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LEHIGH UNIVERSITY INSTITUTE OF RESEARCH

Open Channel Research

THE EFFECTS OF SIMPLE TRANSITIONS ON SUPERCRITICAL FLOW IN AN INCLINED OPEN CHANNEL

by
Gunnar Bagge

Fritz Engineering Laboratory Report No. 326.1

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Dr. L. S. Beedle is Acting Chairman of the Department of Civil Engineering and Director of Fritz Engineering Laboratory.

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LIST OF SYMBOLS

B = Initial width of the channel

d = Depth of flow, measured perpendicular to the bottom

F' = The Froude number, modified for inclined channel ($=V^2/gd\cos\beta$)

g = Acceleration of gravity

p = Pressure

u = Velocity in the x-direction, parallel to the undisturbed flow

v = Velocity in the y-direction, perpendicular to the undisturbed flow

V = Total velocity

w = Velocity in the z-direction, perpendicular to the bottom

β = Slope of the bottom

θ = Change of angle in a straight flaring transition

ρ = Density of the liquid

ABSTRACT

This paper presents a theoretical analysis of the effects of disturbances on supercritical flows in an inclined channel, based on the Method of Characteristics.

Since the practical application of the method would be severely limited by the amount of manual computations involved, the adaptation of the results for computer treatment has been emphasized. A number of numerical examples concerning a straight flaring transition is presented.

1. INTRODUCTION

In many hydraulic construction projects, one is required to design transitions for water flows in open channels at high velocities, that is for supercritical flows. An accurate description of such a flow will be significantly more complex than that of a subcritical flow case, because of the formation of standing waves generated at the points of disturbance.

The theory of these flows was adapted from the analogous theory of supersonic flow of gases by von Karman(1)* and Preiswerk(2). The latter analysis was based on the Method of Characteristics and resulted in a fairly simple graphical solution for horizontal channels (explained in detail by Blaisdell(3)). Most of the later work on the subject was similarly limited to horizontal or slightly inclined channels.(4,5,6,7)

The present paper is an attempt to analyze supercritical flow in an inclined channel. The Method of Characteristics which is explained in detail by Owczarek(8) is used. The method can be summarized as follows:

Using certain limiting assumptions (frictionless fluid, irrotational flow, hydrostatic pressure distribution) two differential equations can be derived, involving the partial derivatives of u and v with respect to y and x, where u and v represent the velocities in the x and y

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* Number in parenthesis refer to references on pages 33 and 34.

direction, respectively. From these equations two sets of characteristics can be found, expressed as:

$$\left(\frac{dy}{dx}\right)_1 = f_1\ (u,\ v,\ d) \quad \text{and}$$

$$\left(\frac{dy}{dx}\right)_2 = f_2\ (u,\ v,\ d)$$

where d indicates the depth of flow, measured perpendicular to the bottom. Along these lines u, v, and d vary in a prescribed manner, given in the compatibility relations:

$$\left(\frac{dv}{du}\right)_1 = g_1\ \left(u,\ v,\ d,\ \left(\frac{dy}{du}\right)\right) \quad \text{and}$$

$$\left(\frac{dv}{du}\right)_2 = g_2\ \left(u,\ v,\ d,\ \left(\frac{dy}{du}\right)\right)$$

These equations, when written in finite difference form, make it possible to analyze the entire flow field, that is to determine values of u, v, and d at various points.

2. DERIVATION OF EQUATIONS

Using the Energy Equation and the Continuity Equation together with the condition for irrotational flow, two differential equations are derived, involving the partial derivations of u and v with respect to x and y.

Assumptions

1. The flow is frictionless, without energy dissipation.

2. Acceleration perpendicular to the bottom is infinitely small compared to the acceleration of gravity (this results in a hydrostatic pressure distribution).

Energy Equation

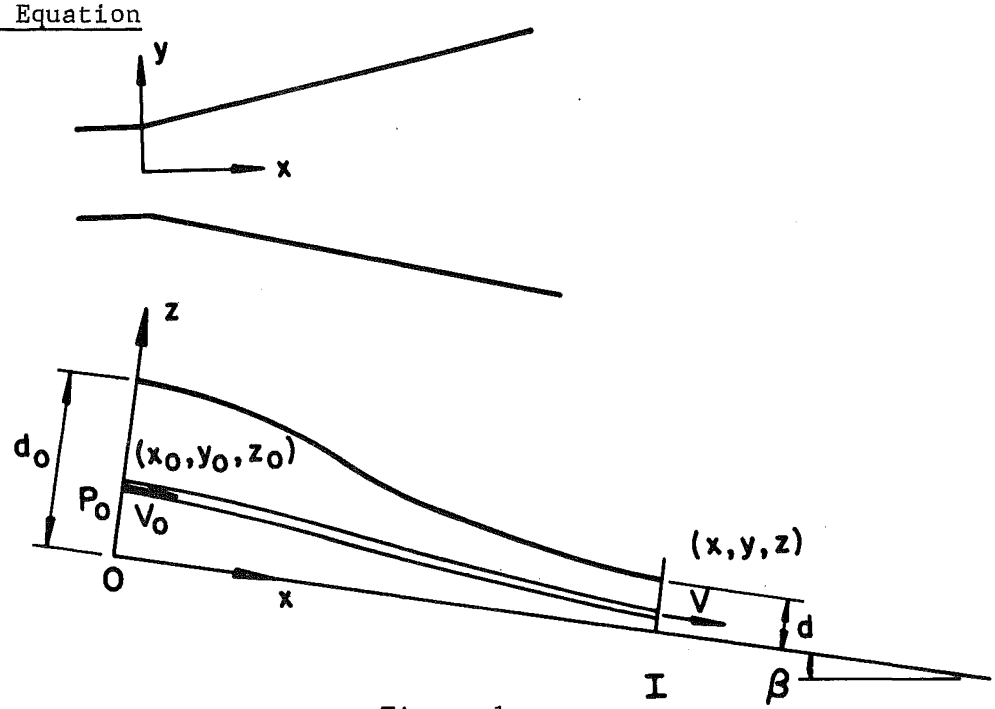


Figure 1

At $x = x_o$ the flow is assumed to be uniform, with constant velocity V_o ($= u_o$). Applying the energy equation from point 0 to I gives:

$$p + \frac{1}{2}\rho V^2 + \rho g\,(z\cos\beta - (x - x_o)\sin\beta) = p_o + \rho g\, z_o\cos\beta + \frac{1}{2}\rho V_o^{\,2} \quad \text{or}$$

$$V^2 - V_o^{\,2} = \frac{2}{\rho}(p_o - p) + 2g\,((z_o - z)\cos\beta + (x - x_o)\sin\beta) \tag{1}$$

With static pressure distribution (Assumption 2)

$$p = \rho g\,(d - z)\cos\beta \tag{2}$$

Differentiation of equation (1) gives:

$$V\frac{\partial V}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g\sin\beta \tag{3}$$

$$V\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\rho y} \tag{4}$$

$$V\frac{\partial V}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\cos\beta \tag{5}$$

while differentiation of (2) results in:

$$\frac{\partial p}{\partial x} = \rho g\,\frac{\partial d}{\partial x}\cos\beta \tag{6}$$

$$\frac{\partial p}{\partial y} = \rho g\,\frac{\partial d}{\partial y}\cos\beta \tag{7}$$

$$\frac{\partial p}{\partial z} = -\rho g\cos\beta \tag{8}$$

Equations (6) and (7) state that the accelerations in the x and y directions are independent of z. Therefore, u and v must also be independent of z, since this was the case during the initial (uniform) flow.

Equations (5) and (8) combined give that $\frac{\partial V}{\partial z} = 0$ or that V is independent of z. Now $w = \sqrt{V^2 - u^2 - v^2}$ must similarly be independent of z, and since it equals 0 at the bottom one may conclude that $w << u$ or v, or that $V^2 = u^2 + v^2$. (9)

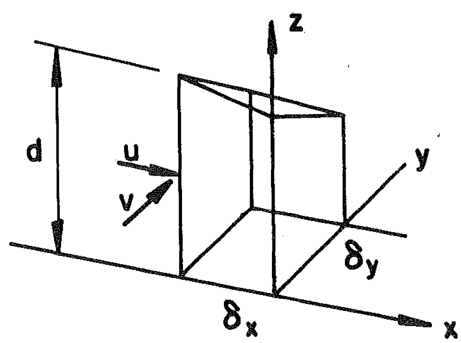
Combining equations (3), (6), and (9) results in:

$$\frac{\partial d}{\partial x} = \tan\beta - \frac{v}{g\cos\beta}\frac{\partial v}{\partial x} - \frac{u}{g\cos\beta}\frac{\partial u}{\partial x} \tag{10}$$

Similarly equations (4), (7), and (9) give:

$$\frac{\partial d}{\partial y} = -\frac{v}{g\cos\beta}\frac{\partial v}{\partial y} - \frac{u}{g\cos\beta}\frac{\partial u}{\partial y} \tag{11}$$

<u>Continuity Equation</u>



<u>Figure 2</u>

As the flow can be considered incompressible, the continuity equation becomes:

$$\delta y\left\{\left[(d)(u) + \frac{1}{2}\frac{\partial[(d)(u)]}{\partial y}\delta y\right] - \left[(d)(u) + \frac{\partial[(d)(u)]}{\partial x}\delta x + \frac{1}{2}\frac{\partial[(d)(u)]}{\partial y}\delta y\right]\right\} +$$

$$+\ \delta x\left\{\left[(d)(v) + \frac{1}{2}\frac{\partial[(d)(v)]}{\partial x}\delta x\right] - \left[(d)(v) + \frac{\partial[(d)(v)]}{\partial y}\delta y + \frac{1}{2}\frac{\partial[(d)(v)]}{\partial x}\delta x\right]\right\} = 0$$

or

$$\frac{\partial[(d)(u)]}{\partial x} + \frac{\partial[(d)(v)]}{\partial y} = 0$$

which can be written as:

$$d \frac{\partial u}{\partial x} + u \frac{\partial d}{\partial x} + d \frac{\partial v}{\partial y} + v \frac{\partial d}{\partial y} = 0 \qquad (12)$$

Final Differential Equations

Introducing the expressions for $\frac{\partial d}{\partial x}$ and $\frac{\partial d}{\partial y}$ from equations (10), (11) into the continuity equation (12) one gets:

$$\frac{\partial u}{\partial x}\left(1 - \frac{u^2}{gd\ \cos\beta}\right) + \frac{\partial v}{\partial y}\left(1 - \frac{v^2}{gd\ \cos\beta}\right) - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{uv}{gd\ \cos\beta} + \frac{u}{d}\tan\beta = 0 \qquad (13)$$

Since the flow is frictionless and initially irrotational it will remain so. Therefore

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (14)$$

These equations are suitable for analysis by the Method of Characteristics as shown in the following section.

3. SOLUTION BY METHOD OF CHARACTERISTICS

From the preceding equations expressions for the slope of the characteristics are derived with compatibility relations controlling the variation of u, v and d along these characteristics. Equations (13) and (14) can be written in the form:

$$A_1 \frac{\partial u}{\partial x} + A_2 \frac{\partial u}{\partial y} + A_3 \frac{\partial v}{\partial x} + A_4 \frac{\partial v}{\partial y} = F_1$$

$$B_1 \frac{\partial u}{\partial x} + B_2 \frac{\partial u}{\partial y} + B_3 \frac{\partial v}{\partial x} + B_4 \frac{\partial v}{\partial y} = F_2$$

where:

$$A_1 = 1 - \frac{u^2}{gd\ \cos\beta}\ ,\ A_2 = A_3 = \frac{-uv}{gd\ \cos\beta},\ A_4 = 1 - \frac{v^2}{gd\ \cos\beta},\ F_1 = \frac{u}{d}\tan\beta$$

$$B_1 = 0,\ B_2 = -1,\ B_3 = +1,\ B_4 = 0,\ F_2 = 0$$

The physical characteristics can now be determined from the following equation *):

$$(A_1B_3-A_3B_1)\left(\frac{dy}{dx}\right)^2 - (A_1B_4+A_2B_3-A_4B_1-A_3B_2)\frac{dy}{dx} + (A_2B_4-A_4B_2) = 0 \qquad (15)$$

Introducing the Froude number (modified for inclined channel):

$$F' = \frac{u^2 + v^2}{gd\ \cos\beta} \qquad (16)$$

*) Further details can be found in reference (8), pp. 278-290.

the solutions of equation (15) become:

$$\frac{dy}{dx} = \frac{-\frac{uv}{gd\ \cos\beta} - \sqrt{F' - 1}}{1 - \frac{u^2}{gd\ \cos\beta}} \tag{17a}$$

and

$$\frac{dy}{dx} = \frac{-\frac{uv}{gd\ \cos\beta} + \sqrt{F' - 1}}{1 - \frac{u^2}{gd\ \cos\beta}} \tag{17b}$$

Solutions corresponding to equation (17a) will be called the C_+ - characteristics while the ones resulting from equation (17b) will be called the C_- - characteristics. The method is obviously applicable for supercritical flow only (where $F' > 1$).

Along the physical characteristics certain compatibility relations must be satisfied. These can be expressed in the following equation*):

$$\frac{dv}{du} = \frac{A_3B_2 - A_2B_3}{A_4B_3 - A_3B_4} - \frac{A_4B_2 - A_2B_4}{(A_4B_3 - A_3B_4)\frac{dy}{dx}} - \frac{(F_1B_4 - A_4F_2)\frac{dy}{du}}{(A_4B_3 - A_3B_4)\frac{dy}{dx}} + \frac{(F_1B_3 - A_3F_2)\frac{dy}{du}}{A_4B_3 - A_3B_4}$$

Inserting the expression for $\frac{dy}{dx}$ given in equation (17a) this reduces to:

$$\frac{dv}{du} = \frac{\frac{uv}{gd\ \cos\beta} + \sqrt{F' - 1} - \frac{u}{d}\tan\beta\ (\frac{dy}{du})}{1 - \frac{v^2}{gd\ \cos\beta}} \tag{18a}$$

*) See footnote on preceding page.

Similarly, for the C_- characteristics (equation (17b)):

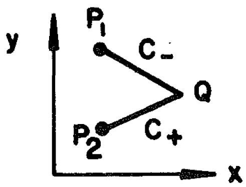
$$\frac{dv}{du} = \frac{\frac{uv}{gd\ \cos\beta} - \sqrt{F' - 1} - \frac{u}{d}\tan\beta\ (\frac{dy}{du})}{1 - \frac{v^2}{gd\ \cos\beta}} \tag{18b}$$

Equations (18a) and (18b) must be satisfied along the C_+ and the C_- characteristics, respectively.

Using the above equations, it is possible to obtain numerical solutions for specific transition problems, as indicated in the following section.

4. ADAPTATION OF SOLUTION FOR COMPUTER TREATMENT

The equations of the characteristics and the compatibility equations will be applied to solve a flow problem involving a straight flaring transition. The method can be summarized as follows:



The solution (u, v and d) is assumed known at P_1 and P_2. Inserting the values of u, v and d from P_1 into equation (17a) the approximate slope of the C_+ characteristic passing through P_1 can be determined. Similarly, using equation (17b) the slope of the C_- characteristic passing through P_2 can be found, after which the approximate location of point Q is determined.

Now equation (18a) can be applied from P_1 to Q, using finite differences and inserting the values of u, v, and d from P_1 while Δy is known as $y_Q = y_{P_1}$. Similarly, equation (18b) is used from P_2 to Q and the two equations yield values of u and v for point Q. Using the energy equation the corresponding value of d at Q is determined.

This procedure is subsequently repeated, using the average values of u, v and d between P_1 and Q and P_2 and Q, respectively, until sufficient accuracy is achieved.

For points on the boundaries the method must be modified, as shown in the following detailed description.

Simple (straight) Flaring Transition

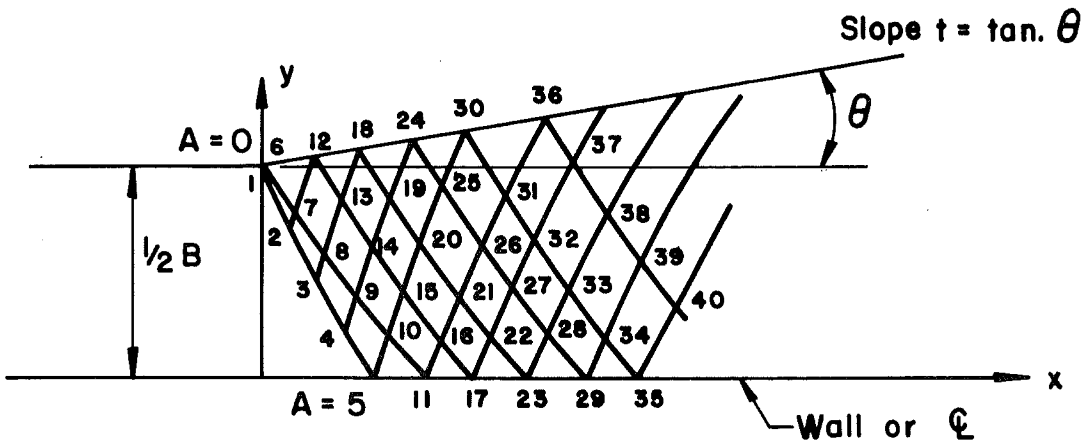


Figure 3

The channel is initially divided into a number of increments, depending on the desired accuracy. Although 5 is chosen here, the approach can easily be modified for a different number of points across the channel.

Four types of node must be considered, that is the lines connecting points 0-5 (I), points 6, 12, 18 etc. (II), points 11, 17, 23 etc. (III) plus the region inside the boundaries, points 7-10, 13-16 etc. (IV).

I. Points 0-5

Figure 4

In this region v = 0, which reduces equations (18b) to:

$$\frac{dv}{du} = 0 = -\sqrt{F' - 1} - \frac{u}{d} \tan\beta \left(\frac{dy}{du}\right)$$

which can be rewritten, using finite differences, as:

$$\Delta u = - \frac{u \tan\beta \ (\Delta y)}{d \sqrt{F' - 1}} \tag{19}$$

in which form it is suitable for iteration. A fixed value of Δy is assumed for each point* and in the first approximation u_{A-1} and d_{A-1} are used to determine Δu. After that

$$u_A = u_{A-1} + \Delta u$$

and

$$d_A = \frac{u_{A-1}}{u_A} d_{A-1} \text{ (from continuity)}$$

In successive approximations $u = \frac{1}{2} (u_A + u_{A-1})$ and $d = \frac{1}{2} (d_A + d_{A-1})$ are used, until sufficient accuracy is achieved.

Since v = 0, equation (17b) can be rewritten as:

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{F' - 1}}{1 - F'}, \text{ or}$$

$$\Delta x = - \Delta y \sqrt{F' - 1} \tag{20}$$

from which equation Δx now can be determined.

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* The most accurate results are obtained if the first Δy increment selected (between point 0 and 1) is very small.

II. Points 6, 12, 18 etc.

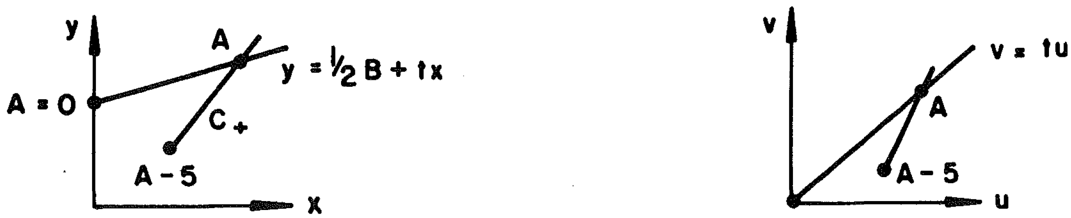

Figure 5

Using initial values of $u = u_{A-5}$, $v = v_{A-5}$ and $d = d_{A-5}$, $\frac{dy}{dx}$ can be determined from equation (17a). The (x,y) - coordinates of A are then found as the intersection between the two lines:

$$0 - (A): \quad y_A = \frac{1}{2} B + tx_A$$

$$(A-5) - (A): \quad y_A = y_{A-5} + \frac{dy}{dx} (x_A - x_{A-5})$$

which give:

$$x_A = \frac{\frac{1}{2} B - y_{A-5} + \frac{dy}{dx} x_{A-5}}{\frac{dy}{dx} - t} \tag{21}$$

and
$$y_A = \frac{1}{2} B + tx_A \tag{22}$$

Now:
$$\Delta x = x_A - x_{A-5} \tag{23}$$

$$\Delta y = y_A - y_{A-5} \tag{24}$$

Equation (18a) can be written as:

$$\frac{\Delta v}{\Delta u} = G_1 + \frac{G_3}{\Delta u} \tag{25}$$

where
$$G_1 = \frac{\frac{uv}{gd \cos\beta} + \sqrt{F' - 1}}{1 - \frac{v^2}{gd \cos\beta}} \tag{26}$$

and $$G_3 = \frac{-\frac{u}{d}\tan\beta\ (\Delta y)}{1 - \frac{v^2}{gd\ \cos\beta}} \tag{27}$$

(Δy is known from equation (24))

The boundary conditions for v_A and u_A can be written as:

$$v_A = t\ u_A$$

$$v_{A-5} + \Delta u = t\ (u_{A-5} + \Delta u) \quad \text{or}$$

$$\Delta v = t\ \Delta u = tu_{A-5} - v_{A-5} \tag{28}$$

Equation (23) and (26) can now be solved for Δv and Δu resulting in:

$$\Delta u = \frac{G_3 - tu_{A-5} + v_{A-5}}{t - G_1} \tag{29}$$

and $$\Delta v = G_1 \Delta u + G_3 \tag{30}$$

At this point, the (approximate) values of u_A and v_A can be determined:

$$u_A = u_{A-5} + \Delta u$$

$$v_A = v_{A-5} + \Delta v$$

The depth at point A is found using the energy equation:

$$d_{A-5}\ \cos\beta + \Delta x\ \sin\beta + \frac{1}{2g}\ V_{A-5}^2 = d_A\ \cos\beta + \frac{1}{2g}\ V_A^2$$

$$d_A = d_{A-5} + \Delta x\ \tan\beta + \frac{(u_{A-5}^2 + v_{A-5}^2 - u_A^2 - v_A^2)}{2g\ \cos\beta} \tag{31}$$

The iteration process can now be repeated, using

$$u = \frac{1}{2}(u_{A-5} + u_A)$$

$$v = \frac{1}{2}(v_{A-5} + v_A)$$

and $$d = \frac{1}{2}(d_{A-5} + d_A)$$

until sufficient accuracy is achieved.

III. Points 11, 17, 23 etc.

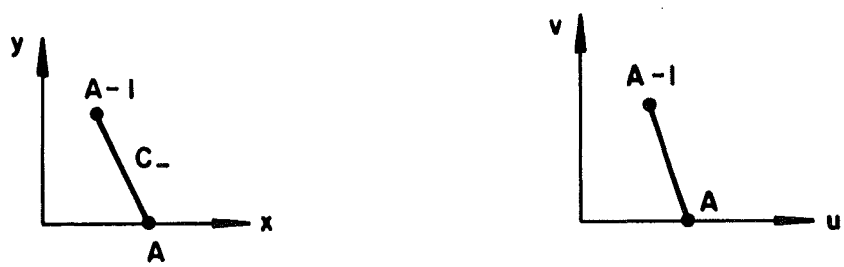

Figure 6

Using $u = u_{A-1}$, $v = v_{A-1}$ and $d = d_{A-1}$, $\frac{dy}{dx}$ can be determined from equation (17b). The (x,y) coordinates of point A will then become:

$$x_A = x_{A-1} - \frac{y_{A-1}}{\frac{dy}{dx}} \tag{32}$$

and $$y_A = 0 \tag{33}$$

$$\Delta x = x_A - x_{A-1} \tag{34}$$

$$\Delta y = y_A - y_{A-1} \tag{35}$$

With known values of u, v, d and Δy, equation (18b) can now be rewritten as:

$$\frac{\Delta v}{\Delta u} = G_2 + \frac{G_3}{\Delta u} \tag{36}$$

where

$$G_2 = \frac{\frac{uv}{gd\ \cos\beta} - \sqrt{F' - 1}}{1 - \frac{v^2}{gd\ \cos\beta}} \tag{37}$$

and G_3 is expressed in equation (27).

In this case the boundary conditions become:

$$v_A = 0 \tag{38}$$

or $$\Delta v = - v_{A-1} \tag{39}$$

Combining equations (36) and (39) gives:

$$\Delta u = - \frac{v_{A-1} + G_3}{G_2} \tag{40}$$

$$u_A = u_{A-1} + \Delta u$$

The depth d_A may now be found using an expression analogous to the one in equation (31) after which the iteration can continue using average values of u, v, and d between point (A) and point (A-1).

IV. Points 7-10, 13-16, etc.

Figure 7

$(\frac{dy}{dx})_p$ can be determined from equation (17a) using values of u, v and d (as u_p, v_p and d_p) corresponding to point (A-5). Similarly, $(\frac{dy}{dx})_n$ is determined from equation (17b) using values (u_n, v_n and d_n) from point (A-1).

Now, the (x,y) coordinates of A can be found as the intersection between the two lines:

$$y - y_{A-5} = (\frac{dy}{dx})_p \ (x - x_{A-5})$$

and
$$y - y_{A-1} = (\frac{dy}{dx})_n \ (x - x_{A-1})$$

This gives:

$$x_A = \frac{y_{A-1} - y_{A-5} + (\frac{dy}{dx})_p \ x_{A-5} - (\frac{dy}{dx})_n \ x_{A-1}}{(\frac{dy}{dx})_p - (\frac{dy}{dx})_n} \tag{41}$$

and
$$y_A = y_{A-5} + (\frac{dy}{dx})_p \ (x_A - x_{A-5}) \tag{42}$$

$$(\Delta x)_p = x_A - x_{A-5}$$

$$(\Delta y)_p = y_A - y_{A-5}$$

$$(\Delta x)_n = x_A - x_{A-1}$$

$$(\Delta y)_n = y_A - y_{A-1}$$

Introducing:

$$G_4 = \frac{\frac{u_p \ v_p}{gd_p \ \cos\beta} + \sqrt{F'_p - 1}}{1 - \frac{v_p^2}{gd_p \ \cos\beta}}$$

$$G_5 = -\frac{\frac{u_p}{d_p}\tan\beta\,(\Delta y)_p}{1 - \frac{v_p^2}{gd_p\cos\beta}}$$

$$G_6 = \frac{\frac{u_n v_n}{gd_n\cos\beta} - \sqrt{F'_n - 1}}{1 - \frac{v_n^2}{gd_n\cos\beta}} \qquad \text{and}$$

$$G_7 = -\frac{\frac{u_n}{d_n}\tan\beta\,(\Delta y)_n}{1 - \frac{v_n^2}{gd_n\cos\beta}}$$

equations (18a) and (18b) can be written as:

$$\frac{(\Delta v)_p}{(\Delta u)_p} = G_4 + \frac{G_5}{(\Delta u)_p} \tag{43}$$

and

$$\frac{(\Delta v)_n}{(\Delta u)_n} = G_6 + \frac{G_7}{(\Delta u)_n} \tag{44}$$

From geometry, the following additional relations are achieved:

$$(\Delta u)_p - (\Delta u)_n = u_{A-1} - u_{A-5} = G_8 \tag{45}$$

$$(\Delta v)_p - (\Delta v)_n = v_{A-1} - v_{A-5} = G_9 \tag{46}$$

Equations (43) through (46) can be solved giving:

$$(\Delta v)_p = \frac{G_8 G_4 - \frac{G_7 G_4}{G_6} - \frac{G_9 G_4}{G_6} + G_5}{1 - \frac{G_4}{G_6}} \tag{47}$$

and $$(\Delta u)_p = \frac{(\Delta v)_p - G_5}{G_4} \tag{48}$$

Now u_A and v_A can be expressed as:

$$u_A = u_{A-5} + (\Delta u)_p$$

and $$v_A = v_{A-5} + (\Delta v)_p$$

For d_A the expression in equation (31) can be used. The iteration can now continue, using the average values of u, v and d between A-5 and A for the C_+ characteristics (subscript p) or between A-1 and A for the C_- characteristics (subscript n)

5. EXAMPLES

The variables in a flow problem involving a straight flaring transition can be expressed in dimensionless form as:

the slope, β

the direction change, θ

the initial (modified) Froude No., F'_o

and the initial depth to width ratio, d_o/B

Since this paper is primarily concerned with the effects of the slope, constant values of the three remaining variables will be chosen for a series of examples.

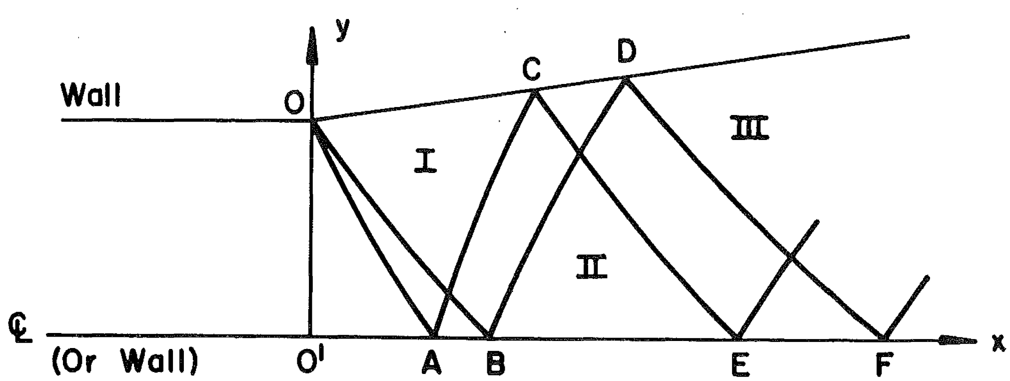

Figure 8

The most significant changes in the flow regime occur along the (expansion) wave generated at point 0 and its reflections. In the regions between the waves (regions, I, II and III) the flow variables remain approximately constant at any cross-section.

Figures 9 through 13 show that the effects of change in slope can be summarized as follows:

(1) With increasing slope the system of waves and reflections will be moved further downstream from the transition.

(2) The increase of the Froude Number and the decrease in depth of the flow in the transition will be more pronounced with increasing slope.

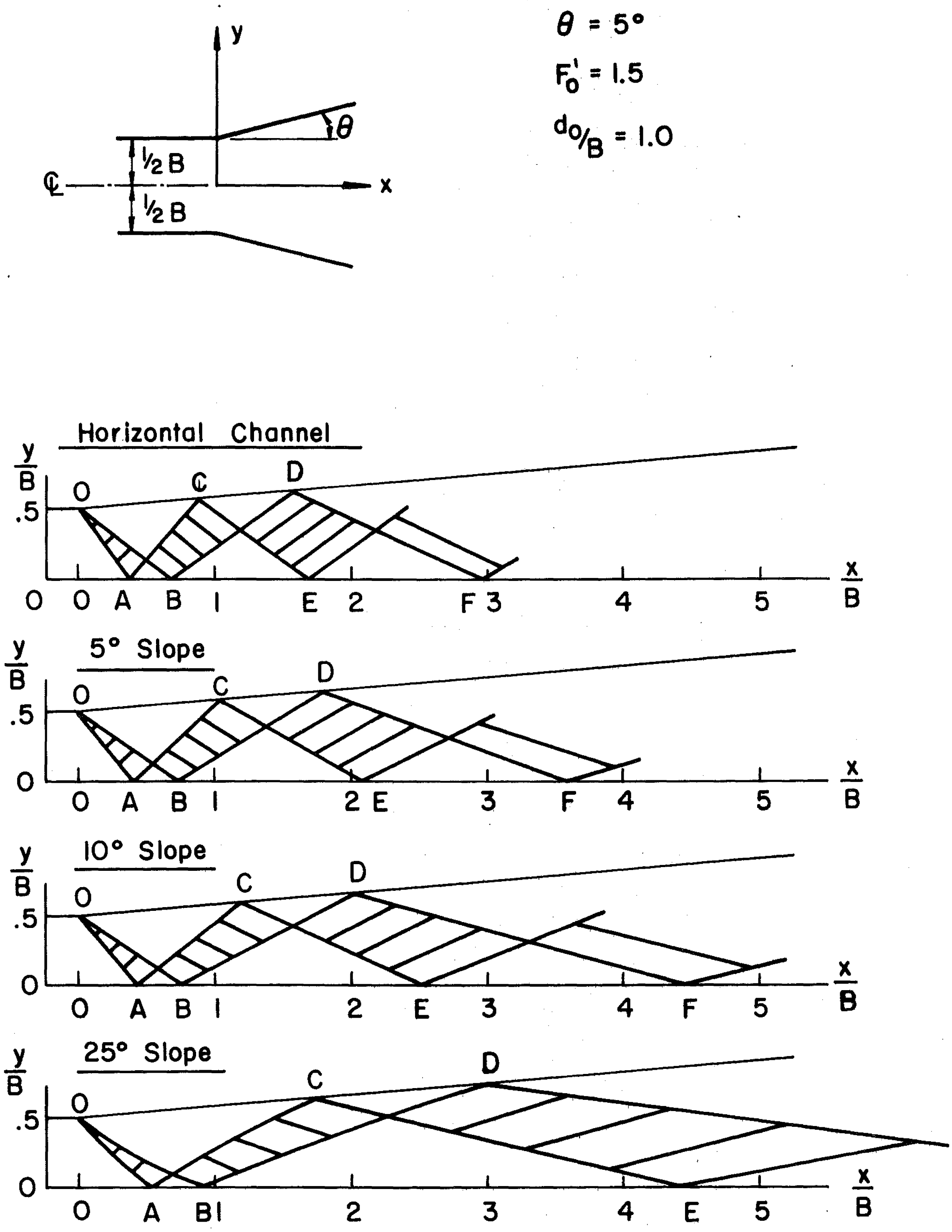

Figure 9 Examples, plots of transition

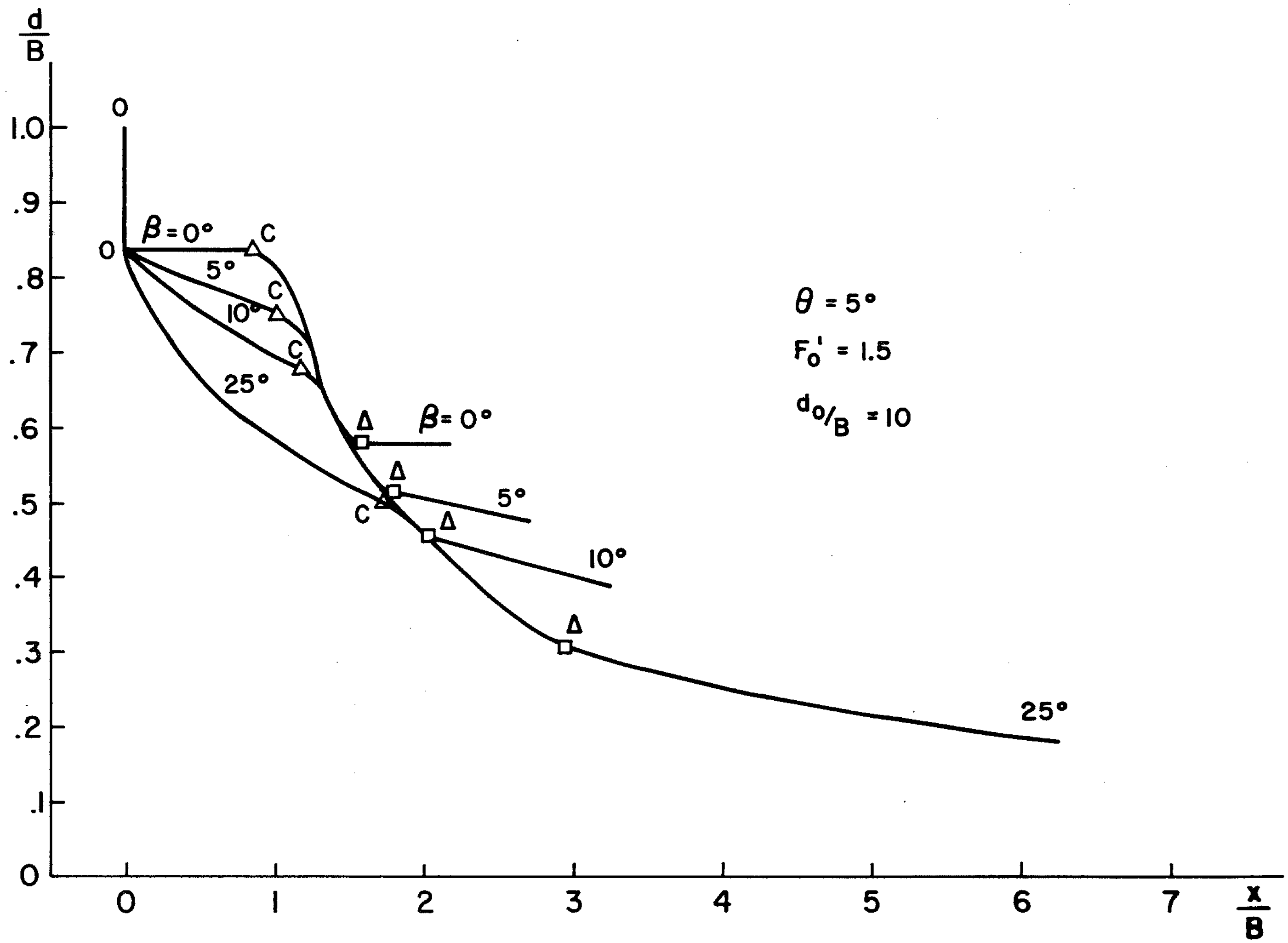

Figure 10 Depth at wall versus distance

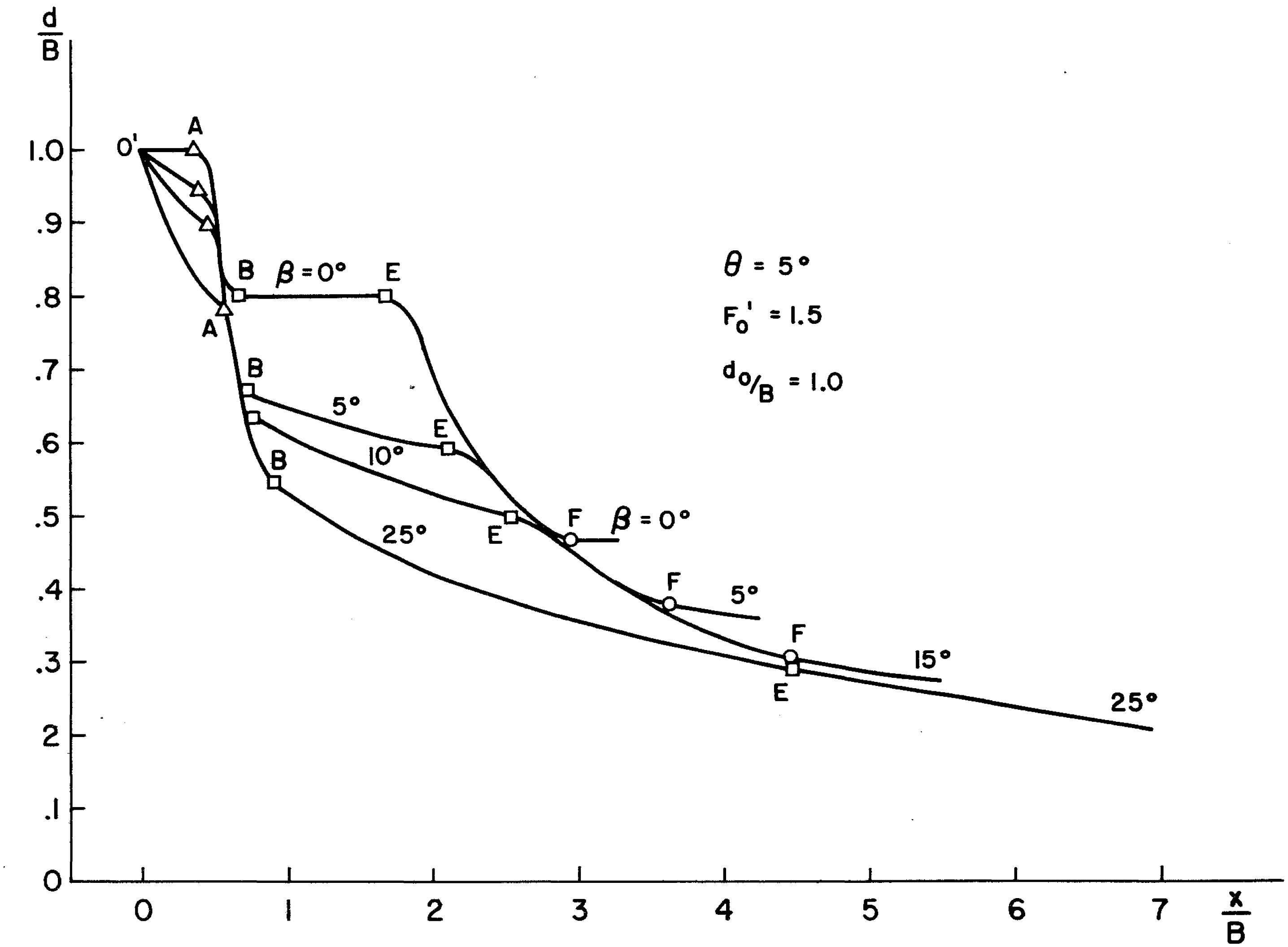

Figure 11 Depth at centerline versus distance

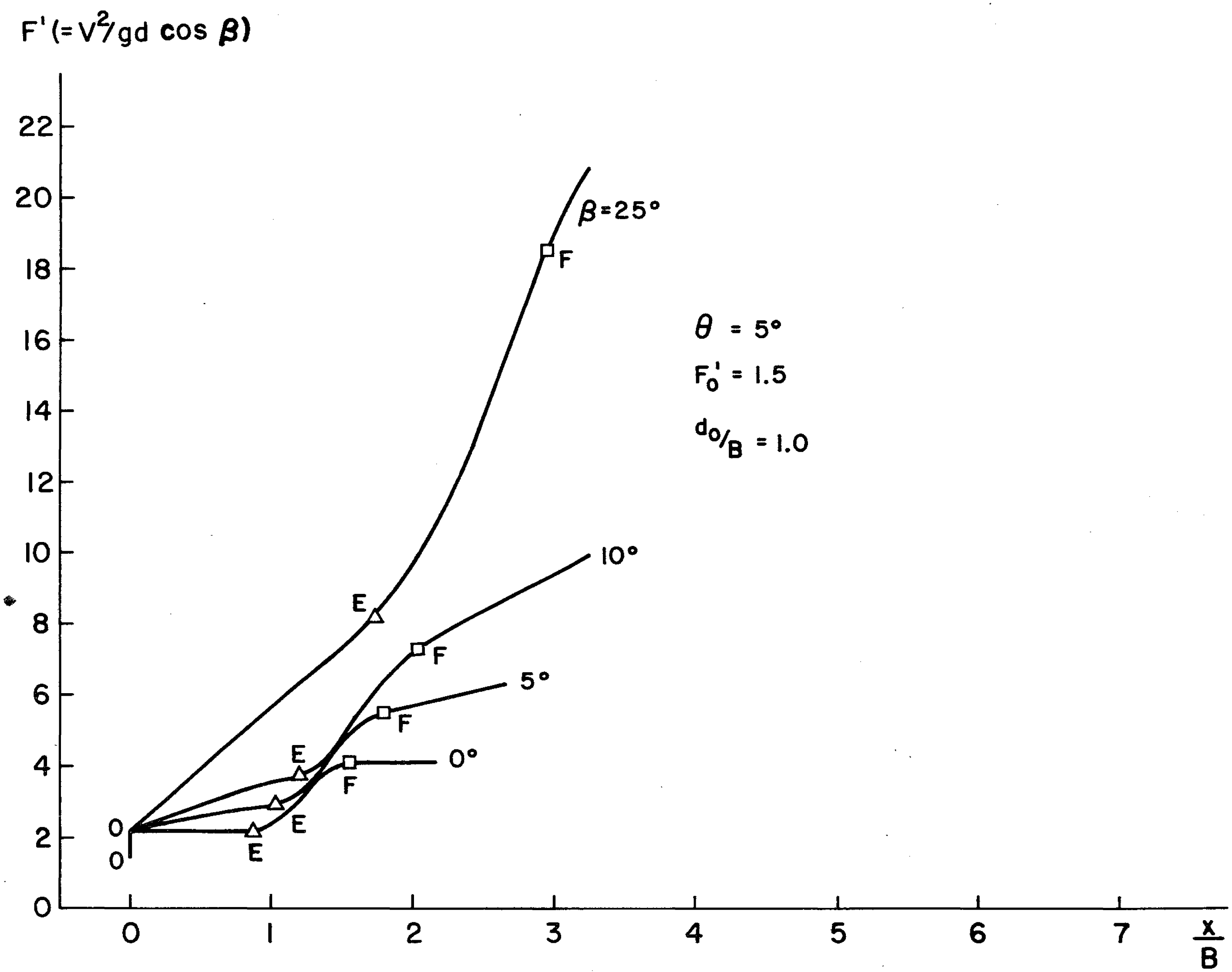

Figure 12 Froude No. (modified) at wall versus distance

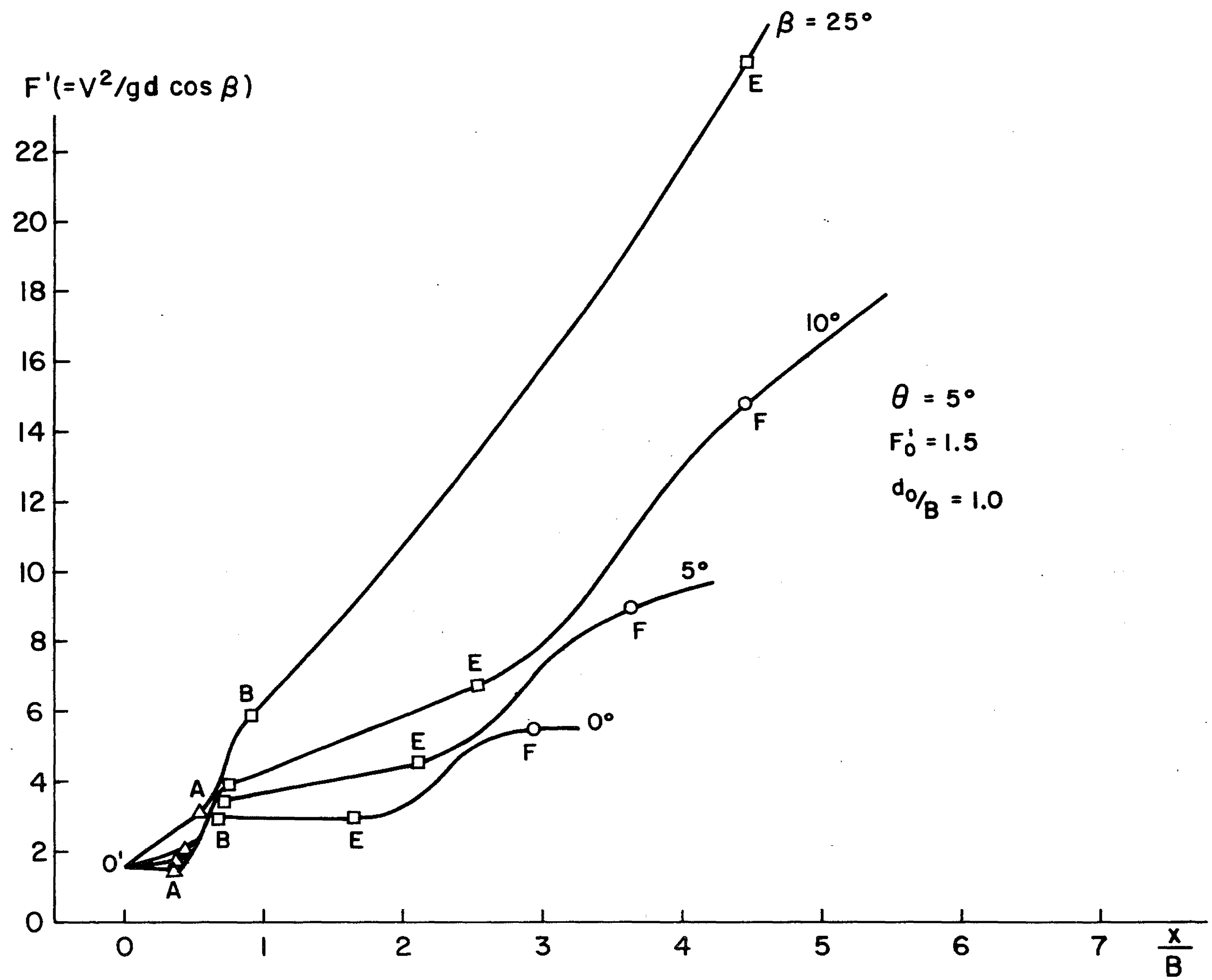

Figure 13 Froude No. (modified) at centerline versus distance

CONCLUSIONS

It is obvious from the preceding examples that even a moderate change in slope will have a significant effect on the supercritical flow in an open channel transition. Although the analysis presented in this paper is rather complicated for direct practical use it would be feasible to design a number of computer programs for various types of transitions, which could then be applied to actual flow problems.

The method is limited by the initial assumptions of frictionless and irrotational flow with hydrostatic pressure distribution. Future experimental studies would be needed to investigate the importance of these limitations.

APPENDIX

Computer Program

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    238  GUNNAR BAGGE  SUPERCRITICAL FLOW TRANSITIONS
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                                                                               PAGE 1.      JUL 29 66

 #     SEQ  LABL TYP                 STATEMENT                      C ZERO     NOT 0     PLUS      MINUS     ELSE

001.    10          D X[50],Y[50],U[50],V[50],D[50]                  [    ]    [    ]    [    ]    [    ]    [    ]
002.    20  ENDE CRDSLOPE,THETA,B,F,D[0]                             [    ]    [    ]    [    ]    [    ]    [    ]
003.    25       PL                                                  [    ]    [    ]    [    ]    [    ]    [    ]
004.    26       PVLSLOPE,THETA,B,F,D[0]                             [    ]    [    ]    [    ]    [    ]    [    ]
005.    30          T=SIN.[THETA*3,14159/180]/COS.[THETA*3,14159/C
   .    31          180]                                             [    ]    [    ]    [    ]    [    ]    [    ]
006.    40          BETA=SLOPE*3,14159/180                           [    ]    [    ]    [    ]    [    ]    [    ]
007.    50       PL                                                  [    ]    [    ]    [    ]    [    ]    [    ]
008.    60       PL         X/B            Y/B            U         C
   .    61                  V              D/B            F         C
   .    62                  A                                        [    ]    [    ]    [    ]    [    ]    [    ]
009.    70       PL                                                  [    ]    [    ]    [    ]    [    ]    [    ]
010.    80          A=0                                              [    ]    [    ]    [    ]    [    ]    [    ]
011.    90          X[0]=0,Y[0]=B/2,V[0]=0                           [    ]    [    ]    [    ]    [    ]    [    ]
012.    94          U[0]=SQRT.[32,2*D[0]*COS.[BETA]*F]               [    ]    [    ]    [    ]    [    ]    [    ]
013.    95  PRIN    F=[V[A]**2+U[A]**2]/[32,2*D[A]*COS.[BETA]]       [    ]    [    ]    [    ]    [    ]    [    ]
014.   100       PV X[A]/B,Y[A]/B,U[A],V[A],D[A]/B,F,A               [    ]    [    ]    [    ]    [    ]    [    ]
015.   105       PL                                                  [    ]    [    ]    [    ]    [    ]    [    ]
016.   110          A=A+1                                            [    ]    [    ]    [    ]    [    ]    [    ]
017.   120          A-5                                              [ AB ]    [    ]    [    ]    [ AB ]    [    ]
018.   130          A-6                                              [ CD ]    [    ]    [    ]    [    ]    [    ]
019.   140          A-12                                             [ CD ]    [    ]    [    ]    [    ]    [    ]
020.   150          A-18                                             [ CD ]    [    ]    [    ]    [    ]    [    ]
021.   160          A-24                                             [ CD ]    [    ]    [    ]    [    ]    [    ]
022.   170          A-30                                             [ CD ]    [    ]    [    ]    [    ]    [    ]
023.   175          A-36                                             [ CD ]    [    ]    [    ]    [    ]    [    ]
024.   176          A-42                                             [ CD ]    [    ]    [    ]    [    ]    [    ]
025.   177          A-48                                             [ CD ]    [    ]    [    ]    [    ]    [    ]
026.   180          A-11                                             [ EF ]    [    ]    [    ]    [    ]    [    ]
027.   190          A-17                                             [ EF ]    [    ]    [    ]    [    ]    [    ]
028.   200          A-23                                             [ EF ]    [    ]    [    ]    [    ]    [    ]
029.   210          A-29                                             [ EF ]    [    ]    [    ]    [    ]    [    ]
030.   220          A-35                                             [ EF ]    [    ]    [    ]    [    ]    [    ]
031.   222          A-41                                             [ EF ]    [    ]    [    ]    [    ]    [    ]
032.   224          A-47                                             [ EF ]    [    ]    [    ]    [    ]    [    ]
033.   230          A-50                                             [ENDE]    [    ]    [    ]    [    ]    [    ]
034.   240          UP=U[A-5],VP=V[A-5],DP=D[A-5]                    [    ]    [    ]    [    ]    [    ]    [    ]
035.   250          UN=U[A-1],VN=V[A-1],DN=D[A-1]                    [    ]    [    ]    [    ]    [    ]    [    ]
036.   260   GH     S=UP,R=VP,C=VN,P=DP,O=DN                         [    ]    [    ]    [    ]    [    ]    [    ]
037.   270          GP=32,2*DP*COS.[BETA]                            [    ]    [    ]    [    ]    [    ]    [    ]
038.   280          GN=GP*DN/DP                                      [    ]    [    ]    [    ]    [    ]    [    ]
039.   290          FP=[UP*UP+VP*VP]/GP                              [    ]    [    ]    [    ]    [    ]    [    ]
040.   300          FN=[UN*UN+VN*VN]/GN                              [    ]    [    ]    [    ]    [    ]    [    ]
041.   310          DYDXP=[-UP*VP/GP-SQRT.[FP-1]]/[1-UP*UP/GP]       [    ]    [    ]    [    ]    [    ]    [    ]
042.   320          DYDXN=[-UN*VN/GN+SQRT.[FN-1]]/[1-UN*UN/GN]       [    ]    [    ]    [    ]    [    ]    [    ]
043.   330          X[A]=[Y[A-1]-Y[A-5]+DYDXP*X[A-5]-DYDXN*X[A-1]C
   .   331          ]/[DYDXP-DYDXN]                                  [    ]    [    ]    [    ]    [    ]    [    ]
044.   340          Y[A]=[X[A-1]-X[A-5]+Y[A-5]/DYDXP-Y[A-1]/DYDXNC
   .   341          ]/[1/DYDXP-1/DYDXN]                              [    ]    [    ]    [    ]    [    ]    [    ]
045.   350          DYP=Y[A]-Y[A-5],DXP=X[A]-X[A-5]                  [    ]    [    ]    [    ]    [    ]    [    ]
046.   360          DYN=Y[A]-Y[A-1]                                  [    ]    [    ]    [    ]    [    ]    [    ]
047.   370          C1=[UP*VP/GP+SQRT.[FP-1]]/[1-VP*VP/GP]           [    ]    [    ]    [    ]    [    ]    [    ]
048.   375          C2=-UP/DP*SIN.[BETA]/COS.[BETA]*DYP/[1-VP*VP/C
   .   376          GP]                                              [    ]    [    ]    [    ]    [    ]    [    ]
049.   380          C3=[UN*VN/GN-SQRT.[FN-1]]/[1-VN*VN/GN]           [    ]    [    ]    [    ]    [    ]    [    ]
```

```
  #      SEQ  LABL TYP             STATEMENT                                  C ZERO     NOT 0      PLUS      MINUS      ELSE

050.     385            C4=-UN/DN*SIN.[BETA]/COS.[BETA]*DYN/[1-VN*VN/C
   .     386            GN]                                               [      ]  [      ]  [      ]  [      ]  [      ]
051.     390            C5=U[A-1]-U[A-5]                                  [      ]  [      ]  [      ]  [      ]  [      ]
052.     395            C6=V[A-1]-V[A-5]                                  [      ]  [      ]  [      ]  [      ]  [      ]
053.     400            DVP=[C5*C1-C4*C1/C3-C6*C1/C3+C2]/[1-C1/C3]        [      ]  [      ]  [      ]  [      ]  [      ]
054.     402            DVN=DVP-C6                                        [      ]  [      ]  [      ]  [      ]  [      ]
055.     404            DUP=[DVP-C2]/C1                                   [      ]  [      ]  [      ]  [      ]  [      ]
056.     406            DUN=[DVN-C4]/C3                                   [      ]  [      ]  [      ]  [      ]  [      ]
057.     408            U[A]=U[A-5]+DUP,V[A]=V[A-5]+DVP                   [      ]  [      ]  [      ]  [      ]  [      ]
058.     410            D[A]=D[A-5]+DXP*SIN.[BETA]/COS.[BETA]+[U[A-5]C
   .     411            **2+V[A-5]**2-U[A]**2-V[A]**2]/[64.4*COS.[BETC
   .     412            A]]                                               [      ]  [      ]  [      ]  [      ]  [      ]
059.     440            DDP=D[A]-D[A-5],DDN=D[A]-D[A-1]                   [      ]  [      ]  [      ]  [      ]  [      ]
060.    450             UP=U[A-5]+DUP/2,UN=U[A-1]+DUN/2                   [      ]  [      ]  [      ]  [      ]  [      ]
061.     460            VP=V[A-5]+DVP/2,VN=V[A-1]+DVN/2                   [      ]  [      ]  [      ]  [      ]  [      ]
062.     470            DP=D[A-5]+DDP/2,DN=D[A-1]+DDN/2                   [      ]  [      ]  [      ]  [      ]  [      ]
063.     486            ABS.[[O-DN]/DN]-.0005                             [      ]  [      ]  [  GH  ]  [      ]  [      ]
064.     487            ABS.[[P-DP]/DP]-.0005                             [      ]  [      ]  [  GH  ]  [      ]  [      ]
065.     488            ABS.[[Q-VN]/VN]-.0005                             [      ]  [      ]  [  GH  ]  [      ]  [      ]
066.     489            ABS.[[R-VP]/VP]-.0005                             [      ]  [      ]  [  GH  ]  [      ]  [      ]
067.     490            ABS.[[S-UP]/UP]-.0005                             [PRIN]    [      ]  [  GH  ]  [PRIN]    [      ]
068.     500   CD       L=U[A-5],M=V[A-5],N=D[A-5]                        [      ]  [      ]  [      ]  [      ]  [      ]
069.     510   IJ       S=L,R=M,Q=N                                       [      ]  [      ]  [      ]  [      ]  [      ]
070.     520            GD=32.2*N*COS.[BETA]                              [      ]  [      ]  [      ]  [      ]  [      ]
071.     530            F=[L*L+M*M]/GD                                    [      ]  [      ]  [      ]  [      ]  [      ]
072.     540            DYDX=[-L*M/GD-SQRT.[F-1]]/[1-L*L/GD]              [      ]  [      ]  [      ]  [      ]  [      ]
073.     550            X[A]=[B/2-Y[A-5]+DYDX*X[A-5]]/[DYDX-T]            [      ]  [      ]  [      ]  [      ]  [      ]
074.     560            Y[A]=B/2+T*X[A]                                   [      ]  [      ]  [      ]  [      ]  [      ]
075.     570            DY=Y[A]-Y[A-5],DX=X[A]-X[A-5]                     [      ]  [      ]  [      ]  [      ]  [      ]
076.     580            C1=[L*M/GD+SQRT.[F-1]]/[1-M*M/GD]                 [      ]  [      ]  [      ]  [      ]  [      ]
077.     585            C2=-L/N*SIN.[BETA]/COS.[BETA]*DY/[1-M*M/GD]       [      ]  [      ]  [      ]  [      ]  [      ]
078.     590            DU=[C2-T*U[A-5]+V[A-5]]/[T-C1]                    [      ]  [      ]  [      ]  [      ]  [      ]
079.     600            DV=C1*DU+C2                                       [      ]  [      ]  [      ]  [      ]  [      ]
080.     610            V[A]=V[A-5]+DV,U[A]=U[A-5]+DU                     [      ]  [      ]  [      ]  [      ]  [      ]
081.     620            D[A]=D[A-5]+DX*SIN.[BETA]/COS.[BETA]+[U[A-5]*C
   .     621            *2+V[A-5]**2-U[A]**2-V[A]**2]/[64.4*COS.[BETAC
   .     622            ]]                                                [      ]  [      ]  [      ]  [      ]  [      ]
082.     630            DD=D[A]-D[A-5]                                    [      ]  [      ]  [      ]  [      ]  [      ]
083.     640            L=U[A-5]+DU/2,M=V[A-5]+DV/2                       [      ]  [      ]  [      ]  [      ]  [      ]
084.     650            N=D[A-5]+DD/2                                     [      ]  [      ]  [      ]  [      ]  [      ]
085.     656            ABS.[[R-M]/M]-.0005                               [      ]  [      ]  [  IJ  ]  [      ]  [      ]
086.     658            ABS.[[Q-N]/N]-.0005                               [      ]  [      ]  [  IJ  ]  [      ]  [      ]
087.     660            ABS.[[S-L]/L]-.0005                               [PRIN]    [      ]  [  IJ  ]  [PRIN]    [      ]
088.     670   EF       L=U[A-1],M=V[A-1],N=D[A-1]                        [      ]  [      ]  [      ]  [      ]  [      ]
089.     680   KL       S=L,R=M,Q=N                                       [      ]  [      ]  [      ]  [      ]  [      ]
090.     690            GD=32.2*N*COS.[BETA]                              [      ]  [      ]  [      ]  [      ]  [      ]
091.     700            F=[L*L+M*M]/GD                                    [      ]  [      ]  [      ]  [      ]  [      ]
092.     710            DYDX=[-L*M/GD+SQRT.[F-1]]/[1-L*L/GD]              [      ]  [      ]  [      ]  [      ]  [      ]
093.     720            X[A]=X[A-1]-Y[A-1]/DYDX                           [      ]  [      ]  [      ]  [      ]  [      ]
094.     730            Y[A]=0.0                                          [      ]  [      ]  [      ]  [      ]  [      ]
095.     740            DX=X[A]-X[A-1],DY=Y[A]-Y[A-1]                     [      ]  [      ]  [      ]  [      ]  [      ]
096.     750            C1=[L*M/GD-SQRT.[F-1]]/[1-M*M/GD]                 [      ]  [      ]  [      ]  [      ]  [      ]
097.     760            C2=-L/N*SIN.[BETA]/COS.[BETA]*DY/[1-M*M/GD]       [      ]  [      ]  [      ]  [      ]  [      ]
098.     770            V[A]=0.0                                          [      ]  [      ]  [      ]  [      ]  [      ]
099.     771            DV=V[A]-V[A-1]                                    [      ]  [      ]  [      ]  [      ]  [      ]
100.     772            DU=[DV-C2]/C1                                     [      ]  [      ]  [      ]  [      ]  [      ]
```

#	SEQ	LABL	TYP	STATEMENT	C ZERO	NOT 0	PLUS	MINUS	ELSE
101.	774			U[A]=U[A-1]+DU	[ ]	[ ]	[ ]	[ ]	[ ]
102.	780			D[A]=D[A-1]+DX*SIN,[BETA]/COS,[BETA]+[U[A-1]*C					
.	781			*2+V[A-1]**2-U[A]**2]/[64,4*COS,[BETA]]	[ ]	[ ]	[ ]	[ ]	[ ]
103.	790			DD=D[A]-D[A-1]	[ ]	[ ]	[ ]	[ ]	[ ]
104.	810			L=U[A-1]+DU/2,M=V[A-1]+DV/2	[ ]	[ ]	[ ]	[ ]	[ ]
105.	820			N=D[A-1]+DD/2	[ ]	[ ]	[ ]	[ ]	[ ]
106.	826			ABS,[[Q-N]/N]-,0005	[ ]	[ ]	[KL]	[ ]	[ ]
107.	828			ABS,[[R-M]/M]-,0005	[ ]	[ ]	[KL]	[ ]	[ ]
108.	830			ABS,[[S-L]/L]-,0005	[PRIN]	[ ]	[KL]	[PRIN]	[ ]
109.	840	AB		L=U[A-1],N=D[A-1]	[ ]	[ ]	[ ]	[ ]	[ ]
110.	850	MN		S=L,R=N,DY=-B/8	[ ]	[ ]	[ ]	[ ]	[ ]
111.	854			A=2	[ ]	[ ]	[BA]	[BU]	[ ]
112.	856			DY=-B/8+B/300	[ ]	[ ]	[ ]	[ ]	[BA]
113.	858	BU		DY=-B/300	[ ]	[ ]	[ ]	[ ]	[ ]
114.	860	BA		F=L*L/[32,2*N*COS,[BETA]]	[ ]	[ ]	[ ]	[ ]	[ ]
115.	870			DU=-L*SIN,[BETA]/COS,[BETA]/N*DY/SQRT,[F-1]	[ ]	[ ]	[ ]	[ ]	[ ]
116.	880			U[A]=U[A-1]+DU	[ ]	[ ]	[ ]	[ ]	[ ]
117.	890			D[A]=U[A-1]/U[A]*D[A-1]	[ ]	[ ]	[ ]	[ ]	[ ]
118.	900			DD=D[A]-D[A-1]	[ ]	[ ]	[ ]	[ ]	[ ]
119.	910			L=U[A-1]+DU/2	[ ]	[ ]	[ ]	[ ]	[ ]
120.	920			N=D[A-1]+DD/2	[ ]	[ ]	[ ]	[ ]	[ ]
121.	928			ABS,[[R-N]/N]-,0005	[ ]	[ ]	[MN]	[ ]	[ ]
122.	930			ABS,[[S-L]/L]-,0005	[ ]	[ ]	[MN]	[ ]	[ ]
123.	940			DX=-DY*SQRT,[F-1]	[ ]	[ ]	[ ]	[ ]	[ ]
124.	950			X[A]=X[A-1]+DX	[ ]	[ ]	[ ]	[ ]	[ ]
125.	960			Y[A]=Y[A-1]+DY	[ ]	[ ]	[ ]	[ ]	[ ]
126.	970			V[A]=0.0	[ ]	[ ]	[ ]	[ ]	[PRIN]
127.	980		END	$END OF PROGRAM	[ ]	[ ]	[ ]	[ ]	[ ]

REFERENCES

(1) Von Karman, T.
EINE PRAKTISCHE ANWENDUNG DER ANALOGIE ZWISCHEN ÜBERSCHALLSTRÖMUNG IN GASEN UND ÜBERKRITISCHER STRÖMUNG IN OFFENEN GERINNEN (A Practical Application of the Analogy Between Supersonic Flow in Gases and Supercritical Flow in Open Channels) Zeitschrift Von Angewandte Mathematik Und Mechanik, Vol. 18, No. 1, pp. 49-56, February 1938.

(2) Preiswerk, E.
APPLICATION OF THE METHODS OF GAS DYNAMICS TO WATER FLOWS WITH FREE SURFACE, PART I. FLOWS WITH NO ENERGY DISSIPATION, Institut für Aerodynamik Eidgenössische Technische Hochschule, Zürich, Translated as Technical Memorandum No. 934, National Advisory Committee for Aeronautics, Washington, D. C., March 1940.

(3) Blaisdell, F. W.
GRAPHICAL CONSTRUCTION OF THE FLOW IN TRANSITION BY THE METHOD OF CHARACTERISTICS (AFTER PREISWERK), U. S. Dept. Agr., Soil Conservation Service, Minneapolis, Minn., May 1944, revised February 1946.

(4) Ippen, Arthur T.
MECHANICS OF SUPERCRITICAL FLOW, 1st Paper in High-Velocity Flow in Open Channels: A Symposium, Transactions, American Society of Civil Engineers, Vol. 116, pp. 268-295, 1951

(5) Knapp, Robert T.
DESIGN OF CHANNEL CURVES FOR SUPERCRITICAL FLOW, 2nd Paper in High-Velocity Flow in Open Channels: A Symposium, Transactions, American Society of Civil Engineers, pp. 296-325, 1951

(6) Ippen, Arthur T. and Dawson, John H.
DESIGN OF CHANNEL CONTRACTIONS, 3rd Paper in High-Velocity Flow in Open Channels: A Symposium, Transactions, American Society of Civil Engineers, pp. 326-346, 1951.

(7) Rouse, Hunter, Bhoota, B. V. and Hsu, En-Yun
DESIGN OF CHANNEL EXPANSIONS, 4th Paper in High-Velocity Flow in Open Channels: A Symposium, Transactions, American Society of Civil Engineers, pp. 347-363, 1951.

REFERENCES (continued)

(8) Owczarek, J. A.
FUNDAMENTALS OF GAS DYNAMICS, pp. 278-290, International Textbook Company, Scranton, Pa., 1964.