integrand $F(\varphi)$ by making use of $\sinh\varphi = 1/2(e^{\varphi} - e^{-\varphi})$ and neglecting higher order terms, the second integral is approximately expressed as

$$\int_{\varphi_1}^\infty F(\varphi)d\varphi \simeq \frac{4}{1+q^2}\int_{\varphi_1}^\infty \varphi e^{-2\varphi}\left[1 - 2e^{-2\varphi}\left\{\frac{2}{1+q^2} - 1 + \left(\frac{\tan\frac{\pi}{2}\alpha}{\tan\frac{\pi}{2}\beta}\right)^2\right\}\right]d\varphi$$

If φ_1 is chosen as $\varphi_1 = 3.0$ then

$$\int_{3.0}^{\infty} \varphi e^{-2\varphi}\, d\varphi \simeq 0.00434, \qquad \int_{3.0}^{\infty} \varphi e^{-4\varphi}\, d\varphi \simeq 0.000005,$$

and in the second term of the integral, since

$$-1 < \frac{2}{1+q^2} - 1 + \left(\frac{\tan \frac{\pi\alpha}{2}}{\tan \frac{\pi\beta}{2}}\right)^2 < 2, \text{ this term is negligible.}$$

Thus $\overline{\delta}_D$ can be written as

$$\left.\begin{aligned}
\overline{\delta}_D &= \overline{\delta}_{D1} + \overline{\delta}_{D2} \\
\overline{\delta}_{D1} &= \frac{4}{\pi} q^2 \tan \frac{\pi}{2}\alpha \int_0^{3.0} F(\varphi)\, d\varphi \\
\overline{\delta}_{D2} &\simeq \frac{4}{\pi} \frac{q^2}{1+q^2} \tan \frac{\pi}{2}\alpha \times 0.01736
\end{aligned}\right\} \quad (37)$$

For the several selected values of α and t, make the approximate calculations of $\overline{\delta}_{D1}$ by Simpson's formula and $\overline{\delta}_{D2}$ by Eqs. 37.

The only case for which $\overline{\delta}_C$ can be found exactly from tabulated functions is the one where $\alpha = 0.5$ and $t = 1$. For this case

$$\overline{\delta}_D = \frac{4}{\pi}\frac{1}{4} \int_0^{\infty} \frac{u\, du}{\cosh u} = \frac{4}{\pi}(0.4580)$$

and the approximate value obtained by the method discussed above is $\overline{\delta}_D = \frac{4}{\pi}(0.4583)$. For other cases it is expected the approximation provides a similar degree of accuracy. The results are shown in Fig. 2.

4.2 The Elastic-Perfectly Plastic Model

The following exact results were obtained by Rice[9] for the linear elastic-perfectly plastic model in Mode III.

By the nondimensional notations given by Eq. 33, the crack opening dislocation is given by

$$\bar{\delta}_R = \frac{\delta_R}{(\tau_Y W/\pi G)} = 2\alpha \left[t(1-\alpha) \log \frac{\sqrt{1-\{t(1-\alpha)s\}^2} + \sqrt{s^2-\{t(1-\alpha)\}^2}}{\sqrt{1-\{t(1-\alpha)s\}^2} - \sqrt{s^2-\{t(1-\alpha)\}^2}} \right.$$
$$+ 2(1+s^2)\left\{ E_1\left(s^2, \sin^{-1}\frac{t(1-\alpha)}{s}\right) \right.$$
$$\left. - E_3\left(s^2, s^2, \sin^{-1}\frac{t(1-\alpha)}{s}\right) \right\}$$
$$\left. + 2(1-\alpha)(1+s^2)\ \{E_1(s^2, \pi/2) - E_3(s^2, s^2, \pi/2)\} \right] \tag{38}$$

where $s = s(\alpha,t)$ is the dimensionless stress at the edge of the plate on the line of the crack, that is

$$s = \frac{\tau_E}{\tau_Y} \tag{39}$$

($t(1-\alpha) < s < 1$ before yield zone completely traverses the plate) and is given implicitly by

$$\alpha = \frac{E_2(s^2,\pi/2) - E_2\left(s^2, \sin^{-1}\frac{t(1-\alpha)}{s}\right) - (1-s^2)\left[E_1(s^2,\pi/2) - E_1\left(s^2,\sin^{-1}\frac{t(1-\alpha)}{s}\right)\right]}{E_2(s^2,\pi/2) - (1-s^2)E_1(s^2,\pi/2)} \tag{40}$$

and

$$E_1(k,\varphi) = \int_0^{\varphi} \frac{d\varphi}{\{1-k^2\sin^2\varphi\}^{1/2}}$$

$$E_2(k,\varphi) = \int_0^{\varphi} \{1-k^2 \sin^2 \varphi\}^{1/2} d\varphi$$

and

$$E_3(\eta,k,\varphi) = \int_0^{\varphi} \frac{d\varphi}{(1+\eta \sin^2 \varphi)\sqrt{1-k^2 \sin^2 \varphi}}$$

are elliptic integrals of the first, second, and third kind, respectively.

For limit load (when the yield zone completely traverses the plate)

$$s = t = 1$$

$$\bar{\delta}_R = \pi(1-\alpha) + 2\alpha \log(\frac{2}{\alpha} - 1) - 4 \tan^{-1}(1-\alpha) - 2(1-\alpha) \log\left(\frac{4-4\alpha + 2\alpha^2}{4-4\alpha + \alpha^2}\right) \quad (41)$$

For the several values of α and t, the approximate calculations of δ_R were made.

4.3 The Comparisons of δ values with $\mathcal{G}/\tau_Y$ values

From Eqs. 25 and 26, and the relation $\mathcal{G} = \frac{K^2}{2G}$, $\mathcal{G}/\tau_Y$ is given by

$$\frac{\mathcal{G}}{\tau_Y} = \frac{\pi^2}{2} \frac{\tau_Y W}{\pi G} (\alpha_1-\alpha) \quad (42)$$

where

$$\alpha_1 = \alpha_1(\alpha,t) = \frac{2a_1}{W}$$

is determined by

$$\frac{\alpha_1 - \alpha}{A(\alpha,t)} = \tan\frac{\pi}{2}\,\alpha_1$$

$$A(\alpha,t) = \frac{1}{\pi}\,\{t(1-\alpha)\}^2 \qquad (43)$$

For given values of α and t, the approximate values of α_1 are calculated graphically by Eq. 43.

The comparisons were made by taking the δ ratio to $\mathcal{G}/\tau_Y$. The results are shown in Fig. 2 for the cases of $\alpha = 0$ (infinite plate), 0.2, 0.5 and 0.8. For a given type plastic zone (Mode III elastic-plastic or strip), the ratio of δ to $\mathcal{G}$ does not differ significantly from a constant value except for the extreme case of a crack of small size in relation to W combined with an average stress on the net section nearly equal to the yield stress. The difference of the proportionality factor relating δ to $\mathcal{G}/\tau_Y$ for the two plasticity models is moderate. Differences of this nature, though probably smaller, might occur in real materials due to differences of yielding behavior. However, the constancy of the proportionality factor for a given material and plate thickness is the noteworthy aspect. There is no apparent reason to expect a smaller degree of constancy of the ratio of $\mathcal{G}$ to δ for similar calculations in the tensile mode (Mode I).

5. ESTIMATES OF δ AFTER GENERAL YIELDING

An example of δ estimation for a condition of general yield is provided by plastic bending of a bar having a sharp notch (to represent a crack) on the tension side. This problem was considered by Wells during his initial investigations of the crack opening dislocation concept.[1] From a slip field analysis by Green and Hundy,[10] Wells concluded that, after the start of general yielding, further movements of the arms of the specimen could be regarded as rotations about a point in the net ligament located 0.45 t_N beneath the root of the notch, where t_N is the size of the net ligament. The equation for δ was

$$\frac{\delta}{0.45t_N} = \theta^{pl} \tag{44}$$

where θ^{pl} was the irreversible portion of the angle between the specimen arms at the load corresponding to δ. Thus the indirect measurement procedure for δ consisted in measurements of θ^{pl} and the use of Eq. 44. No comparisons of δ from this method with δ from the $\mathcal{G}/\sigma_Y$ ratio have been published. However, Yukawa (GE, Schenectady) discussed unpublished comparisons of this nature with Irwin and Wells. In the case of a number of large spin-disk measurements of $\mathcal{G}_c$ for rotor steels, similar steel in the same thickness was broken in notched-bend tests. Selecting those bend specimens for which a measurement of θ^{pl}

seemed feasible from the broken specimen halves, Yukawa noted that the agreement between δ from Eq. 44 and $\mathcal{G}_c/\sigma_{YS}$ was good enough to support Eq. 44 as at least plausible. Only a limited precision was possible.

A second analysis plan, illustrative of calculations of δ after general yielding, appeared in the appendix section of a report on hydrotesting of gas transmission pipe-lines, prepared by Irwin and Corten for Northern Natural Gas Company and El Paso Natural Gas Company (October 1968).[11]

The problem considered was a part-through crack with a surface length longer than the depth by a factor of 10 or more and with a sufficient depth to cause general yielding of the net ligament. It was assumed that the crack depth ($B - B_N$) and net ligament (B_N) could be regarded as nearly constant along the surface length of the crack. Since the force per unit area across the plane containing the crack was limited nearly to a constant value by the yield strength of the metal and since the net ligament was assumed constant from a generalized plane stress viewpoint, this problem is equivalent to a central crack in a very large plate with a remote stress σ (the hoop tension) and a uniform closing pressure at the crack plane equal in magnitude to $\frac{B_N}{B}\sigma_Y$, where B is the plate thickness. For a problem of this nature, the plasticity correction to the crack length, $2a$, can be neglected. The generalized plane-stress analysis predicts the crack opening to be an ellipse with major axis, a, and minor axis v, where

$$2v = \frac{4a}{E}\left(\sigma - \frac{B_N}{B}\sigma_Y\right) \qquad (45)$$

Next, one assumes that the pattern of plastic strain across the net ligament, in central regions of the crack, transfers all of the opening displacement, 2v, into the crack opening dislocation, δ. A schematic view of the yield pattern is shown in Fig. 3. Thus Eq. 45 can be used to calculate values of δ. Experimental trials of part-through cracks in line-pipe have indicated a probable usefulness of Eq. 45. However, additional experiments are needed to check initial ideas and to assist development of analysis refinements.

6. INVESTIGATIONS WITH FACE-GROOVED DOUBLE-CANTILEVER-BEND SPECIMENS

This section discusses an additional experimental arrangement which permits a theoretical estimation of δ after general yielding.

Consider the face-grooved double-cantilever-bend (DCB) specimen shown in Fig. 4. The condition for the specimen arms to remain elastic is

$$\frac{6Pa}{Bh^2} < \sigma_{YS} \qquad (46)$$

The condition for the net section near the leading edge of the crack to yield plastically is

$$\delta > B_N \frac{\sigma_{YS}}{E}$$

where

P = applied load

a = distance from leading edge of crack to **P**

h = beam length of each loading arm

B = (un-notched) specimen thickness

B_N = net section thickness between the face grooves

A theoretical estimate of δ can be made using Eq. 7 and the simple beam theory value of G.

Thus

$$\delta = \frac{12\ P^2\ a^2}{E\ \sigma_{YS}\ B\ h^3\ B_N} \tag{48}$$

For later work a more accurate method of calculating δ can be used. For these preliminary trials comparisons using Eq. 48 were considered suitable.

A trial specimen, 12 inches long, h = 1 inch, B = 1 inch, was made using a plate of A441 structural steel for which σ_{YS} = 50 ksi. B_N was 0.25 inches. A crack simulating notch was cut producing an a value of about 5 inches and this was extended about 1/4 inch by low amplitude fatigue. Cycles of plastic tensile strain were then applied to the region near the leading edge of the crack at a slow rate of 3 cycles per minute using an Instron Machine adjusted to cycle between 2000 lbs. and 200 lbs. of load.

Two methods of observing crack extension per cycle were used. Periodic visual measurements of the a value was one method. The second method was based upon change of compliance. Since the specimen compliance, C, was approximately proportional to a^3, it followed that

$$\frac{dC}{C} = 3\ \frac{da}{a} \tag{49}$$

An agreement between these two methods was observed although the evidence for fractional change of compliance was taken from the load-time record assuming separation of the specimen arms to be the dominant contribution to C. This agreement must have been largely fortuitous since a similar degree of agreement was not found for subsequent specimens. In any case the observed crack extension was very similar to the trend shown for the compliance in Fig. 5. The calculated value of applied δ from Eq. 6 was

$$\delta = 0.69 \times 10^{-3} \text{ in.} \quad (50)$$

The average rate of crack extension across 220 cycles of loading was

$$\frac{da}{dN} = 0.34 \times 10^{-3} \text{ in.} \quad (51)$$

For the simple models discussed in sections 2 and 3 above, the applied cyclic change of δ would be based upon the load reversal change of K termed Δ K and upon an effective yield strength equal to 2 σ_{YS}. On this basis, from Eq. 7, one obtains

$$\Delta \delta = 0.28 \times 10^{-3} \text{ in.} \quad (52)$$

It is well-known that the crack growth rate values, da/dN tend to increase rapidly during the last cycles of fatigue prior to onset of fast fracturing. Furthermore at very small values of Δ K, the crack growth rate is very small in comparison to $\Delta \delta$. Thus when fatigue crack growth rate measurements are extended close enough to onset of rapid fracturing, a range of Δ K would be expected to develop for which $\Delta \delta$ is equal to da/dN.

Examples of fatigue crack growth rate measurements across a wide range of Δ K were examined using results reported by Frankford Arsenal. The specimens were single edge notched plates of high strength steel and aluminum alloys. It was noted that the position where da/dN was equal to $\Delta\ \delta$ occurred in a region for which a substantial amount of thickness reduction yielding would be expected. In addition, for these data, the match of da/dN to $\Delta\ \delta$ occurred on upturn portion of log da/dN relative to log ΔK which is customary in the approach toward onset of rapid fracture.

Investigations of the use of the δ concept in studies of fatigue are, so far, only preliminary and exploratory. The results are encouraging relative to potential usefulness of the concept to low cycle fatigue studies for two reasons. (1) The behavior trend of da/dN relative to $\Delta\ \delta$ may possess sufficient regularity so that knowledge of this relationship can be organized in a manner which assists predictions of crack growth rate and low cycle fatigue life. (2) The fact that a near unity ratio of $\Delta\ \delta$ to da/dN occurs in the range where the crack growth rate is associated with low cycle fatigue (either because of section yielding or because of a close approach to onset of rapid fracturing) suggests that the δ concept will assist efforts toward fundamental understanding.

7. SUMMARY

This paper reports on one phase of an overall study, "Low Cycle Fatigue Behavior of Joined Structures". The purpose of this phase is to develop and use fracture mechanics analysis and measurement techniques so as to assist understanding of low cycle fatigue.

In real structures, conditions suitable for the term "low-cycle fatigue" imply relatively high stress and, frequently, the development of yielding across large sections of material. For this reason an appropriate fracture mechanics approach is one in which the characterization used to indicate severity of cyclic straining near the leading edge of a crack is a plasticity type characterization, which can be employed after general yielding as well as before. The crack opening dislocation, δ, is the simplest available characterization having such capabilities. A close relationship can be retained to characterizations using K and $\mathscr{G}$ if we can rely on the existence of a simple proportionality between the r_Y corrected $\mathscr{G}$ value and δ. This report presented analytical and experimental evidence which support this viewpoint and which suggest that the proportionality remains valid up to stress levels nearly large enough for general yielding. After general yielding of the remaining net ligament (beyond the crack), calculation methods for δ are so far available only for a few special situations. Laboratory studies can be done using various simple geometries which do permit calculation of δ. The example which has been given experimental trial is a double-cantilever bend (DCB) specimen with relatively deep face grooves of half-circle shape. The specimen design permits plastic yielding of the net section near the leading edge of the crack while the specimen arm deflections are mainly elastic and reversible. The preliminary trials discussed in this report provide a basis for use of the specimen in various low cycle fatigue investigations. The development of δ calculation methods suitable for other situations including fatigue fracturing of welded beams requires additional study.

8. ACKNOWLEDGMENTS

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Lynn S. Beedle is Director of Fritz Engineering Laboratory, and Joseph F. Libsch is Vice President for Research, Lehigh University.

9. NOMENCLATURE

BWRA	British Welding Research Association, now re-named The Welding Institute
GE Schenectady	General Electric Company, Large Steam Turbine Generator Department at Schenectady
NRL	U. S. Naval Research Laboratory
a	crack size factor
b_o	length of the yield zone of the strip type model of yielding near a crack
B	plate thickness
B_N	net section between the leading edge of a long part-through crack in a plate and the opposing plate surface; also (in Section 6) net section between the face grooves of a DCB specimen
c	a plus the length of the yield zone
DCB	double-cantilever-bend type fracture testing specimen
E	Young's Modulus
E'	E for plane-stress and $E/1-\mu^2$ for plane-strain
E_1, E_2, E_3	symbols for elliptic integral functions of the first, second, and third kinds
G	elastic shear modulus
$\mathcal{G}$	K^2/E', the crack extension force
$\mathcal{G}_c$	the critical value of $\mathcal{G}$, for onset of rapid crack extension
ImZ	real functions of x and y defined by the equation $Z(\zeta) = \text{ReZ} + i\ \text{ImZ}$
ζ	complex variable, $\zeta = x + iy$
K	stress intensity factor for the leading edge stress field of a crack

r_Y	crack size plasticity adjustment factor
t	relative size of τ_N, $\tau_N = t\ \tau_Y$
v	displacement of a material point in the direction normal to the crack plane
w	displacement of a material point parallel to the leading edge of the crack
W	plate width
α	relative crack size, $2a/W$
β	relative size of the factor 2c, $2c = \beta W$
δ	crack opening dislocation (or crack opening stretch)
γ	shear strain
μ	Poisson's ratio
ξ	relative size of the position factor 2x, $2x = \xi W$
σ_Y	tensile yield point stress
τ_N	average shear stress on the net section
τ_Y	shear yield point stress

10. FIGURES

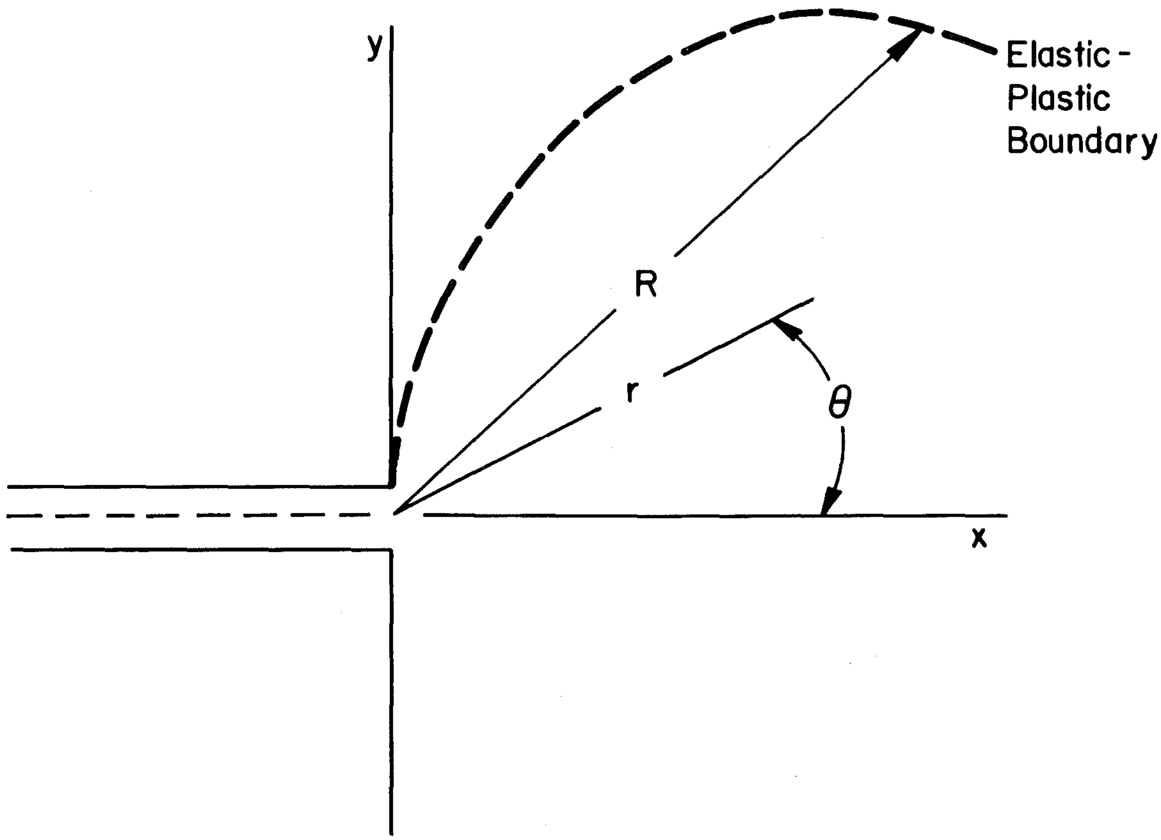

Fig. 1 Rectangular Coordinates x, y, and Cylindrical Coordinates r, θ at the Apparent Leading Edge of a Crack

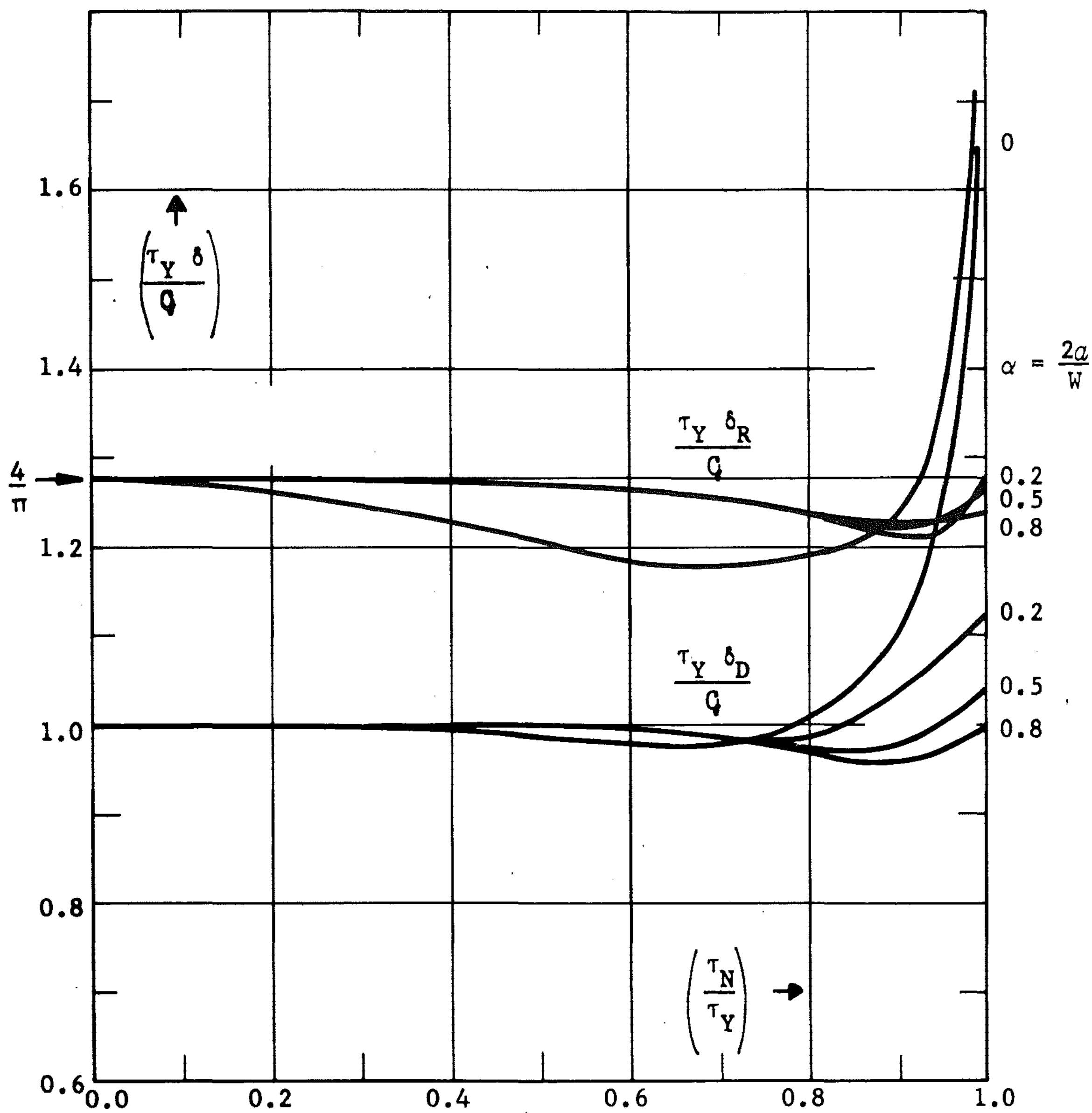

Fig. 2 Values are shown for the Ratio of δ from the Mode III Elastic-Plastic Model to δ computed as G/τ_Y where G is obtained from an r_y corrected Linear-Elastic Analysis. The subscript D indicates use of the Dugdale or Strip Model. The subscript R indicates use of Rice's Formulation of the McClintock Mode III Elastic-Plastic Model.

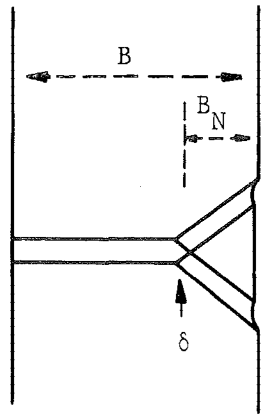

Fig. 3 Schematic View of Plastic Deformation Near the Leading Edge of a Long Part-Through Crack After Substantial Plastic Yielding of the Net Ligament

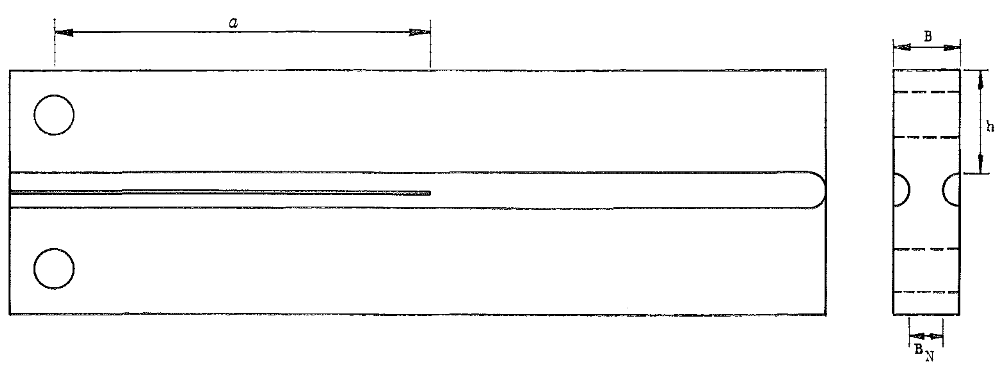

Fig. 4 Face-Grooved Double-Cantilever-Bend Specimen

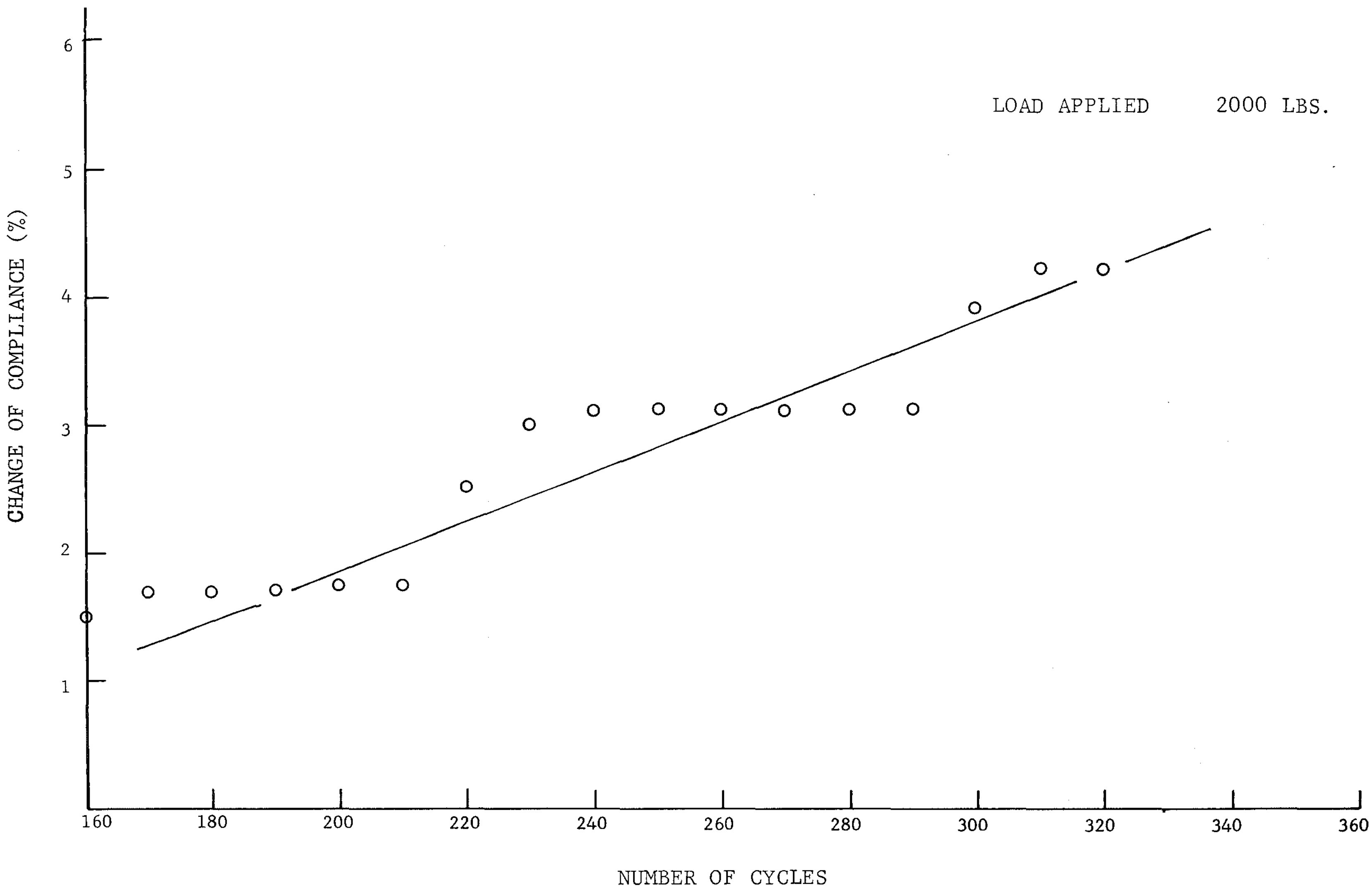

Fig. 5 . Relationship Between the Change in Compliance and Cycle Life

11. REFERENCES

1. Wells, A. A.
NOTCHED BAR TESTS, FRACTURE MECHANICS, AND THE BRITTLE STRENGTH OF WELDED STRUCTURES,
British Welding, Vol. 12, No. 2, 1965

2. Burdekin, F. M.
DISCUSSION,
ASTM STP 381, p. 400, 1965

3. Wells, A. A.
Informal Communication

4. Barenblatt, G. I.
MATHEMATICAL THEORY OF EQUILIBRIUM CRACKS IN BRITTLE FRACTURE,
Advances in Applied Mechanics, Vol. 7, p. 55, 1962

5. Bilby, B. A., Cottrell, A. H. and Swinden, K. H.
THE SPREAD OF PLASTIC YIELD FROM A NOTCH,
Proc. Roy. Soc. A272, p. 304, 1963

6. Dugdale, D. S.
YIELDING OF STEEL SHEETS CONTAINING SLITS,
Journal Mech. Phys. Solids, Vol. 9, p. 100, 1960

7. McClintock, F. A. and Hult, J. A.
Proc. of IXth Int. Congr. of Applied Mechanics,
Vol. VIII, pp. 51-58, Brussels, 1956

8. McClintock F. A. and Irwin, G. R.
PLASTICITY ASPECTS OF FRACTURE MECHANICS,
ASTM STP 381, p. 84, 1965

9. Rice. J. R.
CONTAINED PLASTIC DEFORMATION NEAR CRACKS AND NOTCHES UNDER LONGITUDINAL SHEAR,
Brown University Technical Report, July 1965; also
Int. Journal of Fracture Mechanics, June 1966

10. Green, A. P. and Hundy, B. B.
INITIAL PLASTIC YIELDING IN NOTCH BENT TESTS,
Journal Mech. Phys. Solids, Vol. 4, p. 128, 1956

11. Meetings of USAS Committee B31.8 at Los Angeles,
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13. ABSTRACT

This report presents historical, analytical, and experimental evidence which support the use of the crack opening dislocation (δ) as a measure of severity of strain near the leading edge of a crack. A close relationship is shown between the δ characterization and characterization in terms of the stress field parameters $\mathcal{G}$ and K in the stress range prior to general yielding of the net section ahead of the crack. Applications of the δ concept to low cycle fatigue are discussed. Test results obtained using face-grooved double-cantilever specimens are given as illustrative of these applications.

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