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Space Frames with Biaxial Loading in Columns

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ABSTRACT

Computations to obtain the ultimate strength of an inelastic beam-column are fairly involved and, at present, only numerical methods are available to get the best possible solutions in most cases. For practical purposes, however, these numerical approaches are often laborious. This paper presents simple approximate forms of solutions by assuming an idealized relationship among moment, curvature and thrust in the ultimate state.

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FIGURES

1. INTRODUCTION

In most reference works [2, 7] on the approximate theory of beam-column problems, the axis of the deflected beam-column is often assumed to be a certain shape of curve, such as a sine curve or a parabolic curve. As a consequence of this, the analysis of the beam-column problems is considerably simplified. Simple interaction equations which define the load carrying capacity of the beam-column can then be obtained. It has been found [2, 5] that this simplification gives satisfactory results for simply supported beam-columns under symmetric loading conditions. It is clear, however, that this type of simplification is not very suitable for the fast determination of beam-column strength for the unsymmetric cases or for the case of beam-column with fixed end supports.

The work reported in this paper is an effort to help fill part of this gap. Toward this purpose, an alternative but extremely simple approximate analysis is developed and applied to various beam-column problems.

In the analysis, the moment-curvature-thrust relationship is idealized as elastic-perfectly plastic. The moment-curvature relationship for a constant thrust is assumed to be linear up to a certain moment level M_{mc}. From here on the section is assumed to flow plastically at the constant moment M_{mc} (Fig. 1). The adoption of this idealized relationship must not be thought of as neglecting curvature work-hardening, but rather as averaging its effect over the entire beam-column. The appropriate average flow moment M_{mc} must lie between the initial yield moment M_{yc} and the plastic limit moment M_{pc} of the cross section (Fig. 1). The choice of the level M_{mc} is dependent on the section used as well as on the geometry and loading of the entire beam-column. Once the proper value of M_{mc} is selected, the maximum load carrying capacity of beam-columns can be computed in a rather simple manner by the elastic analysis. The subsequent discussion in this paper shows how this average flow moment M_{mc} may be determined.

2. ESTIMATION OF AVERAGE FLOW MOMENT

All beam-columns to be considered are assumed to be made of an ideally plastic material which is elastic up to the yield point and then flows under constant stress. The corresponding moment-curvature-thrust relationship of a common structural section with or without the influence of residual stress is shown by the curve O-E-F in Fig. 1. The curve may be divided into two parts: Linear elastic part (O-E) with an initial yield moment M_{yc} and curvature work-hardening part (E-F), with the moment asymptotically approaching the limit value M_{pc} as curvature ϕ tends to infinity. If M_{yc} is used for the idealized flow moment, the ultimate strength of a beam-column will be lower than the actual one, on the other hand, if M_{pc} is used, the solution will be an upper bound. The exact solution is thus bounded by the two extreme solutions. A satisfactory selection of the average flow moment M_{mc} will therefore enable the estimation of the ultimate strength of the beam-column with high accuracy.

The yield moment M_{yc} and the plastic limit moment M_{pc} for a constant thrust P have been obtained in Refs. 3, 4 for several commonly used structural sections. As an example, the expression for strong axis bending of a wide flange section including the influence of residual stress is

$$\frac{M_{yc}}{M_y} = \begin{cases} 0.9 - \left(\frac{P}{P_y}\right) & \left(\frac{P}{P_y} \leq 0.8\right) \\ -1.1 + 3.1\left(\frac{P}{P_y}\right) - 2\left(\frac{P}{P_y}\right)^2 & \left(\frac{P}{P_y} \geq 0.8\right) \end{cases}$$

$$\frac{M_{pc}}{M_y} = \begin{cases} 1.11 - 2.64\left(\frac{P}{P_y}\right)^2 & \left(\frac{P}{P_y} \leq 0.225\right) \\ 1.238 - 1.143\left(\frac{P}{P_y}\right) - 0.095\left(\frac{P}{P_y}\right)^2 & \left(\frac{P}{P_y} \geq 0.225\right) \end{cases} \tag{1}$$

where P is the applied thrust, P_y is the yield thrust, and M_y is the yield moment in the absence of the thrust P.

Since the average flow moment M_{mc} must lie between the values M_{yc} and M_{pc}, hence, the value of M_{mc} may be represented by

$$M_{mc} = M_{pc} - f\,(M_{pc} - M_{yc}) \tag{2}$$

where f is the parameter function

$f = 0$ corresponds to $M_{mc} = M_{pc}$ (the upper bound solution)

$f = 1$ corresponds to $M_{mc} = M_{yc}$ (the lower bound solution)

The parameter f will be a function of the thrust P, the length L and the boundary conditions of a beam-column. For simplicity, the parameter function f is assumed to have the form

$$f = f_1\left(\frac{P}{P_y}\right)\ f_2\left(\frac{L}{r}\right)\ f_3(\text{B.C.}) \tag{3}$$

the functions f_1, f_2, and f_3 need to be determined for each type of beam-column.

<u>Example 1 Beam-Column with an Uniformly Distributed Lateral Load (Fig. 2a)</u>

If $P/P_y = 0$, it is a beam problem and the plastic limit moment M_{pc} will govern the ultimate state, i.e., $f = 0$. If $P/P_y \approx 1$, it is an axially loaded short column problem and the yield moment M_{yc} will be the governing one, i.e., $f = 1$. The elastic solution [see Eq. 7] using $f = 0$ and $f = 1$ then gives the upper- and lower-bound interaction plot shown in Fig. 3. The solution $f = f_1 = P/P_y$ (assuming $f_2 = 1.0$, $f_3 = 1.0$) is found to be in good agreement with the exact solution reported in Ref. 6. The approximate solution can be improved by taking $f = f_1 = (P/P_y)^{0.6}$. The improved result is plotted as small circles in Fig. 3 and given a very good approximation to the exact solution. Therefore, it seems reasonable to assume that the function f_1 has the general form

$$f_1\left(\frac{P}{P_y}\right) = \left(\frac{P}{P_y}\right)^N \tag{4}$$

Consider, next, the second parameter $f_2(L/r)$, where L/r is slenderness ratio of the beam-column. If the member is very short, it will lose the nature of a column, and the value of f should be close to 0. By comparison with the exact solution, the following formula is an appropriate one as correction for short beam-columns

$$f_2\left(\frac{L}{r}\right) = \begin{cases} 1 & \left(\frac{L}{r} \geq 60\right) \\ \frac{1}{40}\left(\frac{L}{r}\right) - \frac{1}{2} & \left(20 \leq \frac{L}{r} < 60\right) \\ 0 & \left(\frac{L}{r} < 20\right) \end{cases} \tag{5}$$

Usually, the slenderness ratio of a column is greater than 60, so that $f_2(L/r)$ may be chosen as unity.

The third parameter $f_3(B.C.)$ is determined based on boundary conditions. If a beam-column is fixed, plastic hinges will form at the ends first as shown in Fig. 4a. Until the third (and the last) plastic hinge forms at center C, large rotations will have been experienced at the previously formed plastic hinges at both ends. At the ultimate state, the moments at both ends will be close to M_{pc} and the moment at center C will be close to M_{yc}. Therefore, the mean value of M_{yc} and M_{pc} will be one of the approximate values of M_{mc}; or f_3(fixed) = 0.5. On the other hand, if the beam-column is simply supported, M_{yc} will be the governing flow moment; f_3(simple) = 1.0.

In summary, for a beam-column of usual length ($L/r \geq 60$), M_{mc} has the form

$$M_{mc} = \begin{cases} M_{pc} - \frac{P}{P_y}(M_{pc} - M_{yc}) & \text{(simple)} \\ M_{pc} - 0.5\frac{P}{P_y}(M_{pc} - M_{yc}) & \text{(fixed)} \end{cases} \qquad (6)$$

Using the average flow moment M_{mc} in Eq. 6, the ultimate load w of the beam-column shown in Fig. 2a can be computed by the formula [7]:

$$Q = wL = k\, M_{mc}\, kL \frac{\lambda + \cos\frac{kL}{2}}{1 - \cos\frac{kL}{2}} \qquad (7)$$

where

$$k^2 = \frac{P}{EI}$$

and

$\lambda = 0$ for simple supports

$\lambda = 1$ for fixed ends

Using the proposed values, a comparison is made with the exact solution of a beam-column of a wide flange section (8 WF 31) with residual stresses and shown in Fig. 5 (simply supported) and Fig. 6 (fixed ends). The solid lines are exact solutions reported in Ref. 6, and the dotted lines are results obtained by the present method. They show a sufficiently good agreement with each other. In case of simply supported beam-columns, however, an approximation by N = 0.6 gives a better result as plotted by small circles in Fig. 5.

Example 2 Beam-Column with a Concentrated Lateral Load (Fig. 2b)

The parameter f for the average flow moment M_{mc} will be the same as in the case of a uniform load. For a fixed end beam-column, since plastic hinges will be formed at both ends and under the load at the same time (Fig. 4b), the governing flow moment will still be the mean value of M_{yc} and M_{pc}, i.e., f_3(fixed) = 0.5.

Using the average flow moment M_{mc} in Eq. 6, the ultimate load Q of the beam-column shown in Fig. 2b can be computed by the formula:

$$Q = 2k\, M_{mc} \frac{\lambda + \cos \frac{kL}{2}}{\sin \frac{kL}{2}} \tag{8}$$

Comparison with the exact solution [6] is shown in Fig. 7 (simply supported) and Fig. 8 (fixed ends). A good agreement is observed in both cases. Eq. 8 can be rewritten in the form:

$$Q_s = 2k\ M_{mc} \cot \frac{kL}{2} \qquad \text{(simply supported)} \tag{9}$$

$$Q_f = 2k\ M_{mc} \cot \frac{kL}{4} \qquad \text{(fixed end)} \tag{10}$$

It is seen that in these two cases equations are analogous to each other (note: the values of M_{mc} are different, see Eq. 6), and the ultimate strength for a beam-column with both ends fixed (Eq. 10) may be computed from the beam-column with hinged ends having a reduced length equal to half the actual length. (It is evident from symmetry that this conclusion is true for the actual situation).

Referring now to the partially distributed load cases represented in Fig. 2(c) and proceeding as for an elastic solution with plastic hinges, one finds the following expressions for the ultimate lateral load:

$$Q = wC = 2k\ M_{mc} \frac{\frac{kC}{4}}{\sin \frac{kC}{4}} \frac{\lambda + \cos\frac{kL}{2}}{\sin\left(\frac{kL}{2} - \frac{kC}{4}\right)} \tag{11}$$

where M_{mc} is given by Eq. 6. As can be seen here, the ultimate load for the fully distributed load case (Eq. 7) and the concentrated load case (Eq. 8) are particular cases of Eq. 11.

Example 3 Beam-Column with End-Moments (Fig. 2d)

Consider, next, a beam-column subjected to end moments M_o and κM_o as shown in Fig. 2d. Average plastic moment M_{mc} is assumed in a similar form as before:

$$M_{mc} = M_{pc} - \left(\frac{P}{P_y}\right)^N (M_{pc} - M_{yc}) \quad . \tag{12}$$

Here, $N = 1/2$ gives a good approximation for M_{mc}. A plastic hinge occurs either within the span or at one of the end supports depending upon the ratio of applied moments κ. The ultimate moment M_o is given by the following formulae [7];

if $\kappa \leq \cos kL$

$$M_o = M_{mc} \tag{13}$$

if $\kappa \geq \cos kL$

$$M_o = M_{mc} \frac{\sin kL}{\sqrt{\sin^2 kL + (\kappa - \cos kL)^2}}$$

Comparison with exact solutions [1] is shown in Fig. 9 ($\kappa = 1$). Results by the present method (dotted lines) are computed using $N = 1/2$. A sufficiently good agreement is observed.

3. UNSYMMETRICALLY LOADED BEAM-COLUMN

The average flow moment M_{mc} for a symmetrically loaded beam-column has been obtained in the previous section. This result (Eq. 6) is considered to be applicable to unsymmetric problems as well, if the unsymmetricity is not very large.

The ultimate concentrated load applied unsymmetrically to a beam-column (Fig. 2e) is computed by assuming that the last plastic hinge is formed under the load. It has the form

$$Q = 2k\ M_{mc} \sin \frac{kL}{2} \frac{\lambda \cos \dfrac{kL_A - kL_B}{2} + \cos \dfrac{kL}{2}}{\sin kL_A \sin kL_B} \tag{14}$$

In case of partially distributed load (Fig. 2f), the expression for ultimate strength becomes lenghty (see Appendix), but a simple form of solution can be analogized based on the results for the symmetrically distributed case (Eq. 11), and the unsymmetrically concentrated load case (Eq. 14) as

$$Q = 2k\ M_{mc} \frac{\dfrac{kC}{4}}{\sin \dfrac{kC}{4}} \sin \left(\frac{kL}{2} - \frac{kC}{4}\right) \frac{\lambda \cos \dfrac{kL_A - kL_B}{2} + \cos \dfrac{kL}{2}}{\sin (kL_A - \dfrac{kC}{4}) \sin (kL_B - \dfrac{kC}{4})} \tag{15}$$

This is considered to be the most general form of solution for a laterally loaded beam-column, as it covers the ultimate value of a symmetrically distributed load ($L_A = L_B =$

L/2), Eq. 11, or a concentrated load (C = 0), Eq. 14.

Accuracy of Eq. 15 is investigated by comparing with the elastic solution (Eqs. 25 to 29) presented in Appendix, where location of plastic hinge is computed exactly. In Fig. 10 to 13, the comparison of interaction relationship between thrust P and lateral load Q is made. Diagrams drawn to the right represent location of the last plastic hinge.

In case of uniform load of the width L/3 (Figs. 10 and 11) some difference is observed between the two solutions. If the eccentricity of the load is less than L/6, the error remains within 5% and for e = L/4, it is 13%. Location of the plastic hinge moves in as P increases, and when P reaches the critical value P_{cr} (Euler's buckling load) the hinge is formed at the center, in this state the allowable lateral load Q is zero as to be obvious.

In case of concentrated load (Figs. 12 and 13), results by Eq. 15 checks very well even for large eccentricity of loading. The location of plastic hinge does not move until P reaches certain values, then it moves in as P increases following a curve shown in the right diagram of Fig. 12. It is interesting to note that this curve is common for all values of eccentricity of loading.

4. CONCLUSION

The ultimate strengths of beam-columns are obtained in simple closed forms. Although they are approximate solutions, their validity has been shown by comparison with exact solutions in symmetrically loaded cases. This validity is considered to be true in unsymmetric cases also as long as the unsymmetricity is not very large. The formulas for the ultimate strength of beam-columns may be considered as suitable bases for a method of design for symmetrically as well as unsymmetrically loaded compression members.

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7. APPENDIX (I)

GENERAL SOLUTION OF ELASTIC BEAM-COLUMN

Consider a member which is subjected to a thrust P, end moments M_A and M_B, and partly distributed uniform load w as shown in Fig. (14). The governing equations for this elastic beam-column problem are

$$\frac{d^4y_1}{dx^4} + k^2 \frac{d^2y_1}{dx^2} = 0 \qquad (0 \leq x \leq x_1)$$

$$\frac{d^4y_2}{dx^4} + k^2 \frac{d^2y_2}{dx^2} = \frac{w}{EI} \qquad (x_1 \leq x \leq x_2) \tag{16}$$

$$\frac{d^4y_3}{dx^4} + k^2 \frac{d^2y_3}{dx^2} = 0 \qquad (x_2 \leq x \leq \ell)$$

where

$$k^2 = P/EI$$

The general solutions are obtained in the following forms:

$$y_i = A_i \cos(kx) + B_i \sin(kx) + C_i x + D_i + f_i(x) \qquad (i = 1, 2, 3) \tag{17}$$

where $f_i(x)$ are particular solution of the above differential equations and

$$f_1(x) = f_3(x) = 0 \qquad f_2(x) = \frac{w}{2k^2 EI} x^2 \tag{18}$$

The twelve integration constants A_i, B_i, C_i, D_i are solved from the following twelve boundary conditions:

$$
\begin{aligned}
x = 0 &: \quad y_1 = 0, \quad y_1'' = -\frac{M_A}{EI} \\
x = x_1 &: \quad y_1 = y_2, \quad y_1' = y_2', \quad y_1'' = y_2'', \quad y_1''' = y_2''' \\
x = x_2 &: \quad y_2 = y_3, \quad y_2' = y_3', \quad y_2'' = y_3'', \quad y_2''' = y_3''' \\
x = \ell &: \quad y_3 = 0, \quad y_3'' = -\frac{M_A}{EI}
\end{aligned}
\tag{19}
$$

These conditions become twelve simultaneous equations as follows:

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & & & & & & & & \\
-1 & 0 & 0 & 0 & & & & & & & & \\
C_1 & S_1 & x_1 & 1 & -C_1 & -S_1 & -x_1 & -1 & & & & \\
-S_1 & C_1 & \frac{1}{k} & 0 & S_1 & -C_1 & -\frac{1}{k} & 0 & & & & \\
-C_1 & -S_1 & 0 & 0 & C_1 & S_1 & 0 & 0 & & & & \\
S_1 & -C_1 & 0 & 0 & -S_1 & C_1 & 0 & 0 & & & & \\
 & & & & C_2 & S_2 & x_2 & 1 & -C_2 & -S_2 & -x_2 & -1 \\
 & & & & -S_2 & C_2 & \frac{1}{k} & 0 & S_2 & -C_2 & -\frac{1}{k} & 0 \\
 & & & & -C_2 & -S_2 & 0 & 0 & C_2 & S_2 & 0 & 0 \\
 & & & & S_2 & -C_2 & 0 & 0 & -S_2 & C_2 & 0 & 0 \\
 & & & & & & & & C_\ell & S_\ell & \ell & 1 \\
 & & & & & & & & -C_\ell & -S_\ell & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
A_1 \\ B_1 \\ C_1 \\ D_1 \\ A_2 \\ B_2 \\ C_2 \\ D_2 \\ A_3 \\ B_3 \\ C_3 \\ D_3
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\
-\frac{1}{k^2}\frac{MA}{EI} \\
\frac{1}{2k^2}\frac{w}{EI}x_1^2 \\
\frac{1}{k^3}\frac{w}{EI}x_1 \\
\frac{1}{k^4}\frac{w}{EI} \\
0 \\
-\frac{1}{2k^2}\frac{w}{EI}x_2^2 \\
-\frac{1}{k^3}\frac{w}{EI}x_2 \\
-\frac{1}{k^4}\frac{w}{EI} \\
0 \\
0 \\
-\frac{1}{k^2}\frac{MB}{EI}
\end{Bmatrix}
\tag{20}
$$

where

$$C_1 = \cos(kx_1) \qquad S_1 = \sin(kx_1)$$

$$C_2 = \cos(kx_2) \qquad S_2 = \sin(kx_2)$$

$$C_\ell = \cos(k\ell) \qquad S_\ell = \sin(k\ell)$$

The constants are solved as follows:

$$A_1 = \frac{M_A}{k^2EI}$$

$$B_1 = \frac{1}{k^2EI}\,\frac{M_B - M_A \cos k\ell}{\sin k\ell} - \frac{w}{k^4EI}\,\frac{\cos k(\ell - x_1) - \cos k(\ell - x_2)}{\sin k\ell}$$

$$C_1 = \frac{M_A - M_B}{k^2\ell EI} - \frac{w}{2k^2\ell EI}(x_2 - x_1)(2\ell - x_1 - x_2)$$

$$D_1 = \frac{1}{k^2}\,\frac{M_A}{EI}$$

$$A_2 = \frac{M_A}{k^2EI} + \frac{w}{k^4EI}\cos kx_1$$

$$B_2 = \frac{1}{k^2EI}\,\frac{M_B - M_A \cos k\ell}{\sin k\ell} - \frac{w}{k^4EI}\,\frac{\cos kx_1 \cos k\ell - \cos k(\ell - x_2)}{\sin k\ell}$$

$$(21)$$

$$C_2 = \frac{M_A - M_B}{k^2\ell EI} - \frac{w}{2k^2\ell EI}(x_1^2 + 2x_2\ell - x_2^2)$$

$$D_2 = \frac{M_A}{k^2EI} - \frac{w}{k^4EI}\left(1 - \frac{k^2x_1^2}{2}\right)$$

$$A_3 = \frac{M_A}{k^2EI} + \frac{w}{k^4EI}(\cos kx_1 - \cos kx_2)$$

$$B_3 = \frac{1}{k^2EI}\,\frac{M_B - M_A \cos k\ell}{\sin k\ell} - \frac{w}{k^4EI}(\cos kx_1 - \cos kx_2)\cot k\ell$$

$$C_3 = \frac{M_A - M_B}{k^2\ell EI} - \frac{w}{2k^2\ell EI}(x_1^2 - x_2^2)$$

$$D_3 = \frac{M_A}{k^2 EI} + \frac{w}{2k^2 EI}(x_1^{\ 2} - x_2^{\ 2}) \tag{21}$$

Let them be expressed in the following form:

$$A_i = \frac{M_{mc}}{k^2 EI}(a_{im} + q\ a_{iw})$$

$$B_i = \frac{M_{mc}}{k^2 EI}(b_{im} + q\ b_{iw})$$

$$C_i = \frac{M_{mc}}{k^2 1EI}(c_{im} + q\ c_{iw}) \tag{22}$$

$$D_i = \frac{M_{mc}}{k^2 EI}(d_{im} + q\ d_{iw})$$

Where

$$q = w/k^2\ M_{mc}$$

Location of the maximum bending moment $x = x_i^{\ *}$ for each portion is obtained from the condition

$$\frac{d^3 y_i}{dx^3} = A_i k^3 \sin(kx_i^{\ *}) - B_i k^3 \cos(kx_i^{\ *}) = 0$$

or

$$kx_i^* = \tan^{-1} \frac{B_i}{A_i} = \tan^{-1} \frac{b_{im} + qb_{iw}}{a_{im} + qa_{iw}} \tag{23}$$

The ultimate state is obtained when the maximum bending moment reaches the plastic moment of the member M_{mc}, ie,

$$\left. \frac{d^2y}{dx^2} \right|_{x = x^*} = -A_i k^2 \cos (kx^*) - B_i k^2 \sin (kx^*)$$

$$+ f_i'' (x^*) = - \frac{M_{mc}}{EI}$$

or

$$A_i \cos (kx^*) + B_i \sin (kx^*) = \frac{M_{mc}}{k^2 EI} - \frac{1}{k^2} f_i'' (x^*) \tag{24}$$

Elimination of x^* from Eq. (23) and Eq. (24) gives the ultimate load q_o. There are five possible cases in the ultimate state according to the location of the plastic hinge (Fig. 15):

<u>Case-I</u> The plastic hinge in the left portion

$$x^* = x_1^* \quad (x_1^* < x_1, \; x_2^* < x_1, \; x_3^* < x_2)$$

$$q_o = \frac{1}{b_{1w}} (\sqrt{1 - a_{1m}^2} - b_{1m}) \tag{25}$$

Case-II The plastic hinge at the left boundary

$$x^* = x_1 \quad (x_1^* > x_1,\ x_2^* < x_1,\ x_3^* < x_2)$$

$$q_o = \frac{1}{b_{1w}} \left[\frac{1 - a_{1m} \cos(kx_1)}{\sin(kx_1)} - b_{1m} \right] \tag{26}$$

Case-III The plastic hinge in the middle portion

$$x^* = x_2^* \quad (x_1^* > x_1,\ x_1 < x_2^* < x_2,\ x_3^* < x_2)$$

$$q_o = -\frac{1-a_{2m}\ a_{2w}-b_{2m}\ b_{2w}}{1-a_{2m}^2-b_{2m}^2} - \sqrt{\left(\frac{1-a_{2m}\ a_{2w}-b_{2m}\ b_{2w}}{1-a_{2m}^2-b_{2w}^2}\right)^2 - \frac{1-a_{2m}^2-b_{2m}^2}{1-a_{2w}^2-b_{2w}^2}} \tag{27}$$

Case-IV The plastic hinge at the right boundary

$$x^* = x_2 \quad (x_1^* > x_1,\ x_2^* > x_2,\ x_3^* < x_2)$$

$$q_o = \frac{1-a_{3m}\ \cos(kx_2)-b_{3m}\ \sin(kx_2)}{a_{3w}\ \mathrm{cox}(kx_2)+b_{3w}\ \sin(kx_2)} \tag{28}$$

Case-V The plastic hinge in the right portion

$$x^* = x_3^* \quad (x_1^* > x_1,\ x_2^* > x_2,\ x_3^* > x_2)$$

$$q_o = - \frac{a_{3m}\ a_{3w} + b_{3m}\ b_{3w}}{a_{3w}^2 + b_{3w}^2} + \sqrt{\left(\frac{a_{3m}\ a_{3w} + b_{3m}\ b_{3w}}{a_{3w}^2 + b_{3w}^2}\right)^2 + \frac{1 - a_{3m}^2 - b_{3m}^2}{a_{3w}^2 + b_{3w}^2}} \tag{29}$$

The location of the plastic hinge x^*_i (Eq. 23) and the ultimate load q_o (Eq. 25 to 29) are functions of each other. Therefore, they have to be solved by iteration. A computer program was made for this purpose.

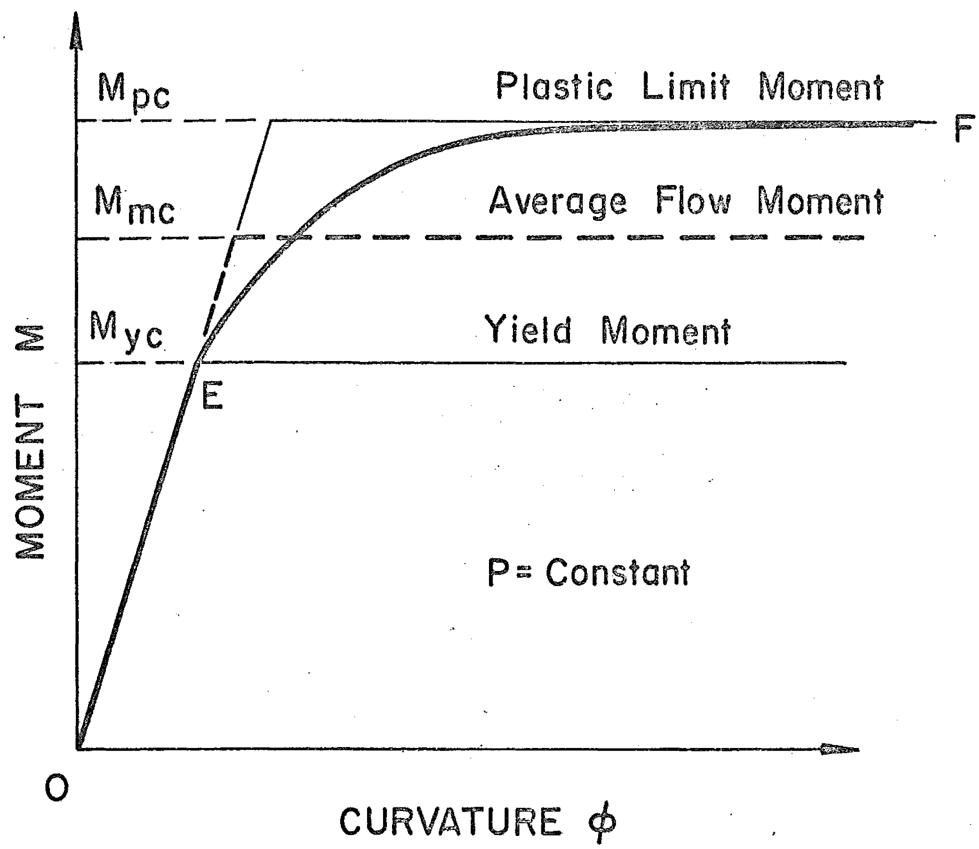


Fig. 1 Idealization of Moment Curvature Relation

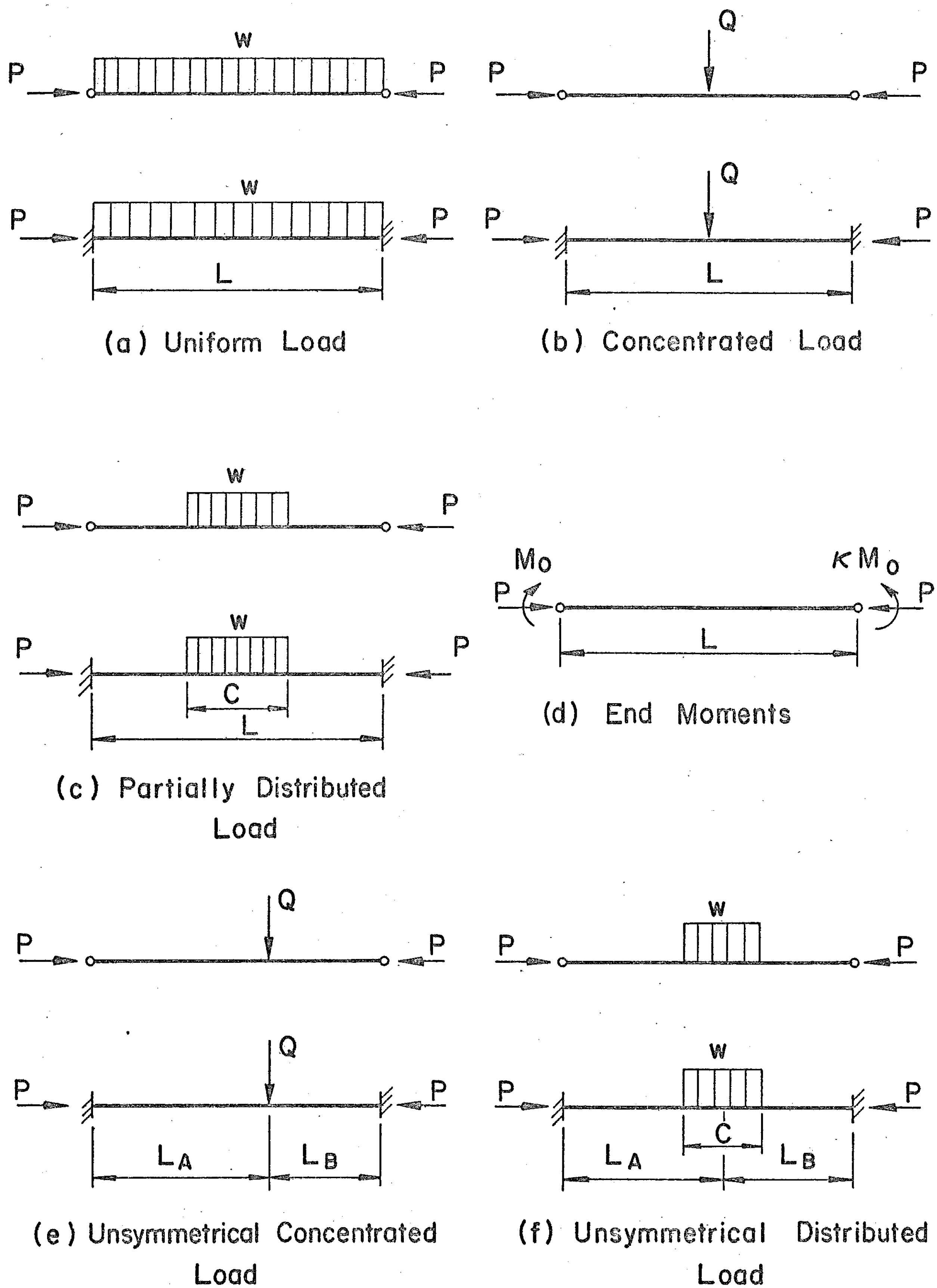


Fig. 2 Laterally Loaded Beam-Columns

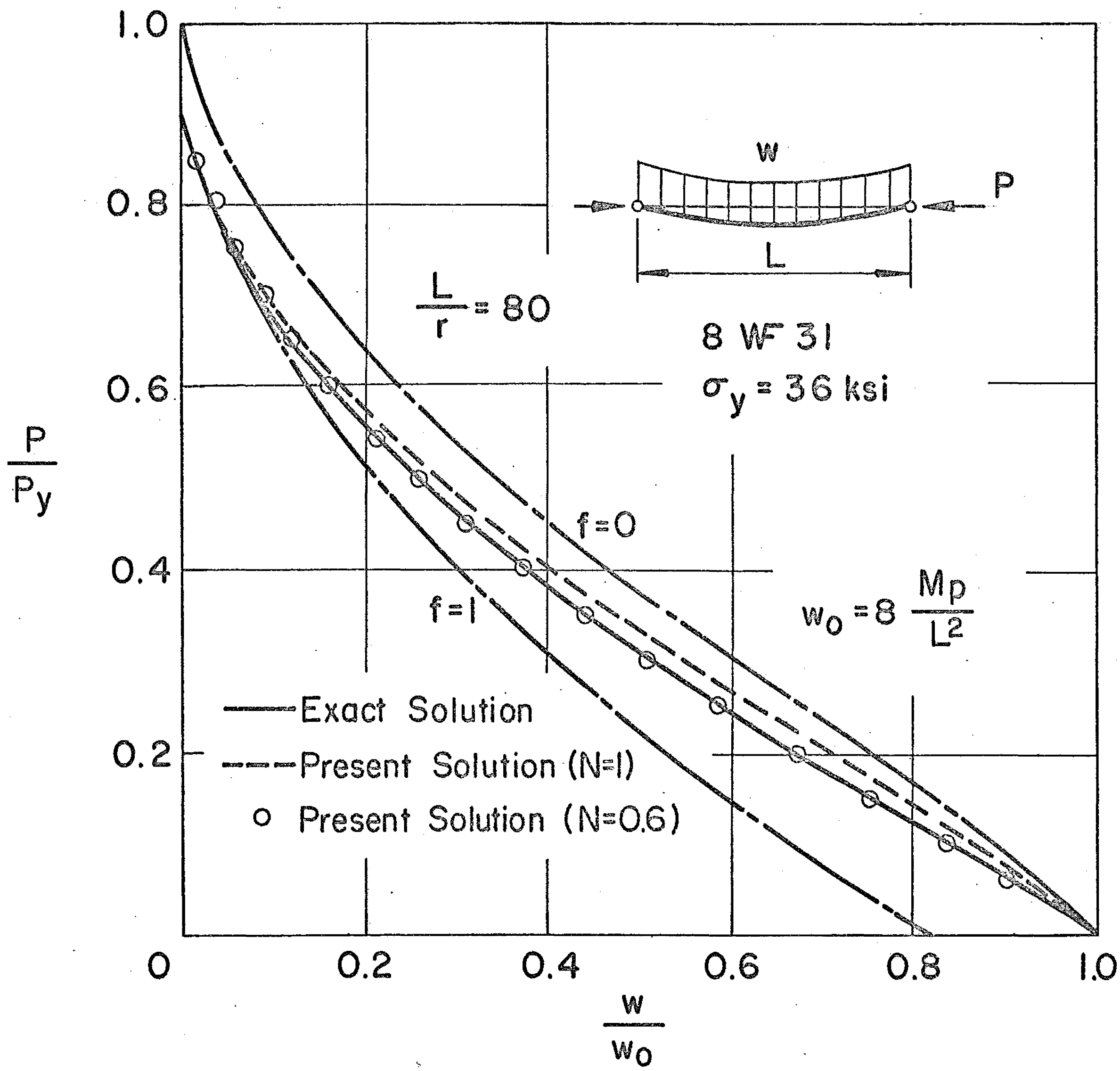
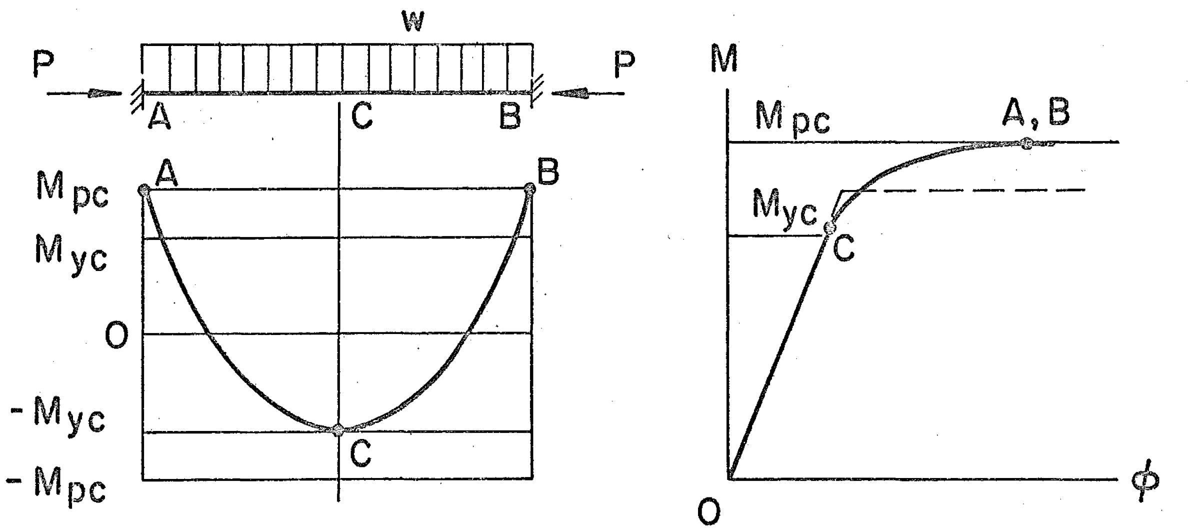
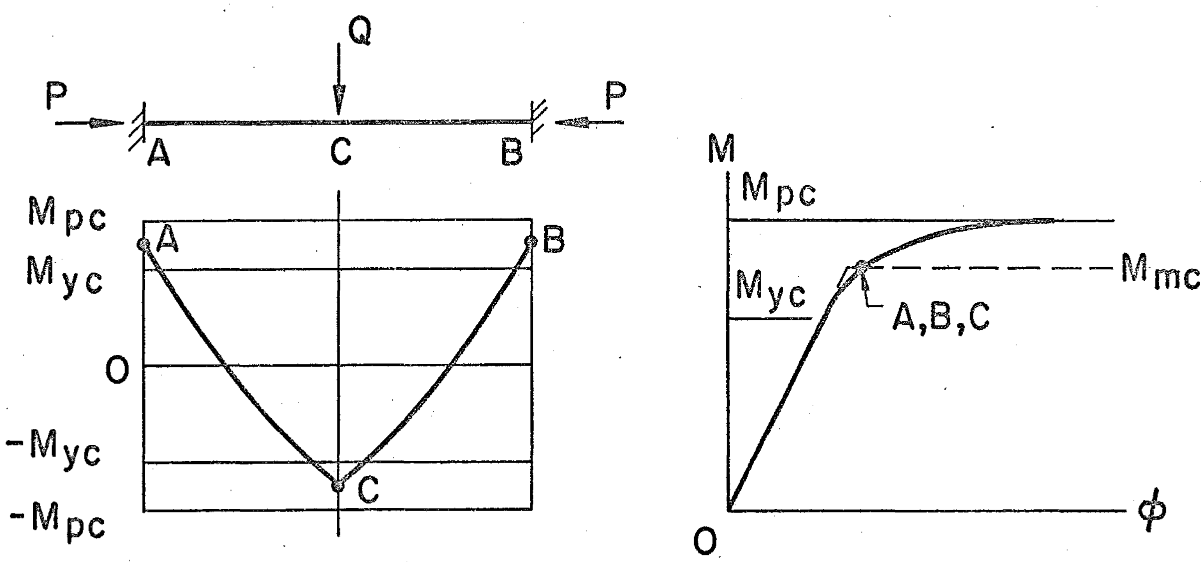


Fig. 3 A Bounded Solution of Beam-Column



(a) Uniform Load



(b) Concentrated Load

Fig. 4 Average Flow Moment of Fixed Beam-Columns

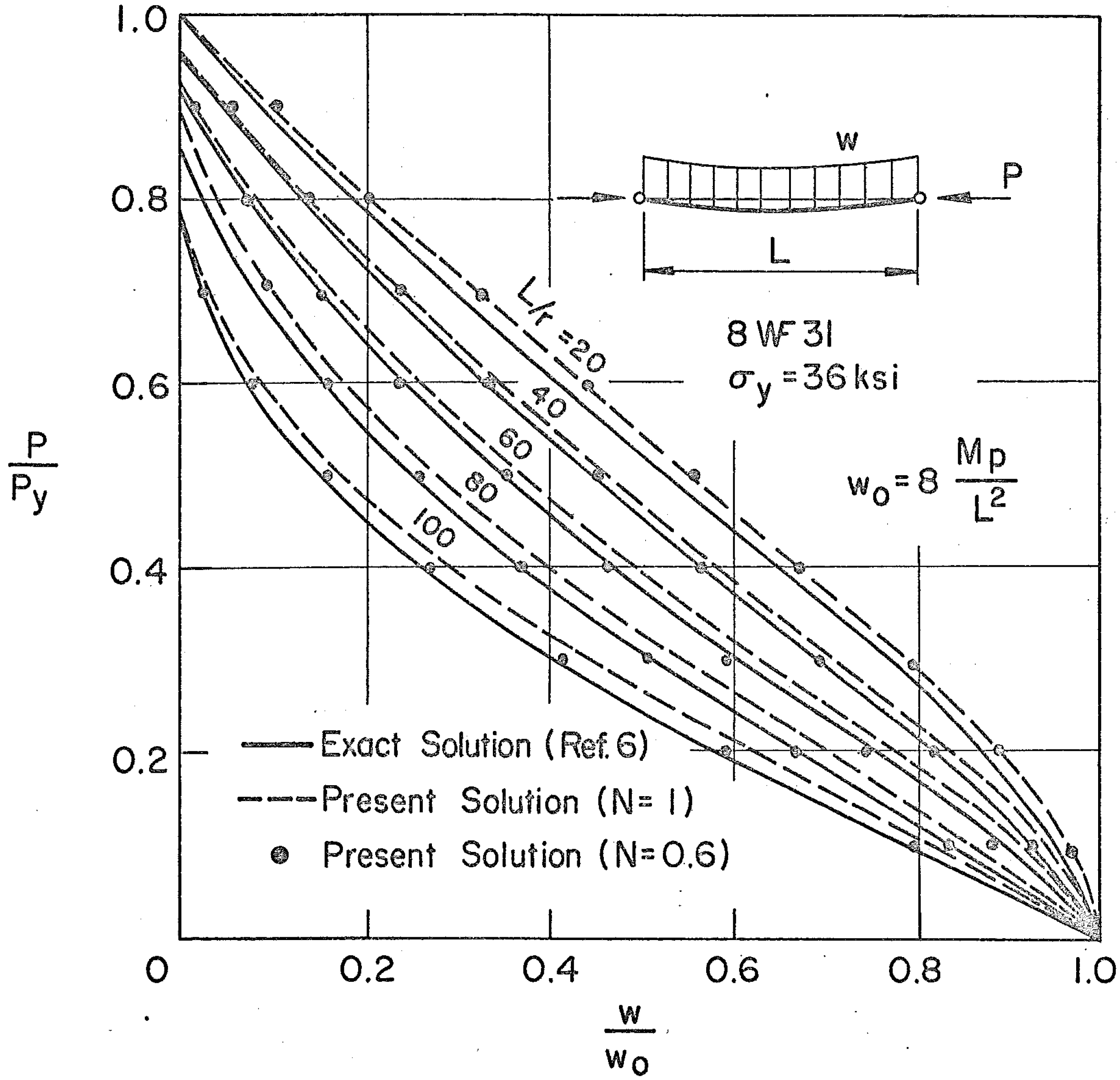


Fig. 5 Simple Beam-Column with Uniform Load

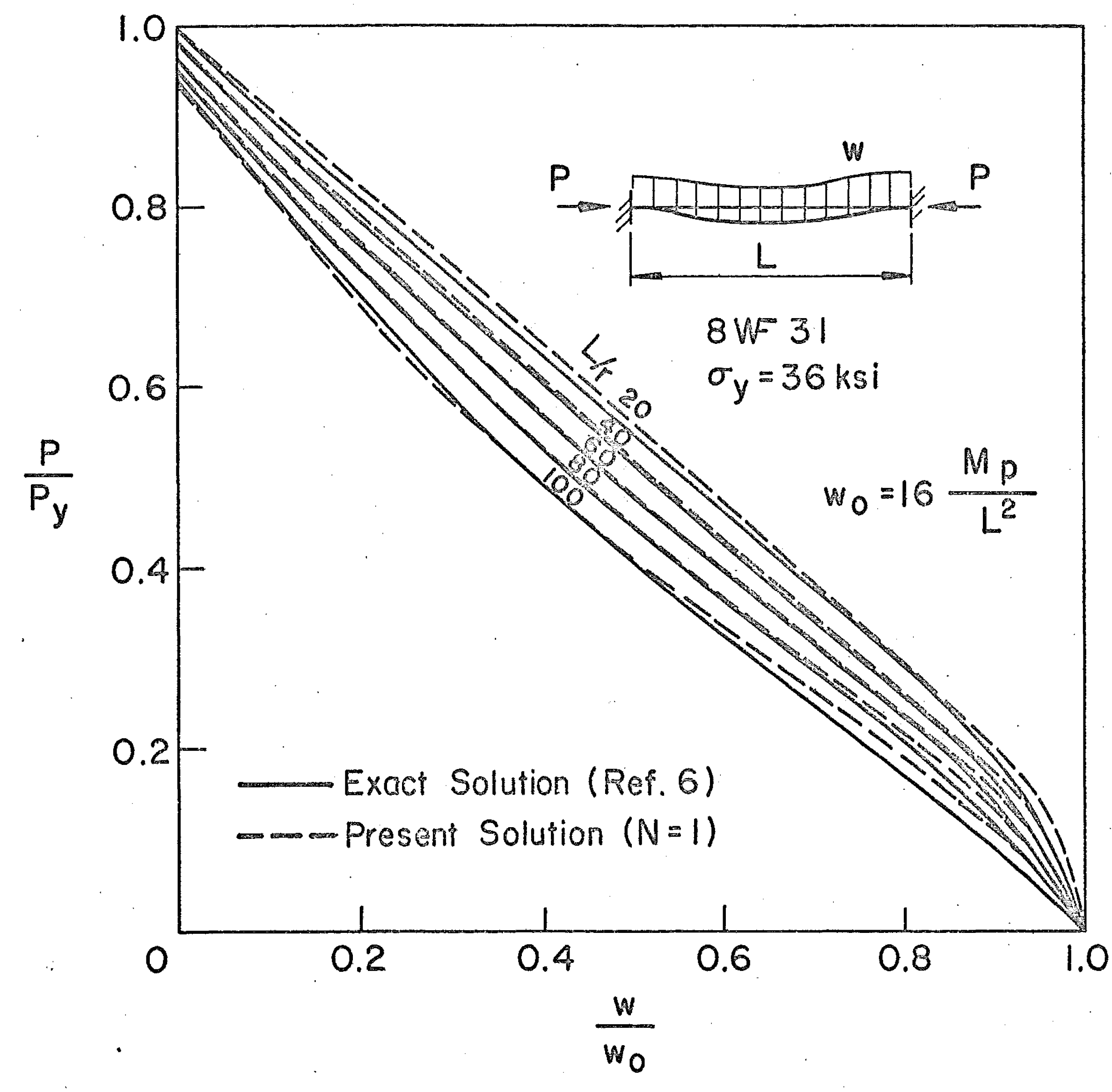


Fig. 6 Fixed Beam-Column with Uniform Load

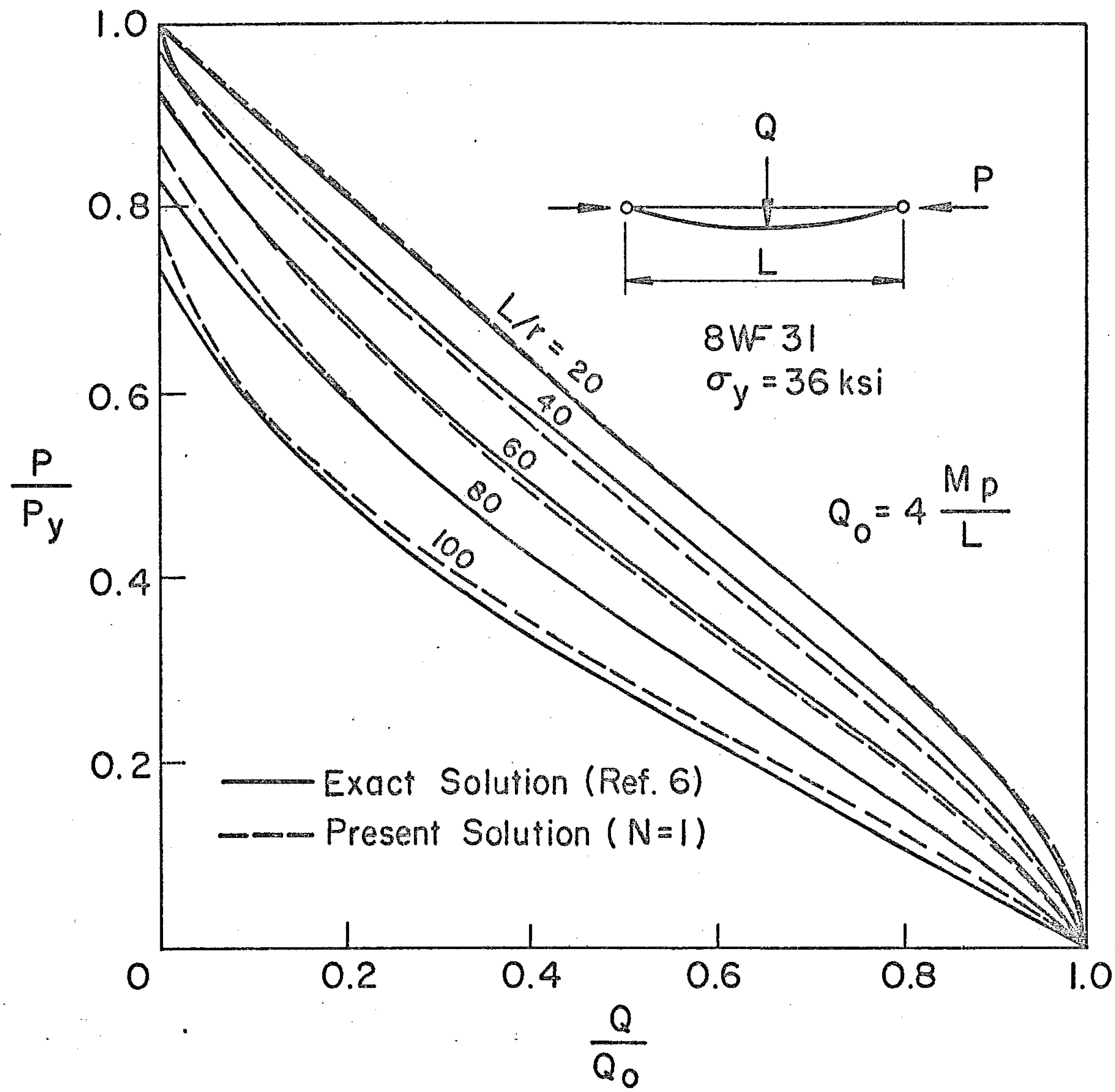


Fig. 7 Simple Beam-Column with Concentrated Load

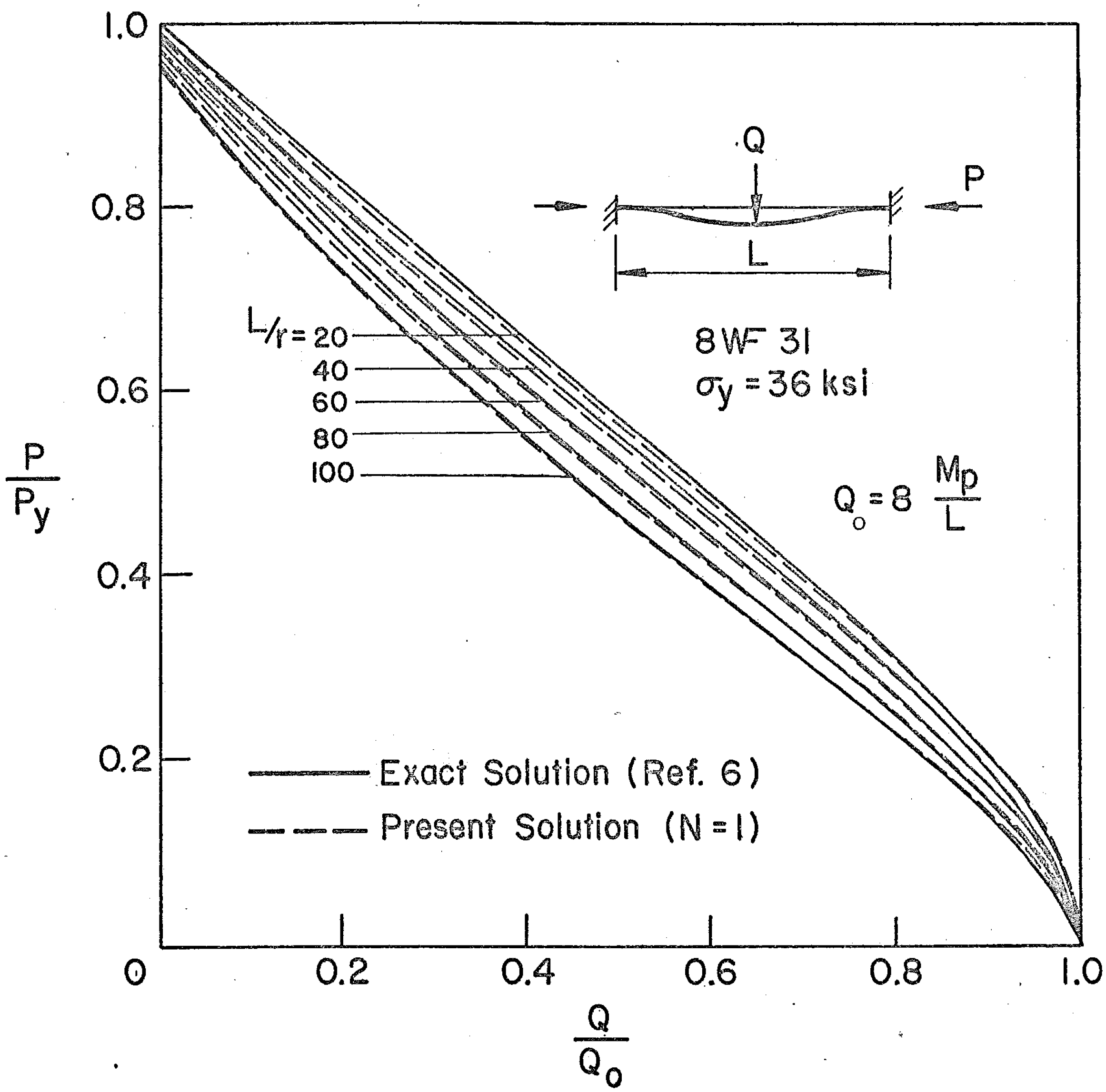


Fig. 8 Fixed Beam-Column with Concentrated Load

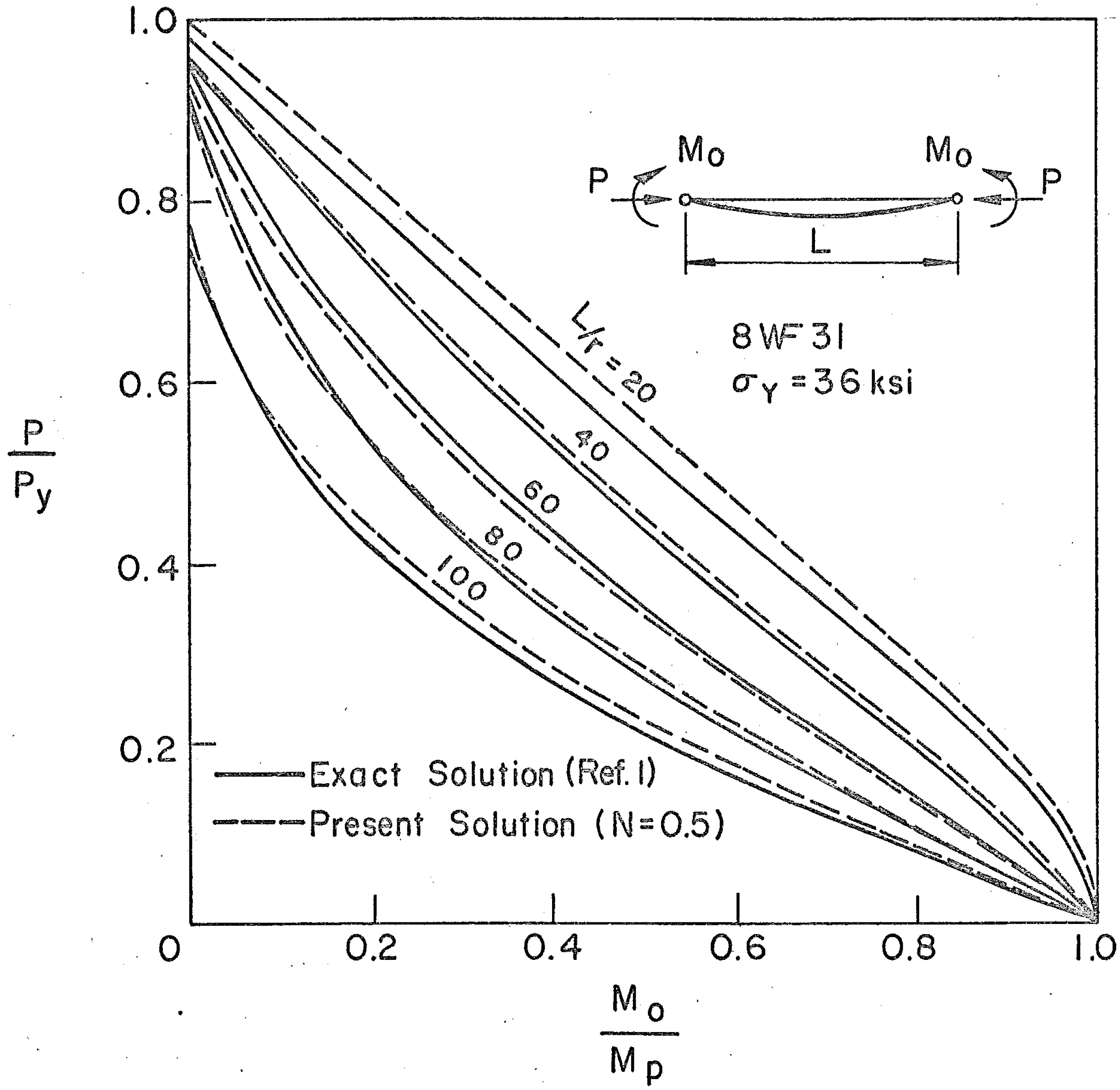


Fig. 9 Beam-Column with End-Moments

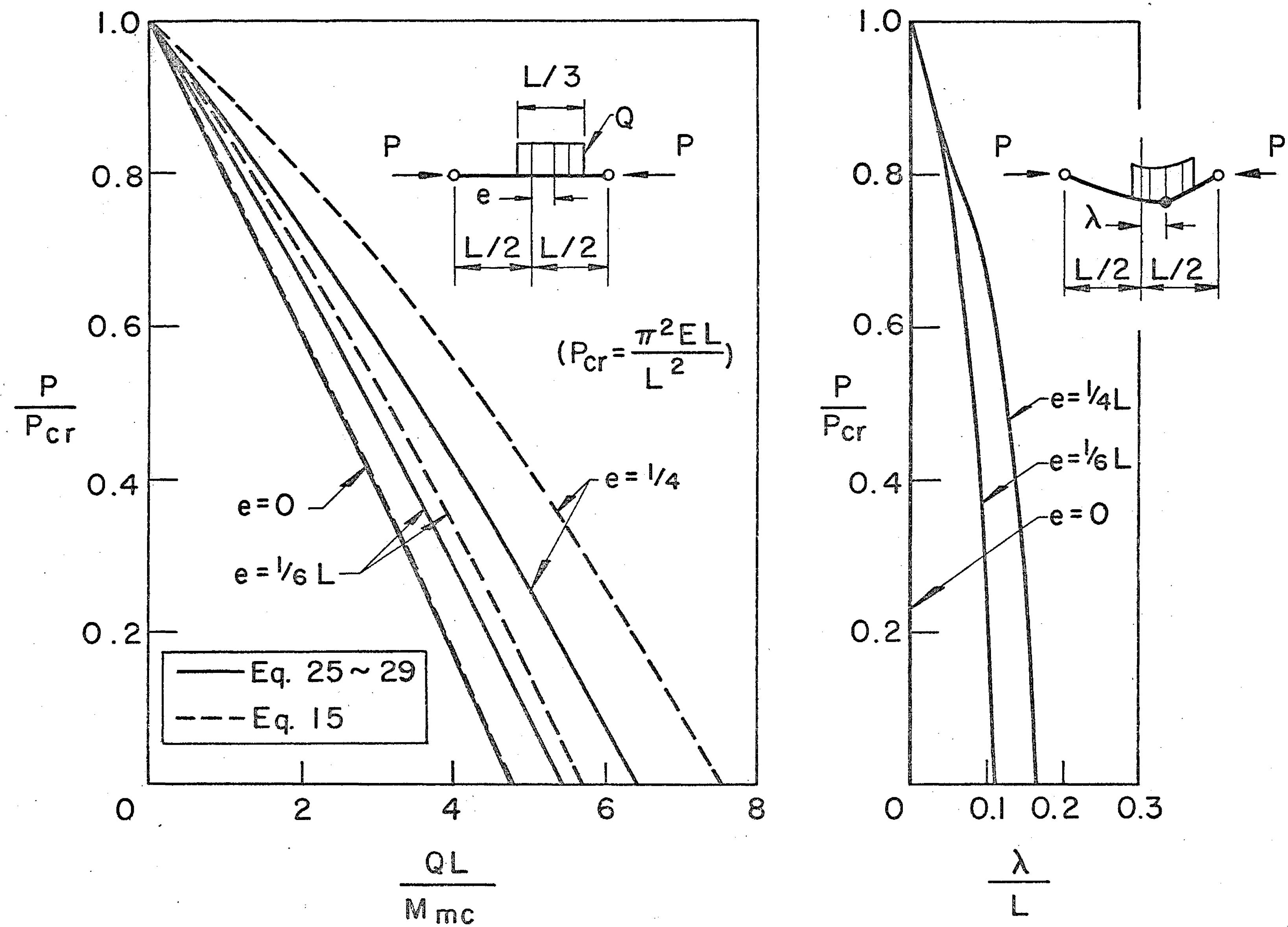


Fig. 10 Accuracy of Eq. 15 (Uniform Load on Simple Beam-Column)

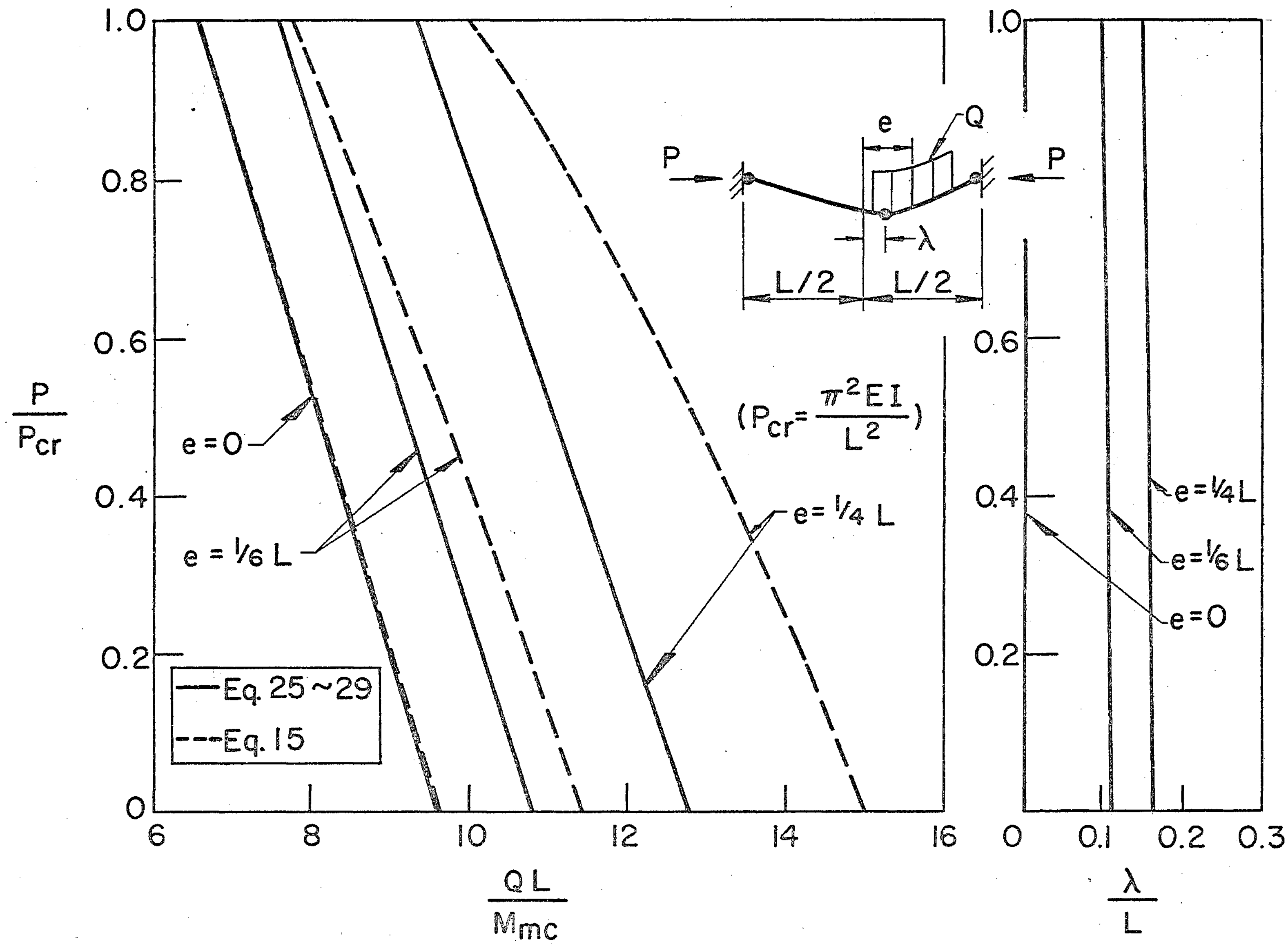


Fig. 11 Accuracy of Eq. 15 (Uniform Load on Fixed Beam-Column)

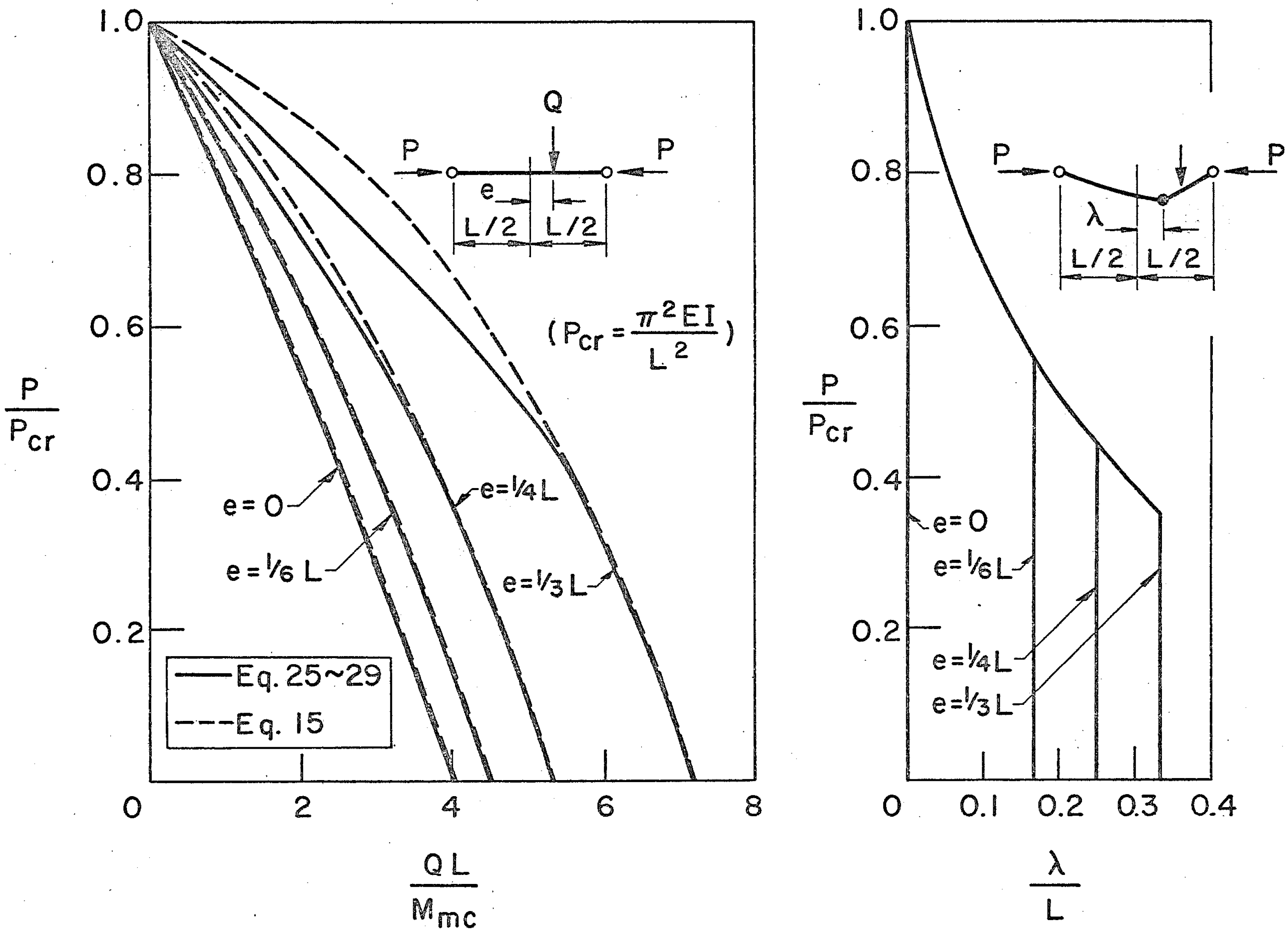


Fig. 12 Accuracy of Eq. 15 (Concentrated Load on Simple Beam-Column)

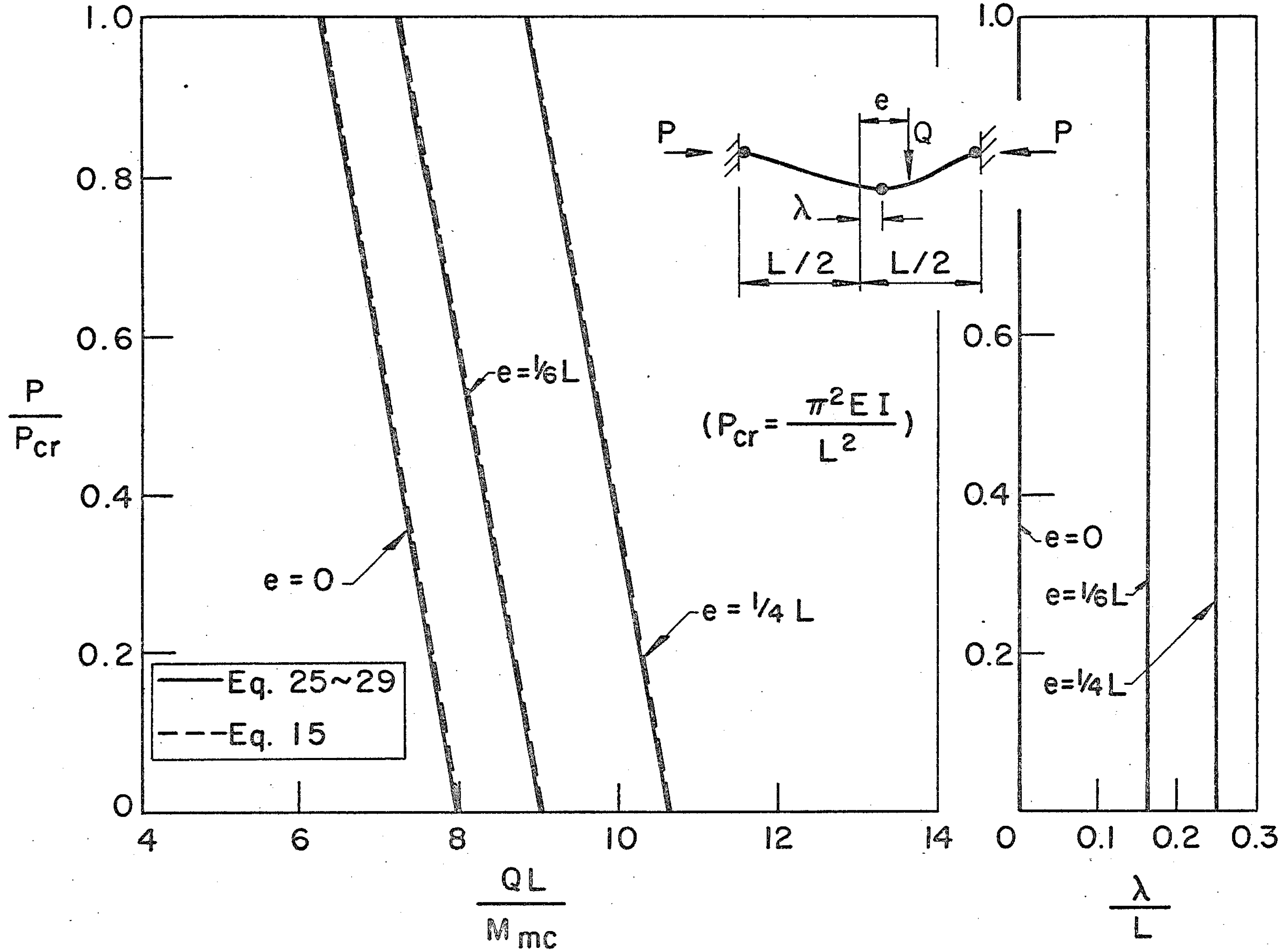


Fig. 13 Accuracy of Eq. 15 (Concentrated Load on Fixed Beam-Column)

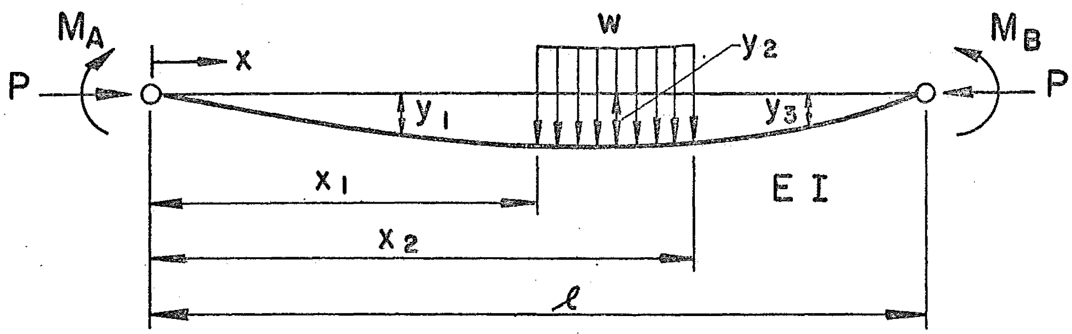


Fig. 14 Laterally Loaded Beam-Column

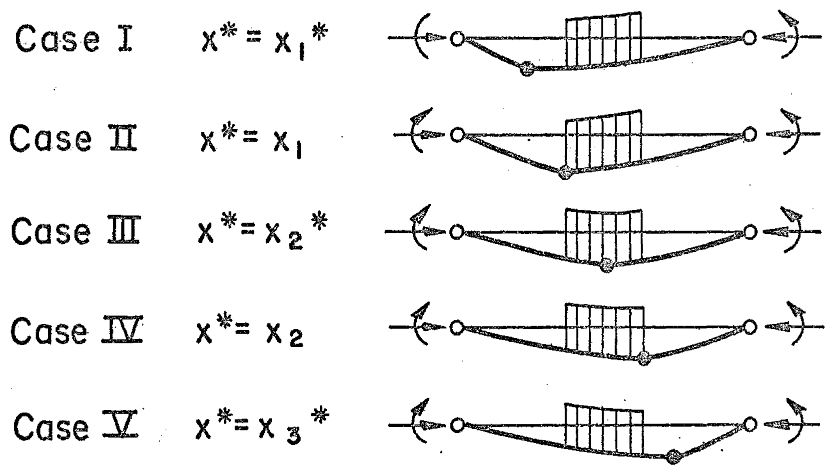


Fig. 15 Ultimate States